A tougher challenge to 3-manifold topologists and group algebraists *

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Abstract

This paper poses some basic questions about instances (hard to find) of a special problem in 3-manifold topology. "Important though the general concepts and propositions may be with the modern industrious passion for axiomatizing and generalizing has presented us ... nevertheless I am convinced that the special problems in all their complexity constitute the stock and the core of mathematics; and to master their difficulty requires on the whole the harder labor." Hermann Weyl 1885-1955, cited in the preface of the first edition (1939) of A. N. Whitehead's book *The classical groups: their invariants and representations* [13].

In this paper I focus on new uncertainties left unanswered in L. Lins thesis [3] on the homemorphism problem of eleven concrete pairs of closed orientable 3-manifolds induced by 3-connected monochromatic *blinks* ([2]). The eleven HG8QI-classes are the only doubts left in the thesis, but first two of the eleven were solved few days ago and I report on the solution.

1 Introduction

In a joint recent paper posted recently in the arXiv ([5]) my son Lauro Lins and me ask some 6 years old questions for which we had no answers about homeomorphisms between closed orientable 3-manifolds. The two pairs of 3-manifolds were the only uncertainties that were left in L. Lins thesis ([3]) under my supervision in the domain of 3-manifolds being induced by blackboard framed links (for short *bfl*'s) up to 9 crossings (9-small 3-manifolds). A subset of relevant 10-crossings bfl's were generated but their topological classification remains untouched. The paper was taken seriously by a few researchers, among them M. Culler, N. Dunfield and C. Hodgson and others that could solve them very quickly using GAP, SnapPy, Sage, tools that were unknown to us. The solutions were obtained by distinct methods and are all consistent (inclusive with BLINK, the program of L. Lins (implementing my theory described in [4]), which support his thesis). Together with my colleague Cristiana Nascimento, here at CIn/UFPE, I am learning fast to operate these wonderful tools. The solutions people found shows that BLINK does a complete job in topologically classifying the 9-small 3-manifolds.

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The first solution that I got, and that blew my mind, was by Craig Hodgson using length spectra techniques, based in his joint paper with J. Weeks entitled *Symmetries, isometries and length spectra of closed hyperbolic three-manifolds* ([1]). By using SnapPy Craig showed that even though the quantum WRT-invariants as well as the volumes of the hyperbolic Z-homology spheres induced by the bfl's), $U[1466]$ and $U[1563]$ are the same, the length of the smallest geodesics of them are distinct. For the other pair of bfl's $U[2125]$ and $U[2165]$ he shows that precisely the same facts apply. Here is a summary of Craig's findings extracted from the SnapPy session that he kindly sent me:

Class 9_{126}:

```
From first geodesic of U[1466]: 1.0152103824828331+0.39992347315914334j.
From first geodesic of U[1563]: 0.9359206605025168+2.333526236965665j.
Volume of both manifolds: 7.36429600733.
```

Class 9_{199}:

```
From first geodesic of U[2125]:  0.8939075859248593+0.761197185679321j.
From first geodesic of U[2165]:  0.7978548001747316+2.9487425029345973j.
Volume of both manifolds: 7.12868652133
```

There were 3 previous versions of [5] that I appeared before, because of annoying mistakes in the presentations and in focusing the challenge in a broader context. By the way, the presentations are redundant because the blink is enough to define the 3-manifold. The time spanned between the first (April 22, 2013) and the last version (May 1, 2013) was a little more than one week. Even so, some people including Nathan Dunfield worked with the wrong presentations. Not without reason he was angry at me, but I think that this is no longer true, since he was willing to answer my sometime naive and stupid questions and send me a solution for the first pair of manifolds of the present work, using Sage and GAP computations, by working with covers. I did not know these tools. Also I did not know that the blog on lower dimensional topology was very active exposing my incorrections and I apologize for my ignorance. I thank Cristiana for having calling the blog to my attention. But I learn fast and do believe that I have something important and different to say in this brave new world of 3-manifolds... After all I have been putting a great amount of time and effort during my scientific carreer, (as basically as an isolated researcher) on (mainly closed) 3-manifods. I seek no longer to be isolated.

Marc Culler was very helpful in answering questions of me and Cristiana and helping her about issues in the downloading and installing SnapPy and Sage and GAP in her machine. After the presentation incorrections out of the way he produced an independent proof of the distinctveness of $(U[1466, U[1563])$ and of $(U[2125, U[2165])$. He also produced instantaneous isomorphic triangulations of the homeomorphic 3-manifolds in the classes 9_{126} and 9_{199}. This fact makes me anxious to compare and timing the performances of BLINK and SnapPy regarding finding homeomorphisms of k-small 3-manifolds when the homeomrphisms exist.

In this paper I put some new challenges (also coming from [3]), that seem harder than the ones considered in the previous paper. The reason I think so is that going from 9 to 14,15,16 crossings in the links, numerical problems start appearing concerning finding the Dirichlet domain and, in these cases, finding isomorphic triangulations might be harder to

SnapPy than to BLINK. At any rate I have hundreds of examples where the performance of these programs in this issue could be compared, if anyone is seriously interested. Currently BLINK is not documented and one of my objectives is to seek for help in doing it and extend its capability. BLINK is hosted at Github under the userid *laurolins* and is open source code project. Unfortunately him (currently a researcher at AT&T) does not have the necessary time to go on with the implementation. But he welcomes and is willing to help collaborators in gettig started. As for me, I am too old for the energy needed to construct good pieces of software. I intend to act as one Scientific Supervisor for the deployment and for the discussions the new algorithms algorithms to be included in BLINK, but only at the mathematical level. The technological and software engineering needed I leave to others.

An example of such new algoriths that we want to include in BLINK is made possible by the theory in Ricardo Machado's thesis under my supervision, defended last March. We got an $O(n^2)$-algorithm for going from a special kind of gem, named resoluble gems, to a blink inducing the same manifold. This work is available, in still rather sketch form (even the definition of resolubility is unecessary complicated), in the three joint papers posted last year in the arXiv, [6, 7, 8]. The algorithm was implemented in Mathematica, but it needs to be improved and re-implemented in Java or C++. We found a rather crude framed link presentation for the hyperbolic dodecahedral space (Weber-Seifert manifold). As far as I know nobody has found such framed link. My interest in it was aroused by J. Weeks in a visit to the Geometry Center in 1994, when he asked me whether I had such framed link. The link inducing the Weber-Seifert 3-manifold is given by a 9-component link embedded into $\mathbb{R}^3$ and having a total of 68 (only) vertices. (It started with more than 600.) In a fourth paper under preparation we will show that every 3-manifold admits a resoluble gem inducing it.

2 The eleven classes of tough 3-manifolds

I assume that the reader has with him a copy of the version 4 of previous challenge paper ([5]) and has learned how to read the manifold either from the blink or from the blackboard framed link, [2]. As for obtaining a presentation of the fundamental group based on the Wirtinger relators ([11]) and the Dehn fillings ([10]) the two detailed examples given in [5] should suffice, if the reader has not available other pieces of softwares to get the presentation by automatic means. I had not and this partially explain my incorrections in the presntations of the 3 firts version: is a messy time consuming task to be done by hand, very much prone to errors.

Actually the first two classes are resolved, only remaining the nine final ones. The complex numbers in polar form which appear at each m_p^t-class are the common quantum WRT-invariants. All except one of the eleven classes are formed by Z-homology spheres. The exception is 16^t_{56} which has no torsion but Betti number 1. These facts are indicated by the small number in parenthesis (which gives the homology of the manifold).

2.1 The $HG8QI_t$ class 14^t_{24}:

```
sage: from snappy import *
sage: M1 = Manifold('1424_T71.tri')
```

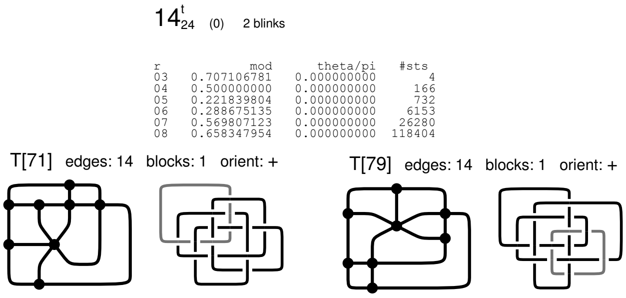


Figure 1: The above two manifolds are not homeomorphic. They are distinguished by the homology of their 5-covers. This was immediately noted by N. Dunfield using Sage and GAP from triangulations obtained by C. Nascimento using SnapPy, which could not find the Dirichlet domain due to numerical instability.

```
sage: M2 = Manifold('1424_T79.tri')
sage: covers1 = M1.covers(5, method='gap')
sage: covers2 = M2.covers(5, method='gap')
sage: [C.homology() for C in covers1]
[Z/132 + Z/132, Z/63 + Z/63, Z/3 + Z/3 + Z/3 + Z/3]
sage: [C.homology() for C in covers2]
[Z/3 + Z/3 + Z/3 + Z/3, Z/213 + Z/213, Z/432 + Z/432]
```

Above, in verbatim style, is Dunfield's Sage session.

2.2 The $HG8QI_t$ class 15^t_{16}:

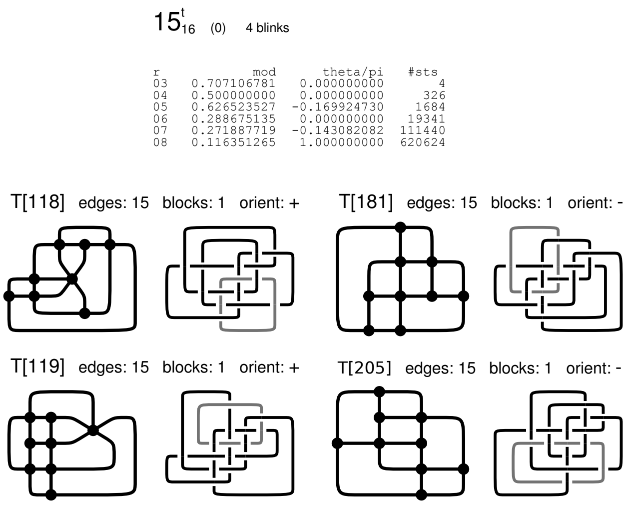


Figure 2: The above two manifolds are also non-homeomorphic. They are also distinguished by the homology of their 5-covers. Relative to the class 15^t_{24} class 15^t_{16} the Sage/GAP software demands much more time. This was obtained by C. Nascimento using SnapPy/Sage/GAP. The software SnapPy could not find the Dirichlet domain due to numerical instability.

2.3 The $HG8QI_t$ class 15^t_{19}:

15^t_{19} (0) 4 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	1.180868794	-0.318489048	1664
06	0.288675135	0.000000000	18957
07	0.344363154	-0.282364319	108168
08	0.990504026	1.000000000	593752

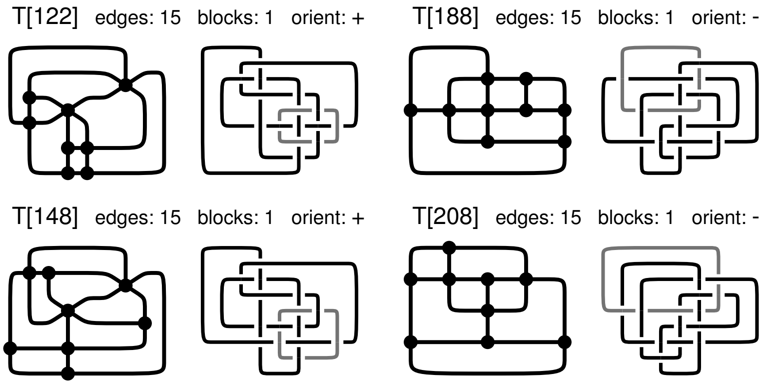


Figure 3: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four.

2.4 The $HG8QI_t$ class 15^t_{22}:

15^t_{22} (0) 4 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.943093903	-0.180036791	1680
06	0.288675135	0.000000000	20945
07	0.242490221	-0.482680140	111968
08	0.641616039	1.000000000	656072

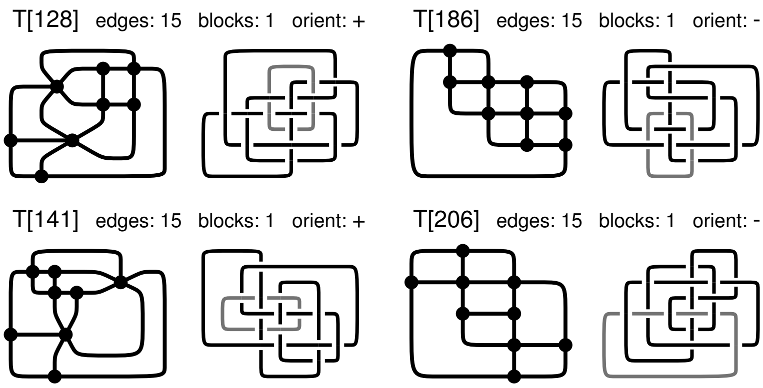


Figure 4: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four.

2.5 The $HG8QI_t$ class 16^t_{42}:

16^t_{42} (0) 2^2 4 blinks

r	mod	theta/pi	#sts
03	0.000000000	0.000000000	4
04	1.224744871	0.195913276	294
05	0.000000000	0.000000000	1900
06	0.577350269	-0.333333333	22883
07	0.000000000	0.000000000	159992
08	1.774989898	0.206598266	1001208

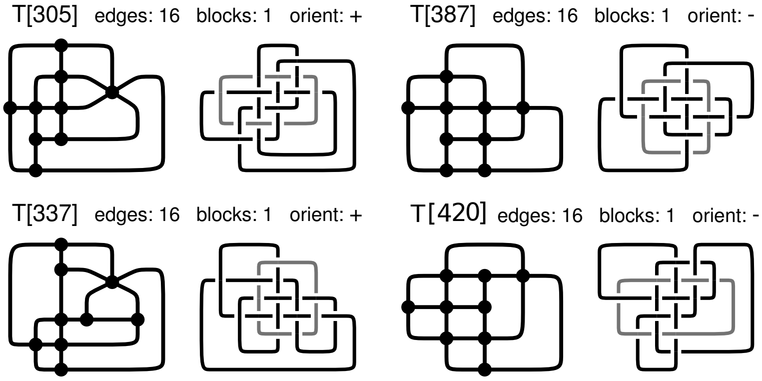


Figure 5: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four.

2.6 The $HG8QI_t$ class 16^t_{56}:

16^t_{56} (1) 4 blinks

r	mod	theta/pi	#sts
03	1.000000000	0.000000000	8
04	1.000000000	0.000000000	420
05	1.249042228	0.093934688	4512
06	2.000000000	0.000000000	61009
07	1.748655511	-0.081646235	501032
08	0.801948466	0.000000000	3620288

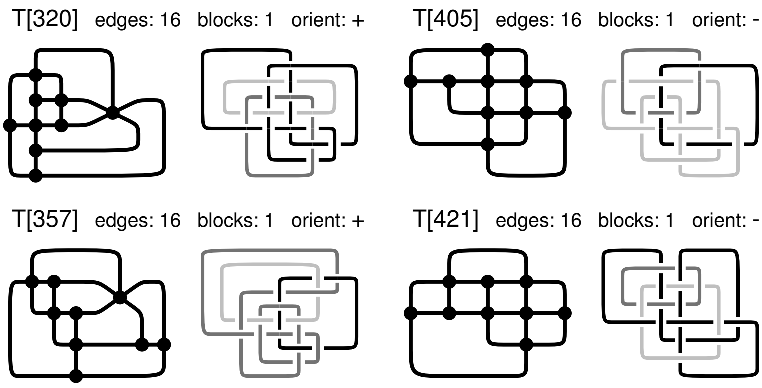


Figure 6: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four.

2.7 The $HG8QI_t$ class 16^t_{140}:

16^t_{140} (0) 4 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.828810492	0.200682317	1736
06	0.288675135	0.000000000	19469
07	0.369735450	0.097853027	114084
08	0.499366312	1.000000000	624200

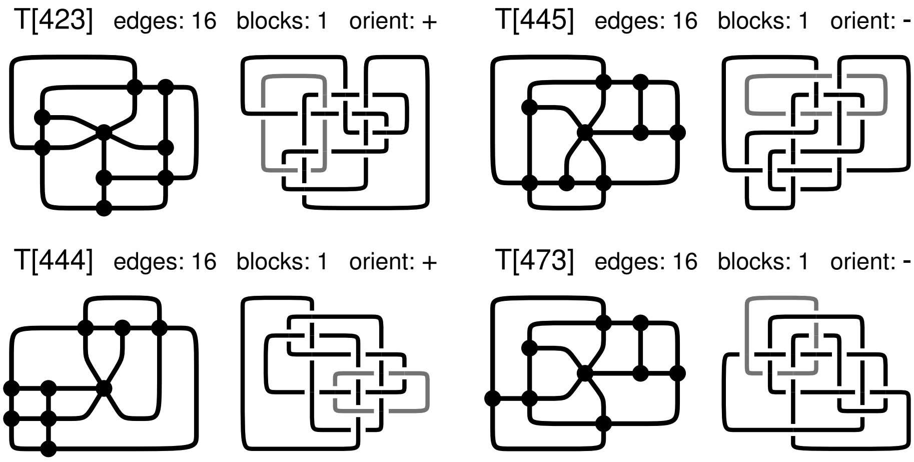


Figure 7: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four.

2.8 The $HG8QI_t$ class 16^t_{141}:

16^t_{141} (0) 2 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.828231334	0.000000000	1736
06	0.288675135	0.000000000	19469
07	0.591962818	0.000000000	114084
08	0.297863204	0.000000000	624200

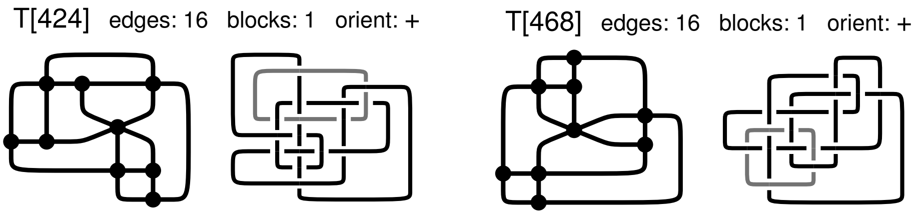


Figure 8: I do not know whether the above two manifolds are homeomorphic or not.

2.9 The $HG8QI_t$ class 16^t_{142}:

16^t_{142} (0) 2 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.314488186	0.000000000	1696
06	0.288675135	0.000000000	21113
07	1.546327427	0.000000000	113232
08	0.931913875	0.000000000	663132

T[425] edges: 16 blocks: 1 orient: + T[435] edges: 16 blocks: 1 orient: +

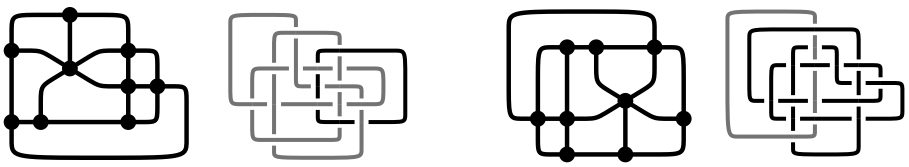


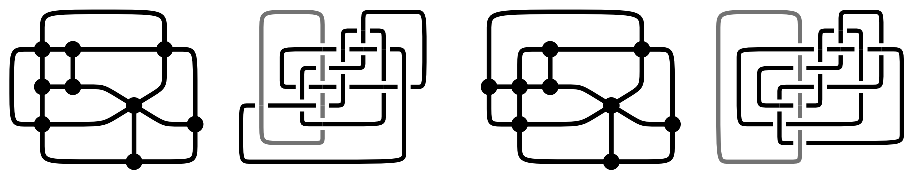
Figure 9: I do not know whether the above two manifolds are homeomorphic or not.

2.10 The $HG8QI_t$ class 16^t_{149}:

16^t_{149} (0) 4 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.314488186	0.000000000	1828
06	0.288675135	0.000000000	21553
07	0.745358881	0.081311555	126032
08	1.620357495	0.000000000	735988

T[437] edges: 16 blocks: 1 orient: + T[464] edges: 16 blocks: 1 orient: +



T[449] edges: 16 blocks: 1 orient: - T[537] edges: 16 blocks: 1 orient: -

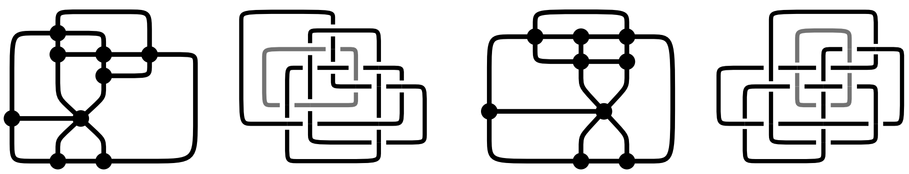


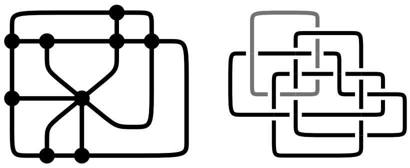
Figure 10: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four.

2.11 The $HG8QI_t$ class 16^t_{233}:

16^t_{233} (0) 2 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.706953028	0.000000000	1420
06	0.288675135	0.000000000	17019
07	0.472161455	0.000000000	72552
08	0.573693534	0.000000000	401400

T[631] edges: 16 blocks: 1 orient: +



T[663] edges: 16 blocks: 1 orient: +

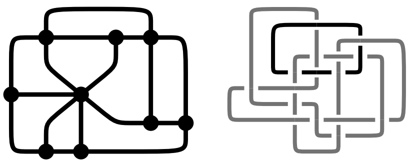


Figure 11: I do not know whether the above two manifolds are homeomorphic or not.

3 A concluding remark

By the way, the elegant drawings of blinks and blackboard framed links produced by BLINK are possible due the groundbreaking algorithm of R. Tamasia [12]. Lauro could implement the drawings very fast because we had at hand the implementation of network flow algorithms he had done for a project to solve timetable problems. This is an example of the unicity in Mathematics, advocated by L. Lovasz in his famous essay [9]. To get the drawings one has to apply three times the full strength of network flow theory. The drawings BLINK presents are in an integer grid and deterministically minimize the number of $\pi/2$-bents in the blackboarded framed links. In particular, it permit us to deal with the unavoidable curls which adjust the integer framing in the best possible way: we do not care about them. The drawings for the companion blinks are a slight modification: it replaces each p-valent vertex $p > 4$, by a p-polygon inducing 3-valent ones. The final result is massaged a bit to produce aesthetically nice and unambiguous drawings.

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