Multiple solutions for 1-D quasilinear indefinite Schrödinger equations

Shibo Liu

Florida Institute of Technology

http://lausb.github.io

sliu@fit.edu

Contents

1. Introduction

To find standing waves $\psi(t,x) = e^{-i\omega t}u(x)$ of QL Schrödinger equation

$$i\partial_t \psi = -\Delta\psi + U(x) - \psi\Delta(|\psi|^2) - \bar{g}(|\psi|^2)\psi \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N,$$

1. Introduction

To find standing waves $\psi(t,x) = e^{-i\omega t}u(x)$ of QL Schrödinger equation

$$i\partial_t\psi = -\Delta\psi + U(x) - \psi\Delta(|\psi|^2) - \bar{g}(|\psi|^2)\psi \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N,$$

we need to solve ($V = U - \omega$)

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad u \in H^1(\mathbb{R}^N), \tag{1}$$

initiated by Poppenberg et al. (2002) for $N = 1$.

1. Introduction

To find standing waves $\psi(t,x) = e^{-i\omega t}u(x)$ of QL Schrödinger equation

$$i\partial_t\psi = -\Delta\psi + U(x) - \psi\Delta(|\psi|^2) - \bar{g}(|\psi|^2)\psi \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N,$$

we need to solve $(V = U - \omega)$

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad u \in H^1(\mathbb{R}^N), \tag{1}$$

initiated by Poppenberg et al. (2002) for $N = 1$. Solutions are critical pts of

$$J(u) = \frac{1}{2}\int\left(1 + 2u^2\right)|\nabla u|^2 + \frac{1}{2}\int V(x)u^2 - \int G(u),$$

which is not well-defined ($J(u) = \infty$ for some $u \in H^1(\mathbb{R}^N)$) unless $N = 1$.

1. Introduction

To find standing waves $\psi(t,x)=e^{-i\omega t}u(x)$ of QL Schrödinger equation

$$i\partial_t\psi=-\Delta\psi+U(x)-\psi\Delta(|\psi|^2)-\bar{g}(|\psi|^2)\psi \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^N,$$

we need to solve ($V=U-\omega$)

$$-\Delta u+V(x)u-u\Delta(u^2)=g(u), \qquad u\in H^1(\mathbb{R}^N), \tag{1}$$

initiated by Poppenberg et al. (2002) for $N=1$. Solutions are critical pts of

$$J(u)=\frac{1}{2}\int\left(1+2u^2\right)|\nabla u|^2+\frac{1}{2}\int V(x)u^2-\int G(u),$$

which is not well-defined ($J(u)=\infty$ for some $u\in H^1(\mathbb{R}^N)$) unless $N=1$.
Liu et al. (2003); Colin & Jeanjean (2004) introduced a nonlinear transform $u=f(v)$ so that $\Phi:H^1(\mathbb{R}^N)\to\mathbb{R}$,

$$\Phi(v)=\frac{1}{2}\int\left(|\nabla v|^2+V(x)f^2(v)\right)-\int G(f(v))$$

is well defined,

1. Introduction

To find standing waves $\psi(t,x) = e^{-i\omega t}u(x)$ of QL Schrödinger equation

$$i\partial_t\psi = -\Delta\psi + U(x) - \psi\Delta(|\psi|^2) - \bar{g}(|\psi|^2)\psi \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N,$$

we need to solve ($V = U - \omega$)

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad u \in H^1(\mathbb{R}^N), \tag{1}$$

initiated by Poppenberg et al. (2002) for $N = 1$. Solutions are critical pts of

$$J(u) = \frac{1}{2}\int \left(1 + 2u^2\right)|\nabla u|^2 + \frac{1}{2}\int V(x)u^2 - \int G(u),$$

which is not well-defined ($J(u) = \infty$ for some $u \in H^1(\mathbb{R}^N)$) unless $N = 1$.
Liu et al. (2003); Colin & Jeanjean (2004) introduced a nonlinear transform $u = f(v)$ so that $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$,

$$\Phi(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x)f^2(v)\right) - \int G(f(v))$$

is well defined, if $\Phi'(v) = 0$ then $u = f(v)$ solves (1).
Then many results for $\inf V > 0$ appear.

1. Introduction

To find standing waves $\psi(t,x) = e^{-i\omega t}u(x)$ of QL Schrödinger equation

$$i\partial_t\psi = -\Delta\psi + U(x) - \psi\Delta(|\psi|^2) - \bar{g}(|\psi|^2)\psi \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N,$$

we need to solve $(V = U - \omega)$

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad u \in H^1(\mathbb{R}^N), \tag{1}$$

initiated by Poppenberg et al. (2002) for $N = 1$. Solutions are critical pts of

$$J(u) = \frac{1}{2}\int \left(1 + 2u^2\right)|\nabla u|^2 + \frac{1}{2}\int V(x)u^2 - \int G(u),$$

which is not well-defined ($J(u) = \infty$ for some $u \in H^1(\mathbb{R}^N)$) unless $N = 1$.
Liu et al. (2003); Colin & Jeanjean (2004) introduced a nonlinear transform $u = f(v)$ so that $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$,

$$\Phi(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x)f^2(v)\right) - \int G(f(v))$$

is well defined, if $\Phi'(v) = 0$ then $u = f(v)$ solves (1).
Then many results for $\inf V > 0$ appear. Until Shen & Han (2015) for $N = 1$ and Liu & Zhou (2018) for $N \geq 1$, no results if $-\Delta + V$ is indefinite.

In Liu & Zhou (2018): $g(t) \approx |t|^{p-2}\, t$ for some $p \in (4, 2 \cdot 2^*)$.

In Liu & Zhou (2018): $g(t) \approx |t|^{p-2}t$ for some $p \in (4, 2 \cdot 2^*)$. It is crucial that

$$\lim_{|t|\to\infty} V(x) = +\infty, \tag{2}$$

so that the working space

$$E = \left\{ u \in H^1(\mathbb{R}^N) \,\middle|\, \|u\| = \left(\int \left(|\nabla u|^2 + V u^2 \right) \right)^{1/2} < \infty \right\} \hookrightarrow\hookrightarrow L^2(\mathbb{R}^N).$$

See also Silva & Silva (2019).

In Liu & Zhou (2018): $g(t) \approx |t|^{p-2}t$ for some $p \in (4, 2 \cdot 2^*)$. It is crucial that

$$\lim_{|t|\to\infty} V(x) = +\infty, \tag{2}$$

so that the working space

$$E = \left\{ u \in H^1(\mathbb{R}^N) \,\middle|\, \|u\| = \left(\int \left(|\nabla u|^2 + Vu^2 \right) \right)^{1/2} < \infty \right\} \hookrightarrow\hookrightarrow L^2(\mathbb{R}^N).$$

See also Silva & Silva (2019).

If $N = 1$ we can study the case that $|V|_\infty < \infty$.

In Liu & Zhou (2018): $g(t) \approx |t|^{p-2}\, t$ for some $p \in (4, 2\cdot 2^*)$. It is crucial that

$$\lim_{|t|\to\infty} V(x) = +\infty, \tag{2}$$

so that the working space

$$E = \left\{ u \in H^1(\mathbb{R}^N) \,\middle|\, \|u\| = \left(\int \left(|\nabla u|^2 + V u^2 \right) \right)^{1/2} < \infty \right\} \hookrightarrow\hookrightarrow L^2(\mathbb{R}^N).$$

See also Silva & Silva (2019).

If $N = 1$ we can study the case that $|V|_\infty < \infty$. We consider

$$-u'' + V(x)u - (u^2)''u = f(x,u), \qquad u \in H^1(\mathbb{R}).$$

In Liu & Zhou (2018): $g(t) \approx |t|^{p-2}\, t$ for some $p \in (4, 2\cdot 2^*)$. It is crucial that

$$\lim_{|t|\to\infty} V(x) = +\infty, \tag{2}$$

so that the working space

$$E = \left\{ u \in H^1(\mathbb{R}^N) \,\middle|\, \|u\| = \left(\int \left(|\nabla u|^2 + V u^2 \right) \right)^{1/2} < \infty \right\} \hookrightarrow\hookrightarrow L^2(\mathbb{R}^N).$$

See also Silva & Silva (2019).

If $N = 1$ we can study the case that $|V|_\infty < \infty$. We consider

$$-u'' + V(x)u - (u^2)''u = f(x,u), \qquad u \in H^1(\mathbb{R}). \tag{3}$$

Assume $V \in C(\mathbb{R})$ is bounded from below. Let

$$\lambda_n = \inf_{X\in\mathscr{X}_n} \sup_{u\in X\setminus\{0\}} \frac{\int \left(\dot{u}^2 + V(x)u^2 \right)}{\int u^2},$$

where $\mathscr{X}_n$ is the collection of all n-dimensional subspaces of $C_0^\infty(\mathbb{R})$.

In Liu & Zhou (2018): $g(t) \approx |t|^{p-2}\, t$ for some $p \in (4, 2\cdot 2^*)$. It is crucial that

$$\lim_{|t|\to\infty} V(x) = +\infty, \tag{2}$$

so that the working space

$$E = \left\{ u \in H^1(\mathbb{R}^N) \,\middle|\, \|u\| = \left(\int \left(|\nabla u|^2 + V u^2 \right) \right)^{1/2} < \infty \right\} \hookrightarrow\hookrightarrow L^2(\mathbb{R}^N).$$

See also Silva & Silva (2019).

If $N = 1$ we can study the case that $|V|_\infty < \infty$. We consider

$$-u'' + V(x)u - (u^2)''u = f(x,u), \qquad u \in H^1(\mathbb{R}). \tag{3}$$

Assume $V \in C(\mathbb{R})$ is bounded from below. Let

$$\lambda_n = \inf_{X\in\mathscr{X}_n} \sup_{u\in X\setminus\{0\}} \frac{\int \left(\dot{u}^2 + V(x)u^2 \right)}{\int u^2},$$

where $\mathscr{X}_n$ is the collection of all n-dimensional subspaces of $C_0^\infty(\mathbb{R})$. Assume

$$\lambda_n \to \lambda_\infty < \infty,$$

then λ_∞ is the bottom of the essential spectrum of $S = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$

In Liu & Zhou (2018): $g(t) \approx |t|^{p-2}t$ for some $p \in (4, 2\cdot 2^*)$. It is crucial that

$$\lim_{|t|\to\infty} V(x) = +\infty, \tag{2}$$

so that the working space

$$E = \left\{ u \in H^1(\mathbb{R}^N) \,\middle|\, \|u\| = \left(\int \left(|\nabla u|^2 + V u^2 \right) \right)^{1/2} < \infty \right\} \hookrightarrow\hookrightarrow L^2(\mathbb{R}^N).$$

See also Silva & Silva (2019).

If $N = 1$ we can study the case that $|V|_\infty < \infty$. We consider

$$-u'' + V(x)u - (u^2)''u = f(x,u), \qquad u \in H^1(\mathbb{R}). \tag{3}$$

Assume $V \in C(\mathbb{R})$ is bounded from below. Let

$$\lambda_n = \inf_{X \in \mathscr{X}_n} \sup_{u \in X\setminus\{0\}} \frac{\int \left(\dot{u}^2 + V(x)u^2 \right)}{\int u^2},$$

where $\mathscr{X}_n$ is the collection of all n-dimensional subspaces of $C_0^\infty(\mathbb{R})$. Assume

$$\lambda_n \to \lambda_\infty < \infty,$$

then λ_∞ is the bottom of the essential spectrum of $S = -\frac{d^2}{dx^2} + V$ and $\lambda_n < \lambda_\infty$ implies that λ_n is an eigenvalue of S of finite multiplicity.

We assume

(V_1) $V \in C(\mathbb{R})$ is such that $0 \in (\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$.

We assume

(V_1) $V \in C(\mathbb{R})$ is such that $0 \in (\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$.

(f_1) $f \in C(\mathbb{R} \times \mathbb{R})$ and there are constants $p > 2$ and $c > 0$ such that

$$|f(x,t)| \leq c\left(1 + |t|^{p-1}\right) \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}.$$

We assume

(V_1) $V \in C(\mathbb{R})$ is such that $0 \in (\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$.

(f_1) $f \in C(\mathbb{R} \times \mathbb{R})$ and there are constants $p > 2$ and $c > 0$ such that

$$|f(x,t)| \le c\left(1 + |t|^{p-1}\right) \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}.$$

(f_2) $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}$.

We assume

(V_1) $V \in C(\mathbb{R})$ is such that $0 \in (\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$.

(f_1) $f \in C(\mathbb{R} \times \mathbb{R})$ and there are constants $p > 2$ and $c > 0$ such that

$$|f(x,t)| \le c\left(1 + |t|^{p-1}\right) \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}.$$

(f_2) $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}$.

(f_3^*) There exists $h \in (0, \lambda_\infty)$ such that $tf(x,t) \le ht^2$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$.

We assume

(V_1) $V \in C(\mathbb{R})$ is such that $0 \in (\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$.

(f_1) $f \in C(\mathbb{R} \times \mathbb{R})$ and there are constants $p > 2$ and $c > 0$ such that

$$|f(x,t)| \le c\left(1 + |t|^{p-1}\right) \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}.$$

(f_2) $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}$.

(f_3^*) There exists $h \in (0, \lambda_\infty)$ such that $tf(x,t) \le ht^2$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$.

Increasing h a little bit we may assume that h in (f_3^*) is not an eigenvalue.

Thm 1. Suppose (V_1), (f_1), (f_2) and (f_3^*) are satisfied, then (3) has at least two nontrivial solutions.

We assume

(V_1) $V \in C(\mathbb{R})$ is such that $0 \in (\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$.

(f_1) $f \in C(\mathbb{R} \times \mathbb{R})$ and there are constants $p > 2$ and $c > 0$ such that

$$|f(x,t)| \le c\left(1 + |t|^{p-1}\right) \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}.$$

(f_2) $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}$.

(f_3^*) There exists $h \in (0, \lambda_\infty)$ such that $tf(x,t) \le ht^2$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$.

Increasing h a little bit we may assume that h in (f_3^*) is not an eigenvalue.

Thm 1. Suppose (V_1), (f_1), (f_2) and (f_3^*) are satisfied, then (3) has at least two nontrivial solutions. If in addition $f(x,\cdot)$ is odd for all $x \in \mathbb{R}$, then (3) has k pairs of nontrivial solutions.

We assume

(V_1) $V \in C(\mathbb{R})$ is such that $0 \in (\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$.

(f_1) $f \in C(\mathbb{R} \times \mathbb{R})$ and there are constants $p > 2$ and $c > 0$ such that
$$|f(x,t)| \le c\left(1 + |t|^{p-1}\right) \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}.$$

(f_2) $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}$.

(f_3^*) There exists $h \in (0, \lambda_\infty)$ such that $tf(x,t) \le ht^2$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$.

Increasing h a little bit we may assume that h in (f_3^*) is not an eigenvalue.

Thm 1. Suppose (V_1), (f_1), (f_2) and (f_3^*) are satisfied, then (3) has at least two nontrivial solutions. If in addition $f(x,\cdot)$ is odd for all $x \in \mathbb{R}$, then (3) has k pairs of nontrivial solutions.

Rem 1. Under (V_1), (f_1), (f_2) and the following condition weaker than (f_3^*):

(f_3) There exists $h \in (0, \lambda_\infty)$ such that $F(x,t) \le \frac{1}{2}ht^2$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$,

a nontrivial solution is obtained by Wang & Yang (2015).

Thm1 is motivated by Chen & Wang (2014) who obtained similar results for

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

Thm1 is motivated by Chen & Wang (2014) who obtained similar results for

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{4}$$

Unlike our case, weak limits of (PS) sequences of

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int \phi_u u^2 - \int F(x,u), \quad \phi_u(x) = \frac{1}{4\pi}\int \frac{u^2(y)}{|x-y|}\mathrm{d}y$$

are critical points of Φ.

Thm1 is motivated by Chen & Wang (2014) who obtained similar results for

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{4}$$

Unlike our case, weak limits of (PS) sequences of

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int \phi_u u^2 - \int F(x,u), \quad \phi_u(x) = \frac{1}{4\pi}\int \frac{u^2(y)}{|x-y|}\mathrm{d}y$$

are critical points of Φ.

Lacking this property is the reason that Wang & Yang (2015) could not get multiplicity results for (3) as in Thm1.

Thm 1 is motivated by Chen & Wang (2014) who obtained similar results for

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{4}$$

Unlike our case, weak limits of (PS) sequences of

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int \phi_u u^2 - \int F(x,u), \quad \phi_u(x) = \frac{1}{4\pi}\int \frac{u^2(y)}{|x-y|}\mathrm{d}y$$

are critical points of Φ.

Lacking this property is the reason that Wang & Yang (2015) could not get multiplicity results for (3) as in Thm 1.

Rem 2. Initiated by Benci & Fortunato (1998), (4) attracted great interest for definite case in which $u = 0$ is loc min of Φ.

Thm 1 is motivated by Chen & Wang (2014) who obtained similar results for

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{4}$$

Unlike our case, weak limits of (PS) sequences of

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int \phi_u u^2 - \int F(x,u), \quad \phi_u(x) = \frac{1}{4\pi}\int \frac{u^2(y)}{|x-y|}\mathrm{d}y$$

are critical points of Φ.

Lacking this property is the reason that Wang & Yang (2015) could not get multiplicity results for (3) as in Thm 1.

Rem 2. Initiated by Benci & Fortunato (1998), (4) attracted great interest for definite case in which $u = 0$ is loc min of Φ.

The first work on the indefinite case that $u = 0$ is saddle pt is due to Chen & Liu (2015).

Thm 1 is motivated by Chen & Wang (2014) who obtained similar results for

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{4}$$

Unlike our case, weak limits of (PS) sequences of

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int \phi_u u^2 - \int F(x,u), \quad \phi_u(x) = \frac{1}{4\pi}\int \frac{u^2(y)}{|x-y|}\mathrm{d}y$$

are critical points of Φ.

Lacking this property is the reason that Wang & Yang (2015) could not get multiplicity results for (3) as in Thm 1.

Rem 2. Initiated by Benci & Fortunato (1998), (4) attracted great interest for definite case in which $u = 0$ is loc min of Φ.

The first work on the indefinite case that $u = 0$ is saddle pt is due to Chen & Liu (2015).

Motivated by Liu & Wu (2017) on indefinite problem (4), we consider (3) when $f(x,\cdot)$ is 4-superlinear:

$$\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = +\infty \quad \text{a.e. } x \in \mathbb{R}, \qquad \text{where } F(x,t) = \int_0^t f(x,\cdot).$$

(f_4) $0 < 4F(x,t) \le tf(x,t)$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$,

$$\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = +\infty \quad \text{a.e. } x \in \mathbb{R}.$$

(f_4) $0 < 4F(x,t) \leq tf(x,t)$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$,

$$\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = +\infty \quad \text{a.e. } x \in \mathbb{R}. \tag{5}$$

(f_5) if $u_n \rightharpoonup u$ in $H^1(\mathbb{R})$, then $\overline{\lim_{n\to\infty}} \int f(x,u_n)(u_n - u) \leq 0$.

(f_4) $0 < 4F(x,t) \leq tf(x,t)$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$,

$$\lim_{|t| \to \infty} \frac{F(x,t)}{t^4} = +\infty \quad \text{a.e. } x \in \mathbb{R}. \tag{5}$$

(f_5) if $u_n \rightharpoonup u$ in $H^1(\mathbb{R})$, then $\overline{\lim_{n\to\infty}} \int f(x,u_n)(u_n - u) \leq 0$.

Thm 2. Suppose (V_1), (f_1), (f_2), (f_4) and (f_5) are satisfied, then (3) has at least one nontrivial solutions.

(f_4) $0 < 4F(x,t) \le tf(x,t)$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$,

$$\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = +\infty \quad \text{a.e. } x \in \mathbb{R}. \tag{5}$$

(f_5) if $u_n \rightharpoonup u$ in $H^1(\mathbb{R})$, then $\overline{\lim_{n\to\infty}} \int f(x,u_n)(u_n - u) \le 0$.

Thm 2. Suppose (V_1), (f_1), (f_2), (f_4) and (f_5) are satisfied, then (3) has at least one nontrivial solutions. If in addition $f(x,\cdot)$ is odd for $x \in \mathbb{R}$, then (3) has a sequence of solutions $\{u_n\}$ such that $\Phi(u_n) \to +\infty$.

(f_4) $0 < 4F(x,t) \le tf(x,t)$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$,

$$\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = +\infty \quad \text{a.e. } x \in \mathbb{R}. \tag{5}$$

(f_5) if $u_n \rightharpoonup u$ in $H^1(\mathbb{R})$, then $\overline{\lim_{n\to\infty}} \int f(x,u_n)(u_n - u) \le 0$.

Thm 2. Suppose (V_1), (f_1), (f_2), (f_4) and (f_5) are satisfied, then (3) has at least one nontrivial solutions. If in addition $f(x,\cdot)$ is odd for $x \in \mathbb{R}$, then (3) has a sequence of solutions $\{u_n\}$ such that $\Phi(u_n) \to +\infty$.

Rem 3. Condition (f_5) holds in e.g. one of the following:

(1) for $\forall r > 0$, $\lim_{|x|\to\infty} \sup_{0<|t|\le r} \left| \frac{f(x,t)}{t} \right| = 0$, Bartsch et al. (2004)

(f_4) $0 < 4F(x,t) \le tf(x,t)$ for all $(x,t) \in \mathbb{R}\times\mathbb{R}$,

$$\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = +\infty \quad \text{a.e. } x \in \mathbb{R}. \tag{5}$$

(f_5) if $u_n \rightharpoonup u$ in $H^1(\mathbb{R})$, then $\overline{\lim_{n\to\infty}} \int f(x,u_n)(u_n - u) \le 0$.

Thm 2. Suppose (V_1), (f_1), (f_2), (f_4) and (f_5) are satisfied, then (3) has at least one nontrivial solutions. If in addition $f(x,\cdot)$ is odd for $x \in \mathbb{R}$, then (3) has a sequence of solutions $\{u_n\}$ such that $\Phi(u_n) \to +\infty$.

Rem 3. Condition (f_5) holds in e.g. one of the following:

(1) for $\forall r > 0$, $\displaystyle \lim_{|x|\to\infty} \sup_{0<|t|\le r} \left| \frac{f(x,t)}{t} \right| = 0$, Bartsch et al. (2004)

Example: $f(x,t) = a(x)\,|t|^{p-2}\,t$, $\displaystyle \lim_{|x|\to\infty} a(x) = 0$.

(f_4) $0 < 4F(x,t) \le tf(x,t)$ for all $(x,t) \in \mathbb{R}\times\mathbb{R}$,

$$\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = +\infty \quad \text{a.e. } x \in \mathbb{R}. \tag{5}$$

(f_5) if $u_n \rightharpoonup u$ in $H^1(\mathbb{R})$, then $\overline{\lim_{n\to\infty}} \int f(x,u_n)(u_n - u) \le 0$.

Thm 2. Suppose (V_1), (f_1), (f_2), (f_4) and (f_5) are satisfied, then (3) has at least one nontrivial solutions. If in addition $f(x,\cdot)$ is odd for $x \in \mathbb{R}$, then (3) has a sequence of solutions $\{u_n\}$ such that $\Phi(u_n) \to +\infty$.

Rem 3. Condition (f_5) holds in e.g. one of the following:

(1) for $\forall r > 0$, $\displaystyle\lim_{|x|\to\infty} \sup_{0<|t|\le r} \left|\frac{f(x,t)}{t}\right| = 0$, Bartsch et al. (2004)

Example: $f(x,t) = a(x)|t|^{p-2}t$, $\displaystyle\lim_{|x|\to\infty} a(x) = 0$.

(2) $|f(x,t)| \le a_+(x)|t|^{p_+-1} + a_-(x)|t|^{p_--1}$, $a_\pm \in L^{q_\pm}(\mathbb{R})$ for some $q_\pm > 1$.

2. Proof of Thm 1

We denote $X = H^1(\mathbb{R})$. By Poppenberg et al. (2002), $N : X \to \mathbb{R}$ given by

$$N(u) = \int \dot{u}^2 u^2$$

is of class C^1,

2. Proof of Thm 1

We denote $X = H^1(\mathbb{R})$. By Poppenberg et al. (2002), $N : X \to \mathbb{R}$ given by

$$N(u) = \int \dot{u}^2 u^2$$

is of class C^1,

$$\langle N'(u), v \rangle = 2 \int \left(\dot{u}^2 u v + u^2 \dot{u} \dot{v} \right).$$

2. Proof of Thm 1

We denote $X = H^1(\mathbb{R})$. By Poppenberg et al. (2002), $N : X \to \mathbb{R}$ given by

$$N(u) = \int \dot{u}^2 u^2$$

is of class C^1,

$$\langle N'(u), v \rangle = 2 \int \left(\dot{u}^2 u v + u^2 \dot{u} \dot{v} \right).$$

Therefore, $\Phi : X \to \mathbb{R}$

$$\Phi(u) = \frac{1}{2} \int \left(\dot{u}^2 + V(x) u^2 \right) + \int \dot{u}^2 u^2 - \int F(x, u)$$

2. Proof of Thm 1

We denote $X = H^1(\mathbb{R})$. By Poppenberg et al. (2002), $N : X \to \mathbb{R}$ given by

$$N(u) = \int \dot{u}^2 u^2$$

is of class C^1,

$$\langle N'(u), v\rangle = 2\int \left(\dot{u}^2 uv + u^2 \dot{u}\dot{v}\right).$$

Therefore, $\Phi : X \to \mathbb{R}$

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int \left(\dot{u}^2 + V(x)u^2\right) + \int \dot{u}^2 u^2 - \int F(x,u)\\ &= \frac{1}{2}\int \left(\dot{u}^2 + V(x)u^2\right) + N(u) - \int F(x,u),\end{aligned}$$

is of class C^1 as well,

2. Proof of Thm 1

We denote $X = H^1(\mathbb{R})$. By Poppenberg et al. (2002), $N : X \to \mathbb{R}$ given by

$$N(u) = \int \dot{u}^2 u^2$$

is of class C^1,

$$\langle N'(u), v\rangle = 2\int \left(\dot{u}^2 uv + u^2 \dot{u}\dot{v}\right).$$

Therefore, $\Phi : X \to \mathbb{R}$

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int \left(\dot{u}^2 + V(x)u^2\right) + \int \dot{u}^2 u^2 - \int F(x,u)\\ &= \frac{1}{2}\int \left(\dot{u}^2 + V(x)u^2\right) + N(u) - \int F(x,u),\end{aligned}$$

is of class C^1 as well, with derivative

$$\left\langle \Phi'(u), v\right\rangle = \int (\dot{u}\dot{v} + V(x)uv) + \langle N'(u), v\rangle - \int f(x,u)v.$$

2. Proof of Thm 1

We denote $X = H^1(\mathbb{R})$. By Poppenberg et al. (2002), $N : X \to \mathbb{R}$ given by

$$N(u) = \int \dot{u}^2 u^2$$

is of class C^1,

$$\langle N'(u), v\rangle = 2\int \left(\dot{u}^2 uv + u^2 \dot{u}\dot{v}\right).$$

Therefore, $\Phi : X \to \mathbb{R}$

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int \left(\dot{u}^2 + V(x)u^2\right) + \int \dot{u}^2 u^2 - \int F(x,u)\\ &= \frac{1}{2}\int \left(\dot{u}^2 + V(x)u^2\right) + N(u) - \int F(x,u),\end{aligned}$$

is of class C^1 as well, with derivative

$$\left\langle \Phi'(u), v\right\rangle = \int (\dot{u}\dot{v} + V(x)uv) + \langle N'(u), v\rangle - \int f(x,u)v.$$

Critical points of Φ are weak solutions of the problem (3).

Lem 1. If $u_n \rightharpoonup u$ in X,

Lem 1. If $u_n \rightharpoonup u$ in X, then

$$\overline{\lim_{n\to\infty}} \left(\langle N'(u_n), u \rangle - 4N(u_n) \right) \le 0.$$

Lem 1. If $u_n \rightharpoonup u$ in X, then

$$\overline{\lim_{n\to\infty}} \left(\langle N'(u_n), u \rangle - 4N(u_n) \right) \le 0. \tag{6}$$

Pf. The inequality (6) is a consequence of

$$\frac{1}{2}\langle N'(u_n), u\rangle = \int \left(\dot{u}_n^2 u_n u + u_n^2 \dot{u}_n \dot{u} \right)$$

Lem 1. If $u_n \rightharpoonup u$ in X, then

$$\varlimsup_{n\to\infty} \left(\langle N'(u_n), u\rangle - 4N(u_n)\right) \le 0. \tag{6}$$

Pf. The inequality (6) is a consequence of

$$\begin{aligned}\frac{1}{2}\langle N'(u_n), u\rangle &= \int \left(\dot{u}_n^2 u_n u + u_n^2 \dot{u}_n \dot{u}\right) \\ &\le 2\int \dot{u}_n^2 u_n^2 + o(1) = 2N(u_n) + o(1),\end{aligned}$$

which has been proven in (Chen, 2014, Eqn (3.16)).

Lem 1. If $u_n \rightharpoonup u$ in X, then

$$\overline{\lim_{n\to\infty}} \left(\langle N'(u_n), u\rangle - 4N(u_n)\right) \le 0. \tag{6}$$

Pf. The inequality (6) is a consequence of

$$\begin{aligned}\frac{1}{2}\langle N'(u_n), u\rangle &= \int \left(\dot{u}_n^2 u_n u + u_n^2 \dot{u}_n \dot{u}\right) \\ &\le 2\int \dot{u}_n^2 u_n^2 + o(1) = 2N(u_n) + o(1),\end{aligned}$$

which has been proven in (Chen, 2014, Eqn (3.16)).

Lem 2. Suppose $g \in C(\mathbb{R}^N \times \mathbb{R})$, $|g(x,t)| \le \Lambda |t|$ for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Lem 1. If $u_n \rightharpoonup u$ in X, then

$$\varlimsup_{n\to\infty} \left(\langle N'(u_n), u\rangle - 4N(u_n)\right) \le 0. \tag{6}$$

Pf. The inequality (6) is a consequence of

$$\begin{aligned}\frac{1}{2}\langle N'(u_n), u\rangle &= \int \left(\dot{u}_n^2 u_n u + u_n^2 \dot{u}_n \dot{u}\right) \\ &\le 2\int \dot{u}_n^2 u_n^2 + o(1) = 2N(u_n) + o(1),\end{aligned}$$

which has been proven in (Chen, 2014, Eqn (3.16)).

Lem 2. Suppose $g \in C(\mathbb{R}^N \times \mathbb{R})$, $|g(x,t)| \le \Lambda|t|$ for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then for all $\phi \in H^1(\mathbb{R}^N)$ we have

$$\int g(x, u_n)\phi \to \int g(x,u)\phi.$$

Lem 1. If $u_n \rightharpoonup u$ in X, then

$$\varlimsup_{n\to\infty} \left(\langle N'(u_n), u\rangle - 4N(u_n)\right) \le 0. \tag{6}$$

Pf. The inequality (6) is a consequence of

$$\begin{aligned} \frac{1}{2}\langle N'(u_n), u\rangle &= \int \left(\dot{u}_n^2 u_n u + u_n^2 \dot{u}_n \dot{u}\right) \\ &\le 2\int \dot{u}_n^2 u_n^2 + o(1) = 2N(u_n) + o(1), \end{aligned}$$

which has been proven in (Chen, 2014, Eqn (3.16)).

Lem 2. Suppose $g \in C(\mathbb{R}^N \times \mathbb{R})$, $|g(x,t)| \le \Lambda|t|$ for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then for all $\phi \in H^1(\mathbb{R}^N)$ we have

$$\int g(x,u_n)\phi \to \int g(x,u)\phi. \tag{7}$$

Rem 4. By Brézis & Lieb (1983) we get

$$g(u_n) \rightharpoonup g(u) \quad \text{in } L^2(\mathbb{R}^N),$$

which implies (7).

Pf (Without using B-L). Since g is of linear growth and $\{u_n\}$ is bounded,

$$\beta := \sup_n |g(x,u_n) - g(x,u)|_2 < \infty.$$

Pf (Without using B-L). Since g is of linear growth and $\{u_n\}$ is bounded,

$$\beta := \sup_n |g(x,u_n) - g(x,u)|_2 < \infty.$$

Given $\varepsilon > 0$, choose $R > 0$ such that

$$\int_{|x|\geq R} \phi^2 \leq \varepsilon^2.$$

Pf (Without using B-L). Since g is of linear growth and $\{u_n\}$ is bounded,

$$\beta := \sup_n |g(x,u_n) - g(x,u)|_2 < \infty.$$

Given $\varepsilon > 0$, choose $R > 0$ such that

$$\int_{|x|\geq R} \phi^2 \leq \varepsilon^2.$$

Using Hölder inequality we have

$$\left| \int g(x,u_n)\phi - \int g(x,u)\phi \right|$$

Pf (Without using B-L). Since g is of linear growth and $\{u_n\}$ is bounded,

$$\beta := \sup_n |g(x,u_n) - g(x,u)|_2 < \infty.$$

Given $\varepsilon > 0$, choose $R > 0$ such that

$$\int_{|x|\geq R} \phi^2 \leq \varepsilon^2.$$

Using Hölder inequality we have

$$\left| \int g(x,u_n)\phi - \int g(x,u)\phi \right|$$
$$\leq \int_{|x|\geq R} |g(x,u_n) - g(x,u)||\phi| + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi|$$

Pf (Without using B-L). Since g is of linear growth and $\{u_n\}$ is bounded,

$$\beta := \sup_n |g(x, u_n) - g(x, u)|_2 < \infty.$$

Given $\varepsilon > 0$, choose $R > 0$ such that

$$\int_{|x|\geq R} \phi^2 \leq \varepsilon^2.$$

Using Hölder inequality we have

$$\begin{aligned}
&\left| \int g(x, u_n)\phi - \int g(x, u)\phi \right| \\
&\qquad \leq \int_{|x|\geq R} |g(x, u_n) - g(x, u)||\phi| + \int_{|x|<R} |g(x, u_n) - g(x, u)||\phi| \\
&\qquad \leq |g(x, u_n) - g(x, u)|_2 \left(\int_{|x|\geq R} \phi^2 \right)^{1/2}
\end{aligned}$$

Pf (Without using B-L). Since g is of linear growth and $\{u_n\}$ is bounded,

$$\beta := \sup_n |g(x,u_n) - g(x,u)|_2 < \infty.$$

Given $\varepsilon > 0$, choose $R > 0$ such that

$$\int_{|x|\geq R} \phi^2 \leq \varepsilon^2.$$

Using Hölder inequality we have

$$\begin{aligned}
&\left| \int g(x,u_n)\phi - \int g(x,u)\phi \right| \\
&\qquad \leq \int_{|x|\geq R} |g(x,u_n) - g(x,u)||\phi| + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi| \\
&\qquad \leq |g(x,u_n) - g(x,u)|_2 \left(\int_{|x|\geq R} \phi^2 \right)^{1/2} + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi|
\end{aligned}$$

Pf (Without using B-L). Since g is of linear growth and $\{u_n\}$ is bounded,
$$\beta := \sup_n |g(x,u_n) - g(x,u)|_2 < \infty.$$
Given $\varepsilon > 0$, choose $R > 0$ such that
$$\int_{|x|\geq R} \phi^2 \leq \varepsilon^2.$$
Using Hölder inequality we have
$$\begin{aligned}
&\left| \int g(x,u_n)\phi - \int g(x,u)\phi \right| \\
&\qquad \leq \int_{|x|\geq R} |g(x,u_n) - g(x,u)||\phi| + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi| \\
&\qquad \leq |g(x,u_n) - g(x,u)|_2 \left(\int_{|x|\geq R} \phi^2 \right)^{1/2} + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi| \\
&\qquad \leq \beta\varepsilon + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi|.
\end{aligned}$$

Pf (Without using B-L). Since g is of linear growth and $\{u_n\}$ is bounded,

$$\beta := \sup_n |g(x,u_n) - g(x,u)|_2 < \infty.$$

Given $\varepsilon > 0$, choose $R > 0$ such that

$$\int_{|x|\geq R} \phi^2 \leq \varepsilon^2.$$

Using Hölder inequality we have

$$\begin{aligned}
&\left| \int g(x,u_n)\phi - \int g(x,u)\phi \right| \\
&\qquad \leq \int_{|x|\geq R} |g(x,u_n) - g(x,u)||\phi| + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi| \\
&\qquad \leq |g(x,u_n) - g(x,u)|_2 \left(\int_{|x|\geq R} \phi^2 \right)^{1/2} + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi| \\
&\qquad \leq \beta\varepsilon + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi|.
\end{aligned}$$

Since $H^1(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^2_{\text{loc}}(B_R)$ we deduce $g(x,u_n) \to g(x,u)$ in $L^2(B_R)$.

Pf (Without using B-L). Since g is of linear growth and $\{u_n\}$ is bounded,

$$\beta := \sup_n |g(x,u_n) - g(x,u)|_2 < \infty.$$

Given $\varepsilon > 0$, choose $R > 0$ such that

$$\int_{|x|\geq R} \phi^2 \leq \varepsilon^2.$$

Using Hölder inequality we have

$$\begin{aligned}
&\left| \int g(x,u_n)\phi - \int g(x,u)\phi \right| \\
&\qquad \leq \int_{|x|\geq R} |g(x,u_n) - g(x,u)||\phi| + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi| \\
&\qquad \leq |g(x,u_n) - g(x,u)|_2 \left(\int_{|x|\geq R} \phi^2 \right)^{1/2} + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi| \\
&\qquad \leq \beta\varepsilon + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi|.
\end{aligned}$$

Since $H^1(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^2_{\mathrm{loc}}(B_R)$ we deduce $g(x,u_n) \to g(x,u)$ in $L^2(B_R)$. Hence

$$\varlimsup_{n\to\infty} \left| \int g(x,u_n)\phi - \int g(x,u)\phi \right| \leq \beta\varepsilon.$$

Let E^- be the negative subspace of $S - h$ and $E^+ = (E^-)^\perp$.

Let E^- be the negative subspace of $S - h$ and $E^+ = (E^-)^\perp$. Then $\dim E^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on $X = E^- \oplus E^+$

Let E^- be the negative subspace of $S - h$ and $E^+ = (E^-)^\perp$. Then $\dim E^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on $X = E^- \oplus E^+$ such that

$$\int \left(\dot{u}^2 + V(x)u^2 - hu^2\right) = \|u^+\|^2 - \|u^-\|^2,$$

Let E^- be the negative subspace of $S - h$ and $E^+ = (E^-)^\perp$. Then $\dim E^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on $X = E^- \oplus E^+$ such that

$$\int \left(\dot{u}^2 + V(x)u^2 - hu^2\right) = \|u^+\|^2 - \|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of u on $E^\pm$.

Let E^- be the negative subspace of $S - h$ and $E^+ = (E^-)^\perp$. Then $\dim E^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on $X = E^- \oplus E^+$ such that

$$\int \left(\dot{u}^2 + V(x)u^2 - hu^2\right) = \|u^+\|^2 - \|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of u on $E^\pm$.

Lem 3. Under conditions (V_1), (f_1) and (f_3^*), Φ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.

Let E^- be the negative subspace of $S-h$ and $E^+=(E^-)^\perp$. Then $\dim E^-<\infty$ and there is an equivalent norm $\|\cdot\|$ on $X=E^-\oplus E^+$ such that

$$\int\left(\dot{u}^2+V(x)u^2-hu^2\right)=\|u^+\|^2-\|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of u on $E^\pm$.

Lem 3. Under conditions (V_1), (f_1) and (f_3^*), Φ satisfies the $(PS)_c$ condition for all $c\in\mathbb{R}$.

Pf. Under (V_1), (f_1) and the following condition weaker than (f_3^*):

(f_3) There exists $h\in(0,\lambda_\infty)$ such that $F(x,t)\le\frac{1}{2}ht^2$ for all $(x,t)\in\mathbb{R}\times\mathbb{R}$, Let E^- be the negative subspace of $S-h$ and $E^+ = (E^-)^\perp$. Then $\dim E^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on $X = E^- \oplus E^+$ such that

$$\int \left(\dot{u}^2 + V(x)u^2 - hu^2\right) = \|u^+\|^2 - \|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of u on $E^\pm$.

Lem 3. Under conditions (V_1), (f_1) and (f_3^*), Φ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.

Pf. Under (V_1), (f_1) and the following condition weaker than (f_3^*):

(f_3) There exists $h \in (0, \lambda_\infty)$ such that $F(x,t) \le \frac{1}{2}ht^2$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$,

it has been shown by Wang & Yang (2015) that Φ is coercive.

Let E^- be the negative subspace of $S-h$ and $E^+ = (E^-)^\perp$. Then $\dim E^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on $X = E^- \oplus E^+$ such that

$$\int \left(\dot{u}^2 + V(x)u^2 - hu^2\right) = \|u^+\|^2 - \|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of u on $E^\pm$.

Lem 3. Under conditions (V_1), (f_1) and (f_3^*), Φ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.

Pf. Under (V_1), (f_1) and the following condition weaker than (f_3^*):

(f_3) There exists $h \in (0, \lambda_\infty)$ such that $F(x,t) \le \frac{1}{2}ht^2$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$,

it has been shown by Wang & Yang (2015) that Φ is coercive.
Let $\{u_n\}$ be a $(PS)_c$ sequence of Φ, that is

$$\Phi(u_n) \to c, \qquad \Phi'(u_n) \to 0.$$

By the coerciveness of Φ, $\{u_n\}$ is bounded in X.

Let E^- be the negative subspace of $S-h$ and $E^+=(E^-)^\perp$. Then $\dim E^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on $X=E^-\oplus E^+$ such that

$$\int \left(\dot{u}^2+V(x)u^2-hu^2\right)=\|u^+\|^2-\|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of u on $E^\pm$.

Lem 3. Under conditions (V_1), (f_1) and (f_3^*), Φ satisfies the $(PS)_c$ condition for all $c\in\mathbb{R}$.

Pf. Under (V_1), (f_1) and the following condition weaker than (f_3^*):

(f_3) There exists $h\in(0,\lambda_\infty)$ such that $F(x,t)\le\frac{1}{2}ht^2$ for all $(x,t)\in\mathbb{R}\times\mathbb{R}$,

it has been shown by Wang & Yang (2015) that Φ is coercive.
Let $\{u_n\}$ be a $(PS)_c$ sequence of Φ, that is

$$\Phi(u_n)\to c,\qquad \Phi'(u_n)\to 0.$$

By the coerciveness of Φ, $\{u_n\}$ is bounded in X.
We may assume $u_n\rightharpoonup u$ in X. Since $\dim E^-<\infty$, we have $u_n^-\to u^-$ and $\|u_n^-\|\to\|u^-\|$.

Since $\Phi'(u_n) \to 0$, we have

$$o(1) = o(\|u_n\|) = \langle \Phi'(u_n), u_n \rangle$$

Since $\Phi'(u_n) \to 0$, we have

$$o(1) = o(\|u_n\|) = \langle \Phi'(u_n), u_n \rangle$$
$$= \int \left(\dot{u}_n^2 + V(x)u_n^2 - hu_n^2\right) + \langle N'(u_n), u_n \rangle - \int \left(f(x, u_n)u_n - hu_n^2\right)$$

Since $\Phi'(u_n) \to 0$, we have

$$\begin{aligned} o(1) = o(\|u_n\|) &= \langle \Phi'(u_n), u_n \rangle \\ &= \int \left(\dot{u}_n^2 + V(x)u_n^2 - hu_n^2\right) + \langle N'(u_n), u_n \rangle - \int \left(f(x,u_n)u_n - hu_n^2\right) \\ &= \|u_n^+\|^2 - \|u_n^-\|^2 + 4N(u_n) + \int \left(hu_n^2 - f(x,u_n)u_n\right). \end{aligned}$$

Since $\Phi'(u_n) \to 0$, we have

$$
\begin{aligned}
o(1) = o(\|u_n\|) &= \langle \Phi'(u_n), u_n \rangle \\
&= \int \left(\dot{u}_n^2 + V(x) u_n^2 - h u_n^2 \right) + \langle N'(u_n), u_n \rangle - \int \left(f(x, u_n) u_n - h u_n^2 \right) \\
&= \|u_n^+\|^2 - \|u_n^-\|^2 + 4N(u_n) + \int \left(h u_n^2 - f(x, u_n) u_n \right). \qquad (8)
\end{aligned}
$$

Applying Lem 2 to $g : (x, t) \mapsto ht - f(x, t)$ yields

$$
\int (h u_n u - f(x, u_n) u) = \int \left(h u^2 - f(x, u) u \right) + o(1).
$$

Since $\Phi'(u_n) \to 0$, we have

$$\begin{aligned} o(1) = o(\|u_n\|) &= \langle \Phi'(u_n), u_n \rangle \\ &= \int \left(\dot{u}_n^2 + V(x)u_n^2 - hu_n^2 \right) + \langle N'(u_n), u_n \rangle - \int \left(f(x,u_n)u_n - hu_n^2 \right) \\ &= \|u_n^+\|^2 - \|u_n^-\|^2 + 4N(u_n) + \int \left(hu_n^2 - f(x,u_n)u_n \right). \end{aligned} \tag{8}$$

Applying Lem 2 to $g : (x,t) \mapsto ht - f(x,t)$ yields

$$\int (hu_n u - f(x,u_n)u) = \int \left(hu^2 - f(x,u)u \right) + o(1).$$

Therefore

$$o(1) = \langle \Phi'(u_n), u \rangle$$

Since $\Phi'(u_n) \to 0$, we have

$$\begin{aligned} o(1) = o(\|u_n\|) = \langle \Phi'(u_n), u_n \rangle \\ &= \int \left(\dot{u}_n^2 + V(x)u_n^2 - hu_n^2\right) + \langle N'(u_n), u_n \rangle - \int \left(f(x,u_n)u_n - hu_n^2\right) \\ &= \|u_n^+\|^2 - \|u_n^-\|^2 + 4N(u_n) + \int \left(hu_n^2 - f(x,u_n)u_n\right). \end{aligned} \tag{8}$$

Applying Lem 2 to $g : (x,t) \mapsto ht - f(x,t)$ yields

$$\int (hu_n u - f(x,u_n)u) = \int \left(hu^2 - f(x,u)u\right) + o(1).$$

Therefore

$$\begin{aligned} o(1) = \langle \Phi'(u_n), u \rangle \\ &= \|u^+\|^2 - \|u^-\|^2 + \langle N'(u_n), u \rangle + \int \left(hu^2 - f(x,u)u\right) + o(1). \end{aligned}$$

Since $\Phi'(u_n) \to 0$, we have

$$\begin{aligned} o(1) = o(\|u_n\|) &= \langle \Phi'(u_n), u_n \rangle \\ &= \int \left(\dot{u}_n^2 + V(x)u_n^2 - hu_n^2 \right) + \langle N'(u_n), u_n \rangle - \int \left(f(x, u_n)u_n - hu_n^2 \right) \\ &= \|u_n^+\|^2 - \|u_n^-\|^2 + 4N(u_n) + \int \left(hu_n^2 - f(x, u_n)u_n \right). \end{aligned} \tag{8}$$

Applying Lem 2 to $g : (x, t) \mapsto ht - f(x, t)$ yields

$$\int (hu_n u - f(x, u_n)u) = \int \left(hu^2 - f(x, u)u \right) + o(1).$$

Therefore

$$\begin{aligned} o(1) &= \langle \Phi'(u_n), u \rangle \\ &= \|u^+\|^2 - \|u^-\|^2 + \langle N'(u_n), u \rangle + \int \left(hu^2 - f(x, u)u \right) + o(1). \end{aligned} \tag{9}$$

From (8), (9), Lem 1, and

$$\int \left(hu^2 - f(x, u)u \right) \leq \varliminf_{n\to\infty} \int \left(hu_n^2 - f(x, u_n)u_n \right), \qquad ((f_3^*)\ \&\ \text{Fatou})$$

Since $\Phi'(u_n) \to 0$, we have

$$\begin{aligned} o(1) = o(\|u_n\|) &= \langle \Phi'(u_n), u_n \rangle \\ &= \int \left(\dot{u}_n^2 + V(x)u_n^2 - hu_n^2 \right) + \langle N'(u_n), u_n \rangle - \int \left(f(x, u_n)u_n - hu_n^2 \right) \\ &= \|u_n^+\|^2 - \|u_n^-\|^2 + 4N(u_n) + \int \left(hu_n^2 - f(x, u_n)u_n \right). \end{aligned} \tag{8}$$

Applying Lem 2 to $g : (x, t) \mapsto ht - f(x, t)$ yields

$$\int (hu_n u - f(x, u_n)u) = \int \left(hu^2 - f(x, u)u \right) + o(1).$$

Therefore

$$\begin{aligned} o(1) &= \langle \Phi'(u_n), u \rangle \\ &= \|u^+\|^2 - \|u^-\|^2 + \langle N'(u_n), u \rangle + \int \left(hu^2 - f(x, u)u \right) + o(1). \end{aligned} \tag{9}$$

From (8), (9), Lem 1, and

$$\int \left(hu^2 - f(x, u)u \right) \leq \varliminf_{n\to\infty} \int \left(hu_n^2 - f(x, u_n)u_n \right), \qquad ((f_3^*) \text{ \& Fatou})$$

as well as $\|u_n^-\| \to \|u^-\|$,

we deduce

$$\varlimsup_{n\to\infty} \|u_n^+\|^2 = \|u^+\|^2 + \int \Big(hu^2 - f(x,u)u\Big) + \varlimsup_{n\to\infty} \Big(\big[\langle N'(u_n), u\rangle - 4N(u_n)\big] - \int (hu_n u - f(x,u_n)u) \Big)$$

we deduce

$$\varlimsup_{n\to\infty}\|u_n^+\|^2 = \|u^+\|^2 + \int\left(hu^2 - f(x,u)u\right) + \varlimsup_{n\to\infty}\left(\left[\langle N'(u_n),u\rangle - 4N(u_n)\right] - \int(hu_nu - f(x,u_n)u)\right)$$

$$\leq \|u^+\|^2 + \int\left(hu^2 - f(x,u)u\right) - \varliminf_{n\to\infty}\int\left(hu_n^2 - f(x,u_n)u_n\right)$$

we deduce

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \|u_n^+\|^2 &= \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) \\
&\quad + \overline{\lim_{n\to\infty}} \left(\left[\langle N'(u_n), u\rangle - 4N(u_n)\right] - \int (hu_n u - f(x,u_n)u) \right) \\
&\le \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) - \underline{\lim_{n\to\infty}} \int \left(hu_n^2 - f(x,u_n)u_n\right) \\
&\le \|u^+\|^2.
\end{aligned}$$

we deduce

$$\overline{\lim_{n\to\infty}} \|u_n^+\|^2 = \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) + \overline{\lim_{n\to\infty}} \left(\left[\langle N'(u_n), u\rangle - 4N(u_n)\right] - \int (hu_n u - f(x,u_n)u) \right)$$
$$\le \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) - \underline{\lim_{n\to\infty}} \int \left(hu_n^2 - f(x,u_n)u_n\right)$$
$$\le \|u^+\|^2.$$

This and the weak lower semi-continuity of norm functional $u \mapsto \|u\|$, i.e.,

$$\|u^+\|^2 \le \underline{\lim_{n\to\infty}} \|u_n^+\|^2, \tag{10}$$

yields $\|u_n^+\| \to \|u^+\|$.

we deduce

$$\begin{aligned}\overline{\lim_{n\to\infty}}\|u_n^+\|^2 &= \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) \\ &\quad + \overline{\lim_{n\to\infty}}\left(\left[\langle N'(u_n),u\rangle - 4N(u_n)\right] - \int (hu_n u - f(x,u_n)u)\right) \\ &\le \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) - \underline{\lim_{n\to\infty}}\int \left(hu_n^2 - f(x,u_n)u_n\right) \\ &\le \|u^+\|^2.\end{aligned}$$

This and the weak lower semi-continuity of norm functional $u \mapsto \|u\|$, i.e.,

$$\|u^+\|^2 \le \underline{\lim_{n\to\infty}}\|u_n^+\|^2, \tag{10}$$

yields $\|u_n^+\| \to \|u^+\|$. Therefore $\|u_n\| \to \|u\|$ and $u_n \to u$ in X.

we deduce

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \|u_n^+\|^2 &= \|u^+\|^2 + \int \big(hu^2 - f(x,u)u\big) \\
&\quad + \overline{\lim_{n\to\infty}} \Big(\big[\langle N'(u_n), u\rangle - 4N(u_n)\big] - \int (hu_n u - f(x,u_n)u) \Big) \\
&\le \|u^+\|^2 + \int \big(hu^2 - f(x,u)u\big) - \underline{\lim_{n\to\infty}} \int \big(hu_n^2 - f(x,u_n)u_n\big) \\
&\le \|u^+\|^2 .
\end{aligned}$$

This and the weak lower semi-continuity of norm functional $u \mapsto \|u\|$, i.e.,

$$\|u^+\|^2 \le \underline{\lim_{n\to\infty}} \|u_n^+\|^2, \tag{10}$$

yields $\|u_n^+\| \to \|u^+\|$. Therefore $\|u_n\| \to \|u\|$ and $u_n \to u$ in X.

Pf of Thm 1. Let $X^\pm$ be $\pm$-spaces of S.

we deduce

$$
\begin{aligned}
\overline{\lim_{n\to\infty}}\, \|u_n^+\|^2 &= \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) \\
&\quad + \overline{\lim_{n\to\infty}} \left(\left[\langle N'(u_n), u\rangle - 4N(u_n)\right] - \int (hu_n u - f(x,u_n)u) \right) \\
&\le \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) - \underline{\lim_{n\to\infty}} \int \left(hu_n^2 - f(x,u_n)u_n\right) \\
&\le \|u^+\|^2.
\end{aligned}
$$

This and the weak lower semi-continuity of norm functional $u \mapsto \|u\|$, i.e.,

$$
\|u^+\|^2 \le \underline{\lim_{n\to\infty}}\, \|u_n^+\|^2, \tag{10}
$$

yields $\|u_n^+\| \to \|u^+\|$. Therefore $\|u_n\| \to \|u\|$ and $u_n \to u$ in X.

Pf of Thm 1. Let $X^\pm$ be $\pm$-spaces of S. As $u \to 0$, Clark (1972/73)

$$
\Phi(u) = \frac{1}{2}\int \left(\dot{u}^2 + V(x)u^2\right) + \int \dot{u}^2 u^2 - \int F(x,u)
$$

we deduce

$$\begin{aligned}\overline{\lim_{n\to\infty}}\,\|u_n^+\|^2 &= \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) \\ &\qquad + \overline{\lim_{n\to\infty}} \left(\left[\langle N'(u_n), u\rangle - 4N(u_n)\right] - \int (hu_n u - f(x,u_n)u) \right) \\ &\le \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) - \underline{\lim_{n\to\infty}} \int \left(hu_n^2 - f(x,u_n)u_n\right) \\ &\le \|u^+\|^2.\end{aligned}$$

This and the weak lower semi-continuity of norm functional $u \mapsto \|u\|$, i.e.,

$$\|u^+\|^2 \le \underline{\lim_{n\to\infty}}\,\|u_n^+\|^2, \tag{10}$$

yields $\|u_n^+\| \to \|u^+\|$. Therefore $\|u_n\| \to \|u\|$ and $u_n \to u$ in X.

Pf of Thm 1. Let $X^\pm$ be $\pm$-spaces of S. As $u \to 0$, Clark (1972/73)

$$\Phi(u) = \frac{1}{2}\int \left(\dot{u}^2 + V(x)u^2\right) + \int \dot{u}^2u^2 - \int F(x,u) = \frac{1}{2}\int \left(\dot{u}^2 + V(x)u^2\right) + o(\|u\|^2),$$

we deduce

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\|u_n^+\|^2 &= \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) \\
&\qquad + \overline{\lim_{n\to\infty}} \left(\left[\langle N'(u_n), u\rangle - 4N(u_n)\right] - \int (hu_n u - f(x,u_n)u) \right) \\
&\le \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) - \underline{\lim_{n\to\infty}} \int \left(hu_n^2 - f(x,u_n)u_n\right) \\
&\le \|u^+\|^2.
\end{aligned}$$

This and the weak lower semi-continuity of norm functional $u \mapsto \|u\|$, i.e.,

$$\|u^+\|^2 \le \underline{\lim_{n\to\infty}} \|u_n^+\|^2, \tag{10}$$

yields $\|u_n^+\| \to \|u^+\|$. Therefore $\|u_n\| \to \|u\|$ and $u_n \to u$ in X.

Pf of Thm 1. Let $X^\pm$ be $\pm$-spaces of S. As $u \to 0$, Clark (1972/73)

$$\Phi(u) = \frac{1}{2}\int \left(\dot u^2 + V(x)u^2\right) + \int \dot u^2 u^2 - \int F(x,u) = \frac{1}{2}\int \left(\dot u^2 + V(x)u^2\right) + o(\|u\|^2),$$

So Φ has a loc link at $u = 0$, hence has **3** critical points (Liu (1989)).

3. Proof of Thm2

Let X^- be the negative subspace of S and $X^+ = (X^-)^\perp$.

3. Proof of Thm2

Let X^- be the negative subspace of S and $X^+ = (X^-)^\perp$. Then $\dim X^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on X such that

$$\int \left(\dot{u}^2 + V(x)u^2\right) = \|u^+\|^2 - \|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of u on $X^\pm$.

3. Proof of Thm2

Let X^- be the negative subspace of S and $X^+ = (X^-)^\perp$. Then $\dim X^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on X such that

$$\int \left(\dot{u}^2 + V(x)u^2\right) = \|u^+\|^2 - \|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of u on $X^\pm$. Hence

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + N(u) - \int F(x,u).$$

3. Proof of Thm 2

Let X^- be the negative subspace of S and $X^+ = (X^-)^\perp$. Then $\dim X^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on X such that

$$\int \left(\dot{u}^2 + V(x)u^2\right) = \|u^+\|^2 - \|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of u on $X^\pm$. Hence

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + N(u) - \int F(x,u).$$

Lem 4. Under the conditions (V_1), (f_1), (f_4) and (f_5), Φ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.

3. Proof of Thm 2

Let X^- be the negative subspace of S and $X^+ = (X^-)^\perp$. Then $\dim X^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on X such that

$$\int \left(\dot{u}^2 + V(x)u^2\right) = \|u^+\|^2 - \|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of u on $X^\pm$. Hence

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + N(u) - \int F(x,u).$$

Lem 4. Under the conditions (V_1), (f_1), (f_4) and (f_5), Φ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.

Pf. Let $\{u_n\}$ be a $(PS)_c$ sequence of Φ.

3. Proof of Thm 2

Let X^- be the negative subspace of S and $X^+ = (X^-)^\perp$. Then $\dim X^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on X such that

$$\int \left(\dot{u}^2 + V(x)u^2\right) = \|u^+\|^2 - \|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of u on $X^\pm$. Hence

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + N(u) - \int F(x,u).$$

Lem 4. Under the conditions (V_1), (f_1), (f_4) and (f_5), Φ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.

Pf. Let $\{u_n\}$ be a $(PS)_c$ sequence of Φ.
Step 1. If $\|u_n\| \to \infty$. Let $v_n = \|u_n\|^{-1}u_n$,

3. Proof of Thm 2

Let X^- be the negative subspace of S and $X^+ = (X^-)^\perp$. Then $\dim X^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on X such that

$$\int \left(\dot{u}^2 + V(x)u^2\right) = \|u^+\|^2 - \|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of u on $X^\pm$. Hence

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + N(u) - \int F(x,u).$$

Lem 4. Under the conditions (V_1), (f_1), (f_4) and (f_5), Φ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.

Pf. Let $\{u_n\}$ be a $(PS)_c$ sequence of Φ.
Step 1. If $\|u_n\| \to \infty$. Let $v_n = \|u_n\|^{-1}u_n$, then $v_n \rightharpoonup v$ in X,

$$v_n^- \to v^- \quad \text{in } X$$

because $\dim X^- < \infty$.

If $v = 0$, then $v^- = 0$, $\|v_n^+\|^2 - \|v_n^-\|^2 \geq \frac{1}{2}$ for large n.

If $v = 0$, then $v^- = 0$, $\|v_n^+\|^2 - \|v_n^-\|^2 \geq \frac{1}{2}$ for large n. By (f_4)

$$1 + c + \|u_n\| \geq \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n \rangle$$

If $v = 0$, then $v^- = 0$, $\|v_n^+\|^2 - \|v_n^-\|^2 \geq \frac{1}{2}$ for large n. By (f_4)

$$1 + c + \|u_n\| \geq \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n \rangle$$

$$= \frac{1}{4}\|u_n\|^2 \left(\|v_n^+\|^2 - \|v_n^-\|^2\right) + \int \left(\frac{1}{4} f(x, u_n) u_n - F(x, u_n)\right)$$

If $v = 0$, then $v^- = 0$, $\|v_n^+\|^2 - \|v_n^-\|^2 \geq \frac{1}{2}$ for large n. By (f_4)

$$\begin{aligned}
1 + c + \|u_n\| &\geq \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n \rangle \\
&= \frac{1}{4}\|u_n\|^2 \left(\|v_n^+\|^2 - \|v_n^-\|^2\right) + \int \left(\frac{1}{4} f(x, u_n)u_n - F(x, u_n)\right) \\
&\geq \frac{1}{8}\|u_n\|^2,
\end{aligned}$$

If $v = 0$, then $v^- = 0$, $\|v_n^+\|^2 - \|v_n^-\|^2 \geq \frac{1}{2}$ for large n. By (f_4)

$$\begin{aligned}
1 + c + \|u_n\| &\geq \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n\rangle \\
&= \frac{1}{4}\|u_n\|^2 \left(\|v_n^+\|^2 - \|v_n^-\|^2\right) + \int \left(\frac{1}{4} f(x, u_n)u_n - F(x, u_n)\right) \\
&\geq \frac{1}{8}\|u_n\|^2,
\end{aligned}$$

contradicting $\|u_n\| \to \infty$.

If $v = 0$, then $v^- = 0$, $\|v_n^+\|^2 - \|v_n^-\|^2 \geq \frac{1}{2}$ for large n. By (f_4)

$$\begin{aligned}
1 + c + \|u_n\| &\geq \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n \rangle \\
&= \frac{1}{4}\|u_n\|^2 \left(\|v_n^+\|^2 - \|v_n^-\|^2 \right) + \int \left(\frac{1}{4} f(x, u_n) u_n - F(x, u_n) \right) \\
&\geq \frac{1}{8}\|u_n\|^2,
\end{aligned}$$

contradicting $\|u_n\| \to \infty$.

If $v \neq 0$, by Fatou's lemma (5) implies

$$\int \frac{F(x, u_n)}{\|u_n\|^4} \geq \int_{v \neq 0} \frac{F(x, u_n)}{\|u_n\|^4} v_n^4 \to +\infty.$$

If $v = 0$, then $v^- = 0$, $\|v_n^+\|^2 - \|v_n^-\|^2 \geq \frac{1}{2}$ for large n. By (f_4)

$$
\begin{aligned}
1 + c + \|u_n\| &\geq \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n\rangle \\
&= \frac{1}{4}\|u_n\|^2\left(\|v_n^+\|^2 - \|v_n^-\|^2\right) + \int\left(\frac{1}{4}f(x,u_n)u_n - F(x,u_n)\right) \\
&\geq \frac{1}{8}\|u_n\|^2,
\end{aligned}
$$

contradicting $\|u_n\| \to \infty$.
If $v \neq 0$, by Fatou's lemma (5) implies

$$\int \frac{F(x,u_n)}{\|u_n\|^4} \geq \int_{v\neq 0} \frac{F(x,u_n)}{\|u_n\|^4} v_n^4 \to +\infty. \tag{11}$$

Using $|N(u)| \leq B\|u\|^4$ we get a contradiction:

$$\frac{c-1}{\|u_n\|^4} \leq \frac{\Phi(u_n)}{\|u_n\|^4}$$

If $v = 0$, then $v^- = 0$, $\|v_n^+\|^2 - \|v_n^-\|^2 \ge \frac{1}{2}$ for large n. By (f_4)

$$\begin{aligned}
1 + c + \|u_n\| &\ge \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n\rangle \\
&= \frac{1}{4}\|u_n\|^2 \left(\|v_n^+\|^2 - \|v_n^-\|^2\right) + \int \left(\frac{1}{4} f(x,u_n)u_n - F(x,u_n)\right) \\
&\ge \frac{1}{8}\|u_n\|^2,
\end{aligned}$$

contradicting $\|u_n\| \to \infty$.

If $v \ne 0$, by Fatou's lemma (5) implies

$$\int \frac{F(x,u_n)}{\|u_n\|^4} \ge \int_{v\ne 0} \frac{F(x,u_n)}{\|u_n\|^4} v_n^4 \to +\infty. \tag{11}$$

Using $|N(u)| \le B\|u\|^4$ we get a contradiction:

$$\frac{c-1}{\|u_n\|^4} \le \frac{\Phi(u_n)}{\|u_n\|^4} \le \frac{\|u_n^+\|^2 - \|u_n^-\|^2}{2\|u_n\|^4} + B - \int \frac{F(x,u_n)}{\|u_n\|^4} \to -\infty.$$

Step 2. Assume $u_n \rightharpoonup u$. We show that $\|u_n\| \to \|u\|$.

Step 2. Assume $u_n \rightharpoonup u$. We show that $\|u_n\| \to \|u\|$. Since $\dim X^- < \infty$, we have $\|u_n^-\| \to \|u^-\|$.

Step 2. Assume $u_n \rightharpoonup u$. We show that $\|u_n\| \to \|u\|$. Since $\dim X^- < \infty$, we have $\|u_n^-\| \to \|u^-\|$. Noting

$$\langle N'(u_n), u_n - u\rangle = 4N(u_n) - \langle N'(u_n), u\rangle,$$

Step 2. Assume $u_n \rightharpoonup u$. We show that $\|u_n\| \to \|u\|$. Since $\dim X^- < \infty$, we have $\|u_n^-\| \to \|u^-\|$. Noting

$$\langle N'(u_n), u_n - u\rangle = 4N(u_n) - \langle N'(u_n), u\rangle,$$

by direct computation we deduce

$$0 = \langle \Phi'(u_n), u_n - u\rangle + o(1)$$

Step 2. Assume $u_n \rightharpoonup u$. We show that $\|u_n\| \to \|u\|$. Since $\dim X^- < \infty$, we have $\|u_n^-\| \to \|u^-\|$. Noting

$$\langle N'(u_n), u_n - u\rangle = 4N(u_n) - \langle N'(u_n), u\rangle,$$

by direct computation we deduce

$$\begin{aligned} 0 &= \langle \Phi'(u_n), u_n - u\rangle + o(1) \\ &= \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) - \left(\|u^+\|^2 - \|u^-\|^2\right) \\ &\qquad + 4N(u_n) - \langle N'(u_n), u\rangle - \int f(x, u_n)(u_n - u)\,. \end{aligned}$$

Step 2. Assume $u_n \rightharpoonup u$. We show that $\|u_n\| \to \|u\|$. Since $\dim X^- < \infty$, we have $\|u_n^-\| \to \|u^-\|$. Noting

$$\langle N'(u_n), u_n - u\rangle = 4N(u_n) - \langle N'(u_n), u\rangle,$$

by direct computation we deduce

$$\begin{aligned} 0 &= \langle \Phi'(u_n), u_n - u\rangle + o(1) \\ &= \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) - \left(\|u^+\|^2 - \|u^-\|^2\right) \\ &\quad + 4N(u_n) - \langle N'(u_n), u\rangle - \int f(x, u_n)(u_n - u). \end{aligned}$$

Now, applying Lem 1, using $\|u_n^-\| \to \|u^-\|$ and condition (f_5) we get

$$\varlimsup_{n\to\infty} \|u_n^+\|^2 = \|u^+\|^2 + \varlimsup_{n\to\infty} \left(\left[\langle N'(u_n), u\rangle - 4N(u_n)\right] + \int f(x, u_n)(u_n - u)\right)$$

Step 2. Assume $u_n \rightharpoonup u$. We show that $\|u_n\| \to \|u\|$. Since $\dim X^- < \infty$, we have $\|u_n^-\| \to \|u^-\|$. Noting

$$\langle N'(u_n), u_n - u\rangle = 4N(u_n) - \langle N'(u_n), u\rangle,$$

by direct computation we deduce

$$\begin{aligned} 0 &= \langle \Phi'(u_n), u_n - u\rangle + o(1) \\ &= \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) - \left(\|u^+\|^2 - \|u^-\|^2\right) \\ &\qquad + 4N(u_n) - \langle N'(u_n), u\rangle - \int f(x, u_n)(u_n - u)\,. \end{aligned}$$

Now, applying Lem 1, using $\|u_n^-\| \to \|u^-\|$ and condition (f_5) we get

$$\begin{aligned} \overline{\lim_{n\to\infty}} \|u_n^+\|^2 &= \|u^+\|^2 + \overline{\lim_{n\to\infty}} \left(\left[\langle N'(u_n), u\rangle - 4N(u_n)\right] + \int f(x, u_n)(u_n - u)\right) \\ &\le \|u^+\|^2. \end{aligned}$$

Hence $\|u_n^+\| \to \|u^+\|$.

Step 2. Assume $u_n \rightharpoonup u$. We show that $\|u_n\| \to \|u\|$. Since $\dim X^- < \infty$, we have $\|u_n^-\| \to \|u^-\|$. Noting

$$\langle N'(u_n), u_n - u\rangle = 4N(u_n) - \langle N'(u_n), u\rangle,$$

by direct computation we deduce

$$\begin{aligned} 0 &= \langle \Phi'(u_n), u_n - u\rangle + o(1) \\ &= \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) - \left(\|u^+\|^2 - \|u^-\|^2\right) \\ &\qquad + 4N(u_n) - \langle N'(u_n), u\rangle - \int f(x, u_n)(u_n - u). \end{aligned}$$

Now, applying Lem1, using $\|u_n^-\| \to \|u^-\|$ and condition (f_5) we get

$$\begin{aligned} \overline{\lim_{n\to\infty}} \|u_n^+\|^2 &= \|u^+\|^2 + \overline{\lim_{n\to\infty}} \left(\left[\langle N'(u_n), u\rangle - 4N(u_n)\right] + \int f(x, u_n)(u_n - u)\right) \\ &\le \|u^+\|^2. \end{aligned}$$

Hence $\|u_n^+\| \to \|u^+\|$. Noting $\|u_n^-\| \to \|u^-\|$, we conclude $\|u_n\| \to \|u\|$.

Lem 5. Assume (V_1), (f_1), (f_4), there exists $A < \inf_{B_2} \Phi$,

Lem 5. Assume (V_1), (f_1), (f_4), there exists $A < \inf_{B_2} \Phi$, s.t. if $\Phi(u) \le A$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu) < 0.$$

Lem 5. Assume (V_1), (f_1), (f_4), there exists $A < \inf_{B_2} \Phi$, s.t. if $\Phi(u) \le A$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu) < 0.$$

Pf. Otherwise, there is a sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \le -n$ but

$$\langle \Phi'(u_n), u_n \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu_n) \ge 0.$$

Lem 5. Assume (V_1), (f_1), (f_4), there exists $A < \inf_{B_2} \Phi$, s.t. if $\Phi(u) \le A$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu) < 0.$$

Pf. Otherwise, there is a sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \le -n$ but

$$\langle \Phi'(u_n), u_n \rangle = \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu_n) \ge 0. \tag{12}$$

Using (f_4), we deduce

$$\|u_n^+\|^2 - \|u_n^-\|^2 \le \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) + \int [f(x,u_n)u_n - 4F(x,u_n)]$$

Lem 5. Assume (V_1), (f_1), (f_4), there exists $A < \inf_{B_2} \Phi$, s.t. if $\Phi(u) \le A$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu) < 0.$$

Pf. Otherwise, there is a sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \le -n$ but

$$\langle \Phi'(u_n), u_n \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu_n) \ge 0. \tag{12}$$

Using (f_4), we deduce

$$\begin{aligned} \|u_n^+\|^2 - \|u_n^-\|^2 &\le \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) + \int [f(x,u_n)u_n - 4F(x,u_n)] \\ &= 4\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle \end{aligned}$$

Lem 5. Assume (V_1), (f_1), (f_4), there exists $A < \inf_{B_2} \Phi$, s.t. if $\Phi(u) \le A$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu) < 0.$$

Pf. Otherwise, there is a sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \le -n$ but

$$\langle \Phi'(u_n), u_n \rangle = \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu_n) \ge 0. \tag{12}$$

Using (f_4), we deduce

$$\begin{aligned} \|u_n^+\|^2 - \|u_n^-\|^2 &\le \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) + \int [f(x,u_n)u_n - 4F(x,u_n)] \\ &= 4\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle \le -4n. \end{aligned}$$

Lem 5. Assume (V_1), (f_1), (f_4), there exists $A < \inf_{B_2} \Phi$, s.t. if $\Phi(u) \le A$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu) < 0.$$

Pf. Otherwise, there is a sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \le -n$ but

$$\langle \Phi'(u_n), u_n \rangle = \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu_n) \ge 0. \tag{12}$$

Using (f_4), we deduce

$$\begin{aligned} \|u_n^+\|^2 - \|u_n^-\|^2 &\le \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) + \int [f(x,u_n)u_n - 4F(x,u_n)] \\ &= 4\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle \le -4n. \end{aligned} \tag{13}$$

Let $v_n = \|u_n\|^{-1} u_n$, then $v_n^- \to v^-$ in X ($\dim X^- < \infty$), $v^- \neq 0$.

Lem 5. Assume (V_1), (f_1), (f_4), there exists $A < \inf_{B_2} \Phi$, s.t. if $\Phi(u) \le A$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu) < 0.$$

Pf. Otherwise, there is a sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \le -n$ but

$$\langle \Phi'(u_n), u_n\rangle = \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu_n) \ge 0. \tag{12}$$

Using (f_4), we deduce

$$\begin{aligned} \|u_n^+\|^2 - \|u_n^-\|^2 &\le \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) + \int [f(x,u_n)u_n - 4F(x,u_n)] \\ &= 4\Phi(u_n) - \langle \Phi'(u_n), u_n\rangle \le -4n. \end{aligned} \tag{13}$$

Let $v_n = \|u_n\|^{-1}u_n$, then $v_n^- \to v^-$ in X ($\dim X^- < \infty$), $v^- \neq 0$. Hence

$$\int \frac{f(x,u_n)u_n}{\|u_n\|^4} \ge 4\int \frac{F(x,u_n)}{\|u_n\|^4} \to +\infty.$$

Lem 5. Assume (V_1), (f_1), (f_4), there exists $A < \inf_{B_2} \Phi$, s.t. if $\Phi(u) \le A$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu) < 0.$$

Pf. Otherwise, there is a sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \le -n$ but

$$\langle \Phi'(u_n), u_n \rangle = \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu_n) \ge 0. \tag{12}$$

Using (f_4), we deduce

$$\begin{aligned}\|u_n^+\|^2 - \|u_n^-\|^2 &\le \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) + \int [f(x,u_n)u_n - 4F(x,u_n)] \\ &= 4\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle \le -4n.\end{aligned} \tag{13}$$

Let $v_n = \|u_n\|^{-1} u_n$, then $v_n^- \to v^-$ in X ($\dim X^- < \infty$), $v^- \ne 0$. Hence

$$\int \frac{f(x,u_n)u_n}{\|u_n\|^4} \ge 4 \int \frac{F(x,u_n)}{\|u_n\|^4} \to +\infty. \qquad \text{(see (11))}$$

$$0 \le \frac{\langle \Phi'(u_n), u_n \rangle}{\|u_n\|^4}$$

Lem 5. Assume (V_1), (f_1), (f_4), there exists $A < \inf_{B_2} \Phi$, s.t. if $\Phi(u) \le A$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu) < 0.$$

Pf. Otherwise, there is a sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \le -n$ but

$$\langle \Phi'(u_n), u_n \rangle = \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu_n) \ge 0. \tag{12}$$

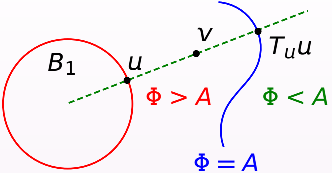
Using (f_4), we deduce

$$\begin{aligned} \|u_n^+\|^2 - \|u_n^-\|^2 &\le \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) + \int [f(x,u_n)u_n - 4F(x,u_n)] \\ &= 4\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle \le -4n. \end{aligned} \tag{13}$$

Let $v_n = \|u_n\|^{-1} u_n$, then $v_n^- \to v^-$ in X ($\dim X^- < \infty$), $v^- \ne 0$. Hence

$$\int \frac{f(x,u_n)u_n}{\|u_n\|^4} \ge 4 \int \frac{F(x,u_n)}{\|u_n\|^4} \to +\infty. \qquad \text{(see (11))}$$

$$0 \le \frac{\langle \Phi'(u_n), u_n \rangle}{\|u_n\|^4} \le \frac{\|u_n^+\|^2 - \|u_n^-\|^2}{\|u_n\|^4} + \frac{\langle N'(u_n), u_n \rangle}{\|u_n\|^4} - \int \frac{f(x,u_n)u_n}{\|u_n\|^4} \to -\infty.$$

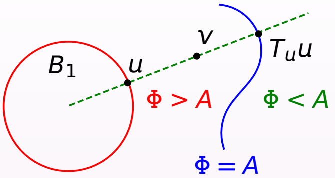


$$\varphi : X \backslash B_1 \to \Phi_A,$$

$$\varphi(v) = \begin{cases} T_{v/\|v\|} \dfrac{v}{\|v\|} & \text{if } \Phi(v) > A \\ v & \text{if } \Phi(v) \le A \end{cases}$$

Pf of Thm 2. For $u \in \partial B_1$, it is clear that

$$\Phi(tu) \to -\infty \qquad \text{as } t \to +\infty.$$



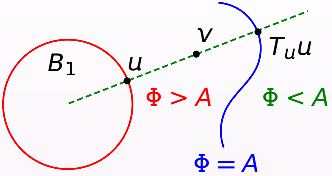
$$\varphi : X \backslash B_1 \to \Phi_A,$$

$$\varphi(v) = \begin{cases} T_{v/\|v\|} \dfrac{v}{\|v\|} & \text{if } \Phi(v) > A \\ v & \text{if } \Phi(v) \le A \end{cases}$$

Pf of Thm 2. For $u \in \partial B_1$, it is clear that

$$\Phi(tu) \to -\infty \qquad \text{as } t \to +\infty.$$

Thus $\exists 1\, T_u > 1$ s.t. $\Phi(T_u u) = A$.



$$\varphi : X \backslash B_1 \to \Phi_A,$$

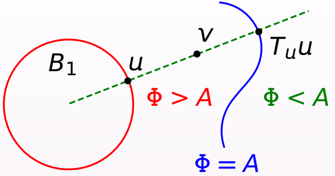
$$\varphi(v) = \begin{cases} T_{v/\|v\|} \dfrac{v}{\|v\|} & \text{if } \Phi(v) > A \\ v & \text{if } \Phi(v) \le A \end{cases}$$

Pf of Thm 2. For $u \in \partial B_1$, it is clear that

$$\Phi(tu) \to -\infty \qquad \text{as } t \to +\infty.$$

Thus $\exists 1\, T_u > 1$ s.t. $\Phi(T_u u) = A$. By Lem 5

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=T_u} \Phi(su)$$



$$\varphi : X \backslash B_1 \to \Phi_A,$$

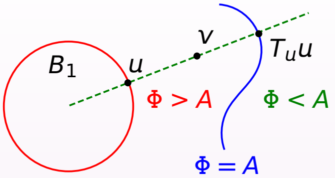
$$\varphi(v) = \begin{cases} T_{v/\|v\|} \dfrac{v}{\|v\|} & \text{if } \Phi(v) > A \\ v & \text{if } \Phi(v) \le A \end{cases}$$

Pf of Thm 2. For $u \in \partial B_1$, it is clear that

$$\Phi(tu) \to -\infty \qquad \text{as } t \to +\infty.$$

Thus $\exists 1\, T_u > 1$ s.t. $\Phi(T_u u) = A$. By Lem 5

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=T_u} \Phi(su) = \frac{1}{T_u} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \Phi(t \cdot T_u u) < 0.$$



$$\varphi : X \backslash B_1 \to \Phi_A,$$

$$\varphi(v) = \begin{cases} T_{v/\|v\|} \dfrac{v}{\|v\|} & \text{if } \Phi(v) > A \\ v & \text{if } \Phi(v) \le A \end{cases}$$

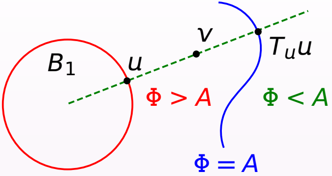
Pf of Thm 2. For $u \in \partial B_1$, it is clear that

$$\Phi(tu) \to -\infty \qquad \text{as } t \to +\infty.$$

Thus $\exists 1\, T_u > 1$ s.t. $\Phi(T_u u) = A$. By Lem 5

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=T_u} \Phi(su) = \frac{1}{T_u} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \Phi(t \cdot T_u u) < 0.$$

Following Wang (1991), by IFT $u \mapsto T_u$ is continuous,



$$\varphi : X\backslash B_1 \to \Phi_A,$$

$$\varphi(v) = \begin{cases} T_{v/\|v\|}\dfrac{v}{\|v\|} & \text{if } \Phi(v) > A \\ v & \text{if } \Phi(v) \le A \end{cases}$$

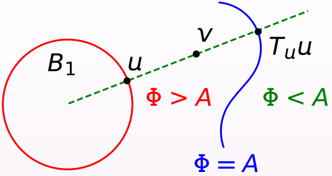
Pf of Thm 2. For $u \in \partial B_1$, it is clear that

$$\Phi(tu) \to -\infty \qquad \text{as } t \to +\infty.$$

Thus $\exists 1\, T_u > 1$ s.t. $\Phi(T_u u) = A$. By Lem 5

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=T_u} \Phi(su) = \frac{1}{T_u}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \Phi(t \cdot T_u u) < 0.$$

Following Wang (1991), by IFT $u \mapsto T_u$ is continuous, and we construct a deformation $\varphi : X\backslash B_1 \to \Phi_A$,



$$\varphi : X\backslash B_1 \to \Phi_A,$$

$$\varphi(v) = \begin{cases} T_{v/\|v\|} \dfrac{v}{\|v\|} & \text{if } \Phi(v) > A \\ v & \text{if } \Phi(v) \le A \end{cases}$$

Pf of Thm 2. For $u \in \partial B_1$, it is clear that

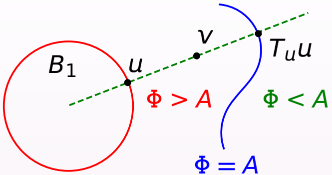
$$\Phi(tu) \to -\infty \qquad \text{as } t \to +\infty.$$

Thus $\exists 1\, T_u > 1$ s.t. $\Phi(T_u u) = A$. By Lem 5

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=T_u} \Phi(su) = \frac{1}{T_u} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \Phi(t \cdot T_u u) < 0.$$

Following Wang (1991), by IFT $u \mapsto T_u$ is continuous, and we construct a deformation $\varphi : X\backslash B_1 \to \Phi_A$, and deduce (critical groups)

$$C_i(\Phi, \infty) = \mathrm{H}_i(X, \Phi_A)$$



$$\varphi : X \backslash B_1 \to \Phi_A,$$

$$\varphi(v) = \begin{cases} T_{v/\|v\|} \dfrac{v}{\|v\|} & \text{if } \Phi(v) > A \\ v & \text{if } \Phi(v) \le A \end{cases}$$

Pf of Thm 2. For $u \in \partial B_1$, it is clear that

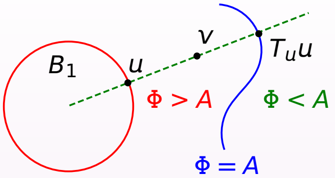
$$\Phi(tu) \to -\infty \qquad \text{as } t \to +\infty.$$

Thus $\exists 1\, T_u > 1$ s.t. $\Phi(T_u u) = A$. By Lem 5

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=T_u} \Phi(su) = \frac{1}{T_u} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \Phi(t \cdot T_u u) < 0.$$

Following Wang (1991), by IFT $u \mapsto T_u$ is continuous, and we construct a deformation $\varphi : X \backslash B_1 \to \Phi_A$, and deduce (critical groups)

$$C_i(\Phi, \infty) = \mathrm{H}_i(X, \Phi_A) = \mathrm{H}_i(X, X \backslash B_1) = 0 \qquad \text{for } i \in \mathbb{N}_0.$$



$$\varphi : X \backslash B_1 \to \Phi_A,$$
$$\varphi(v) = \begin{cases} T_{v/\|v\|} \dfrac{v}{\|v\|} & \text{if } \Phi(v) > A \\ v & \text{if } \Phi(v) \le A \end{cases}$$

Pf of Thm 2. For $u \in \partial B_1$, it is clear that

$$\Phi(tu) \to -\infty \qquad \text{as } t \to +\infty.$$

Thus $\exists 1\, T_u > 1$ s.t. $\Phi(T_u u) = A$. By Lem 5

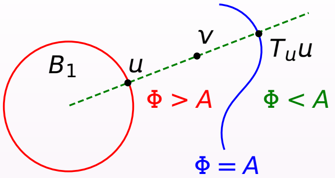
$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=T_u} \Phi(su) = \frac{1}{T_u} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \Phi(t \cdot T_u u) < 0.$$

Following Wang (1991), by IFT $u \mapsto T_u$ is continuous, and we construct a deformation $\varphi : X \backslash B_1 \to \Phi_A$, and deduce (critical groups)

$$C_i(\Phi, \infty) = \mathrm{H}_i(X, \Phi_A) = \mathrm{H}_i(X, X \backslash B_1) = 0 \qquad \text{for } i \in \mathbb{N}_0.$$

Since Φ has a loc link at $u = 0$, by Liu (1989) (for $\ell = \dim X^-$)

$$C_\ell(\Phi, 0) \neq 0.$$



$$\varphi : X\backslash B_1 \to \Phi_A,$$

$$\varphi(v) = \begin{cases} T_{v/\|v\|}\dfrac{v}{\|v\|} & \text{if } \Phi(v) > A \\ v & \text{if } \Phi(v) \le A \end{cases}$$

Pf of Thm 2. For $u \in \partial B_1$, it is clear that

$$\Phi(tu) \to -\infty \qquad \text{as } t \to +\infty.$$

Thus $\exists 1\, T_u > 1$ s.t. $\Phi(T_u u) = A$. By Lem 5

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=T_u} \Phi(su) = \frac{1}{T_u}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \Phi(t \cdot T_u u) < 0.$$

Following Wang (1991), by IFT $u \mapsto T_u$ is continuous, and we construct a deformation $\varphi : X\backslash B_1 \to \Phi_A$, and deduce (critical groups)

$$C_i(\Phi,\infty) = \mathrm{H}_i(X,\Phi_A) = \mathrm{H}_i(X, X\backslash B_1) = 0 \qquad \text{for } i \in \mathbb{N}_0.$$

Since Φ has a loc link at $u = 0$, by Liu (1989) (for $\ell = \dim X^-$)

$$C_\ell(\Phi, 0) \neq 0. \qquad \text{Hence } C_\ell(\Phi,0) \neq C_\ell(\Phi,\infty),$$

Applying Bartsch & Li (1997), Φ has a crt pt $u \neq 0$.

References

Poppenberg M, Schmitt K, Wang ZQ (2002). On the existence of soliton solutions to quasilinear Schrödinger equations. Calc. Var. Partial Differential Equations, 14(3) 329–344.

Liu Jq, Wang Yq, Wang ZQ (2003). Soliton solutions for quasilinear Schrödinger equations. II. J. Differential Equations, 187(2) 473–493.

Colin M, Jeanjean L (2004). Solutions for a quasilinear Schrödinger equation: a dual approach. Nonlinear Anal., 56(2) 213–226.

Shen Z, Han Z (2015). Existence of solutions to quasilinear Schrödinger equations with indefinite potential. Electron. J. Differential Equations, No. 91, 9.

Liu S, Zhou J (2018). Standing waves for quasilinear Schrödinger equations with indefinite potentials. J. Differential Equations, 265(9) 3970–3987.

Silva ED, Silva JS (2019). Quasilinear Schrödinger equations with nonlinearities interacting with high eigenvalues. J. Math. Phys., 60(8) 081504, 24.

Wang DB, Yang K (2015). Existence of solutions for a class of quasilinear Schrödinger equations on $\mathbb{R}$. Bound. Value Probl., 2015:215, 5.

Chen S, Wang C (2014). Existence of multiple nontrivial solutions for a Schrödinger-Poisson system. J. Math. Anal. Appl., 411(2) 787–793.

Benci V, Fortunato D (1998). An eigenvalue problem for the Schrödinger-Maxwell equations. Topol. Methods Nonlinear Anal., 11(2) 283–293.

Chen H, Liu S (2015). Standing waves with large frequency for 4-superlinear Schrödinger-Poisson systems. Ann. Mat. Pura Appl. (4), 194(1) 43–53.

Liu S, Wu Y (2017). Standing waves for 4-superlinear Schrödinger-Poisson systems with indefinite potentials. Bull. Lond. Math. Soc., 49(2) 226–234.

Bartsch T, Liu Z, Weth T (2004). Sign changing solutions of superlinear Schrödinger equations. Comm. Partial Differential Equations, 29(1-2) 25–42.

Chen J (2014). Multiple positive solutions to a class of modified nonlinear Schrödinger equations. J. Math. Anal. Appl., 415(2) 525–542.

Brézis H, Lieb E (1983). A relation between pointwise convergence of functions and convergence of functionals. Proc. Amer. Math. Soc., 88(3) 486–490.

Clark DC (1972/73). A variant of the Lusternik-Schnirelman theory. Indiana Univ. Math. J., 22 65–74.

Liu JQ (1989). The Morse index of a saddle point. Systems Sci. Math. Sci., 2(1) 32–39.

Wang ZQ (1991). On a superlinear elliptic equation. Ann. Inst. H. Poincaré Anal. Non Linéaire, 8(1) 43–57.

Bartsch T, Li S (1997). Critical point theory for asymptotically quadratic functionals and applications to problems with resonance. Nonlinear Anal., 28(3) 419–441.

Thank you!

http://lausb.github.io