

A challenge to 3-manifold topologists *

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Abstract

This paper is an instance of a special problem in 3-manifold topology. “Important though the general concepts and propositions may be with the modern industrious passion for axiomatizing and generalizing has presented us . . . nevertheless I am convinced that the special problems in all their complexity constitute the stock and the core of mathematics; and to master their difficulty requires on the whole the harder labor.” Hermann Weyl 1885-1955, cited in the preface of the first edition (1939) of [4].

1 The first of various doubts in the classification of 3-manifolds

The objective of this short note is to pinpoint an aspect of the classification of 3-manifolds which is very important and has been essentially neglected in the last more than three decades of success with the work of W. Thurston, G. Perelman, I. Agol and many others. In despite of enormous progress, the classification problem remains, to our eyes, very difficult. The aspect we want to pinpoint is asking basic questions on hard to find tough instances of the general theory.

As an example consider the two closed 3-manifolds obtained from surgery on the 2-component framed links below. Both are homology spheres, so their fundamental groups are perfect. Snappea [3] tells us that they are both hyperbolic and have the same volume. Moreover, their Witten-Reshetiken-Turaev invariants with 10 decimal places agree up to $r = 12$. These facts seem to imply that the manifolds are homeomorphic. However, computations based on the methodology of [1] and [2], which were up to this point successful in finding homeomorphism between pairs of 3-manifolds, appear to fail for the first time. Our bet is that the methodology does not fail: the manifolds are not homeomorphic. In the last 5 years we have asked the help of various distinguished topologists in trying to settle this example. None of them succeeded in answering our question. So, we believe the time is ripe to bring our doubt to the whole community of mathematicians dealing with 3-manifolds.

Being hyperbolic replaces the difficult topological question of homeomorphism between the manifolds into the possibly equally difficult algebraic question of isomorphism between their fundamental groups. So, as long as the general associated question is not settled, we have replaced a problem which we do not know how to solve into another, which we also do not know how to solve. This might be, in some aspects, progress, but hardly a definitive one. In general, how to prove that the fundamental groups of hyperbolic 3-manifolds are not isomorphic? Start by proving that there is no isomorphism between the fundamental groups of the 3-manifolds below. Or find one.

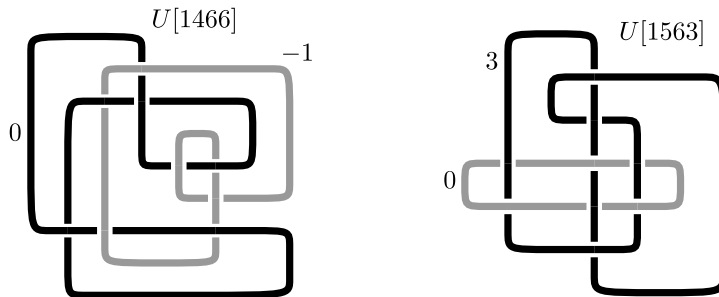


Figure 1: Are these 3-manifolds homeomorphic or not?

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References

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