

A tougher challenge to 3-manifold topologists and group algebraists *

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Abstract

This paper poses some basic questions about instances (hard to find) of a special problem in 3-manifold topology. “Important though the general concepts and propositions may be with the modern industrious passion for axiomatizing and generalizing has presented us . . . nevertheless I am convinced that the special problems in all their complexity constitute the stock and the core of mathematics; and to master their difficulty requires on the whole the harder labor.” Hermann Weyl 1885-1955, cited in the preface of the first edition (1939) of A. N. Whitehead’s book *The classical groups: their invariants and representations* [17].

In this paper I focus on new uncertainties left unanswered in L. Lins thesis [6] on the homomorphism problem of eleven concrete pairs of closed orientable 3-manifolds induced by 3-connected monochromatic *blinks* ([4]). The eleven HG8QI-classes are the only doubts left in the thesis, but the first two of them were solved few days ago and in this work I report on their solutions. We also include an appendix which can be used to import all the links of this paper into SnapPy. The appendix was obtained by drawing the links in SnapPy, work performed by C. Nascimento.

1 Introduction

In a joint recent paper posted recently in the arXiv ([8]) my son Lauro Lins and myself ask some 6 years old questions for which we had no answers about homeomorphisms between closed orientable 3-manifolds. The two pairs of 3-manifolds were the only uncertainties that were left in L. Lins thesis ([6]) under my supervision in the domain of 3-manifolds being induced by arbitrary connected blinks up to 9 edges (9-small 3-manifolds). A subset of relevant 10-crossings blinks were generated but their topological classification remains untouched. The paper was taken seriously by a few researchers, among them M. Culler, N. Dunfield, C. Hodgson and others that could solve them very quickly using GAP ([2]), Sage ([14]) and SnapPy ([1]), tools that (except for GAP) were basically unknown to us. The solutions were obtained by distinct methods and are all consistent (inclusive with BLINK, the program of L. Lins (implementing my theory described in [7]), which support his thesis). Together with my colleague Cristiana Nascimento, here at CIN/UFPE, I am learning fast

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to operate these wonderful tools. The solutions people found shows that BLINK does a complete job in topologically classifying the 9-small 3-manifolds. This is the subject of a joint paper with Lauro, currently under preparation.

The first solution that I got, and that still blows my mind, was by Craig Hodgson using length spectra techniques, based in his joint paper with J. Weeks entitled *Symmetries, isometries and length spectra of closed hyperbolic three-manifolds* ([3]). By using SnapPy Craig showed that even though the quantum WRT-invariants as well as the volumes of the hyperbolic Z -homology spheres induced by the bfl's, $U[1466]$ and $U[1563]$ are the same, the length of the smallest geodesics of them are distinct. For the other pair of bfl's $U[2125]$ and $U[2165]$ he shows that precisely the same facts apply. Here is a summary of Craig's findings extracted from the SnapPy session that he kindly sent me. As Craig writes: *"The output of the length spectrum command shows the complex lengths of closed geodesics — the real part is the actual length and the imaginary part is the rotation angle as you go once around the geodesic."*

Class 9_{126} :

First geodesic of $U[1466]$: $1.0152103824828331+0.39992347315914334i$.
 First geodesic of $U[1563]$: $0.9359206605025168+2.333526236965665i$.
 Volume of both manifolds: 7.36429600733 .

Class 9_{199} :

First geodesic of $U[2125]$: $0.8939075859248593+0.761197185679321i$.
 First geodesic of $U[2165]$: $0.7978548001747316+2.9487425029345973i$.
 Volume of both manifolds: 7.12868652133 .

I posted 4 versions of [8] correcting annoying mistakes in the presentations of the fundamental groups, putting a second pair of links, and in focusing the challenge in a broader context. I computed the presentations manually and I had a hard time making them correct. Even though the presentations are redundant because the blink is enough to define the 3-manifold, as explained in [4], my objective was to facilitate the work for those wanting to use GAP. The time spanned between the first (April 22, 2013) and the last version (May 1, 2013) was a little more than one week. During these revisions I was completely unaware that the paper had called the attention of many people. I did not know that the blog on lower dimensional topology was very active exposing my incorrections and I apologize for my ignorance. I thank Cristiana for having calling the blog to my attention. Worse, some people did not see the follow up versions. This was the case of Nathan Dunfield who worked with the wrong presentations. Not without reason he was angry at me, but I think that this is no longer true, since he was willing to answer my sometimes naives and stupid questions and send me a solution for the first pair of manifolds of the present work, using SnapPy, Sage and GAP computations, by working with covers. I did not know these tools. But, when properly motivated, I can learn fast and in general I do believe that I have something important and different to say in this brave new world of 3-manifolds: see the wonderful essay of E. Klarreich published by the Simons Foundation (march 2012), [5]. I have been putting a great amount of time and effort during my scientific career, (most of the time as

an isolated researcher) on (mainly closed) 3-manifolds. I seek no longer to be isolated: my team is the World, my compromise is with Truth (independently of whom first found it).

Marc Culler was very helpful in answering questions of myself and Cristiana and helping her about issues in the downloading and installing SnapPy and Sage and GAP in her machine. With the presentation incorrections out of the way he produced an independent proof of the distinctiveness of $(U[1466], U[1563])$ and of $(U[2125], U[2165])$. He also produced instantaneous isomorphic triangulations of the homeomorphic 3-manifolds in the classes 9_{126} and 9_{199} . This fact makes me anxious to compare and timing the performances of BLINK (which also produces instantaneous solutions for the same problems) and SnapPy regarding finding homeomorphisms of k -small 3-manifolds, given that the homeomorphisms exist.

2 Objective of this work: help to make BLINK known

In this paper I put some new challenges (also coming from [6]), that seem harder than the ones considered in the previous paper. The reason I think so is that going from 9 to 14,15,16 crossings in the links, numerical problems start appearing concerning finding the Dirichlet domain and, in these cases, finding isomorphic triangulations might be harder to SnapPy than to BLINK. At any rate I have hundreds of examples where the performance of these programs in this issue could be compared, if anyone is seriously interested. Currently BLINK is not documented and one of my objectives is to seek for help in doing it and extend its capability. BLINK is hosted at Github under the userid *laurobins* and is open source code project. Unfortunately Lauro (currently a researcher at AT&T) does not have the necessary time to go on with the implementation. But he welcomes and is willing to help collaborators in getting started. As for myself, I am too old for the energy needed to construct good pieces of software. I intend to act as one of some Scientific Supervisors for the deployment and for the discussions of the new algorithms to be included in BLINK, but only at the mathematical level. The technological and software engineering screws and bolts needed, I leave to others.

An algorithm that I want to attach to BLINK is finding a uniformly distributed random closed orientable 3-manifold induced by a blink with an arbitrary number (even thousands) of edges. I want to gather evidence for the truth of some important conjectures that depend on this capability. Another example of such new algorithms that I want to include in BLINK is made possible by the theory in Ricardo Machado's thesis under my supervision, defended in March, 2013. We got an $O(n^2)$ -algorithm for going from a special kind of gem, named *resoluble gem*, to a blink inducing the same manifold. This work is available, in still rather sketchy form (even the definition of resolvability is unnecessary complicated), in the three joint papers posted last year in the arXiv, [9, 10, 11]. The algorithm was implemented in Mathematica, but it needs to be improved and re-implemented in Java or C++. We found a rather crude framed link presentation for the hyperbolic dodecahedral space (Weber-Seifert manifold). As far as I know nobody has found such a framed link. My interest in it was aroused by J. Weeks in a visit to the Geometry Center in April 1993, when he asked me whether I had such framed link. The link inducing the Weber-Seifert 3-manifold is a 9-component link embedded into \mathbb{R}^3 , with an integer attached to each component (its framing) and having a total of 68 (only) vertices with a projection having 142 crossings. (It started with a PL-link with more than 600 vertices.) In a fourth joint paper with R. Machado,

currently under preparation, we will show that every 3-manifold admits a resolvable gem inducing it.

3 The 11 $HG8QI_t$ -classes of blinks left unresolved in [6]

I assume that the reader has with him a copy of the version 4 of previous challenge paper ([8]) and has learned how to read the manifold either from the blink or from the blackboard framed link, [4]. As for obtaining a presentation of the fundamental group based on the Wirtinger relators ([15]) and the Dehn fillings ([13]) the two detailed examples given in [8] should suffice, if the reader has not available other pieces of softwares to get the presentation by automatic means. Actually, the best way to enter these manifolds is to draw the blackboard framed link using SnapPy and informing the w of each component as its self-writhe in the projection. The framing of that component to be informed to SnapPy is $(w, 1)$. The complex numbers in polar form which appear at each m_p^t -class are the common quantum WRT-invariants. All except one of the eleven classes are formed by Z -homology spheres. The exception is 16_{56}^t which has no torsion but Betti number 1. These facts are indicated by the small number in parenthesis (which gives the homology of the manifold). Actually the first two classes were recently resolved, only remaining the nine final ones. Here is N. Dunfield's Sage session distinguishing the two manifolds induced by the two blinks in 14_{24}^t :

```
sage: from snappy import *
sage: M1 = Manifold('1424_T71.tri')
sage: M2 = Manifold('1424_T79.tri')
sage: covers1 = M1.covers(5, method='gap')
sage: covers2 = M2.covers(5, method='gap')
sage: [C.homology() for C in covers1]
[Z/132 + Z/132, Z/63 + Z/63, Z/3 + Z/3 + Z/3 + Z/3]
sage: [C.homology() for C in covers2]
[Z/3 + Z/3 + Z/3 + Z/3, Z/213 + Z/213, Z/432 + Z/432]
```

and Cristiana's Sage session also distinguishing the two manifolds induced by the two blinks in 14_{24}^t and, in conjunction with BLINK, topologically classifying the manifolds induced by the four blinks in 15_{16}^t :

```
M=1424_T71, N=1424_T79
sage: [C.homology() for C in coversM]
[Z/3 + Z/3 + Z/3 + Z/3, Z/63 + Z/63, Z/132 + Z/132]
sage: [C.homology() for C in coversN]
[Z/3 + Z/3 + Z/3 + Z/3, Z/213 + Z/213, Z/432 + Z/432]
-----
A=1516_T118, B=1516_T119, C=1516_T181, D=1516_T205
sage: [X.homology() for X in coversA]
[Z/229773, Z/1110327, Z/3699687, Z/3018207]
sage: [X.homology() for X in coversC]
[Z/1110327, Z/229773, Z/3018207, Z/3699687]
sage: [X.homology() for X in coversB]
[Z/1052067, Z/3 + Z/1299909, Z/4117827, Z/126627]
```

```
sage: [X.homology() for X in coversD]  
[Z/4117827, Z/1052067, Z/3 + Z/1299909, Z/126627]
```

3.1 The $HG8QI_t$ class 14_{24}^t :

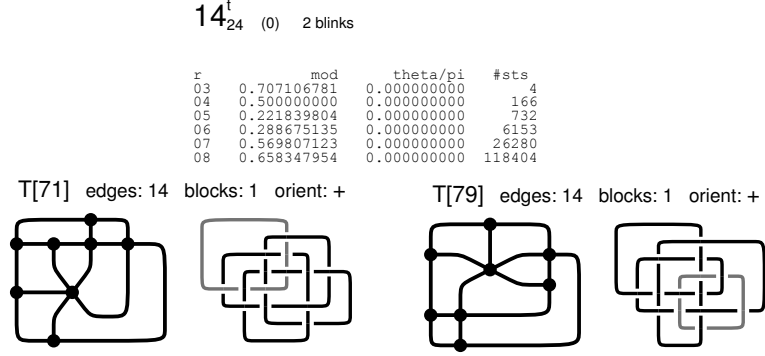


Figure 1: The above two manifolds are not homeomorphic. They are distinguished by the homology of their 5-covers. This was immediately noted by N. Dunfield using Sage and GAP from triangulations obtained by C. Nascimento using SnapPy, which could not find the Dirichlet domain due to numerical instability.

3.2 The $HG8QI_t$ class 15_{16}^t :

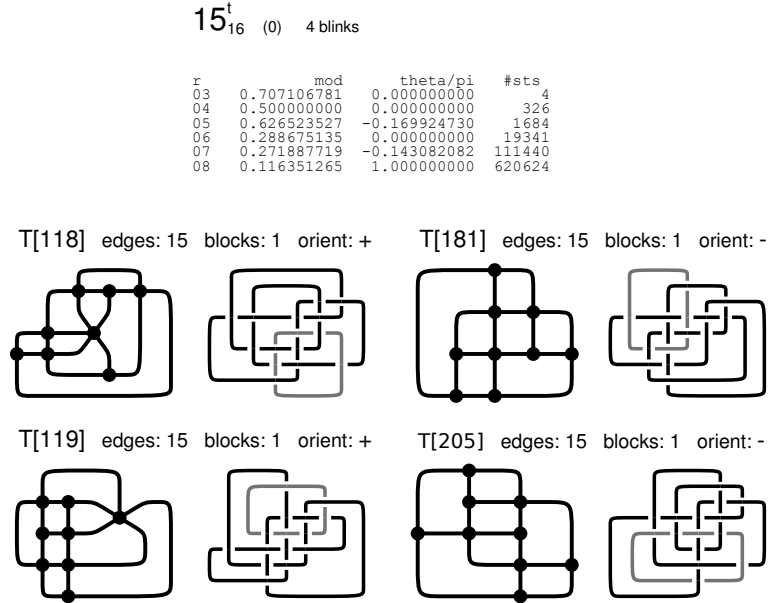


Figure 2: The above two manifolds are also non-homeomorphic. They are also distinguished by the homology of their 5-covers. Relative to the class 15_{24}^t class 15_{16}^t the Sage/GAP software demands much more time. This was obtained by C. Nascimento using SnapPy/Sage/GAP. The software SnapPy could not find the Dirichlet domain due to numerical instability.

3.3 The $HG8QI_t$ class 15_{19}^t :

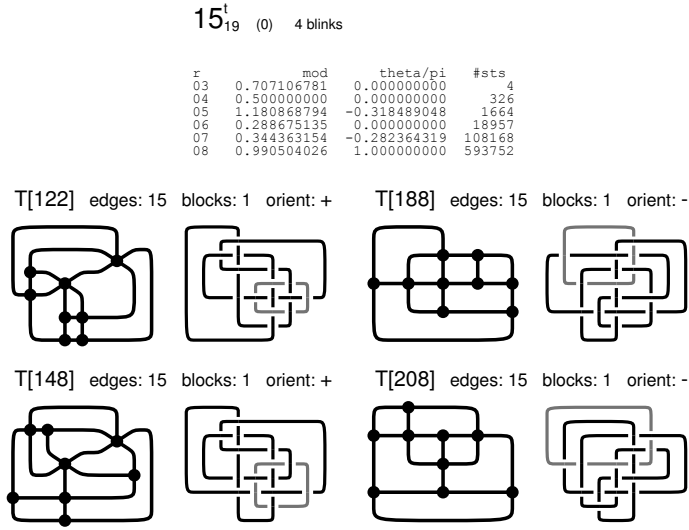


Figure 3: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

3.4 The $HG8QI_t$ class 15_{22}^t :

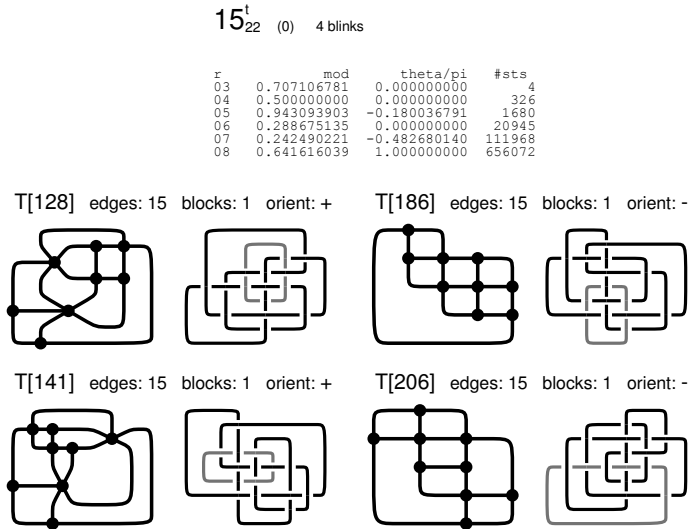


Figure 4: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

3.5 The $HG8QI_t$ class 16_{42}^t :

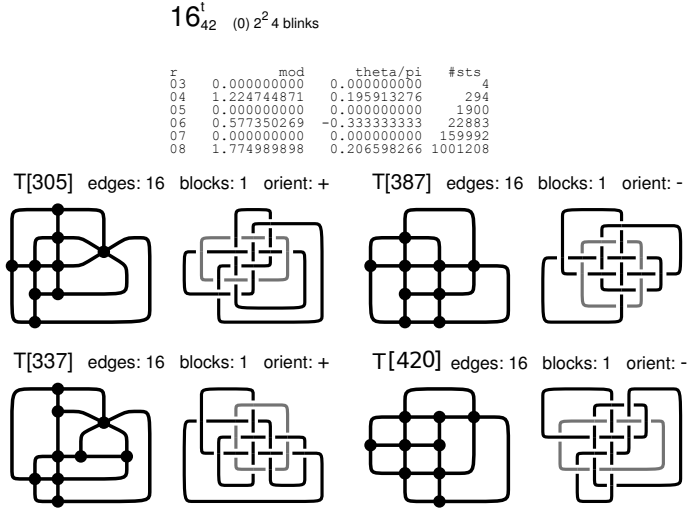


Figure 5: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

3.6 The $HG8QI_t$ class 16_{56}^t :

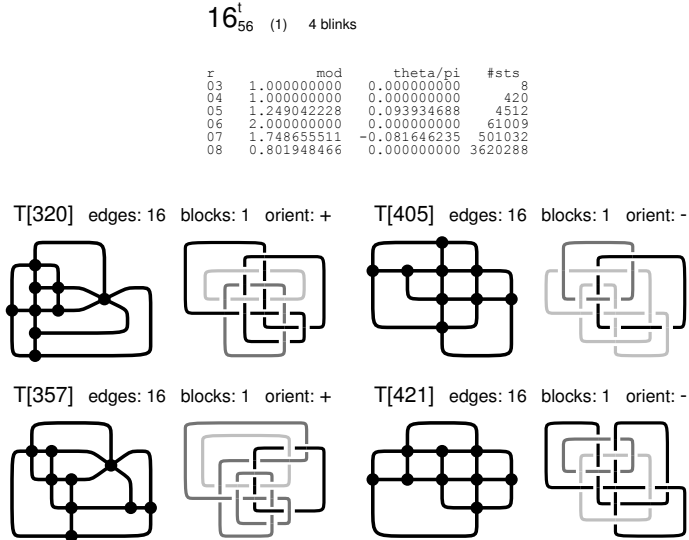


Figure 6: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

3.7 The $HG8QI_t$ class 16_{140}^t :

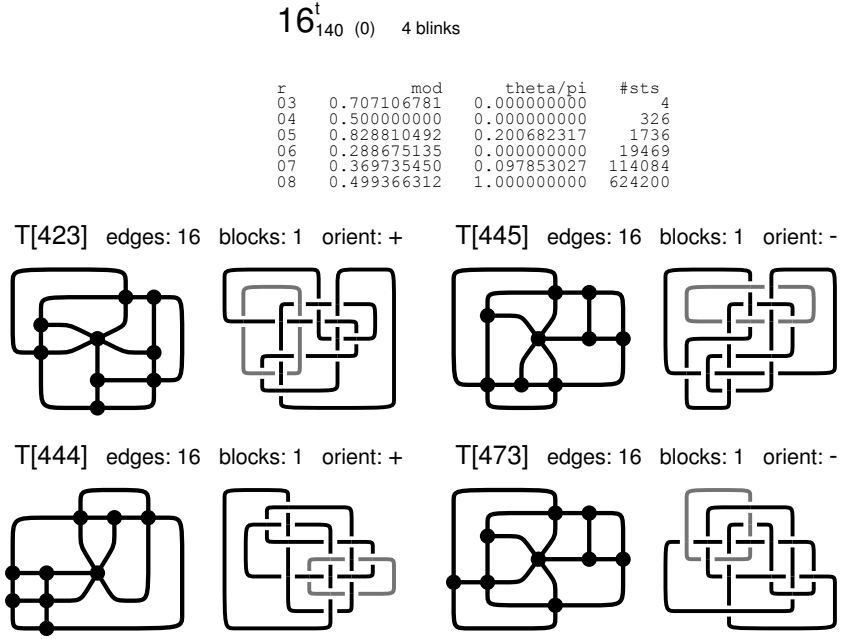


Figure 7: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

3.8 The $HG8QI_t$ class 16_{141}^t :

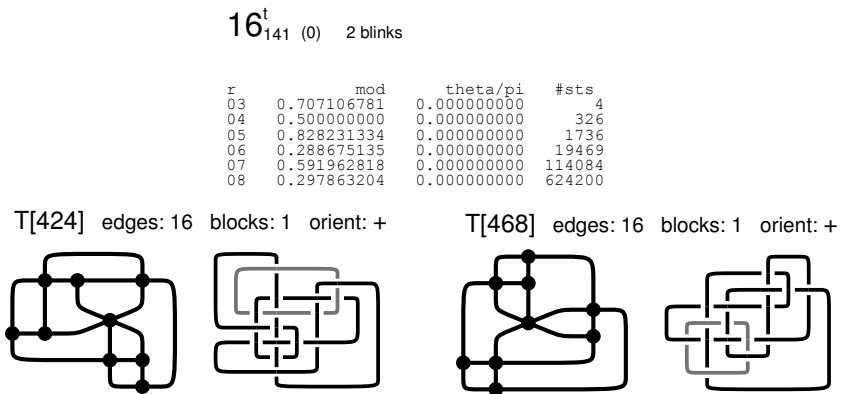


Figure 8: I do not know whether the above two manifolds are homeomorphic or not.

3.9 The $HG8QI_t$ class 16_{142}^t :

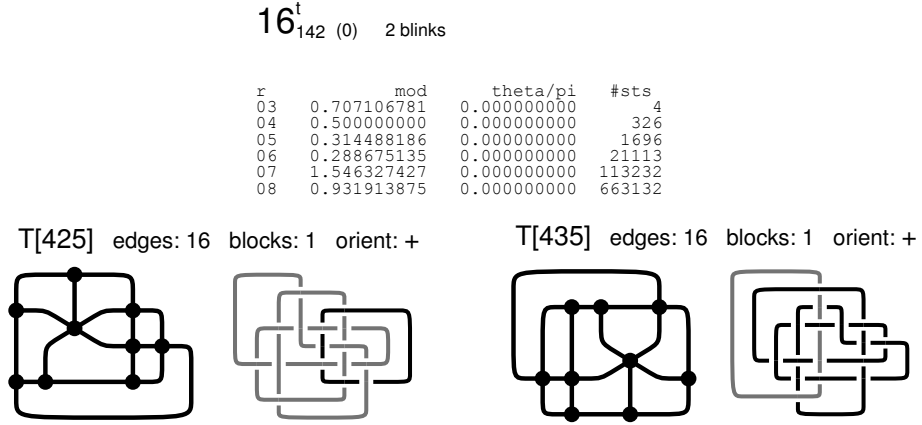


Figure 9: I do not know whether the above two manifolds are homeomorphic or not.

3.10 The $HG8QI_t$ class 16_{149}^t :

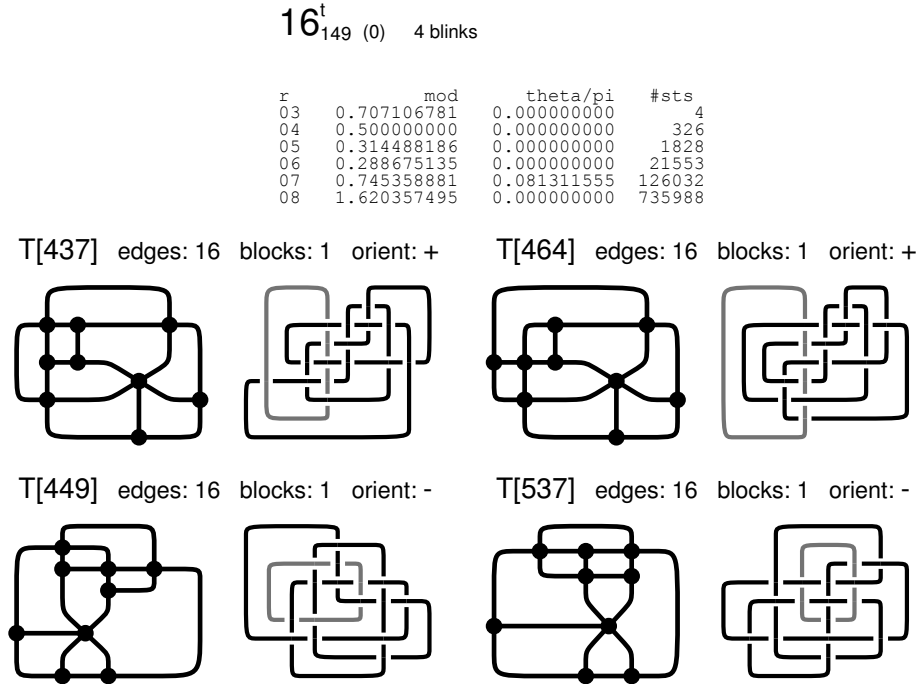


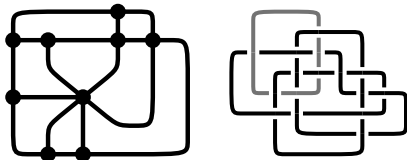
Figure 10: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

3.11 The $HG8QI_t$ class 16^t_{233} :

16^t_{233} (0) 2 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.706953028	0.000000000	1420
06	0.288675135	0.000000000	17019
07	0.472161455	0.000000000	72552
08	0.573693534	0.000000000	401400

T[631] edges: 16 blocks: 1 orient: +



T[663] edges: 16 blocks: 1 orient: +

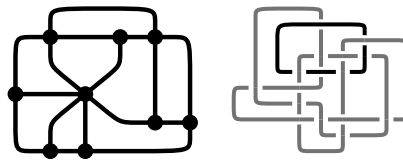


Figure 11: I do not know whether the above two manifolds are homeomorphic or not.

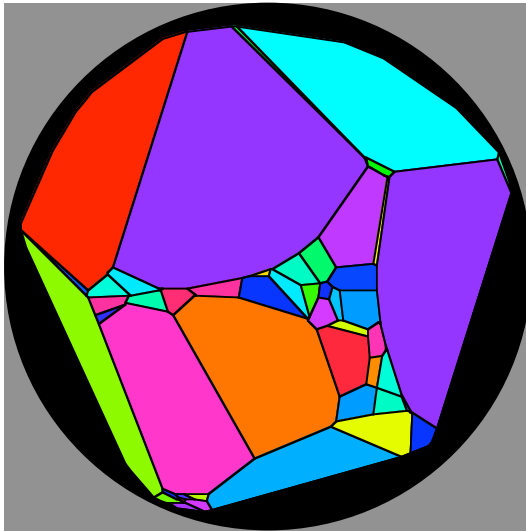
4 Concluding remarks

The elegant drawings of blinks and blackboard framed links produced by BLINK are possible due the groundbreaking algorithm of R. Tamassia [16]. Lauro could implement the drawings very fast because we had at hand the implementation of network flow algorithms he had done for a project to solve *practical timetable (!) problems*. This is an example of the unicity in Mathematics, advocated by L. Lovasz in his famous essay [12]. To get the drawings one has to apply three times the full strength of network flow theory. The drawings BLINK presents are in an integer grid and deterministically minimize the number of $\pi/2$ -bents in the blackboarded framed links. In particular, it permit us to deal with the unavoidable curls which adjust the integer framings in the best possible way: we do not care about them. The drawings for the companion blinks require a slight modification: it replaces each p -valent vertex $p > 4$, by a p -polygon inducing 3-valent ones. The final result is massaged a bit to produce aesthetically pleasing and unambiguous drawings.

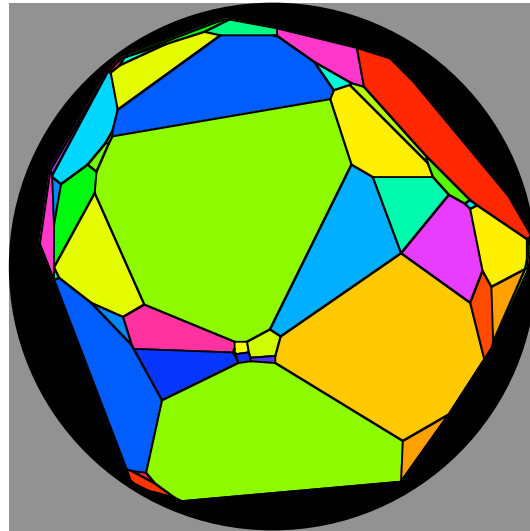
As of this writing, C. Hodgson sent me some puzzling information (computed with a stronger version of SnapPy) about the first pair of manifolds. These are induced by $T[71]$ and $T[79]$, forming the $HG8QI$ -class 14_{24}^t . They are non-homeomorphic 3-manifolds as first shown by N. Dunfield. They are homology \mathbb{Z} -spheres which have the same WRT-invariants (according to BLINK), and quoting Craig “*the same volume (around 24.8) and the same lenght spectra (up to 12 decimals): the (complex) length of the first geodesic of $T[71]$ is $0.4749346632398791 + 0i$ (of multiplicity 1) and that of $T[79]$ is $0.4749346632399361 + 0i$ (of multiplicity 1).*”

Here are the Dirichlet domains:

A view of the Dirichlet domain for 1424_T71



A view of the Dirichlet domain for 1424_T79



Craig found another proof that $T[71]$ and $T[79]$ are non-homeomorphic: “*Now we can drill out the shortest geodesics using SnapPea to obtain one-cusped manifolds (the manifold files are attached). Then SnapPea’s isometry checker (which uses the canonical cell decompositions) shows that these cusped manifolds are not isometric. Hence the original closed manifolds are not homeomorphic. This gives another proof that $T[71]$ and $T[79]$ are distinct!*”

A final challenge: From the data I could get so far, if a closed orientable 3-manifold is hyperbolic, it seems that the WRT-invariants determine its volume. Prove it or disproved

it.

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5 Appendix: text file to import all links of the paper into SnapPy. Prepared by Cristiana Nascimento

% Link Projection 14-24_T71

```

2
    3    3
    15   15
16
    64    90
    59   232
    247   231
    247    91
    325   115
    208   116
    207   278
    366   275
    365   199
    162   204
    162   333
    274   333
    276   150
    108   149
    104   255
    327   253
16
    0     1
    1     2
    2     3
    3     0
    4     5
    5     6
    6     7
    7     8
    8     9
    9    10
   10    11
   11    12
   12    13
   13    14
   14    15
   15     4
14
    2     4
    5     1
    2     8

```

8	5
1	9
6	11
8	11
12	2
5	12
13	1
14	5
9	14
11	14
15	8

-1

% Link Projection 14-24_T71

2

3	3
15	15

16

235	205
236	298
363	296
361	206
192	120
193	416
279	416
278	63
56	62
57	235
311	234
311	169
111	173
111	278
406	272
403	113

16

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11

11	12
12	13
13	14
14	15
15	4

14

1	6
6	3
0	9
9	4
9	6
3	10
4	11
11	6
12	9
13	0
13	2
4	13
6	13
6	15

-1

% Link Projection 15-16_T118

2

3	3
17	17

18

301	217
304	350
435	350
428	213
223	280
457	271
456	145
335	146
345	322
137	323
140	185
410	175
408	72
177	79
181	237
379	240
376	100
225	110

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	4

15

4	0
4	2
7	3
7	4
0	8
10	7
11	6
10	13
0	14
14	7
3	15
6	15
15	10
17	10
14	17

-1

% Link Projection 15-16_T119

2

3	3
17	17

18

203	95
205	183
378	182
378	93
342	114

448	110
450	386
248	386
244	156
422	158
422	224
97	222
95	331
285	328
283	69
156	72
152	289
345	288

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	4

15

2	4
1	7
8	2
7	10
7	12
13	1
13	3
8	13
10	13
10	15
16	7
13	16
1	17

17	8
17	10

-1

% Link Projection 15-16_T181

2		
	3	3
	17	17

18

	80	41
	84	250
	234	252
	228	50
	120	342
	118	213
	346	214
	341	81
	147	87
	152	392
	406	385
	404	123
	263	131
	267	301
	32	298
	28	170
	373	166
	378	338

18

	0	1
	1	2
	2	3
	3	0
	4	5
	5	6
	6	7
	7	8
	8	9
	9	10
10		11
11		12
12		13
13		14
14		15
15		16
16		17
17		4

15

1	4
2	5
2	7
8	1
5	8
6	11
5	12
4	13
13	8
15	0
15	2
15	6
8	15
12	15
8	17

-1

% Link Projection 15-16_T205

2

3	3
17	17

18

148	284
146	384
396	384
391	284
388	237
95	241
95	448
262	446
259	177
453	175
455	354
183	357
181	106
322	102
320	326
215	325
216	152
392	146

18

0	1
1	2
2	3
3	0

4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	4

15

1	7
7	3
4	7
10	2
7	10
11	3
4	11
3	13
13	4
8	13
14	7
3	15
15	4
13	16
17	8

-1

% Link Projection 15-19_T122

2

3	3
17	17

18

279	281
276	378
445	376
445	280
245	144
247	422
394	420
388	234
209	238

207	175
492	171
490	338
157	348
162	210
325	204
331	477
103	474
100	155

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	4

15

1	6
3	6
4	7
4	9
0	11
11	2
11	4
6	11
13	4
8	13
14	1
14	3
5	14
7	14
11	14

-1

% Link Projection 15-19_T148

2

3	3
17	17

18

315	285
318	422
460	421
457	285
278	139
279	450
368	449
363	204
189	206
193	354
416	349
415	244
234	247
235	174
481	168
483	380
142	391
143	143

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	4

15

1	6
6	3
7	4

0	9
9	4
9	6
3	10
4	11
11	6
12	7
4	13
15	0
15	2
4	15
6	15

-1

% Link Projection 15-19_T188

2

3	3
17	17

18

62	41
62	155
231	154
231	43
39	84
42	232
406	228
401	54
194	56
198	304
120	305
114	116
279	115
281	273
157	277
156	195
342	190
343	87

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8

8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	4

15

2	7
8	1
8	5
1	10
10	5
2	11
11	8
5	12
13	8
5	14
15	8
12	15
17	0
17	2
8	17

-1

% Link Projection 15-19_T208

2

3	3
17	17

18

52	68
53	205
297	202
296	70
78	89
80	263
421	263
419	120
115	123
115	299
233	296
232	155
384	157

385	226
165	223
167	343
267	341
267	91

18

1	0
2	1
3	2
0	3
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	4

15

4	1
2	7
1	8
8	5
10	1
10	5
11	2
13	10
5	14
14	9
1	16
5	16
7	16
16	11
16	13

-1

% Link Projection 15-22_T128

2

3	3
17	17

18

289	111
289	274
406	273
406	109
231	186
441	188
437	78
169	78
174	318
472	320
471	148
344	149
356	412
108	409
108	246
430	237
436	365
233	367

18

0	1
1	2
2	3
3	0
5	4
6	5
7	6
8	7
9	8
10	9
11	10
12	11
13	12
14	13
15	14
16	15
17	16
4	17

15

0	4
2	4
10	2
5	10
1	11
4	11
11	8

14	0
14	2
14	7
11	14
8	15
16	11
8	17
17	14

-1

% Link Projection 1522-T141

2	
3	3
17	17

18

110	219
110	297
280	299
280	217
171	89
49	90
51	348
450	350
445	121
135	122
135	261
397	260
398	387
235	386
236	185
330	185
333	417
175	417

18

0	1
1	2
2	3
3	0
5	4
6	5
7	6
8	7
9	8
10	9
11	10
12	11

13	12
14	13
15	14
16	15
17	16
4	17

15

9	3
2	10
6	11
13	1
13	3
6	13
10	13
15	6
10	15
12	15
1	17
3	17
17	6
17	8
17	10

-1

% Link Projection 15-22_T186

2

3	3
17	17

18

236	239
238	382
325	382
322	240
272	60
276	355
418	355
417	164
201	163
201	90
459	87
463	319
152	318
150	133
384	134
384	282
186	280

181	59
-----	----

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	4

15

3	4
5	2
4	7
4	9
0	11
2	11
11	4
11	6
13	4
8	13
7	14
15	0
15	2
4	15
13	16

-1

% Link Projection 15-22_T206

2

3	3
17	17

18

86	265
87	406
407	406

401	262
119	143
127	366
320	366
312	48
399	49
404	184
235	182
239	292
363	292
358	91
180	92
186	334
445	335
438	131

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	4

15

3	4
6	3
9	6
3	10
11	6
3	12
12	9
13	6
14	3
15	2
6	15

6	17
8	17
17	12
17	14

-1

% Link Projection 16-42_T305

2

3	3
17	17

18

94	114
91	204
295	208
298	113
335	135
54	131
55	226
202	232
204	94
386	98
389	285
124	287
121	178
275	178
273	63
160	60
156	264
331	262

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16

16	17
17	4
16	
4	0
4	2
1	7
7	3
4	7
1	11
11	6
7	12
3	13
13	4
13	8
15	1
3	15
15	4
6	15
12	15
-1	

% Link Projection 16-42_T337

2	
3	3
17	17
18	
259	100
259	258
378	260
377	105
308	321
304	60
193	62
196	223
438	222
444	392
147	388
142	140
343	131
346	320
412	316
406	171
225	180
230	316
18	
0	1

1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	4

16	
4	1
4	3
0	7
7	2
7	4
0	11
11	4
11	6
1	12
12	7
14	7
15	0
2	15
4	15
15	12
7	16

-1

% Link Projection 16-42_T387

2

3	3
15	15

16

224	129
223	281
366	295
364	133
303	352

302	64
169	62
166	212
438	215
438	95
261	96
271	247
415	255
410	172
128	165
130	347
16	
0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	4
16	
1	4
3	4
7	0
2	7
7	4
4	9
10	3
10	7
11	2
4	11
7	12
0	13
13	2
4	13
13	6
13	10
-1	

% Link Projection 16-42_T420

2		
	3	3
	17	17
18		
	192	177
	195	280
	392	279
	388	175
	223	252
	226	338
	292	340
	281	131
	126	138
	126	217
	349	215
	341	152
	250	157
	260	313
	427	310
	422	110
	323	122
	328	256
18		
	0	1
	1	2
	2	3
	3	0
	4	5
	5	6
	6	7
	7	8
	8	9
	9	10
	10	11
	11	12
	12	13
	13	14
	14	15
	15	16
	16	17
	17	4
16		
	4	1
	6	1

3	6
0	9
6	9
3	10
6	11
1	12
12	3
9	12
13	6
16	3
9	16
11	16
17	6
12	17

-1

% Link Projection 16-56_T320

3	
3	3
13	13
17	17

18

215	187
216	346
331	344
324	185
260	102
264	316
371	316
368	270
113	274
111	56
291	53
291	295
408	295
402	101
173	142
372	143
370	217
167	215

18

0	1
1	2
2	3
3	0
4	5

5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	4
14	15
15	16
16	17
17	14

16	
3	4
5	2
0	7
7	2
7	4
10	3
10	7
2	11
11	6
10	13
4	14
14	10
16	0
2	16
4	16
16	10

-1

% Link Projection 16-56_T357

3	
11	11
15	15
19	19

20	
83	46
84	226
343	227
345	303
227	301
229	171
300	172
302	335

170	334
170	129
387	128
389	42
273	90
267	269
433	263
429	95
125	62
130	194
346	193
342	66

20

1	0
2	1
3	2
4	3
5	4
6	5
7	6
8	7
9	8
10	9
11	10
0	11
12	13
13	14
14	15
15	12
16	17
17	18
18	19
19	16

16

1	4
1	6
3	6
8	1
12	1
12	5
9	12
13	2
6	13
10	15
4	17
6	17

17	8
17	12
18	9
15	18

-1

% Link Projection 16-56_T405

3

3	3
7	7
17	17

18

88	75
89	200
270	201
269	78
198	106
199	297
428	302
428	103
392	257
38	265
32	134
313	134
319	338
135	347
128	166
238	162
248	384
388	387

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	4
8	9
9	10
10	11
11	12
12	13
13	14
14	15

15	16
16	17
17	8
16	
4	1
2	7
8	4
10	0
10	2
4	10
11	5
8	11
1	13
13	8
14	4
1	15
5	15
15	8
15	12
5	17
-1	

% Link Projection 16-56_T421

3	
3	3
13	13
17	17
18	
87	110
91	187
267	179
261	105
150	286
140	69
65	70
69	243
354	230
364	355
193	362
183	64
321	63
333	275
110	146
116	325
302	311
285	134

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	4
14	15
15	16
16	17
17	14

16

4	1
4	3
7	4
1	10
3	10
10	7
12	7
13	10
1	14
14	7
10	15
7	16
16	13
17	2
17	4
10	17

-1

% Link Projection 16-140_T423

2

3	3
19	19

20

132	101
129	284
225	286

227	96
78	67
74	172
339	170
342	243
162	251
158	317
307	311
305	64
431	64
425	365
192	365
191	131
388	130
392	207
262	204
265	64

20

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	18
18	19
19	4

16

0	5
2	5
7	2
8	1
10	5
10	7
1	14

5	14
14	7
14	9
15	2
15	10
6	17
17	10
5	18
18	15

-1

% Link Projection 16-140_T444

2		
	3	3
	17	17

18

275	293
274	372
442	369
441	287
219	416
342	414
337	203
141	206
138	267
407	263
406	328
85	337
85	243
296	237
297	452
43	450
45	165
225	160

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11

11	12
12	13
13	14
14	15
15	16
16	17
17	4

16

1	5
3	5
5	8
9	3
0	10
5	10
7	12
13	1
13	3
4	13
8	13
10	13
17	6
17	8
10	17
12	17

-1

% Link Projection 16-140_T445

2

3	3
19	19

20

177	77
177	158
409	158
403	79
124	49
125	298
463	297
459	39
336	38
335	353
207	351
204	270
380	271
379	128
247	129

255	418
159	416
156	231
286	227
283	44

20

0	1
1	2
2	3
3	0
5	4
6	5
7	6
8	7
9	8
10	9
11	10
12	11
13	12
14	13
15	14
16	15
17	16
18	17
19	18
4	19

16

1	8
3	8
5	8
5	10
8	11
1	12
8	13
1	14
5	14
9	14
11	14
5	16
14	17
1	18
3	18
13	18

-1

% Link Projection 16-140_473

2		
	3	3
	17	17
18		
	103	72
	107	221
	233	228
	233	65
	82	163
	339	162
	347	300
	276	302
	270	126
	189	134
	203	375
	387	371
	372	258
	151	262
	141	99
	303	95
	315	333
	82	326
18		
	0	1
	1	2
	2	3
	3	0
	4	5
	5	6
	6	7
	7	8
	8	9
	9	10
	10	11
	11	12
	12	13
	13	14
	14	15
	15	16
	16	17
	17	4
16		
	4	0
	2	4
	4	7
	8	2

9	1
4	9
5	12
7	12
12	9
1	13
13	4
2	14
15	4
15	6
12	15
9	16

-1

% Link Projection 16-141_T424

2

3	3
19	19

20

141	83
147	191
337	190
326	79
214	59
228	332
416	328
400	105
289	106
307	301
93	306
90	242
378	239
369	136
185	137
192	280
262	274
256	207
90	211
82	61

20

0	1
1	2
2	3
3	0
4	5
5	6

6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	18
18	19
19	4

16

4	1
4	3
2	7
1	8
4	9
4	11
8	11
13	2
13	4
8	13
1	14
11	14
15	4
11	16
17	4
14	17

-1

% Link Projection 16-141_T468

2

3	3
17	17

18

59	290
61	386
216	387
211	286
33	256
33	320
354	313
349	159

269	157
274	352
183	357
178	204
407	201
417	430
86	428
82	176
316	177
315	247

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	4

16

0	5
5	2
8	5
2	9
3	10
10	5
6	11
8	11
14	1
14	3
5	14
15	8
11	16
17	8
10	17
17	14

-1

% Link Projection 16-142_T425

2

13	13
17	17

18

87	60
86	244
387	237
383	165
122	177
119	364
303	363
301	89
154	90
150	393
331	387
336	199
192	208
190	60
276	116
271	260
452	259
451	114

18

0	1
1	2
2	3
3	4
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	0
14	15
15	16
16	17
17	14

16

4	1
---	---

1	6
3	6
1	8
8	3
8	5
10	1
6	11
3	12
12	7
14	1
14	3
11	14
6	15
15	10
6	17

-1

% Link Projection 16-142_T435

2		
	3	3
	17	17
18		
	90	77
	90	387
	253	387
	250	77
	127	101
	123	307
	439	305
	436	207
	158	217
	158	353
	482	354
	482	264
	331	270
	328	149
	200	148
	205	428
	370	426
	366	100
18		
	0	1
	1	2
	2	3
	3	0
	4	5

5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	4

16

5	2
2	7
5	8
2	9
6	11
7	12
13	2
14	1
14	5
7	14
9	14
16	5
16	7
9	16
11	16
2	17

-1

% Link Projection 16-149_T437

2

3	3
19	19

20

85	88
91	341
200	338
195	89
33	375
29	267
221	270
216	115
274	117

282	303
148	303
148	183
251	183
243	88
341	90
347	223
120	224
119	147
307	144
317	373

20

0	1
1	2
2	3
3	0
5	4
6	5
7	6
8	7
9	8
10	9
11	10
12	11
13	12
14	13
15	14
16	15
17	16
18	17
19	18
4	19

16

5	0
5	2
2	9
10	5
11	2
6	11
7	12
2	15
15	6
8	15
15	10
2	17
17	6

17	8
12	17
15	18

-1

% Link Projection 16-149_T449

2	
3	3
17	17

18

71	199
76	287
242	290
239	192
153	159
158	369
364	368
366	255
194	252
189	118
26	127
29	324
312	322
307	216
112	224
116	409
285	405
271	160

18

0	1
1	2
2	3
3	0
5	4
6	5
7	6
8	7
9	8
10	9
11	10
12	11
13	12
14	13
15	14
16	15
17	16

4	17
16	
4	1
4	3
2	7
3	8
11	4
12	7
13	2
13	4
8	13
1	14
14	11
5	16
7	16
16	11
16	13
8	17

-1

% Link Projection 16-149_T464

2	
3	3
19	19
20	
83	78
83	373
240	372
236	82
117	137
118	274
365	271
365	99
298	100
296	247
161	252
165	182
337	181
339	76
415	74
413	211
191	214
193	317
440	315
442	124

20

0	1
1	2
2	3
3	0
5	4
6	5
7	6
8	7
9	8
10	9
11	10
12	11
13	12
14	13
15	14
16	15
17	16
18	17
19	18
4	19

16	
2	5
9	2
11	2
8	11
7	12
2	15
6	15
15	8
5	16
16	9
17	2
2	19
19	6
19	8
12	19
14	19

-1

% Link Projection 16-149_T537

2	
3	3
17	17
18	
196	113
194	299

306	301
304	108
53	187
376	186
377	83
158	76
161	389
367	392
369	251
119	259
116	443
261	441
259	146
413	141
415	349
54	341

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14
14	15
15	16
16	17
17	4

16

0	4
2	4
4	7
10	0
10	2
7	10
1	13
8	13
13	10
4	13

14	2
5	14
16	7
16	9
11	16
13	16

-1

% Link Projection 16-233_T631

2

3	3
17	17

18

77	59
74	193
193	196
189	63
39	119
235	115
238	200
376	201
381	276
154	272
151	93
289	92
296	324
120	322
120	157
344	160
346	237
39	241

18

0	1
1	2
2	3
3	0
4	5
5	6
6	7
7	8
8	9
9	10
10	11
11	12
12	13
13	14

14	15
15	16
16	17
17	4

16

4	0
4	2
9	1
9	4
2	10
6	11
8	11
1	13
2	14

14 5

14 9

11 14

15 6

16 9

11 16

13 16

-1

% Link Projection 16-233_T663

2

15	15
19	19

20

51	94
198	94
201	255
17	254
18	318
440	312
438	147
284	147
286	419
171	419
170	181
366	179
365	366
230	367
231	283
53	287
104	121
104	228

	324	227
	320	120
20		
	0	1
	1	2
	2	3
	3	4
	4	5
	5	6
	6	7
	7	8
	8	9
	9	10
	10	11
	11	12
	12	13
	13	14
	14	15
	15	0
	16	17
	17	18
	18	19
	19	16
16		
	7	4
	9	2
	9	4
	10	1
	7	10
	4	11
	12	7
	4	13
	14	9
	2	15
	1	17
	17	7
	17	9
	18	6
	10	18
	1	19
-1		