A tougher challenge to 3-manifold topologists and group algebraists *

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Abstract

In [?] we posed some questions for which we had no answers about homeomorphisms between closed orientable 3-manifolds. The two pairs of 3-manifolds were the only uncertainties that were left in L. Lins Thesis under my supervision in the realm of 3-manifolds being induced by blackboard framed links up to 9 crossings (9-small 3-manifolds). The paper was taken seriously by a few researchers that very quick could solve them. The solution shows that the BLINK program of L. Lins does a complete job in topologically classifying the 9-small 3-manifolds. In this paper I put some new challenges that seem harder than the ones considered in the previous paper.

1 A doubt in the classification of 3-manifolds: $U[1466]$ and $U[1563]$

The objective of this note is to pinpoint an aspect of the classification of 3-manifolds which is very important and has been essentially neglected in the last 35 years of successes with the work of W. Thurston, G. Perelman, I. Agol and many others. In despite of enormous progress, the classification problem remains, to our eyes, very difficult. The aspect we want to pinpoint is asking basic questions on hard to find tough instances of the general theory.

14^t_{24} (0) 2 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	166
05	0.221839804	0.000000000	732
06	0.288675135	0.000000000	6153
07	0.569807123	0.000000000	26280
08	0.658347954	0.000000000	118404

T[71] edges: 14 blocks: 1 orient: + T[79] edges: 14 blocks: 1 orient: +

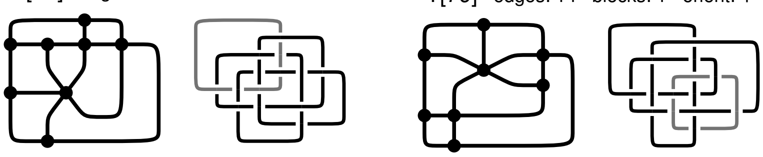


Figure 1: Are the closed orientable 3-manifolds obtained from surgery on $\mathbb{S}^3$ of the above blackboard framed links followed by the canonical Dehn fillings homeomorphic, or not?

In a fundamental paper W. B. R. Lickorish proves that each closed orientable 3-manifold can be encoded as a link in $\mathbb{S}^3$ with integers in 1-1 correspondence with its components, [3], the so called *framed links.*

Consider the two closed orientable 3-manifolds obtained from surgery and canonical Dehn fillings on the 2-component blackboard framed [1] link of Fig. 1. Both are homology spheres, so their fundamental groups are perfect. SnapPea [9] tells us, according to S. Matveev [6], that they are both hyperbolic and have the same volume up to 10 decimal places,. Moreover, their Witten-Reshetiken-Turaev invariants with 10 decimal places agree up to $r = 12$, [4]. These facts seem to imply that the manifolds are homeomorphic. However, computations based on the methodology of [4] and [5], which were up to this point successful in finding homeomorphism between pairs of 3-manifolds, appear to fail for the first time. Our bet is that the

*2010 Mathematics Subject Classification: 57M25 and 57Q15 (primary), 57M27 and 57M15 (secondary) methodology does not fail, that is, the manifolds are not homeomorphic. In the last 5 years we have asked the help of various distinguished topologists in trying to settle this example. None of them succeeded in answering our question. So, we believe the time is ripe to bring our doubt to the broader community of mathematicians dealing with 3-manifolds and/or combinatorial group theory. This example corresponds to the pair of blackboard framed links $U[1466]$ and $U[1563]$ of [4]. The numbers attached to the components (framing) coincide with their self-writhes in the given projection and, so, can be discarded. Note that by introducing an appropriate number of positive or negative curls we can obtain any framed link as a blackboard framed link (and discard the framings). In a blackboard framed link we do not need nor use the framing to obtain a presentation of the fundamental group.

If the manifolds being compared are hyperbolic, then the difficult topological question of homeomorphism between the manifolds transforms into the possibly equally difficult algebraic question of isomorphism between their fundamental groups. So, as long as the general associated question is not settled, we have replaced a problem which we do not know how to solve into another, which we also do not know how to solve. This might be, in some aspects, progress, but hardly a definitive one. In general, how to prove that the fundamental groups of hyperbolic 3-manifolds are not isomorphic? Start by proving that there is no isomorphism between the fundamental groups of the above 3-manifolds. Or find one.

15^{t}_{16} (0) 4 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.626523527	-0.169924730	1684
06	0.288675135	0.000000000	19341
07	0.271887719	-0.143082082	111440
08	0.116351265	1.000000000	620624

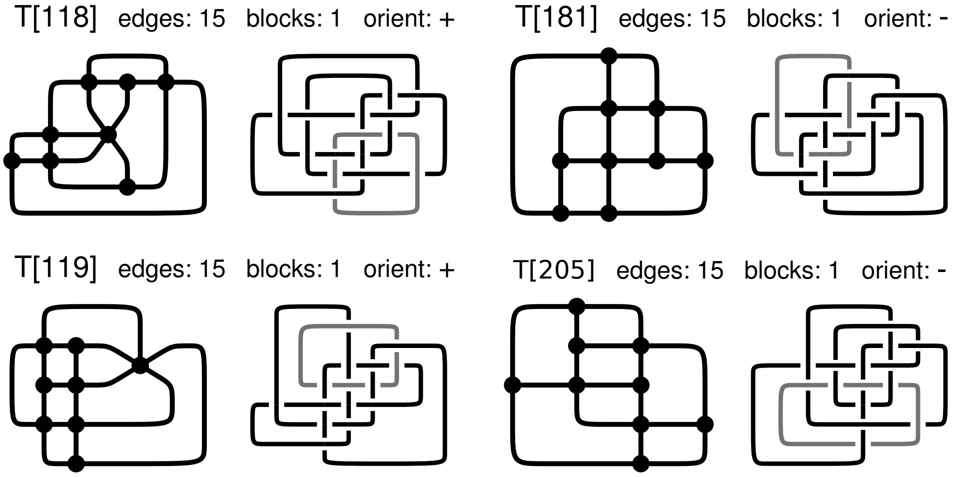


Figure 2: Finding presentations for the fundamental groups of $M^3[1466]$ and $M^3[1563]$: we arbitrarily orient the links, write the transition generators, t_{xy}'s, in terms of the Wirtinger generators ([8]), write the Dehn fillings relators ([7]) in terms of the transition generators and, finally, write the Wirtinger relations for the fundamental groups of the exterior of the links.

enddocument

The presentations for the fundamental groups of the manifolds $M^3[1466]$ and $M^3[1563]$ are:

$$\begin{gathered}\pi_1[1466] = \langle \{t_{ab}, t_{bc}, t_{cd}, t_{de}, t_{ea}, t_{fg}, t_{gh}, t_{hi}, t_{if}, a, b, c, d, e, f, g, h, i\},\\ \{t_{ab} = g^{-1}, t_{bc} = h^{-1}, t_{cd} = b^{-1}, t_{de} = a, t_{ea} = f,\\ t_{fg} = d^{-1}, t_{gh} = i^{-1}, t_{hi} = c^{-1}, t_{if} = e,\\ t_{ab}t_{bc}t_{cd}t_{de}t_{ea} = 1, t_{fg}t_{gh}t_{hi}t_{if} = 1,\\ bg = ga, ch = hb, db = bc, da = ae, ef = fa, gd = df, hi = ig, ic = ch, ie = ef\}\rangle,\end{gathered}$$

$$\pi_1[1563] = \langle \{t_{jk}, t_{kl}, t_{lm}, t_{mn}, t_{no}, t_{oj}, t_{pq}, t_{qr}, t_{rp}, j, k, l, m, n, o, p, q, r\}$$
$$\{t_{jk} = r, t_{kl} = q^{-1}, t_{lm} = o, t_{mn} = k, t_{no} = p, t_{oj} = m,$$
$$t_{pq} = l^{-1}, t_{qr} = n, t_{rp} = j,$$
$$t_{jk}t_{kl}t_{lm}t_{mn}t_{no}t_{oj} = 1, t_{pq}t_{qr}t_{rp} = 1,$$
$$jr = rk, lq = qk, lo = om, mk = kn, np = po, om = mj, ql = lp, qn = nr, rj = jp\}\rangle.$$

2 Another doubt: $U[2125]$ and $U[2165]$

It is important also to distinguish the pair 3-manifolds induced by the blackboard framed links of Fig. ??. As the previous pair, they are closed hyperbolic homology spheres and their WRT-invariants agree up to

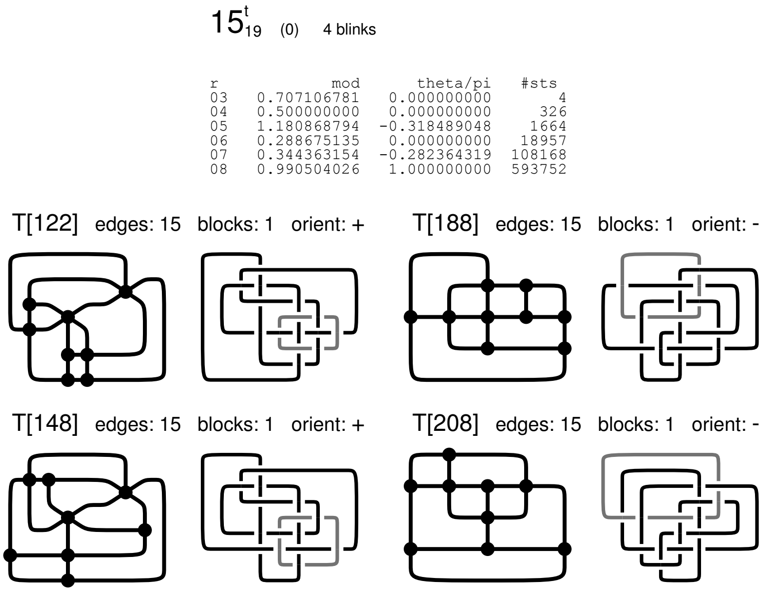


Figure 3: Are the closed orientable 3-manifolds obtained from surgery on $\mathbb{S}^3$ of the above blackboard framed links followed by canonical Dehn fillings homeomorphic, or not? The framing of a component in the above links is its self-writhe in the given projection.

$r = 12$ with 10 decimal places, [4]. The presentations for the fundamental groups of the manifolds $M^3[2125]$ and $M^3[2165]$ are:

$$\pi_1[2125] = \langle \{t_{ab}, t_{bc}, t_{cd}, t_{de}, t_{ef}, t_{fa}, t_{gh}, t_{hi}, t_{ig}, a, b, c, d, e, f, g, h, i\},$$
$$\{t_{ab} = h^{-1}, t_{bc} = d, t_{cd} = g^{-1}, t_{de} = b, t_{ef} = a, t_{fa} = i,$$
$$t_{gh} = c^{-1}, t_{hi} = f, t_{ig} = e^{-1},$$
$$t_{ab}t_{bc}t_{cd}t_{de}t_{ef}t_{fa} = 1, t_{gh}t_{hi}t_{ig} = 1,$$
$$bh = ha, bd = dc, dg = gc, db = be, ea = af, fi = ia, hc = cg, hf = fi, ge = ei\}\rangle,$$

$$\pi_1[2165] = \langle \{t_{jk}, t_{kl}, t_{lm}, t_{mn}, t_{no}, t_{oj}, t_{pq}, t_{qr}, t_{rp}, j, k, l, m, n, o, p, q, r\},$$
$$\{t_{jk} = r^{-1}, t_{kl} = q, t_{lm} = j^{-1}, t_{mn} = k, t_{no} = p^{-1}, t_{oj} = l^{-1},$$
$$t_{pq} = n^{-1}, t_{qr} = m, t_{rp} = o^{-1},$$
$$t_{jk}t_{kl}t_{lm}t_{mn}t_{no}t_{oj} = 1, t_{pq}t_{qr}t_{rp} = 1,$$
$$\{kr = rj, kq = ql, mj = jl, mk = kn, op = pn, jl = lo, qn = np, qm = mr, po = or\}\rangle.$$

These are read directly from Fig. 4, in a way similar to the previous pair of links.

15^t_{22} (0) 4 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.943093903	-0.180036791	1680
06	0.288675135	0.000000000	20945
07	0.242490221	-0.482680140	111968
08	0.641616039	1.000000000	656072

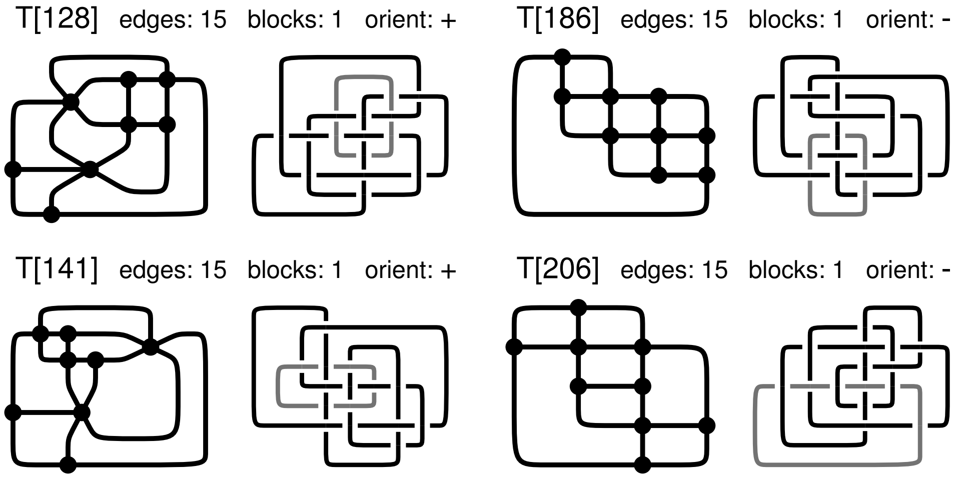


Figure 4: Finding presentations for the fundamental groups of $M^3[2125]$ and $M^3[2165]$

3 A more general question: the $hgqi^d_u$-classes of 3-manifolds

The 3-manifolds of [4] are classified by homology and the quantum WRT_r-invariants $r = 3, \ldots, u$, up to d decimal digits forming $hgqi^d_u$-classes. Our algorithm for computing the WRT^d_r-invariants are based on the theory developed in [2]. The actual values rely on independent implementations which coincide throughout [2] and [4]. The main domain of links in [4] (there are others) is formed by the so called *representative g-blinks*, $U[p]$'s $p = 1, 2, \ldots$, which is a highly filtered class of blackboard framed links indexed by lexicography. An important result of the work is that the $U[p]$'s form a universal class of 3-manifolds, in the sense that no closed orientable 3-manifold is missing. The examples of the previous section embed into two $hgqi_{12}$-classes: 9_{126} (page 201 of [4]) and 9_{199} (page 213 of [4]). The $hgqi^{10}_{12}$-class 9_{126} is formed by 5 links $U[1466], U[1563], U[1738], U[2233]$ and $U[2866]$. The $hgqi^{10}_{12}$-class 9_{199} is formed by 3 links: $U[2125], U[2165]$ and $U[3089]$. In Fig. **??**, we display 9_{126} and 9_{199}. This note's final challenge is to classify topologically 9_{126} and 9_{199}, in the sense given in the caption of Fig. **??**.

16^t_{42} (0) 2^2 4 blinks

r	mod	theta/pi	#sts
03	0.000000000	0.000000000	4
04	1.224744871	0.195913276	294
05	0.000000000	0.000000000	1900
06	0.577350269	-0.333333333	22883
07	0.000000000	0.000000000	159992
08	1.774989898	0.206598266	1001208

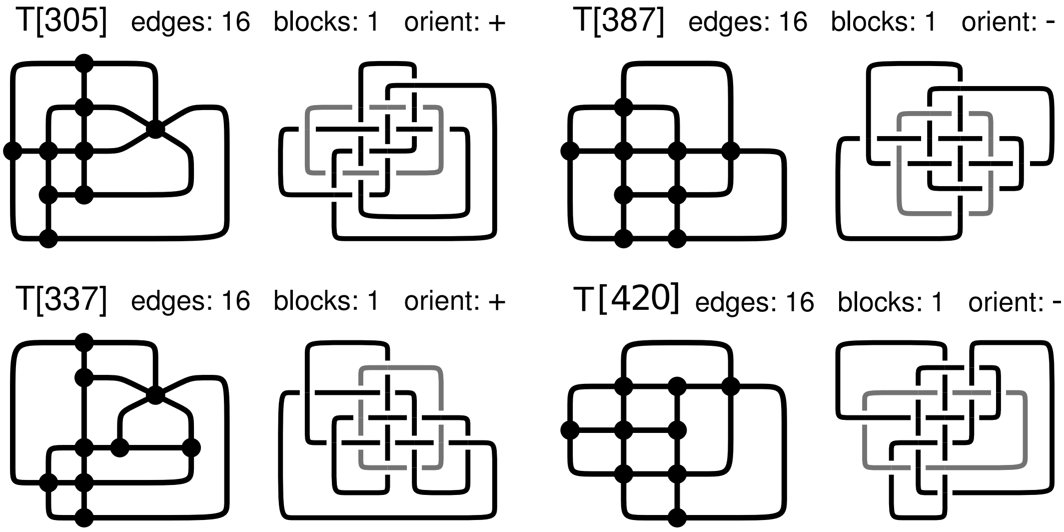


Figure 5: In the case of gems the corresponding theory

16^{t}_{56} (1) 4 blinks

r	mod	theta/pi	#sts
03	1.000000000	0.000000000	8
04	1.000000000	0.000000000	420
05	1.249042228	0.093934688	4512
06	2.000000000	0.000000000	61009
07	1.748655511	-0.081646235	501032
08	0.801948466	0.000000000	3620288

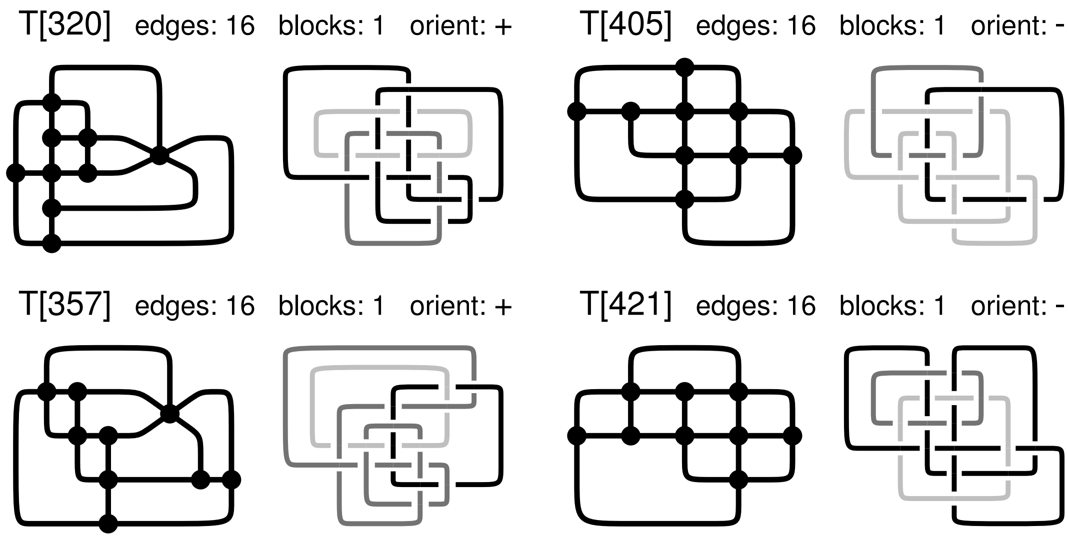
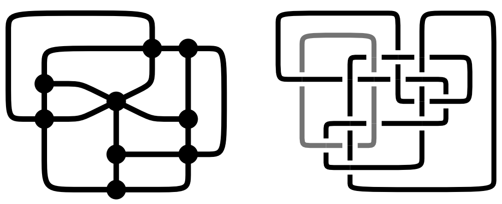


Figure 6: In the case of gems the corresponding theory

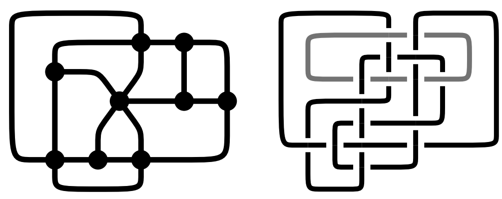
16^{t}_{140} (0) 4 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.828810492	0.200682317	1736
06	0.288675135	0.000000000	19469
07	0.369735450	0.097853027	114084
08	0.499366312	1.000000000	624200

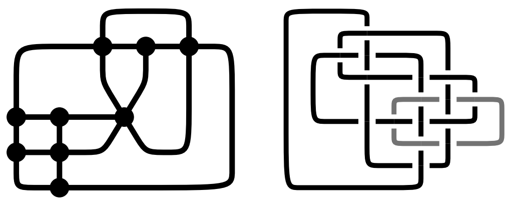
T[423] edges: 16 blocks: 1 orient: +



T[445] edges: 16 blocks: 1 orient: -



T[444] edges: 16 blocks: 1 orient: +



T[473] edges: 16 blocks: 1 orient: -

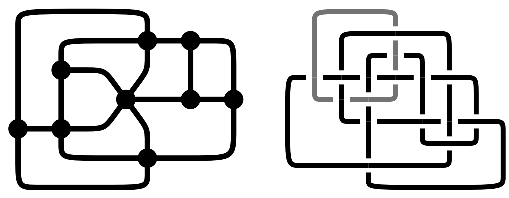
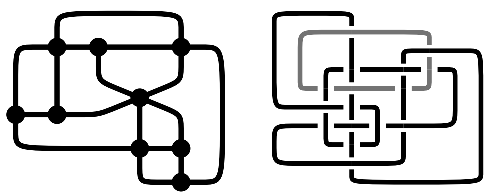


Figure 7: In the case of gems the corresponding theory

16^{t}_{141} (0) 2 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.828231334	0.000000000	1736
06	0.288675135	0.000000000	19469
07	0.591962818	0.000000000	114084
08	0.297863204	0.000000000	624200

T[424] edges: 16 blocks: 1 orient: +



T[468] edges: 16 blocks: 1 orient: +

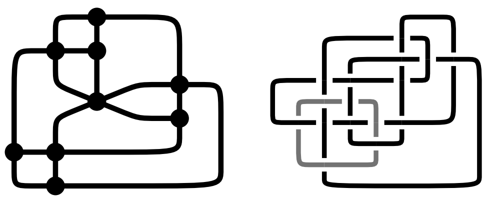
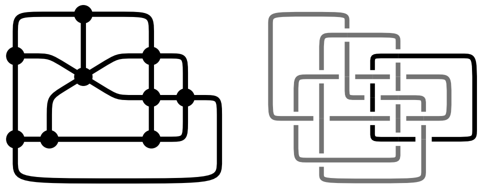


Figure 8: In the case of gems the corresponding theory

16^{t}_{142} (0) 2 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.314488186	0.000000000	1696
06	0.288675135	0.000000000	21113
07	1.546327427	0.000000000	113232
08	0.931913875	0.000000000	663132

T[425] edges: 16 blocks: 1 orient: +



T[435] edges: 16 blocks: 1 orient: +

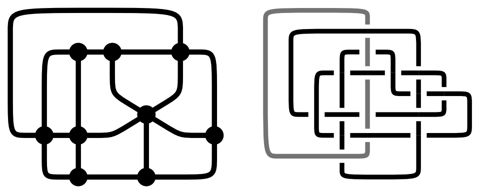
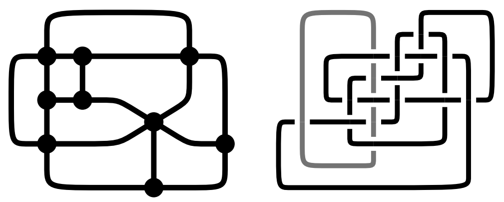


Figure 9: In the case of gems the corresponding theory

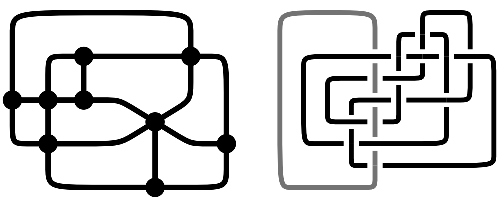
16^{t}_{149} (0) 4 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.314488186	0.000000000	1828
06	0.288675135	0.000000000	21553
07	0.745358881	0.081311555	126032
08	1.620357495	0.000000000	735988

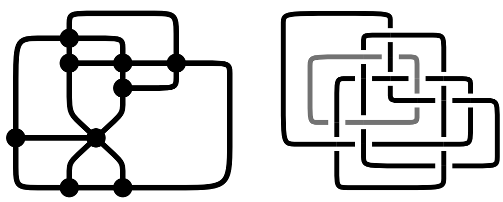
T[437] edges: 16 blocks: 1 orient: +



T[464] edges: 16 blocks: 1 orient: +



T[449] edges: 16 blocks: 1 orient: -



T[537] edges: 16 blocks: 1 orient: -

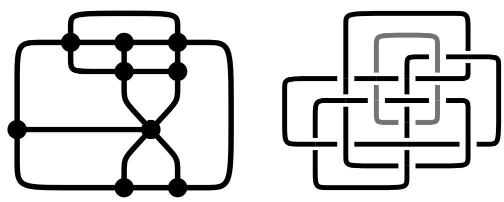
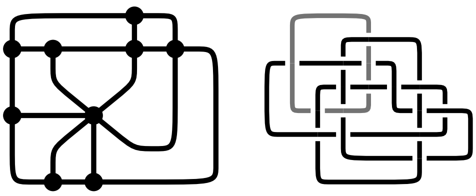


Figure 10: In the case of gems the corresponding theory

16^{t}_{233} (0) 2 blinks

r	mod	theta/pi	#sts
03	0.707106781	0.000000000	4
04	0.500000000	0.000000000	326
05	0.706953028	0.000000000	1420
06	0.288675135	0.000000000	17019
07	0.472161455	0.000000000	72552
08	0.573693534	0.000000000	401400

T[631] edges: 16 blocks: 1 orient: +



T[663] edges: 16 blocks: 1 orient: +

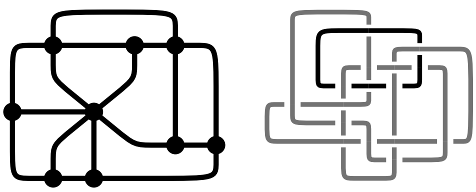


Figure 11: In the case of gems the corresponding theory

4 Conclusion

A closed orientable 3-manifold is denoted *n-small* if it is induced by surgery on a blackboard framed link with at most n crossings. Our bet is that both pairs of 3-manifolds in the 2 first sections of this short note are not homeomorphic. This would mean that the 9-small manifolds are completely classified and that the combinatorial dynamics of Chapter 4 in [5] based on TS-moves which leads to a (small, in the case of hyperbolic 3-manifolds) number of minimal gems, named the *attractor of the 3-manifold* is successful. This induces an efficient algorithm which is capable of classifying topologically all the 3-manifolds given as a blackboard framed link with up to (so far) 9 crossings and maintains live the two Conjectures of page 15 of [5]: the TS- and u^n-moves yield an efficient algorithm to classify n-small 3-manifolds by explicitly displaying homeomorphisms, whenever they exist.

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