Surjections between Euclidean spaces, changing variable formula and Brouwer fixed point theorem

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1. Differentiation for vector functions

The Euclidean norm of a vector $x \in \mathbb{R}^m$ is

$$|x| = \sqrt{(x^1)^2 + \cdots + (x^m)^2}.$$

Let $U \subset \mathbb{R}^m$, $f : U \to \mathbb{R}^n$, $a \in U^\circ$. If there is $n \times m$ matrix A s.t.

$$f(a+h) - f(a) - Ah = o(|h|), \qquad \text{as } |h| \to 0, \tag{1}$$

we call A the derivative of f at a, write $A = f'(a)$ or $A = Df(a)$.
Let $f = \left(f^1, \ldots, f^n\right)$, denote the i^{th}-row of A by A^i, the i component of (1) is

$$f^i(a+h) - f^i(a) - A^i h = o(|h|).$$

Thus f^i is differentiable at a, and

$$A^i = \nabla f^i(a) = \left(\partial_1 f^i, \ldots, \partial_m f^i\right).$$

Therefore

$$A = \begin{pmatrix} \partial_1 f^1 & \cdots & \partial_m f^1 \\ \vdots & & \vdots \\ \partial_1 f^n & \cdots & \partial_m f^n \end{pmatrix}, \qquad \text{the Jacobian matrix of } f \text{ at } a.$$

When $m=n$, the Jacobian determinant of A is

$$\frac{\partial(f^1,\ldots,f^m)}{\partial(x^1,\ldots,x^m)} = \det\begin{pmatrix} \partial_1 f^1 & \cdots & \partial_m f^1 \\ \vdots & & \vdots \\ \partial_1 f^n & \cdots & \partial_m f^n \end{pmatrix}. \tag{2}$$

Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$. $f: U \to \mathbb{R}^n$, $g: V \to \mathbb{R}^\ell$. If $f(U) \subset V$, we have

$$g \circ f : U \to \mathbb{R}^\ell.$$

Thm1 (Chain role). If f is differentiable at $a \in U$, g is differentiable at $b = f(a)$, then $g \circ f$ is differentiable at a and $(g \circ f)'(a) = g'(b) f'(a)$.

If $y = g(u)$, $u = f(x)$, then

$$\begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^m} \\ \vdots & & \vdots \\ \frac{\partial y^\ell}{\partial x^1} & \cdots & \frac{\partial y^\ell}{\partial x^m} \end{pmatrix}_a = \begin{pmatrix} \frac{\partial y^1}{\partial u^1} & \cdots & \frac{\partial y^1}{\partial u^n} \\ \vdots & & \vdots \\ \frac{\partial y^\ell}{\partial u^1} & \cdots & \frac{\partial y^\ell}{\partial u^n} \end{pmatrix}_b \begin{pmatrix} \frac{\partial u^1}{\partial x^1} & \cdots & \frac{\partial u^1}{\partial x^m} \\ \vdots & & \vdots \\ \frac{\partial u^n}{\partial x^1} & \cdots & \frac{\partial u^n}{\partial x^m} \end{pmatrix}_a, \quad \frac{\partial y^i}{\partial x^j} = \sum_{k=1}^n \frac{\partial y^i}{\partial u^k} \frac{\partial u^k}{\partial x^j}.$$

Thm2 (Inverse function). Let Ω be open in $\mathbb{R}^m$, $f : \Omega \to \mathbb{R}^m$ be C^1, $a \in \Omega$, $b = f(a)$. If $\det f'(a) \neq 0$, there are $U \in \mathscr{U}_a$ and $V \in \mathscr{U}_b$, s.t. $f : U \to V$ is diffeomorphism. (illustrating the basic idea of differential calculus)

Rek1. $\det f'(a) \neq 0$ means $f'(a) : \mathbb{R}^m \to \mathbb{R}^m$ is a linear isomorphism; then f is locally invertible.

Cor1 (Local surjection). Let Ω be open in $\mathbb{R}^m$, $f : \Omega \to \mathbb{R}^n$ be C^1, $a \in \Omega$. If $\operatorname{rank} f'(a) = n$, then $b = f(a)$ is an interior pt of $f(\Omega)$. $b \in [f(\Omega)]^\circ$

Rek2. $\operatorname{rank} f'(a) = n$ means $f'(a) : \mathbb{R}^m \to \mathbb{R}^n$ is linear surjection; then f is locally surjective.

Pf(Cor1). .

$$f'(a) = \begin{pmatrix} \partial_1 f^1 & \cdots & \partial_n f^1 & \cdots & \partial_m f^1 \\ \vdots & & \vdots & & \vdots \\ \partial_1 f^n & \cdots & \partial_n f^n & \cdots & \partial_m f^n \end{pmatrix}, \quad \det \begin{pmatrix} \partial_1 f^1 & \cdots & \partial_n f^1 \\ \vdots & & \vdots \\ \partial_1 f^n & \cdots & \partial_n f^n \end{pmatrix} \neq 0.$$

Let $F : \Omega \to \mathbb{R}^m$, $F(x) = (f^1(x), \ldots, f^n(x), x^{n+1}, \ldots, x^m)$. then

$$F'(a) = \begin{pmatrix} \left(\partial_i f^j\right)_{i,j=1,\ldots,n} & \left(\partial_i f^j\right)_{i>n} \\ 0 & I_{m-n} \end{pmatrix}$$

is invertible. We apply the Inverse Function Theorem to F.

2. Suejectivity of $f: \mathbb{R}^m \to \mathbb{R}^n$ and FTA

Thm3 (FTA). Let $a_i \in \mathbb{C}$, $n \geq 1$,

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

be a polynomial, then $\exists \xi \in \mathbb{C}$ s.t. $p(\xi) = 0$.

- Gauss gave the first proof more than 300 years ago. Even in 21st century, new proofs keep emerging [Sen00, LL10].
- Usual proofs use Complex Analysis or Algebraic Topology, many proofs are collected in [FR97].
- A proof via Green theorem can be found in the textbook [OY03] by G. Ouyang (need multi-valued function Arg, but can avoid).
- [LL10] applied Fourier inverse transform to prove FTA.
- [Sen00] applied Inverse Fun Thm, the proof need topological concepts such as open sets (and connectivity) in subspace of $\mathbb{C}$.
- Our motivation was to avoid subspace topology.

Let $z = x + iy$, $p(z) = u(x, y) + iv(x, y)$. View p as a map $p : \mathbb{R}^2 \to \mathbb{R}^2$,

$$p(x, y) = (u(x, y), v(x, y)) .$$

From Cauchy-Riemann equation we know

$$p'(z) = 0 \iff \det p'(x, y) = 0, \quad (x, y) \text{ is critical point of } p : \mathbb{R}^2 \to \mathbb{R}^2$$

Since $p'(z)$ is polynomial of order $(n-1)$, the map $p : \mathbb{R}^2 \to \mathbb{R}^2$ has finitely many critical points. The fact

$$\lim_{|(x,y)|\to\infty} |p(x, y)| = +\infty,$$

brings our attention to a classical result (advanced calculus exercise):

Pro1 ([Dei85, Page 24]). If the C^1-map $f : \mathbb{R}^n \to \mathbb{R}^n$ is coercive:

$$\lim_{|x|\to\infty} |f(x)| = +\infty, \tag{3}$$

and $\det Df(x) \neq 0$ for $\forall x \in \mathbb{R}^n$, then $f(\mathbb{R}^n) = \mathbb{R}^n$.

Rek3. (1) If we can weaken $\det Df(x) \neq 0$ for $\forall x \in \mathbb{R}^n$ to $\det Df(x) \neq 0$ except finitely many x, then FTA follows immediately.

(2) (3) means that $f(\mathbb{R}^n)$ is closed in $\mathbb{R}^n$. This motivates our Thm5.

Def1. Let $f:\mathbb{R}^m\to\mathbb{R}^n$ be C^1, $a\in\mathbb{R}^m$. If $Df(a):\mathbb{R}^m\to\mathbb{R}^n$ is not surjective (i.e., $\operatorname{rank} Df(a)<n$), we call a a critical point of f.

Thm4 (Local surjection). Let $\Omega\subset\mathbb{R}^m$, $f:\Omega\to\mathbb{R}^n$ be C^1, $a\in\Omega^\circ$. If $Df(a):\mathbb{R}^m\to\mathbb{R}^n$ is surjective (i.e. $\operatorname{rank} Df(a)=n$), then $f(a)\in[f(\Omega)]^\circ$.

Rek4. [Ma[11]] used this to prove the Lagrange multiplier theorem.

Thm5 ([LL[18]]). If the C^1-map $f:\mathbb{R}^m\to\mathbb{R}^n$ has only finitely many critical points, $n\geq 2$, $f(\mathbb{R}^m)$ is closed in $\mathbb{R}^n$, then $f(\mathbb{R}^m)=\mathbb{R}^n$. (necessary condition)

Pf. Let K be the critical set of f, then $f(K)$ is also finite.

- $\mathbb{R}^m\backslash K$ is open in $\mathbb{R}^m$, $\forall x\in\mathbb{R}^m\backslash K$, $Df(x):\mathbb{R}^m\to\mathbb{R}^n$ is surjective.

  By Thm4, $f(x)$ is interior to $A=f(\mathbb{R}^m\backslash K)$. hence A is open in $\mathbb{R}^n$.

- By assumption,
$$A\cup f(K)=f(\mathbb{R}^m\backslash K)\cup f(K)=f(\mathbb{R}^m)\quad\text{is closed.}$$

- Noting that $f(\mathbb{R}^m)=\overline{f(\mathbb{R}^m)}\supset\overline{A}$, it suffices to prove the intuitive result:

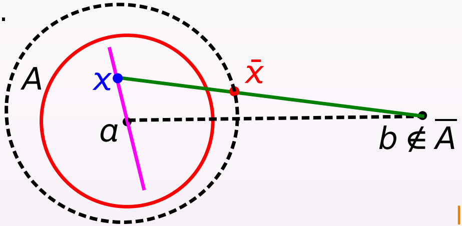
  if the union of open set A and a finite set is closed, then $\overline{A}=\mathbb{R}^n$.

Lem1. Let $n \geq 2$, A be nonempty open set in $\mathbb{R}^n$. If there are k points $p_i \in \mathbb{R}^n$ s.t. $A \cup \{p_i\}_{i=1}^k$ is closed, then $\overline{A} = \mathbb{R}^n$.

Pf. Since A is open, $A \cap \partial A = \emptyset$.

$$A \cup \{p_i\} = \overline{A \cup \{p_i\}}$$
$$= A \cup \partial A \cup \{p_i\}$$
$$\implies \partial A \subset \{p_i\},$$

So ∂A is finite.

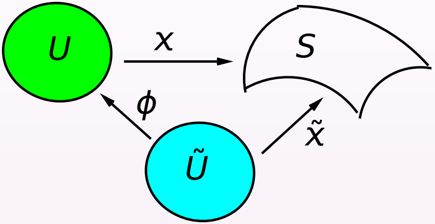


Cor2. Assume M is m-dimensional smooth manifold without boundary, C^1-map $f : M \to \mathbb{R}^n$ has only finitely many critical points, $n \geq 2$. If $f(M)$ is closed in $\mathbb{R}^n$, then $f(M) = \mathbb{R}^n$.

Cor3. Let $n \geq 2$, M be m-dimensional compact manifold without boundary. If $f : M \to \mathbb{R}^n$ is C^1-map, then f has infinitely many critical points.

3. Changing variable for multiple integrals

Motivated by [dC[76]] (CVF for double integral via Green Theorem), we assume CVF for $(m-1)$-integrals, define **surface integral** in $\mathbb{R}^m$ and prove the **Divergence Theorem**, then prove CVF for m-integrals.



3.1. Surface integral and Divergence Theorem

- Let U be Jordan measurable closed domain in $\mathbb{R}^{m-1}$, a C^1-parametrized surface is a C^1-map $x : U \to \mathbb{R}^m$ satisfying $\operatorname{rank}\left(\partial x^i/\partial u^j\right) = m-1$ and injective in U°.
- x is equivalent with $\tilde{x} : \tilde{U} \to \mathbb{R}^m$ if there is diffism $\phi : \tilde{U} \to U$ s.t. $\tilde{x} = x \circ \phi$.

  Then $\tilde{x}(\tilde{U}) = x(U)$, we can identify the equivalent class $[x]$ with $x(U)$, called **smooth surface**, and denoted by $S = x(U)$ or $S = [x : U \to \mathbb{R}^m]$.

Using Cramer we know that a **normal vector** of $S = [x : U \to \mathbb{R}^m]$ at $x(u)$ is

$$N(u) = \left(\frac{\partial(x^2, \ldots, x^m)}{\partial(u^1, \ldots, u^{m-1})}, \ldots, (-1)^{m+1} \frac{\partial(x^1, \ldots, x^{m-1})}{\partial(u^1, \ldots, u^{m-1})} \right).$$

The **surface integral** of cont $f : S \to \mathbb{R}$ on $S = [x : U \to \mathbb{R}^m]$ is defined by

$$\int_S f(x) \mathrm{d}\sigma = \int_U f(x(u)) \, |N(u)| \, \mathrm{d}u. \tag{4}$$

By $(m-1)$-dim CVF, the RHS is **indept** with the parametrization of S.

For piece-wise smooth surface $\Sigma = \bigcup_{i=1}^{\ell} S_i$, where $S_i = x_i(U_i)$ interiorly-disjoint, set

$$\int_\Sigma f \mathrm{d}\sigma = \sum_{i=1}^{\ell} \int_{S_i} f \mathrm{d}\sigma. \qquad x_i(U_i^\circ) \cap x_j(U_j^\circ) = \emptyset.$$

Thm6 (Divergence Theorem). Let $D \subset \mathbb{R}^m$ be bounded closed domain, ∂D piece-wise smooth, $F \in C^1(D, \mathbb{R}^m)$, n is unit outer normal of ∂D, then

$$\int_D \operatorname{div} F \, \mathrm{d}x = \int_{\partial D} F \cdot n \, \mathrm{d}\sigma. \qquad \int_D \partial_i f \, \mathrm{d}x = \int_{\partial D} f n^i \, \mathrm{d}\sigma \quad \text{for } f \in C^1(D).$$

3.2. Simple domain

A bdd domain Ω is **simple**, if there is $(m-1)$-dim C^1–parametrized surface $x: U \to \mathbb{R}^m$ s.t. $\partial\Omega = x(U)$. Note that U is closed, x is injective in U°.

Exm1. Balls B in $\mathbb{R}^m$ is simple. For $m=3$, we take

$$x:[0,\pi]\times[0,2\pi]\to\mathbb{R}^3,\quad (\varphi,\theta)\mapsto(\sin\varphi\cos\theta,\sin\varphi\sin\theta,\cos\varphi).$$

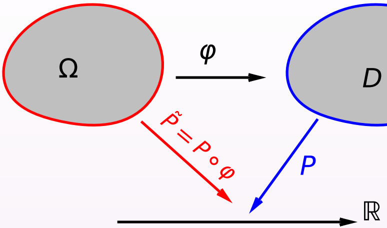
Thm7 ([LZ17]). Let D and Ω be bdd open domain in $\mathbb{R}^m$ with C^1-boundary, Ω simple, the C^2-map $\varphi:\overline{\Omega}\to\overline{D}$ maps $\partial\Omega$ onto ∂D diffeomly, $f\in C(\overline{D})$, then

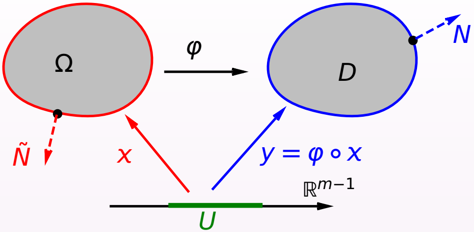
$$\int_D f(y)\mathrm{d}y = \pm\int_\Omega f(\varphi(x))J_\varphi(x)\mathrm{d}x, \qquad \text{where } J_\varphi(x)=\det\varphi'(x).$$

Rek5. (1) Using mollifier, we may assume that f is the restriction to $\overline{D}$ of smooth fun on $\mathbb{R}^m$. Thus we can take $P\in C^1(\mathbb{R}^m)$ s.t. $\partial P/\partial y^1 = f$.
(2) φ only good on bdry, this leads to Brouwer Fixed Point Theorem (Thm9).
(3) P. Lax *et. al.* have papers on this, [Lax99, Lax01, Tay02, Iva05].

Lax 99: $f\in C_0(\mathbb{R}^m)$
φ is identity outside some ball

$$\int_{\mathbb{R}^m} f(y)\mathrm{d}y = \int_{\mathbb{R}^m} f(\varphi(x))\det\left(\frac{\partial y}{\partial x}\right)\mathrm{d}x.$$





Pf. Let $\varphi : y^i = y^i(x^1, \ldots, x^m)$. Take P s.t. $\partial P/\partial y^1 = f$, set $\tilde{P} = P \circ \varphi$.
Let $x : U \to \mathbb{R}^m$ be parametrization of $\partial\Omega$, then $y = \varphi \circ x$ is parametrion of ∂D,

$$N = \left(\frac{\partial(y^2, \ldots, y^m)}{\partial(u^1, \ldots, u^{m-1})}, \ldots, (-1)^{m+1} \frac{\partial(y^1, \ldots, y^{m-1})}{\partial(u^1, \ldots, u^{m-1})} \right)$$

is normal vector of ∂D, $n = \pm N/|N| = (n^1, \ldots, n^m)$ unit normal.

Let $\quad A = (A_1, \ldots, A_m), \quad \tilde{N} = (\tilde{N}^1, \ldots, \tilde{N}^m), \quad$ where

$$A_i = (-1)^{i+1} \frac{\partial(y^2, \ldots, y^m)}{\partial(x^1, \ldots, \hat{x}^i, \ldots, x^m)}, \quad \tilde{N}^i = (-1)^{i+1} \frac{\partial(x^1, \ldots, \hat{x}^i, \ldots, x^m)}{\partial(u^1, \ldots, u^{m-1})}.$$

then $\tilde{n} = \pm\tilde{N}/|\tilde{N}|$ is unit normal vec of $\partial\Omega$.

By Cauchy-Binet formula,

$$\pm n^1|N| = \frac{\partial(y^2,\ldots,y^m)}{\partial(u^1,\ldots,u^{m-1})}$$

$$= \sum_{i=1}^{m} \frac{\partial(y^2,\ldots,y^m)}{\partial(x^1,\ldots,\hat{x}^i,\ldots x^m)} \frac{\partial(x^1,\ldots,\hat{x}^i,\ldots x^m)}{\partial(u^1,\ldots,u^{m-1})} = A\cdot\tilde{N}.$$

By Divergence Theorem, $P\circ y = P\circ(\varphi\circ x) = (P\circ\varphi)\circ x = \tilde{P}\circ x$

$$\int_D f(y)\mathrm{d}y = \int_D \frac{\partial P}{\partial y^1}\mathrm{d}y = \int_{\partial D} P n^1 \mathrm{d}\sigma$$

$$= \pm\int_U P(y(u))n^1(u)\,|N(u)|\,\mathrm{d}u \qquad \tilde{N} = \pm|\tilde{N}|\tilde{n}$$

$$= \pm\int_U \tilde{P}(x(u))(A\cdot\tilde{N})\,\mathrm{d}u = \pm\int_U(\tilde{P}A\cdot\tilde{n})|\tilde{N}|\,\mathrm{d}u$$

$$= \pm\int_{\partial\Omega}\tilde{P}A\cdot\tilde{n}\,\mathrm{d}\sigma = \pm\int_\Omega \mathrm{div}(\tilde{P}A)\,\mathrm{d}x. \tag{5}$$

To compute $\operatorname{div}(\tilde{P}A)$, let $y_i^k = \partial y^k/\partial x^i$, then A_i is algebraic cofactor of y_i^1 in

$$\frac{\partial(y^1,\dots,y^m)}{\partial(x^1,\dots,x^m)} = \det \begin{pmatrix} y_1^1 & y_2^1 & \cdots & y_i^1 & \cdots & y_m^1 \\ y_1^2 & y_2^2 & \cdots & y_i^2 & \cdots & y_m^2 \\ \vdots & \vdots & & \vdots & & \vdots \\ y_1^m & y_2^m & \cdots & y_i^m & \cdots & y_m^m \end{pmatrix}.$$

By Hadamard identity we get $\operatorname{div} A = 0$, so

$$\sum_{i=1}^m y_i^j A_i = \delta_1^j \frac{\partial(y^1,\dots,y^m)}{\partial(x^1,\dots,x^m)} = \delta_1^j J_\varphi(x),$$

$$\tilde{P}(x) = P(\varphi(x))$$

$$\operatorname{div}(\tilde{P}A) = \nabla\tilde{P}\cdot A + \tilde{P}\operatorname{div} A = \nabla\tilde{P}\cdot A = \sum_{i=1}^m \frac{\partial\tilde{P}}{\partial x^i}A_i = \sum_{i=1}^m \left(\sum_{j=1}^m \frac{\partial P}{\partial y^j}\frac{\partial y^j}{\partial x^i}\right)A_i$$

$$= \sum_{j=1}^m \frac{\partial P}{\partial y^j}\left(\sum_{i=1}^m y_i^j A_i\right) = (\partial_{y^1}P)J_\varphi(x) = f(\varphi(x))J_\varphi(x).$$

From (5) we get

$$\int_D f(y)\mathrm{d}y = \pm\int_\Omega f(\varphi(x))J_\varphi(x)\mathrm{d}x.$$

Cor4. Under assumptions of Thm7 , if J_φ does not change sign on $\overline{\Omega}$, then

$$\int_D f(y)\mathrm{d}y = \int_\Omega f(\varphi(x))\,|J_\varphi(x)|\mathrm{d}x.$$

3.3. General domain

Thm8. Let D and Ω be Jordan measurable bounded open domains in $\mathbb{R}^m$, $f \in C(\overline{D})$, $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^m)$, $\varphi : \Omega \to D$ is diffeomorphism, then

$$\int_D f(y)\mathrm{d}y = \int_\Omega f(\varphi(x))|J_\varphi(x)|\mathrm{d}x.$$

Pf. Since $f = f_+ - f_-$, $f_\pm = \frac{1}{2}(|f| \pm f) \in C(\overline{D})$, we may assume $f \geq 0$.
Set $\tilde{f}(x) = f(\varphi(x))|J_\varphi(x)|$.

(1) $\forall \varepsilon > 0$, choose disjoint balls $B_i \subset \Omega$ s.t.

$$\int_\Omega \tilde{f}(x)\mathrm{d}x - \varepsilon \leq \sum_i \int_{B_i} \tilde{f}(x)\mathrm{d}x = \sum_i \int_{\varphi(B_i)} f(y)\mathrm{d}y \leq \int_D f(y)\mathrm{d}y.$$

(2) Letting $\varepsilon \to 0$ we get $\int_\Omega \tilde{f}(x)\mathrm{d}x \leq \int_D f(y)\mathrm{d}y$.

(3) Similarly, $\int_D f(y)\mathrm{d}y \leq \int_\Omega \tilde{f}(x)\mathrm{d}x$.

4. Brouwer fixed point theorem (BFPT)

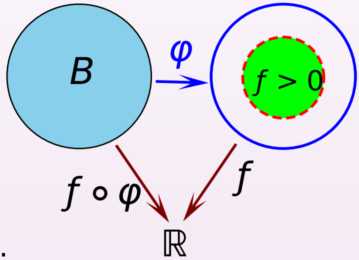
Thm9 (Brouwer). Let B be the unit closed ball in $\mathbb{R}^m$, $g : B \to B$ be continuous. Then g has a fixed point.

It is well known that to prove Thm 9 it suffices to prove

Pro2. There does not exist $\varphi \in C^2(B, \mathbb{R}^m)$ s.t. $\varphi(B) \subset \partial B$ and $\varphi|_{\partial B} = 1_{\partial B}$.

Pf(Motivated by [BD93]). Take

$$f(y) = \begin{cases} \sqrt{1 - 4|y|^2}, & |y| \le \frac{1}{2}, \\ 0, & \frac{1}{2} < |y| \le 1. \end{cases}$$



Then $f(\varphi(x)) = 0$ for $x \in B$.

View $\varphi : x \mapsto y$ as transformation, Thm 7 yields

$$0 < \int_B f(y)\mathrm{d}y$$

$$= \pm \int_B f(\varphi(x)) \det\left(\frac{\partial y}{\partial x}\right) \mathrm{d}x = 0, \text{ a contradiction.}$$

Rek6. ✳ Most people learn the proof of BFPT for the first time as application of homology theory in Algebraic Topology.

✳ BFPT can also proved as application of Stokes formula on Differentiable manifolds.

✳ Elementary proofs can also be found in [Mil[78], Rog[80], Kan[81]].

Rek7. Advantages of our proof of Changing variable formula:

(1) It is just clever computation (Cauchy-Binet, etc), more easy to follow.

(2) Theory of surface integral (including Divergence Theorem) is developed during the proof.

(3) As by product we get Brouwer fixed point theorem.

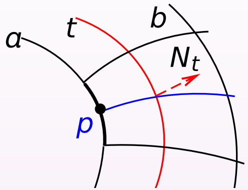
Exm2 (Application of BFPT). Let A be invertible, $f : \mathbb{R}^n \to \mathbb{R}^n$ verify

$$\lim_{|x|\to\infty} \frac{|f(x)|}{|x|} = 0, \qquad \sum_{j=1}^{n} a_i^j x^j = f^i(x^1, \dots, x^n) \qquad \begin{cases} -\Delta u = \lambda u + f(u), \\ u|_{\partial\Omega} = 0. \end{cases} \qquad \lim_{|t|\to\infty} \frac{f(t)}{t} = 0.$$

Then the nonlinear equation $Ax = f(x)$ is solvable ($Ax = b$, Cramer rule).

5. Coarea formula and applications

Influenced by [Mei[11]], I present coarea formula.



Thm10. Let $G \subset \mathbb{R}^m$ be bdd open, $f \in C^2(G)$, $\nabla f(x) \neq 0$ for $\forall x \in G$. $\Omega = f^{-1}[a,b] \subset G$.

If $g \in C(\Omega)$, then $$\int_\Omega g = \int_a^b \mathrm{d}t \int_{f^{-1}(t)} \frac{g}{|\nabla f|} \mathrm{d}\sigma.$$

Pf(Motivated by [WSY[89], §11]). For $p \in f^{-1}(a)$, let $x(\cdot, p)$ be solution of

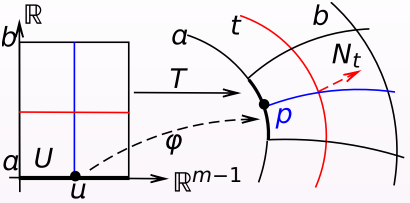
$$x' = \frac{\nabla f(x)}{|\nabla f(x)|^2}, \quad x(a) = p. \tag{6}$$

Then $x(b,p) \in f^{-1}(b)$. (ODE is applied in Calculus)

Let $\varphi : U \to \mathbb{R}^m$ be par of $f^{-1}(a)$, then the C^1-map $T : U \times [a,b] \to \mathbb{R}^m$,

$$T(u,t) = x(t-a, \varphi(u))$$ is interiorly injective,

$T(\cdot, t) : U \to \mathbb{R}^m$ is par of $f^{-1}(t)$, with normal $N_t(u)$.



Expanding $\det T'(u,t)$ and using (6) yield

$$\left|\det T'(u,t)\right| = \frac{|N_t(u)|}{|\nabla f(T(u,t))|} \neq 0.$$

So T is diffeomorphism on $U^\circ \times (a,b)$.
By changing variable and Fubini

$$\int_\Omega g(x)\mathrm{d}x = \int_{T(U\times(a,b))} g(x)\mathrm{d}x \qquad x = T(u,t)$$

$$= \int_{U\times(a,b)} g\left(T(u,t)\right)\left|\det T'(u,t)\right| \mathrm{d}u\mathrm{d}t$$

$$= \int_a^b \mathrm{d}t \int_U \frac{g\left(T(u,t)\right)}{|\nabla f(T(u,t))|}\,|N_t(u)|\,\mathrm{d}u$$

$$= \int_a^b \mathrm{d}t \int_{f^{-1}(t)} \frac{g}{|\nabla f|}\mathrm{d}\sigma.$$

Exm3. Let B be unit ball in $\mathbb{R}^m$, $f \in C^1(B)$, $f|_{\partial B} = 0$. Find

$$I = \lim_{\varepsilon \to 0^+} \int_{B \setminus B_\varepsilon} \frac{x \cdot \nabla f(x)}{|x|^m} \mathrm{d}x, \qquad \text{where } B_\varepsilon : |x| \le \varepsilon.$$

Pf. By defn of surface integrals (4), $\int_{|x|=t} g(x)\,\mathrm{d}\sigma = t^{m-1} \int_{|y|=1} g(ty)\,\mathrm{d}\sigma.$

$$\int_{B \setminus B_\varepsilon} \frac{x \cdot \nabla f(x)}{|x|^m} \mathrm{d}x = \int_\varepsilon^1 \mathrm{d}t \int_{|x|=t} \frac{x \cdot \nabla f(x)}{|x|^m} \mathrm{d}\sigma$$

$$= \int_\varepsilon^1 \left(t^{m-1} \int_{|y|=1} \frac{(ty) \cdot \nabla f(ty)}{|ty|^m} \mathrm{d}\sigma \right) \mathrm{d}t$$

$$= \int_\varepsilon^1 \mathrm{d}t \int_{|y|=1} \nabla f(ty) \cdot y \,\mathrm{d}\sigma = \int_{|y|=1} \mathrm{d}\sigma \int_\varepsilon^1 \frac{d}{dt} f(ty)\,\mathrm{d}t$$

$$= \int_{|y|=1} [-f(\varepsilon y)]\,\mathrm{d}\sigma \to -f(0)\omega_m.$$

Rek8. This can also be solved using divergence theorem

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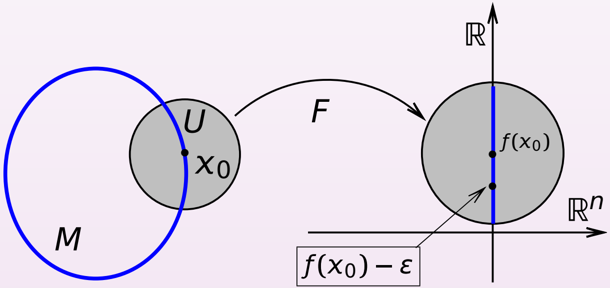
Appendix

[✱] Pf of L-multipliers, $F(x) = (f(x), g^1(x), \ldots, g^n(x))$. By Thm 4

$$\min\{f : \mathbb{R}^m \to \mathbb{R}\} \qquad F : U \to \mathbb{R}^{n+1}$$

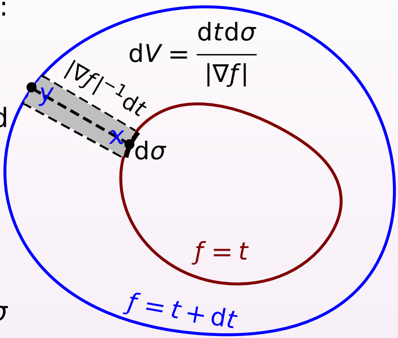
$$\begin{aligned} g^1(x) &= 0 \\ &\vdots \\ g^n(x) &= 0 \end{aligned} \qquad \operatorname{rank} F'(x_0) = \operatorname{rank} \begin{pmatrix} \nabla f(x_0) \\ \nabla g^1(x_0) \\ \vdots \\ \nabla g^n(x_0) \end{pmatrix} = n,$$

thus $\nabla f(x_0) \in \operatorname{span}\left\{\nabla g^1(x_0), \ldots, \nabla g^n(x_0)\right\}$.



[✱] Coarea and method of element

Let $G \subset \mathbb{R}^m$, $f : G \to \mathbb{R}$. $\Omega = f^{-1}[a,b]$, $g : \Omega \to \mathbb{R}$.



Take surface element $\mathrm{d}\sigma$ at $x \in f^{-1}(t)$. Let y be the intersection of $f^{-1}(t+\mathrm{d}t)$ and normal line of $f^{-1}(t)$ at x. Then

$$\mathrm{d}t = f(y) - f(x) \approx \nabla f(x)\cdot(y-x),$$

$$|y-x| = \frac{\mathrm{d}t}{|\nabla f(x)|}.$$

Volume of the gray column with base $\mathrm{d}\sigma$ and high $|y-x|$ is

$$\mathrm{d}V = \frac{\mathrm{d}t\mathrm{d}\sigma}{|\nabla f(x)|}, \quad \mathrm{d}m = \frac{g(x)}{|\nabla f(x)|}\mathrm{d}t\mathrm{d}\sigma. \quad \int_{f=t}\mathrm{d}m = \text{mass of } f^{-1}[t, t+dt]$$

Hence mass of Ω is

$$\int_\Omega g(x)\mathrm{d}x = \int_a^b \mathrm{d}t \int_{f=t} \frac{g(x)}{|\nabla f(x)|}\mathrm{d}\sigma.$$

Exm4. Let $g \in C^1(B_R \times [0,R])$, then for $r \in (0,R)$,

$$\frac{d}{dr}\int_{B_r} g(x,r)\,dx = \int_{B_r} \frac{\partial}{\partial r} g(x,r)\,dx + \int_{\partial B_r} g(x,r)\,d\sigma.$$

Pf. By coarea formula,

$$\int_{B_r} g(x,r)\,dx = \int_0^r dt \int_{|x|=t} g(x,r)\,d\sigma.$$

Applying

$$\frac{d}{dr}\int_0^r F(r,t)\,dt = F(r,r) + \int_0^r \partial_r F(r,t)\,dt$$

to

$$F(r,t) = \int_{|x|=t} g(x,r)\,d\sigma,$$

we deduce

$$\begin{aligned}\frac{d}{dr}\int_{B_r} g(x,r)\,dx &= \int_{|x|=r} g(x,r)\,d\sigma + \int_0^r dt \int_{|x|=t} \partial_r g(x,r)\,d\sigma \\ &= \int_{\partial B_r} g(x,r)\,d\sigma + \int_{B_r} \partial_r g(x,r)\,dx.\end{aligned}$$

[✱] Chain rule and Cauchy-Binet

$$\begin{pmatrix} \frac{\partial y^2}{\partial u^1} & \cdots & \frac{\partial y^2}{\partial u^{m-1}} \\ \vdots & & \vdots \\ \frac{\partial y^m}{\partial u^1} & \cdots & \frac{\partial y^m}{\partial u^{m-1}} \end{pmatrix} = \begin{pmatrix} \frac{\partial y^2}{\partial x^1} & \cdots & \frac{\partial y^2}{\partial x^i} & \cdots & \frac{\partial y^2}{\partial x^m} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial y^m}{\partial x^1} & \cdots & \frac{\partial y^m}{\partial x^i} & \cdots & \frac{\partial y^m}{\partial x^m} \end{pmatrix} \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^1}{\partial u^{m-1}} \\ \vdots & & \vdots \\ \frac{\partial x^i}{\partial u^1} & \cdots & \frac{\partial x^i}{\partial u^{m-1}} \\ \vdots & & \vdots \\ \frac{\partial x^m}{\partial u^1} & \cdots & \frac{\partial x^m}{\partial u^{m-1}} \end{pmatrix}$$

$(m-1)\times(m-1)$ $\quad$ $(m-1)\times m$ $\quad$ $m\times(m-1)$

Cauchy-Binet yields

$$\frac{\partial(y^2,\ldots,y^m)}{\partial(u^1,\ldots,u^{m-1})} = \sum_{i=1}^{m} \frac{\partial(y^2,\ldots,y^m)}{\partial(x^1,\ldots,\hat{x}^i,\ldots x^m)} \frac{\partial(x^1,\ldots,\hat{x}^i,\ldots x^m)}{\partial(u^1,\ldots,u^{m-1})}$$

Thank you!

http://lausb.github.io