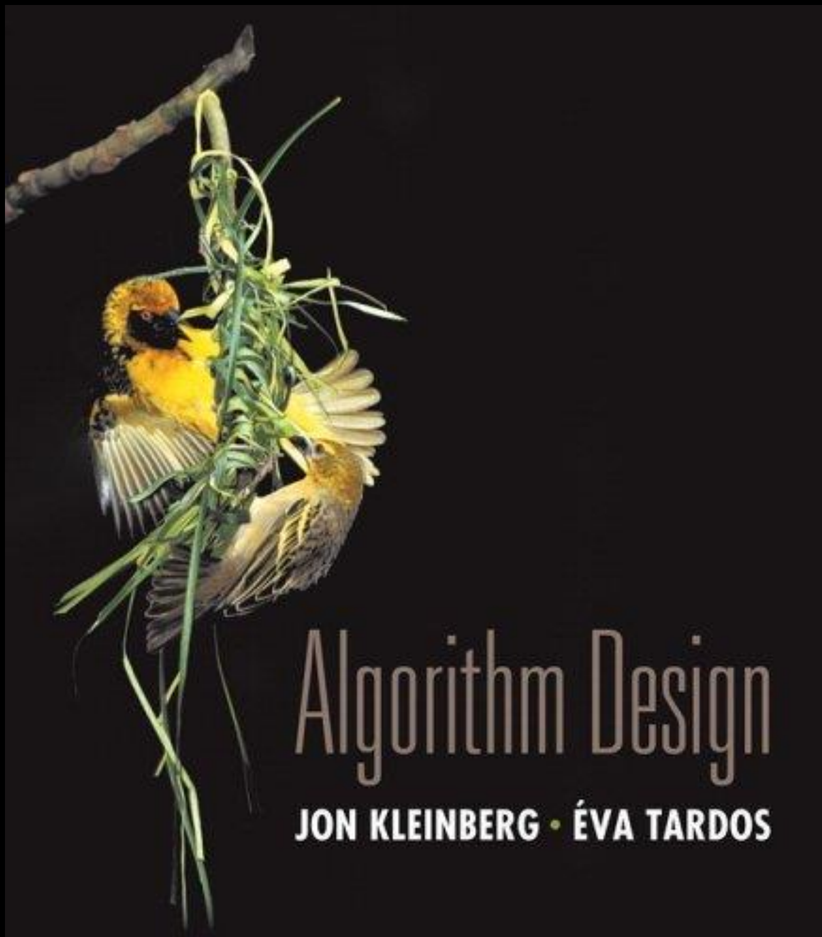


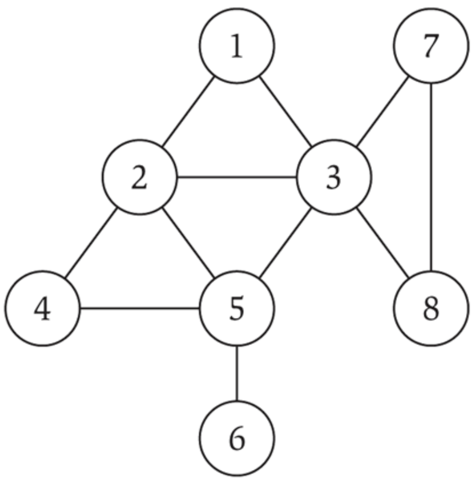
Chapter 3

Graphs

3.1 Basic Definitions and Applications

Undirected Graphs

Undirected graph. G = (V, E)
- V = nodes (non-empty)
- E = edges between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters: n = |V|, m = |E|.



V = { 1, 2, 3, 4, 5, 6, 7, 8 }

E = { 1-2, 1-3, 2-3, 2-4, 2-5, 3-5, 3-7, 3-8, 4-5, 5-6,7-8}

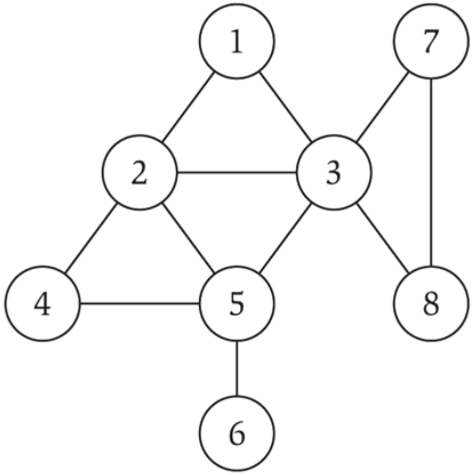
n = 8

m = 11

Undirected Graphs

Undirected graph. $G = (V, E)$

- u and v are adjacent (neighbors) in G iff there is an edge between u and v in G
- The degree d(u) of a vertex u is the number of neighbors of u



1 and 3 are adjacent

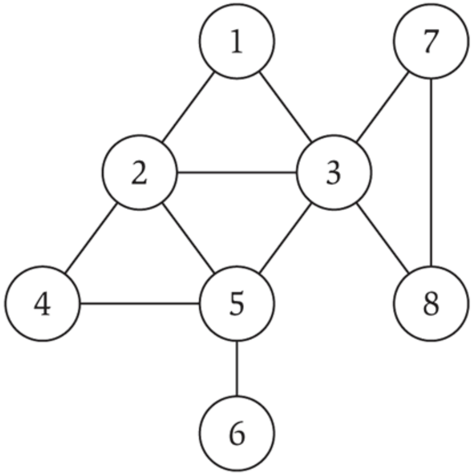
2 and 8 are not adjacent

d(3)=5

d(4)=2

Undirected Graphs

Important Property: For every graph G, the sum of degrees of G equals twice the number of edges.



m=11

Sum of degrees =22

Undirected Graphs

Loops

- Edge whose two endpoints are the same

Parallel edges

- Two Edges with the same endpoints

Simple Graph

- A simple graph is a graph with neither loops nor parallel edges

- Most of the time we'll be considering simple graphs

Q: What is max number of edges a simple graph on n nodes can have?

A: $m \le n(n-1)/2$ for simple graphs

- Bound is tight for complete graphs

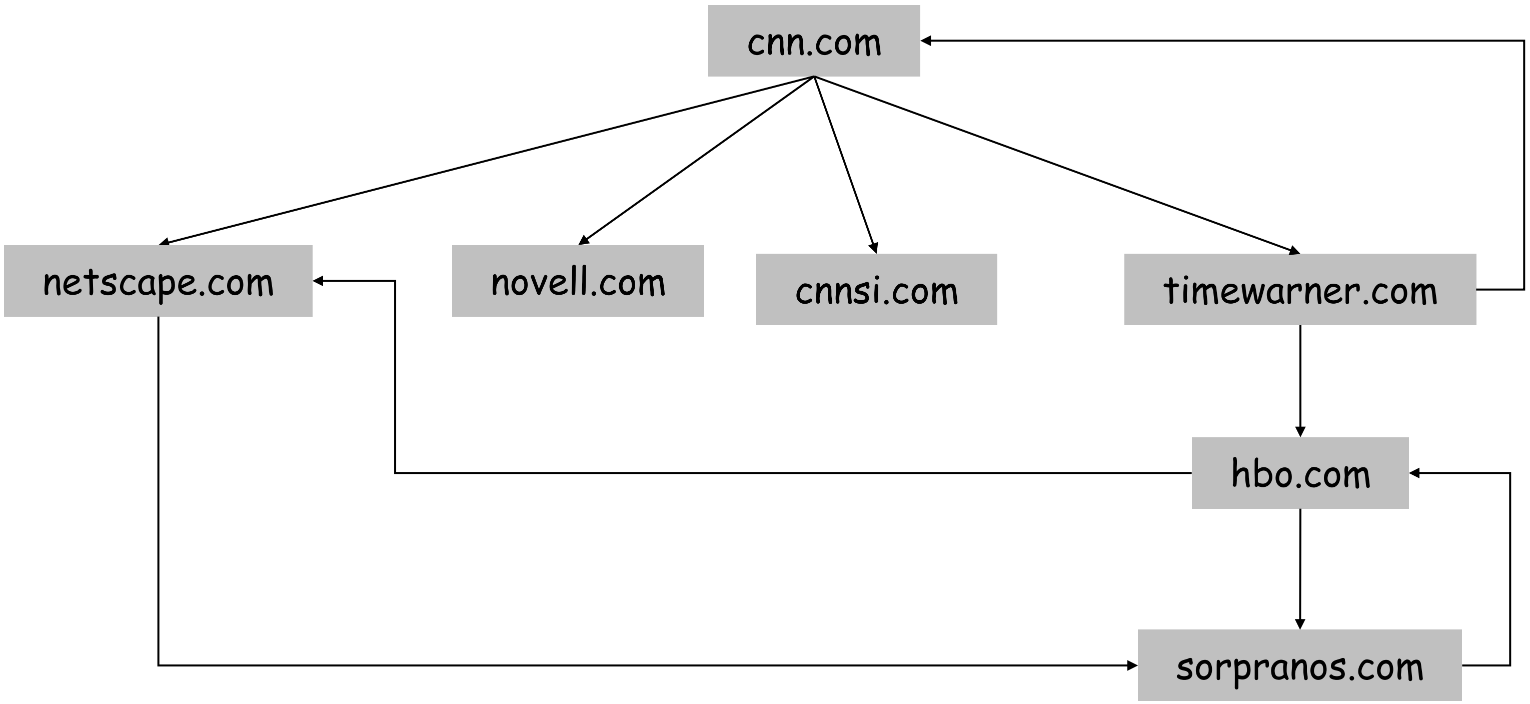
Some Graph Applications

Graph	Nodes	Edges
transportation	street intersections	highways
communication	computers	fiber optic cables
World Wide Web	web pages	hyperlinks
social	people	relationships
food web	species	predator-prey
software systems	functions	function calls
scheduling	tasks	precedence constraints
circuits	gates	wires
kidney exchange	patient+relative	compatibility

World Wide Web

Web graph.

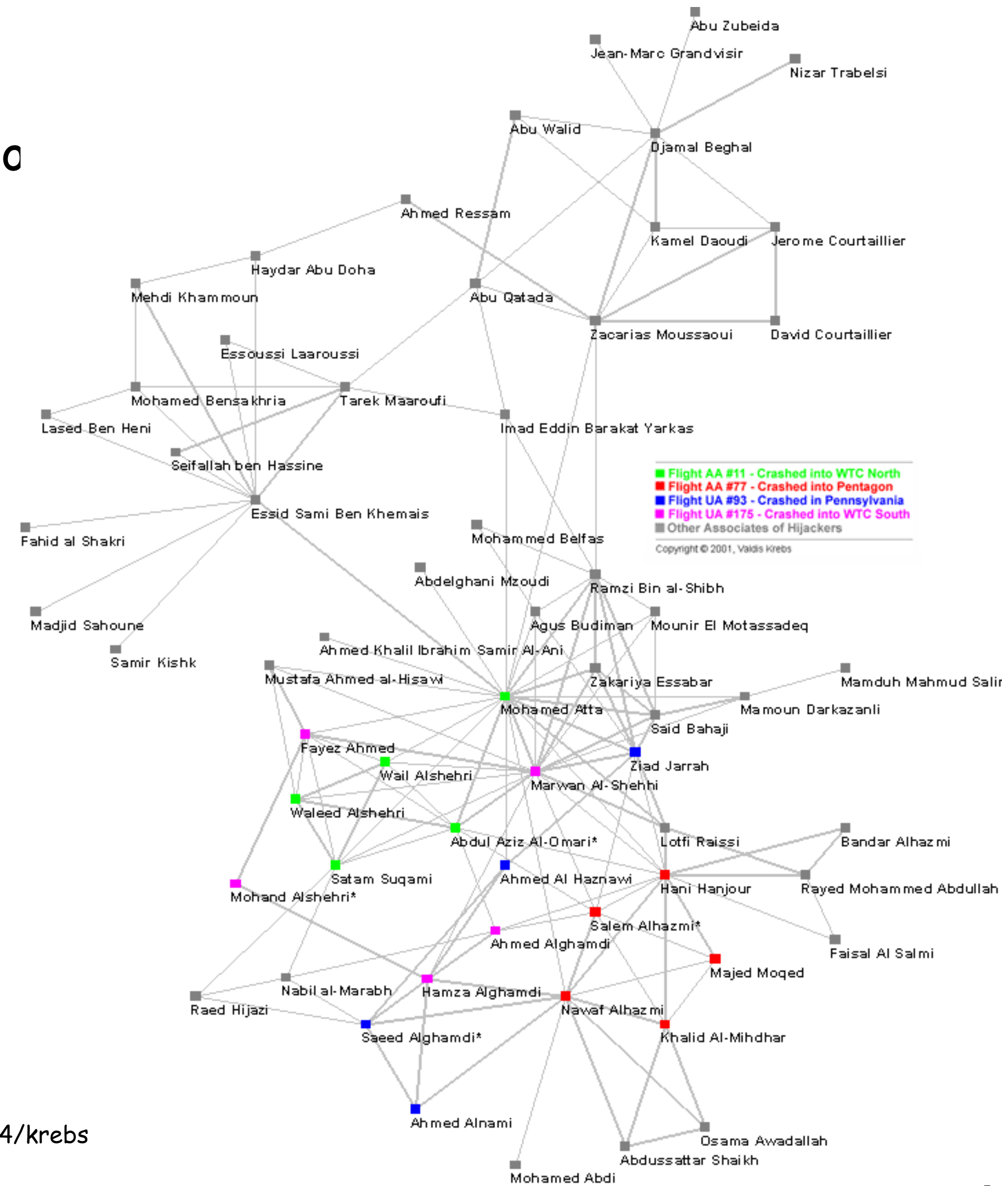
- Node: web page.
- Edge: hyperlink from one page to another.



9-11 Terrorist Network

Social network graph.

- Node: people.
- Edge: relationship between two people.

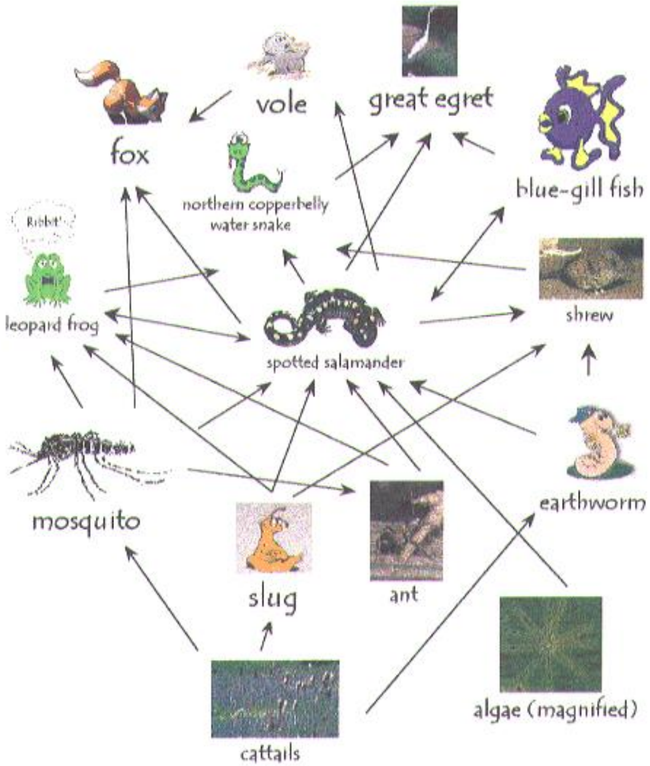


Reference: Valdis Krebs, http://www.firstmonday.org/issues/issue7_4/krebs

Ecological Food Web

Food web graph.

- Node = species.
- Edge = from prey to predator.

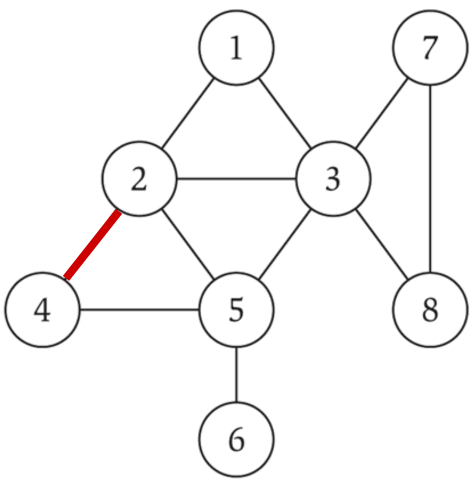


Reference: http://www.twingroves.district96.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.giff

Graph Representation: Adjacency Matrix

Adjacency matrix. n-by-n matrix with $A_{uv} = 1$ if (u, v) is an edge.

- Two representations of each edge.
- Space proportional to
- Checking if (u, v) is an edge takes time.
- Identifying all edges takes time.

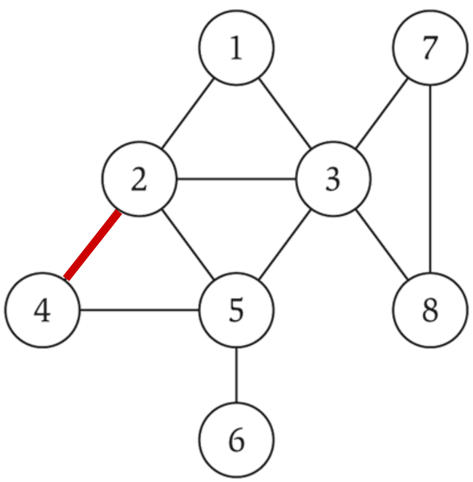


	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

Graph Representation: Adjacency Matrix

Adjacency matrix. n-by-n matrix with A_{uv} = 1 if (u, v) is an edge.

- Two representations of each edge.
- Space proportional to n^2.
- Checking if (u, v) is an edge takes $\Theta(1)$ time.
- Identifying all edges takes $\Theta(n^2)$ time.



	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

Graph Representation: Adjacency Matrix

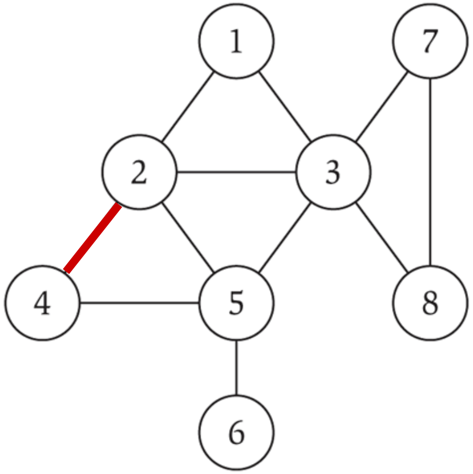
Drawback: independent of number of edges

- In line graph (n vertices and n-1 edges) adjacency matrix is full of 0's

Facebook

- 750M vertices
- Assumption: each person has 130 friends in average

➔ 550 Petabytes to store approximately 50 Billion edges;



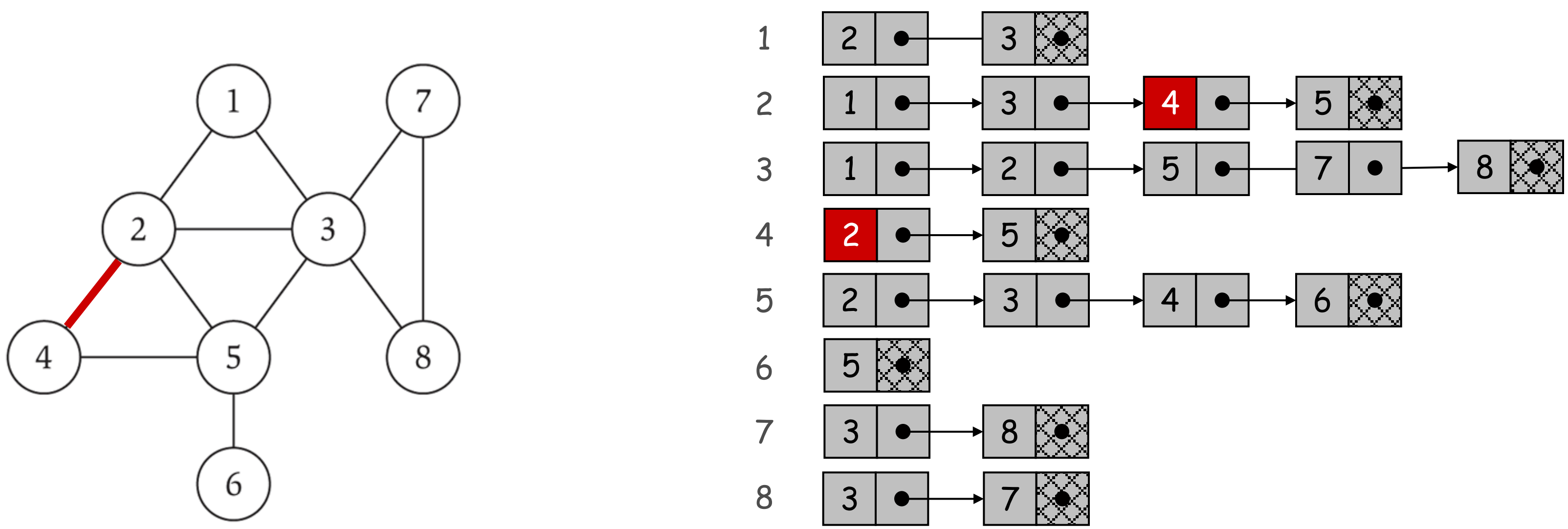
	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

Graph Representation: Adjacency List

Adjacency list. List of neighbors of each node

- Two representations of each edge.
- Space proportional to
- Checking if (u, v) is an edge takes time.
- Identifying all edges takes time.

degree = number of neighbors of u



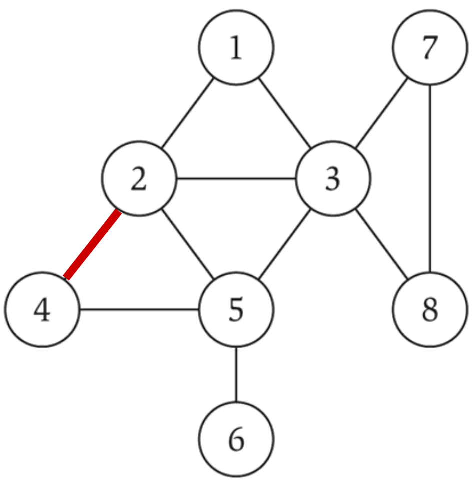
Graph Representation: Adjacency List

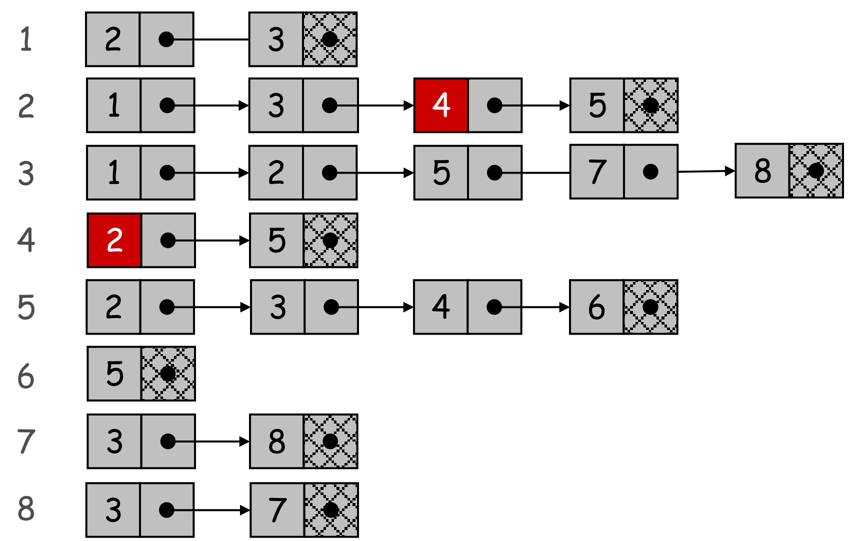
Advantage: sensitive to the number of edges

Facebook

- 750M vertices
- Assumption: each person has 130 friends in average

➔ 100 Gigabytes to store approximately 50 Billion edges;



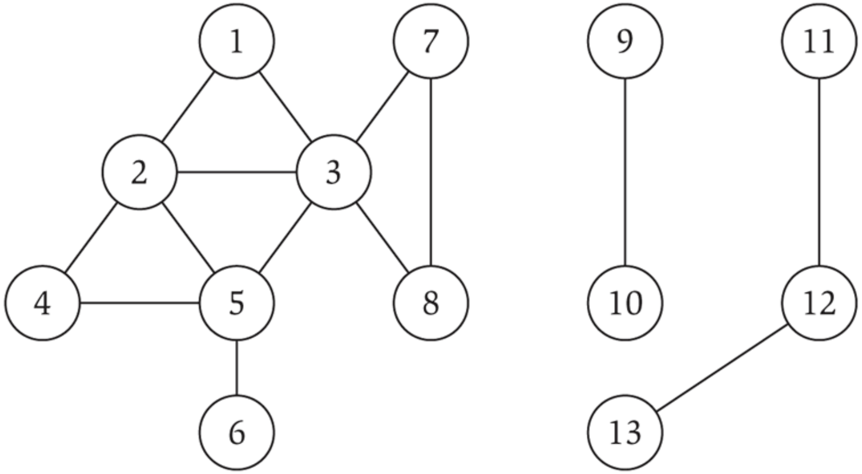


Paths and Connectivity

Def. A path in an undirected graph $G = (V, E)$ is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k$ with the property that each consecutive pair v_i, v_{i+1} is joined by an edge in E.

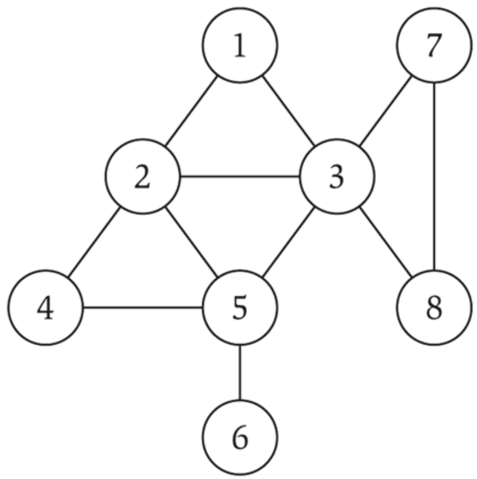
Def. A path is simple if all nodes are distinct.

Def. An undirected graph is connected if for every pair of nodes u and v, there is a path between u and v.



Cycles

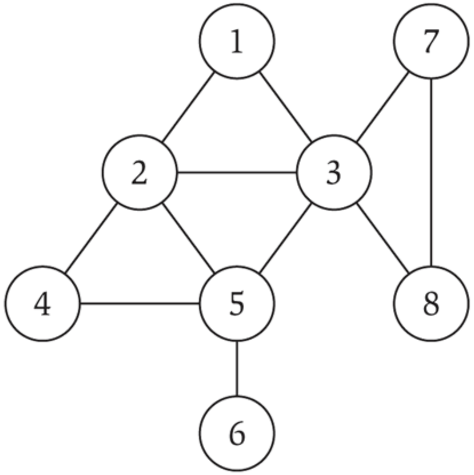
Def. A cycle is a path $v_1, v_2, \ldots, v_{k-1}, v_k$ in which $v_1 = v_k$, $k > 3$, and the first k-1 nodes are all distinct.



cycle C = 1-2-4-5-3-1

Distance

Def. The distance between vertices s and t in a graph G is the number of edges of the shortest path connecting s to t in G.



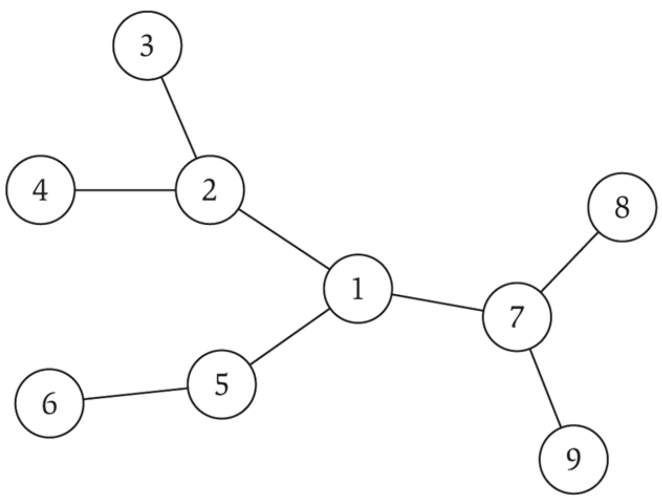
Distance(1,4) =2
Distance(6,3)= 2
Distance(7,8) =1

Trees

Def. An undirected graph is a tree if it is connected and does not contain a cycle.

Theorem. Let G be an undirected graph on n nodes. Any two of the following statements imply the third.

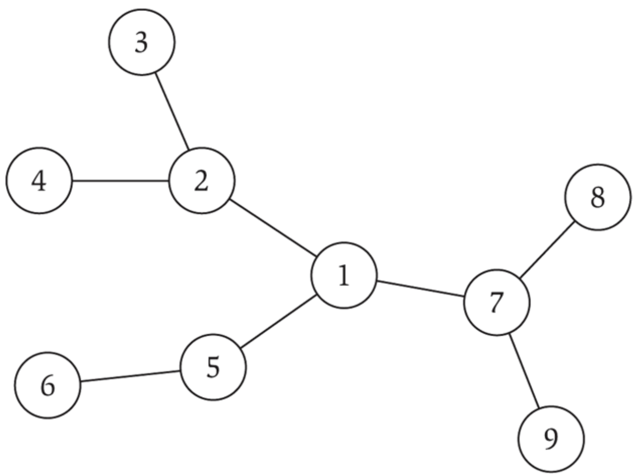
- G is connected.
- G does not contain a cycle.
- G has n-1 edges.



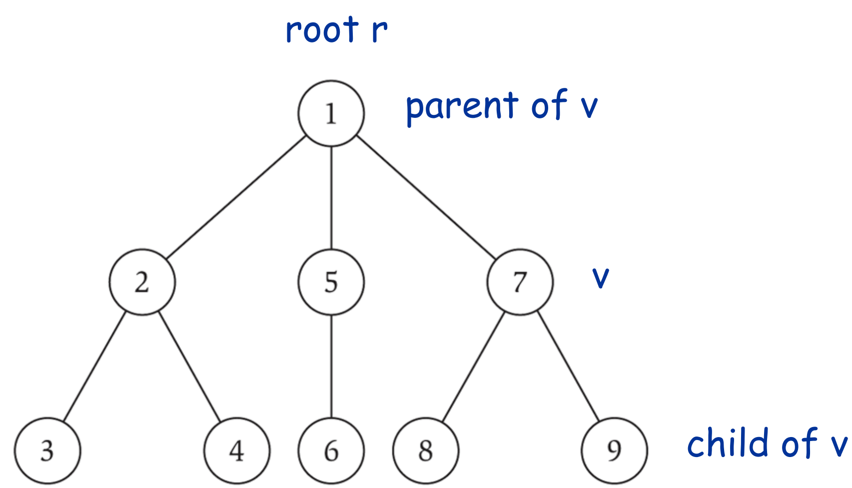
Rooted Trees

Rooted tree. Given a tree T, choose a root node r and "orient" each edge away from r.

Importance. Models hierarchical structure.



a tree



the same tree, rooted at 1

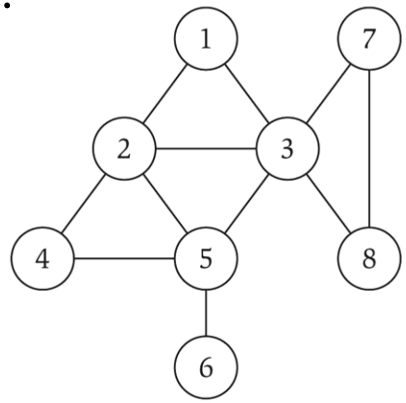
3.2 Graph Traversal

Connectivity

s-t connectivity problem. Given two node s and t, is there a path between s and t?

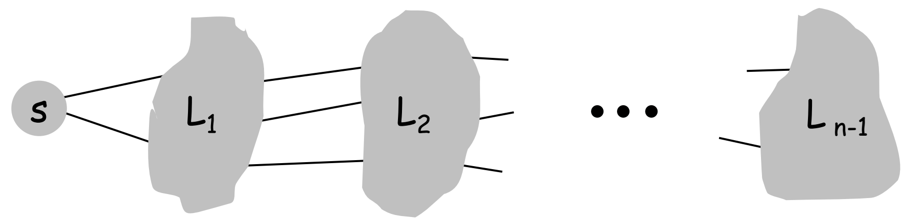
s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?

Applications.
- Maze traversal
- Fastest route
- Minimum number of connections to reach a person on LinkedIn
- Fewest number of hops in a communication network.



Breadth First Search

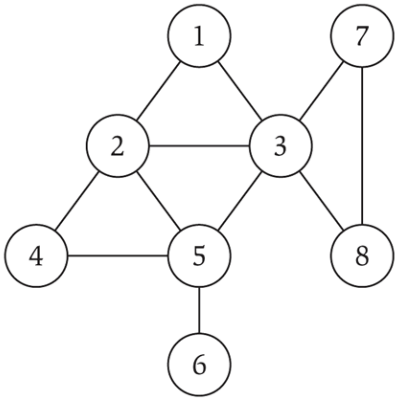
BFS intuition. Explore outward from s in all possible directions, adding nodes one "layer" at a time.



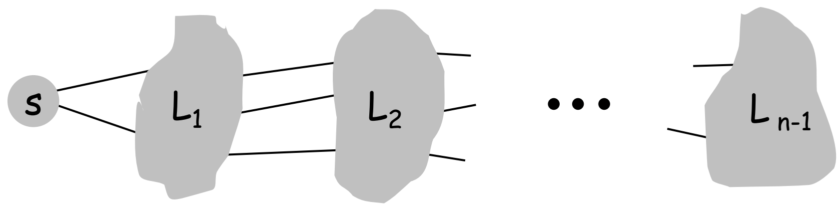
Algorithm BFS(G, s).

- $L_0 = \{ s \}$.
- L_1 = all neighbors of L_0.
- L_2 = all nodes that do not belong to L_0 or L_1, and that have an edge to a node in L_1.
- L_{i+1} = all nodes that do not belong to an earlier layer, and that have an edge to a node in L_i.

Ex: Run BFS(G,1) on this graph



Breadth First Search

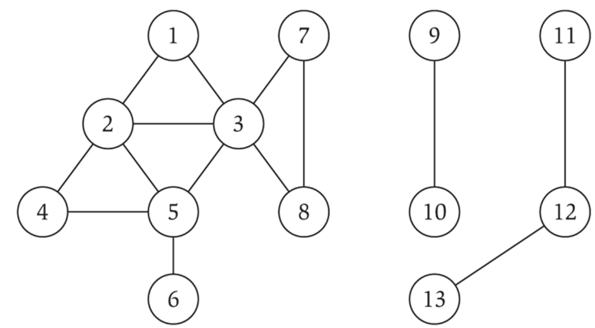


Q: What is the distance of a node in Li from s?

Theorem. For each i, L_i consists of all nodes at distance exactly i from s. Also, there is a path from s to t iff t appears in some layer.

Q: If G is the graph in the right, which nodes does BFS(G,1) visit?

A: Nodes 1,2,...,8



Q: How can we use BFS(G,s) to visit all nodes in the graph?

A: For each node s in G

If s has not been visited, do BFS(G,s)

End for

Breadth First Search: Implementation

Implementation: Maintain list of frontier of nodes in the last level explored, use them to define the next level of nodes

Breadth First Search: Implementation

```
BFS(G, s)     //does BFS starting from node s
Initialize vector of level 0: L[0] = {s}
Mark s as visited
for i = 1 to …
        if all nodes are visited, Return
        L[i]  = [] //level i
        for each u in L[i-1]
              for each v in Adj[u]        ← set of vertices adjacent to u
                    if v has not been visited
                          add v to level L[i]
                          parent[v] = u
                          mark v as visited
```

```
BFS(G)     //does BFS visiting everyone
Mark all nodes as unvisited
for every vertex s of G not visited yet
        do BFS(G,s)
```

Breadth First Search: Implementation

Obs: Cormen's book (and other) have a different code, with a queue (FIFO)

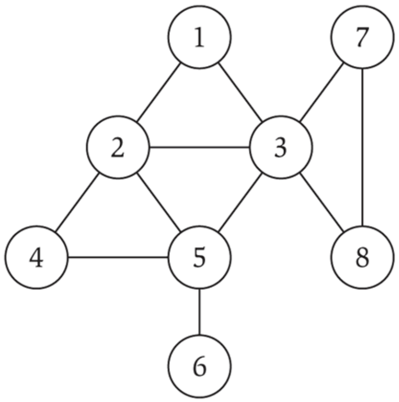
- Gives the same result
- Uses just one queue to keep track of "frontier" and "next"
- Makes sure that all nodes öf the "frontier" come before in the queue than the "next" nodes, so they do not mix

Breadth First Search: Analysis

BFS can "touch" a node many times

- In graph below, BFS(G,1) touches node 3 when looking at neighbors of 1, neighbors of 2, neighbors of 5...

But only touches each edge twice (once in each direction)



Breadth First Search: Analysis

Analysis $O(n^2)$:

- Initialization part costs in total $O(n)$
- Each vertex only apears once as the "u" in green **for** (only apears in 1 level)

  => total of n iterations of green **for** over the whole execution
- But each node has at most n adjacent nodes

  => each iteration of green **for** takes at most n iterations
- Total is $O(n^2)$

```
BFS(G, s)     //does BFS starting from node s
Initialize vector of level 0: L[0] = {s}
Mark s as visited
for i = 1 to …
        if all nodes are visited, Return
        L[i]  = [] //level i
        for each u in L[i-1]
                for each v in Adj[u]
                        if v has not been visited
                                add v to level L[i]
                                parent[v] = u
                                mark v as visited
```

```
BFS(G)     //does BFS visiting everyone
Mark all nodes as unvisited
for every vertex s of G  not explored yet
        do BFS(G,s)
```

Breadth First Search: Analysis

Analysis $O(n + m)$:

- Initialization part costs $O(n)$
- Each vertex only apears once as the “u” in green **for** (only apears in 1 level)

  => total of n iterations of green **for** over the whole execution
- Cost of red **for** is degree(u)
- Total is $\Sigma_{u \in V}$ degree(u) = 2m
- Total cost is $O(n + m)$

```
BFS(G, s)     //does BFS starting from node s
Initialize vector of level 0: L[0] = {s}
Mark s as visited
for i = 1 to …
        if all nodes are visited, Return
        L[i]  = [] //level i
        for each u in L[i-1]
                for each v in Adj[u]
                        if v has not been visited
                                add v to level L[i]
                                parent[v] = u
                                mark v as visited
```

```
BFS(G)     //does BFS visiting everyone
Mark all nodes as unvisited
for every vertex s of G  not explored yet
        do BFS(G,s)
```

Breadth First Search: Applications

Application 1: Finding if there is a path from node s to node t

- Just run BFS(G, s); if there is path from s to t, this BFS visits t, otherwise it does not

Application 2: Length of the shortest path from s to t

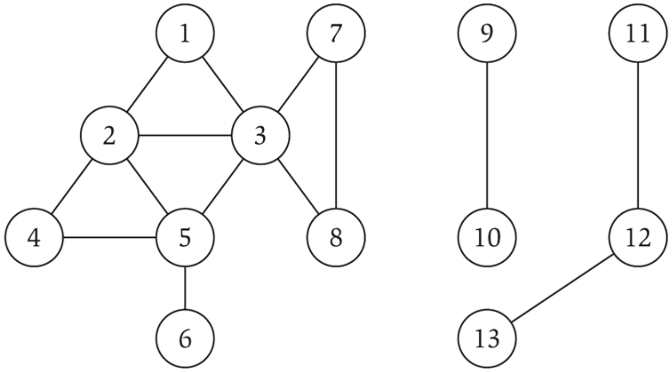
- It's the level[t] computed by BFS(G,s) (if there is a path from s to t)

Application: Connected Component

Definition: Connected set. S is a connected set if v is reachable from u and u is reachable from v for every u,v in S

Definition: Connected Component: The connected "blocks" that compose the graph

More precisely, S is a connected component if is a connected set and for every u in V-S, S $\cup$ {u} is not connected



Application: Connected Component

Since BFS(G,s) visits exactly the nodes in the connected component containing s, we can use it to determine such connected component

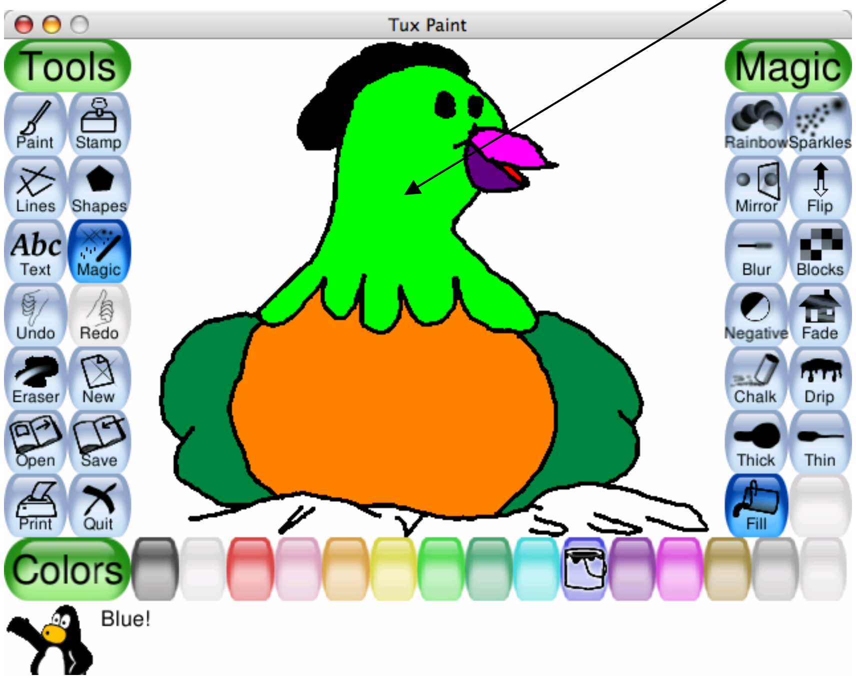
Exercise: Use BFS to output all the connected components of a graph

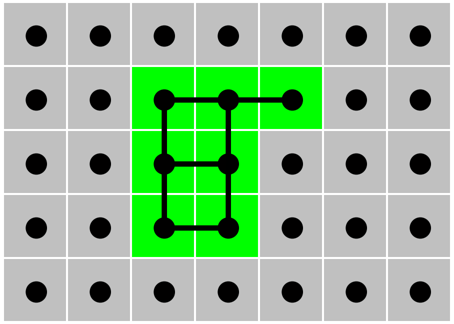
Application: Flood Fill

Flood fill. Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue.

- Node: pixel.
- Edge: two neighboring lime pixels.
- Blob: connected component of lime pixels.

recolor lime green blob to blue





Application: Flood Fill

Flood fill. Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue.

- Node: pixel.
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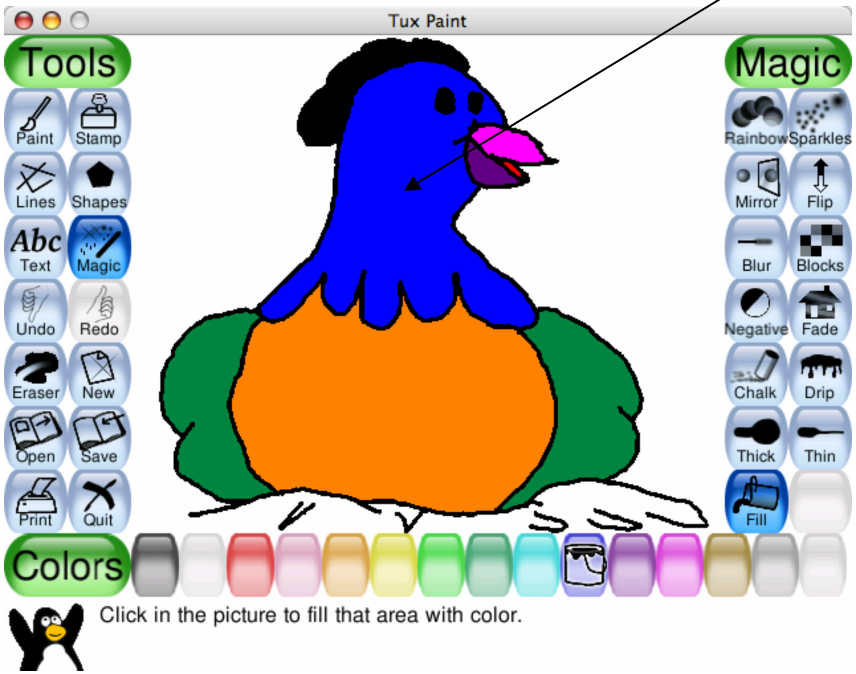
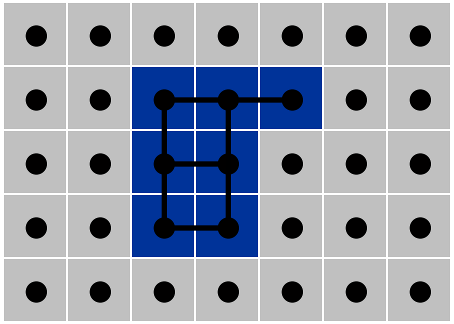
recolor lime green blob to blue





Breadth First Search: Applications

Application: Length of the shortest path from s to t

- It's the level[t] computed by BFS(G,s) (if there is a path from s to t)

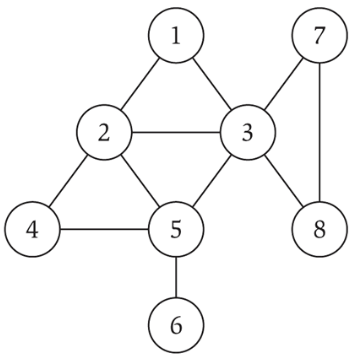
Q: How to get the shortest path, not just length?

Breadth First Search: BFS tree

Definition: A BFS tree of G = (V, E), is the tree induced by a BFS search on G.

- The root of the tree is the starting point of the BFS
- A node u is a parent of v if v is first visited when the BFS traverses the neighbors of u

Ex: BFS(G,1)



L_0

L_1

L_2

L_3

Breadth First Search: BFS tree

Definition: A BFS tree of G = (V, E), is the tree induced by a BFS search on G.

- The root of the tree is the starting point of the BFS
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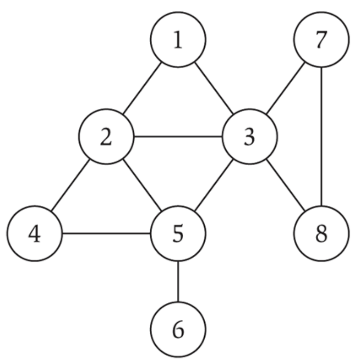
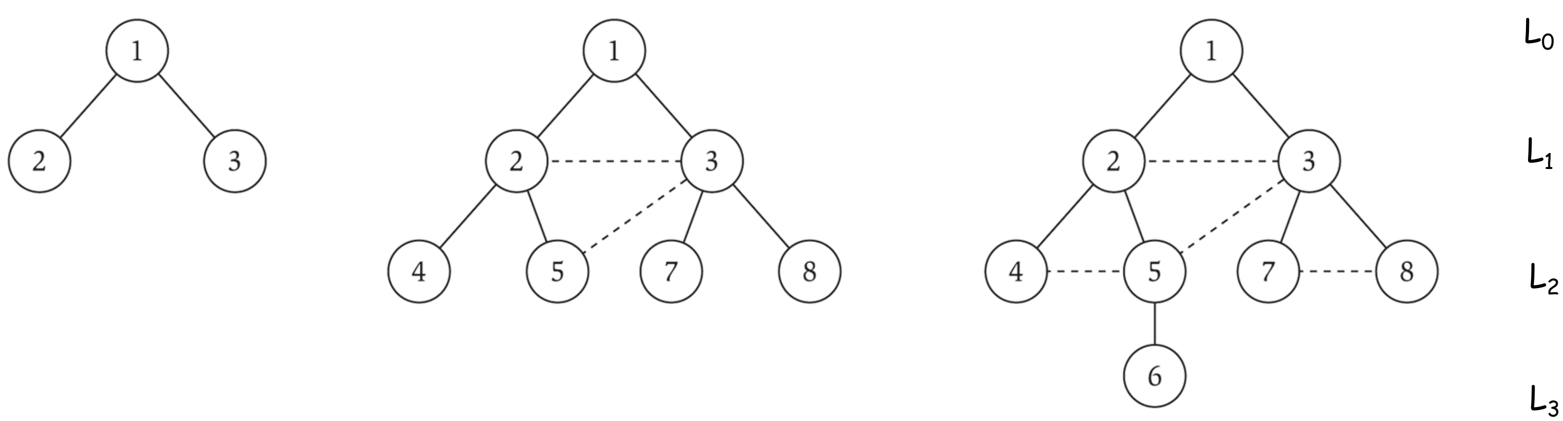
Ex: BFS(G,1)





Breadth First Search: BFS tree

Our BFS algorithm (implicitly) finds a BFS tree: the variable parent[v] indicates the parent of node v in the BFS tree

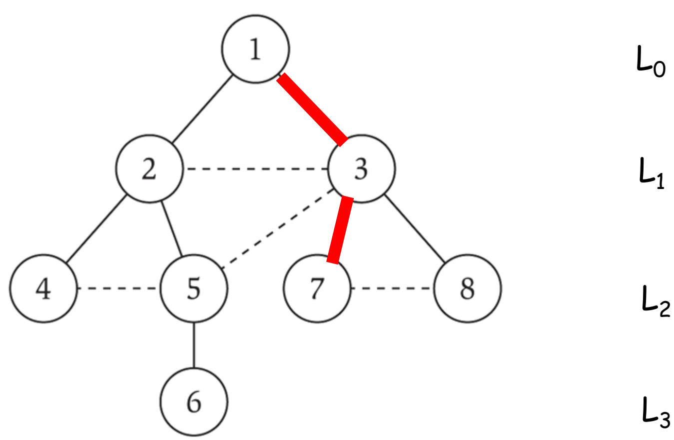
Observation: For the same graph there can be different BFS trees. The BFS tree topology depends on the starting point of the BFS and the rule employed to break ties

Breadth First Search: BFS tree

Q: How do we get the shortest path from s to t using BFS(G,s) ?

A: Run BFS(G,s) and follow the path in the BFS tree from s to t
(or better, start at t and follow to its parent, and then its parent,... until reach s, getting the reverse shortest path from s to t)

Shortest path from 1 to 7



Breadth First Search

Exercise. Let T be a BFS tree of G = (V, E), and let (x, y) be an edge of G. Show that the level of x and y differ by at most 1.

Proof: Cannot be that level(y) >= level(x) + 1: when exploring x, either:

- y has been visited by someone at level <= level(x), so y it put at level <= level(x) + 1
- y has not been visited yet, so x himself add y to level(x) + 1

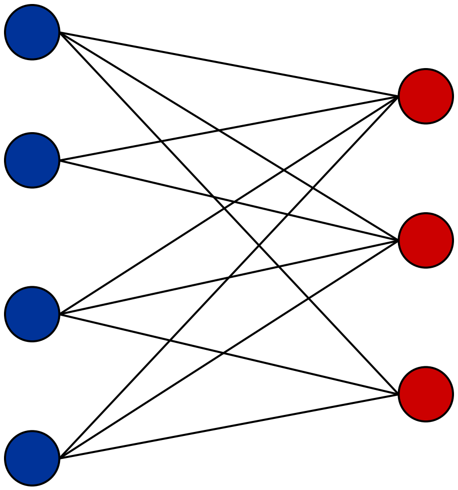
Another application:
Testing Bipartiteness

Bipartite Graphs

Def. An undirected graph G = (V, E) is bipartite if the nodes can be colored red or blue such that every edge has one red and one blue end.

Applications.

- Stable marriage: men = red, women = blue.
- Scheduling: machines = red, jobs = blue.

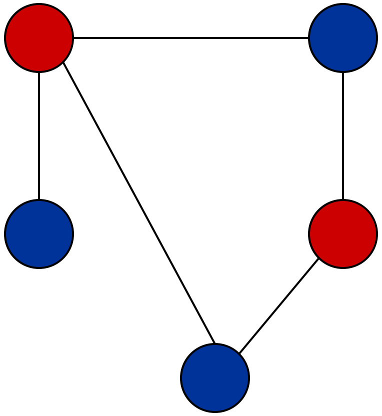


a bipartite graph

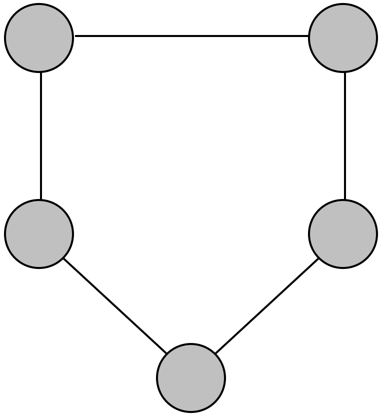
An Obstruction to Bipartiteness

Lemma. If a graph G is bipartite, it cannot contain an odd length cycle.

Pf. Not possible to 2-color the odd cycle, let alone G.



bipartite
(2-colorable)



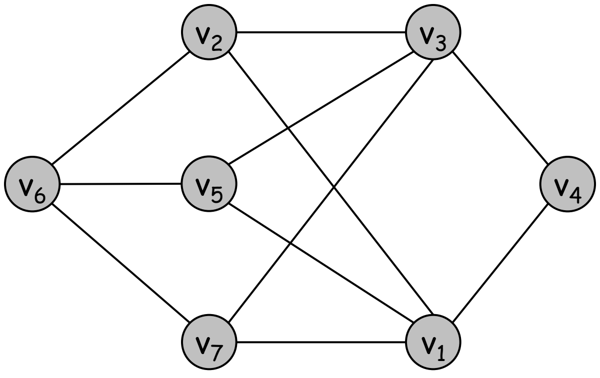
not bipartite
(not 2-colorable)

Testing Bipartiteness

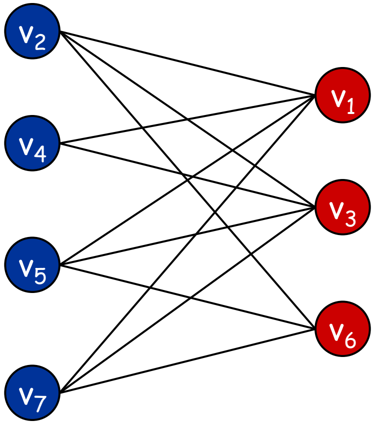
Testing bipartiteness. Given a graph G, is it bipartite?

- Many graph problems become:
  - easier if the underlying graph is bipartite (matching)
  - tractable if the underlying graph is bipartite (independent set)

- So if we detect our graph is bipartite, we may be able to use better algorithms



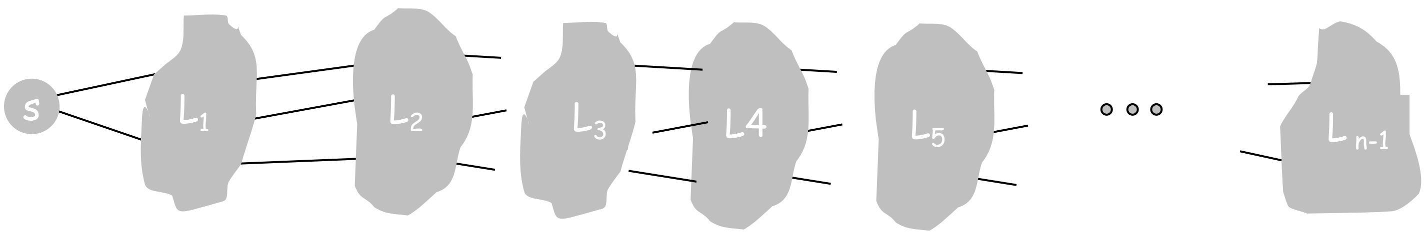
a bipartite graph G



another drawing of G

Testing Bipartiteness

Q: Can we use BFS to test if a graph is bipartite/try to color it?



Testing Bipartiteness

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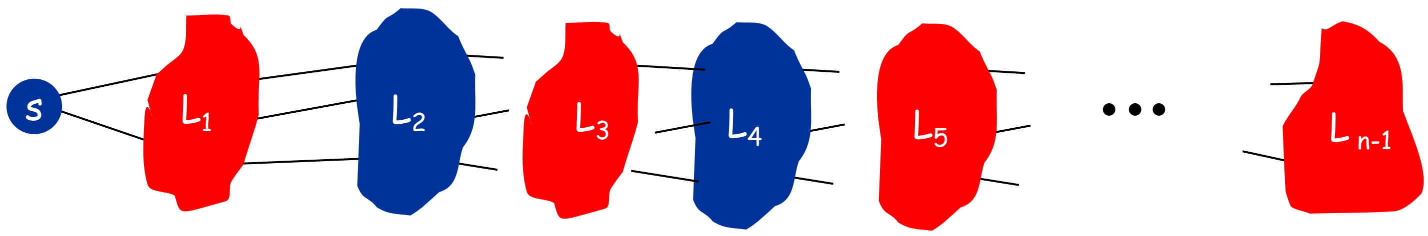
Idea: Color the levels of a BFS(G,s) tree with alternate colors



If there are no edges of G between blue/blue or red/red: done, bipartite

Testing Bipartiteness

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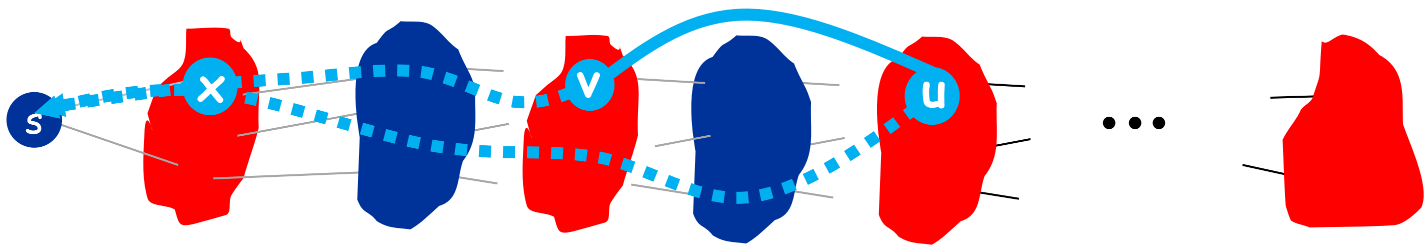


If there are no edges of G between blue/blue or red/red: done, bipartite

If there is an edge of G betweeen blue/blue or red/red:

- Suppose this edge is between nodes u and v
- Walk back from u and from v in the BFS tree; at some point you reach a common node x (it can be the root s)
- The cycle u—x—v-u is odd:
  - Since u and v have the same color, the length of segments u—x and x—v have the same parity (either both odd or both even)
- So graph is not bipartite

Bipartite Graphs

We have just proved the following

Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS(G,s). If we color the layers alternately blue and red, exactly one of the following holds:

(i) There is no blue/blue or red/red edge, and so G is bipartite

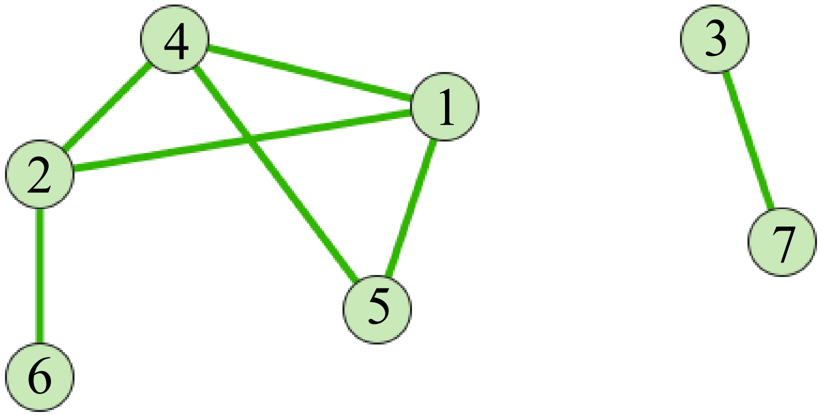
(ii) There is a blue/blue or red/red edge, and G contains an odd-length cycle (and hence is not bipartite).

So the only way we cannot color the graph is if it has an odd cycle

Corollary. (Konig 1916) A graph G is bipartite if and only if it contains no odd length cycle.

Depth first search

Depth First Search (DFS)



DFS(*G*)

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For v in G
    If v not visited then
        DFS-Visit(G, v)
```

DFS-Visit(*G*, u)

```
Mark u as visited
For v in Adj(u)
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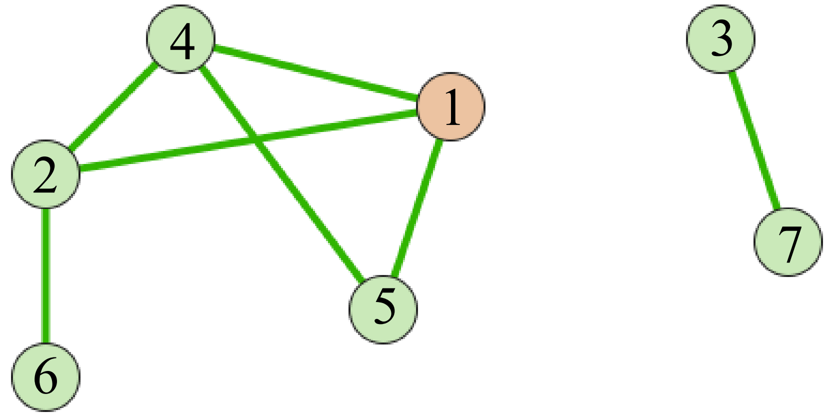

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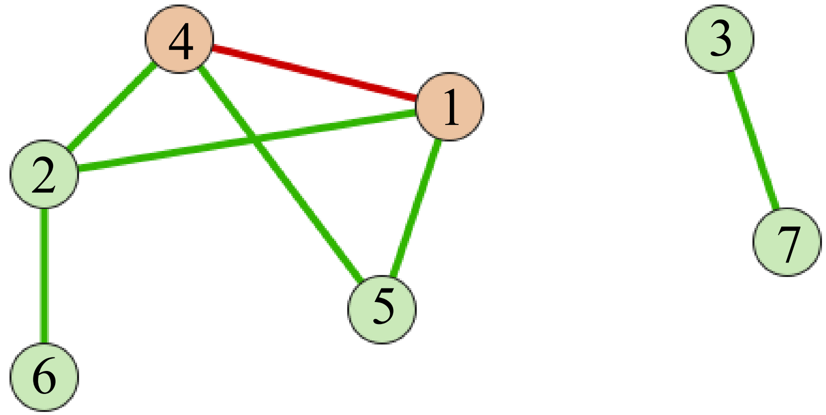

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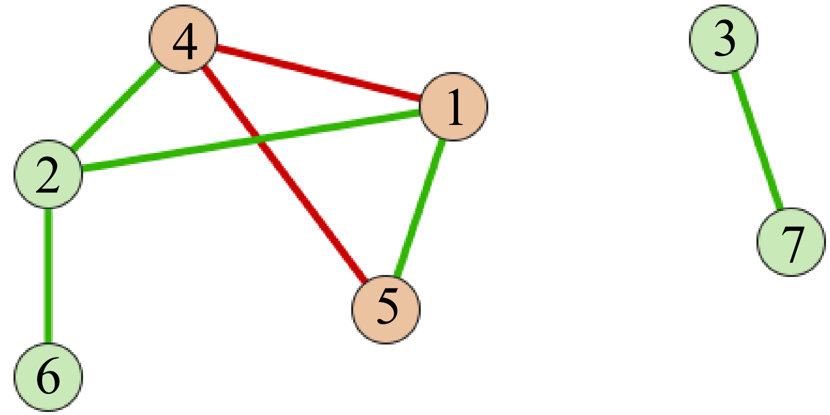

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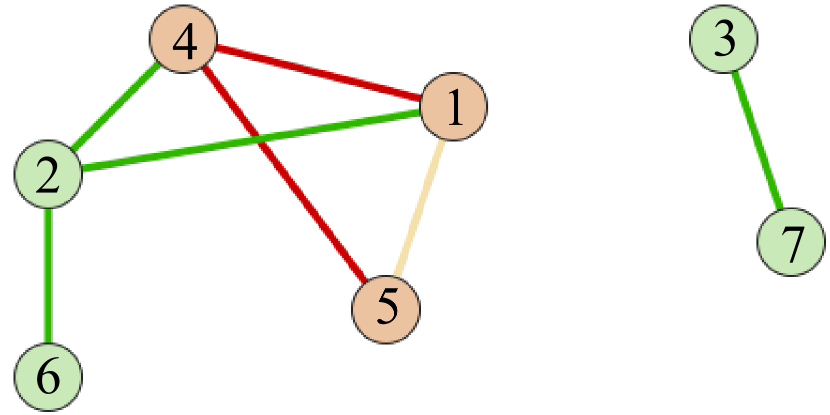


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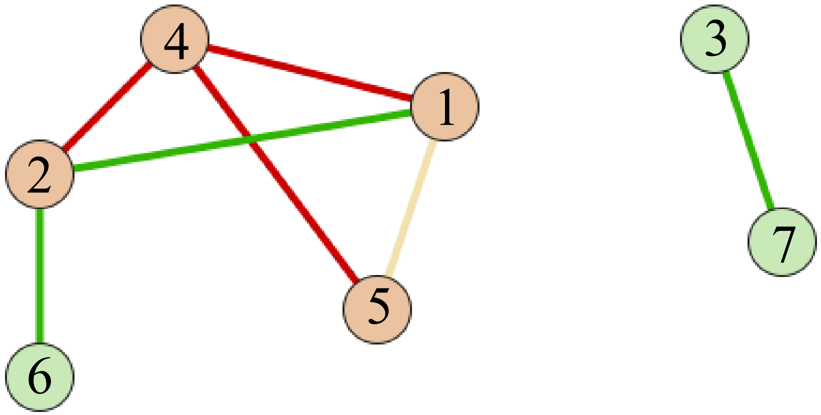

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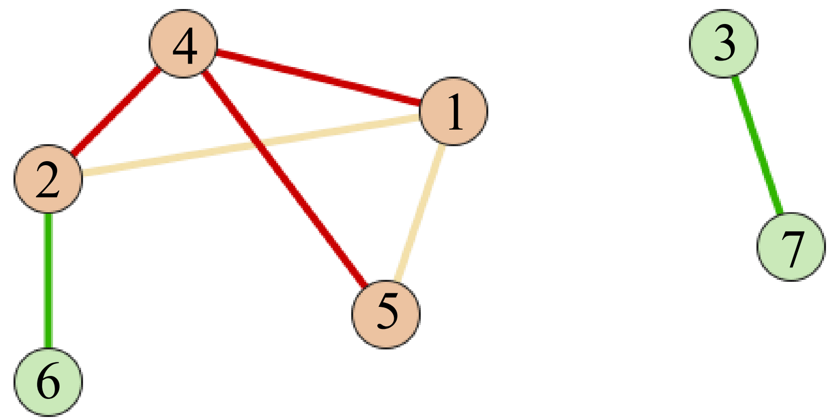

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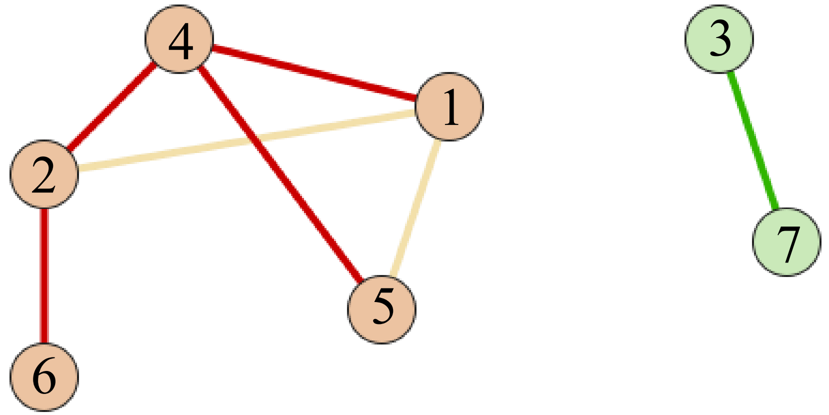


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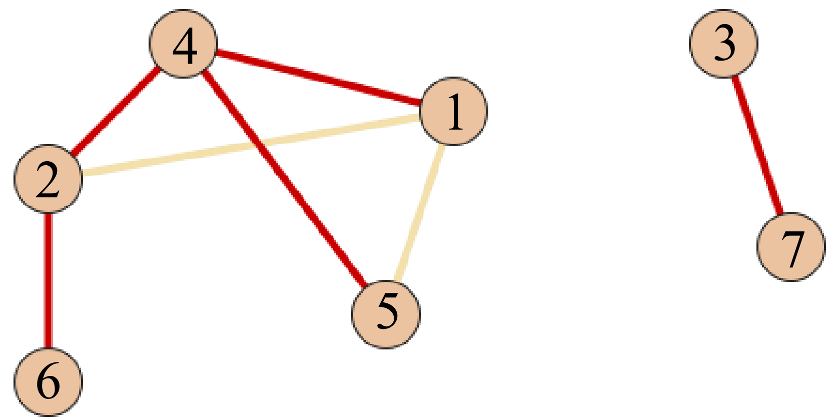

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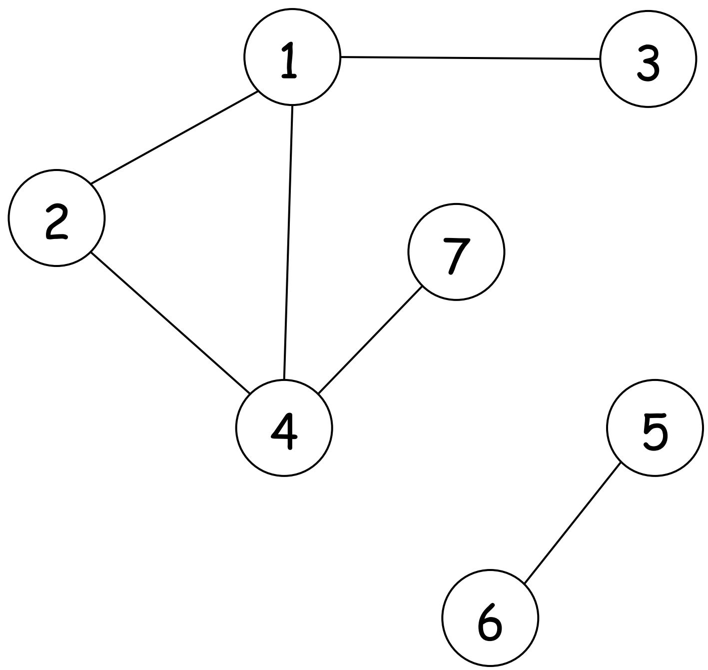
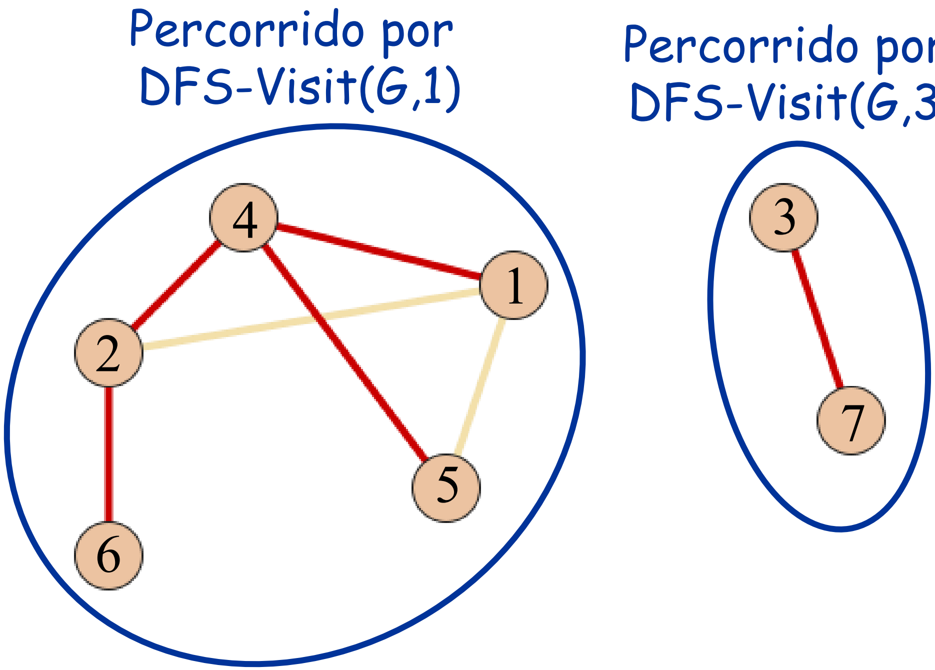
Depth First Search (DFS)

Exercise: Run DFS for the following graph



Depth First Search

Assim como na busca em largura, DFS-Visit(G, u) visita apenas o componente conexo contendo o no de inicio u



Depth First Search: Analysis

DFS-Visit(G,u) tem complexidade

O(#nos no comp. conexo de u + #arestas no comp. conexo de u)

Justificativa:

- O número de blocos na arvore de recursao é exatamente o #nos no comp. conexo de u, pois cada no e visitado uma unica vez
- O custo de cada bloco da arvore de recursao (sem contar as chamadas recursivas) é ~(1 + número de vizinhos do nó associado):
  - Checa pra cada vizinho se ja foi visitado
- Somando o custo de todos os blocos, temos

~ #nos no comp. conex. de u + $\sum_{v: v\ em\ comp\ conexo} \deg(v)$

~ #nos no comp. conex. de u + 2#arestas no comp conexo

Depth First Search: Analysis

A busca completa DFS(G) tem complexidade O(n + m)

Justificativa: Lança uma busca por componente conexo. Somando o custo de cada uma dessas buscas, obtemos o resultado:

custo = $O(\sum_{comp\ conexo}$ (#nos no comp. conexo + #arestas no comp. conexo))

= O(#nos grafo + #arestas grafo)

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    Mark u as visited
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        If v not visited then
            ......
```

Depth First Search: Analysis

Resumo: DFS(G) tem complexidade O(n + m)

Depth First Search

Just like for BFS, we have a DFS tree

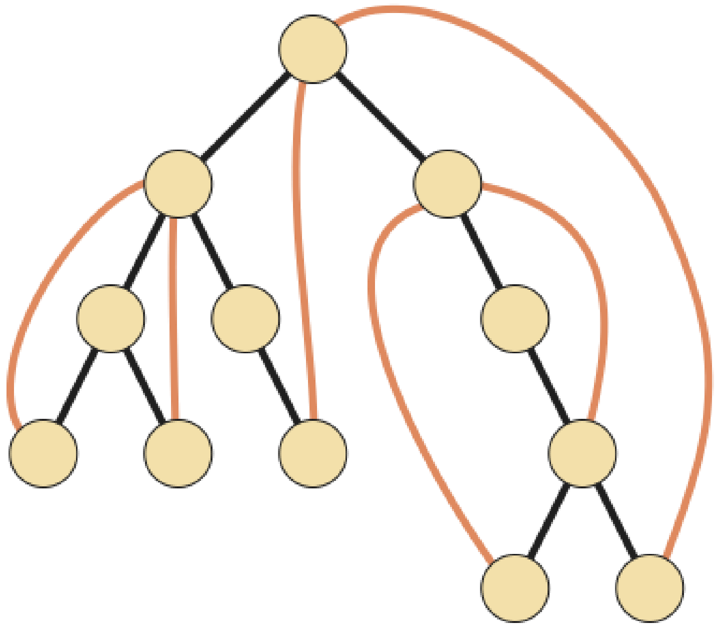
Definition A DFS tree of $G = (V, E)$, is the tree induced by a DFS search on G.

- The root of the tree is the starting point of the DFS
- A node u is a parent of v if v is first visited when the DFS traverses the neighbors of u

Exactly the recursion tree of the algorithm

Properties of DFS

Theorem: Consider a graph G and let T be a DFS tree. Then for any edge vw of G, if v is visited before w then v is an ancestor of w in T



Edges in black: DFS tree
Edges in orange: other graph edges

Properties of DFS

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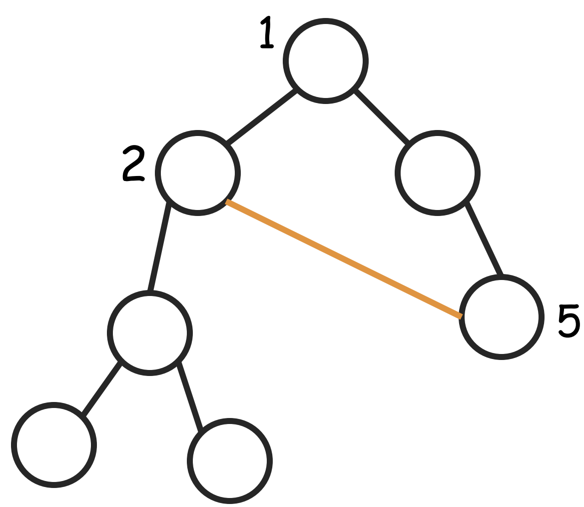
Ex: We cannot "crossing edges" like in the following situation (numbers indicate order in which nodes are visited)



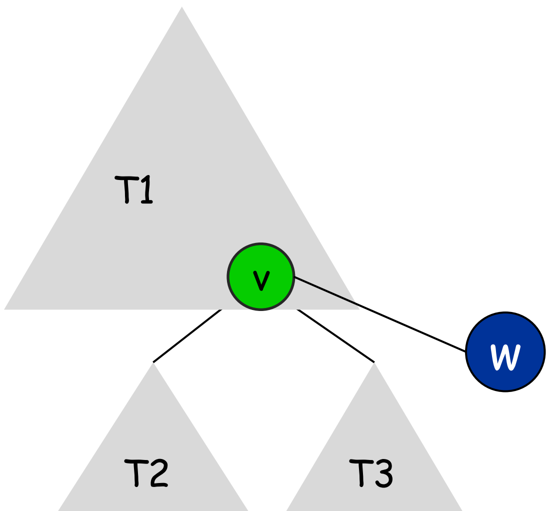
Properties of DFS

Theorem: Consider a graph G and let T be a DFS tree. Then for any edge vw of G, if **v** is visited before **w** then **v** is an ancestor of **w** in T

Proof: Consider the exploration of **v**

- Before stareted exploring **v**, did not visit **w** (so w not in T1)
- Then explored some neighbors of **v** (visiting T2 and T3)
- Now **v** tries to explore neighbor **w**
    - If **w** has not been explored, then **v** is the parent of **w**
    - If **w** has been explored, it must be in T2 or T3, **v** is an ancestor of **w**

(recall **w** not in T1)



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[Write this on the board, we'll use in the next application]

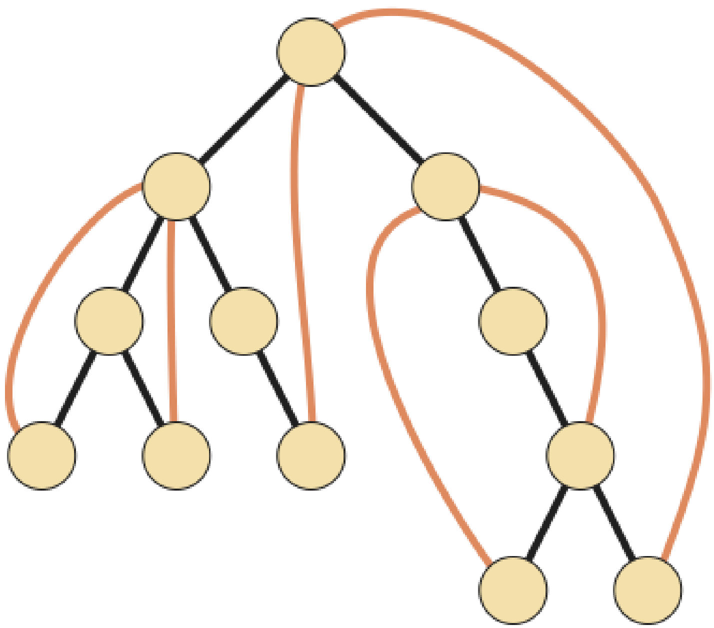
Obs: This is not true for BFS

Exercise: Construct a graph that shows this

Application of DFS: Finding cycles

Q: How can we use DFS to find a cycle in the graph?

A: If tries to revisit nodes in DFS => cycle (only exclude case where trying to revisit parent)



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            Se  w<>pai(v)
                Return Existe Ciclo
        Fim Se
    Fim Para
```

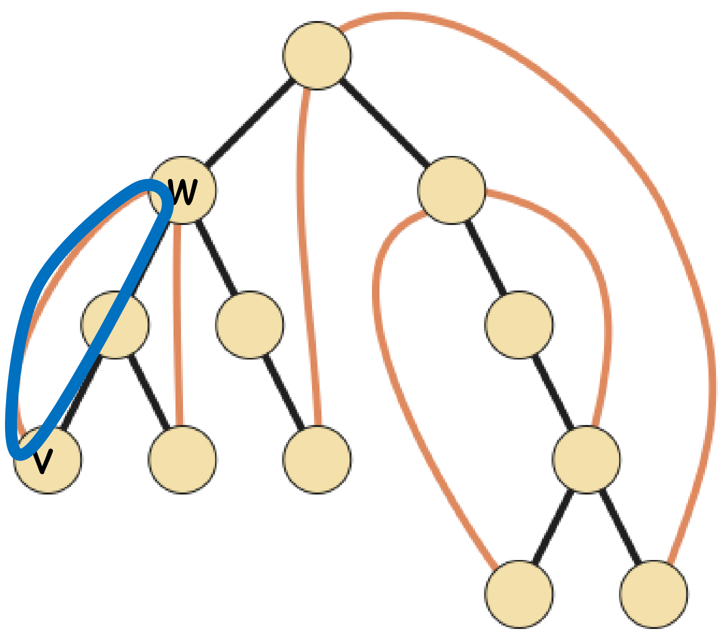
Application of DFS: Finding cycles

Need to show it actually works

Claim 1: If returned "Existe ciclo", then there is a cycle in the graph

Proof: If **w** was already visited and is a neighbor of **v**, then is an ancestor of in DFS tree

If **w** is not the parent of v in the tree,
have cycle w ---- v - w

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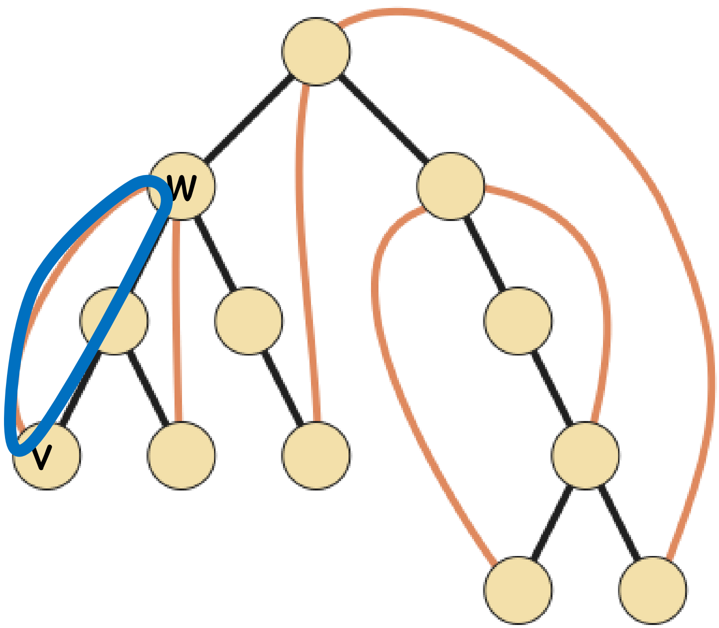

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```

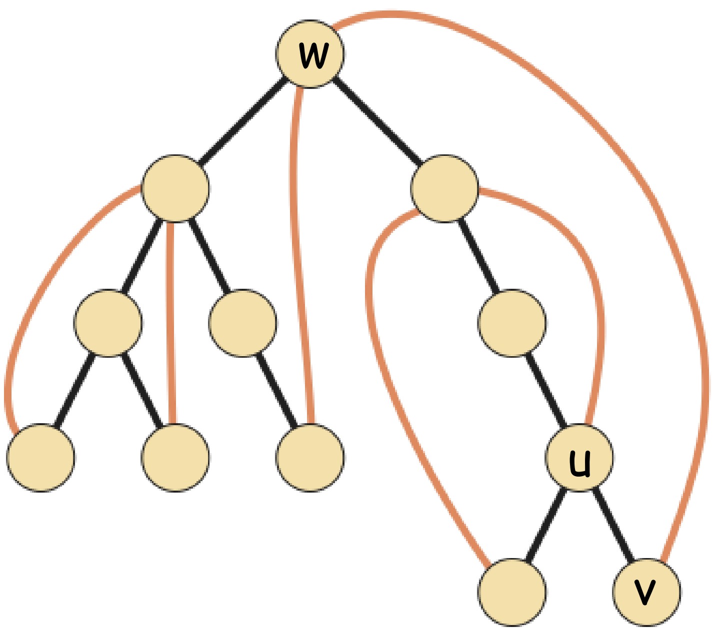
Application of DFS: Finding cycles

Claim 2: If there is cycle in the graph, algo returns “Existe ciclo”

Proof: Let **v** be the last vertex of the cycle visited by the DFS

So both neighbors of **v** in the cycle are of **v**

At least one of them is not the parent of **v** => DFS returns “Existe ciclo”

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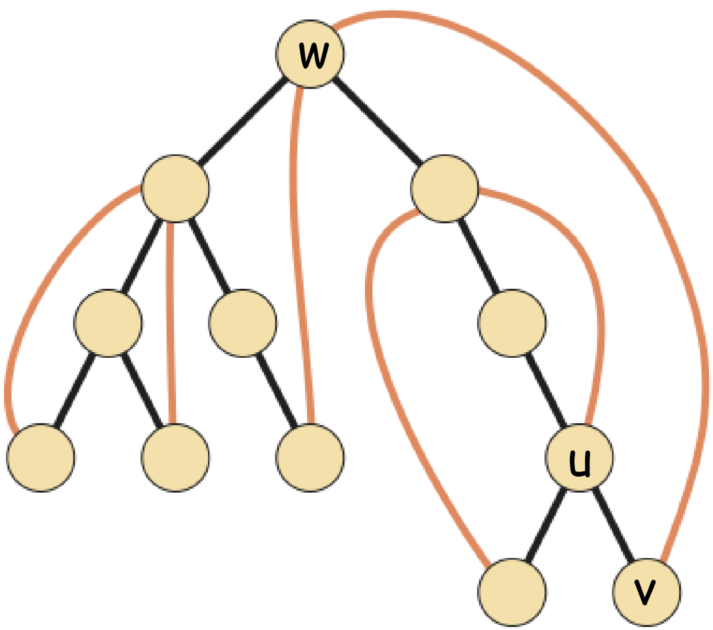


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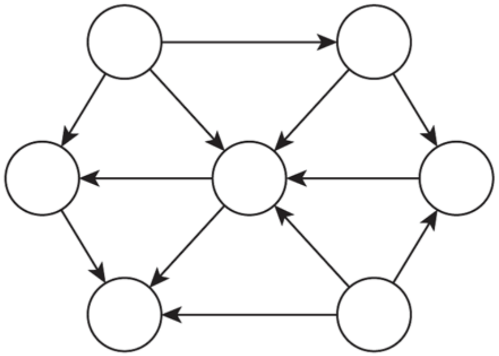
3.5 Connectivity in Directed Graphs

Directed Graphs

Directed graph. $G = (V, E)$

- Edge (u, v) goes from node u to node v.

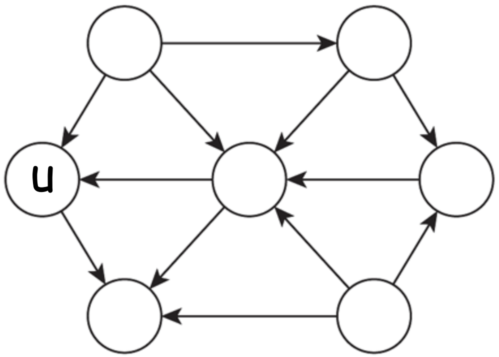


Ex. Web graph - hyperlink points from one web page to another.

- Directedness of graph is crucial.
- Modern web search engines exploit hyperlink structure to rank web pages by importance.

Directed Graphs

- The in-degree $d^{-}(u)$ of a vertex u is the number of edges that arrive at u
- The out-degree $d^{+}(u)$ of a vertex u is the number of edges that leave u



$d^{-}(u)=2$

$d^{+}(u)=1$

Important property:

sum of indegrees sum of outdegre

Directed Graphs

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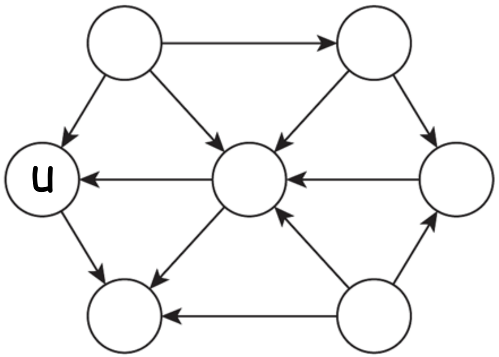


$d^-(u)=2$

$d^+(u)=1$

Important property:

sum of indegrees = sum of outdegre = m

Representation via Adjacency List

Undirected Graph

Directed Graph

Graph Search

Directed reachability. Given a node s, find all nodes reachable from s. (need to use arcs in the right direction)

Directed s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?

Graph search. BFS and DFS extend naturally to directed graphs.

Exercise: Check that you know how to do BFS and DFS in directed graphs!

Application: Web crawler. Start from web page s. Find all web pages linked from s, either directly or indirectly.

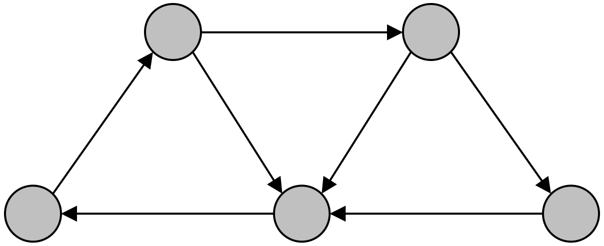
Strong Connectivity

Def. A graph is strongly connected if for every pair of nodes **u**,**v** there is a path from **u to v** and from **v to u**

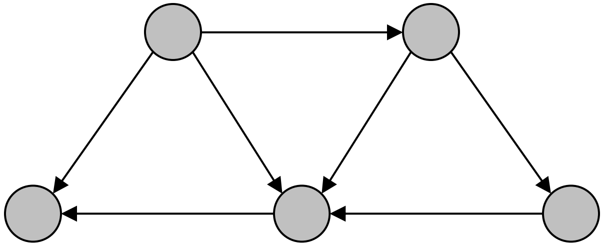
How to decide whether a given graph is strongly connected?

Q: Applications?

A: Road/bus connectivity: no one gets stuck
User interface: make sure user can navigate to/from everywhere



strongly connected



not strongly connected

Strong Connectivity

Q: Give a simple algorithm to decide where a graph is strongly connected or not

Algorithm 1

```
SC ← true
For all u,v  in V
        Run DFS(u)
        If the search does not reach v
                SC ← False
        End If
End
Return SC
```

Analysis:

$O(n^2(m+n))$

Strong Connectivity

Q: Can we do better?

A: Can use 1 search to check if everyone is reachable from u

Algorithm 2

SC ← true
For all u in V
  Run DFS(u)
  If the search does not visit all nodes
    SC ← False
  End If
End
Return SC

Analysis:

O(n (m+n))

Strong Connectivity

Q: Even better??

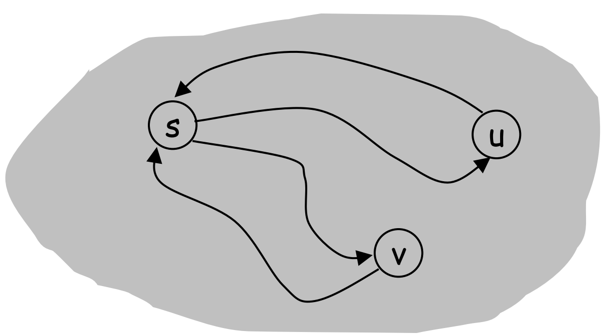
Lemma. Consider a node **s**. G is strongly connected $\Leftrightarrow$ every node is reachable from **s**, and **s** is reachable from every node.

Pf. $\Rightarrow$ Follows from definition.

Pf. $\Leftarrow$ Can go from any node u to v (in both directions):

Path from u to v: concatenate u-s path with s-v path.

Path from v to u: concatenate v-s path with s-u path. ▪



ok if paths overlap

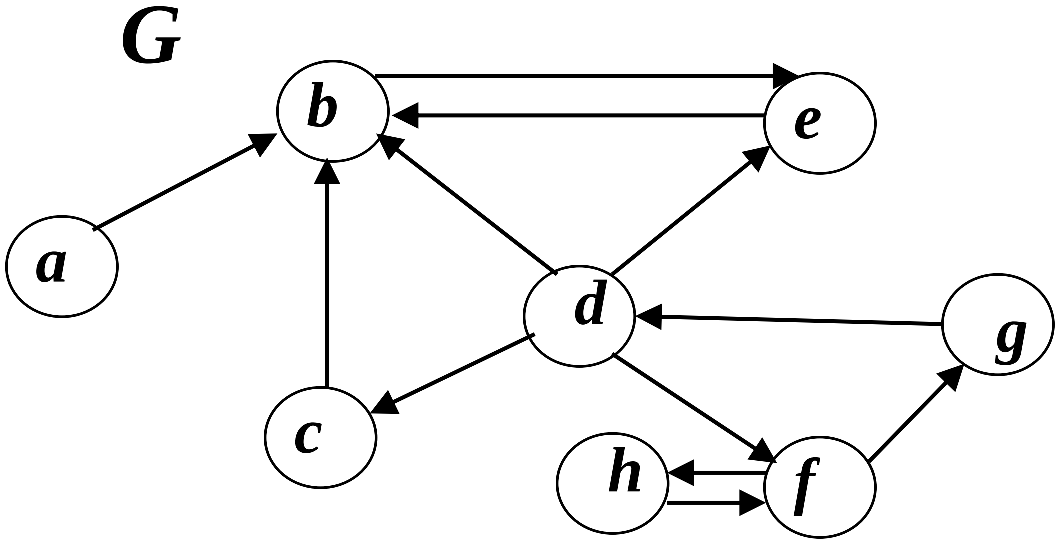
Strong Connectivity

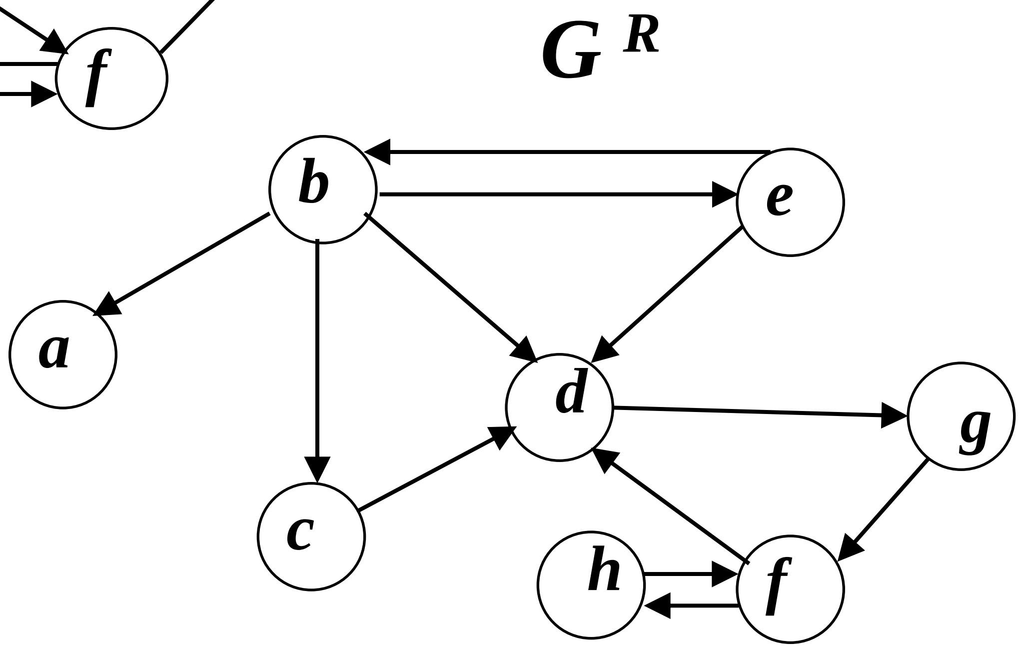
Def. The reverse graph of a graph G is obtained by reversing the directions of all the edges

Observation: The reverse graph of a graph G can be constructed in $O(m+n)$ time

Strong Connectivity

Example:



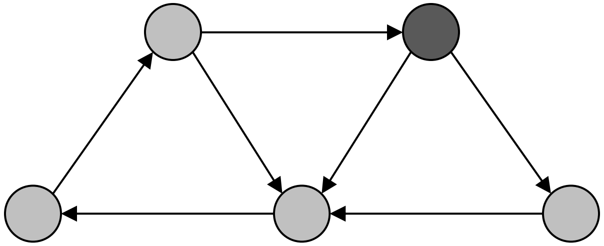


Strong Connectivity: Algorithm

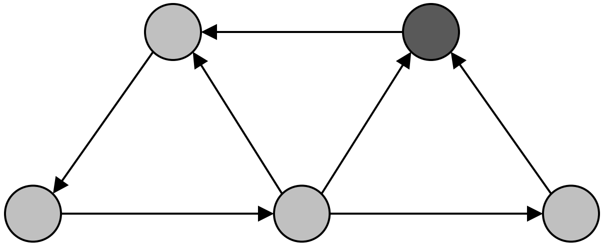
Theorem. Can determine if G is strongly connected in $O(m + n)$ time.

Pf.

- Pick any node s.
- (s reaches everyone?) Run BFS/DFS from **s** in G.
- (everyone reaches s?) Run BFS/DFS from **s** in reverse graph G^R.
- Return true iff all nodes reached in both BFS/DFS executions.
- Correctness follows immediately from previous lemma. ▪



Graph G



Reverse graph G^R

Using graphs to model state space

Problema

- Seja um grafo G=(V,E) com n vértices representando a planta de um edifício. Inicialmente temos dois robos localizados em dois vértices *a* e *b*, que devem alcançar os vértices *c* e *d* respectivamente. Queremos manter sempre uma distancia de seguranca r entre eles.

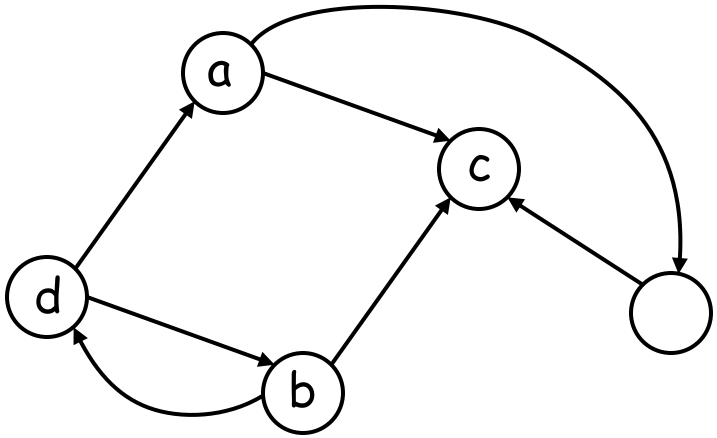
- No passo i+1 um dos dois robos deve caminhar para um vértice adjacente ao vértice que ele se encontra no momento i. Exiba um algoritmo polinomial para resolver o seguinte problema:

- **Entrada:** Grafo G=(V,E) , quatro vértices: *a*,*b*,*c* e *d* e um inteiro r.

- **Saída: SIM** se é possível os robos partirem dos vértices *a* e *b* e chegarem em *c* e *d*, respectivamente, sem que em nenhum momento eles estejam a distância menor do que r. **NÃO**, caso contrário.

Example graph

r = 2



Solução

Seja H=(V',E') um grafo representando as configurações possíveis (posições dos robos) do problema. Cada nó de H corresponde a um par ordenado de vértices do grafo original G cuja distância é menor ou igual a r. Logo existem no máximo $|V|^2$ vértices em H.

Um par de nós u e v de H tem uma aresta se e somente em um passo é possível alcançar a configuração v a partir da configuração u. Mais formalmente, se uv é uma aresta de E', com u=(u1,u2) e v=(v1,v2), então uma das alternativas é válida

(i) u1=v1 e (u2,v2) pertence a E

(ii) u2=v2 e (u1,v1) pertence a E

O problema, portanto, consiste em decidir se existe um camìnho entre o nó x=(a,b) e o nó y=(c,d) em H.

Solução

Para construir o grafo H basta realizar **n** BFS's no grafo G, cada uma delas partindo de um vértice diferente. Ao realizar uma BFS a partir de um nó s obtemos o conjunto de todos os vértices que estão a distância maior ou igual a r de s. A obtenção do conjunto V' tem custo $O(n(m+n))$ e a do conjunto de arestas E' tem custo $O(n^3)$.

Decidir se existe um camìnho entre o nó $x=(a,b)$ e o nó $y=(c,d)$ em H tem complexidade $O(|V'|+|E'|)$. Como $|V'|$ tem $O(n^2)$ vértices e $|E'|$ tem $O(n^3)$ arestas, o algoritmo executa em $O(n^3)$. Note que $|E'|$ é $O(n^3)$ porque cada vértice de H tem no máximo $2(n-1)$ vizinhos

BFS/DFS exercises

Exercises:

1. Suppose your graph is an undirected tree. If run BFS starting from the root of the tree, in which order are the nodes explored? What about in DFS?

2. Using the BFS/DFS tree, show that every connected undirected graph has a node that can be removed keeping the graph still connected [show example]

3. Suppose your undirected graph has a value x(v) for each node. Modify DFS to compute

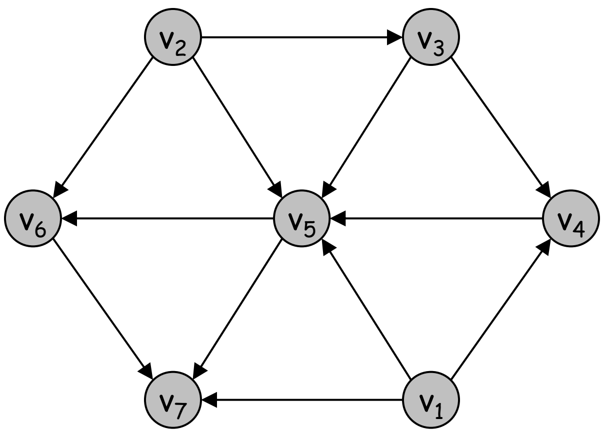
z(v)=sum of values of all descendants of v in the DFS tree,

for all nodes. The algorithm should still run in O(n + m)

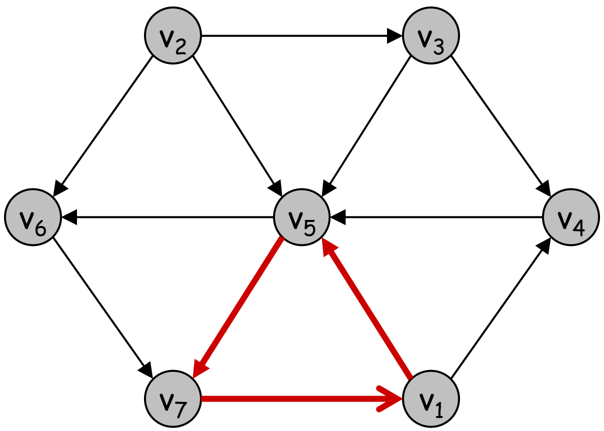
3.6 DAGs and Topological Ordering

Directed Acyclic Graphs

Def. An DAG is a directed graph that contains no directed cycles.



a DAG



Not a DAG

Precedence Constraints

Precedence constraints. Edge (v_i, v_j) means task v_i must occur before v_j.

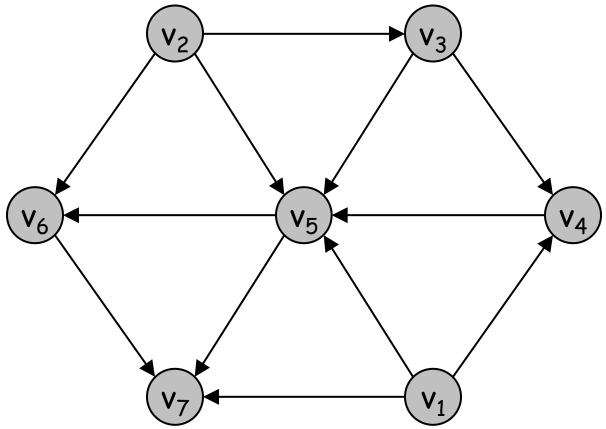
Applications.

- Course prerequisite graph: course v_i must be taken before v_j.
- Compilation: module v_i must be compiled before v_j.

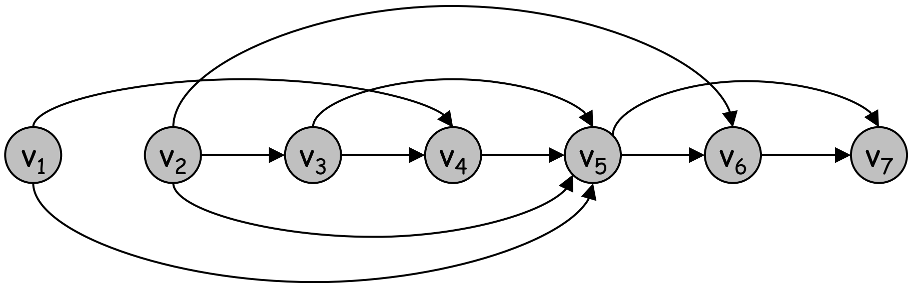
Q: What is a feasible sequence of courses?
What is a feasible order to compile the jobs?

Directed Acyclic Graphs

Def. A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.



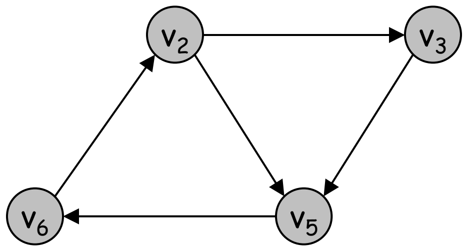
G



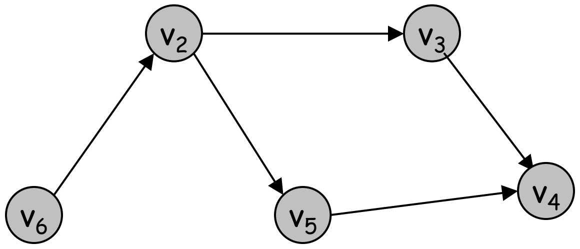
a topological ordering for G

Directed Acyclic Graphs

Def. A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.



Has no topological order



Topological orders:

v6->v2->v3->v5->v4

v6->v2->v5->v3->v4

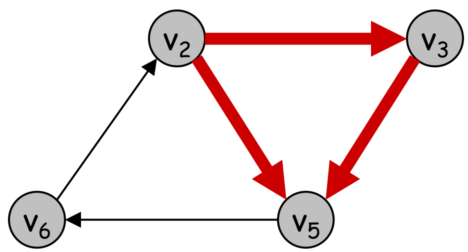
What is the relation between DAG's and topological orderings?

Directed Acyclic Graphs

Obs: Directed cycle does not have a topological order

Since we cannot topologically order a directed cycle, we cannot do it for any graph containing a directed cycle

Lemma. If G has a topological order, then G is a DAG.



Has no topological order

Q. Does every DAG have a topological ordering?

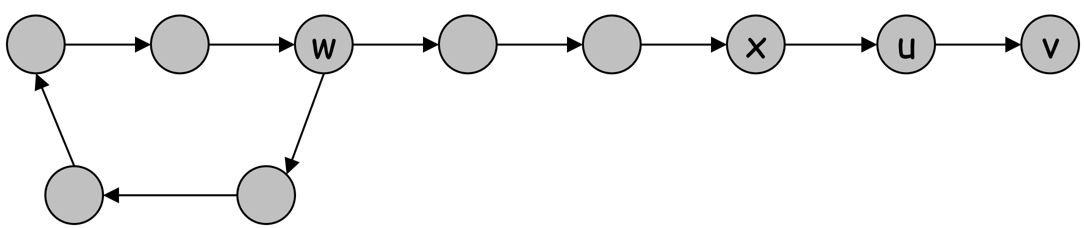
Q. If so, how do we compute one?

Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)

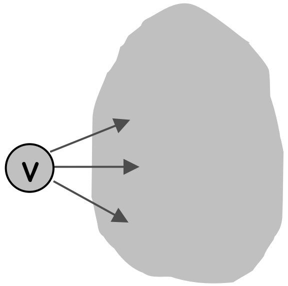
- Suppose that G is a DAG and every node has at least one incoming edge. Let's see what happens.
- Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.
- Then, since u has at least one incoming edge (x, u), we can walk backward to x.
- Repeat until we visit a node, say w, twice.
- Let C denote the sequence of nodes encountered between successive visits to w. C is a cycle. ▪



Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a topological ordering.

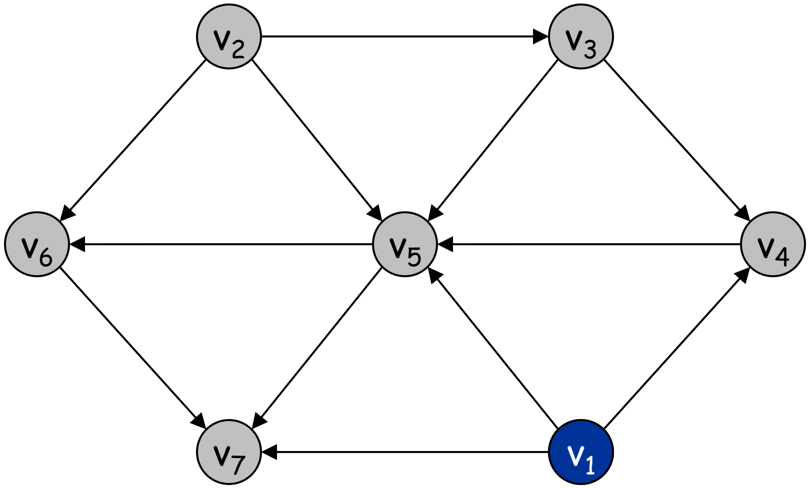
```
To compute a topological ordering of G:
Find a node v with no incoming edges and order it first
Delete v from G
Recursively compute a topological ordering of G−{v}
  and append this order after v
```



Proof that it works: (by induction on n)

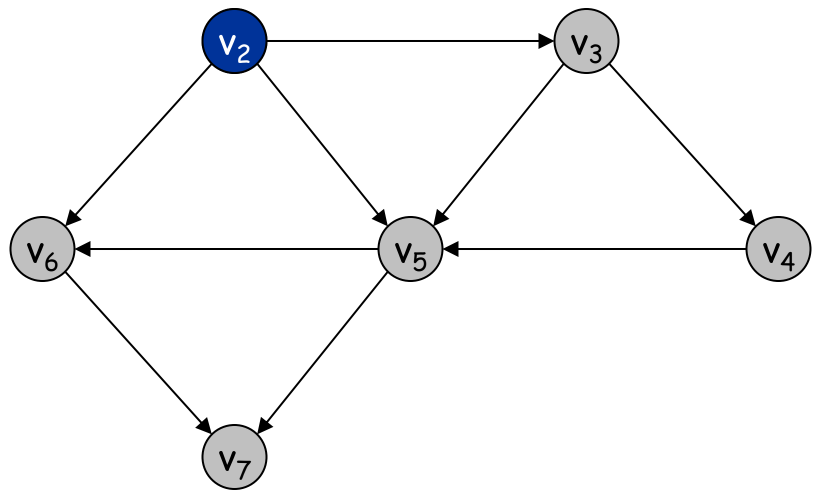
- Base case: true if n = 1.
- Given DAG on n > 1 nodes, find a node v with no incoming edges.
- G - { v } is a DAG, since deleting v cannot create cycles.
- By inductive hypothesis, G - { v } has a topological ordering.
- Place v first in topological ordering; then append nodes of G - { v }
- in topological order. This is valid since v has no incoming edges. ▪

Topological Ordering Algorithm: Example



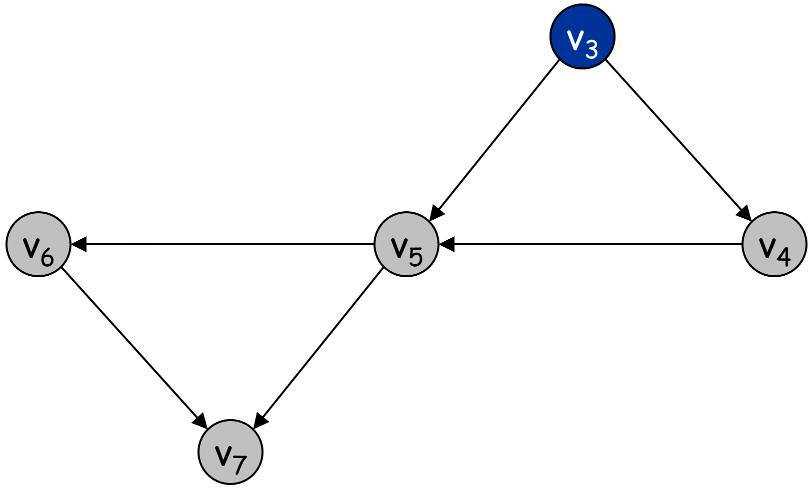
Topological order:

Topological Ordering Algorithm: Example



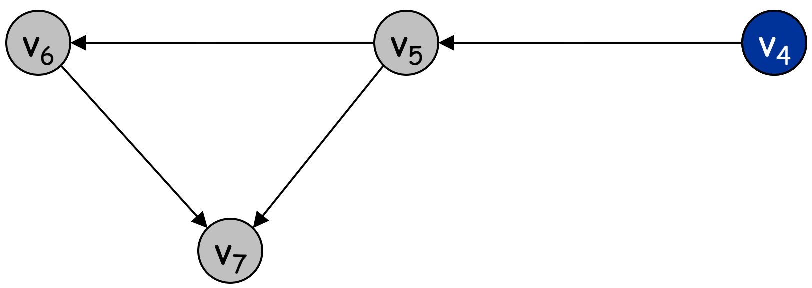
Topological order: v_1

Topological Ordering Algorithm: Example



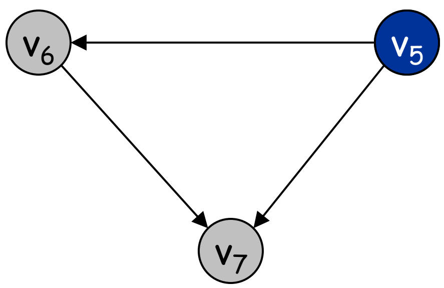
Topological order: v_1, v_2

Topological Ordering Algorithm: Example



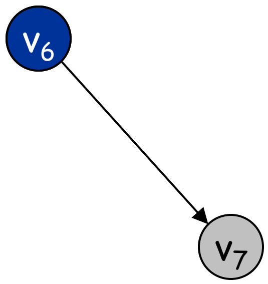
Topological order: v_1, v_2, v_3

Topological Ordering Algorithm: Example



Topological order: v_1, v_2, v_3, v_4

Topological Ordering Algorithm: Example

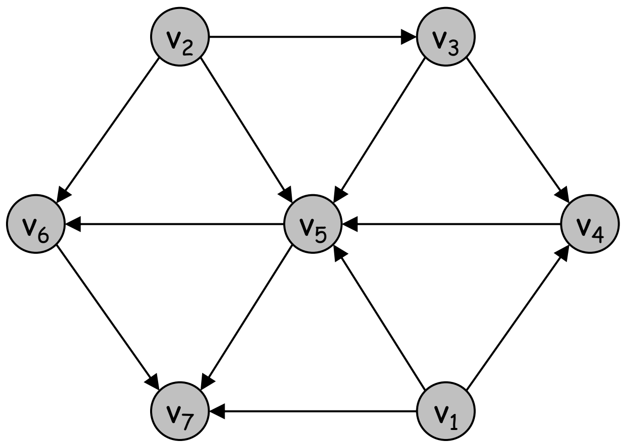


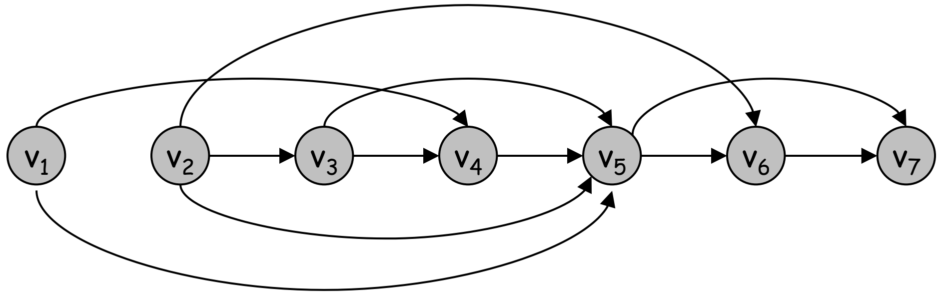
Topological order: v_1, v_2, v_3, v_4, v_5

Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3, v_4, v_5, v_6$

Topological Ordering Algorithm: Example





Topological order: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$.

Topological Sorting Algorithm: Running Time

Q: How to implement this algorithm with fast running time?

Implementation idea: keep a vector `count` that stores for each node v the number of remaining edges that are incident in v

Implementation 1:

```
i←0
While i< n
        v ←node with minimum value in count
        i++
        If v has value larger than 0
                Return G is not a DAG
        End If
        Add v to the topological order
        Remove v from count
        Update the vector count for the nodes adjacent to v
End
```

Topological Sorting Algorithm: Running Time

Analysis : `count` stored as a vector

O(n+m) to compute the `count`

The loop executes at most n times

O(n) to find the node v with minimum degree

O(1) to remove v

O(d^+(u)) to update the neighbors of v

➔ O(n^2 + m)

Analysis : `count` stored as a heap

O(n+m) to compute the vector `count`

The loop executes at most n times

O(1) to find the node v with minimum degree

O(log n) to remove v

O(d^+(u) log n) to update the neighbors of v

➔ O(n log n + m log n)

Topological Sorting Algorithm: Running Time

Theorem. We can implement the algorithm to find a topological order in O(m + n) time.

Pf.

- Maintain the following information:
  - `count[w]` = remaining number of incoming edges
  - S = set of remaining nodes with no incoming edges
- Initialization: O(m + n) via single scan through graph.
- Update: to delete v
  - remove v from S
  - decrement `count[w]` for all edges from v to w, and add w to S if `count[w]` hits 0
  - this is O(1) per edge ▪

Detecting if a directed graph is DAG

Q: We How can we detect if a directed graph G has a directed cycle or not?

A: Try to run topological ordering algorithm on G. Works ⇔ G does not have cycle

- G does not have cycle => works
- G does have a cycle => cannot work, since G does not have top. Order

Q: Where does algorithm does not work if graph has cycle?

A: At some point it will not find a node with in-degree 0

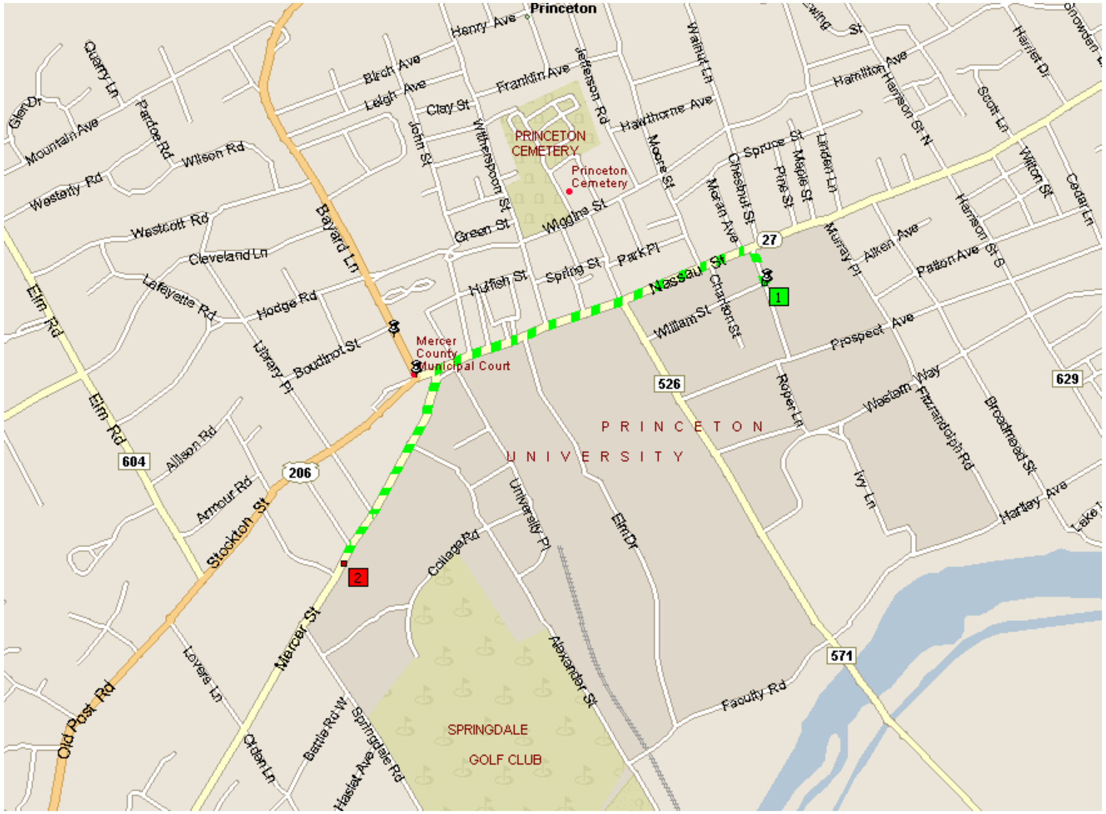
Exercises topological order

1. Suppose you have a DAG where each node has a price p(v). Let cost(u) be the smallest price of all nodes reachable from u. Use topological order to compute cost(u) for all nodes in the graph in O(n + m).

2. Given a list of courses a student needs to take and the prerequisites between then, give an algorithm that finds the minimum number of semesters needed for the student to finish all the courses.

[give concrete example on the board]

4.4 Weighted Shortest Paths in a Graph



shortest path from Princeton CS department to Einstein's house

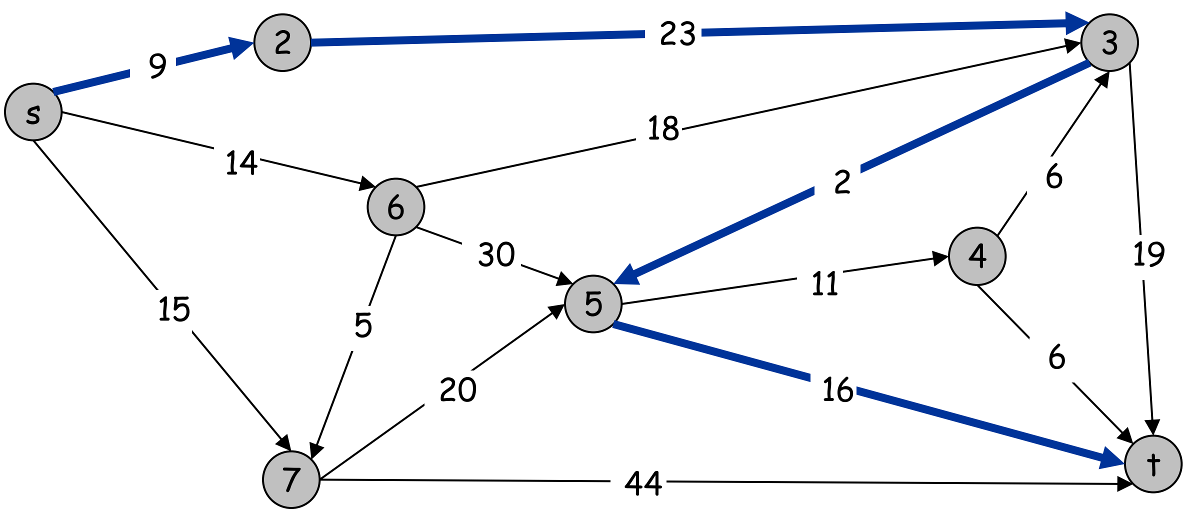
Shortest Path Problem

Shortest path network.

- Directed graph $G = (V, E)$.
- Source s, destination t.
- Length c_e = length of edge e. (non-negative numbers)

Shortest path problem: find shortest directed path from s to t.

Length of path = sum of lengths in path

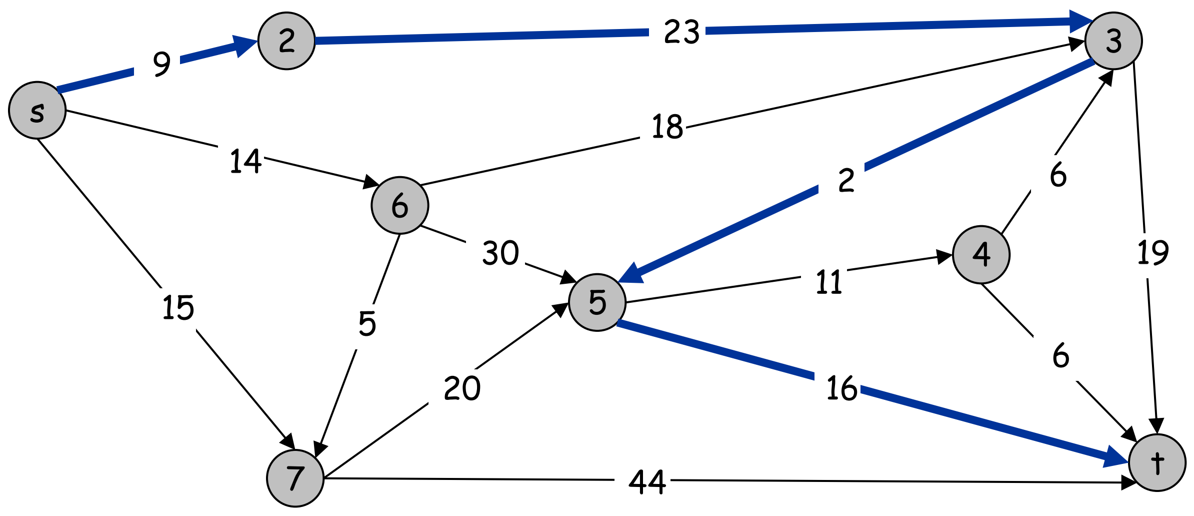


Length of path s-2-3-5-t
= 9 + 23 + 2 + 16
= 50.

Shortest Path Problem

Q: Does BFS give shortest path now that we have different lengths?

A: No



Shortest Path Problem

Q: Suppose all lengths are integers. Can we use BFS on a modified graph to find shortest path?

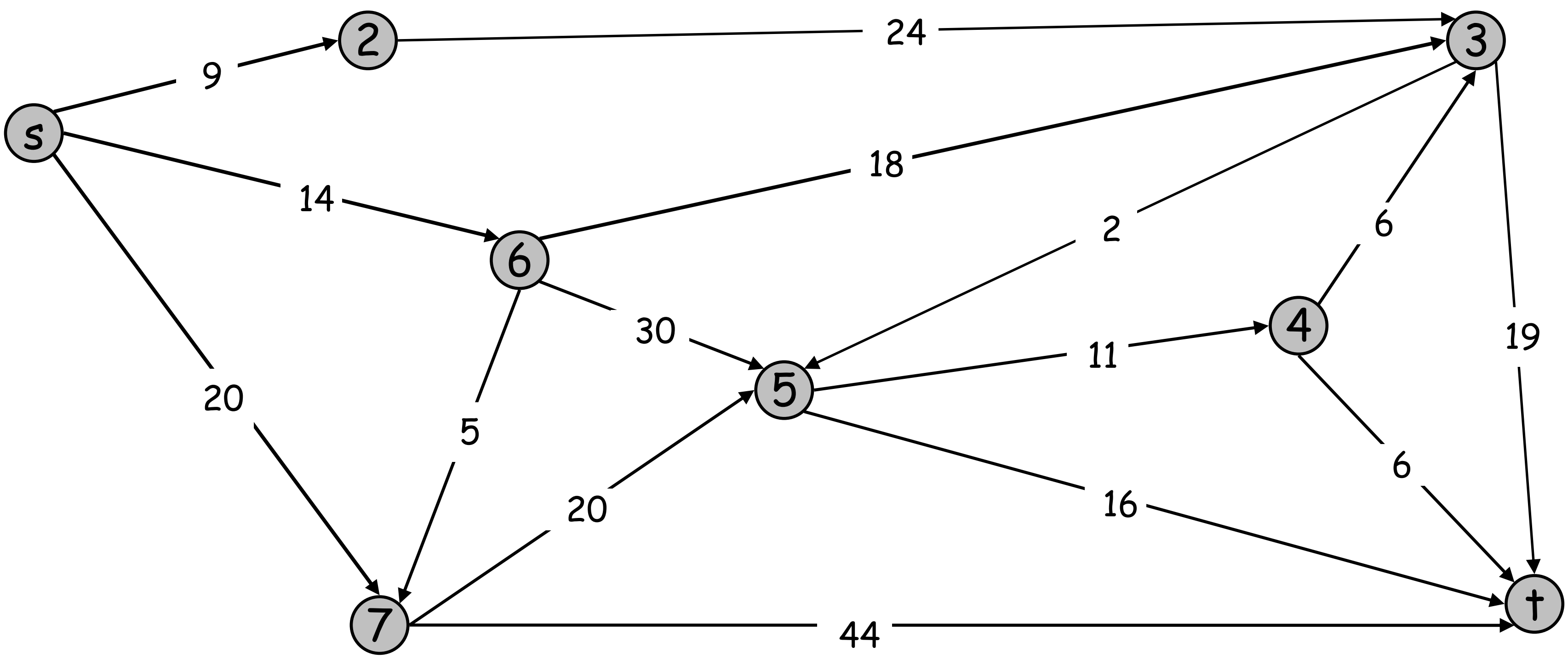
A: Replace each arc of length x by a path with x-1 intermediate nodes, run BFS in the new graph.

Dijkstra's Algorithm

Approach

- Find the node closest to **s**, then the second closest, then the third closest, and so on …, computing their distances from **s** (similar to BFS)

Find closest node to **s**, second closest, etc.



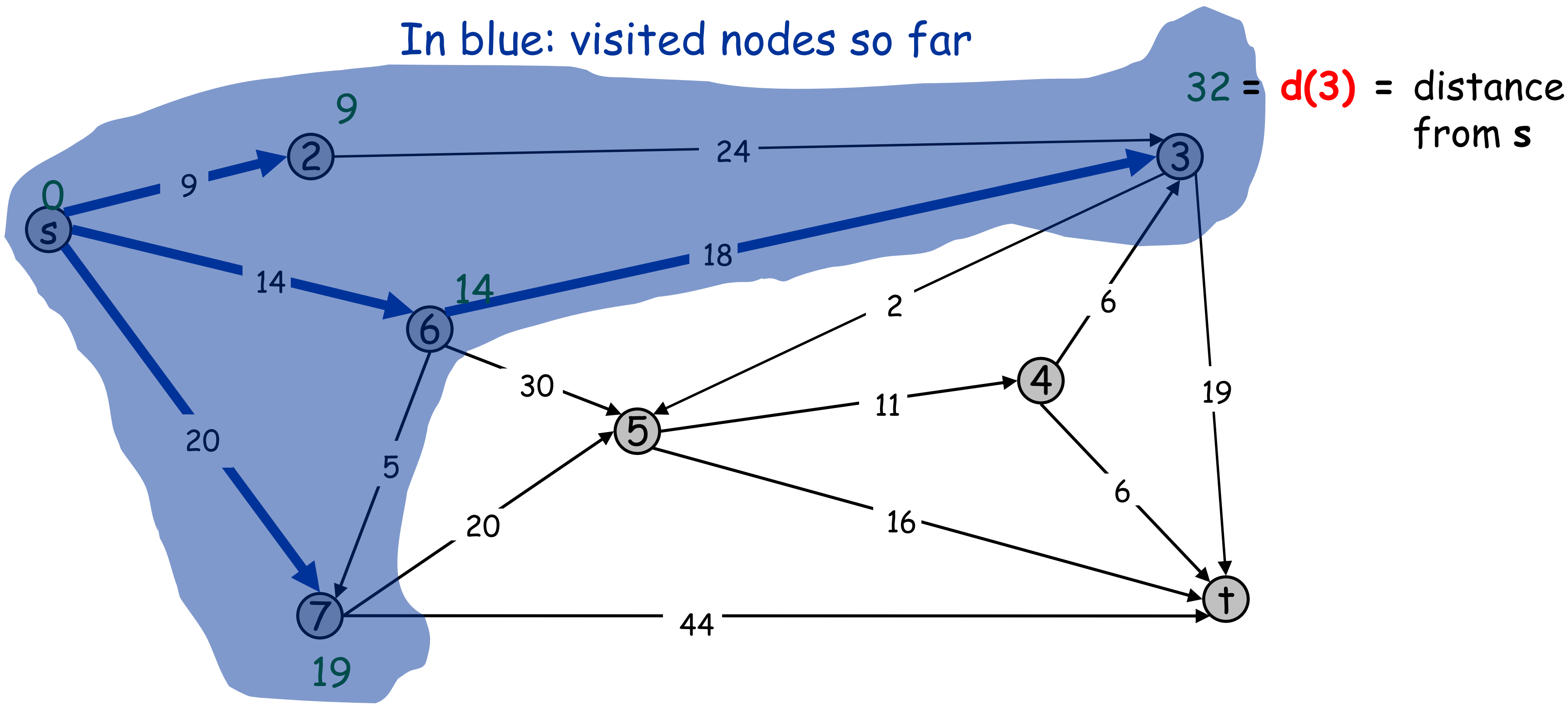
Dijkstra's Algorithm

Observation: The k-th closest node to s must be a neighbor of one of the visited nodes (i.e. closest, second closest, …. (k-1)-th closest)

Dijkstra's Algorithm

Q: Suppose we have visited the closest, second closest, ... (k-1)-th closest nodes, and have computed their distances from s. How to find k-th closest?

Ex: Find 6th closest node to s



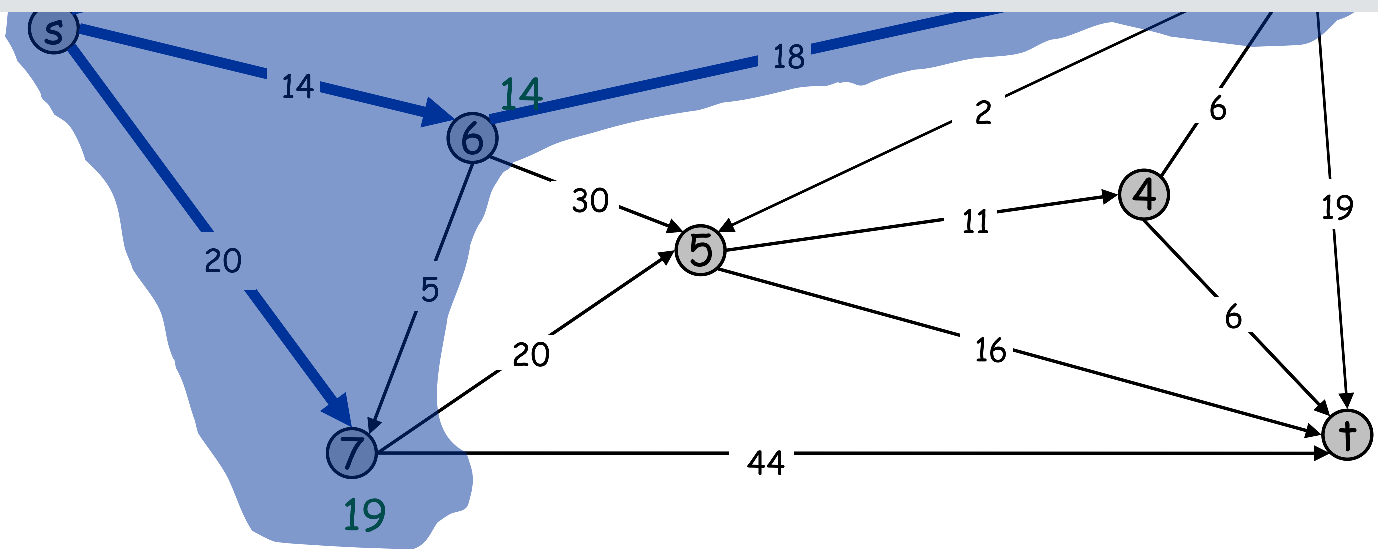
Dijkstra's Algorithm

Q: Suppose we have visited the closest, second closest, ... (k-1)-th closest nodes, and have computed their distances from s. How to find k-th closest?

A: For each unvisited node u compute the shortest distance to the visited nodes

$$\pi(u) = \min_{(v,u)\in G, v \text{ visited}} (d(v) + c_{vu})$$

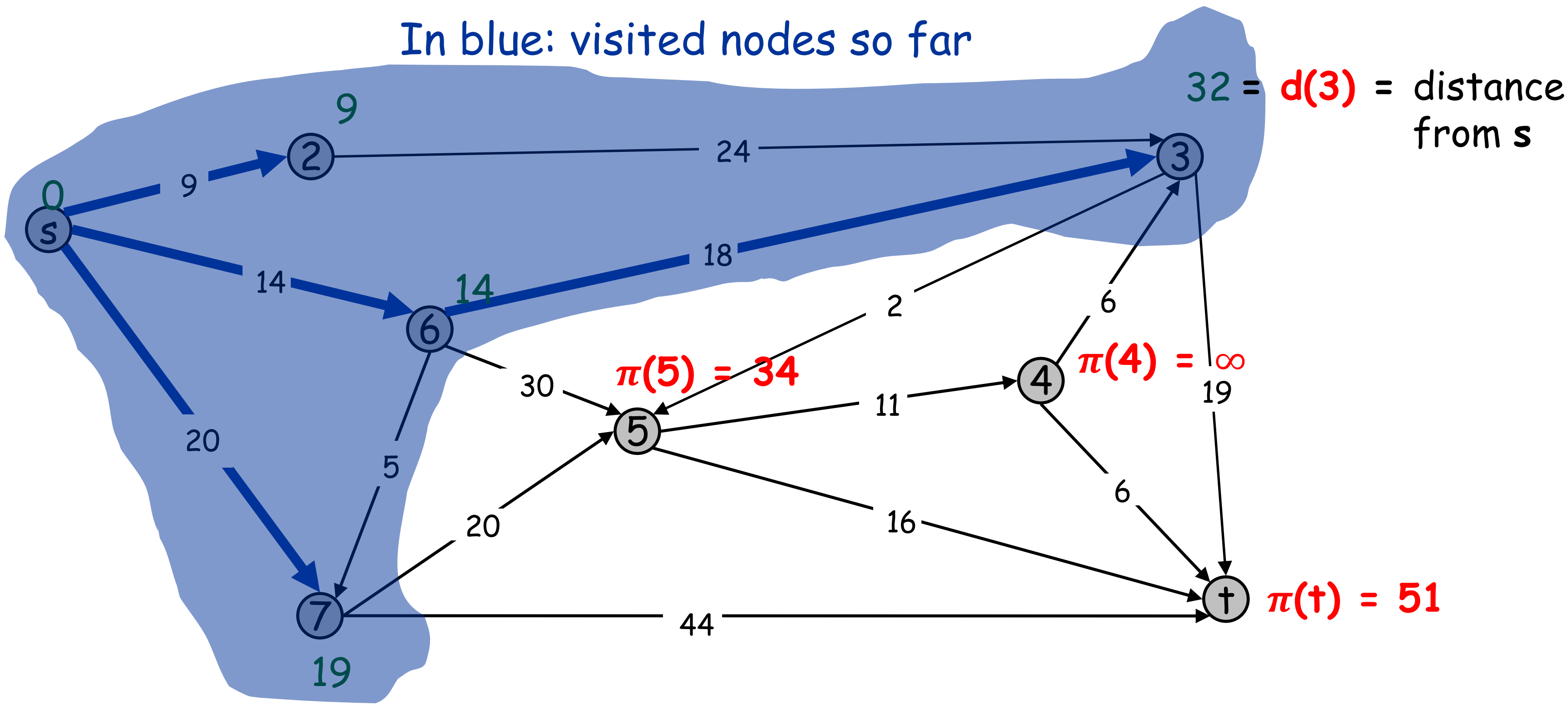
Pick unvisited node with smallest $\pi(u)$



Dijkstra's Algorithm

Q: Suppose we have visited the closest, second closest, ... (k-1)-th closest nodes, and have computed their distances from s. How to find k-th closest?

Ex: Find 6th closest node to s



Dijkstra's Algorithm

SlowDijkstra Algorithm

d(s)=0

Visited = {s}

For i=1 to n-1

- Compute $\pi(u) = \min_{(v,u)\in G, v\ visited} (d(v) + c_{vu})$ for all unvisited node u
- Pick node u with smallest $\pi(u)$
- Set d(u) = π(u)
- Add u to Visited

SlowDijkstra computes the distance from s to all other nodes

Q: Complexity of SlowDijkstra?

A: Each iteration takes at most O($\sum_u deg-in(u)$)=O(m) [computing pi] + O(n) [picking smallest]

=> Total: $O(n^2 + nm)$

Dijkstra's Algorithm

Idea to make faster: Only need to update the pi's for the neighbors of the node visited in the previous iteration

Dijkstra's Algorithm

d(s)=0, Visited = {s}

Initialize pi:

- Set π(u) = c_{su} for all out-neighbors u of s
- Set π(u) = ∞ for other nodes u

For i=1 to n-1

- Pick node u with smallest $\pi(u)$
- Set d(u) = π(u)
- Add u to Visited
- For all out-neighbors v of u #update pi's
  - If $\pi(u) + c_{uv} < \pi(v)$
    - $\pi(v) = \pi(u) + c_{uv}$

Dijkstra's Algorithm: Analysis

Dijkstra's Algorithm

[initialization]

For i=1 to n-1

- Pick node u with smallest $\pi(u)$
- Set d(u) = π(u)
- Add u to Visited
- For all out-neighbors v of u #update pi's
  - If $\pi(u) + c_{uv} < \pi(v)$
    - $\pi(v) = \pi(u) + c_{uv}$

Q: Complexity of Dijkstra if we keep pi in a vector?

A: Each iteration:

- O(n) for picking node with smallest $\pi(u)$
- O(out-deg(u)) for updating pi's
- O(1) for all else

Total (including initialization): $O(n^2 + m)$

Dijkstra's Algorithm: Analysis

Dijkstra's Algorithm

[initialization] ← MakeHeap

For i=1 to n-1

- Pick node u with smallest $\pi(u)$
- Set d(u) = π(u)
- Add u to Visited
- For all out-neighbors v of u #update pi's
  - If $\pi(u) + c_{uv} < \pi(v)$
    - $\pi(v) = \pi(u) + c_{uv}$

Q: Complexity of Dijkstra if we keep pi in a heap?

A: Initialization: O(n) to make heap

Each iteration:

- O(log n) for finding and removing from heap node with smallest $\pi(u)$
- O(out-deg(u) * log(n)) for updating pi's
- O(1) for all else

Total (including initialization): O($(n + m) \log n$)

Dijkstra's Algorithm: Getting the Path

Q: How to get shortest path from s to t, not just distance?

A: Similar to BFS:

- Keep track of who caused the visit to node u, call it the parent of u
- Starting from t, follow its parent, and its parent, etc.

Exercises: Weighted Shortest Paths

Exericise 1: Run Dijkstra's algorithm on the following graph, starting from node **s**

Exercise 2: Can we run Dijkstra's algorithm on undirected graphs? How?

Exercise 3: Show that Dijkstra's algorithm may not return the correct distance if there are negative lengths (construct a graph)

Exercise 4: Consider a slightly different problem: You are given a directed graph and costs on the **nodes**. You want to find the shortest cost path from s to t, where the cost of a path is the sum of the costs of the nodes in the path.

Find an algorithm to solve this problem.
(Hint: run Dijkstra on a modified graph)