ON DENSITY, MASS AND MULTIPLE INTEGRALS

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CONTENTS

1. MASS AND DOUBLE INTEGRALS

Let's recall the 1-D case. Consider a thin rod occupying $I = [0, a]$ on the x-axis with 1-D density $f(x)$ (*mass per unit length*). To find the mass of I, we consider an infinitesimal segment $[x, x + \mathrm{d}x]$ at $x \in I$, whose mass is approximately $\mathrm{d}M \approx f(x)\mathrm{d}x$. Therefore the mass of the rod I is

$$M = \int \mathrm{d}M = \int_0^a f(x)\mathrm{d}x.$$

We get the mass of the whole rod by integrating the *mass element* $\mathrm{d}M$. This is what we have learned as application of definite integral in Calculus 1.

Similar idea applies to higher dimensional problems. For simplicity, it is standard to denote a rectangle

$$R: \quad a \le x \le b, c \le y \le d$$

by product of intervals $R = [a, b] \times [c, d]$. Similarly, $[a, b] \times [c, d] \times [e, f]$ is a box in 3-D space.

Let $Q \subset \mathbb{R}^2$ be a thin sheet on the xy-plane with 2-D density $f(x, y)$ (*mass per unit area*). Then the mass of the infinitesimal rectangle

$$\mathrm{d}A = [x, x + \mathrm{d}x] \times [y, y + \mathrm{d}y]$$

based at $(x, y) \in Q$ is approximately

$$\mathrm{d}M \approx f(x, y)\mathrm{d}A,$$

here we use the same notation $\mathrm{d}A$ to denote the area of the small rectangle. Thus the mass of Q is

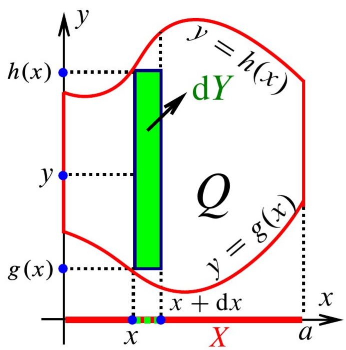
$$M = \iint \mathrm{d}M = \iint_Q f(x,y)\mathrm{d}A. \tag{1.1}$$

This is the simplest interpretation of double integrals. Since $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$, we also write $\iint_Q f\mathrm{d}x\mathrm{d}y$ for $\iint_Q f\mathrm{d}A$.

2. Double integrals and iterated integrals

Now we assume that the thin sheet Q is bounded by the curves $y = g(x)$ and $y = h(x)$, $x \in [0,a]$, i.e.,

$$Q = \{(x,y) \mid g(x) \le y \le h(x), x \in [0,a]\}.$$



At $x \in [0,a]$ we consider infinitesimal increment $\mathrm{d}x$. The part of Q above $[x, x+\mathrm{d}x]$ is approximately the thin strip (*the vertical green region in the picture*)

$$\mathrm{d}Y = [x, x+\mathrm{d}x] \times [g(x), h(x)].$$

Since $\mathrm{d}x$ is very small, $\mathrm{d}Y$ can be viewed as a vertical line with 1-D density[(1)] $f(x,y)\mathrm{d}x$ at $y \in [g(x), h(x)]$. Thus its mass is

$$\mathrm{d}M = \int_{g(x)}^{h(x)} [f(x,y)\mathrm{d}x]\,\mathrm{d}y = \left(\int_{g(x)}^{h(x)} f(x,y)\mathrm{d}y\right)\mathrm{d}x,$$

here we are integrating with respect to y, so $\mathrm{d}x$ is constant and can be moved out of the integral sign.

At this moment, we can already integrate $\mathrm{d}M$ against $x \in [0,a]$ to get $\iint_Q f\mathrm{d}A$, the mass of Q, thus deduce the equality

$$\int_0^a \left(\int_{g(x)}^{h(x)} f(x,y)\mathrm{d}y\right)\mathrm{d}x = \iint_Q f(x,y)\mathrm{d}A. \tag{2.1}$$

If this does not convince you, read the next paragraph for more details. That paragraph plays the same role explaining *why integrating the area of cross section yields the volume* in the volume interpretation of the iterated integral formula (2.1).

Imaging that we squeeze Q vertically to the x-axis. Then the squeezed Q becomes the 1-D segment $X = [0,a]$ on the x-axis (*the thick red segment lying on the x-axis*

(1) Multiplying the horizontal width $\mathrm{d}x$ to the 2-D density $f(x,y)$, we get the 1-D density in the vertical direction.

in the picture), with ***the same mass***. The mass of the small segment (*the dotted green segment on* X) $[x, x + \mathrm{d}x]$ on X equals the mass of $\mathrm{d}Y$ we obtained above:

$$\mathrm{d}M = \left(\int_{g(x)}^{h(x)} f(x, y)\mathrm{d}y \right) \mathrm{d}x,$$

Thus, the 1-D density of X at $x \in [0, a]$ is

$$\rho(x) = \frac{\mathrm{d}M}{\mathrm{d}x} = \int_{g(x)}^{h(x)} f(x, y)\mathrm{d}y. \tag{2.2}$$

Hence the mass of the squeezed Q (that is the segment X) is

$$\int_0^a \left(\int_{g(x)}^{h(x)} f(x, y)\mathrm{d}y \right) \mathrm{d}x = \iint_Q f(x, y)\mathrm{d}A.$$

The equality here is because: LHS is the mass of X obtained by integrating its 1-D density given in (2.2); RHS is the mass of Q, which equals that of X (the squeezed Q).

3. Triple integrals and iterated integrals

Let $D \subset \mathbb{R}^3$ be a solid body in 3-D space, $f(x, y, z)$ be the 3-D density (*mass per unit volume*) at point $(x, y, z) \in D$. Then the mass of the infinitesimal box

$$\mathrm{d}V = [x, x + \mathrm{d}x] \times [y, y + \mathrm{d}y] \times [z, z + \mathrm{d}z]$$

based at $(x, y, z) \in D$ is approximately

$$\mathrm{d}M \approx f(x, y, z)\mathrm{d}V,$$

here we use the same notation $\mathrm{d}V$ to denote the volume of the small box. Thus the mass of D is

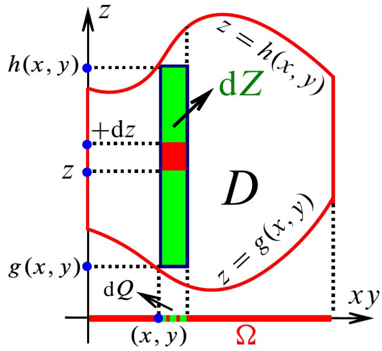
$$M = \iiint \mathrm{d}M = \iiint_D f(x, y, z)\mathrm{d}V.$$

Since $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$, we also write $\iiint_D f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ for the triple integral $\iiint_D f\,\mathrm{d}V$.

Assume that D is bounded by the surfaces $z = g(x, y)$ and $z = h(x, y)$, $(x, y) \in \Omega$, i.e.,

$$D = \{(x, y, z) \mid g(x, y) \le z \le h(x, y), (x, y) \in \Omega\},$$

see the picture, where the xy-plane is *represented as the horizontal axis* to avoid a messy 3-D illustration (this trick can be applied to the volume interpretation of (2.1)).



At $(x, y) \in \Omega$ we consider infinitesimal increments $\mathrm{d}x$ and $\mathrm{d}y$. The thin vertical bar (*the vertical green region in the picture*) $\mathrm{d}Z$ with base (*the dotted green segment on* Ω) $\mathrm{d}Q = [x, x + \mathrm{d}x] \times [y, y + \mathrm{d}y]$ can be view as a vertical line bounded by $z = g(x, y)$ and $z = h(x, y)$, with 1-D density $f(x, y, z)\mathrm{d}x\mathrm{d}y$; because the portion (*the small solid red rectangle*) between z and $z+\mathrm{d}z$ has mass $\mathrm{d}m \approx f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$, dividing the infinitesimal length $\mathrm{d}z$ we get the desired 1-D density $f(x, y, z)\mathrm{d}x\mathrm{d}y$.

Thus the mass of $\mathrm{d}Z$ is

$$\mathrm{d}M = \int_{g(x,y)}^{h(x,y)} [f(x, y, z)\mathrm{d}x\mathrm{d}y]\,\mathrm{d}z = \mathrm{d}x\mathrm{d}y \int_{g(x,y)}^{h(x,y)} f(x, y, z)\mathrm{d}z. \tag{3.1}$$

At this moment, we can already integrate $\mathrm{d}M$ against $(x, y) \in \Omega$ to get $\iiint_D f\,\mathrm{d}V$, the mass of D, thus deduce

$$\iiint_D f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_\Omega \mathrm{d}x\mathrm{d}y \int_{g(x,y)}^{h(x,y)} f(x, y, z)\mathrm{d}z. \tag{3.2}$$

If this does not convince you, read the next paragraph for more details.

We compress D vertically onto Ω, the mass M does not change. Now Ω is a planar region on xy-plane with 2-D density at $(x, y) \in \Omega$

$$\rho(x, y) = \frac{\mathrm{d}M}{\mathrm{d}x\mathrm{d}y} = \int_{g(x,y)}^{h(x,y)} f(x, y, z)\mathrm{d}z, \tag{3.3}$$

because the mass of the infinitesimal rectangle (*the dotted green segment on* Ω) $\mathrm{d}Q$ is the $\mathrm{d}M$ given in (3.1) and the area is $\mathrm{d}x\mathrm{d}y$.

Therefore we deduce

$$\begin{aligned}\iiint_D f(x, y, z)\mathrm{d}V &= \iint_\Omega \rho(x, y)\mathrm{d}x\mathrm{d}y \\ &= \iint_\Omega \left(\int_{g(x,y)}^{h(x,y)} f(x, y, z)\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y = \iint_\Omega \mathrm{d}x\mathrm{d}y \int_{g(x,y)}^{h(x,y)} f(x, y, z)\mathrm{d}z.\end{aligned}$$

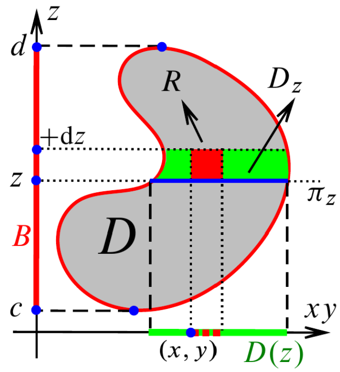
Here, the first equality is because: the LHS is the mass of D, the RHS is the mass of Ω obtained by integrating its 2-D density given in (3.3); Ω, being the vertical compression of D on the xy-plane, has the same mass as D.

After deducing (3.2), we may further evaluate the outer integral on Ω using iterated integral: Assuming Ω is bounded by two curves $y = \varphi_\pm(x)$, $x \in [a, b]$, then

$$\iiint_D f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_\Omega \mathrm{d}x\mathrm{d}y \int_{g(x,y)}^{h(x,y)} f(x, y, z)\mathrm{d}z$$

$$= \int_a^b \mathrm{d}x \int_{\varphi_-(x)}^{\varphi_+(x)} \mathrm{d}y \int_{g(x,y)}^{h(x,y)} f(x,y,z)\mathrm{d}z. \tag{3.4}$$

4. Another iterated triple integral formula



If D is bounded by $z = c$ and $z = d$, D_z (the thick blue segment) is the intersection of D and the horizontal plane π_z at level $z \in [c, d]$. Given infinitesimal increment $\mathrm{d}z$, the part $\mathrm{d}Z$ (the green region in the picture) of D between π_z and $\pi_{z+\mathrm{d}z}$ can be view as planar region with 2-D density[(2)]

$$\rho(x, y) = f(x, y, z)\mathrm{d}z, \quad (x, y) \in D(z), \tag{4.1}$$

where $D(z)$ (the thick green segment on the horizontal axis) is the projection of D_z on the xy-plane.

Therefor the mass of $\mathrm{d}Z$ is

$$\mathrm{d}M = \iint_{D(z)} [f(x, y, z)\mathrm{d}z]\,\mathrm{d}x\mathrm{d}y = \mathrm{d}z \iint_{D(z)} f(x, y, z)\mathrm{d}x\mathrm{d}y.$$

Imaging that we compress D horizontally onto the z-axis. Then D becomes a thin bar B (the red vertical segment in the picture) on the z-axis with the same mass. The 1-D density of B at z is

$$\varrho(z) = \frac{\mathrm{d}M}{\mathrm{d}z} = \iint_{D(z)} f(x, y, z)\mathrm{d}x\mathrm{d}y.$$

Thus, the mass of B (which equals $\iiint_D f\,\mathrm{d}V$, the mass of D), is

$$\begin{aligned}\iiint_D f(x, y, z)\mathrm{d}V &= \int_c^d \varrho(z)\mathrm{d}z \\ &= \int_c^d \mathrm{d}z \iint_{D(z)} f(x, y, z)\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{4.2}$$

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(2) To get (4.1) we take infinitesimal increments $\mathrm{d}x$ and $\mathrm{d}y$ at $(x, y) \in D_z$. Then the 2-D density

$$\rho(x, y) = \frac{\mathrm{d}m}{\mathrm{d}x\mathrm{d}y} = f(x, y, z)\mathrm{d}z, \qquad (x, y) \in D(z),$$

where $\mathrm{d}m = f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ is the mass of the small box (the red solid rectangle)

$$R = [x, x + \mathrm{d}x] \times [y, y + \mathrm{d}y] \times [z, z + \mathrm{d}z]$$

based at (x, y, z).