# A tougher challenge to 3-manifold topologists and group algebraists *

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## Abstract

This paper poses some basic questions about instances (hard to find) of a special problem in 3-manifold topology. "Important though the general concepts and propositions may be with the modern industrious passion for axiomatizing and generalizing has presented us ...nevertheless I am convinced that the special problems in all their complexity constitute the stock and the core of mathematics; and to master their difficulty requires on the whole the harder labor." Hermann Weyl 1885-1955, cited in the preface of the first edition (1939) of A. N. Whitehead's book *The classical groups: their invariants and representations* [17].

In this paper I focus on new uncertainties left unanswered in L. Lins thesis [6] on the homemorphism problem of eleven concrete pairs of closed orientable 3-manifolds induced by 3-connected monochromatic *blinks* ([4]). The eleven HG8QI-classes are the only doubts left in the thesis, but the first two of them were solved few days ago and in this work I report on their solutions.

## 1 Introduction

In a joint recent paper posted recently in the arXiv ([8]) my son Lauro Lins and myself ask some 6 years old questions for which we had no answers about homeomorphisms between closed orientable 3-manifolds. The two pairs of 3-manifolds were the only uncertainties that were left in L. Lins thesis ([6]) under my supervision in the domain of 3-manifolds being induced by monochromatic blinks up to 9 edges (9-small 3-manifolds). A subset of relevant 10-crossings blinks were generated but their topological classification remains untouched. The paper was taken seriously by a few researchers, among them M. Culler, N. Dunfield, C. Hodgson and others that could solve them very quickly using GAP ([2]), Sage ([14]) and SnapPy ([1]), tools that (except for GAP) were basically unknown to us. The solutions were obtained by distinct methods and are all consistent (inclusive with BLINK, the program of L. Lins (implementing my theory described in [7]), which support his thesis). Together with my colleague Cristiana Nascimento, here at CIn/UFPE, I am learning fast to operate these wonderful tools. The solutions people found shows that BLINK does a complete job

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in topologically classifying the 9-small 3-manifolds. This is the subject of a joint paper with Lauro, currently under preparation.

The first solution that I got, and that still blows my mind, was by Craig Hodgson using length spectra techniques, based in his joint paper with J. Weeks entitled *Symmetries, isometries and length spectra of closed hyperbolic three-manifolds* ([3]). By using SnapPy Craig showed that even though the quantum WRT-invariants as well as the volumes of the hyperbolic $Z$-homology spheres induced by the bfl's, $U[1466]$ and $U[1563]$ are the same, the length of the smallest geodesics of them are distinct. For the other pair of bfl's $U[2125]$ and $U[2165]$ he shows that precisely the same facts apply. Here is a summary of Craig's findings extracted from the SnapPy session that he kindly sent me. As Craig writes: *"The output of the length spectrum command shows the* complex lengths *of closed geodesics — the real part is the actual length and the imaginary part is the rotation angle as you go once around the geodesic."*

<div align="center">Class $9_{126}$:</div>

```
First geodesic of U[1466]: 1.0152103824828331+0.39992347315914334i.
First geodesic of U[1563]: 0.9359206605025168+2.333526236965665i.
Volume of both manifolds: 7.36429600733.
```

<div align="center">Class $9_{199}$:</div>

```
First geodesic of U[2125]:  0.8939075859248593+0.761197185679321i.
First geodesic of U[2165]:  0.7978548001747316+2.94874250293459973i.
Volume of both manifolds: 7.12868652133.
```

I posted 4 versions of [8] correcting annoying mistakes in the presentations of the fundamental groups, putting a second pair of links, and in focusing the challenge in a broader context. I computed the presentations manually and I had a hard time making them correct. Even though the presentations are redundant because the blink is enough to define the 3-manifold, as explained in [4], my objective was to facilitate the work for those wanting to use GAP. The time spanned between the first (April 22, 2013) and the last version (May 1, 2013) was a little more than one week. During these revisions I was completely unaware that the paper had called the attention of many people. I did not know that the blog on lower dimensional topology was very active exposing my incorrections and I apologize for my ignorance. I thank Cristiana for having calling the blog to my attention. Worse, some people did not see the follow up versions. This was the case of Nathan Dunfield who worked with the wrong presentations. Not without reason he was angry at me, but I think that this is no longer true, since he was willing to answer my sometimes naives and stupid questions and send me a solution for the first pair of manifolds of the present work, using SnapPy, Sage and GAP computations, by working with covers. I did not know these tools. But, when properly motivated, I can learn fast and in general I do believe that I have something important and different to say in this brave new world of 3-manifolds: see the wonderful essay of E. Klarreich published by the Simons Foundation (march 2012), [5]. I have been putting a great amount of time and effort during my scientific carreer, (most of the time as an isolated researcher) on (mainly closed) 3-manifods. I seek no longer to be isolated: my team is the World, my compromisse is with Truth (independently of whom first found it).

Marc Culler was very helpful in answering questions of myself and Cristiana and helping her about issues in the downloading and installing SnapPy and Sage and GAP in her machine. With the presentation incorrections out of the way he produced an independent proof of the distinctveness of $(U[1466, U[1563])$ and of $(U[2125, U[2165])$. He also produced instantaneous isomorphic triangulations of the homeomorphic 3-manifolds in the classes $9_{126}$ and $9_{199}$. This fact makes me anxious to compare and timing the performances of BLINK (which also produces instantaneous solutions for the same problems) and SnapPy regarding finding homeomorphisms of $k$-small 3-manifolds, given that the homeomorphisms exist.

# 2 Objective of this work: help to make BLINK known

In this paper I put some new challenges (also coming from [6]), that seem harder than the ones considered in the previous paper. The reason I think so is that going from 9 to 14,15,16 crossings in the links, numerical problems start appearing concerning finding the Dirichlet domain and, in these cases, finding isomorphic triangulations might be harder to SnapPy than to BLINK. At any rate I have hundreds of examples where the performance of these programs in this issue could be compared, if anyone is seriously interested. Currently BLINK is not documented and one of my objectives is to seek for help in doing it and extend its capability. BLINK is hosted at Github under the userid *laurolins* and is open source code project. Unfortunately Lauro (currently a researcher at AT&T) does not have the necessary time to go on with the implementation. But he welcomes and is willing to help collaborators in getting started. As for myself, I am too old for the energy needed to construct good pieces of software. I intend to act as one of some Scientific Supervisors for the deployment and for the discussions of the new algorithms to be included in BLINK, but only at the mathematical level. The technological and software engineering screws and bolts needed, I leave to others.

An algorithm that I want to attach to BLINK is finding a uniformly distributed random closed orientable 3-manifold induced by a blink with an arbitrary number (even thousands) of edges. I want to gather evidence for the truth of some important conjectures that depend on this capability. Another example of such new algorithms that I want to include in BLINK is made possible by the theory in Ricardo Machado's thesis under my supervision, defended in March, 2013. We got an $O(n^2)$-algorithm for going from a special kind of gem, named *resoluble gem*, to a blink inducing the same manifold. This work is available, in still rather sketchy form (even the definition of resolubility is unecessary complicated), in the three joint papers posted last year in the arXiv, [9, 10, 11]. The algorithm was implemented in Mathematica, but it needs to be improved and re-implemented in Java or C++. We found a rather crude framed link presentation for the hyperbolic dodecahedral space (Weber-Seifert manifold). As far as I know nobody has found such a framed link. My interest in it was aroused by J. Weeks in a visit to the Geometry Center in April 1993, when he asked me whether I had such framed link. The link inducing the Weber-Seifert 3-manifold is a 9-component link embedded into $\mathbb{R}^3$, with an integer attached to each component (its framing) and having a total of 68 (only) vertices with a projection having 142 crossings. (It started with a PL-link with more than 600 vertices.) In a fourth joint paper with R. Machado, currently under preparation, we will show that every 3-manifold admits a resoluble gem inducing it.

# 3 The eleven classes of tough 3-manifolds

I assume that the reader has with him a copy of the version 4 of previous challenge paper ([8]) and has learned how to read the manifold either from the blink or from the blackboard framed link, [4]. As for obtaining a presentation of the fundamental group based on the Wirtinger relators ([15]) and the Dehn fillings ([13]) the two detailed examples given in [8] should suffice, if the reader has not available other pieces of softwares to get the presentation by automatic means. Actually, the best way to enter these manifolds is to draw the blackboard framed link using SnapPy and informing the $w$ of each component as its self-writhe in the projection. The framing of that component to be informed to SnapPy is $(w, 1)$. Actually the first two classes are resolved, only remaining the nine final ones. The complex numbers in polar form which appear at each $m_p^t$-class are the common quantum WRT-invariants. All except one of the eleven classes are formed by $Z$-homology spheres. The exception is $16_{56}^t$ which has no torsion but Betti number 1. These facts are indicated by the small number in parenthesis (which gives the homology of the manifold).

Nathan's Sage session distinguishing the two manifolds induced by the two blinks in $14_{24}^t$:

```
sage: from snappy import *
sage: M1 = Manifold('1424_T71.tri')
sage: M2 = Manifold('1424_T79.tri')
sage: covers1 = M1.covers(5, method='gap')
sage: covers2 = M2.covers(5, method='gap')
sage: [C.homology() for C in covers1]
[Z/132 + Z/132, Z/63 + Z/63, Z/3 + Z/3 + Z/3 + Z/3]
sage: [C.homology() for C in covers2]
[Z/3 + Z/3 + Z/3 + Z/3, Z/213 + Z/213, Z/432 + Z/432]
```

Cristiana's Sage session distinguishing the two manifolds induced by the two blinks in $14_{24}^t$ and, in conjuction with BLINK, topologically classifying the manifolds induced by the four blinks in $15_{16}^t$:

```
M=1424_T71, N=1424_T79
sage: [C.homology() for C in coversM]
[Z/3 + Z/3 + Z/3 + Z/3, Z/63 + Z/63, Z/132 + Z/132]
sage: [C.homology() for C in coversN]
[Z/3 + Z/3 + Z/3 + Z/3, Z/213 + Z/213, Z/432 + Z/432]
-------------------------------------------------------
A=1516_T118, B=1516_T119, C=1516_T181, D=1516_T205
sage: [X.homology() for X in coversA]
[Z/229773, Z/1110327, Z/3699687, Z/3018207]
sage: [X.homology() for X in coversC]
[Z/1110327, Z/229773, Z/3018207, Z/3699687]
sage: [X.homology() for X in coversB]
[Z/1052067, Z/3 + Z/1299909, Z/4117827, Z/126627]
sage: [X.homology() for X in coversD]
[Z/4117827, Z/1052067, Z/3 + Z/1299909, Z/126627]
```

## 3.1 The $HG8QI_t$ class $14^t_{24}$:

$14^t_{24}$ (0)   2 blinks

```
r           mod      theta/pi     #sts
03   0.707106781   0.000000000        4
04   0.500000000   0.000000000      166
05   0.221839804   0.000000000      732
06   0.288675135   0.000000000     6153
07   0.569807123   0.000000000    26280
08   0.658347954   0.000000000   118404
```

T[71]  edges: 14  blocks: 1  orient: +          T[79]  edges: 14  blocks: 1  orient: +
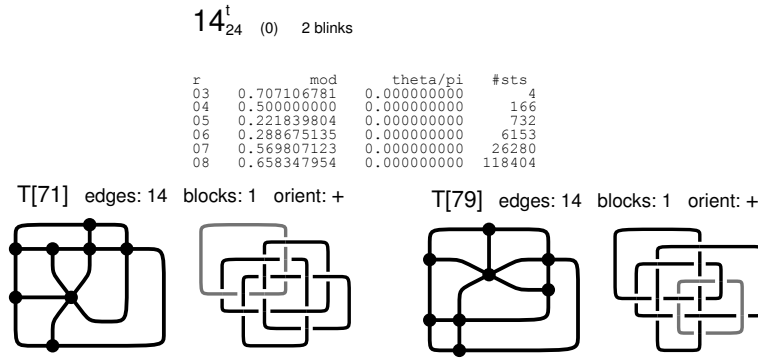
Figure 1: The above two manifolds are not homeomorphic. They are distinguished by the homology of their 5-covers. This was immediately noted by N. Dunfield using Sage and GAP from triangulations obtained by C. Nascimento using SnapPy, which could not find the Dirichlet domain due to numerical instability.

## 3.2 The $HG8QI_t$ class $15^t_{16}$:

$15^t_{16}$ (0)   4 blinks

```
r           mod       theta/pi      #sts
03   0.707106781    0.000000000        4
04   0.500000000    0.000000000      326
05   0.626523527   -0.169924730     1684
06   0.288675135    0.000000000    19341
07   0.271887719   -0.143082082   111440
08   0.116351265    1.000000000   620624
```

T[118]  edges: 15  blocks: 1  orient: +          T[181]  edges: 15  blocks: 1  orient: -

T[119]  edges: 15  blocks: 1  orient: +          T[205]  edges: 15  blocks: 1  orient: -

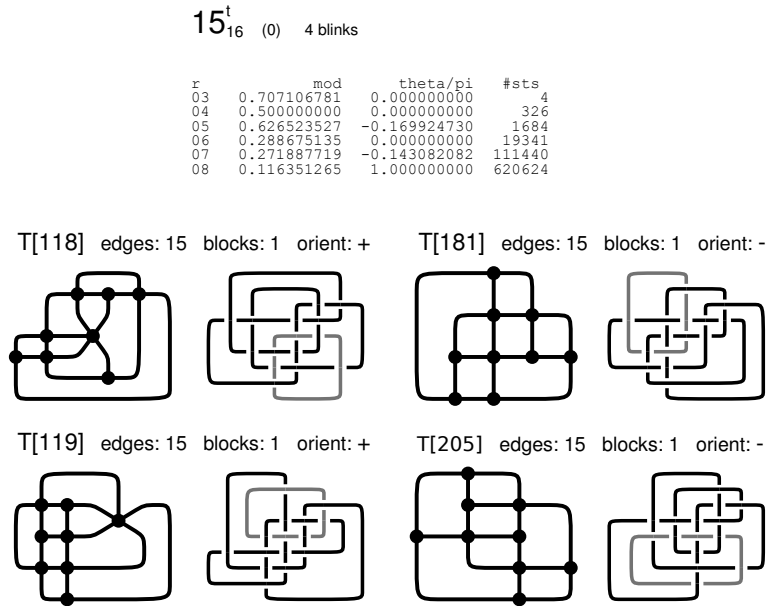Figure 2: The above two manifolds are also non-homeomorphic. They are also distinguished by the homology of their 5-covers. Relative to the class $15^t_{24}$ class $15^t_{16}$ the Sage/GAP software demands much more time. This was obtained by C. Nascimento using SnapPy/Sage/GAP. The software SnapPy could not find the Dirichlet domain due to numerical instability.

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## 3.3 The $HG8QI_t$ class $15_{19}^t$:



$15_{19}^t$ (0) 4 blinks

| r | mod | theta/pi | #sts |
|---|---|---|---|
| 03 | 0.707106781 | 0.000000000 | 4 |
| 04 | 0.500000000 | 0.000000000 | 326 |
| 05 | 1.180868794 | -0.318489048 | 1664 |
| 06 | 0.288675135 | 0.000000000 | 18957 |
| 07 | 0.344363154 | -0.282364319 | 108168 |
| 08 | 0.990504026 | 1.000000000 | 593752 |

T[122]  edges: 15  blocks: 1  orient: +    T[188]  edges: 15  blocks: 1  orient: -

T[148]  edges: 15  blocks: 1  orient: +    T[208]  edges: 15  blocks: 1  orient: -
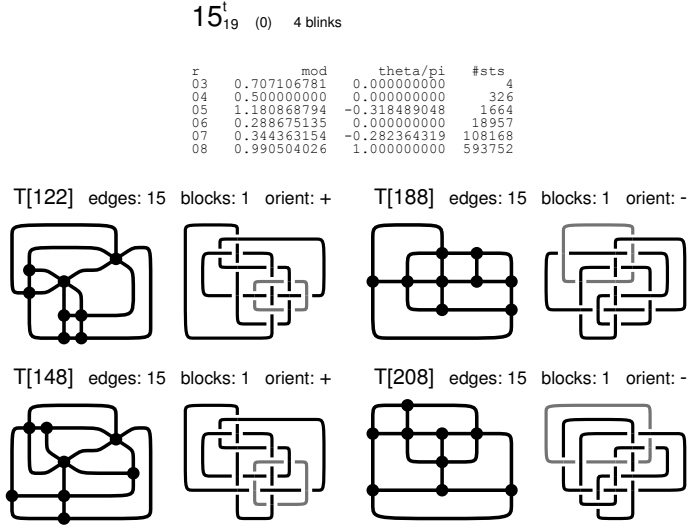
Figure 3: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

## 3.4 The $HG8QI_t$ class $15_{22}^t$:



$15_{22}^t$ (0) 4 blinks

| r | mod | theta/pi | #sts |
|---|---|---|---|
| 03 | 0.707106781 | 0.000000000 | 4 |
| 04 | 0.500000000 | 0.000000000 | 326 |
| 05 | 0.943093903 | -0.180036791 | 1680 |
| 06 | 0.288675135 | 0.000000000 | 20945 |
| 07 | 0.242490221 | -0.482680140 | 111968 |
| 08 | 0.641616039 | 1.000000000 | 656072 |

T[128]  edges: 15  blocks: 1  orient: +    T[186]  edges: 15  blocks: 1  orient: -

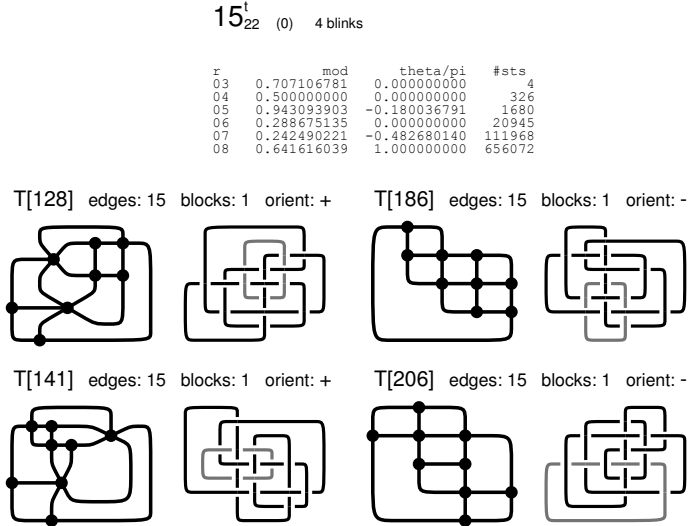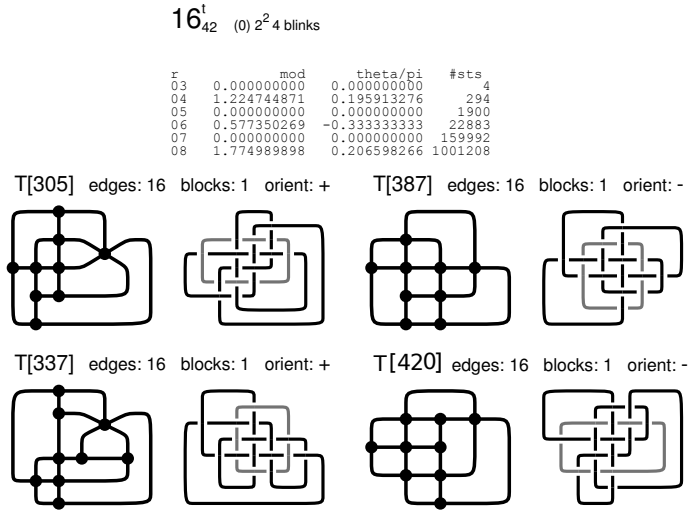T[141]  edges: 15  blocks: 1  orient: +    T[206]  edges: 15  blocks: 1  orient: -

Figure 4: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

## 3.5  The $HG8QI_t$ class $16_{42}^t$:

$16_{42}^t$  (0) $2^2$ 4 blinks

```
r           mod        theta/pi       #sts
03   0.000000000   0.000000000              4
04   1.224744871   0.195913276            294
05   0.000000000   0.000000000           1900
06   0.577350269  -0.333333333          22883
07   0.000000000   0.000000000         159992
08   1.774989898   0.206598266        1001208
```

T[305]  edges: 16  blocks: 1  orient: +          T[387]  edges: 16  blocks: 1  orient: -

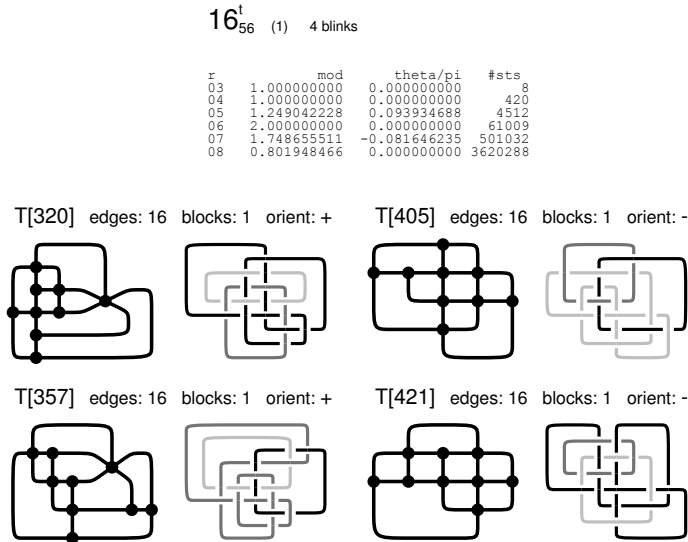T[337]  edges: 16  blocks: 1  orient: +          T[420]  edges: 16  blocks: 1  orient: -

Figure 5: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

## 3.6  The $HG8QI_t$ class $16_{56}^t$:

$16_{56}^t$  (1)  4 blinks

```
r           mod        theta/pi       #sts
03   1.000000000   0.000000000              8
04   1.000000000   0.000000000            420
05   1.249042228   0.093934688           4512
06   2.000000000   0.000000000          61009
07   1.748655511  -0.081646235         501032
08   0.801948466   0.000000000        3620288
```

T[320]  edges: 16  blocks: 1  orient: +          T[405]  edges: 16  blocks: 1  orient: -

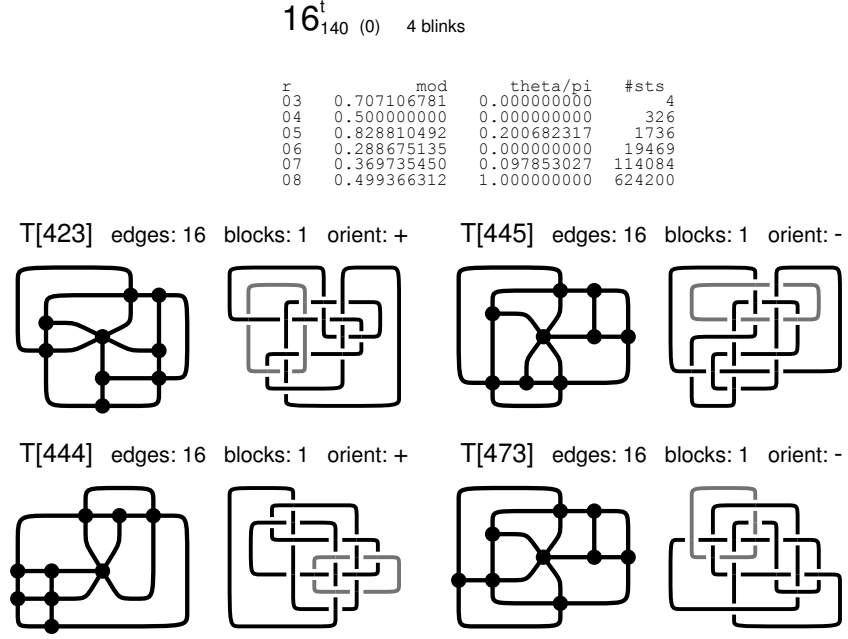T[357]  edges: 16  blocks: 1  orient: +          T[421]  edges: 16  blocks: 1  orient: -

Figure 6: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.
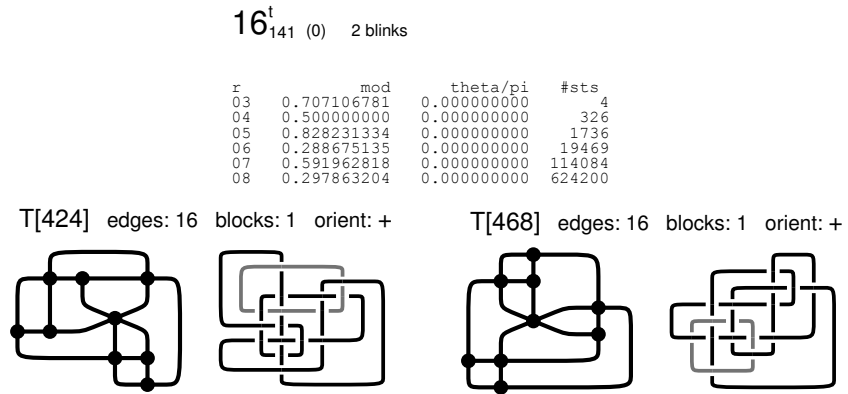
## 3.7 The $HG8QI_t$ class $16^t_{140}$:

$16^t_{140}$ (0)    4 blinks

```
r         mod        theta/pi    #sts
03   0.707106781   0.000000000        4
04   0.500000000   0.000000000      326
05   0.828810492   0.200682317     1736
06   0.288675135   0.000000000    19469
07   0.369735450   0.097853027   114084
08   0.499366312   1.000000000   624200
```

T[423]  edges: 16  blocks: 1  orient: +      T[445]  edges: 16  blocks: 1  orient: -

T[444]  edges: 16  blocks: 1  orient: +      T[473]  edges: 16  blocks: 1  orient: -

Figure 7: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

## 3.8 The $HG8QI_t$ class $16^t_{141}$:

$16^t_{141}$ (0)    2 blinks

```
r         mod        theta/pi    #sts
03   0.707106781   0.000000000        4
04   0.500000000   0.000000000      326
05   0.828231334   0.000000000     1736
06   0.288675135   0.000000000    19469
07   0.591962818   0.000000000   114084
08   0.297863204   0.000000000   624200
```

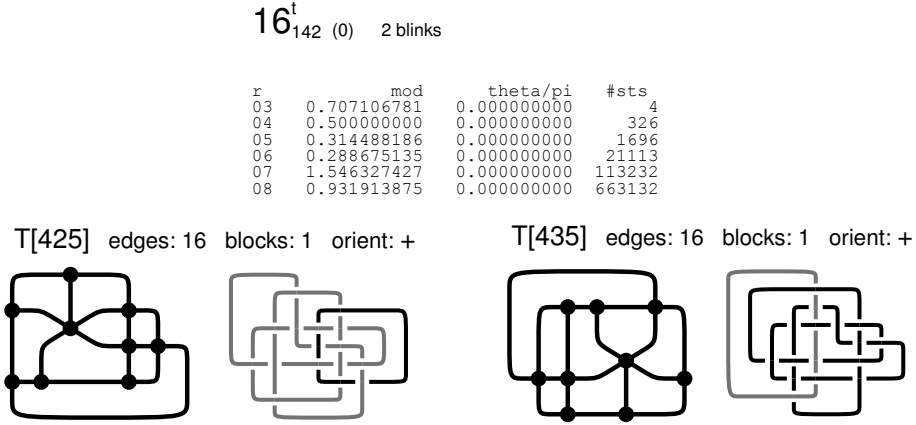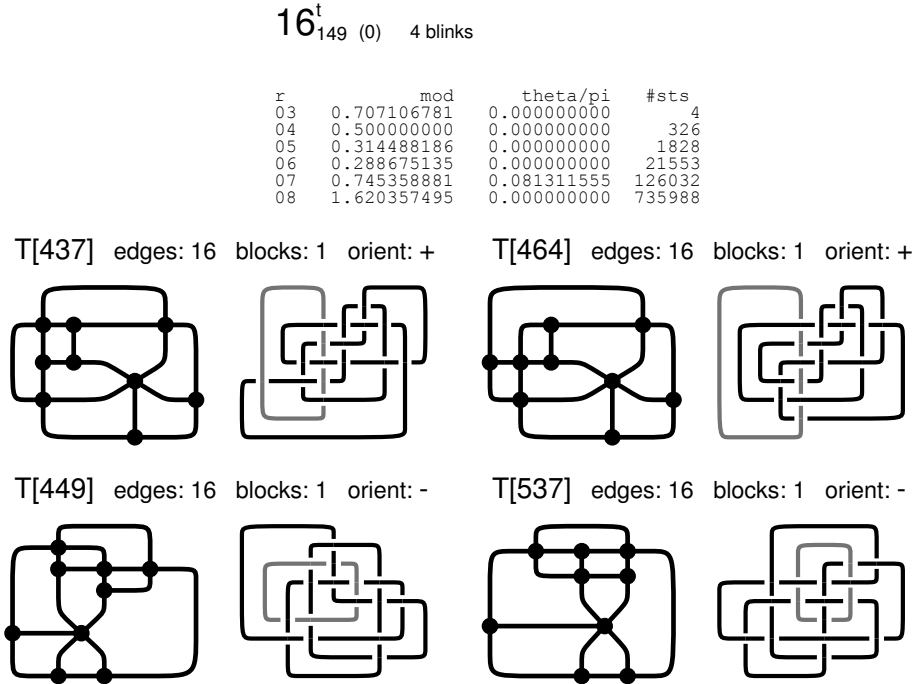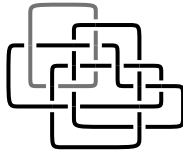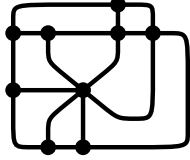T[424]  edges: 16  blocks: 1  orient: +        T[468]  edges: 16  blocks: 1  orient: +

Figure 8: I do not know whether the above two manifolds are homeomorphic or not.

## 3.9 The $HG8QI_t$ class $16^t_{142}$:

$16^t_{142}$ (0)    2 blinks

```
r          mod       theta/pi      #sts
03   0.707106781   0.000000000         4
04   0.500000000   0.000000000       326
05   0.314488186   0.000000000      1696
06   0.288675135   0.000000000     21113
07   1.546327427   0.000000000    113232
08   0.931913875   0.000000000    663132
```

T[425]  edges: 16  blocks: 1  orient: +        T[435]  edges: 16  blocks: 1  orient: +



Figure 9: I do not know whether the above two manifolds are homeomorphic or not.

## 3.10 The $HG8QI_t$ class $16^t_{149}$:

$16^t_{149}$ (0)    4 blinks

```
r          mod       theta/pi      #sts
03   0.707106781   0.000000000         4
04   0.500000000   0.000000000       326
05   0.314488186   0.000000000      1828
06   0.288675135   0.000000000     21553
07   0.745358881   0.081311555    126032
08   1.620357495   0.000000000    735988
```

T[437]  edges: 16  blocks: 1  orient: +        T[464]  edges: 16  blocks: 1  orient: +

T[449]  edges: 16  blocks: 1  orient: -        T[537]  edges: 16  blocks: 1  orient: -



Figure 10: I do not know whether the above four manifolds are homeomorphic or not. BLINK says that there are at most two homeomorphisms classes among the four and I bet that this bound is attained.

## 3.11  The $HG8QI_t$ class $16^t_{233}$:

$16^t_{233}$ (0)    2 blinks

```
r              mod       theta/pi    #sts
03    0.707106781    0.000000000         4
04    0.500000000    0.000000000       326
05    0.706953028    0.000000000      1420
06    0.288675135    0.000000000     17019
07    0.472161455    0.000000000     72552
08    0.573693534    0.000000000    401400
```

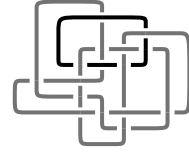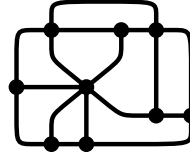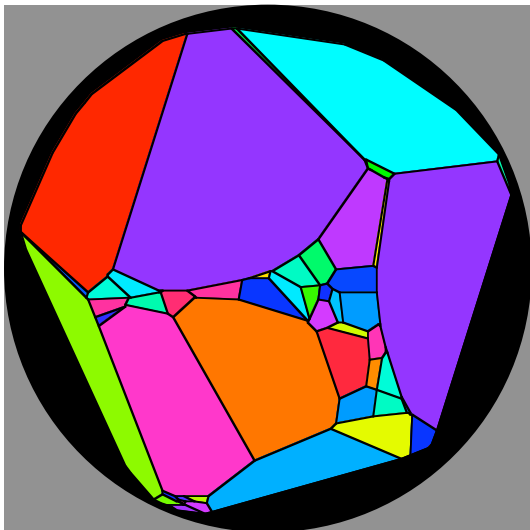T[631]  edges: 16  blocks: 1  orient: +          T[663]  edges: 16  blocks: 1  orient: +

Figure 11: I do not know whether the above two manifolds are homeomorphic or not.

# 4 Concluding remarks

The elegant drawings of blinks and blackboard framed links produced by BLINK are possible due the groundbreaking algorithm of R. Tamasia [16]. Lauro could implement the drawings very fast because we had at hand the implementation of network flow algorithms he had done for a project to solve *practical timetable (!) problems*. This is an example of the unicity in Mathematics, advocated by L. Lovasz in his famous essay [12]. To get the drawings one has to apply three times the full strength of network flow theory. The drawings BLINK presents are in an integer grid and deterministically minimize the number of $\pi/2$-bents in the blackboarded framed links. In particular, it permit us to deal with the unavoidable curls which adjust the integer framings in the best possible way: we do not care about them. The drawings for the companion blinks require a slight modification: it replaces each $p$-valent vertex $p > 4$, by a $p$-polygon inducing 3-valent ones. The final result is massaged a bit to produce aesthetically pleasing and unambiguous drawings.

As of this writing, C. Hodgson sent me some puzzling information (computed with a stronger version of SnapPy) about the first pair of manifolds. These are induced by $T[71]$ and $T[79]$, forming the $HG8QI$-class $14_{24}^t$. They are non-homeomorphic 3-manifolds as first shown by N. Dunfield. They are homology $\mathbb{Z}$-spheres which have the same WRT-invariants (according to BLINK), and quoting Craig *"the same volume (around 24.8) and the same lenght spectra (up to 12 decimals): the (complex) length of the first geodesic of $T[71]$ is 0.4749346632398791 + 0i (of multiplicity 1) and that of $T[79]$ is 0.4749346632399361 + 0 i (of multiplicity 1)."*
Here are the Dirichlet domains:

A view of the Dirichlet domain for 1424_T71          A view of the Dirichlet domain for 1424_T79
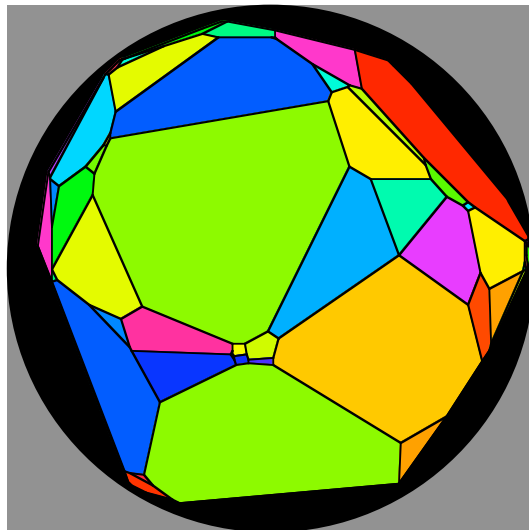


Craig found another proof that $T[71]$ and $T[79]$ are non-homemorphic: *"Now we can drill out the shortest geodesics using SnapPea to obtain one-cusped manifolds (the manifold files are attached). Then SnapPea's isometry checker (which uses the canonical cell decompositions) shows that these cusped manifolds are not isometric. Hence the original closed manifolds are not homeomorphic. This gives another proof that $T[71]$ and $T[79]$ are distinct!"*

**A final challenge:** From the data I could get so far, if a closed orientable 3-manifold is hyperbolic, it seems that the WRT-invariants determine its volume. Prove it or disproved

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it.

# References

[1] M. Culler, N. M. Dunfield, and J. Weeks. Snappy, a computer program for studying the geometry and topology of 3-manifolds, 2012.

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