Extensions on the Moving Sofa Problem

By Lauryn Harvey, Tenki Ko, Sam Huynh
Advised by Dr. Ryan Card

Abstract

What is the shape of the sofa with maximal area that can move through 2-dimensional right-angled hallway with a width of one unit? In this paper we summarize the methods of Romik's work investigating this question and pursue new questions inspired by this paper. Using differential equations and linear algebra we describe the conditions on the shape of the sofa and solve the system of conditions for an explicit shape. Our new question: What is the shape of a bed with maximal area that can fit in an ℓ by m room with a door of one unit width?

Contents

1 Introduction

Leo Moser originally proposed the question in 1966, 'What is the shape of the sofa with maximal area that can move through 2-dimensional right-angled hallway with a width of one unit?' [3] The first few attempts to answer this question were constructions from simple geometric shapes. Then, Hammersley combined geometric figures to create a sofa with larger area [2]. Hammersley's sofa discovered in 1968 cuts out a semicircle from a rectangle and adds two quarter-circles to the end. This sofa has an area$= \frac{2}{\pi} + \frac{\pi}{2}$, shown in figure 1.

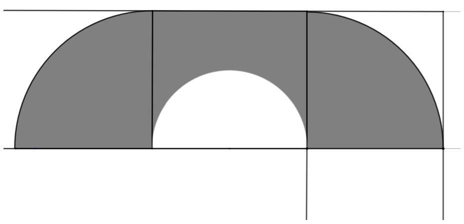

Figure 1: Hammersley's Sofa

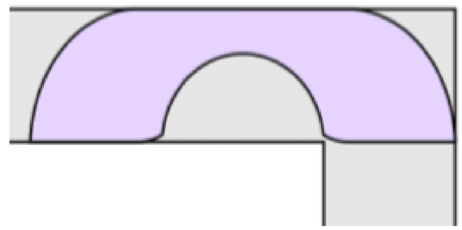

Figure 2: Gerver's Sofa

The current proposed sofa with maximal area is Gerver's sofa discovered in 1992 [1] shown in figure 2, although this sofa is not proven to be maximal. This sofa has an area ≈ 2.2195 In section 2, we will discuss Romik's paper "Differential equations and exact solutions in the moving sofa problem" [3] deriving exact solution to this question. We will lay out the processes in Romik's paper in a way that is accessible to students who have taken introductory courses in differential equations and linear algebra. We will provide the set up for major computations in Romik's paper and fill in intermediate steps.

In section 3, we will discuss a new question inspired by the Moving Sofa Problem: *What is the shape of a bed with maximal area that can fit in an ℓ by m room with a door of one unit width?* We will call this the "Bedroom Problem". We will introduce a few simplifications for the "Bedroom Problem" and introduce our geometric construction. One simplification we will consider is a centered door shown in figure 3.

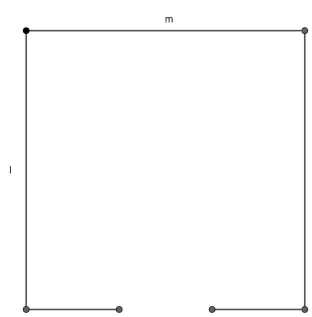

Figure 3: Bedroom Problem with Centered Door

2 Set-up for Moving Sofas

In this section, we will introduce the mathematical arguments of Romik's paper providing exact solutions for the original moving sofa question. We will provide the clear set-up and explanation of mathematical computations done in Romik's paper.

2.1 The Hallway

The hallway introduced in the original question has a width of one unit between parallel walls and has a right angled corner as shown in figures 1 and 2. We define the hallway as the union between two sets of ordered pairs given by

$$\begin{aligned} L_{horiz} &= \{(x, y) \in \mathbb{R}^2 : x \leq 1, 0 \leq y \leq 1\}, \\ L_{vert} &= \{(x, y) \in \mathbb{R}^2 : y \leq 1, 0 \leq x \leq 1\}, \\ L &= L_{horiz} \cup L_{vert}. \end{aligned}$$

The sofa must stay within our hallway, L. In addition, the shape is further restricted by it moving from L_{horiz} to L_{vert} continuously while still being in L. Intuitively, this means the boundary of the sofa will not have points that are isolated from those on the rest of the shape. So, the shape needs to be able to rotate from 0 to $\frac{\pi}{2}$ radians around the corner. Let S denote the shape of the sofa and let $\mathbf{x}(t)$ denote the path that is traced out by the point that touches both the inner corner of the hallway and the sofa. Then the shape S can go through the corner if it satisfies this condition:

$$S \subseteq L_{horiz} \cap \bigcap_{0 \leq t \leq \frac{\pi}{2}} \Big(\mathbf{x}(t) + R_t(L)\Big) \cap \Big(\mathbf{x}\Big(\frac{\pi}{2}\Big) + R_{\frac{\pi}{2}}(L_{vert})\Big). \tag{1}$$

Mathematically, we will be moving the hallway about the unknown shape by using a rotation matrix R_t. We use the vectors μ_t and ν_t to describe the rotation matrix R_t which rotates the hallway from $0 \leq t \leq \frac{\pi}{2}$.

Definition 1. Define $\mu_t = [\cos t, \sin t]^\top$ and $\nu_t = [-\sin t, \cos t]^\top$.

As an example, figure 4 shows the hallway with vectors μ_t and ν_t at $t = 0$.

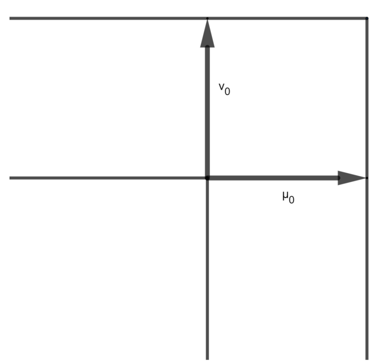

Figure 4: Hallway with μ_0 and ν_0

The vectors μ_t and ν_t in $\mathbb{R}^2$ will form the columns of our rotation matrix,

$$R_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}. \tag{2}$$

Then, we have a system for describing the position of the walls of the hallway at angle t.

2.2 Contact Points

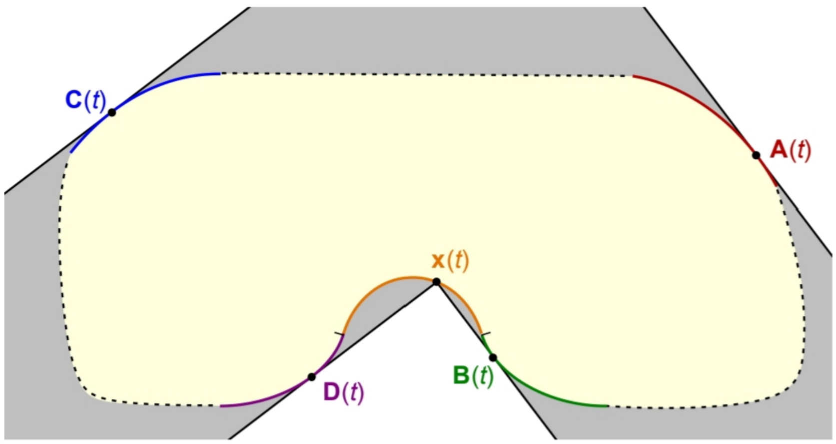

Figure 5: Contact points and paths [3]

The important idea behind most of the following mathematical computations is that a maximal sofa will touch each of the four walls and the corner at least once while t ranges from 0 to $\frac{\pi}{2}$. The sofa that does not touch at least one of the walls would not be maximal. Figure 5 shows the contact points $\mathbf{x}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$. Note that $\mathbf{x}$ is the point where the sofa touches the inner corner, $\mathbf{D}$ and $\mathbf{B}$ are the points where the sofa touches the inner walls, and $\mathbf{C}$ and $\mathbf{A}$ are the points where the sofa touches the outer walls of the hallway. The path that each respective point traces out as t ranges from $t = 0$ to $t = \frac{\pi}{2}$ will be denoted by the following functions $\mathbf{x}(t), \mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t), \mathbf{D}(t)$. As seen from 5, $\mathbf{A}(t)$ and $\mathbf{C}(t)$ touch the outer wall of the hallway, while $\mathbf{x}(t)$, $\mathbf{B}(t)$ and $\mathbf{D}(t)$ touch the inner wall of the hallway. We can describe the path of each point using only $\mathbf{x}(t)$ and our rotation vectors μ_t, ν_t. Let S_x denote the maximal sofa. The problem is defining a map from $\mathbf{x} \to S_x$ in order to relate a rotation path to the shape.

2.3 Conditions for Contact Paths

In this section, we will introduce a method of deriving a relationship between each contact path with respect to only the inner corner $\mathbf{x}(t)$ and our unit, vectors μ_t and ν_t. This will simplify later conditions. In **Theorem 1** we will describe the wall of the hallway at time t with the line ℓ_t. Then we will describe the wall of the hallway at time s with another line, ℓ_s. We use the fact that $\mathbf{A}(t)$ was required to touch each of these lines and look at what happens as the angles s and t become arbitrarily close. In the following sections we use $\langle \mathbf{a}, \mathbf{b} \rangle$ to define the dot product of the vectors $\mathbf{a}$ and $\mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$.

Theorem 1. The contact paths can be described by the following:

$$\begin{aligned}
\mathbf{A}(t) =& \mathbf{x}(t) + \langle \mathbf{x}'(t), \mu_t \rangle \nu_t + \mu_t && (3)\\
\mathbf{B}(t) =& \mathbf{x}(t) + \langle \mathbf{x}'(t), \mu_t \rangle \nu_t && (4)\\
\mathbf{C}(t) =& \mathbf{x}(t) - \langle \mathbf{x}'(t), \nu_t \rangle \mu_t + \nu_t && (5)\\
\mathbf{D}(t) =& \mathbf{x}(t) - \langle \mathbf{x}'(t), \nu_t \rangle \mu_t && (6)
\end{aligned}$$

Proof. We will derive (3) in this proof. We can describe the contact path for $\mathbf{A}(t)$ as lying on the line ℓ_t where

$$\ell_t := \{\mathbf{p} : \langle \mathbf{p}, \mu_{\mathrm{t}} \rangle = \langle \mathbf{x}(\mathrm{t}), \mu_{\mathrm{t}} \rangle + 1\}.$$

Let $s = t + \delta$ where δ is a small positive number. Define $\mathbf{p}(t, s)$ as the point of intersection of the lines ℓ_t and ℓ_s. Then,

$$\mathbf{A}(t) = \lim_{\delta \to 0} \mathbf{p}(t, s)$$

given that $\mathbf{A}(t)$ lies on both ℓ_t and ℓ_s and is continuous at t. Note that $\mathbf{p}(t, s)$ satisfies

$$\langle \mathbf{p}(t, s), \mu_t \rangle = \langle \mathbf{x}(t), \mu_t \rangle + 1 \tag{7}$$

$$\langle \mathbf{p}(t, s), \mu_s \rangle = \langle \mathbf{x}(s), \mu_s \rangle + 1 \tag{8}$$

Since $s = t + \delta$, the rotation a matrix satisfies

$$\begin{aligned}
R_s =& R_{t+\delta}\\
=& R_t R_\delta.
\end{aligned}$$

Then for μ_s we have

$$\begin{aligned}
\mu_s =& R_t R_\delta \begin{bmatrix} 1 \\ 0 \end{bmatrix}\\
&= \cos\delta \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + \sin\delta \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}\\
&= \cos\delta \mu_t + \sin\delta \nu_t.
\end{aligned}$$

Using this relation for μ_s we can rewrite (8) as

$$\cos\delta \langle \mathbf{p}(t, s), \mu_t \rangle + \sin\delta \langle \mathbf{p}(t, s), \nu_t \rangle = \cos\delta \langle \mathbf{x}(s), \mu_t \rangle + \sin\delta \langle \mathbf{x}(s), \nu_t \rangle + 1$$

Using (7), we can show that

$$\sin\delta\langle \mathbf{p}(t,s),\nu_t\rangle = \cos\delta\langle \mathbf{x}(s)-\nu_t\rangle - \cos\delta + 1 \tag{9}$$

Next, we divide by δ

$$\frac{\sin\delta}{\delta}\langle \mathbf{p}(t,s),\nu_t\rangle = \cos\delta\langle\frac{\mathbf{x}(s)-\mathbf{x}(t)}{\delta},\mu_t\rangle + \frac{\sin\delta}{\delta}\langle \mathbf{x}(s),\nu_t\rangle + \frac{1-\cos\delta}{\delta} \tag{10}$$

Now, we will take the $\lim_{\delta\to 0}$ of this expression. Using L'Hospital's rule[1], we have (10) becomes

$$\langle \mathbf{A}(t),\nu_t\rangle = \langle \mathbf{x}'(t),\mu_t\rangle + \langle \mathbf{x}(t),\nu_t\rangle. \tag{11}$$

Additionally, taking the limit as delta goes to zero of (7) we have

$$\langle \mathbf{A}(\mathbf{t}),\mu_{\mathbf{t}}\rangle = \langle \mathbf{x}(\mathbf{t}),\mu_{\mathbf{t}}\rangle + 1 \tag{12}$$

Combining (11) and (12) we have $\mathbf{A}(t) = \mathbf{x}(t) + \langle \mathbf{x}'(t),\mu_t\rangle\nu_t + \mu_t$. □

The reader may verify by taking the dot product of $\mathbf{A}(t)$ with ν_t and μ_t to obtain (11) and (12) respectively using properties of parallel and perpendicular vectors to simplify.

Similarly, we can define a variation of ℓ_t for each contact path to show the relations for $\mathbf{B}(t), \mathbf{C}(t)$, and $\mathbf{D}(t)$. By representing each contact path in terms of $\mathbf{x}(t), \mu_t$ and ν_t the conditions for the sofa with maximal area will be simpler to discuss.

2.4 Example: Generalized Hammersley Sofa

In this section, we will investigate a generalized example of Hammersley's Sofa finding the relations $\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t)$, and $\mathbf{D}(t)$ where the semicircle cut out of the sofa has a radius of r shown in figure 6. Specifically, we will look at the contact path $\mathbf{A}(t)$ in depth and verify it matches the geometric description of Hammersley's sofa.

Let $\mathbf{A}^r(t), \mathbf{B}^r(t), \mathbf{C}^r(t)$, and $\mathbf{D}^r(t)$ denote the contact paths of a Hammersley's sofa with radius r. Fix $0 \le r \le 1$. Consider a semicircular rotation path $\mathbf{x}^r(t) = r[\cos(2t)-1, \sin(2t)]^T$ of radius r counter-clockwise traveling from origin $(0,0)$ to $(-2r, 0)$.

[1] Recall the following limits: $\lim_{\delta\to 0}\frac{\sin\delta}{\delta} = 1$ and $\lim_{\delta\to 0}\frac{1-\cos\delta}{\delta} = 0$. Additionally, recall the limit definition of the derivative: $\lim_{s\to t}\frac{\mathbf{x}(s)-\mathbf{x}(t)}{s-t} = \lim_{\delta\to 0}\frac{x(t+\delta)-\mathbf{x}(t)}{\delta} = \mathbf{x}'(t)$.

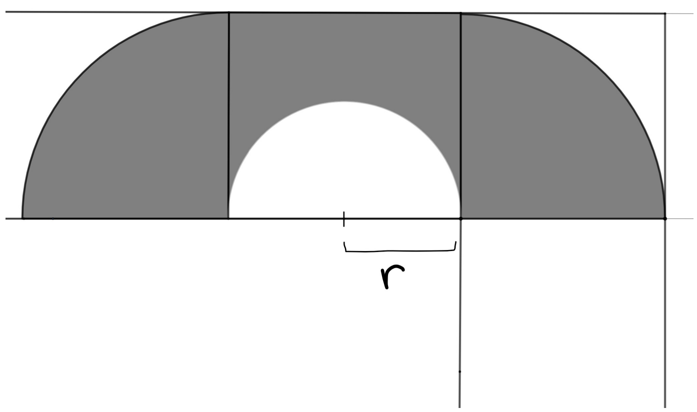

Figure 6: Generalized Hammersley's Sofa

$\mathbf{A}^r(t)$,$\mathbf{B}^r(t)$,$\mathbf{C}^r(t)$,$\mathbf{D}^r(t)$ are the associated contact paths of a Hammersley's sofa whose radius is generalized to be r. Using **Theorem 1**, we may derive the contact paths below

$$\begin{aligned}
\mathbf{A}^r(t) &= [\cos t, \sin t]^\top \\
\mathbf{B}^r(t) &= [0, 0]^\top \\
\mathbf{C}^r(t) &= [-2r - \sin t, \cos t]^\top \\
\mathbf{D}^r(t) &= [-2r, 0]^\top
\end{aligned}$$

We will now use **Theorem 1** equation (3) to find $\mathbf{A}^r(t)$ for the generalized Hammersley's sofa. Since we know a semicircular rotation path $\mathbf{x}^r(t) = r[\cos 2t - 1, \sin 2t]^\top$ of radius r counter-clockwise traveling from origin $[0, 0]^\top$ to $[-2r, 0]^\top$ from the example of Hammersley sofa. First, recall the path of $\mathbf{x}(t)$:

$$\mathbf{x}(t) = \begin{bmatrix} r\cos 2t - r \\ r\sin 2t \end{bmatrix}$$

$$\mathbf{x}'(t) = \begin{bmatrix} -2r\sin 2t \\ 2r\cos 2t \end{bmatrix}$$

$$\mathbf{A}^r(t) = \mathbf{x}(t) + \langle \mathbf{x}'(t), \mu_t \rangle \nu_t + \mu_t$$

Next, solve $\langle \mathbf{x}'(t), \mu_t \rangle$,

$$\begin{aligned}
\langle \mathbf{x}'(t), \mu_t \rangle &= -2r\sin(2t)\cos(t) + 2r\cos(2t)\sin(t) \\
&= -4r\sin(t)\cos^2(t) + 2r(2\cos^2(t) - 1)\sin(t) \\
&= -2r\sin(t)
\end{aligned}$$

Substituting $\langle \mathbf{x}'(t), \mu_t \rangle = -2r\sin(t)$ into (3) and using double angle identities [2] to simplify, we have,

$$\begin{aligned}
\mathbf{A}^r(t) &= \begin{bmatrix} r\cos(2t) - r \\ r\sin(2t) \end{bmatrix} - (-2r\sin(t)) \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \\
&= \begin{bmatrix} r(1 - 2\sin^2(t)) - 1 \\ 2r\sin(t)\cos(t) \end{bmatrix} - (2r\sin(t)) \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \\
&= \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}
\end{aligned}$$

This computation of $\mathbf{A}^r(t) = [\cos t, \sin t]^\top$ through **Theorem 1** matches our intuitive idea that the inner corner of Hammersley's sofa with radius r will trace out the path of a semicircle. Note that $\mathbf{B}^r(t), \mathbf{C}^r(t)$, and $\mathbf{D}^r(t)$ can be found in a similar way.

Thus, the contact paths and **B** and **D** are fixed at the corners of the semicircle hole traced out by the rotation path, and the contact paths **A** and **C** trace out two quarter-circles of unit radius. The sofa shape S^r therefore consists of two unit quarter-circles separated by a $2r \times 1$ rectangular block from which a semicircular piece of radius r has been removed. This family of shapes was considered by Hammersley, who noticed that the area $f(r) = \frac{\pi}{2} + r(2 - \frac{\pi}{2}r)$ of the shape takes its maximum value at r_* in the following:

First, take the derivative of $f(r)$ and we will get

$$\begin{aligned}
f'(r) = 2 - \pi r &= 0 \\
r &= \frac{2}{\pi}
\end{aligned}$$

Since maximum value at $r_* = \frac{2}{\pi}$, then plug into $f(x)$

$$\begin{aligned}
f\left(\frac{2}{\pi}\right) &= \frac{\pi}{2} + 2\left(\frac{2}{\pi}\right) - \frac{\pi}{2}\left(\frac{2}{\pi}\right)^2 \\
f\left(\frac{2}{\pi}\right) &= \frac{\pi}{2} + \frac{2}{\pi} \approx 2.2074
\end{aligned}$$

The shape S^{r*} was the one proposed by Hammersley as a possible solution to the moving sofa problem.

2.5 System of Differential Equations

Before we go ahead, we must present a assumption about our rotation path $\mathbf{x}$. We say $\mathbf{x}$ is **well-behaved** if for $0 \le t \le \frac{\pi}{2}$, $\mathbf{x}$ is twice continuously differentiable and if the following conditions hold:

1. If $\mathbf{x}(t)$ is a contact point, then $\langle \mathbf{x}'(t), \nu_t \rangle \ge 0$ and $\langle \mathbf{x}'(t), \mu_t \rangle \le 0$.
2. If $\mathbf{A}(t)$ is defined, then $\langle \mathbf{A}'(t), \nu_t \rangle \ge 0$.
3. If $\mathbf{B}(t)$ is defined, then $\langle \mathbf{B}'(t), \nu_t \rangle \le 0$.

[2] The double angle identities are $\sin(2x) = 2\sin x\cos x$ and $\cos(2x) = 1 - 2\sin^2 x = 2\cos^2 x - 1 = \cos^2 x - \sin^2 x$.

4. If $\mathbf{C}(t)$ is defined, then $\langle \mathbf{C}'(t), \mu_t \rangle \leq 0$.

5. If $\mathbf{D}(t)$ is defined, then $\langle \mathbf{D}'(t), \mu_t \rangle \geq 0$.

Intuitively, it may be helpful to visualize each respective statement above as

1. $\mathbf{x}(t)$ is moving in the direction of ν_t and in the opposite direction of μ_t.

2. $\mathbf{A}(t)$ is moving in direction of ν_t.

3. $\mathbf{B}(t)$ is moving opposite direction of ν_t.

4. $\mathbf{C}(t)$ is moving opposite direction of μ_t.

5. $\mathbf{D}(t)$ is moving in direction of μ_t.

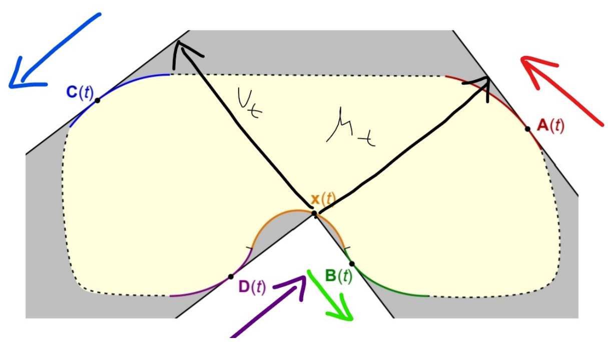

Figure 7: Direction of Contact Paths

Figure 7 is a visual representation of conditions 2-5 above for well-behavedness. The vectors μ_t and ν_t are included for reference to the intuitive versions of conditions 1-5. The arrow denotes the direction of the corresponding contact path (color-coded) as t ranges from 0 to $\frac{\pi}{2}$. This can be visualized by the counter-clockwise rotation of the hallway. Note that this figure is simply for visual illustration and does not imply each contact point will be touching the wall at the same time for all values of t. As we will see in **Theorem 2**, we will consider 6 cases for combinations of contact points touching the walls at a given value for t.

The well-behavedness assumption is the reason Gerver's sofa has not been proven to be a maximal sofa for the original moving sofa problem. In other words, no one has shown that a maximal sofa need be well-behaved. Additionally, a key insight of how Gerver's sofa was derived was condition of being a **balanced polygon**[1]. A **balanced polygon** is a shape whose parallel sides are one unit apart. For example, the unit square is a balanced polygon. As pictured in 8, the unit square's parallel sides are a distance of 1 unit apart.

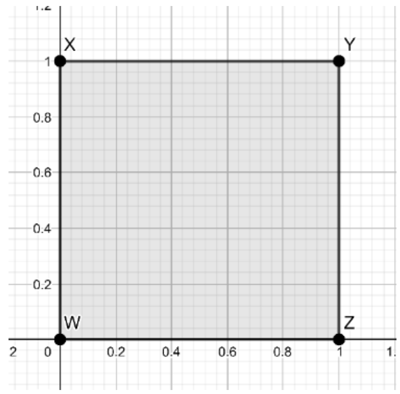

Figure 8: Unit Square Balanced Polygon

Although the sofa is not a polygon, this idea can be translated into an additional assumption of the sofa. We call this **balanced-ness**. It is the assumption that when contact points from parallel wall are touching the maximal sofa, they must be one unit apart. We will use the balanced-ness assumption to derive a the differential equations general sofa shape.

Theorem 2. If we let $\mathbf{x}(t)$ be a rotation path of a shape S_x with a set of contact points, $\Gamma_{\mathbf{x}}$ where $t \in (0, \frac{\pi}{2})$ allows $\mathbf{x}(t)$ to be well-behaved, then the shape has to follow the condition that $\mathbf{x}$ has to satisfy one of the differential equations at t depending on the set of contact points that is appropriate.

- Case 1: $\Gamma_{\mathbf{x}}(t) = \{\mathbf{A}, \mathbf{C}, \mathbf{D}\}$
  $$\mathbf{x}''(t) = R_t\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}2\sin t & -2\cos t\\ 2\cos t & 2\sin t\end{bmatrix}\mathbf{x}'(t)\right)$$
- Case 2: $\Gamma_{\mathbf{x}}(t) = \{\mathbf{x}, \mathbf{A}, \mathbf{C}, \mathbf{D}\}$
  $$\mathbf{x}''(t) = R_t\left(\begin{bmatrix}-1\\-1/2\end{bmatrix} + \begin{bmatrix}\sin t & -\cos t\\ (3/2)\cos t & (3/2)\sin t\end{bmatrix}\mathbf{x}'(t)\right)$$
- Case 3: $\Gamma_{\mathbf{x}}(t) = \{\mathbf{x}, \mathbf{A}, \mathbf{C}\}$
  $$\mathbf{x}''(t) = R_t\left(\begin{bmatrix}-1\\-1\end{bmatrix} + \begin{bmatrix}\sin t & -\cos t\\ \cos t & \sin t\end{bmatrix}\mathbf{x}'(t)\right)$$
- Case 4: $\Gamma_{\mathbf{x}}(t) = \{\mathbf{x}, \mathbf{A}, \mathbf{B}, \mathbf{C}\}$
  $$\mathbf{x}''(t) = R_t\left(\begin{bmatrix}-1/2\\-1\end{bmatrix} + \begin{bmatrix}(3/2)\sin t & -(3/2)\cos t\\ \cos t & \sin t\end{bmatrix}\mathbf{x}'(t)\right)$$
- Case 5: $\Gamma_{\mathbf{x}}(t) = \{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$
  $$\mathbf{x}''(t) = R_t\left(\begin{bmatrix}-1/2\\-1\end{bmatrix} + \begin{bmatrix}2\sin t & -2\cos t\\ 2\cos t & 2\sin t\end{bmatrix}\mathbf{x}'(t)\right)$$
- Case 6: $\Gamma_{\mathbf{x}}(t) = \{\mathbf{x}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$
  $$\mathbf{x}''(t) = R_t\left(\begin{bmatrix}-1/2\\-1/2\end{bmatrix} + \begin{bmatrix}(3/2)\sin t & -(3/2)\cos t\\ (3/2)\cos t & (3/2)\sin t\end{bmatrix}\mathbf{x}'(t)\right)$$

Proof. Let's start off with Case 3 since it is simpler than the others. First, let δ be some small positive value and $t^* = t + \delta$. We will use this to create a modified sequence of operations:

1. When time s is between $[0, t]$, drag the inside corner, $(0, 0)$ of the hallway along path $\mathbf{x}(s)$ and rotate the hallway around that corner.

2. Slide the hallway in δ amount by the direction of μ_t without rotating.

3. When s is between $[t, t^*]$, drag the inside corner of the hallway which is right now at $\mathbf{x}(t)+\delta\mu_t$ next to the translated copy of $\mathbf{x}(s) + \delta\mu_t$ of the segment $\mathbf{x}(s)$, $t \leq s \leq t^*$ of $\mathbf{x}$, while rotating such that for each s the angle of rotation is equal to s in the original motion.

4. Slide the hallway in δ amount by the direction of $-\mu_{t^*}$. Now the inner corner of the hallway is at $\mathbf{x}(t^*)$.

5. Continue the rotation where s is between $[t^*, \frac{\pi}{2}]$ by rotation path $\mathbf{x}$ in a similar manner to step 1.

Then, denote a new shape $S^*_{\mathbf{x}}$ by the intersections of the copies of the hallway being translated and rotated in the modified sequence.

$$\begin{aligned}
S^*_{\mathbf{x}} = L_{horiz} \cap \left(\mathbf{x}\left(\frac{\pi}{2}\right) + R_{\frac{\pi}{2}}(L_{vert})\right) &\cap \bigcap_{0\leq s\leq t} \Big(\mathbf{x}(s) + R_s(L)\Big) \\
&\cap \bigcap_{0\leq r\leq \delta} \Big(\mathbf{x}(t) + r\mu_t + R_t(L)\Big) \\
&\cap \bigcap_{t\leq s\leq t^*} \Big(\mathbf{x}(s) + \delta\mu_t + R_s(L)\Big) \\
&\cap \bigcap_{0\leq r\leq \delta} \Big(\mathbf{x}(t^*) + r\mu_t + R_{t^*}(L)\Big) \\
&\cap \bigcap_{t^*\leq s\leq \frac{\pi}{2}} \Big(\mathbf{x}(s) + R_s(L)\Big).
\end{aligned}$$

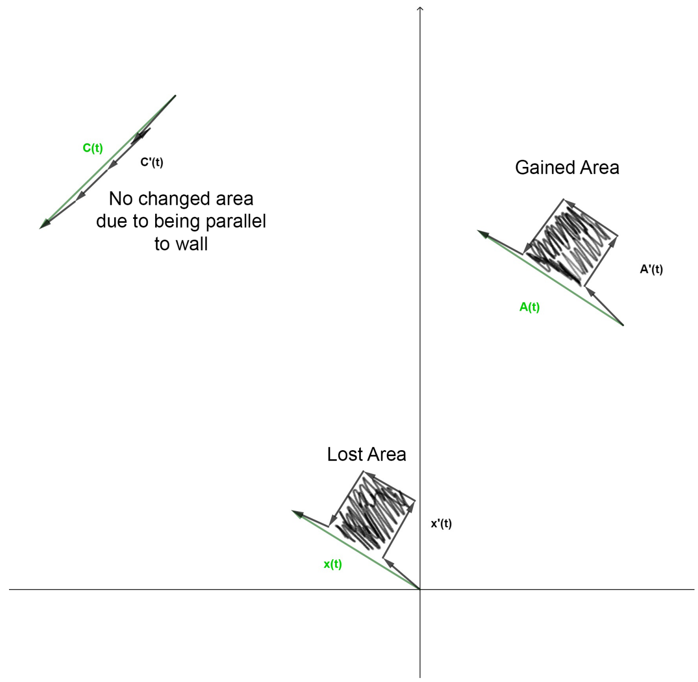

Figure 9: Comparing area between original and modified shape

If we compare our original shape, S_x, to $S^*_{\mathbf{x}}$, we can see something different. As illustrated in figure 9 near $\mathbf{x}(t)$, we lost some area. Near $\mathbf{A}(t)$, there is some area that is gained. And by $\mathbf{C}(t)$, nothing much has changed. Now, we need to calculate the two areas that we have.

Since the path that $\mathbf{x}(t)$ and $\mathbf{A}(t)$ took was slightly curved, it would be very complicated to calculate the area exactly. However, we did say that $\mathbf{x}(t)$ was well-behaved, so $\mathbf{x}(t)$ is twice continuously differentiable at t. If we take the derivative of $\mathbf{x}(t)$, the result would be a rectangular shape whose area is easier to calculate shown in figure 10

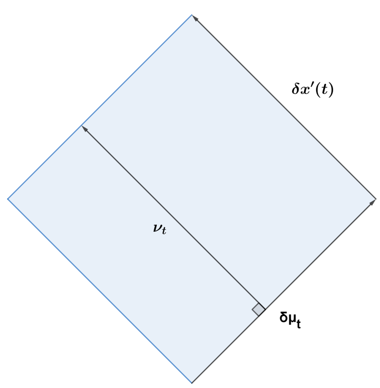

Figure 10: Area lost and area gained

In summary, we described that the area lost near $\mathbf{x}(t)$, area gained near $\mathbf{A}(t)$ can be approximated by the rectangle shown in figure 10. These visualizations are key to understanding the derivations of the system of differential equations. We want to answer the question: What conditions must a maximal well-behaved sofa require?

To continue our discussion, we will use this idea to derive the case where only $\mathbf{x}, \mathbf{A}$ and $\mathbf{C}$ are touching the walls of the hallway. This is Case 3 from Theorem 2 where $\Gamma_{\mathbf{x}}(t) = \{\mathbf{x}, \mathbf{A}, \mathbf{C}\}$. See figure 5 for a visual reference of the contact points. In the following derivation, we will use Little-o notation to simplify the computations of the area.

Definition 2 Let f and g be real valued functions. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then $f(x) = o(g(x))$. Then we can approximate the area lost near $\mathbf{x}(t)$ with

$$\begin{aligned}|\langle \delta\mathbf{x}'(t), \nu_t\rangle\delta| + o(\delta^2) &= |\langle \mathbf{x}'(t), \nu_t\rangle\delta^2| + o(\delta^2)\\ &= \langle \mathbf{x}'(t), \nu_t\rangle\delta^2 + o(\delta^2).\end{aligned}$$

Likewise, we can approximate the area gained by $\mathbf{A}(t)$ with

$$\begin{aligned}|\langle \delta\mathbf{A}'(t), \nu_t\rangle\delta| + o(\delta^2) &= |\langle \mathbf{A}'(t), \nu_t\rangle\delta^2| + o(\delta^2)\\ &= \langle \mathbf{A}'(t), \nu_t\rangle\delta^2 + o(\delta^2).\end{aligned}$$

Then,

$$\begin{aligned}\langle \mathbf{x}'(t), \nu_t\rangle\delta^2 - \langle \mathbf{A}'(t), \nu_t\rangle\delta^2 + o(\delta^2) &\geq 0\\ \langle \mathbf{x}'(t) - \mathbf{A}'(t), \nu_t\rangle\delta^2 + o(\delta^2) &\geq 0.\end{aligned}$$

We would then divide by δ^2 and let $\delta \to 0$ to get

$$\begin{aligned}\langle \mathbf{x}'(t) - \mathbf{A}'(t), \nu_t\rangle + \frac{o(\delta^2)}{\delta^2} &\geq 0\\ \langle \mathbf{x}'(t) - \mathbf{A}'(t), \nu_t\rangle &\geq 0.\end{aligned}$$

If we make the same argument but for the shift of $-\delta\mu_t$, we get $\langle \mathbf{x}'(t) - \mathbf{A}'(t), \nu_t\rangle \leq 0$ so we have

$$\langle \mathbf{x}'(t) - \mathbf{A}'(t), \nu_t\rangle = 0. \tag{13}$$

Then we will get the exact same argument for the shift in $\delta\nu_t$ and $-\delta\nu_t$,

$$\langle \mathbf{x}'(t) - \mathbf{C}'(t), \mu_t\rangle = 0. \tag{14}$$

From Theorem 1, we can take the derivative of $\mathbf{A}(t)$ and $\mathbf{C}(t)$ so that

$$\mathbf{A}'(t) = \mathbf{x}'(t) + \langle \mathbf{x}'', \mu_t\rangle\nu_t + \langle \mathbf{x}'(t), \nu_t\rangle\nu_t - \langle \mathbf{x}'(t), \mu_t\rangle\mu_t + \nu_t \tag{15}$$
$$\mathbf{C}'(t) = \mathbf{x}'(t) - \langle \mathbf{x}'', \nu_t\rangle\mu_t + \langle \mathbf{x}'(t), \mu_t\rangle\mu_t - \langle \mathbf{x}'(t), \nu_t\rangle\nu_t - \mu_t \tag{16}$$

Then, let's substitute these derivatives into (13) and (14) so that

$$\begin{aligned}\langle -\mathbf{A}'(t), \nu_t\rangle &= \langle -\langle \mathbf{x}''(t), \mu_t\rangle \nu_t, \nu_t\rangle \\ &= -\langle \mathbf{x}''(t), \mu_t\rangle\langle \nu_t, \nu_t\rangle \\ &= -\langle \mathbf{x}''(t), \mu_t\rangle - \langle \mathbf{x}'(t), \nu_t\rangle + \langle \mathbf{x}'(t), \mu_t\rangle\langle \mu_t, \nu_t\rangle - \langle \nu_t, \nu_t\rangle \\ &= -\langle \mathbf{x}''(t), \mu_t\rangle - \langle \mathbf{x}'(t), \nu_t\rangle - 1 = 0 \end{aligned} \tag{17}$$

and

$$\langle -\mathbf{C}'(t), \mu_t\rangle = \langle \mathbf{x}''(t), \nu_t\rangle - \langle \mathbf{x}'(t), \mu_t\rangle + 1 = 0. \tag{18}$$

From (17) and (18), we get a pair of two differential equations,

$$\langle \mathbf{x}''(t), \mu_t\rangle = -\langle \mathbf{x}'(t), \nu_t\rangle - 1 \tag{19}$$

$$\langle \mathbf{x}''(t), \nu_t\rangle = \langle \mathbf{x}'(t), \mu_t\rangle - 1 \tag{20}$$

These two differential equations form

$$\begin{aligned}\mathbf{x}''(t) &= (-\langle \mathbf{x}'(t), \nu_t\rangle - 1)\mu_t + (\langle \mathbf{x}'(t), \mu_t\rangle - 1)\nu_t \\ &= \langle \mathbf{x}'(t), -\nu_t\rangle \mu_t + \langle \mathbf{x}'(t), \mu_t\rangle \nu_t - \mu_t - \nu_t \\ &= \langle \mathbf{x}'(t), -\nu_t\rangle \begin{bmatrix}\cos t \\ \sin t\end{bmatrix} + \langle \mathbf{x}'(t), \mu_t\rangle \begin{bmatrix}-\sin t \\ \cos t\end{bmatrix} - \begin{bmatrix}\cos t \\ \sin t\end{bmatrix} - \begin{bmatrix}-\sin t \\ \cos t\end{bmatrix} \\ &= R_t\left(\begin{bmatrix}-1 \\ -1\end{bmatrix} + \begin{bmatrix}\sin t & -\cos t \\ \cos t & \sin t\end{bmatrix}\mathbf{x}'(t)\right). \end{aligned} \tag{21}$$

The differential equations for the five remaining cases from Theorem 2 can be derived by similar methods.

□

2.6 Solving the System of Differential Equations

Now that we have discussed the reasoning behind the derivation of the system of differential equations from **Theorem 2**, we will dive into solving one of the ODE introduced in **Theorem 2**.

Consider the Differential Equation in Case 1 where $\Gamma_x(t) = \{\mathbf{A}, \mathbf{C}, \mathbf{D}\}$:

$$\mathbf{x}''(t) = R_t\left(\begin{bmatrix}-1 \\ -1/2\end{bmatrix} + \begin{bmatrix}2\sin t & -2\cos t \\ 2\cos t & 2\sin t\end{bmatrix}\mathbf{x}'(t)\right).$$

Recall that

$$R_t = \begin{bmatrix}\cos t & -\sin t \\ \sin t & \cos t\end{bmatrix}$$

and

$$R_{-t} = \begin{bmatrix}\cos t & \sin t \\ -\sin t & \cos t\end{bmatrix}.$$

We use the substitution $\mathbf{y}(t) = R_{-t}\mathbf{x}'(t)$ to aid in the integration of the differential equations for each case. By the product rule,

$$\begin{aligned}
\mathbf{y}'(t) =& \frac{d}{dt}[R_{-t}\mathbf{x}'(t)] \\
=& \frac{d}{dt}[R_{-t}]\mathbf{x}'(t) + R_{-t}\frac{d}{dt}[\mathbf{x}'(t)] \\
=& \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix}\mathbf{x}'(t) + R_{-t}\mathbf{x}''(t) \\
=& \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix}\mathbf{x}'(t) + R_{-t}R_t\left(\begin{bmatrix} -1 \\ -1/2 \end{bmatrix} + \begin{bmatrix} 2\sin t & -2\cos t \\ 2\cos t & 2\sin t \end{bmatrix}\mathbf{x}'(t)\right) \\
=& \begin{bmatrix} -1 \\ -1/2 \end{bmatrix} + \begin{bmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{bmatrix}\mathbf{x}'(t).
\end{aligned}$$

Additionally, rearranging the substitution $\mathbf{y}(t) = R_{-t}\mathbf{x}'(t)$, we have $x'(t) = R_t\mathbf{y}(t)$. This yields

$$\begin{aligned}
\mathbf{y}'(t) =& \begin{bmatrix} -1 \\ -1/2 \end{bmatrix} + \begin{bmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{bmatrix}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\mathbf{y}(t) \\
=& \begin{bmatrix} -1 \\ -1/2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{y}(t).
\end{aligned}$$

Notice the form $\mathbf{y}'(t) = v + T\mathbf{y}(t)$ where v is a constant vector and T is a constant 2x2 matrix. The general solution, y_g, can be found by solving for the eigenvectors of the homogenous equation $\mathbf{y}'(t) = T\mathbf{y}(t)$. Then, we will add the particular solution found using the method of undetermined coefficients.

To find the eigenvalues, we use
$\det(T - \lambda I) = 0$ results in $\lambda = \pm i$.

In this case, the corresponding *eigenvectors* are $\begin{bmatrix} i \\ 1 \end{bmatrix}$, $\begin{bmatrix} -i \\ 1 \end{bmatrix}$. Recall that the form of the general solution, $\mathbf{y_g}(t)$, for the differential equation must be

$$\mathbf{y_g}(t) = \gamma_1 e^{it}\begin{bmatrix} i \\ 1 \end{bmatrix} + \gamma_2 e^{-it}\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

for constants γ_1, γ_2. Using **Euler's Formula** $e^{ix} = \cos x + i\sin x$, we substitute this relation for the general solution, $\mathbf{y_g}(t)$. Rearranging we have

$$\mathbf{y_g}(t) = \begin{bmatrix} b_1\cos t + b_2\sin t \\ -b_2\cos t + b_1\sin t \end{bmatrix}$$

Where b_1, b_2 are constants. Now that we have the general solution, we will find the particular solution. Note the form of

$$\mathbf{y_p} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where $\mathbf{y_p}$ is a constant vector. Solving

$$\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}-1\\-1/2\end{bmatrix} + \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$$

we have $\mathbf{y_p} = \begin{bmatrix}1/2\\-1\end{bmatrix}$.

Adding the general solution to the particular solution we have

$$\mathbf{y}(t) = \begin{bmatrix}b_1\cos t + b_2\sin t + 1/2\\-b_2\cos t + b_1\sin t - 1\end{bmatrix}.$$

Then using our substitution for $\mathbf{x}'(t) = R_t\mathbf{y}(t)$ we get

$$\mathbf{x}'(t) = \begin{bmatrix}\cot t & -\sin t\\ \sin t & \cos t\end{bmatrix}\begin{bmatrix}b_1\cos t + b_2\sin t + 1/2\\-b_2\cos t + b_1\sin t - 1\end{bmatrix}.$$

Simplifying we have

$$\mathbf{x}'(t) = \begin{bmatrix}b_2\sin 2t + b_1\cos 2t + (1/2)\cos t + \sin t\\b_1\sin 2t - b_2\cos 2t + (1/2)\sin t - \cos t\end{bmatrix}.$$

To simplify, we replace the arbitrary constants $b_1/2 = a_1$ and $b_2/2 = a_2$. Integrating and rearranging, we have

$$\mathbf{x}(t) = \begin{bmatrix}a_1\sin 2t - a_2\cos 2t + (1/2)\sin t - \cos t\\-a_2\sin 2t - a_1\cos 2t - (1/2)\cos t - \sin t\end{bmatrix}$$

Using trigonometric identities[3] we can recognize this as the product.

$$\mathbf{x}(t) = R_t\begin{bmatrix}a_1\sin t - a_2\cos t - 1\\-a_1\cos t - a_2\sin t - 1/2\end{bmatrix} + \kappa_1.$$

Now we have a solution to the differential equation derived in Case 3. Solutions for Cases 2-6 can be shown in a similar way. For each case, we use the substitution $\mathbf{y}(t) = R_{-t}\mathbf{x}'(t)$. Then solve for the general and particular solutions. Combining all of these solutions, we can talk about the path of $\mathbf{x}(t)$ explicitly for each case.

$$\mathbf{x}_1(t) = R_t\begin{bmatrix}a_1\cos t + a_2\sin t - 1\\-a_2\cos t + a_1\sin t - 1/2\end{bmatrix} + \kappa_1 \tag{22}$$

$$\mathbf{x}_2(t) = R_t\begin{bmatrix}-\frac{1}{4}t^2 + b_1 t + b_2\\\frac{1}{2}t - b_1 - 1\end{bmatrix} + \kappa_2 \tag{23}$$

$$\mathbf{x}_3(t) = R_t\begin{bmatrix}c_1 - t\\c_2 + t\end{bmatrix} + \kappa_3 \tag{24}$$

$$\mathbf{x}_4(t) = R_t\begin{bmatrix}-\frac{1}{2}t + d_1 - 1\\-\frac{1}{4}t^2 + d_1 t + d_2\end{bmatrix} + \kappa_4 \tag{25}$$

[3] Sum of angles formula

$$\mathbf{x}_5(t) = R_t \begin{bmatrix} e_1 \cos t + e_2 \sin t - 1/2 \\ -e_2 \cos t + e_1 \sin t - 1/2 \end{bmatrix} + \kappa_5 \tag{26}$$

$$\mathbf{x}_6(t) = R_t \begin{bmatrix} f_1 \cos(t/2) + f_2 \sin(t/2) - 1 \\ -f_2 \cos(t/2) + f_1 \sin(t/2) - 1 \end{bmatrix} + \kappa_6 \tag{27}$$

where $a_i, b_i, c_i, d_i, e_i, f_i$ for $i = 1, 2$ and κ_j, a constant vector in $\mathbb{R}^2$ for the j^{th} case are arbitrary constants. [3]

2.7 Constants in Gerver's Sofa

Now that we have shown the process for solving the system of differential equations, we need to derive the relationship between the unknown constants. Given that we have the left to right symmetry seen on Gerver's sofa across the line $t - \pi/2$, we will show the cases that mirror one another across $t - \pi/2$ are related. This symmetry condition can be expressed by the following

$$\mathbf{x}'(\pi/2 - t) \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}'(t) \tag{28}$$

From this relation, we can derive more specific relationships between the constants in Gerver's exact sofa.

We will derive the relationship between the constants in solution for case for Case 3 $\Gamma_{\mathbf{x}}(t) = \{\mathbf{x}, \mathbf{A}, \mathbf{C}\}$ as listed in . Let $\mathbf{x_3}(t)$ denote the path of $\mathbf{x}(t)$ restricted to Case 3 when $\Gamma_{\mathbf{x}}(t) = \{x, A, C\}$. The reader may verify that

$$\mathbf{x_3'}(t) = \begin{bmatrix} -\sin t(c_1 - t) - \cos t(c_2 + t) - \cos t - \sin t \\ \cos t(c_1 - t) - \sin t(c_2 + t) - \sin t + \cos t \end{bmatrix}$$

and

$$\mathbf{x_3'}(\pi/2 - t) = \begin{bmatrix} -\cos t(c_1 - \pi/2 + t) - \sin t(c_2 + \pi/2 - t) - \sin t - \cos t \\ \sin t(c_1 - \pi/2 + t) - \cos t(c_2 - \pi/2 - t) + \sin t - \cos t \end{bmatrix}.$$

Specifically our symmetry condition says

$$\mathbf{x_3'}(\pi/2 - t) \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x_3'}(\mathbf{t})$$

Equivalently, we can evaluate

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x_3'}(\pi/2 - t) \equiv \mathbf{x_3'}(\mathbf{t}).$$

So,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x_3'}(\pi/2 - t) \equiv \begin{bmatrix} -\sin t(c_1 - t) - \cos t(c_2 + t) - \cos t - \sin t \\ \cos t(c_1 - t) - \sin t(c_2 + t) - \sin t + \cos t \end{bmatrix}$$

$$\begin{bmatrix} -\cos t(c_1 - \pi/2 + t) - \sin t(c_2 + \pi/2 - t) - \sin t - \cos t \\ -\sin t(c_1 - \pi/2 + t) + \cos t(c_2 - \pi/2 - t) - \sin t + \cos t \end{bmatrix} \equiv \begin{bmatrix} -\sin t(c_1 - t) - \cos t(c_2 + t) - \cos t - \sin t \\ \cos t(c_1 - t) - \sin t(c_2 + t) - \sin t + \cos t \end{bmatrix}$$

Looking at the coefficients of the sine and cosine functions of the matrix, we can see the relation $c_2 = c_1 - \pi/2$. Similarly, for the rest of the solutions to the system of six differential equations, we can use this symmetry condition to establish relations between constants. Let ρ be the angle at which the sofa transitions from Case 1 to Case 2. Let θ be the angle at which the sofa transitions from Case 2 to Case 3. Then $\pi/2 - \theta$ is the angle at which the sofa transitions from Case 3 to Case 4, and $\pi/2 - \rho$ is the angle at which the sofa transitions from Case 4 to Case 5.

$$\mathbf{x}(t) = \begin{cases} \mathbf{x_1}(t) & \text{if } 0 \leq t < \rho, \\ \mathbf{x_2}(t) & \text{if } \rho \leq t < \theta, \\ \mathbf{x_3}(t) & \text{if } \theta \leq t \leq \pi/2 - \theta, \\ \mathbf{x_4}(t) & \text{if } \pi/2 - \theta < t \leq \pi/2 - \rho, \\ \mathbf{x_5}(t) & \text{if } \pi/2 - \rho < t \leq \pi/2. \end{cases}$$

From (28) we obtain the following relations

$$e_1 = a_1 \tag{29}$$

$$e_2 = -a_2 \tag{30}$$

$$d_1 = \frac{\pi}{4} - b_1 \tag{31}$$

$$d_2 = b_2 + \frac{\pi}{4}\left(2b_1 - \frac{\pi}{4}\right) \tag{32}$$

$$c_2 = c_1 - \frac{\pi}{2} \tag{33}$$

From the assumption that $\mathbf{x}(t)$ is twice continuously differentiable, we have the following relations

$$\mathbf{x_1}(\rho) = \mathbf{x_2}(\rho), \tag{34}$$

$$\mathbf{x'_1}(\rho) = \mathbf{x'_2}(\rho), \tag{35}$$

$$\mathbf{x_2}(\theta) = \mathbf{x_3}(\theta), \tag{36}$$

$$\mathbf{x'_2}(\theta) = \mathbf{x'_3}(\theta), \tag{37}$$

$$\mathbf{x_3}(\pi/2 - \theta) = \mathbf{x_4}(\pi/2 - \theta), \tag{38}$$

$$\mathbf{x'_3}(\pi/2 - \theta) = \mathbf{x'_4}(\pi/2 - \theta), \tag{39}$$

$$\mathbf{x_4}(\pi/2 - \rho) = \mathbf{x_5}(\pi/2 - \rho), \tag{40}$$

$$\mathbf{x'_4}(\pi/2 - \rho) = \mathbf{x'_5}(\pi/2 - \rho), \tag{41}$$

As a result of combining the relations between the constants and the symmetry conditions, we have a system of 28 equations with 22 unknowns. As referenced in Romik's paper, this system can be solved using the computer algebra system Mathematica using the "FindRoot[...]" command [3]

3 Bedroom Problem

We were inspired to investigate a new variation on the moving sofa problem. We propose a new question: What is the bed with maximal area which can move into an ℓ by m bedroom.

3.1 Set-Up

For the bedroom problem, we are assuming that the room is a rectangle where it is ℓ units long and m units wide. There is a door on one of the walls of the room where it can allow one unit wide shapes through. A shape must be able to go through the door and into the room where it does not pass through any of the walls.

3.2 Parallelograms

Proposition 1. *The maximal area for a parallelogram bed in an $\ell \times m$ room with a door width of one unit is ℓ square units.*

Proof. The area A of a parallelogram is defined by $A = bh$ where b is the base and h is the height. The maximum base of a parallelogram bed in the $\ell \times m$ bedroom is one unit. Since the parallelogram must be able to fit inside the room, its maximum height is ℓ. Therefore, the maximum area of any parallelogram bed is ℓ square units. □

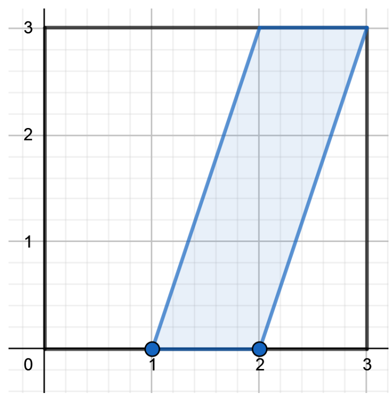

Figure 11: The largest parallelogram in a 3x3 room with center door

Given *Proposition* 1, we know if we want to construct a bed larger than the $\ell \times 1$ rectangle, we will need to introduce non-linear beds. First we investigated combining sectors of circles with rectangles.

3.3 Geometric Examples

Our best geometric example is a 1×1 square attached to a quarter circle with radius 1 joined with a 1×2 rectangle shown below for a 3×3 room. We have labeled the square x, the quarter circle y, and the rectangle z. We propose this is the largest shape constructed of simple geometric figures.

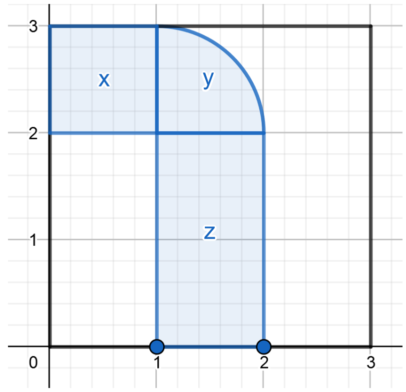

Figure 12: Our current largest shape in a 3 by 3 room

From this example, we can extrapolate and try to generalize the bedroom we are in. To do a generalization, we should look at other examples of different sized bedrooms and observe how the bed shape changes. From figure 13, we have a 5×3 and 3×2 bedroom where the doorway is on the left.

First off, since the doorway was on the left, the x square would be on the right side of the shape and vice versa. Next, the height of the shape depends on ℓ in which the height increases as ℓ increases due to the fact that the height equals ℓ.

For m, it depends on the placement of the doorway. If the doorway was on the left or right side of the bedroom, values of m that are bigger than 2 would cause the square x to become a rectangle to fill the added width of the bedroom. Then, as the value of m is less than 2, the square x becomes more squished until as $m = 1$, the square x disappears in which the semicircle y transforms into a square, where the our new bed shape fills the whole room.

If the doorway was in the middle of the room, values of m that are bigger than 3 would also have the square x transform into a rectangle to fill that empty space. However, the rectangle wouldn't be able to fill the other side of the room, meaning that there is still some empty space. When we reduce the value of m, the side with the empty space moves towards the shape until we hit $m = 2$ where it touches it.

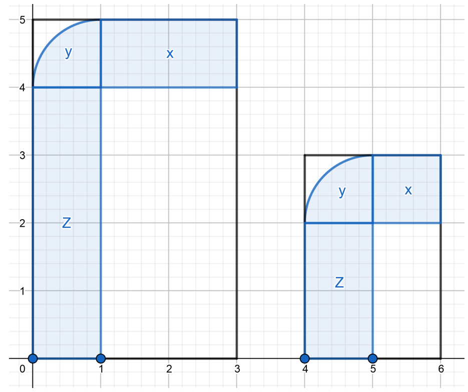

Figure 13: A 5×3 and 3×2 room side by side with left doorway

3.4 Link Between the Couch and Bedroom

During our investigation on our bedroom problem, we found a connection between it and the original sofa problem. If we have the original hallway, we can take out the left vertical wall which would make the hallway into a similar look to a bedroom. Then, Gerver's sofa is still able to go through the hallway as seen in figure 14. However, due to that wall being removed, the contact point $\mathbf{B}(t)$ no longer exists. Therefore, our theorems could be rewritten to not include that contact point. So, that would mean that theorem 2 would have a reduced number of cases due to the removal of cases that included $\mathbf{B}(t)$ before. Possibly, there could be an increase of the area for this shape for this variation on the hallway.

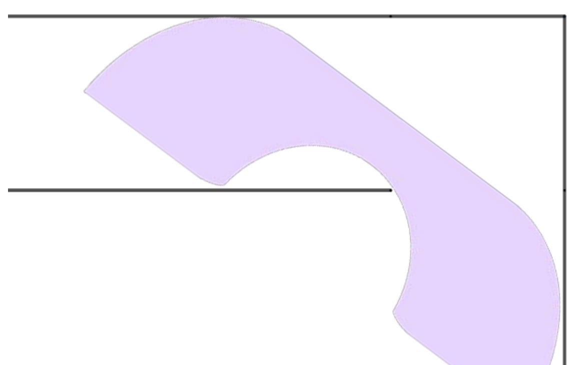

Figure 14: Gerver's sofa fitting through the modified hallway

4 Future Investigations

So far, we explained the exact solutions of the sofa with maximal area for the original moving sofa question. We have bridged gaps in effort to make the derivation of the exact solutions for the more accessible to students who have only taken introductory courses in differential equations and linear algebra. To extend the original moving sofa problem, we asked the bedroom question. Clearly, there is still work to do as far as exact solutions to the bedroom problems, which may be an adventure to the reader.

Additional future investigations may include restrictions of the bedroom problem. For instance the reader may consider a bedroom of height 1 with a doorway on the far right with an arbitrary length. This clearly resembles the original moving sofa problem, to which many of the computations above may be applied and extended.

5 Acknowledgement

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