On the integral formula of the Jacobian determinant

Shibo Liu

Department of Mathematics & Systems Engineering, Florida Institute of Technology
Melbourne, FL 32901, USA

ABSTRACT. It is known that the integral of the Jacobian determinant of a smooth map $f : \bar{\Omega} \to \mathbb{R}^n$ depends only on $f|_{\partial\Omega}$ and this result leads to an analytic proof of the Brouwer fixed point theorem. In this note we provide two new proofs of this result, one by classical analysis and one by differential forms and Stokes formula.

1. Introduction

In a recent paper Krylov (2024), to respond inquiry from some readers about the solution of Exercise 1.2.1 in his book Krylov (2008), Krylov gives a proof of the following theorem, see Krylov (2024, Lemma 1), where $f_\pm$ only need to be C^1.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $f_\pm : \bar{\Omega} \to \mathbb{R}^n$ be C^2-maps such that $f_+|_{\partial\Omega} = f_-|_{\partial\Omega}$. then*

$$\int_\Omega \det f_+'(x)\mathrm{d}x = \int_\Omega \det f_-'(x)\mathrm{d}x. \tag{1.1}$$

Using this theorem, the *no-retraction theorem* follows immediately, see Krylov (2024, Corollary 1). Let B be the closed unit ball in $\mathbb{R}^n$. If there was a smooth retraction $R : B \to \partial B$, since $R|_{\partial B} = \mathrm{id}|_{\partial B}$, Theorem 1.1 with $f_+ = R$, $f_- = \mathrm{id}$ yields a contradiction

$$\int_B \det R'(x)\mathrm{d}x = \int_B \mathrm{d}x = \mathrm{Vol}(B) > 0, \tag{1.2}$$

because $\det R'(x) = 0$, a consequence of $R(B) \subset \partial B$. That the two integrals in (1.2) are equal also follows from a version of the changing variable formula given in Liu & Zhang (2017, Theorem 3.1): *For smooth closed bounded domain D in $\mathbb{R}^n$ and smooth map $\varphi : B \to D$, if $\varphi : \partial B \to \partial D$ is a diffeomorphism, then for continuous $f : D \to \mathbb{R}$ there holds*

$$\int_D f(y)\mathrm{d}y = \pm \int_B f(\varphi(x)) \det \varphi'(x)\mathrm{d}x.$$

The first equality in (1.2) follows by letting $f = 1$, $D = B$ and $\varphi = R$ in this formula. It is well known that the no-retraction theorem is equivalent to the famous Brouwer fixed point theorem.

Krylov's proof of Theorem 1.1 in Krylov (2024) is based on the observation: for small t, $f_\pm^t = \mathrm{id} + tf_\pm$ are diffeomorphisms and $f_+^t(\Omega) = f_-^t(\Omega)$ because $f_+|_{\partial\Omega} = f_-|_{\partial\Omega}$, thus by the changing variable formula

$$\int_\Omega \det(I + tf_+'(x))\mathrm{d}x = \mathrm{Vol}(f_+^t(\Omega)) = \mathrm{Vol}(f_-^t(\Omega)) = \int_\Omega \det(I + tf_-'(x))\mathrm{d}x.$$

Emails: **sliu@fit.edu** (S. Liu).

Since the two sides are polynomials in t, comparing the coefficients of t^n gives (1.1). To make the argument rigorous some issues including why $f_{\pm}^t(\partial\Omega) = \partial f_{\pm}^t(\Omega)$ and why $\partial f_{+}^t(\Omega) = \partial f_{-}^t(\Omega)$ implies $f_{+}^t(\Omega) = f_{-}^t(\Omega)$ need to be handled. These are clarified via a careful point set analysis in a long remark following the proof. One of the advantages of this argument is that the maps $f_{\pm}$ only need to be C^1.

Theorem 1.1 also appears in Evans (2010); Kulpa (1989). In Kulpa (1989), Kulpa defines

$$I(h^1, \dots, h^n) = \int_{\Omega} \det h'(x) \mathrm{d}x$$

for smooth $h : \bar{\Omega} \to \mathbb{R}^n$ and manages to show

$$I(f_+^1, f_+^2, \dots, f_+^n) = I(f_-^1, f_+^2, \dots, f_+^n) \tag{1.3}$$

using Fubini theorem and projections $\Pi_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ given by deleting the i-th component. Then (1.1) follows by applying (1.3) successively to replace the remaining columns of $\det f_+'(x)$ by those of $\det f_-'(x)$. While in Evans (2010, pp. 464), (1.1) is proved using the facts that $L(P) = \det P$ is a null Lagrangian and the energy $\int_{\Omega} L(w'(x), w(x), x)\mathrm{d}x$ of null Lagrangian $L : \mathbb{M}^{m\times n} \times \mathbb{R}^m \times \bar{\Omega} \to \mathbb{R}$ depends only on boundary conditions.

The purpose of this note is to present another two proofs of Theorem 1.1. The first one is performed by playing classical analysis, which is written for readers not familiar with differential forms and in our opinion is very elementary and transparent. The second one is quite short, which depends on differential form and Stokes formula on manifolds. As demonstrated in the paragraph after Theorem 1.1, our note is useful for understanding the Brouwer fixed point theorem from the analytic point of view. In addition to Evans (2010); Krylov (2024); Kulpa (1989); Liu & Zhang (2017) mentioned above, other analytic proofs of the Brouwer fixed point theorem can be found in (Dunford & Schwartz, 1958, pp. 467–470) and Kannai (1981); Lax (1999); Milnor (1978); Rogers (1980).

2. Proof of Theorem 1.1

Let $f_{\pm}^i$ be the i-th component of $f_{\pm}$ and $\Phi_{\pm} = (f_{\pm}^2, \dots, f_{\pm}^n)$. Then $\Phi_+ = \Phi_-$ on $\partial\Omega$. Let

$$A_{\pm} = \left(\frac{\partial(f_{\pm}^2, \dots, f_{\pm}^n)}{\partial(x^2, \dots, x^n)}, -\frac{\partial(f_{\pm}^2, f_{\pm}^3, \dots, f_{\pm}^n)}{\partial(x^1, x^3, \dots, x^n)}, \dots, (-1)^{n+1} \frac{\partial(f_{\pm}^2, \dots, f_{\pm}^n)}{\partial(x^1, \dots, x^{n-1})} \right),$$

then the i-th component of $A_{\pm}$ is just the cofactor of $\partial_{x^i} f_{\pm}^1$ in $\det f_{\pm}'(x)$. By Jacobi identity (see e.g. Kannai (1981, Eq. (5)) or Brezis *et al.* (2024, Eq. (0.3))), we have $\operatorname{div} A_{\pm} = 0$. Thus

$$\operatorname{div}(f_{\pm}^1 A_{\pm}) = \nabla f_{\pm}^1 \cdot A_{\pm} + f_{\pm}^1 \operatorname{div} A_{\pm} = \nabla f_{\pm}^1 \cdot A_{\pm} = \det f_{\pm}'(x). \tag{2.1}$$

For $a \in \partial\Omega$, let $\eta : (u^1, \dots, u^{n-1}) \mapsto x$ be a local parametrization of $\partial\Omega$ at $a = \eta(\alpha)$ such that

$$N(a) = \left(\frac{\partial(x^2, \dots, x^n)}{\partial(u^1, \dots, u^{n-1})}, -\frac{\partial(x^1, x^3, \dots, x^n)}{\partial(u^1, \dots, u^{n-1})}, \dots, (-1)^{n+1} \frac{\partial(x^1, \dots, x^{n-1})}{\partial(u^1, \dots, u^{n-1})} \right)_{\alpha}$$

is an outward normal vector of $\partial\Omega$ at a. Applying the chain rule to the composition $y_\pm = \Phi_\pm(\eta(u))$, we see that

$$\begin{pmatrix} \frac{\partial y_\pm^2}{\partial u^1} & \cdots & \frac{\partial y_\pm^2}{\partial u^{n-1}} \\ \frac{\partial y_\pm^n}{\partial u^1} & \cdots & \frac{\partial y_\pm^n}{\partial u^{n-1}} \end{pmatrix}_\alpha = \begin{pmatrix} \frac{\partial y_\pm^2}{\partial x^1} & \cdots & \frac{\partial y_\pm^2}{\partial x^n} \\ \frac{\partial y_\pm^n}{\partial x^1} & \cdots & \frac{\partial y_\pm^n}{\partial x^n} \end{pmatrix}_a \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^1}{\partial u^{n-1}} \\ \frac{\partial x^n}{\partial u^1} & \cdots & \frac{\partial x^n}{\partial u^{n-1}} \end{pmatrix}_\alpha,$$

where the matrix on the left is the Jacobian matrix $(\Phi_\pm \circ \eta)'(\alpha)$.

Because $\Phi_+ = \Phi_-$ on $\partial\Omega$ and $\eta(u) \in \partial\Omega$ for u near α, we have $\Phi_+ \circ \eta = \Phi_- \circ \eta$ near α. Using the Cauchy-Binet formula we deduce

$$\begin{aligned}(A_+ \cdot N)(a) &= \sum_{i=1}^n \left.\frac{\partial(f_+^2, \dots, f_+^n)}{\partial(x^1, \dots, \hat{x}^i, \dots, x^n)}\right|_a \cdot \left.\frac{\partial(x^1, \dots, \hat{x}^i, \dots, x^n)}{\partial(u^1, \dots, u^{n-1})}\right|_\alpha \\ &= \det(\Phi_+ \circ \eta)'(\alpha) = \det(\Phi_- \circ \eta)'(\alpha) = (A_- \cdot N)(a).\end{aligned}$$

Therefore

$$A_+ \cdot \nu = \frac{A_+ \cdot N}{|N|} = \frac{A_- \cdot N}{|N|} = A_- \cdot \nu \tag{2.2}$$

on $\partial\Omega$, where $\nu = N/|N|$ is outward unit normal vector field on $\partial\Omega$.

Since $f_+^1 = f_-^1$ on $\partial\Omega$, using the divergence theorem we deduce from (2.1) and (2.2)

$$\begin{aligned}\int_\Omega \det f_+'(x)\mathrm{d}x &= \int_\Omega \operatorname{div}(f_+^1 A_+)\mathrm{d}x = \int_{\partial\Omega} f_+^1 A_+ \cdot \nu \mathrm{d}\sigma \\ &= \int_{\partial\Omega} f_-^1 A_- \cdot \nu \mathrm{d}\sigma = \int_\Omega \det f_-'(x)\mathrm{d}x.\end{aligned}$$

Example 2.1. Let A be an $n \times n$ matrix, $\varphi \in C_0^\infty(\Omega, \mathbb{R}^n)$. Set $f_+(x) = Ax + \varphi(x)$, $f_-(x) = Ax$, we get the formula stated in Ambrosio *et al.* (2018, pp. 21):

$$\int_\Omega \det(A + \varphi'(x))\mathrm{d}x = \det(A) \cdot m(\Omega).$$

3. Another proof using differential forms

To conclude this note, we present another proof using differential forms. Let f^i be the components of $f : \bar{\Omega} \to \mathbb{R}^n$, then it is well known that

$$\begin{aligned}\det f'(x)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n &= \mathrm{d}f^1 \wedge \mathrm{d}f^2 \wedge \cdots \wedge \mathrm{d}f^n. \\ &= \mathrm{d}(f^1 \mathrm{d}f^2 \wedge \cdots \wedge \mathrm{d}f^n),\end{aligned}$$

where in the second equality we used $\mathrm{d}^2 = 0$. Recall the Stokes formula

$$\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} i^*\omega,$$

where ω is an $(n-1)$-form on Ω and i^* is the pull back induced by the embedding $i : \partial\Omega \to \Omega$. Because $i^* \circ \mathrm{d} = \mathrm{d} \circ i^*$, we conclude

$$\begin{aligned}\int_\Omega \det f'(x)\mathrm{d}x &= \int_\Omega \det f'(x)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ &= \int_\Omega \mathrm{d}(f^1 \mathrm{d}f^2 \wedge \cdots \wedge \mathrm{d}f^n) = \int_{\partial\Omega} i^*(f^1 \mathrm{d}f^2 \wedge \cdots \wedge \mathrm{d}f^n)\end{aligned}$$

$$= \int_{\partial\Omega} (f^1 \circ i)\mathrm{d}(f^2 \circ i) \wedge \cdots \wedge \mathrm{d}(f^n \circ i).$$

Clearly the right hand side depends only on $f \circ i = f|_{\partial\Omega}$.

References

Ambrosio L, Carlotto A, Massaccesi A (2018). *Lectures on elliptic partial differential equations*, vol. 18 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa.

Brezis H, Mawhin J, Mironescu P (2024). *A brief history of the Jacobian*. Commun. Contemp. Math., 26(2) Paper No. 2330001, 20.

Dunford N, Schwartz JT (1958). *Linear Operators. I. General Theory*, vol. Vol. 7 of *Pure and Applied Mathematics*. Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London. With the assistance of W. G. Bade and R. G. Bartle.

Evans LC (2010). *Partial differential equations*, vol. 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2nd ed.

Kannai Y (1981). *An elementary proof of the no-retraction theorem*. Amer. Math. Monthly, 88(4) 264–268.

Krylov NV (2008). *Lectures on elliptic and parabolic equations in Sobolev spaces*, vol. 96 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI.

Krylov NV (2024). *On a new proof of the key step in the proof of Brouwer's fixed point theorem*. J. Math. Sci. (N.Y.), 279(4) 468–471.

Kulpa W (1989). *An integral theorem and its applications to coincidence theorems*. vol. 30, 83–90. 17th Winter School on Abstract Analysis (Srní, 1989).

Lax PD (1999). *Change of variables in multiple integrals*. Amer. Math. Monthly, 106(6) 497–501.

Liu S, Zhang Y (2017). *On the change of variables formula for multiple integrals*. J. Math. Study, 50(3) 268–276.

Milnor J (1978). *Analytic proofs of the "hairy ball theorem" and the Brouwer fixed-point theorem*. Amer. Math. Monthly, 85(7) 521–524.

Rogers CA (1980). *A less strange version of Milnor's proof of Brouwer's fixed-point theorem*. Amer. Math. Monthly, 87(7) 525–527.