# Sorting Algorithms

# Sorting

Sorting. Given n elements, rearrange in ascending order.

Obvious sorting applications.
- List files in a directory.
- Organize an MP3 library.
- List names in a phone book.
- Display Google PageRank results.

Problems become easier once sorted.
- Find the median.
- Find the closest pair.
- Binary search in a database.
- Identify statistical outliers.
- Find duplicates in a mailing list.

Non-obvious sorting applications.
- Data compression.
- Computer graphics.
- Interval scheduling.
- Computational biology.
- Minimum spanning tree.
- Supply chain management.
- Simulate a system of particles.
- Book recommendations on Amazon.
- Load balancing on a parallel computer.
- . . .

# 1. Basic Algorithms:
# Bubble Sort and Selection Sort

# Bubble sort

Compare each element (except the last **one**) with its neighbor to the right

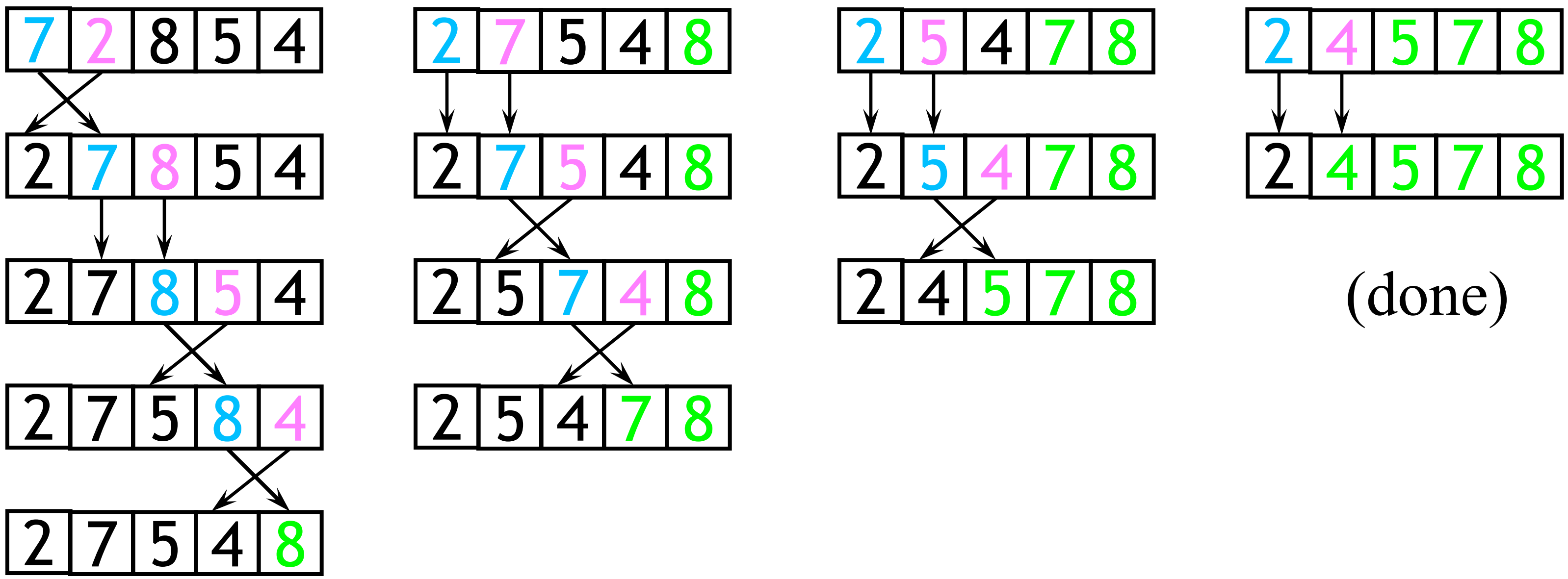
- If they are out of order, swap them
- This puts the largest element at the very end
- The last element is now in the correct and final place

Compare each element (except the last ***two***) with its neighbor to the right

- If they are out of order, swap them
- This puts the second largest element next to last
- The last two elements are now in their correct and final places

Continue as above until no elements are swapped during a scan

# Example of bubble sort



# Analysis of bubble sort

Claim: The time complexity T(n) of Bubble Sort is O($n^2$)

- n operations in the first scan
- (n-1) operations in the second scan
- .
- .
- .
- (n-(i-1)) operations in i-th scan

- Adding the number of operations over all scans we have
    - $n+(n-1)+(n-2) + \ldots + 1 = O(n^2)$

# Analysis of bubble sort

Lower bound:

In a list sorted from the largest to the smallest element we spend $n+(n-1)+(n-2) + \ldots + 1 = \Omega(n^2)$

So Bubble Sort has time complexity $\theta(n^2)$

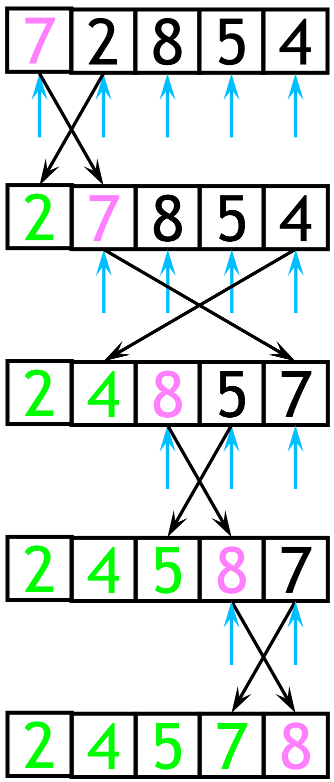
Obs: In the best case (sorted list) we just scan the list once: $O(n)$ time

# Selection sort

Given an array of length **n**,

- Search elements **1** through **n** and select the smallest
  - Swap it with the element in location 1
- Search elements **2** through **n** and select the smallest
  - Swap it with the element in location **2**
- Search elements **3** through **n** and select the smallest
  - Swap it with the element in location **3**
- Continue in this fashion until there’s nothing left to search

# Example and analysis of selection sort




## Analysis:

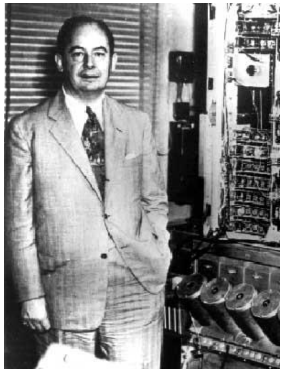
- The outer loop executes **n-1** times
- The i-th internal loop executes (n-i) operations
- Work done in the inner loop is constant (swap two array elements)
- Time complexity: $O(n^2)$
- Also $\Omega(n^2)$ (for every instance)
- So it is $\Theta(n^2)$

# 2. Mergesort

# Mergesort

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole (in linear time)



Jon von Neumann (1945)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| A | L | G | O | R | I | T | H | M | S |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A | L | G | O | R | | I | T | H | M | S | divide |
| A | G | L | O | R | | H | I | M | S | T | sort 2T(n/2) |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| A | G | H | I | L | M | O | R | S | T | merge O(n) |

(see last class)

# A Useful Recurrence Relation

Mergesort recurrence.

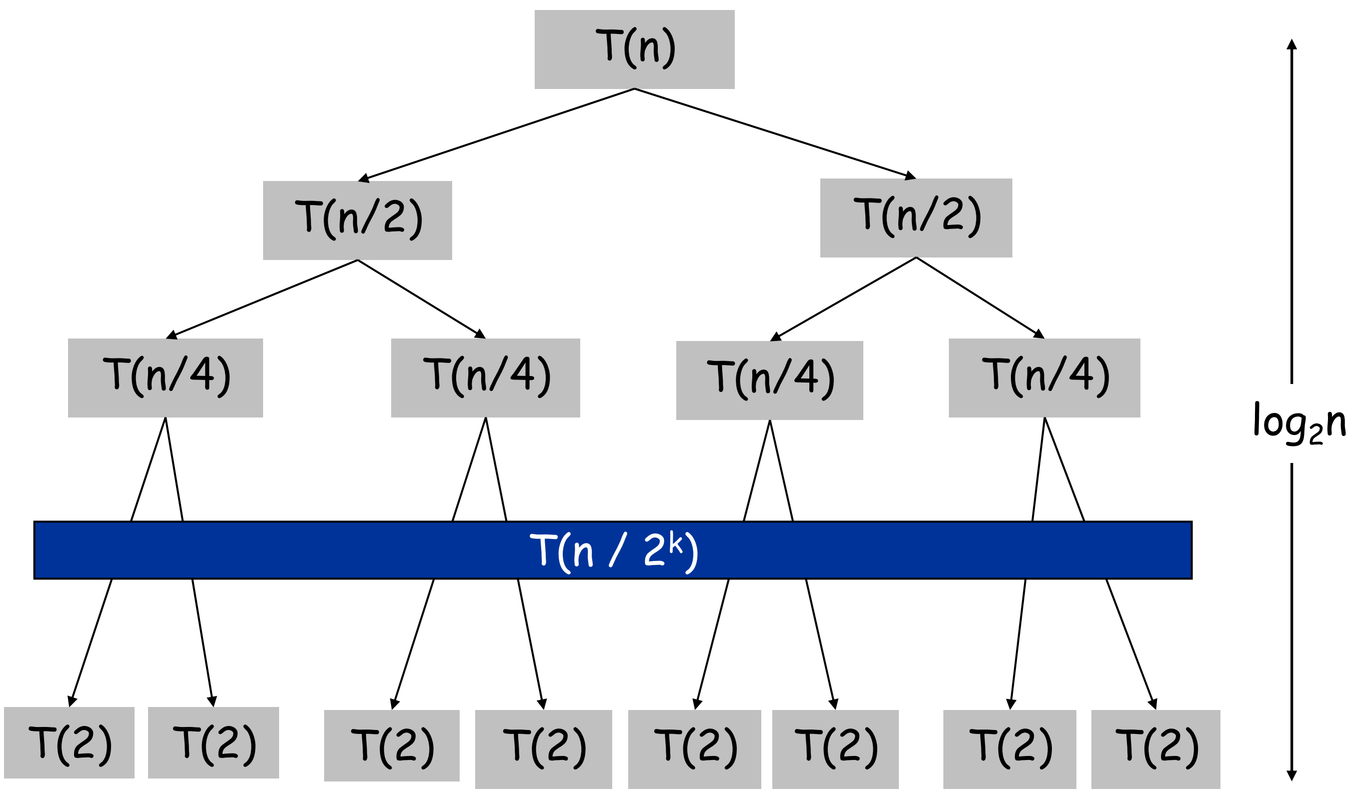
$$\mathrm{T}(n) \le \begin{cases} 1 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Solution. T(n) is $O(n \log_2 n)$.

Proofs. We describe several ways to prove this recurrence. Initially we ignore floor/ceiling

# Proof by Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$




n

$2(n/2) = n$

$4(n/4) = n$

. . .

$2^k (n / 2^k) = n$

. . .

$n/2\,(2) = n$

$n \log_2 n$

# Proof by Induction

Claim. If T(n) satisfies this recurrence, then $T(n) = n \log_2 n$.

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on n)

- Base case: $n = 1$.
- Inductive hypothesis: for $n' \leq n-1$, $T(n') = n' \log_2 n'$.
- Goal: show that $T(n) = n \log_2 (n)$.

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &= (2n/2)(\log_2(n/2)) + n \\ &= (2n/2)\big(\log_2(n) - 1\big) + n \\ &= n\log_2(n) \end{aligned}$$

# Proof Keeping the Floor/Ceiling

(This is just so you can see that everything works out if you keep floor/ceiling, do not worry about it)

**Claim.** If T(n) satisfies the following recurrence, then $T(n) \le n \lceil \lg n \rceil$.

$\log_2 n$

$$T(n) \le \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

**Pf.** (by induction on n)

- Base case: $n = 1$.
- Define $n_1 = \lfloor n / 2 \rfloor$, $n_2 = \lceil n / 2 \rceil$.
- Induction step: assume true for 1, 2, ... , n–1.

$$\begin{aligned} T(n) &\le T(n_1) + T(n_2) + n \\ &\le n_1 \lceil \lg n_1 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &\le n_1 \lceil \lg n_2 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &= n \lceil \lg n_2 \rceil + n \\ &\le n(\lceil \lg n \rceil - 1) + n \\ &= n \lceil \lg n \rceil \end{aligned}$$

$$\begin{aligned} n_2 &= \lceil n/2 \rceil \\ &\le \left\lceil 2^{\lceil \lg n \rceil} / 2 \right\rceil \\ &= 2^{\lceil \lg n \rceil} / 2 \\ \Rightarrow \lg n_2 &\le \lceil \lg n \rceil - 1 \end{aligned}$$

# Mergesort

In conclusion, Mergesort take time O(n log n)

Interesting: Does not need to compare all pairs of items (that would be $\Omega(n^2)$)

# 2. Quicksort

# Quicksort

- Sorts $O(n \lg n)$ in the average case (we will not prove)
- Sorts $\theta(n^2)$ in the worst case

*So why would people use it instead of Mergesort?*

- Very simple to implement
- In practice is very fast (worst case is quite rare and constants are low)

# Quicksort

Input: list of numbers A[1], A[2], …, A[n]

Quicksort algorithm:

- Choose a number *pivot* in the list; lets say we always choose *pivot* = A[1]
- Partition the list A into the list L containing all numbers $< pivot$ , the list R containing all numbers $> pivot$ , and list P containing all numbers $= pivot$
- Sort the list L and R recursively using Quicksort
- Concatenate L, P, R (in this order)

| 4 | 8 | 1 | 7 | 5 | 3 | 2 | 9 | 4 | 1 |
|---|---|---|---|---|---|---|---|---|---|

L= | 1 | 3 | 2 | 1 |   P= | 4 | 4 |   R= | 8 | 7 | 5 | 9 |   *partition*

| 1 | 1 | 2 | 3 |   | 4 | 4 |   | 5 | 7 | 8 | 9 |   *recurse*

| 1 | 1 | 2 | 3 | 4 | 4 | 5 | 7 | 8 | 9 |   *concatenate*

# Quicksort

Input: list of numbers A[1], A[2], …, A[n]

Quicksort algorithm:

- Choose a number *pivot* in the list; lets say we always choose *pivot* = A[1]
- Partition the list A into the list L containing all numbers $< pivot$ , the list R containing all numbers $> pivot$ , and list P containing all numbers $= pivot$
- Sort the list L and R recursively using Quicksort
- Concatenate L, P, R (in this order)

```
Quicksort(A)
    if (A.len > 1)
        L, R, P = Partition(A);
        Quicksort(L);
        Quicksort(R);
     end if
    Return the concatenation of L,P,R
```

# Partition

We can implement partition step in O(n) using a auxiliary vectors

Can even do *in place* without any auxiliary memory

# Analyzing Quicksort

Claim: The time complexity of Quicksort is O($n^2$)

Proof:

- The height of execution tree of QuickSort is at most n (lists L and R are stricly smaller than A; that's why we put pivot in P).

- At each level of the tree the algorithm spends O(n) due to the partition precedure

- 

  So the algorithm spends at most $cn^2$ operations

# Analyzing Quicksort

Second proof (by induction): We have the recurrence

$T(0) = c$
$T(1) = c$
$T(n) <= \max_{i=0\ldots n-1} \{T(i) + T(n - i - 1) + cn\}$

We do not know the size of the subarrays

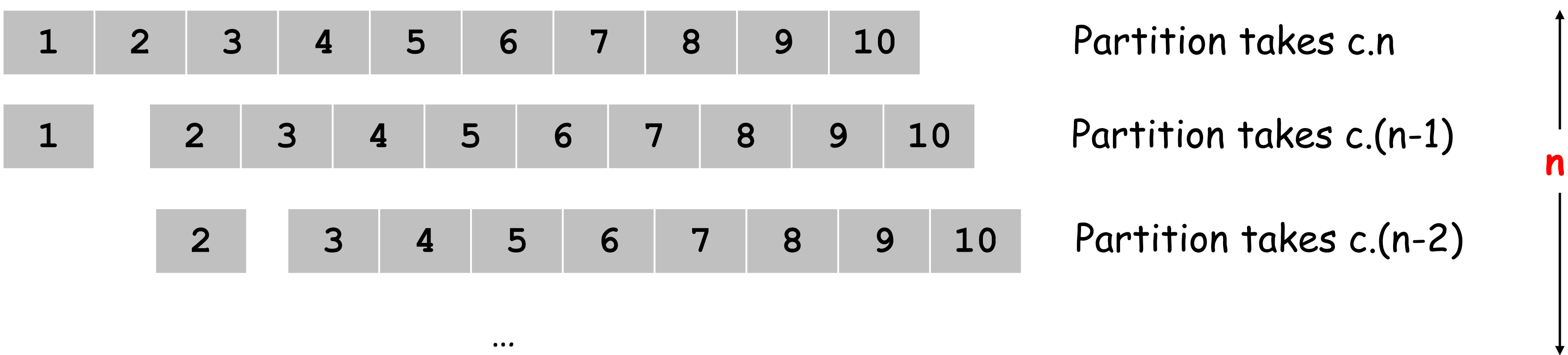
Proof by induction that $T(n) <= cn^2$

- Base case n=0,1 is ok
- Let i* be the value of i that maximizes the expression. By induction $T(n) <= c(i^*)^2 + c(n-i^*-1)^2 + cn$
- The right-hand side is a parabola in i*. So it takes maximum value with i*=0 or i*=n-1. Plugging in these values we get the induction hypothesis

## Analyzing Quicksort

Is this the best analysis analysis we can do?

Yes: If the input is sorted, the partition is always unbalanced: height n




Algorithm takes c.n + c.(n-1) + c.(n-2) + … + c = c.n.(n+1)/2 = $\Omega(n^2)$

So Quicksort is $\Omega(n^2)$

So Quicksort is $\Theta(n^2)$

# Analyzing Quicksort: Average Case

(Nao 'e cobrado)

A(n): time complexity of Quicksort assuming instance is a random permutation

So partition generates splits (1:n-1, 2:n-2, 3:n-3, ... , n-2:2, n-1:1) each with probability 1/n

So it is often balanced

$$A(n) = \frac{1}{n-1} \sum_{k=1}^{n-1} \left[ A(k) + A(n-k) \right] + \Theta(n)$$

We can solve the recurrence to get that A(n) is O(n log n)

# Improving Quicksort

The real liability of quicksort is that it runs in $O(n^2)$ on already-sorted input

Book discusses two solutions:

- Randomize the input array, OR
- Pick a random pivot element

*How will these solve the problem?*

- By insuring that no particular input can be chosen to make quicksort run in $O(n^2)$ expected time

# Exercício

Descreva um algoritmo com complexidade O(n log n) com a seguinte especificação

Entrada: Uma lista de *n* números reais

Saida: SIM se existem números repetidos na lista e NÃO caso contrário

# Exercício - Solução

Ordene a lista lista L
**Para** i=1 até |L|-1
    Se L[i]=L[i+1] **Devolva** SIM
**Fim Para**
**Devolva NÃO**

- ANÁLISE
  - A ordenação da lista L requer O (n log n) utilizando o MergeSort
  - O loop **Para** requer tempo linear

# Exercício

Descreva um algoritmo com complexidade O(n log n) com a seguinte especificação

Entrada: conjunto $S$ de $n$ números reais e um número real $x$

Saida: SIM se existem dois elementos distintos em $S$ com soma x e NÃO caso contrário

# Exercício - Solução

L <- conjunto S em ordem crescente

**Enquanto** a lista L tiver mais que um elemento **faça**

    Some o menor e o maior elemento de L

    Se a soma é x

        **Devolva** SIM

    Se a soma é maior que x

        Retire o maior elemento de L

    Se a soma é menor que x

        Retire o menor elemento de L

**Fim Enquanto**

**Devolva NÃO**

# Exercício - Solução

L <- conjunto S em ordem crescente
**Enquanto** a lista L tiver mais que um elemento **faça**
    Some o menor e o maior elemento de L
    Se a soma é x
        **Devolva** SIM
    Se a soma é maior que x
        Retire o maior elemento de L
    Se a soma é menor que x
        Retire o menor elemento de L
**Fim Enquanto**
**Devolva NÃO**

- ANÁLISE
  - A ordenação do conjunto S requer O (n log n) utilizando o o MergeSort
  - O loop enquanto requer tempo linear

# 3 How fast can we sort?

# How Fast Can We Sort?

Can we sort a list of n numbers in constant time (independent of n)?

How can we prove that no algorithm can sort in constant time?

- This is very hard, research question. We don't know how to do it....

All of the sorting algorithms so far are *comparison sorts*

- The only operation used to gain ordering information about a sequence is the pairwise comparison of two elements

Easier question: How many comparisons are necessary for any comparison sort algorithm?

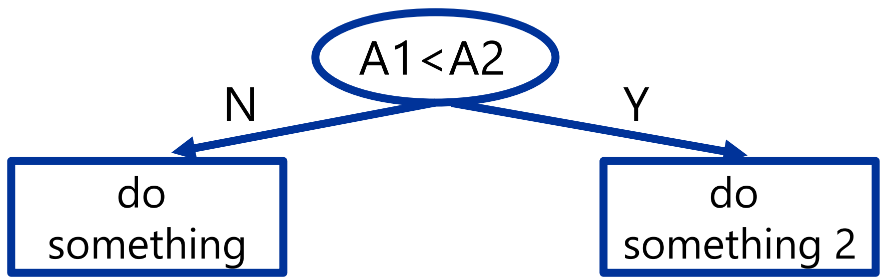
- Just counting comparisons, reordering of items is free

# How Fast Can We Sort?

Ex: Can we sort 3 numbers with just 1 comparison?

$$A = \boxed{?}\boxed{?}\boxed{?}$$

Answer: No: Suppose just compare A[1] with A[2]




But consider $A = \boxed{2}\boxed{3}\boxed{4}$ and $A = \boxed{2}\boxed{3}\boxed{1}$ : both execute "do something 2"

- If "do something 2" moves A[3] then it makes a mistake in instance 2,3,4
- If "do something 2" does not move A[3] then it makes a mistake in instance 2,3,1

*Need to make enough comparisons to find out order of the instance*

# How Fast Can We Sort?

We will show that no comparison algorithm can beat MergeSort

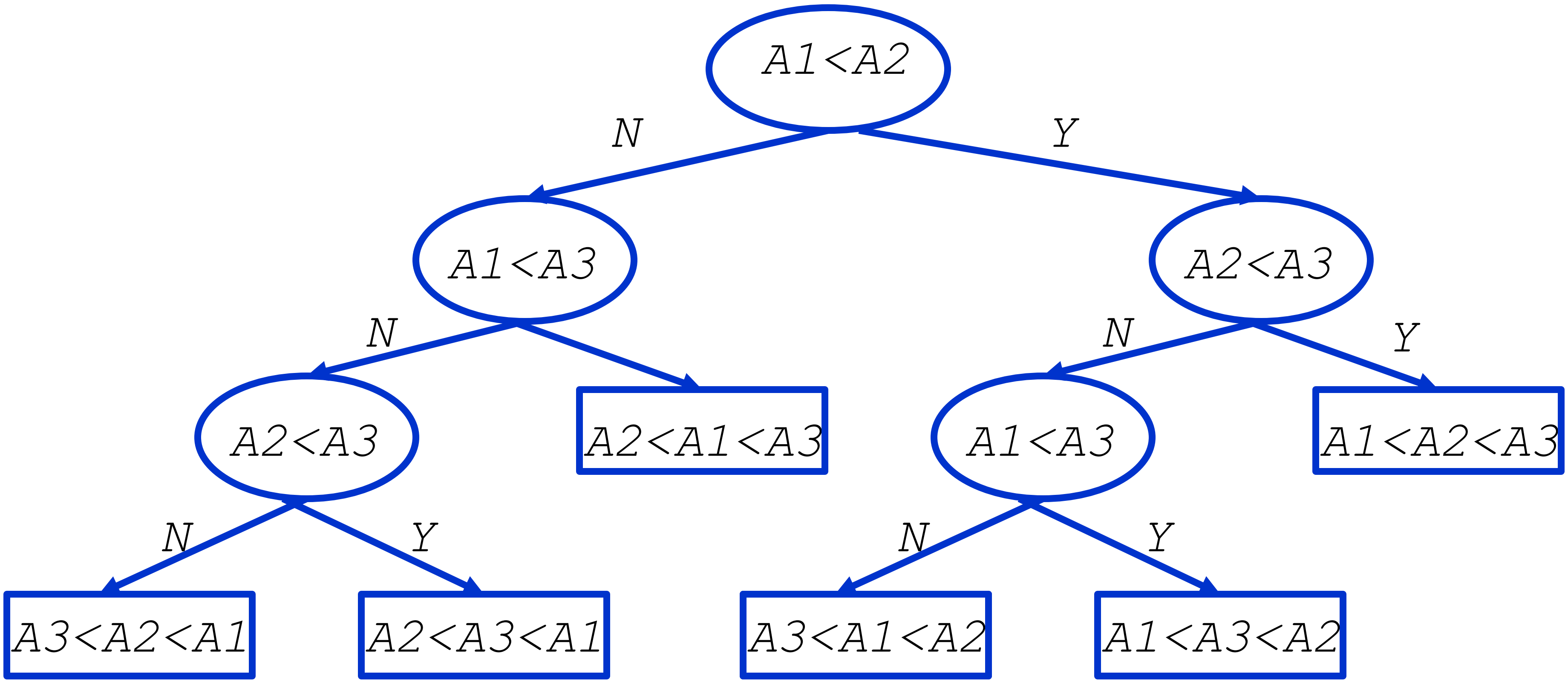
Theorem: Any comparison based sorting algorithm needs at least $\Omega(n \log n)$ comparisons

# Decision Trees

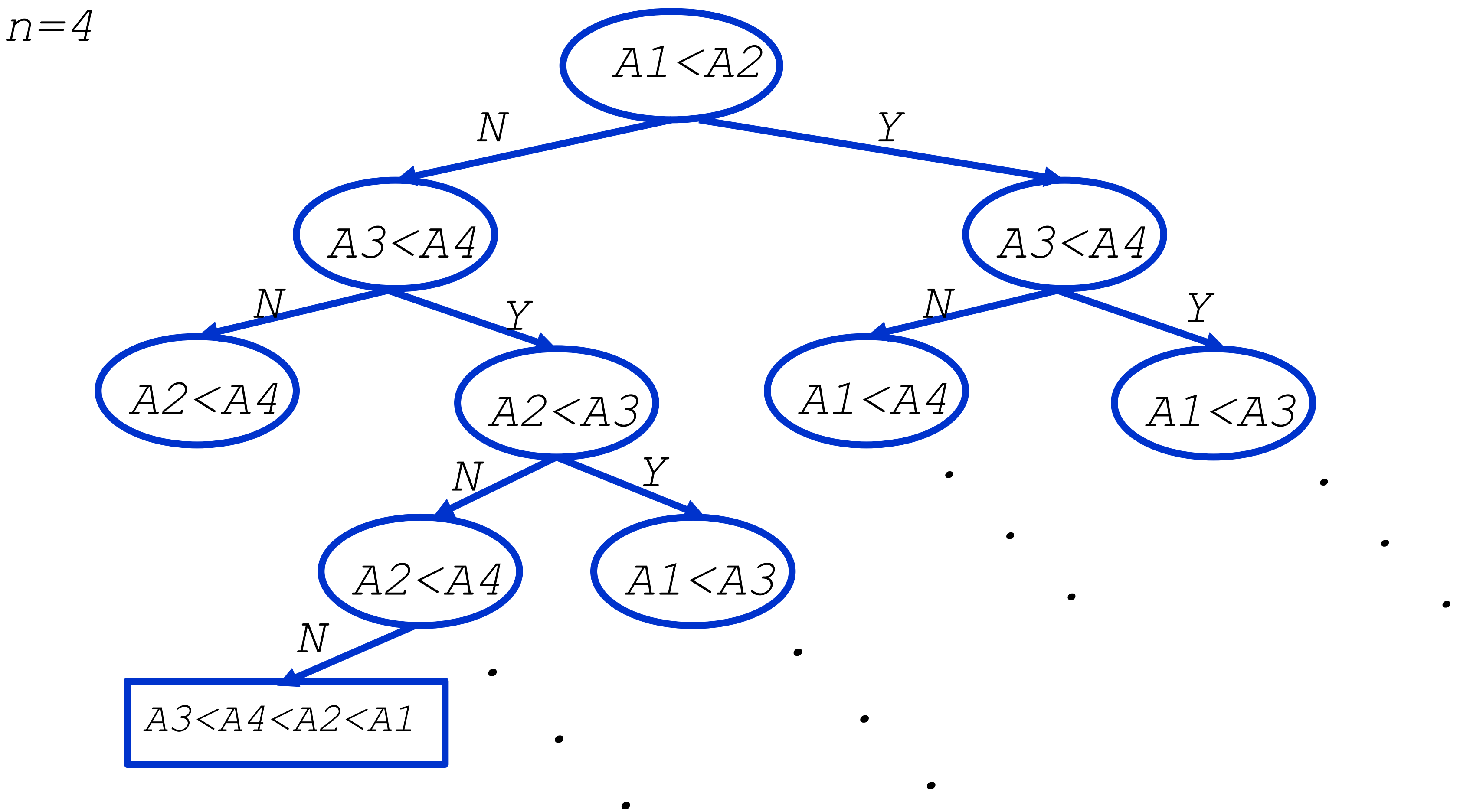
- Decision trees provide an abstraction of comparison sorts
  - A decision tree represents the comparisons made by a comparison sort. Every thing else is ignored
  - Each node corresponds to a comparison
  - Each branch corresponds to an outcome of the comparison
  - There is one leaf for each possible ordering of the numbers

# Decision tree for BubbleSsort

Bubble sort for 3 numbers A1, A2, A3



## Decision tree for MergeSort



# Decision Trees

- Decision trees provide an abstraction of comparison sorts

Q: How many leaves does a decision tree have?

Q: What does the height correspond to?

# Decision Trees

- Decision trees provide an abstraction of comparison sorts

Q: How many leaves does a decision tree have?

A: n!

Q: What does the height correspond to?

A: Number of comparisons for worst case instance

What can we say about the height of any decision tree?

# Lower bound for comparison sorting

Theorem: The height of any decision tree that sorts n numbers is at least $\Omega(n \log n)$

Proof:

- Any decision tree has n! leaves
- The height of a decision tree with n! leaves is at least log(n!) because the decision tree is a binary tree
- log(n!) ≥ (n/2) log n – (n/2):

log(n!) = log(n*(n-1)*…*2*1) = log n + log (n-1) + … log 2 + log 1
≥ log n + log (n-1) + … + log (n/2) >= (n/2) log (n/2)
= (n/2) log n – (n/2)

# Lower bound for comparison sorting

Theorem: The height of any decision tree that sorts n numbers is at least $\Omega(n \log n)$

Corollary: No comparison sort can beat MergeSort asymptotically

# 5.4 Sorting in linear time

# Sorting In linear time

- As notas do vestibular são números entre 1 e 100
- 10.000 alunos prestaram vestibular. Como ordernar os alunos conforme sua classificação?
- Versão reduzida para exemplo
  - 10 alunos com notas {1,2,2,3,4,1,2,2,4,3}

# Sorting In linear time

We will see Counting sort

No comparisons between elements!

**But**…depends on assumption about the numbers being sorted

- We assume numbers are in the range *1..k*

# Counting Sort

Input: List of integers A[1], A[2], ..., A[n] with values between 1 and k

Main idea: Compute vector C where C[i] is number of elements in the list at most i

How can we use this to sort?

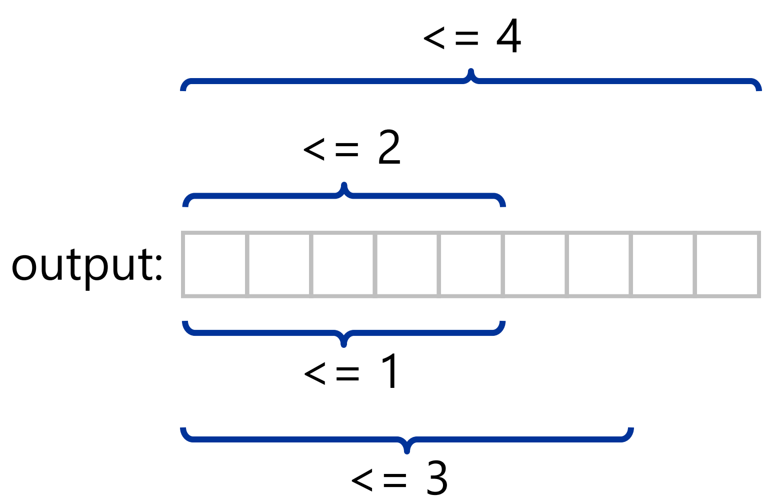
Ex 1: Consider distinct numbers A = [3, 1, 5, 10, 7]

- Counting vector is C = [1, 1, 2, 2, 3, 3, 4, 4, 4, 5]
- Put number "3" in position C[3], number "1" in position C[1], ...

# Counting Sort

Ex 2: Consider non-distinct numbers A = [1,3,3,1,1,4,1,1,4]

- Counting vector is C = [5, 5, 7, 9]
- Put (say) second “3” in position C[3], put first “3” in position C[3] – 1,…



# Counting Sort

```
CountingSort(A, B, k)
     for i=1 to k
          C[i]= 0;
     for j=1 to n
          C[A[j]] += 1;
     for i=2 to k
          C[i] = C[i]+C[i-1];
     for j=n downto 1
          B[C[A[j]]] = A[j];
          C[A[j]] = C[A[j]]- 1;
```

Initialize

C[i] stores the number of elements equal to i,i =1,…,k

C[i] stores the number of elements smaller or equal to i,i =1,…,k

The position of element A[j] in B is equal to the number of integers that are at most A[j], which is C[A[j]]

Ex: A = {1,2,2,3,4,1,2,2,4,3}

# Counting Sort

Q: What is the running time of Counting Sort?

```
CountingSort(A, B, k)
    for i=1 to k
        C[i]= 0;
    for j=1 to n
        C[A[j]] += 1;
    for i=2 to k
        C[i] = C[i] + C[i-1];
    for j=n downto 1
        B[C[A[j]]] = A[j];
        C[A[j]] -= 1;
```

*Takes time O(k)*

*Takes time O(n)*

# Counting Sort

- Total time: $O(n + k)$
  - In many cases, $k = O(n)$; when this happens, Counting sort is $O(n)$
- But sorting is $\Omega(n \lg n)$!
  - No contradiction--this is not a comparison sort (in fact, there are *no* comparisons at all!)
- Notice that this algorithm is *stable*
  - If x and y are two identical numbers and x is before y in the input vector then x is also before y in the output vector (Important for Radix sort)
  - That is the only reason why the last "For" is in reverse order

# Counting Sort

Cool! *Why don't we always use counting sort?*

Because it depends on range $k$ of elements

*Could we use counting sort to sort 32 bit integers? Why or why not?*

Answer: no, $k$ too large ($2^{32}$ = 4,294,967,296)

# Radix Sort

You have a list of numbers to sort

One idea is to sort them based on most significant digit, then the second m.s.d., etc.

Q: Does it work?

A: Not directly: consider input 91,19, 55, 54

- Sorting by most significant digit: 19, 55, 54, 91
- Then sorting by second most significant: 91, 54, 44, 91 **wrong order!**

Key idea: sort the least significant digit first

```
RadixSort(A, d)
    for i=1 to d
        Use a stable sorting algorithm
        to sort based on digit i
```

# Radix Sort

Ex: (329,457,657,839,436,720)

Step 1: third digit

(720, 436, 457,657, 329,839)

Step 2: second digit

(720, 329,436, 839,457,657)

Step 3: first digit

(329,436,457,657, 720, 839)

That is why we need a stable sorting algorithm

# Radix Sort

- *Can we prove it will work?*

- Sketch of an inductive argument (induction on the number of passes):
  - Assume lower-order digits {j: j<i}are sorted
  - Show that sorting next digit i leaves array correctly sorted
    - If two digits at position i are different, ordering numbers by that digit is correct (lower-order digits irrelevant)
    - If they are the same, numbers are already sorted on the lower-order digits. Since we use a stable sort, the numbers stay in the right order

# Radix Sort

*What sort will we use to sort on digits?*

Counting sort is obvious choice:

- Sort $n$ numbers on digits that range from 0..9
- Time: $O(n)$

Each pass over $n$ numbers with $d$ digits takes time $O(n)$, so total time $O(dn)$

Obs: Does not need to use digits (writing numbers in base 10), can use any other base

# Radix Sort

In general, radix sort based on counting sort is:

- Fast
- Good worst-case guarantee O(nd)
- Simple to code
- A good choice

# Radix Sort

**Exercise:** Suppose we want to sort a list of n numbers with values in $\{1,2,\ldots,n^2\}$. Show we can do this in time O(n).

Hint: Consider Radix Sort (with Counting Sort) but don't use digits to represent the numbers, use a different base (with more symbols).

**A:** Suppose numbers are represented in base k. Then Radix Sort with Counting Sort takes time

$$O\big((\log_k n^2) * (n + k)\big) = O((\log_k n) * (n + k))$$

Setting the base $k = n$, this becomes O(n).