A tougher challenge to 3-manifold topologists and group algebraists *

Sóstenes L. Lins

May 16, 2013

Abstract

This paper poses some basic questions about instances (hard to find) of a special problem in 3-manifold topology. "Important though the general concepts and propositions may be with the modern industrious passion for axiomatizing and generalizing has presented us . . . nevertheless I am convinced that the special problems in all their complexity constitute the stock and the core of mathematics; and to master their difficulty requires on the whole the harder labor." Hermann Weyl 1885-1955, cited in the preface of the first edition (1939) of A. N. Whitehead's book *The classical groups: their invariants and representations* [17].

In this paper I focus on new uncertainties left unanswered in L. Lins thesis [6] on the homemorphism problem of eleven concrete pairs of closed orientable 3-manifolds induced by 3-connected monochromatic *blinks* ([4]). The eleven HG8QI-classes are the only doubts left in the thesis, but the first two of them were solved few days ago and in this work I report on their solutions. We also include an appendix which can be used to import all the links of this paper into SnapPy. The appendix was obtained by drawing the links in SnapPy, work performed by C. Nascimento.

1 Introduction

In a joint recent paper posted recently in the arXiv ([8]) my son Lauro Lins and myself ask some 6 years old questions for which we had no answers about homeomorphisms between closed orientable 3-manifolds. The two pairs of 3-manifolds were the only uncertainties that were left in L. Lins thesis ([6]) under my supervision in the domain of 3-manifolds being induced by arbitrary connected blinks up to 9 edges (9-small 3-manifolds). A subset of relevant 10-crossings blinks were generated but their topological classification remains untouched. The paper was taken seriously by a few researchers, among them M. Culler, N. Dunfield, C. Hodgson and others that could solve them very quickly using GAP ([2]), Sage ([14]) and SnapPy ([1]), tools that (except for GAP) were basically unknown to us. The solutions were obtained by distinct methods and are all consistent (inclusive with BLINK, the program of L. Lins (implementing my theory described in [7]), which support his thesis). Together with my colleague Cristiana Nascimento, here at CIn/UFPE, I am learning fast

*2010 Mathematics Subject Classification: 57M25 and 57Q15 (primary), 57M27 and 57M15 (secondary) to operate these wonderful tools. The solutions people found shows that BLINK does a complete job in topologically classifying the 9-small 3-manifolds. This is the subject of a joint paper with Lauro, currently under preparation.

The first solution that I got, and that still blows my mind, was by Craig Hodgson using length spectra techniques, based in his joint paper with J. Weeks entitled *Symmetries, isometries and length spectra of closed hyperbolic three-manifolds* ([3]). By using SnapPy Craig showed that even though the quantum WRT-invariants as well as the volumes of the hyperbolic Z-homology spheres induced by the bfl's, $U[1466]$ and $U[1563]$ are the same, the length of the smallest geodesics of them are distinct. For the other pair of bfl's $U[2125]$ and $U[2165]$ he shows that precisely the same facts apply. Here is a summary of Craig's findings extracted from the SnapPy session that he kindly sent me. As Craig writes: *"The output of the length spectrum command shows the* complex lengths *of closed geodesics — the real part is the actual length and the imaginary part is the rotation angle as you go once around the geodesic."*

Class 9_{126}:

```
First geodesic of U[1466]: 1.0152103824828331+0.39992347315914334i.
First geodesic of U[1563]: 0.9359206605025168+2.333526236965665i.
Volume of both manifolds: 7.36429600733.
```

Class 9_{199}:

```
First geodesic of U[2125]:  0.8939075859248593+0.761197185679321i.
First geodesic of U[2165]:  0.7978548001747316+2.9487425029345973i.
Volume of both manifolds: 7.12868652133.
```

I posted 4 versions of [8] correcting annoying mistakes in the presentations of the fundamental groups, putting a second pair of links, and in focusing the challenge in a broader context. I computed the presentations manually and I had a hard time making them correct. Even though the presentations are redundant because the blink is enough to define the 3-manifold, as explained in [4], my objective was to facilitate the work for those wanting to use GAP. The time spanned between the first (April 22, 2013) and the last version (May 1, 2013) was a little more than one week. During these revisions I was completely unaware that the paper had called the attention of many people. I did not know that the blog on lower dimensional topology was very active exposing my incorrections and I apologize for my ignorance. I thank Cristiana for having calling the blog to my attention. Worse, some people did not see the follow up versions. This was the case of Nathan Dunfield who worked with the wrong presentations. Not without reason he was angry at me, but I think that this is no longer true, since he was willing to answer my sometimes naives and stupid questions and send me a solution for the first pair of manifolds of the present work, using SnapPy, Sage and GAP computations, by working with covers. I did not know these tools. But, when properly motivated, I can learn fast and in general I do believe that I have something important and different to say in this brave new world of 3-manifolds: see the wonderful essay of E. Klarreich published by the Simons Foundation (march 2012), [5]. I have been putting a great amount of time and effort during my scientific carreer, (most of the time as an isolated researcher) on (mainly closed) 3-manifods. I seek no longer to be isolated: my team is the World, my compromisse is with Truth (independently of whom first found it).

Marc Culler was very helpful in answering questions of myself and Cristiana and helping her about issues in the downloading and installing SnapPy and Sage and GAP in her machine. With the presentation incorrections out of the way he produced an independent proof of the distinctveness of $(U[1466, U[1563])$ and of $(U[2125, U[2165])$. He also produced instantaneous isomorphic triangulations of the homeomorphic 3-manifolds in the classes 9_{126} and 9_{199}. This fact makes me anxious to compare and timing the performances of BLINK (which also produces instantaneous solutions for the same problems) and SnapPy regarding finding homeomorphisms of k-small 3-manifolds, given that the homeomorphisms exist.

2 Objective of this work: help to make BLINK known

In this paper I put some new challenges (also coming from [6]), that seem harder than the ones considered in the previous paper. The reason I think so is that going from 9 to 14,15,16 crossings in the links, numerical problems start appearing concerning finding the Dirichlet domain and, in these cases, finding isomorphic triangulations might be harder to SnapPy than to BLINK. At any rate I have hundreds of examples where the performance of these programs in this issue could be compared, if anyone is seriously interested. Currently BLINK is not documented and one of my objectives is to seek for help in doing it and extend its capability. BLINK is hosted at Github under the userid *laurolins* and is open source code project. Unfortunately Lauro (currently a researcher at AT&T) does not have the necessary time to go on with the implementation. But he welcomes and is willing to help collaborators in getting started. As for myself, I am too old for the energy needed to construct good pieces of software. I intend to act as one of some Scientific Supervisors for the deployment and for the discussions of the new algorithms to be included in BLINK, but only at the mathematical level. The technological and software engineering screws and bolts needed, I leave to others.

An algorithm that I want to attach to BLINK is finding a uniformly distributed random closed orientable 3-manifold induced by a blink with an arbitrary number (even thousands) of edges. I want to gather evidence for the truth of some important conjectures that depend on this capability. Another example of such new algorithms that I want to include in BLINK is made possible by the theory in Ricardo Machado's thesis under my supervision, defended in March, 2013. We got an $O(n^2)$-algorithm for going from a special kind of gem, named *resoluble gem*, to a blink inducing the same manifold. This work is available, in still rather sketchy form (even the definition of resolubility is unecessary complicated), in the three joint papers posted last year in the arXiv, [9, 10, 11]. The algorithm was implemented in Mathematica, but it needs to be improved and re-implemented in Java or C++. We found a rather crude framed link presentation for the hyperbolic dodecahedral space (Weber-Seifert manifold). As far as I know nobody has found such a framed link. My interest in it was aroused by J. Weeks in a visit to the Geometry Center in April 1993, when he asked me whether I had such framed link. The link inducing the Weber-Seifert 3-manifold is a 9-component link embedded into $\mathbb{R}^3$, with an integer attached to each component (its framing) and having a total of 68 (only) vertices with a projection having 142 crossings. (It started with a PL-link with more than 600 vertices.) In a fourth joint paper with R. Machado, currently under preparation, we will show that every 3-manifold admits a resoluble gem inducing it.

3 The 11 $HG8QI_t$-classes of blinks left unresolved in [6]

I assume that the reader has with him a copy of the version 4 of previous challenge paper ([8]) and has learned how to read the manifold either from the blink or from the blackboard framed link, [4]. As for obtaining a presentation of the fundamental group based on the Wirtinger relators ([15]) and the Dehn fillings ([13]) the two detailed examples given in [8] should suffice, if the reader has not available other pieces of softwares to get the presentation by automatic means. Actually, the best way to enter these manifolds is to draw the blackboard framed link using SnapPy and informing the w of each component as its self-writhe in the projection. The framing of that component to be informed to SnapPy is $(w, 1)$. The complex numbers in polar form which appear at each m_p^t-class are the common quantum WRT-invariants. All except one of the eleven classes are formed by Z-homology spheres. The exception is 16^t_{56} which has no torsion but Betti number 1. These facts are indicated by the small number in parenthesis (which gives the homology of the manifold). Actually the first two classes were recently resolved, only remaining the nine final ones. Here is N. Dunfield's Sage session distinguishing the two manifolds induced by the two blinks in 14^t_{24}:

```
sage: from snappy import *
sage: M1 = Manifold('1424_T71.tri')
sage: M2 = Manifold('1424_T79.tri')
sage: covers1 = M1.covers(5, method='gap')
sage: covers2 = M2.covers(5, method='gap')
sage: [C.homology() for C in covers1]
[Z/132 + Z/132, Z/63 + Z/63, Z/3 + Z/3 + Z/3 + Z/3]
sage: [C.homology() for C in covers2]
[Z/3 + Z/3 + Z/3 + Z/3, Z/213 + Z/213, Z/432 + Z/432]
```

and Cristiana's Sage session also distinguishing the two manifolds induced by the two blinks in 14^t_{24} and, in conjuction with BLINK, topologically classifying the manifolds induced by the four blinks in 15^t_{16}:

```
M=1424_T71, N=1424_T79
sage: [C.homology() for C in coversM]
[Z/3 + Z/3 + Z/3 + Z/3, Z/63 + Z/63, Z/132 + Z/132]
sage: [C.homology() for C in coversN]
[Z/3 + Z/3 + Z/3 + Z/3, Z/213 + Z/213, Z/432 + Z/432]
-----------------------------------------------------
A=1516_T118, B=1516_T119, C=1516_T181, D=1516_T205
sage: [X.homology() for X in coversA]
[Z/229773, Z/1110327, Z/3699687, Z/3018207]
sage: [X.homology() for X in coversC]
[Z/1110327, Z/229773, Z/3018207, Z/3699687]
sage: [X.homology() for X in coversB]
[Z/1052067, Z/3 + Z/1299909, Z/4117827, Z/126627]
```

```
sage: [X.homology() for X in coversD]
[Z/4117827, Z/1052067, Z/3 + Z/1299909, Z/126627]
```

3.1 The $HG8QI_t$ class 14^t_{24}:

14^t_{24} (0) 2 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	166
05	0.221839804	0.000000000	732
06	0.288675135	0.000000000	6153
07	0.569807123	0.000000000	26280
08	0.658347954	0.000000000	118404

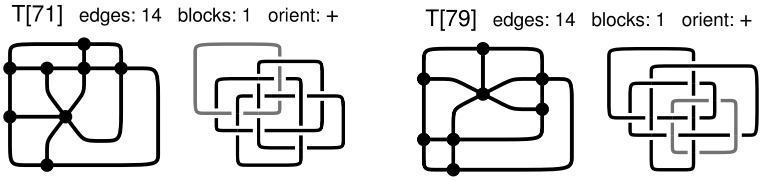


Figure 1: The above two manifolds are not homeomorphic. They are distinguished by the homology of their 5-covers. This was immediately noted by N. Dunfield using Sage and GAP from triangulations obtained by C. Nascimento using SnapPy, which could not find the Dirichlet domain due to numerical instability.

3.2 The $HG8QI_t$ class 15^t_{16}:

15^t_{16} (0) 4 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.626523527	-0.169924730	1684
06	0.288675135	0.000000000	19341
07	0.271887719	-0.143082082	111440
08	0.116351265	1.000000000	620624

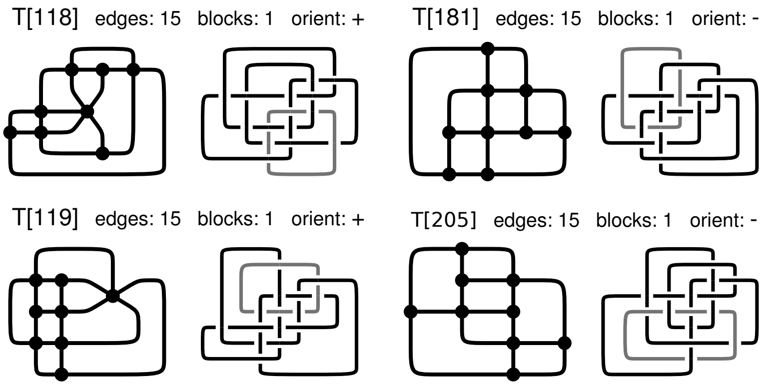


Figure 2: The above two manifolds are also non-homeomorphic. They are also distinguished by the homology of their 5-covers. Relative to the class 15^t_{24} class 15^t_{16} the Sage/GAP software demands much more time. This was obtained by C. Nascimento using SnapPy/Sage/GAP. The software SnapPy could not find the Dirichlet domain due to numerical instability.

3.3 The $HG8QI_t$ class 15^t_{19}:

15^t_{19} (0) 4 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	1.180868794	-0.318489048	1664
06	0.288675135	0.000000000	18957
07	0.344363154	-0.282364319	108168
08	0.990504026	1.000000000	593752

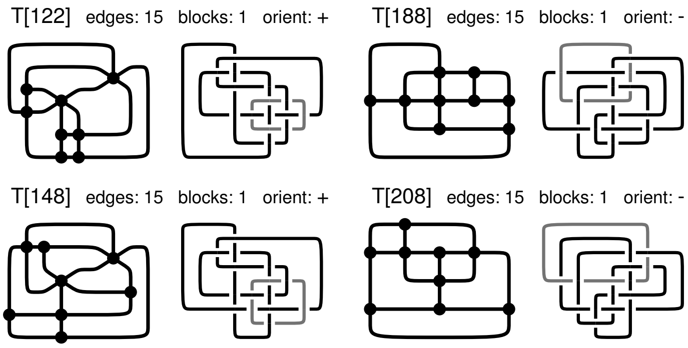


Figure 3: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

3.4 The $HG8QI_t$ class 15^t_{22}:

15^t_{22} (0) 4 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.943093903	-0.180036791	1680
06	0.288675135	0.000000000	20945
07	0.242490221	-0.482680140	111968
08	0.641616039	1.000000000	656072

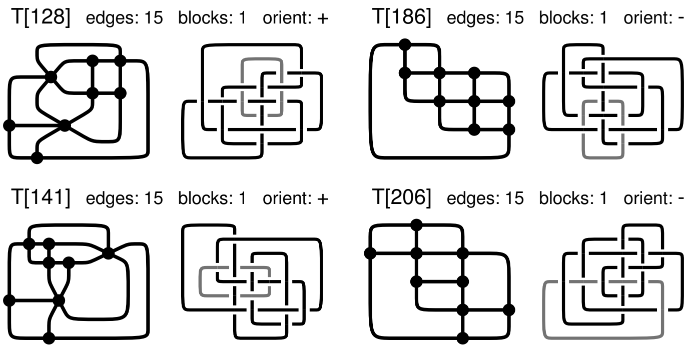


Figure 4: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

3.5 The $HG8QI_t$ class 16^t_{42}:

16^t_{42} (0) 2^2 4 blinks

r	mod	theta/pi	#sts
03	0.000000000	0.000000000	4
04	1.224744871	0.195913276	294
05	0.000000000	0.000000000	1900
06	0.577350269	-0.333333333	22883
07	0.000000000	0.000000000	159992
08	1.774989898	0.206598266	1001208

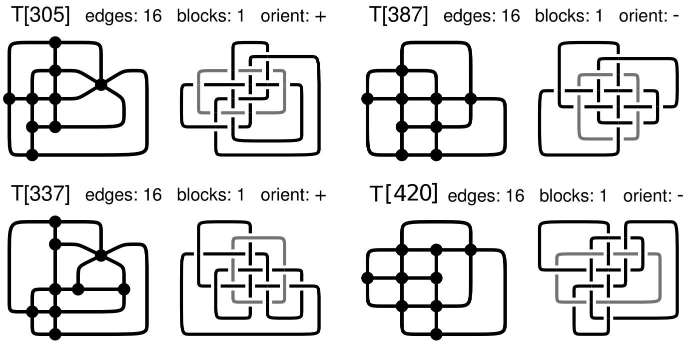


Figure 5: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

3.6 The $HG8QI_t$ class 16^t_{56}:

16^t_{56} (1) 4 blinks

r	mod	theta/pi	#sts
03	1.000000000	0.000000000	8
04	1.000000000	0.000000000	420
05	1.249042228	0.093934688	4512
06	2.000000000	0.000000000	61009
07	1.748655511	-0.081646235	501032
08	0.801948466	0.000000000	3620288

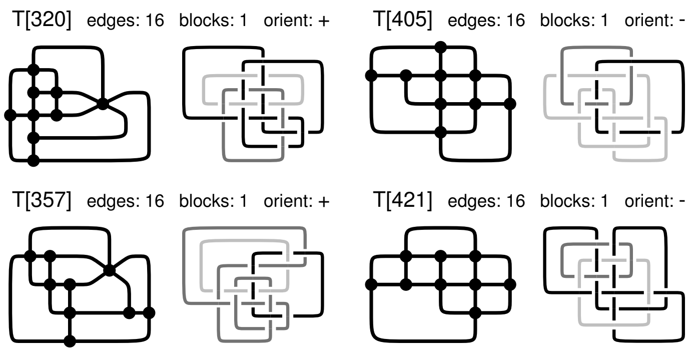


Figure 6: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

3.7 The $HG8QI_t$ class 16^t_{140}:

16^t_{140} (0)   4 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.828810492	0.200682317	1736
06	0.288675135	0.000000000	19469
07	0.369735450	0.097853027	114084
08	0.499366312	1.000000000	624200

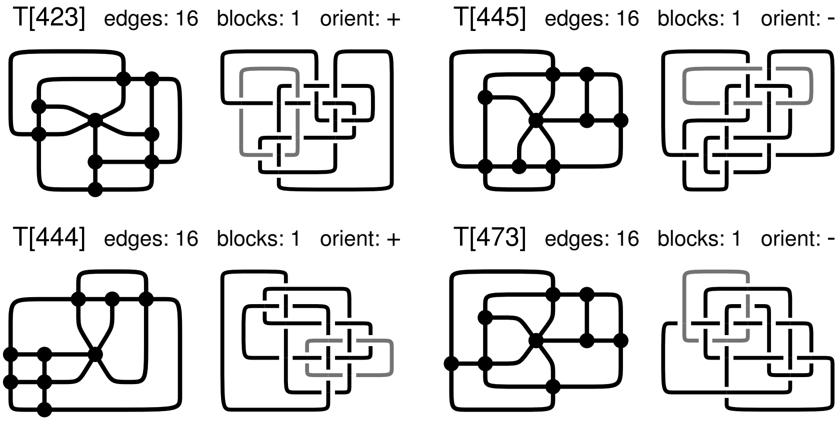


Figure 7: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

3.8 The $HG8QI_t$ class 16^t_{141}:

16^t_{141} (0)   2 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.828231334	0.000000000	1736
06	0.288675135	0.000000000	19469
07	0.591962818	0.000000000	114084
08	0.297863204	0.000000000	624200

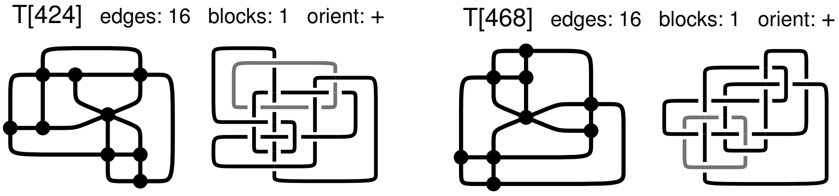


Figure 8: I do not know whether the above two manifolds are homeomorphic or not.

3.9 The $HG8QI_t$ class 16^t_{142}:

16^t_{142} (0) 2 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.314488186	0.000000000	1696
06	0.288675135	0.000000000	21113
07	1.546327427	0.000000000	113232
08	0.931913875	0.000000000	663132

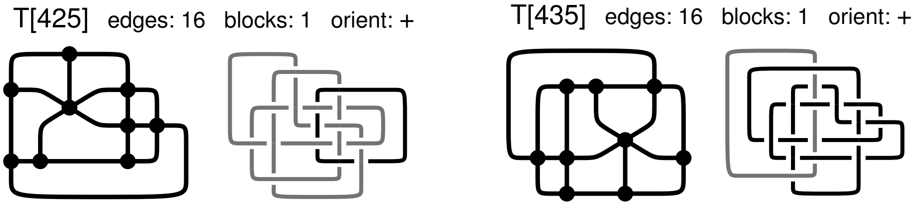


Figure 9: I do not know whether the above two manifolds are homeomorphic or not.

3.10 The $HG8QI_t$ class 16^t_{149}:

16^t_{149} (0) 4 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.314488186	0.000000000	1828
06	0.288675135	0.000000000	21553
07	0.745358881	0.081311555	126032
08	1.620357495	0.000000000	735988

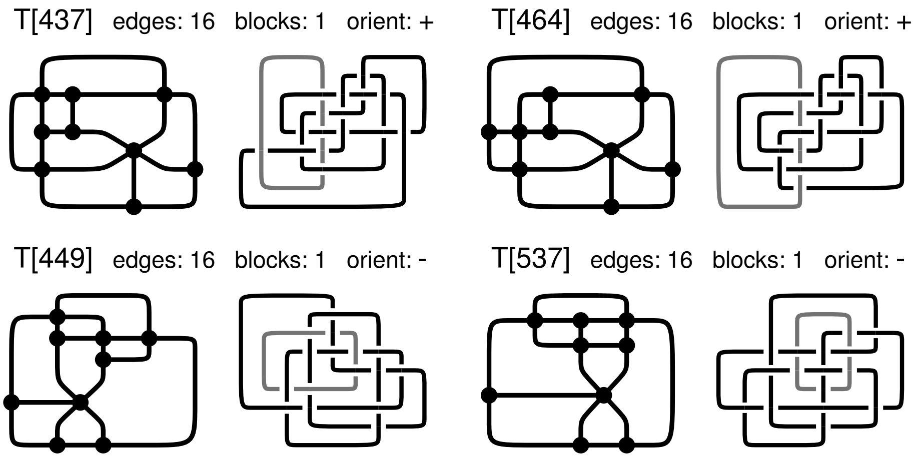


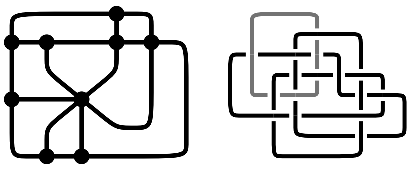
Figure 10: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

3.11 The $HG8QI_t$ class 16^t_{233}:

16^t_{233} (0) 2 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.706953028	0.000000000	1420
06	0.288675135	0.000000000	17019
07	0.472161455	0.000000000	72552
08	0.573693534	0.000000000	401400

T[631] edges: 16 blocks: 1 orient: +



T[663] edges: 16 blocks: 1 orient: +

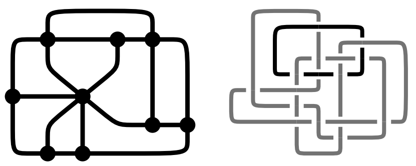


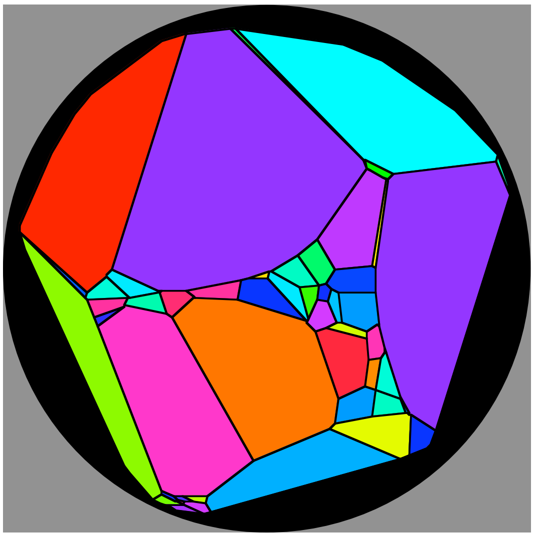
Figure 11: I do not know whether the above two manifolds are homeomorphic or not.

4 Concluding remarks

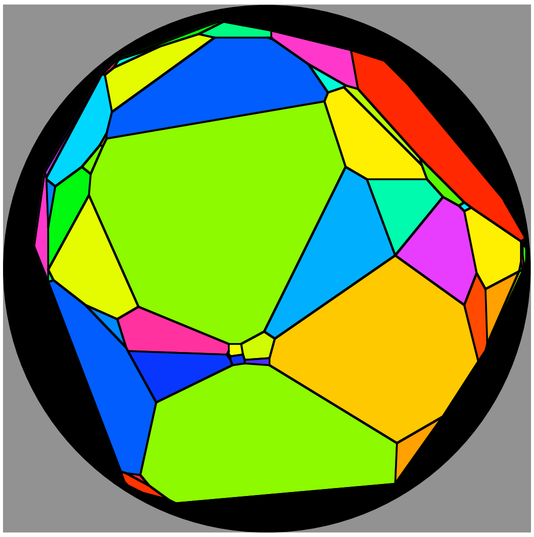
The elegant drawings of blinks and blackboard framed links produced by BLINK are possible due the groundbreaking algorithm of R. Tamasia [16]. Lauro could implement the drawings very fast because we had at hand the implementation of network flow algorithms he had done for a project to solve *practical timetable (!) problems*. This is an example of the unicity in Mathematics, advocated by L. Lovasz in his famous essay [12]. To get the drawings one has to apply three times the full strength of network flow theory. The drawings BLINK presents are in an integer grid and deterministically minimize the number of $\pi/2$-bents in the blackboarded framed links. In particular, it permit us to deal with the unavoidable curls which adjust the integer framings in the best possible way: we do not care about them. The drawings for the companion blinks require a slight modification: it replaces each p-valent vertex $p > 4$, by a p-polygon inducing 3-valent ones. The final result is massaged a bit to produce aesthetically pleasing and unambiguous drawings.

As of this writing, C. Hodgson sent me some puzzling information (computed with a stronger version of SnapPy) about the first pair of manifolds. These are induced by $T[71]$ and $T[79]$, forming the $HG8QI$-class 14^t_{24}. They are non-homeomorphic 3-manifolds as first shown by N. Dunfield. They are homology $\mathbb{Z}$-spheres which have the same WRT-invariants (according to BLINK), and quoting Craig *"the same volume (around 24.8) and the same lenght spectra (up to 12 decimals): the (complex) length of the first geodesic of* $T[71]$ *is 0.47493466632398791 + 0i (of multiplicity 1) and that of* $T[79]$ *is 0.47493466632399361 + 0 i (of multiplicity 1)."*
Here are the Dirichlet domains:

A view of the Dirichlet domain for 1424_T71



A view of the Dirichlet domain for 1424_T79



Craig found another proof that $T[71]$ and $T[79]$ are non-homemorphic: *"Now we can drill out the shortest geodesics using SnapPea to obtain one-cusped manifolds (the manifold files are attached). Then SnapPea's isometry checker (which uses the canonical cell decompositions) shows that these cusped manifolds are not isometric. Hence the original closed manifolds are not homeomorphic. This gives another proof that* $T[71]$ *and* $T[79]$ *are distinct!"*
A final challenge: From the data I could get so far, if a closed orientable 3-manifold is hyperbolic, it seems that the WRT-invariants determine its volume. Prove it or disproved

it.

References

[1] M. Culler, N. M. Dunfield, and J. Weeks. Snappy, a computer program for studying the geometry and topology of 3-manifolds, 2012.

[2] GAP Group et al. Gap – Groups, Algorithms and Programming, version 4.3, 2002.

[3] C. D. Hodgson and J. R. Weeks. Symmetries, isometries and length spectra of closed hyperbolic three-manifolds. *Experimental Mathematics*, 3(4):261–274, 1994.

[4] L.H. Kauffman and S. Lins. Temperley-Lieb Recoupling Theory and Invariants of 3-manifolds. *Annals of Mathematical Studies, Princeton University Press*, 134:1–296, 1994.

[5] E Klarreich. Getting into shapes: from hyperbolic geometry to cube complexes. *Simons Foundation*, October, 2012.

[6] L.D. Lins. Blink: a language to view, recognize, classify and manipulate 3D-spaces. *Arxiv preprint math/0702057*, 2007.

[7] S. Lins. *Gems, Computers, and Attractors for 3-Manifolds.* World Scientific, 1995.

[8] S. Lins and S. Lins. A challenge to 3-manifold topologists and group algebraists. *arXiv:1304.5964v4 [math.GT]*, 2013.

[9] S. Lins and R. Machado. Framed link presentations of 3-manifolds by an $O(n^2)$ algorithm, I: gems and their duals. *arXiv:1211.1953v2 [math.GT]*, 2012.

[10] S. Lins and R. Machado. Framed link presentations of 3-manifolds by an $O(n^2)$ algorithm, II: colored complexes and boundings in their complexity. *arXiv:1212.0826v2 [math.GT]*, 2012.

[11] S. Lins and R. Machado. Framed link presentations of 3-manifolds by an $O(n^2)$ algorithm, III: geometric complex $\mathcal{H}_n^\star$ embedded into $\mathbb{R}^3$. *arXiv:1212.0827v2 [math.GT]*, 2012.

[12] L. Lovasz. One Mathematics. *The Berliner Intelligencer, Berlin*, pages 10–15, 1998.

[13] D. Rolfsen. *Knots and links.* American Mathematical Society, 2003.

[14] W. Stein. *Sage: Open Source Mathematical Software (Version 2.10.2).* The Sage Group, 2008. `http://www.sagemath.org`.

[15] J. Stillwell. *Classical Topology and Combinatorial Group Theory.* Springer Verlag, 1993.

[16] R. Tamassia. On Embedding a Graph in the Grid with the Minimum Number of Bends. *SIAM Journal on Computing*, 16:421, 1987.

[17] A. N. Whitehead. *The classical groups: their invariants and representations.* Princeton University Press, 1997.

Sóstenes L. Lins
Centro de Informática, UFPE
Av. Jornalista Anibal Fernandes s/n
Recife, PE 50740-560
Brazil
sostenes@cin.ufpe.br

5 Appendix: text file to import all links of the paper into SnapPy. Prepared by Cristiana Nascimento

```
% Link Projection 14-24_T71
2
   3    3
  15   15
16
   64    90
   59   232
  247   231
  247    91
  325   115
  208   116
  207   278
  366   275
  365   199
  162   204
  162   333
  274   333
  276   150
  108   149
  104   255
  327   253
16
   0    1
   1    2
   2    3
   3    0
   4    5
   5    6
   6    7
   7    8
   8    9
   9   10
  10   11
  11   12
  12   13
  13   14
  14   15
  15    4
14
   2    4
   5    1
   2    8
```

```
    8    5
    1    9
    6   11
    8   11
   12    2
    5   12
   13    1
   14    5
    9   14
   11   14
   15    8
-1

% Link Projection 14-24_T71
2
    3    3
   15   15
16
   235   205
   236   298
   363   296
   361   206
   192   120
   193   416
   279   416
   278    63
    56    62
    57   235
   311   234
   311   169
   111   173
   111   278
   406   272
   403   113
16
    0    1
    1    2
    2    3
    3    0
    4    5
    5    6
    6    7
    7    8
    8    9
    9   10
   10   11
```

```
  11   12
  12   13
  13   14
  14   15
  15    4
14
   1    6
   6    3
   0    9
   9    4
   9    6
   3   10
   4   11
  11    6
  12    9
  13    0
  13    2
   4   13
   6   13
   6   15
-1

% Link Projection  15-16_T118
2
   3    3
  17   17
18
  301   217
  304   350
  435   350
  428   213
  223   280
  457   271
  456   145
  335   146
  345   322
  137   323
  140   185
  410   175
  408    72
  177    79
  181   237
  379   240
  376   100
  225   110
18
```

```
    0    1
    1    2
    2    3
    3    0
    4    5
    5    6
    6    7
    7    8
    8    9
    9   10
   10   11
   11   12
   12   13
   13   14
   14   15
   15   16
   16   17
   17    4
15
    4    0
    4    2
    7    3
    7    4
    0    8
   10    7
   11    6
   10   13
    0   14
   14    7
    3   15
    6   15
   15   10
   17   10
   14   17
-1

% Link Projection 15-16_T119
2
    3    3
   17   17
18
  203    95
  205   183
  378   182
  378    93
  342   114
```

```
   448   110
   450   386
   248   386
   244   156
   422   158
   422   224
    97   222
    95   331
   285   328
   283    69
   156    72
   152   289
   345   288
18
    0     1
    1     2
    2     3
    3     0
    4     5
    5     6
    6     7
    7     8
    8     9
    9    10
   10    11
   11    12
   12    13
   13    14
   14    15
   15    16
   16    17
   17     4
15
    2     4
    1     7
    8     2
    7    10
    7    12
   13     1
   13     3
    8    13
   10    13
   10    15
   16     7
   13    16
    1    17
```

```
  17    8
  17   10
-1

% Link Projection 15-16_T181
2
   3    3
  17   17
18
    80    41
    84   250
   234   252
   228    50
   120   342
   118   213
   346   214
   341    81
   147    87
   152   392
   406   385
   404   123
   263   131
   267   301
    32   298
    28   170
   373   166
   378   338
18
   0    1
   1    2
   2    3
   3    0
   4    5
   5    6
   6    7
   7    8
   8    9
   9   10
  10   11
  11   12
  12   13
  13   14
  14   15
  15   16
  16   17
  17    4
```

```
15
   1    4
   2    5
   2    7
   8    1
   5    8
   6   11
   5   12
   4   13
  13    8
  15    0
  15    2
  15    6
   8   15
  12   15
   8   17
-1

% Link Projection 15-16_T205
2
   3    3
  17   17
18
  148   284
  146   384
  396   384
  391   284
  388   237
   95   241
   95   448
  262   446
  259   177
  453   175
  455   354
  183   357
  181   106
  322   102
  320   326
  215   325
  216   152
  392   146
18
   0    1
   1    2
   2    3
   3    0
```

```
   4   5
   5   6
   6   7
   7   8
   8   9
   9  10
  10  11
  11  12
  12  13
  13  14
  14  15
  15  16
  16  17
  17   4
15
   1   7
   7   3
   4   7
  10   2
   7  10
  11   3
   4  11
   3  13
  13   4
   8  13
  14   7
   3  15
  15   4
  13  16
  17   8
-1

% Link Projection 15-19_T122
2
   3   3
  17  17
18
  279   281
  276   378
  445   376
  445   280
  245   144
  247   422
  394   420
  388   234
  209   238
```

```
   207    175
   492    171
   490    338
   157    348
   162    210
   325    204
   331    477
   103    474
   100    155
18
     0      1
     1      2
     2      3
     3      0
     4      5
     5      6
     6      7
     7      8
     8      9
     9     10
    10     11
    11     12
    12     13
    13     14
    14     15
    15     16
    16     17
    17      4
15
     1      6
     3      6
     4      7
     4      9
     0     11
    11      2
    11      4
     6     11
    13      4
     8     13
    14      1
    14      3
     5     14
     7     14
    11     14
-1
```

```
% Link Projection 15-19_T148
2
   3    3
  17   17
18
  315   285
  318   422
  460   421
  457   285
  278   139
  279   450
  368   449
  363   204
  189   206
  193   354
  416   349
  415   244
  234   247
  235   174
  481   168
  483   380
  142   391
  143   143
18
   0    1
   1    2
   2    3
   3    0
   4    5
   5    6
   6    7
   7    8
   8    9
   9   10
  10   11
  11   12
  12   13
  13   14
  14   15
  15   16
  16   17
  17    4
15
   1    6
   6    3
   7    4
```

```
   0    9
   9    4
   9    6
   3   10
   4   11
  11    6
  12    7
   4   13
  15    0
  15    2
   4   15
   6   15
-1

% Link Projection 15-19_T188
2
   3    3
  17   17
18
   62    41
   62   155
  231   154
  231    43
   39    84
   42   232
  406   228
  401    54
  194    56
  198   304
  120   305
  114   116
  279   115
  281   273
  157   277
  156   195
  342   190
  343    87
18
   0    1
   1    2
   2    3
   3    0
   4    5
   5    6
   6    7
   7    8
```

```
   8    9
   9   10
  10   11
  11   12
  12   13
  13   14
  14   15
  15   16
  16   17
  17    4
15
   2    7
   8    1
   8    5
   1   10
  10    5
   2   11
  11    8
   5   12
  13    8
   5   14
  15    8
  12   15
  17    0
  17    2
   8   17
-1

% Link Projection 15-19_T208
2
   3    3
  17   17
18
   52    68
   53   205
  297   202
  296    70
   78    89
   80   263
  421   263
  419   120
  115   123
  115   299
  233   296
  232   155
  384   157
```

```
  385   226
  165   223
  167   343
  267   341
  267    91
18
    1     0
    2     1
    3     2
    0     3
    4     5
    5     6
    6     7
    7     8
    8     9
    9    10
   10    11
   11    12
   12    13
   13    14
   14    15
   15    16
   16    17
   17     4
15
    4     1
    2     7
    1     8
    8     5
   10     1
   10     5
   11     2
   13    10
    5    14
   14     9
    1    16
    5    16
    7    16
   16    11
   16    13
-1

% Link Projection 15-22_T128
2
    3     3
   17    17
```

```
18
  289   111
  289   274
  406   273
  406   109
  231   186
  441   188
  437    78
  169    78
  174   318
  472   320
  471   148
  344   149
  356   412
  108   409
  108   246
  430   237
  436   365
  233   367
18
    0    1
    1    2
    2    3
    3    0
    5    4
    6    5
    7    6
    8    7
    9    8
   10    9
   11   10
   12   11
   13   12
   14   13
   15   14
   16   15
   17   16
    4   17
15
    0    4
    2    4
   10    2
    5   10
    1   11
    4   11
   11    8
```

```
  14    0
  14    2
  14    7
  11   14
   8   15
  16   11
   8   17
  17   14
-1

% Link Projection 1522-T141
2
   3    3
  17   17
18
  110   219
  110   297
  280   299
  280   217
  171    89
   49    90
   51   348
  450   350
  445   121
  135   122
  135   261
  397   260
  398   387
  235   386
  236   185
  330   185
  333   417
  175   417
18
   0    1
   1    2
   2    3
   3    0
   5    4
   6    5
   7    6
   8    7
   9    8
  10    9
  11   10
  12   11
```

```
  13   12
  14   13
  15   14
  16   15
  17   16
   4   17
15
   9    3
   2   10
   6   11
  13    1
  13    3
   6   13
  10   13
  15    6
  10   15
  12   15
   1   17
   3   17
  17    6
  17    8
  17   10
-1

% Link Projection 15-22_T186
2
   3    3
  17   17
18
  236   239
  238   382
  325   382
  322   240
  272    60
  276   355
  418   355
  417   164
  201   163
  201    90
  459    87
  463   319
  152   318
  150   133
  384   134
  384   282
  186   280
```

```
  181    59
18
    0     1
    1     2
    2     3
    3     0
    4     5
    5     6
    6     7
    7     8
    8     9
    9    10
   10    11
   11    12
   12    13
   13    14
   14    15
   15    16
   16    17
   17     4
15
    3     4
    5     2
    4     7
    4     9
    0    11
    2    11
   11     4
   11     6
   13     4
    8    13
    7    14
   15     0
   15     2
    4    15
   13    16
-1

% Link Projection 15-22_T206
2
    3     3
   17    17
18
    86   265
    87   406
   407   406
```

```
  401   262
  119   143
  127   366
  320   366
  312    48
  399    49
  404   184
  235   182
  239   292
  363   292
  358    91
  180    92
  186   334
  445   335
  438   131
18
    0    1
    1    2
    2    3
    3    0
    4    5
    5    6
    6    7
    7    8
    8    9
    9   10
   10   11
   11   12
   12   13
   13   14
   14   15
   15   16
   16   17
   17    4
15
    3    4
    6    3
    9    6
    3   10
   11    6
    3   12
   12    9
   13    6
   14    3
   15    2
    6   15
```

```
   6   17
   8   17
  17   12
  17   14
-1

% Link Projection 16-42_T305
2
   3    3
  17   17
18
    94   114
    91   204
   295   208
   298   113
   335   135
    54   131
    55   226
   202   232
   204    94
   386    98
   389   285
   124   287
   121   178
   275   178
   273    63
   160    60
   156   264
   331   262
18
   0    1
   1    2
   2    3
   3    0
   4    5
   5    6
   6    7
   7    8
   8    9
   9   10
  10   11
  11   12
  12   13
  13   14
  14   15
  15   16
```

```
  16   17
  17    4
16
   4    0
   4    2
   1    7
   7    3
   4    7
   1   11
  11    6
   7   12
   3   13
  13    4
  13    8
  15    1
   3   15
  15    4
   6   15
  12   15
-1

% Link Projection 16-42_T337
2
   3    3
  17   17
18
  259   100
  259   258
  378   260
  377   105
  308   321
  304    60
  193    62
  196   223
  438   222
  444   392
  147   388
  142   140
  343   131
  346   320
  412   316
  406   171
  225   180
  230   316
18
   0    1
```

```
    1    2
    2    3
    3    0
    4    5
    5    6
    6    7
    7    8
    8    9
    9   10
   10   11
   11   12
   12   13
   13   14
   14   15
   15   16
   16   17
   17    4
16
    4    1
    4    3
    0    7
    7    2
    7    4
    0   11
   11    4
   11    6
    1   12
   12    7
   14    7
   15    0
    2   15
    4   15
   15   12
    7   16
-1

% Link Projection 16-42_T387
2
    3    3
   15   15
16
  224   129
  223   281
  366   295
  364   133
  303   352
```

```
   302    64
   169    62
   166   212
   438   215
   438    95
   261    96
   271   247
   415   255
   410   172
   128   165
   130   347
16
     0     1
     1     2
     2     3
     3     0
     4     5
     5     6
     6     7
     7     8
     8     9
     9    10
    10    11
    11    12
    12    13
    13    14
    14    15
    15     4
16
     1     4
     3     4
     7     0
     2     7
     7     4
     4     9
    10     3
    10     7
    11     2
     4    11
     7    12
     0    13
    13     2
     4    13
    13     6
    13    10
-1
```

```
% Link Projection 16-42_T420
2
   3    3
  17   17
18
  192   177
  195   280
  392   279
  388   175
  223   252
  226   338
  292   340
  281   131
  126   138
  126   217
  349   215
  341   152
  250   157
  260   313
  427   310
  422   110
  323   122
  328   256
18
   0    1
   1    2
   2    3
   3    0
   4    5
   5    6
   6    7
   7    8
   8    9
   9   10
  10   11
  11   12
  12   13
  13   14
  14   15
  15   16
  16   17
  17    4
16
   4    1
   6    1
```

```
   3    6
   0    9
   6    9
   3   10
   6   11
   1   12
  12    3
   9   12
  13    6
  16    3
   9   16
  11   16
  17    6
  12   17
-1

% Link Projection 16-56_T320
3
   3    3
  13   13
  17   17
18
  215   187
  216   346
  331   344
  324   185
  260   102
  264   316
  371   316
  368   270
  113   274
  111    56
  291    53
  291   295
  408   295
  402   101
  173   142
  372   143
  370   217
  167   215
18
   0    1
   1    2
   2    3
   3    0
   4    5
```

```
    5    6
    6    7
    7    8
    8    9
    9   10
   10   11
   11   12
   12   13
   13    4
   14   15
   15   16
   16   17
   17   14
16
    3    4
    5    2
    0    7
    7    2
    7    4
   10    3
   10    7
    2   11
   11    6
   10   13
    4   14
   14   10
   16    0
    2   16
    4   16
   16   10
-1

% Link Projection 16-56_T357
3
   11   11
   15   15
   19   19
20
    83    46
    84   226
   343   227
   345   303
   227   301
   229   171
   300   172
   302   335
```

```
    170    334
    170    129
    387    128
    389     42
    273     90
    267    269
    433    263
    429     95
    125     62
    130    194
    346    193
    342     66
20
      1      0
      2      1
      3      2
      4      3
      5      4
      6      5
      7      6
      8      7
      9      8
     10      9
     11     10
      0     11
     12     13
     13     14
     14     15
     15     12
     16     17
     17     18
     18     19
     19     16
16
      1      4
      1      6
      3      6
      8      1
     12      1
     12      5
      9     12
     13      2
      6     13
     10     15
      4     17
      6     17
```

```
  17    8
  17   12
  18    9
  15   18
-1

% Link Projection 16-56_T405
3
   3    3
   7    7
  17   17
18
   88    75
   89   200
  270   201
  269    78
  198   106
  199   297
  428   302
  428   103
  392   257
   38   265
   32   134
  313   134
  319   338
  135   347
  128   166
  238   162
  248   384
  388   387
18
   0    1
   1    2
   2    3
   3    0
   4    5
   5    6
   6    7
   7    4
   8    9
   9   10
  10   11
  11   12
  12   13
  13   14
  14   15
```

```
  15  16
  16  17
  17   8
16
   4   1
   2   7
   8   4
  10   0
  10   2
   4  10
  11   5
   8  11
   1  13
  13   8
  14   4
   1  15
   5  15
  15   8
  15  12
   5  17
-1

% Link Projection 16-56_T421
3
   3   3
  13  13
  17  17
18
   87  110
   91  187
  267  179
  261  105
  150  286
  140   69
   65   70
   69  243
  354  230
  364  355
  193  362
  183   64
  321   63
  333  275
  110  146
  116  325
  302  311
  285  134
```

```
18
   0    1
   1    2
   2    3
   3    0
   4    5
   5    6
   6    7
   7    8
   8    9
   9   10
  10   11
  11   12
  12   13
  13    4
  14   15
  15   16
  16   17
  17   14
16
   4    1
   4    3
   7    4
   1   10
   3   10
  10    7
  12    7
  13   10
   1   14
  14    7
  10   15
   7   16
  16   13
  17    2
  17    4
  10   17
-1

% Link Projection 16-140_T423
2
   3    3
  19   19
20
  132   101
  129   284
  225   286
```

```
   227    96
    78    67
    74   172
   339   170
   342   243
   162   251
   158   317
   307   311
   305    64
   431    64
   425   365
   192   365
   191   131
   388   130
   392   207
   262   204
   265    64
20
     0     1
     1     2
     2     3
     3     0
     4     5
     5     6
     6     7
     7     8
     8     9
     9    10
    10    11
    11    12
    12    13
    13    14
    14    15
    15    16
    16    17
    17    18
    18    19
    19     4
16
     0     5
     2     5
     7     2
     8     1
    10     5
    10     7
     1    14
```

```
   5   14
  14    7
  14    9
  15    2
  15   10
   6   17
  17   10
   5   18
  18   15
-1

% Link Projection 16-140_T444
2
   3    3
  17   17
18
  275   293
  274   372
  442   369
  441   287
  219   416
  342   414
  337   203
  141   206
  138   267
  407   263
  406   328
   85   337
   85   243
  296   237
  297   452
   43   450
   45   165
  225   160
18
   0    1
   1    2
   2    3
   3    0
   4    5
   5    6
   6    7
   7    8
   8    9
   9   10
  10   11
```

```
  11   12
  12   13
  13   14
  14   15
  15   16
  16   17
  17    4
16
   1    5
   3    5
   5    8
   9    3
   0   10
   5   10
   7   12
  13    1
  13    3
   4   13
   8   13
  10   13
  17    6
  17    8
  10   17
  12   17
-1

% Link Projection 16-140_T445
2
   3    3
  19   19
20
  177    77
  177   158
  409   158
  403    79
  124    49
  125   298
  463   297
  459    39
  336    38
  335   353
  207   351
  204   270
  380   271
  379   128
  247   129
```

```
  255   418
  159   416
  156   231
  286   227
  283    44
20
    0     1
    1     2
    2     3
    3     0
    5     4
    6     5
    7     6
    8     7
    9     8
   10     9
   11    10
   12    11
   13    12
   14    13
   15    14
   16    15
   17    16
   18    17
   19    18
    4    19
16
    1     8
    3     8
    5     8
    5    10
    8    11
    1    12
    8    13
    1    14
    5    14
    9    14
   11    14
    5    16
   14    17
    1    18
    3    18
   13    18
-1

% Link Projection 16-140_473
```

```
2
     3     3
    17    17
18
   103      72
   107     221
   233     228
   233      65
    82     163
   339     162
   347     300
   276     302
   270     126
   189     134
   203     375
   387     371
   372     258
   151     262
   141      99
   303      95
   315     333
    82     326
18
     0     1
     1     2
     2     3
     3     0
     4     5
     5     6
     6     7
     7     8
     8     9
     9    10
    10    11
    11    12
    12    13
    13    14
    14    15
    15    16
    16    17
    17     4
16
     4     0
     2     4
     4     7
     8     2
```

```
   9   1
   4   9
   5  12
   7  12
  12   9
   1  13
  13   4
   2  14
  15   4
  15   6
  12  15
   9  16
-1

% Link Projection 16-141_T424
2
   3   3
  19  19
20
  141   83
  147  191
  337  190
  326   79
  214   59
  228  332
  416  328
  400  105
  289  106
  307  301
   93  306
   90  242
  378  239
  369  136
  185  137
  192  280
  262  274
  256  207
   90  211
   82   61
20
   0   1
   1   2
   2   3
   3   0
   4   5
   5   6
```

```
   6   7
   7   8
   8   9
   9  10
  10  11
  11  12
  12  13
  13  14
  14  15
  15  16
  16  17
  17  18
  18  19
  19   4
16
   4   1
   4   3
   2   7
   1   8
   4   9
   4  11
   8  11
  13   2
  13   4
   8  13
   1  14
  11  14
  15   4
  11  16
  17   4
  14  17
-1

% Link Projection 16-141_T468
2
   3   3
  17  17
18
    59   290
    61   386
   216   387
   211   286
    33   256
    33   320
   354   313
   349   159
```

```
  269   157
  274   352
  183   357
  178   204
  407   201
  417   430
   86   428
   82   176
  316   177
  315   247
18
    0    1
    1    2
    2    3
    3    0
    4    5
    5    6
    6    7
    7    8
    8    9
    9   10
   10   11
   11   12
   12   13
   13   14
   14   15
   15   16
   16   17
   17    4
16
    0    5
    5    2
    8    5
    2    9
    3   10
   10    5
    6   11
    8   11
   14    1
   14    3
    5   14
   15    8
   11   16
   17    8
   10   17
   17   14
```

```
-1

% Link Projection 16-142_T425
2
  13   13
  17   17
18
   87    60
   86   244
  387   237
  383   165
  122   177
  119   364
  303   363
  301    89
  154    90
  150   393
  331   387
  336   199
  192   208
  190    60
  276   116
  271   260
  452   259
  451   114
18
   0    1
   1    2
   2    3
   3    4
   4    5
   5    6
   6    7
   7    8
   8    9
   9   10
  10   11
  11   12
  12   13
  13    0
  14   15
  15   16
  16   17
  17   14
16
   4    1
```

```
   1    6
   3    6
   1    8
   8    3
   8    5
  10    1
   6   11
   3   12
  12    7
  14    1
  14    3
  11   14
   6   15
  15   10
   6   17
-1

% Link Projection 16-142_T435
2
   3    3
  17   17
18
   90    77
   90   387
  253   387
  250    77
  127   101
  123   307
  439   305
  436   207
  158   217
  158   353
  482   354
  482   264
  331   270
  328   149
  200   148
  205   428
  370   426
  366   100
18
   0    1
   1    2
   2    3
   3    0
   4    5
```

```
   5   6
   6   7
   7   8
   8   9
   9  10
  10  11
  11  12
  12  13
  13  14
  14  15
  15  16
  16  17
  17   4
16
   5   2
   2   7
   5   8
   2   9
   6  11
   7  12
  13   2
  14   1
  14   5
   7  14
   9  14
  16   5
  16   7
   9  16
  11  16
   2  17
-1

% Link Projection 16-149_T437
2
   3   3
  19  19
20
   85   88
   91  341
  200  338
  195   89
   33  375
   29  267
  221  270
  216  115
  274  117
```

```
  282   303
  148   303
  148   183
  251   183
  243    88
  341    90
  347   223
  120   224
  119   147
  307   144
  317   373
20
    0     1
    1     2
    2     3
    3     0
    5     4
    6     5
    7     6
    8     7
    9     8
   10     9
   11    10
   12    11
   13    12
   14    13
   15    14
   16    15
   17    16
   18    17
   19    18
    4    19
16
    5     0
    5     2
    2     9
   10     5
   11     2
    6    11
    7    12
    2    15
   15     6
    8    15
   15    10
    2    17
   17     6
```

```
  17   8
  12  17
  15  18
-1

% Link Projection 16-149_T449
2
   3   3
  17  17
18
   71   199
   76   287
  242   290
  239   192
  153   159
  158   369
  364   368
  366   255
  194   252
  189   118
   26   127
   29   324
  312   322
  307   216
  112   224
  116   409
  285   405
  271   160
18
   0   1
   1   2
   2   3
   3   0
   5   4
   6   5
   7   6
   8   7
   9   8
  10   9
  11  10
  12  11
  13  12
  14  13
  15  14
  16  15
  17  16
```

```
   4   17
16
   4    1
   4    3
   2    7
   3    8
  11    4
  12    7
  13    2
  13    4
   8   13
   1   14
  14   11
   5   16
   7   16
  16   11
  16   13
   8   17
-1

% Link Projection 16-149_T464
2
   3    3
  19   19
20
   83    78
   83   373
  240   372
  236    82
  117   137
  118   274
  365   271
  365    99
  298   100
  296   247
  161   252
  165   182
  337   181
  339    76
  415    74
  413   211
  191   214
  193   317
  440   315
  442   124
20
```

```
    0    1
    1    2
    2    3
    3    0
    5    4
    6    5
    7    6
    8    7
    9    8
   10    9
   11   10
   12   11
   13   12
   14   13
   15   14
   16   15
   17   16
   18   17
   19   18
    4   19
16
    2    5
    9    2
   11    2
    8   11
    7   12
    2   15
    6   15
   15    8
    5   16
   16    9
   17    2
    2   19
   19    6
   19    8
   12   19
   14   19
-1

% Link Projection 16-149_T537
2
    3    3
   17   17
18
   196   113
   194   299
```

```
   306   301
   304   108
    53   187
   376   186
   377    83
   158    76
   161   389
   367   392
   369   251
   119   259
   116   443
   261   441
   259   146
   413   141
   415   349
    54   341
18
     0     1
     1     2
     2     3
     3     0
     4     5
     5     6
     6     7
     7     8
     8     9
     9    10
    10    11
    11    12
    12    13
    13    14
    14    15
    15    16
    16    17
    17     4
16
     0     4
     2     4
     4     7
    10     0
    10     2
     7    10
     1    13
     8    13
    13    10
     4    13
```

```
  14   2
   5  14
  16   7
  16   9
  11  16
  13  16
-1

% Link Projection 16-233_T631
2
   3   3
  17  17
18
    77    59
    74   193
   193   196
   189    63
    39   119
   235   115
   238   200
   376   201
   381   276
   154   272
   151    93
   289    92
   296   324
   120   322
   120   157
   344   160
   346   237
    39   241
18
   0   1
   1   2
   2   3
   3   0
   4   5
   5   6
   6   7
   7   8
   8   9
   9  10
  10  11
  11  12
  12  13
  13  14
```

```
  14   15
  15   16
  16   17
  17    4
16
   4    0
   4    2
   9    1
   9    4
   2   10
   6   11
   8   11
   1   13
   2   14
  14    5
  14    9
  11   14
  15    6
  16    9
  11   16
  13   16
-1

% Link Projection 16-233_T663
2
  15   15
  19   19
20
   51    94
  198    94
  201   255
   17   254
   18   318
  440   312
  438   147
  284   147
  286   419
  171   419
  170   181
  366   179
  365   366
  230   367
  231   283
   53   287
  104   121
  104   228
```

```
   324    227
   320    120
20
     0      1
     1      2
     2      3
     3      4
     4      5
     5      6
     6      7
     7      8
     8      9
     9     10
    10     11
    11     12
    12     13
    13     14
    14     15
    15      0
    16     17
    17     18
    18     19
    19     16
16
     7      4
     9      2
     9      4
    10      1
     7     10
     4     11
    12      7
     4     13
    14      9
     2     15
     1     17
    17      7
    17      9
    18      6
    10     18
     1     19
-1
```