

A challenge to 3-manifold topologists and group algebraists *

Sóstenes L. Lins and Lauro D. Lins

April 26, 2013

Abstract

This paper consists in a question about an instance (hard to find) of a special problem in 3-manifold topology. “Important though the general concepts and propositions may be with the modern industrious passion for axiomatizing and generalizing has presented us . . . nevertheless I am convinced that the special problems in all their complexity constitute the stock and the core of mathematics; and to master their difficulty requires on the whole the harder labor.” Hermann Weyl 1885-1955, cited in the preface of the first edition (1939) of [5].

1 A doubt in the classification of 3-manifolds

The objective of this short note is to pinpoint an aspect of the classification of 3-manifolds which is very important and has been essentially neglected in the last 35 years of successes with the work of W. Thurston, G. Perelman, I. Agol and many others. In despite of enormous progress, the classification problem remains, to our eyes, very difficult. The aspect we want to pinpoint is asking basic questions on hard to find tough instances of the general theory.

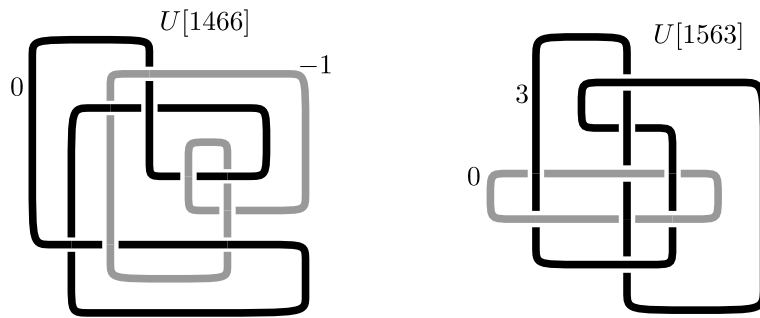


Figure 1: Are the 3-manifolds obtained from surgery on these blackboard framed link homeomorphic, or not?

Consider the two closed 3-manifolds obtained from surgery on the 2-component framed links in Fig. 1. Both are homology spheres, so their fundamental groups are perfect. Snappea [4] tells us that they are both hyperbolic and have the same volume up to many decimal places. Moreover, their Witten-Reshetik-Turaev invariants with 10 decimal places agree up to $r = 12$. These facts seem to imply that the manifolds are homeomorphic. However, computations based on the methodology of [1] and [2], which were up to this point successful in finding homeomorphism between pairs of 3-manifolds, appear to fail for the first time. Our bet is that the methodology does not fail: the manifolds are not homeomorphic. In the last 5 years we have asked the help of various distinguished topologists in trying to settle this example. None of them succeeded in answering our question. So, we believe the time is ripe to bring our doubt to the broader community of mathematicians dealing with 3-manifolds and/or combinatorial group theory. This example corresponds to the pair of blackboard framed links $U[1466]$ and $U[1563]$ of [1]. The numbers attached to the components (framings) coincide with their self-writhe in the given projection and, so, can be discarded.

Being hyperbolic replaces the difficult topological question of homeomorphism between the manifolds into the possibly equally difficult algebraic question of isomorphism between their fundamental groups. So, as long as the general associated question is not settled, we have replaced a problem which we do not know how to solve into another, which we also do not know how to solve. This might be, in some aspects, progress, but hardly a definitive one. In general, how to prove that the fundamental groups of hyperbolic 3-manifolds are not isomorphic? Start by proving that there is no isomorphism between the fundamental groups of the above 3-manifolds. Or find one.

*2010 Mathematics Subject Classification: 57M25 and 57Q15 (primary), 57M27 and 57M15 (secondary)

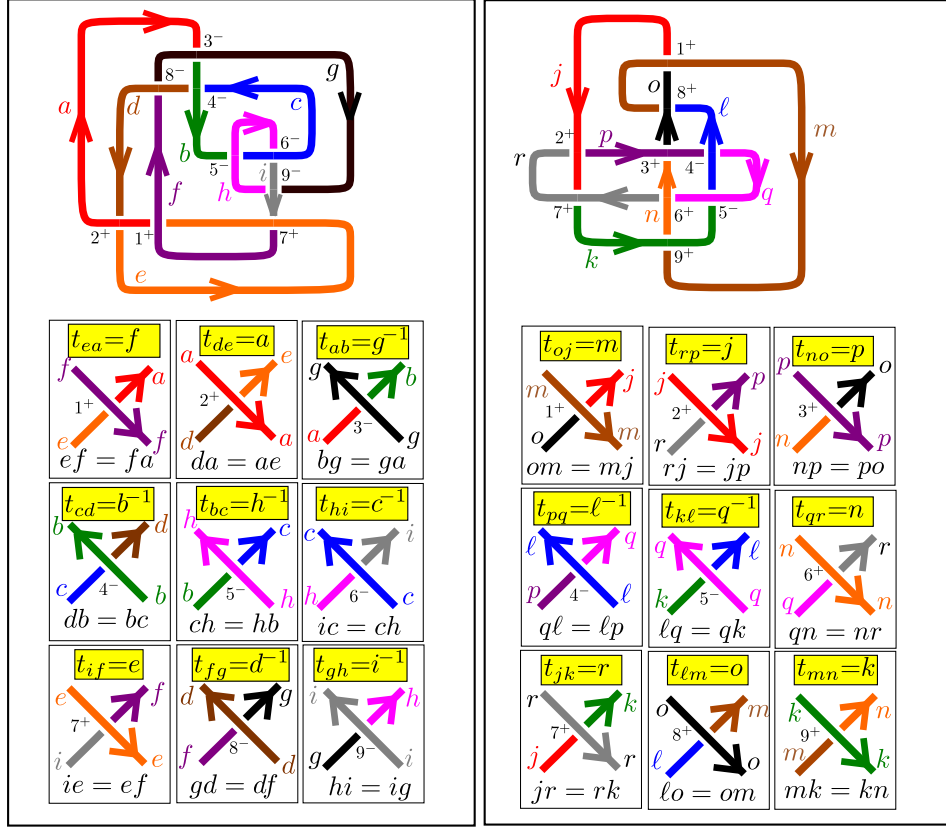


Figure 2: Finding presentations for the fundamental groups

In Fig. 2 we arbitrarily orient the links, display the Wirtinger presentations ([3]) for the exterior of the links and the pair of relations for the Dehn fillings. The presentations for the fundamental groups of the manifolds are:

$$G_1 = \langle \{a, b, c, d, e, f, g, h, i\}, \{ef = fa, da = ae, bg = ga, db = bc, ch = hb, ic = ch, ie = ef, gd = df, hi = ig, g^{-1}h^{-1}b^{-1}af = 1, d^{-1}i^{-1}c^{-1}e = 1\} \rangle,$$

$$G_2 = \langle \{j, k, \ell, m, n, o, p, q, r\}, \{om = mj, rj = jp, np = po, ql = \ell p, \ell q = qk, qn = nr, jr = rk, \ell o = om, mk = kn, rq^{-1}okpm = 1, \ell^{-1}nj = 1\} \rangle.$$

2 Another doubt: $U[2125]$ and $[2165]$

It is important also to distinguish the pair of blackboard framed links of Fig. 3. As the previous pair, they are homology spheres and their WRT-invariants agree up to $r = 12$ with 10 decimal places.

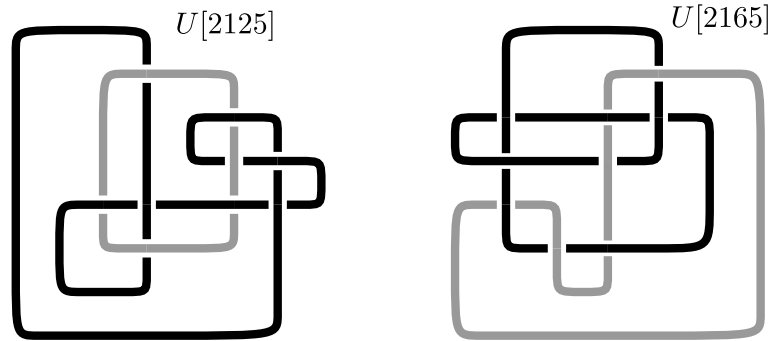


Figure 3: Are the 3-manifolds obtained from surgery on these blackboard framed link homeomorphic, or not?

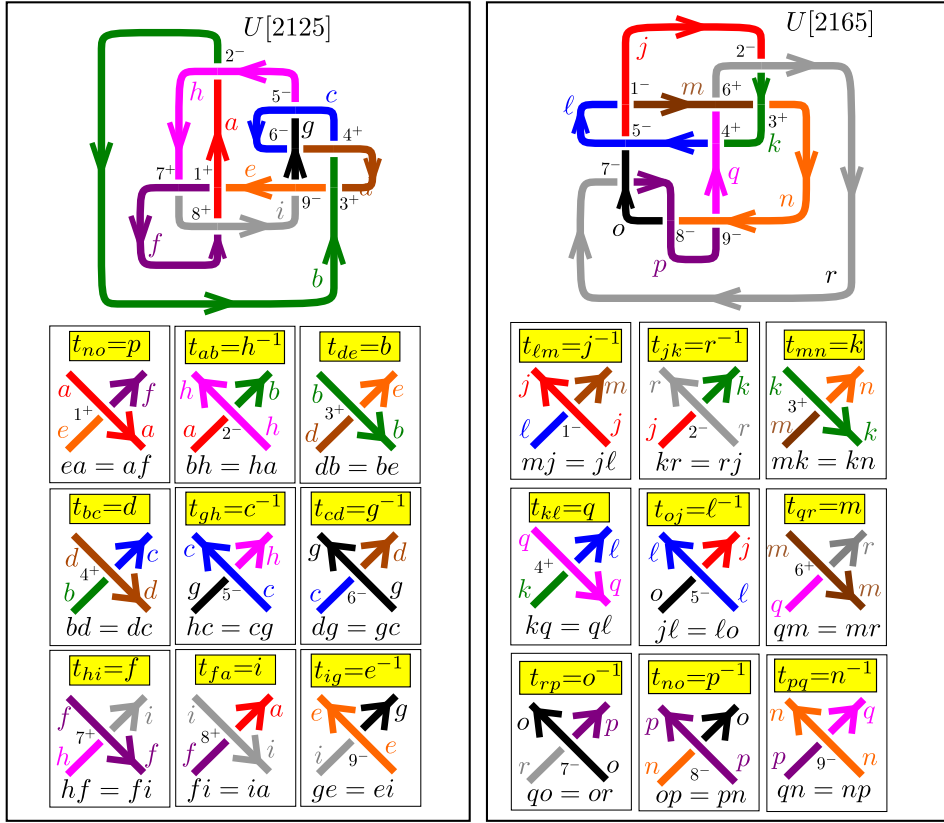


Figure 4: Finding presentations for the fundamental groups

The presentations for the fundamental groups of the manifolds are:

$$G_3 = \langle \{a, b, c, d, e, f, g, h, i\}, \{ef = fa, da = ae, bg = ga, db = bc, ch = hb, ic = ch, ie = ef, gd = df, hi = ig, g^{-1}h^{-1}b^{-1}af = 1, d^{-1}i^{-1}c^{-1}e = 1\} \rangle,$$

$$G_4 = \langle \{j, k, \ell, m, n, o, p, q, r\}, \{om = mj, rj = jp, np = po, q\ell = \ell p, \ell q = qk, qn = nr, jr = rk, lo = om, mk = kn, rq^{-1}okpm = 1, \ell^{-1}nj = 1\} \rangle.$$

This is read directly from Fig. 4.

3 Conclusion

Our bet is that both pairs of 3-manifolds in this short note are not homeomorphic. This would mean that the combinatorial dynamics of Chapter 4 in [2] based on TS -moves induces an efficient algorithm which is capable of topologically classify all 3-manifolds given as a blackboard frame link with up to 9 crossings and maintains live the two Conjectures of page 15 of [2] that the TS - and u^n -moves yield an algorithm to effectively classify 3-manifolds.

References

- [1] L.D. Lins. Blink: a language to view, recognize, classify and manipulate 3D-spaces. *Arxiv preprint math/0702057*, 2007.
- [2] S. Lins. *Gems, Computers, and Attractors for 3-Manifolds*. World Scientific, 1995.
- [3] J. Stillwell. *Classical Topology and Combinatorial Group Theory*. Springer Verlag, 1993.
- [4] J. Weeks. SnapPea: a computer program for creating and studying hyperbolic 3-manifolds, 2001.
- [5] A. N. Whitehead. *The classical group: their invariants and representations*. Princeton University Press, 1997.

Sóstenes L. Lins
Centro de Informática, UFPE
Av. Jornalista Anibal Fernandes s/n
Recife, PE 50740-560
Brazil
sostenes@cin.ufpe.br

Lauro D. Lins
AT&T Labs Research
180 Park Avenue
Florham Park, NJ 07932
USA
llins@research.att.com