13. L. Molnár, *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces*, Lecture Notes in Mathematics, vol. 1895, Springer-Verlag, Berlin, 2007.
14. Á. Münnich, Gy. Maksa, and R. J. Mokken, n-variable bisection, *J. Math. Psychol.* **44** (2000) 569–581. doi:10.1006/jmps.1999.1262
15. Gy. Nagy, ed., *KöMaL—Mathematical and Physical Journal for Secondary Schools*, available at http://www.komal.hu.
16. Zs. Páles, On the characterization of means defined on a linear space, *Publ. Math. Debrecen* **31** (1984) 19–27.
17. S. Presić, Méthode de résolution d'une classe d'équations fonctionnelles linéaries, *Univ. Beograd, Publ. Elektrotechn. Fak. Scr. Math. Fiz.* **115–121** (1963) 21–28.
18. ———, Sur l'équation fonctionnelle $f(x) = H(x, f(x), f(\theta_2 x), \ldots, f(\theta_n x))$, *Univ. Beograd, Publ. Elektrotechn. Fak. Scr. Math. Fiz.* **115–121** (1963) 17–20.
19. C. G. Small, *Functional Equations and How to Solve Them*, Springer Science New York, 2007.
20. L. Székelyhidi, *Discrete Spectral Synthesis and Its Applications*, Springer Monographs in Mathematics, Springer-Verlag, Dordrecht, 2006.

Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary
besse@math.klte.hu

On the Regularity of Operators Near a Regular Operator

Shibo Liu

Abstract. Using the Riesz theorem, we give a new proof that the linear operators near a regular operator are regular.

Dedicated to Professor Shujie Li on the occasion of his 70th birthday

1. INTRODUCTION. Let X be a Banach space and Y be a normed linear space. Recall that $A \in \mathcal{L}(X, Y)$ is a regular operator if $A : X \to Y$ is invertible and the inverse operator $A^{-1} : Y \to X$ is bounded. A classical result in operator theory is the following.

Theorem 1. *Let $A \in \mathcal{L}(X, Y)$ be a regular operator. Then there is some $\varepsilon > 0$ such that if $B \in \mathcal{L}(X, Y)$ and $\|B - A\| < \varepsilon$, then B is regular.*

Let $\mathcal{R}(X, Y)$ denotes the set of all regular operators from X to Y. Then this theorem means that $\mathcal{R}(X, Y)$ is open in $\mathcal{L}(X, Y)$.

Theorem 1 plays a major role in various area in mathematics. For example, it is used in the proof of the inverse function theorem in Banach spaces; see, e.g., [**1**, Theorem 4.1.1]. In the traditional proof of Theorem 1, one considers the case $X = Y$ and uses the fact that if $T \in \mathcal{L}(X, X)$ and $\|T\| < 1$ then $1_X - T$ is regular [**2**, Theorem 17.1.2]. Here 1_X is the identity operator in X. The general case is reduced to the above setting by considering the operator $A^{-1}B$.

The above proof relies on the convergence of series in $\mathcal{L}(X, X)$. In this note, we provide a different proof, which is more geometric in nature, and illustrates another application of the Riesz theorem.

doi:10.4169/000298910X523425

2. PROOF OF THEOREM 1. Since A is regular, we can let $\varepsilon = \|A^{-1}\|^{-1}$. Then

$$\varepsilon \|x\| \leq \|Ax\|, \qquad \text{for all } x \in X.$$

If $\|B - A\| < \varepsilon$, then $\delta = \varepsilon - \|B - A\| > 0$. For all $x \in X$ we have

$$\begin{aligned}\|Bx\| &= \|Ax - (Ax - Bx)\| \\ &\geq \|Ax\| - \|Ax - Bx\| \geq \delta \|x\|. \end{aligned} \tag{1}$$

This inequality implies that B is an injection.

Moreover, the range $\operatorname{Im} B$ is closed in Y. In fact, let $y_n = Bx_n$ be a sequence in $\operatorname{Im} B$ such that $y_n \to y$. By (1) it follows that x_n is a Cauchy sequence in X. Since X is a Banach space, $x_n \to x$ for some $x \in X$. Now it is easy to show that $y = Bx \in \operatorname{Im} B$.

If we can prove that $\operatorname{Im} B = Y$, then B is invertible and (1) implies that $B^{-1} : Y \to X$ is bounded and the proof is completed.

Assume for a contradiction that $\operatorname{Im} B \neq Y$. Since $\operatorname{Im} B$ is closed, by the Riesz theorem [**2**, Lemma 5.2.7], for any $n \in \mathsf{N}$, there exists $y_n \in Y$ such that $\|y_n\| = 1$ and

$$1 - \frac{1}{n} < \inf_{y \in \operatorname{Im} B} \|y_n - y\|. \tag{2}$$

Since A is regular, in particular surjective, there exists a (unique) $x_n \in X$ such that $y_n = Ax_n$, so

$$\|x_n\| = \|A^{-1} y_n\| \leq \|A^{-1}\| \, \|y_n\| = \|A^{-1}\|.$$

Noting that $Bx_n \in \operatorname{Im} B$, we deduce from (2) that

$$\begin{aligned} 1 - \frac{1}{n} < \inf_{y \in \operatorname{Im} B} \|y_n - y\| &\leq \|y_n - Bx_n\| \\ &= \|Ax_n - Bx_n\| \\ &\leq \|A - B\| \, \|x_n\| \leq \|B - A\| \, \|A^{-1}\|. \end{aligned}$$

Hence $\|B - A\| \geq \|A^{-1}\|^{-1} = \varepsilon$. This contradicts $\|B - A\| < \varepsilon$, and the proof is concluded.

ACKNOWLEDGMENTS. The author was supported by National Natural Science Foundation of China (10601041).

REFERENCES

1. P. Drábek and J. Milota, *Methods of Nonlinear Analysis*, Birkhäuser, Basel, 2007.
2. P. D. Lax, *Functional Analysis*, Wiley-Interscience, New York, 2002.

Department of Mathematics, Shantou University, Shantou 515063, P.R. China
liusb@stu.edu.cn