

A challenge to 3-manifold topologists and group algebraists *

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April 30, 2013

Abstract

This paper poses some basic questions about instances (hard to find) of a special problem in 3-manifold topology. “Important though the general concepts and propositions may be with the modern industrious passion for axiomatizing and generalizing has presented us ... nevertheless I am convinced that the special problems in all their complexity constitute the stock and the core of mathematics; and to master their difficulty requires on the whole the harder labor.” Hermann Weyl 1885-1955, cited in the preface of the first edition (1939) of [15].

1 A doubt in the classification of 3-manifolds: $U[1466]$ and $U[1563]$

The objective of this note is to pinpoint an aspect of the classification of 3-manifolds which is very important and has been essentially neglected in the last 35 years of successes with the work of W. Thurston, G. Perelman, I. Agol and many others. In despite of enormous progress, the classification problem remains, to our eyes, very difficult. The aspect we want to pinpoint is asking basic questions on hard to find tough instances of the general theory.

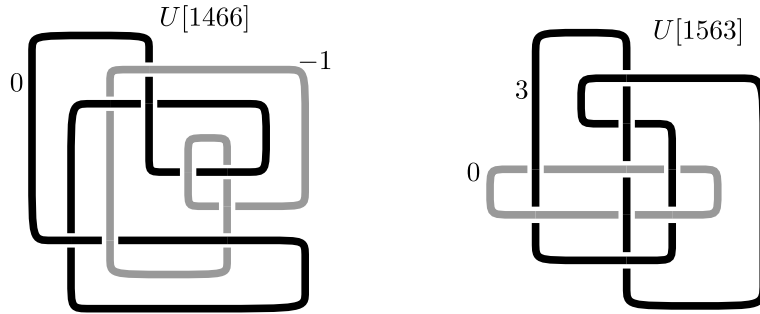


Figure 1: Are the closed orientable 3-manifolds obtained from surgery on \mathbb{S}^3 of the above blackboard framed links followed by the canonical Dehn fillings homeomorphic, or not?

Consider the two closed orientable 3-manifolds obtained from surgery and canonical Dehn fillings on the 2-component blackboard framed [4] link of Fig. 1. Both are homology spheres, so their fundamental groups are perfect. SnapPea [14] tells us that they are both hyperbolic and have the same volume up to many decimal places. Moreover, their Witten-Reshetikin-Turaev invariants with 10 decimal places agree up to $r = 12$. These facts seem to imply that the manifolds are homeomorphic. However, computations based on the methodology of [7] and [8], which were up to this point successful in finding homeomorphism between pairs of 3-manifolds, appear to fail for the first time. Our bet is that the methodology does not fail: the manifolds are not homeomorphic. In the last 5 years we have asked the help of various distinguished topologists in trying to settle this example. None of them succeeded in answering our question. So, we believe the time is ripe to bring our doubt to the broader community of mathematicians dealing with 3-manifolds and/or combinatorial group theory. This example corresponds to the pair of blackboard framed links $U[1466]$ and $U[1563]$ of [7]. The numbers attached to the components (framing) coincide with their self-writhes in

*2010 Mathematics Subject Classification: 57M25 and 57Q15 (primary), 57M27 and 57M15 (secondary)

the given projection and, so, can be discarded. Note that by introducing an appropriate number of positive or negative curls we can obtain any framed link as a blackboard framed link (and discard the framings). In a blackboard framed link we do not use the framing to obtain a presentation of the fundamental group.

If the manifolds being compared are hyperbolic, then the difficult topological question of homeomorphism between the manifolds transforms into the possibly equally difficult algebraic question of isomorphism between their fundamental groups. So, as long as the general associated question is not settled, we have replaced a problem which we do not know how to solve into another, which we also do not know how to solve. This might be, in some aspects, progress, but hardly a definitive one. In general, how to prove that the fundamental groups of hyperbolic 3-manifolds are not isomorphic? Start by proving that there is no isomorphism between the fundamental groups of the above 3-manifolds. Or find one.

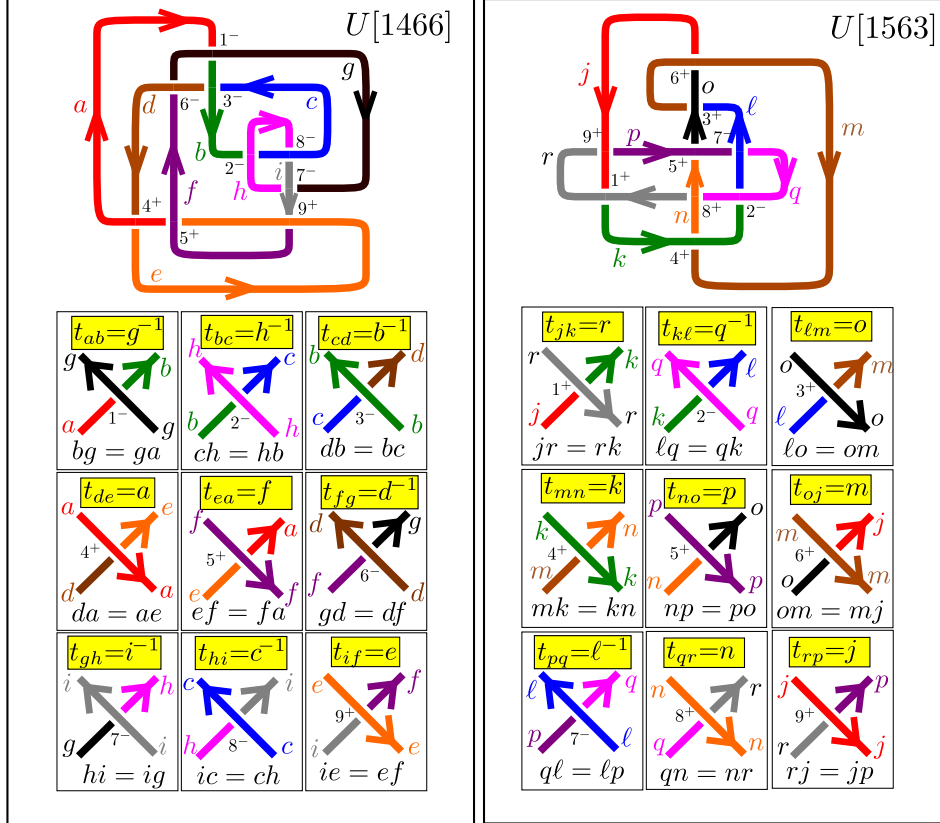


Figure 2: Finding presentations for the fundamental groups of $M^3[1466]$ and $M^3[1563]$: we arbitrarily orient the links, write the transition generators, t_{xy} 's, in terms of the Wirtinger generators ([13]), write the Dehn fillings relators ([12]) in terms of the transition generators and, finally, write the Wirtinger relations for the fundamental groups of the exterior of the links.

The presentations for the fundamental groups of the manifolds $M^3[1466]$ and $M^3[1563]$ are:

$$\begin{aligned}
\pi_1[1466] = \langle & \{t_{ab}, t_{bc}, t_{cd}, t_{de}, t_{ea}, t_{fg}, t_{gh}, t_{hi}, t_{if}, a, b, c, d, e, f, g, h, i\}, \\
& \{t_{ab} = g^{-1}, t_{bc} = h^{-1}, t_{cd} = b^{-1}, t_{de} = a, t_{ea} = f, \\
& t_{fg} = d^{-1}, t_{gh} = i^{-1}, t_{hi} = c^{-1}, t_{if} = e, \\
& t_{ab}t_{bc}t_{cd}t_{de}t_{ea} = 1, t_{fg}t_{gh}t_{hi}t_{if} = 1, \\
& bg = ga, ch = hb, db = bc, da = ae, ef = fa, gd = df, hi = ig, ic = ch, ie = ef\} \rangle, \\
\pi_1[1563] = \langle & \{t_{jk}, t_{kl}, t_{lm}, t_{mn}, t_{no}, t_{oj}, t_{pq}, t_{qr}, t_{rp}, j, k, l, m, n, o, p, q, r\} \\
& \{t_{jk} = r, t_{kl} = q^{-1}, t_{lm} = o, t_{mn} = k, t_{no} = p, t_{oj} = m, \\
& t_{pq} = l^{-1}, t_{qr} = n, t_{rp} = j, \\
& t_{jk}t_{kl}t_{lm}t_{mn}t_{no}t_{oj} = 1, t_{pq}t_{qr}t_{rp} = 1, \\
& jr = rk, lq = qk, lo = om, mk = kn, np = po, om = mj, ql = lp, qn = nr, rj = jp\} \rangle.
\end{aligned}$$

2 Another doubt: $U[2125]$ and $U[2165]$

It is important also to distinguish the pair 3-manifolds induced by the blackboard framed links of Fig. 3. As the previous pair, they are closed hyperbolic homology spheres and their WRT-invariants agree up to

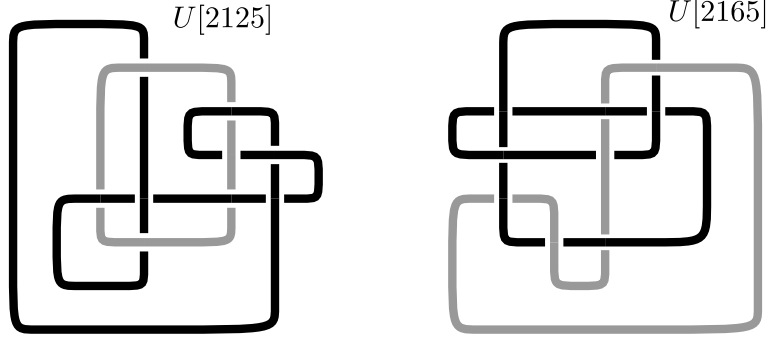


Figure 3: Are the closed orientable 3-manifolds obtained from surgery on \mathbb{S}^3 of the above blackboard framed links followed by canonical Dehn fillings homeomorphic, or not? The framing of a component in the above links is its self-writhe in the given projection.

$r = 12$ with 10 decimal places, [7]. The presentations for the fundamental groups of the manifolds $M^3[2125]$ and $M^3[2165]$ are:

$$\begin{aligned} \pi_1[2125] = \langle & \{t_{ab}, t_{bc}, t_{cd}, t_{de}, t_{ef}, t_{fa}, t_{gh}, t_{hi}, t_{ig}, a, b, c, d, e, f, g, h, i\}, \\ & \{t_{ab} = h^{-1}, t_{bc} = d, t_{cd} = g^{-1}, t_{de} = b, t_{ef} = a, t_{fa} = i, \\ & \quad t_{gh} = c^{-1}, t_{hi} = f, t_{ig} = e^{-1}, \\ & \quad t_{ab}t_{bc}t_{cd}t_{de}t_{ef}t_{fa} = 1, t_{gh}t_{hi}t_{ig} = 1, \\ & \quad bh = ha, bd = dc, dg = gc, db = be, ea = af, fi = ia, hc = cg, hf = fi, ge = ei\} \rangle, \end{aligned}$$

$$\begin{aligned} \pi_1[2165] = \langle & \{t_{jk}, t_{kl}, t_{lm}, t_{mn}, t_{no}, t_{oj}, t_{pq}, t_{qr}, t_{rp}, j, k, l, m, n, o, p, q, r\}, \\ & \{t_{jk} = r^{-1}, t_{kl} = q, t_{lm} = j^{-1}, t_{mn} = k, t_{no} = p^{-1}, t_{oj} = l^{-1}, \\ & \quad t_{pq} = n^{-1}, t_{qr} = m, t_{rp} = o^{-1}, \\ & \quad t_{jk}t_{kl}t_{lm}t_{mn}t_{no}t_{oj} = 1, t_{pq}t_{qr}t_{rp} = 1, \\ & \quad \{kr = rj, kq = ql, mj = jl, mk = kn, op = pn, jl = lo, qn = np, qm = mr, po = or\} \} \rangle. \end{aligned}$$

These are read directly from Fig. 4, in a way similar to the previous pair of links.

3 A more general question: the $hgqi_u^d$ -classes of 3-manifolds

The 3-manifolds of [7] are classified by homology and the quantum WRT_r -invariants $r = 3, \dots, u$, up to d decimal digits forming $hgqi_u^d$ -classes. Our algorithm for computing the WRT_r^d -invariants are based on the theory developed in [5]. The actual values rely on independent implementations which coincide throughout [5] and [7]. The main domain of links in [7] (there are others) is formed by the so called *representative g-blinks*, $U[p]$'s $p = 1, 2, \dots$, which is a highly filtered class of blackboard framed links indexed by lexicography. An important result of the work is that the $U[p]$'s form a universal class of 3-manifolds, in the sense that no closed orientable 3-manifold is missing. The examples of the previous section embed into two $hgqi_{12}$ -classes: 9_{126} (page 201 of [7]) and 9_{199} (page 213 of [7]). The $hgqi_{12}^{10}$ -class 9_{126} is formed by 5 links $U[1466], U[1563], U[1738], U[2233]$ and $U[2866]$. The $hgqi_{12}^{10}$ -class 9_{199} is formed by 3 links: $U[2125], U[2165]$ and $U[3089]$. In Fig. 5, we display 9_{126} and 9_{199} . This note's final challenge is to classify topologically 9_{126} and 9_{199} , in the sense given in the caption of Fig. 5.

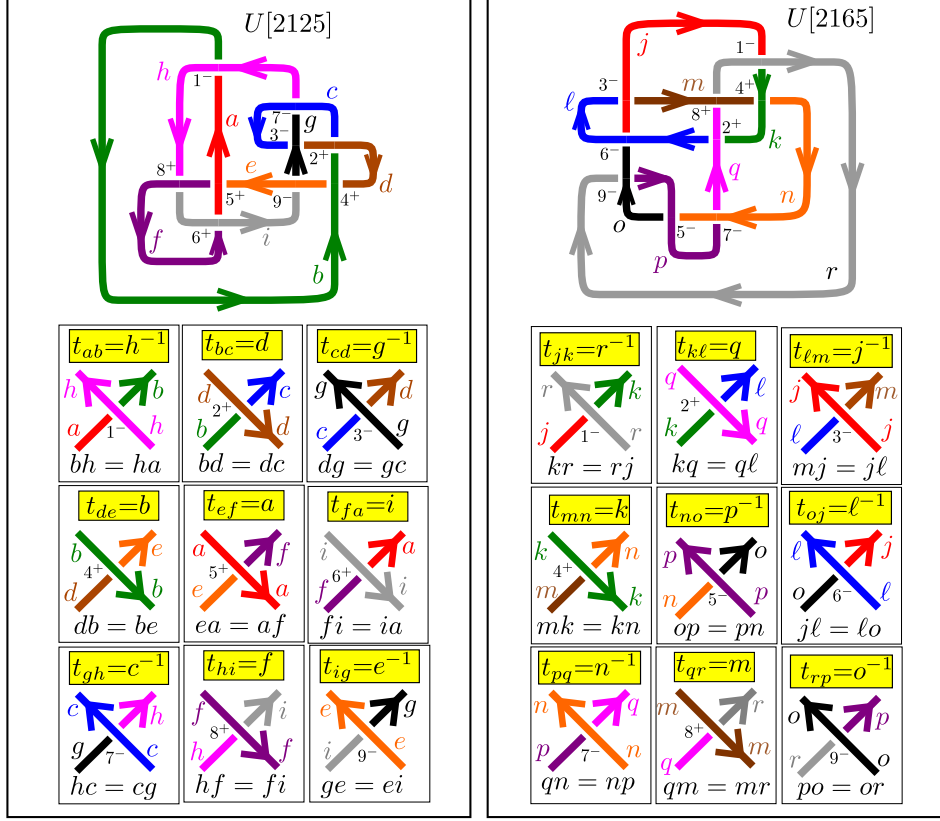


Figure 4: Finding presentations for the fundamental groups of $M^3[2125]$ and $M^3[2165]$

4 Definition of Gem

For completeness we briefly recall the basic definitions of gem theory, leading to its definition, [8]. A *4-graph* G is a finite bipartite 4-regular graph whose edges are partitioned into 4 colors, 0,1,2, and 3, so that at each vertex there is an edge of each color, a proper edge-coloration, [1]. For each $i \in \{0, 1, 2, 3\}$, let E_i denote the set of i -colored edges of G . A $\{j, k\}$ -residue in a 4-graph G is a connected component of the subgraph induced by $E_j \cup E_k$. A 2-residue is a $\{j, k\}$ -residue, for some distinct colors j and k . A *gem* is a 4-graph G such that for each color i , $G \setminus E_i$ can be embedded in the plane such that the boundary of each face is a 2-residue. From a gem there exists a straightforward algorithm to obtain a closed orientable 3-manifold, in two different, dual ways. Every such a manifold is obtainable in this way. An unnecessary big gem is obtained from a triangulation T for a manifold by taking the dual of the barycentric subdivision of T . Here the colors corresponds to the dimensions. Doing simplifications in the gem completely destroys this correspondence.

5 Conclusion

A closed orientable 3-manifold is denoted *n-small* if it is induced by surgery on a blackboard framed link with at most n crossings. Our bet is that both pairs of 3-manifolds in the 2 first sections of this short note are not homeomorphic. This would mean that the 9-small manifolds are completely classified and that the combinatorial dynamics of Chapter 4 in [8] based on *TS*-moves which leads to a (small, in the case of hyperbolic 3-manifolds) number of minimal gems, named the *attractor of the 3-manifold* is successful. This induces an efficient algorithm which is capable of classifying topologically all the 3-manifolds given as a blackboard framed link with up to (so far) 9 crossings and maintains live the two Conjectures of page 15 of [8]: the *TS*- and *u*^{*n*}-moves yield an efficient algorithm to classify *n*-small 3-manifolds by explicitly displaying homeomorphisms, whenever they exist.

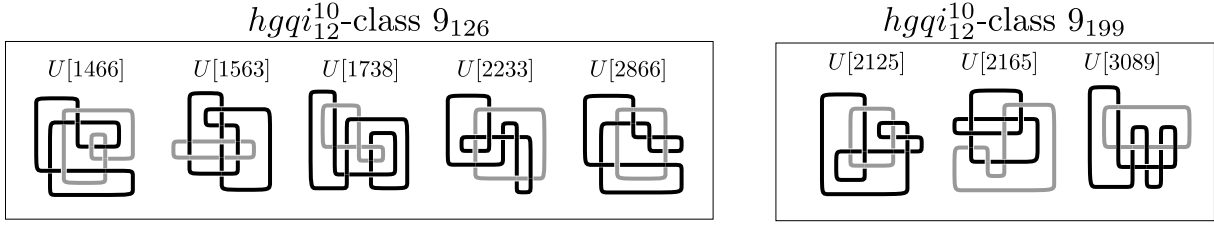


Figure 5: **Note's final challenge:** classify topologically 9_{126} and 9_{199} . That has the following meaning: for each pair of closed oriented hyperbolic 3-manifolds induced by links in one of these classes, either make available a homeomorphism between them or, in the hyperbolic case, make available an isomorphism between their fundamental groups, or else make available an invariant which distinguishes them. Such a proof of the coincidence or distinctiveness must be computationally short and reproducible by other researchers. The given projections define blackboard framed links and the framing of each component is its self-writhe. In a blackboard framed link the presentation of the fundamental group of the associated 3-manifold does not need the framing. GAP [3] and SnapPea ([14]) are good softwares to distinguish manifolds, but we personally have not tried them yet. It is a simple matter to obtain a canonical gem with $8n$ vertices from a blackboard framed link with n crossings, [7]. Gems are good at displaying homeomorphism via TS - and u^n -moves, [8]. It factors the homeomorphism as a sequence of blob cancellations and valid flips ([9]), never increasing the number of vertices of the gems. Because of the lexicography inherent to graph with edges properly colored, a gem-based homeomorphism between two 3-manifolds will coincide in any independent implementation of the algorithm given in Chapter 4 of [8]: the sequences of blob and flips turn out to be exactly the same. Observe that if the manifolds are homeomorphic, each possible invariant will fail to distinguish them. Therefore, to prove that two framed links are indeed manifestation of the same manifold we must make available a homeomorphism; or in the hyperbolic case, to make available an isomorphism between the fundamental groups. To establish an explicit homeomorphism, what else could be used beyond a (short) path in a graph whose vertices are gems and whose edges are either a blob cancellation or a valid flip? Moreover, such an answer has the virtue of being quickly verifiable by independent implementations. Is there a substitute for gems to accomplish this task? Kirby's moves [6] and their variants by Fenn and Rourke [2] and more recently Martelli [10], are, with tailored exceptions, unusable and so, helplessly inferior to gems in this regard. The presentation of 3-manifolds based on the special spines of Matveev [11] seems to be a possibility, but first a theory to deal with isomorphism of such spines, as well as using some filter on them to decrease redundancy, had to be established. In the case of gems the corresponding theory is simpler and is available since 1995, [8].

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