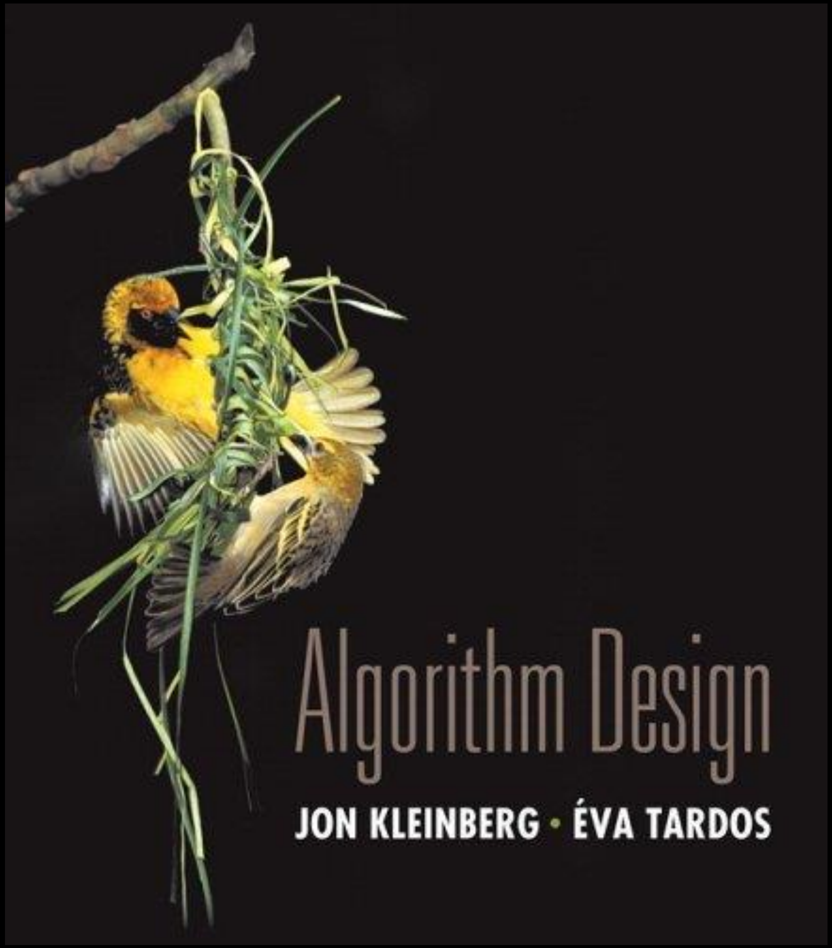


Chapter 4

Greedy Algorithms

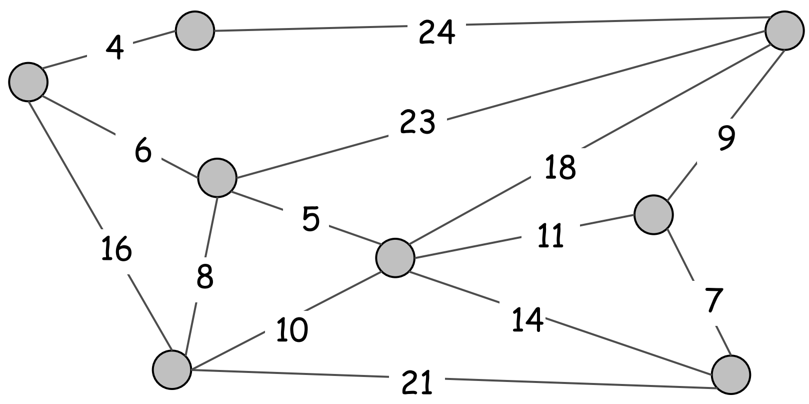
Slides by Kevin Wayne.

4.5 Minimum Spanning Tree

Minimum Spanning Tree

Minimum spanning tree. Given a connected graph $G = (V, E)$ with real-valued edge **positive** weights c_e, find a subset $E' \subseteq E$ such that

(i) the graph $G'=(V,E')$ is connected
(ii) smallest possible cost



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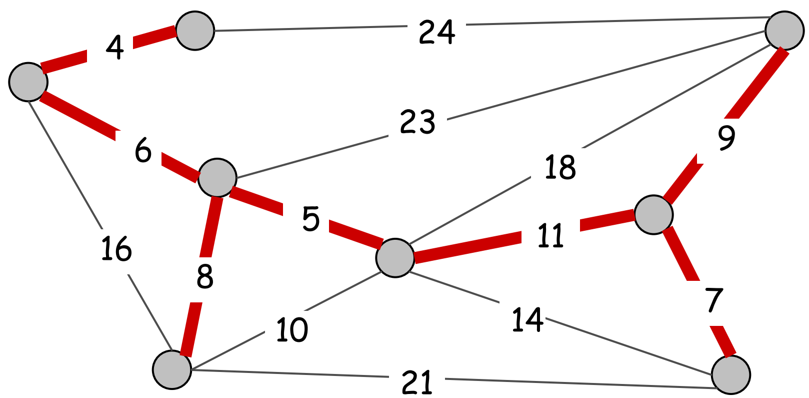


Key Observation. The optimal solution does not contain cycles => it is a tree

Applications

MST is fundamental problem with diverse applications.

- Network design.
  - telephone, electrical, hydraulic, TV cable, computer, road
- Approximation algorithms for NP-hard problems.
  - traveling salesperson problem, Steiner tree
- Indirect applications.
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - learning salient features for real-time face verification
  - reducing data storage in sequencing amino acids in a protein
  - model locality of particle interactions in turbulent fluid flows
  - autoconfig protocol for Ethernet bridging to avoid cycles in a network
- Cluster analysis.

Greedy Algorithms

Kruskal's algorithm. Start with $T = \phi$. Consider edges in ascending order of cost. Insert edge e in T unless doing so would create a cycle.

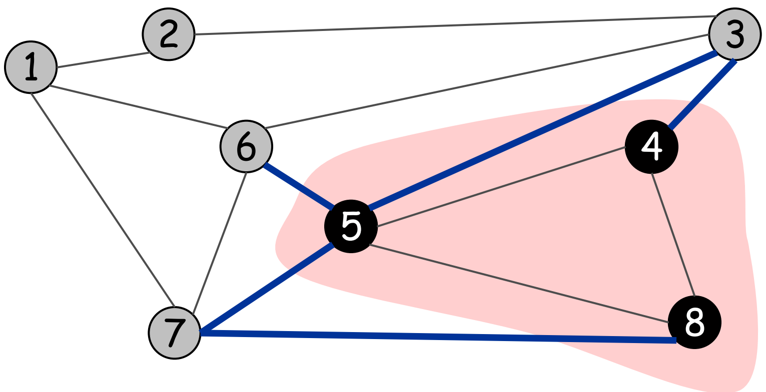
Reverse-Delete algorithm. Start with $T = E$. Consider edges in descending order of cost. Delete edge e from T unless doing so would disconnect T.

Prim's algorithm. Start with some root node s and greedily grow a tree T from s outward. At each step, add the cheapest edge e to T that has exactly one endpoint in T.

Remark. All three algorithms produce an MST.

Cycles and Cuts

Cut. A cut for a graph G=(V,E) is a subset of nodes S.



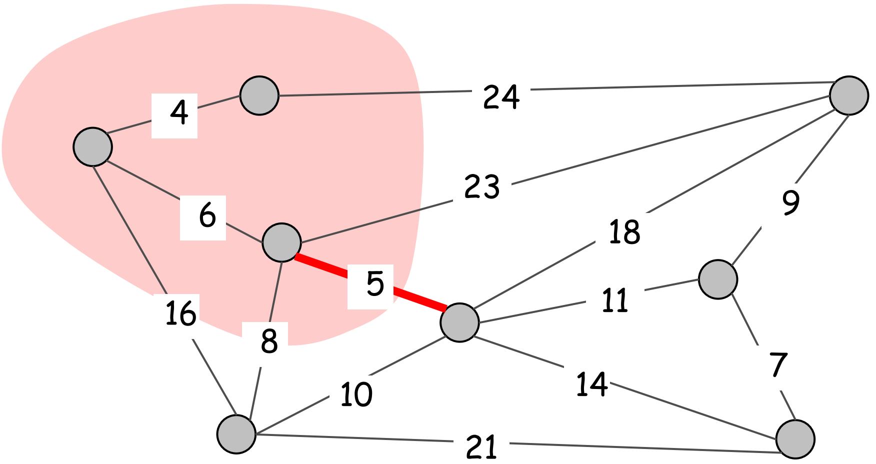
Cut S = { 4, 5, 8 }
Crossing edges = 5-6, 5-7, 3-4, 3-5, 7-8

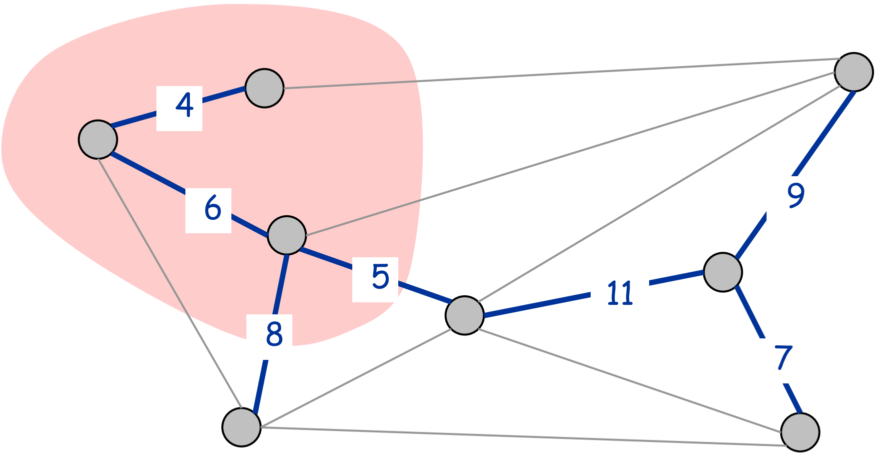
- An edge **e** crosses a cut S if **e** has an edpoint in S and the other one in V-S

Greedy Algorithms

To simplify, assume all edge weights are different => there is unique MST

Cut property (baby version). Consider a graph G. Pick any cut S. If **e** is the lightest edge that crosses S, then **e** belongs to the MST





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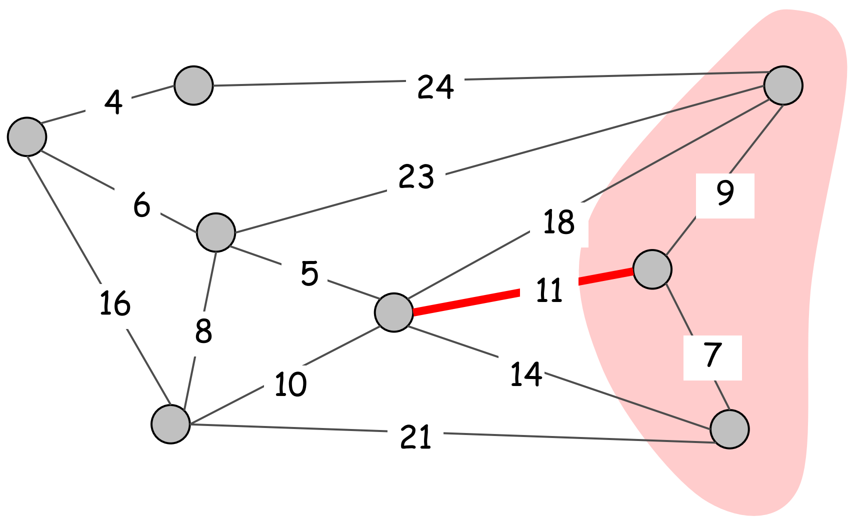
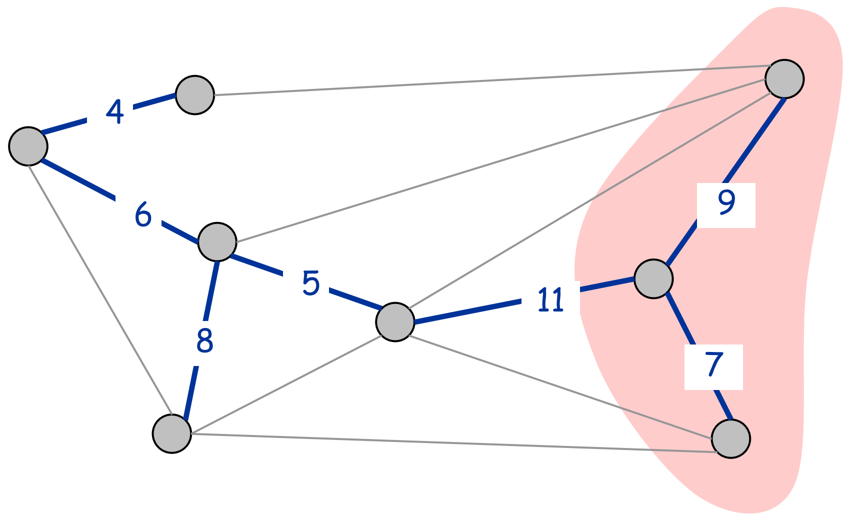




Greedy Algorithms

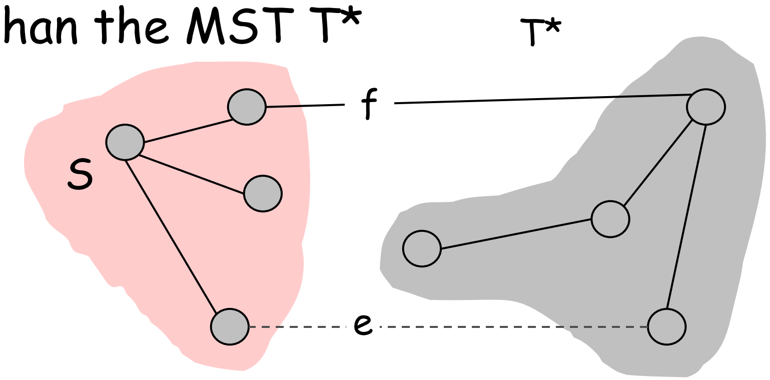
To simplify, assume all edge weights are different => there is unique MST

Cut property (baby version). Consider a graph G. Pick any cut S. If **e** is the lightest edge that crosses S, then **e** belongs to the MST

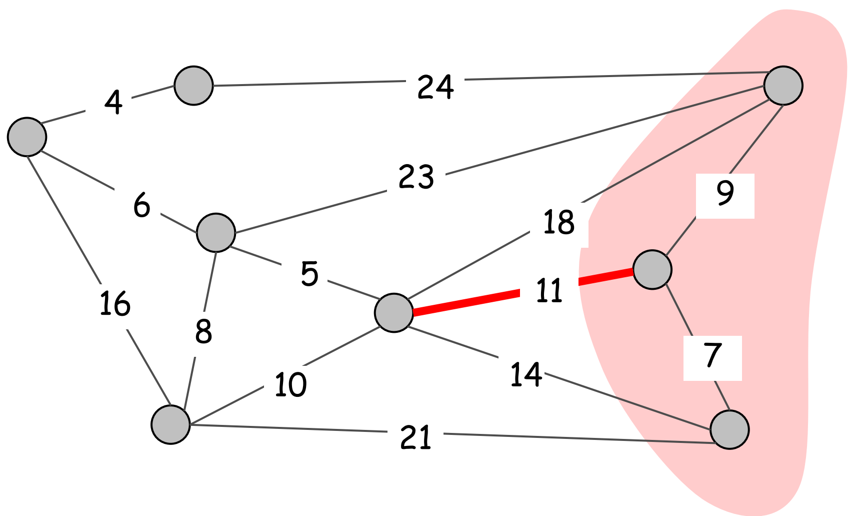
Pf. (exchange argument)

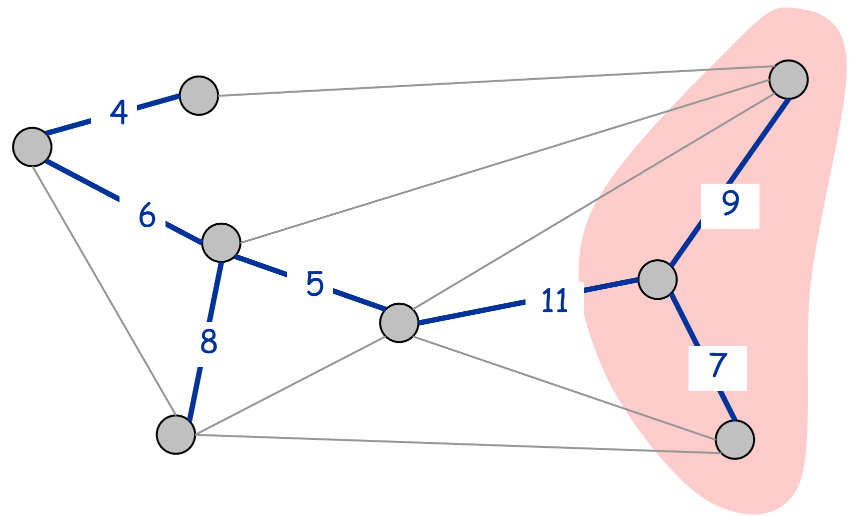
- Consider the MST T*
- Suppose lightest edge **e** does not belong to T* => there is another edge **f** connecting cut to outside of the cut
- Adding **e** to T* creates a cycle containing **f**
- $T' = T^* \cup \{e\} - \{f\}$ is also a spanning tree
- Since $c_e < c_f$, the new tree T' is cheaper than the MST T*

  => contradiction



This property help us to start building the MST: just look at any cut, add the lightest edge to the solution



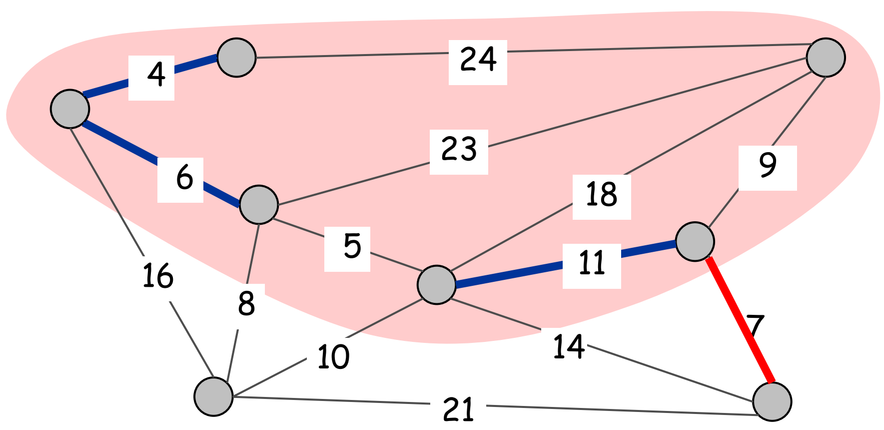


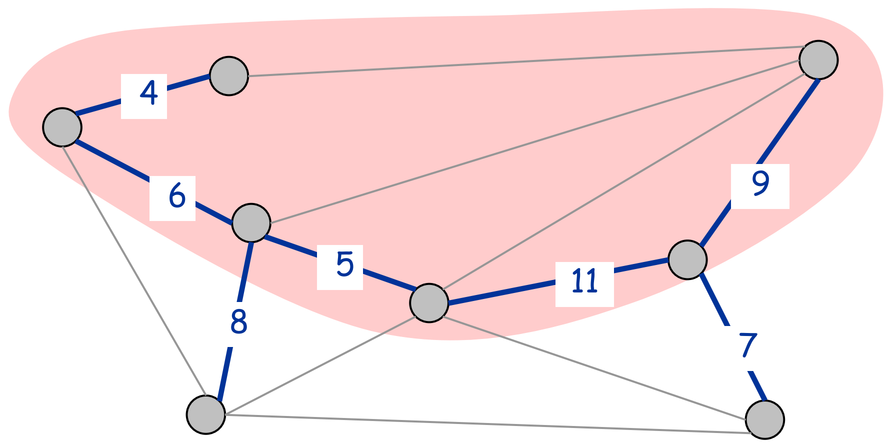
We will prove a stronger version the allows us to continue this process to add other edges until we get the MST

Greedy Algorithms

Cut property. Suppose you have already found a set of edges X that belongs to the MST

Pick a cut S containing all edges in X. If **e** is the lightest edge that crosses S then **e** also belongs to the MST





Greedy Algorithms

Cut property. Suppose you have already found a set of edges X that belongs to the MST

Pick a cut S containing all edges in X. If **e** is the lightest edge that crosses S then **e** also belongs to the MST

Proof: Exactly the same as the "baby version"

Cut property.

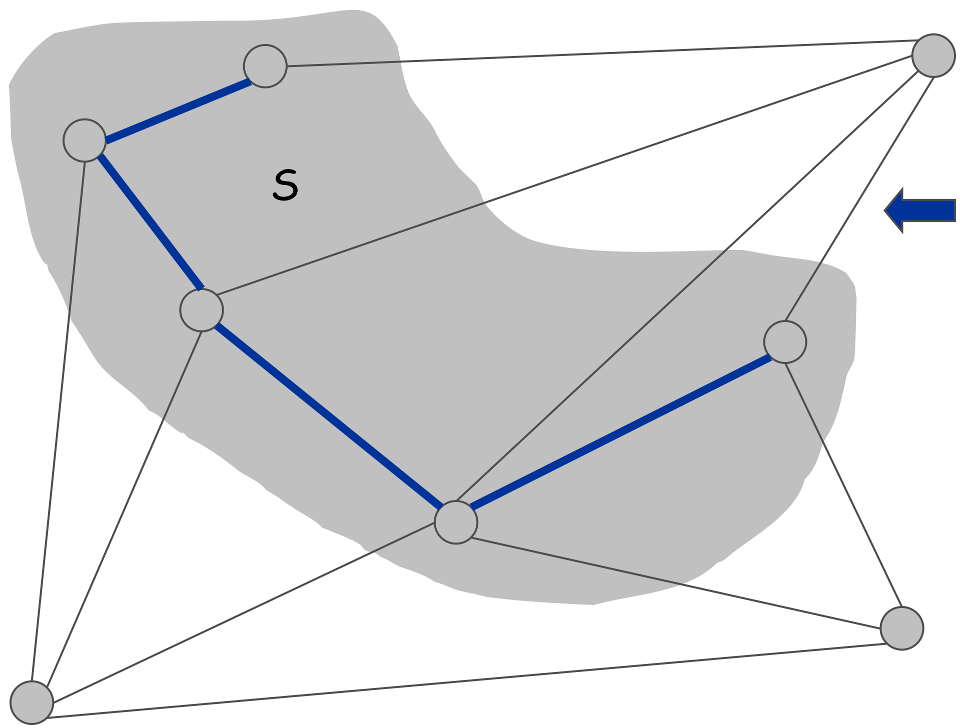
- Different applications of Cut property lead to different algorithms for constructing Minimal Spanning Trees.
- Prim and Kruskal algorithm construct a MST applying the Cut property n-1 times.

Prim's Algorithm

Idea: keep growing the same cut

Prim's algorithm. [Jarník 1930, Dijkstra 1957, Prim 1959]

- Initialize S = any node, tree T = empty
- Apply cut property to S.
- Add min cost edge that crosses S to T, and add one new explored node u to S.



Bad Implementation: Prim's Algorithm

Implementation (Naïve)

- Maintain set of explored nodes S.
- Find the lightest edge that crosses S in O(m) time
- Total complexity O(m.n)

Good Implementation: Prim's Algorithm

Implementation. Use a priority queue.
- Maintain set of explored nodes S.
- For each unexplored node v, maintain attachment cost a[v] = cost of cheapest edge v to a node in S.
- $O(n^2)$ with an array; O(m log n) with a binary heap.

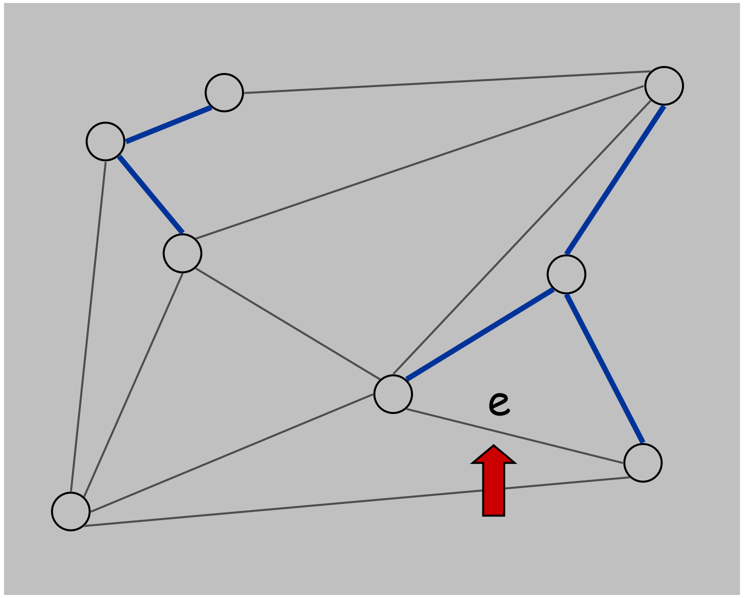
```
Prim(G, c) {
   foreach (v ∈ V) a[v] ← ∞
   Initialize an empty priority queue Q
   foreach (v ∈ V) insert v onto Q
   Initialize set of explored nodes S ← φ

   while (Q is not empty) {
      u ← delete min element from Q
      S ← S ∪ { u }
      foreach (edge e = (u, v) incident to u)
          if ((v ∉ S) and (c_e < a[v]))
             decrease priority a[v] to c_e
}
```

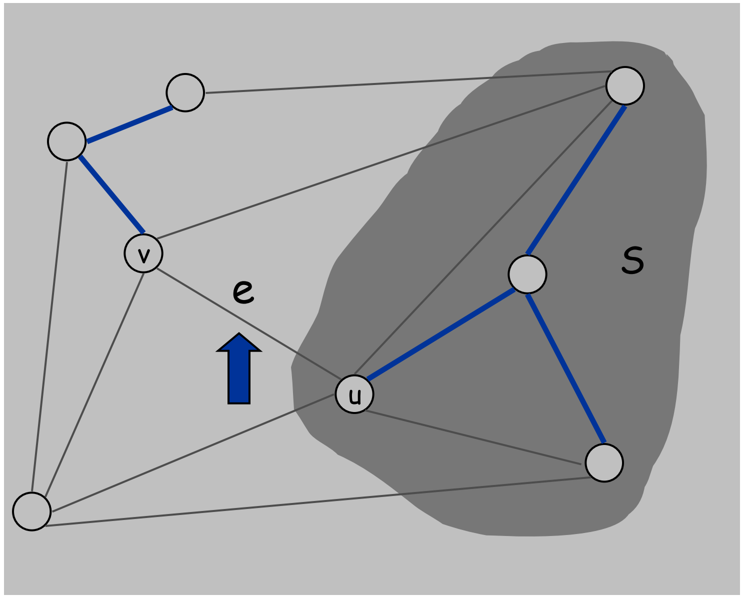
Kruskal's Algorithm

Kruskal's algorithm. [Kruskal, 1956]

- Consider edges in ascending order of weight.
- Case 1: If adding e to T creates a cycle, discard e
- Case 2: Otherwise, insert e = (u, v) into T
  (set S to be the connected component containing u)



Case 1

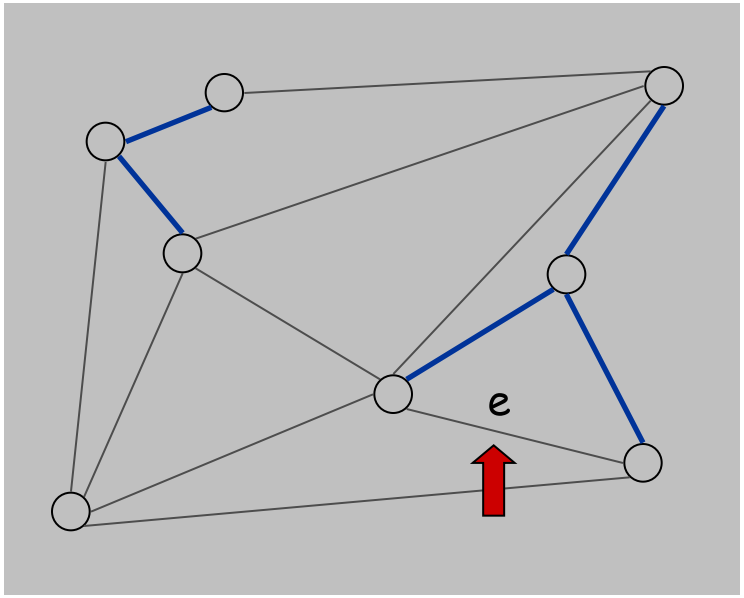


Case 2

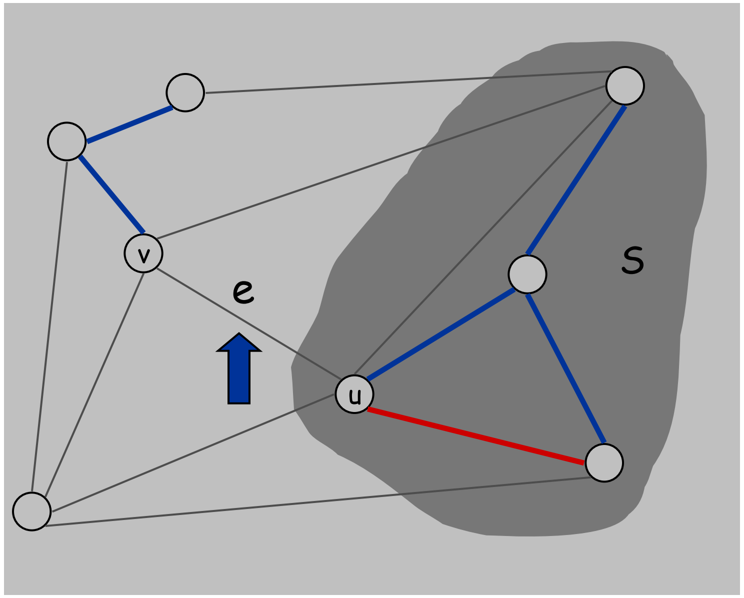
Kruskal's Algorithm: Proof of correctness

Kruskal's algorithm. [Kruskal, 1956]

- Case 1: If adding e to T creates a cycle, discard e
  - Optimal solution does not have a cycle
- Case 2: Otherwise, insert e = (u, v) into T
  - Pick the cut S as the nodes that are reachable from u in T



Case 1



Case 2

Kruskal's Algorithm: Bad Implementation

Kruskal's algorithm. [Kruskal, 1956]

- Sorting the edges O(m log m)
- Testing the existence of a cycle while considering edge e: O(n) via a DFS(BFS). Note that a tree has at most n edges.
- For all edges O(m.n)
- Total complexity O(m log m) +O(m n) = O(n.m)

Implementation: Kruskal's Algorithm

Implementation. Use the union-find data structure.

- Build set T of edges in the MST.
- Maintain set for each connected component.
- $O(m \log n)$ for sorting and $O(m \underbrace{\alpha(m, n)}_{\text{essentially a constant}})$ for union-find.

$m \leq n^2 \Rightarrow \log m$ is $O(\log n)$

```
Kruskal(G, c) {
   Sort edges weights so that c1 ≤ c2 ≤ ... ≤ cm.
   T ← φ

   foreach (u ∈ V) make a set containing singleton u

   for i = 1 to m        are u and v in different connected components?
      (u,v) = ei
      if (u and v are in different sets) {
         T ← T ∪ {ei}
         merge the sets containing u and v
      }                        merge two components
   return T
}
```

MST Algorithms: Theory

Deterministic comparison based algorithms.

- $O(m \log n)$ [Jarník, Prim, Dijkstra, Kruskal, Boruvka]
- $O(m \log \log n)$. [Cheriton-Tarjan 1976, Yao 1975]
- $O(m\, \beta(m, n))$. [Fredman-Tarjan 1987]
- $O(m \log \beta(m, n))$. [Gabow-Galil-Spencer-Tarjan 1986]
- $O(m\, \alpha\, (m, n))$. [Chazelle 2000]

Holy grail. $O(m)$.

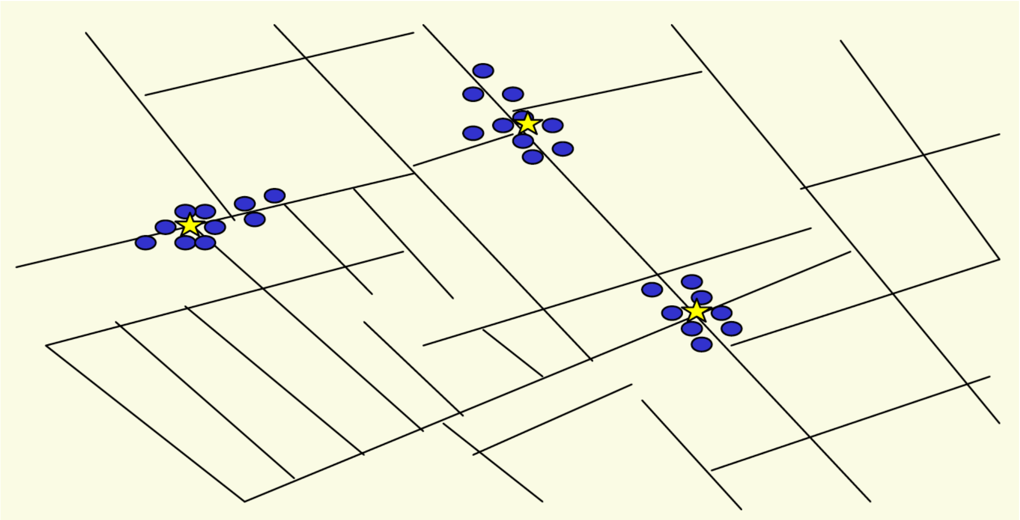
Notable.

- $O(m)$ randomized. [Karger-Klein-Tarjan 1995]
- $O(m)$ verification. [Dixon-Rauch-Tarjan 1992]

Euclidean.

- 2-d: $O(n \log n)$. compute MST of edges in Delaunay
- k-d: $O(k\, n^2)$. dense Prim

4.7 Clustering



Outbreak of cholera deaths in London in 1850s.
Reference: Nina Mishra, HP Labs

Clustering

Clustering. Given a set U of n objects labeled $p_1, \ldots, p_n$, classify into coherent groups.

photos, documents. micro-organisms

Distance function. Numeric value specifying "closeness" of two objects.

number of corresponding pixels whose intensities differ by some threshold

Fundamental problem. Divide into clusters so that points in different clusters are far apart.

- Routing in mobile ad hoc networks.
- Identify patterns in gene expression.
- Document categorization for web search.
- Similarity searching in medical image databases
- Skycat: cluster 10^9 sky objects into stars, quasars, galaxies.

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photos, documents. micro-organisms

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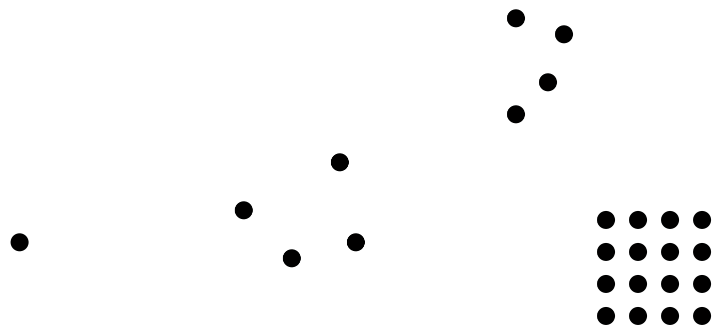
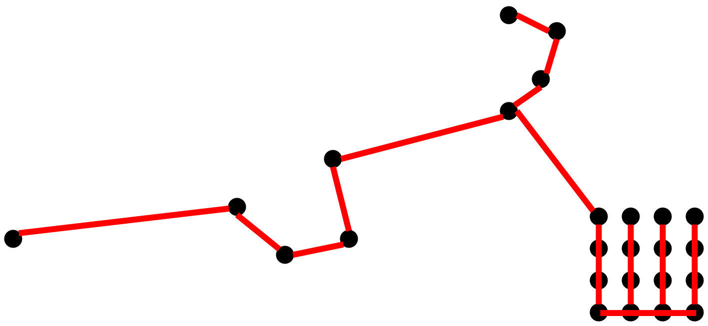


Clustering of Maximum Spacing

k-clustering. Divide objects into k non-empty groups.

Q: Can we use an MST to perform k-clustering?

A: Start with MST, keep removing heaviest edge until we get k connected components



k = 4

Clustering of Maximum Spacing

k-clustering. Divide objects into k non-empty groups.

Q: Can we use an MST to perform k-clustering?

A: Start with MST, keep removing heaviest edge until we get k connected components

Guarantees:

1. This algorithm gives cheapest way of forming k connected components (generalizes MST, which gives cheapest 1 conn. comp)
2. Maximizes spacing: minimum space between different classes