# Multiple solutions for 1-D quasilinear indefinite Schrödinger equations

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# 1. Introduction

To find standing waves $\psi(t,x) = e^{-i\omega t}u(x)$ of QL Schrödinger equation

$$i\partial_t\psi = -\Delta\psi + U(x) - \psi\Delta(|\psi|^2) - \bar{g}(|\psi|^2)\psi \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N,$$

we need to solve ($V = U - \omega$)

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad u \in H^1(\mathbb{R}^N), \tag{1}$$

initiated by Poppenberg et al. (2002) for $N = 1$. Solutions are critical pts of

$$J(u) = \frac{1}{2}\int \left(1 + 2u^2\right)|\nabla u|^2 + \frac{1}{2}\int V(x)u^2 - \int G(u),$$

which is not well-defined ($J(u) = \infty$ for some $u \in H^1(\mathbb{R}^N)$) unless $N = 1$.
Liu et al. (2003); Colin & Jeanjean (2004) introduced a nonlinear transform $u = f(v)$ so that $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$,

$$\Phi(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x)f^2(v)\right) - \int G(f(v))$$

is well defined, if $\Phi'(v) = 0$ then $u = f(v)$ solves (1).
Then many results for $\inf V > 0$ appear. Until Shen & Han (2015) for $N = 1$ and Liu & Zhou (2018) for $N \geq 1$, no results if $-\Delta + V$ is indefinite.

In Liu & Zhou (2018): $g(t) \approx |t|^{p-2}t$ for some $p \in (4, 2\cdot 2^*)$. It is crucial that

$$\lim_{|t|\to\infty} V(x) = +\infty, \tag{2}$$

so that the working space

$$E = \left\{ u \in H^1(\mathbb{R}^N) \,\middle|\, \|u\| = \left( \int \left( |\nabla u|^2 + Vu^2 \right) \right)^{1/2} < \infty \right\} \hookrightarrow\hookrightarrow L^2(\mathbb{R}^N).$$

See also Silva & Silva (2019).

If $N = 1$ we can study the case that $|V|_\infty < \infty$. We consider

$$-u'' + V(x)u - (u^2)''u = f(x,u), \qquad u \in H^1(\mathbb{R}). \tag{3}$$

Assume $V \in C(\mathbb{R})$ is bounded from below. Let

$$\lambda_n = \inf_{X \in \mathscr{X}_n} \sup_{u \in X\setminus\{0\}} \frac{\int \left( \dot{u}^2 + V(x)u^2 \right)}{\int u^2},$$

where $\mathscr{X}_n$ is the collection of all $n$-dimensional subspaces of $C_0^\infty(\mathbb{R})$. Assume

$$\lambda_n \to \lambda_\infty < \infty,$$

then $\lambda_\infty$ is the bottom of the essential spectrum of $S = -\frac{d^2}{dx^2} + V$ and $\lambda_n < \lambda_\infty$ implies that $\lambda_n$ is an eigenvalue of $S$ of finite multiplicity.

We assume

($V_1$) $V \in C(\mathbb{R})$ is such that $0 \in (\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$.

($f_1$) $f \in C(\mathbb{R} \times \mathbb{R})$ and there are constants $p > 2$ and $c > 0$ such that

$$|f(x,t)| \le c\left(1 + |t|^{p-1}\right) \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}.$$

($f_2$) $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}$.

($f_3^*$) There exists $h \in (0, \lambda_\infty)$ such that $tf(x,t) \le ht^2$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$.

Increasing $h$ a little bit we may assume that $h$ in ($f_3^*$) is not an eigenvalue.

**Thm 1.** Suppose ($V_1$), ($f_1$), ($f_2$) and ($f_3^*$) are satisfied, then (3) has at least two nontrivial solutions. If in addition $f(x,\cdot)$ is odd for all $x \in \mathbb{R}$, then (3) has $k$ pairs of nontrivial solutions.

**Rem 1.** Under ($V_1$), ($f_1$), ($f_2$) and the following condition weaker than ($f_3^*$):

($f_3$) There exists $h \in (0, \lambda_\infty)$ such that $F(x,t) \le \frac{1}{2}ht^2$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$,

a nontrivial solution is obtained by Wang & Yang (2015).

Thm 1 is motivated by Chen & Wang (2014) who obtained similar results for

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{4}$$

Unlike our case, weak limits of $(PS)$ sequences of

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int \phi_u u^2 - \int F(x,u), \quad \phi_u(x) = \frac{1}{4\pi}\int \frac{u^2(y)}{|x-y|}\mathrm{d}y$$

are critical points of $\Phi$.

Lacking this property is the reason that Wang & Yang (2015) could not get multiplicity results for (3) as in Thm 1.

**Rem 2.** Initiated by Benci & Fortunato (1998), (4) attracted great interest for definite case in which $u = 0$ is loc min of $\Phi$.

The first work on the indefinite case that $u = 0$ is saddle pt is due to Chen & Liu (2015).

Motivated by Liu & Wu (2017) on indefinite problem (4), we consider (3) when $f(x,\cdot)$ is 4-superlinear:

$$\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = +\infty \quad \text{a.e. } x \in \mathbb{R}, \qquad \text{where } F(x,t) = \int_0^t f(x,\cdot).$$

$(f_4)$ $0 < 4F(x,t) \le tf(x,t)$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$,

$$\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = +\infty \quad \text{a.e. } x \in \mathbb{R}. \tag{5}$$

$(f_5)$ if $u_n \rightharpoonup u$ in $H^1(\mathbb{R})$, then $\overline{\lim_{n\to\infty}} \int f(x,u_n)(u_n - u) \le 0.$

**Thm 2.** Suppose $(V_1)$, $(f_1)$, $(f_2)$, $(f_4)$ and $(f_5)$ are satisfied, then (3) has at least one nontrivial solutions. If in addition $f(x,\cdot)$ is odd for $x \in \mathbb{R}$, then (3) has a sequence of solutions $\{u_n\}$ such that $\Phi(u_n) \to +\infty$.

**Rem 3.** Condition $(f_5)$ holds in e.g. one of the following:

(1) for $\forall r > 0$, $\displaystyle\lim_{|x|\to\infty} \sup_{0<|t|\le r} \left|\frac{f(x,t)}{t}\right| = 0$, Bartsch et al. (2004)

Example: $f(x,t) = a(x)|t|^{p-2}t$, $\displaystyle\lim_{|x|\to\infty} a(x) = 0.$

(2) $|f(x,t)| \le \alpha_+(x)|t|^{p_+-1} + \alpha_-(x)|t|^{p_--1}$, $\alpha_\pm \in L^{q_\pm}(\mathbb{R})$ for some $q_\pm > 1$.

# 2. Proof of Thm 1

We denote $X = H^1(\mathbb{R})$. By Poppenberg et al. (2002), $N : X \to \mathbb{R}$ given by

$$N(u) = \int \dot{u}^2 u^2$$

is of class $C^1$,

$$\langle N'(u), v\rangle = 2\int \left(\dot{u}^2 uv + u^2 \dot{u}\dot{v}\right).$$

Therefore, $\Phi : X \to \mathbb{R}$

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int \left(\dot{u}^2 + V(x)u^2\right) + \int \dot{u}^2 u^2 - \int F(x,u) \\ &= \frac{1}{2}\int \left(\dot{u}^2 + V(x)u^2\right) + N(u) - \int F(x,u),\end{aligned}$$

is of class $C^1$ as well, with derivative

$$\left\langle \Phi'(u), v\right\rangle = \int (\dot{u}\dot{v} + V(x)uv) + \langle N'(u), v\rangle - \int f(x,u)v.$$

Critical points of $\Phi$ are weak solutions of the problem (3).

**Lem 1.** If $u_n \rightharpoonup u$ in $X$, then
$$\overline{\lim_{n\to\infty}} \left(\langle N'(u_n), u\rangle - 4N(u_n)\right) \le 0. \tag{6}$$

**Pf.** The inequality (6) is a consequence of
$$\begin{aligned}\frac{1}{2}\langle N'(u_n), u\rangle &= \int \left(\dot{u}_n^2 u_n u + u_n^2 \dot{u}_n \dot{u}\right)\\ &\le 2\int \dot{u}_n^2 u_n^2 + o(1) = 2N(u_n) + o(1),\end{aligned}$$
which has been proven in (Chen, 2014, Eqn (3.16)).

**Lem 2.** Suppose $g \in C(\mathbb{R}^N \times \mathbb{R})$, $|g(x,t)| \le \Lambda|t|$ for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then for all $\phi \in H^1(\mathbb{R}^N)$ we have
$$\int g(x,u_n)\phi \to \int g(x,u)\phi. \tag{7}$$

**Rem 4.** By Brézis & Lieb (1983) we get
$$g(u_n) \rightharpoonup g(u) \quad \text{in } L^2(\mathbb{R}^N),$$
which implies (7).

**Pf (Without using B-L).** Since $g$ is of linear growth and $\{u_n\}$ is bounded,

$$\beta := \sup_n |g(x,u_n) - g(x,u)|_2 < \infty.$$

Given $\varepsilon > 0$, choose $R > 0$ such that

$$\int_{|x|\geq R} \phi^2 \leq \varepsilon^2.$$

Using Hölder inequality we have

$$\begin{aligned}
&\left| \int g(x,u_n)\phi - \int g(x,u)\phi \right| \\
&\quad \leq \int_{|x|\geq R} |g(x,u_n) - g(x,u)||\phi| + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi| \\
&\quad \leq |g(x,u_n) - g(x,u)|_2 \left( \int_{|x|\geq R} \phi^2 \right)^{1/2} + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi| \\
&\quad \leq \beta\varepsilon + \int_{|x|<R} |g(x,u_n) - g(x,u)||\phi|.
\end{aligned}$$

Since $H^1(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^2_{\text{loc}}(B_R)$ we deduce $g(x,u_n) \to g(x,u)$ in $L^2(B_R)$. Hence

$$\overline{\lim_{n\to\infty}} \left| \int g(x,u_n)\phi - \int g(x,u)\phi \right| \leq \beta\varepsilon.$$

Let $E^-$ be the negative subspace of $S-h$ and $E^+=(E^-)^\perp$. Then $\dim E^-<\infty$ and there is an equivalent norm $\|\cdot\|$ on $X=E^-\oplus E^+$ such that

$$\int \left(\dot{u}^2+V(x)u^2-hu^2\right)=\|u^+\|^2-\|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of $u$ on $E^\pm$.

**Lem 3.** Under conditions $(V_1)$, $(f_1)$ and $(f_3^*)$, $\Phi$ satisfies the $(PS)_c$ condition for all $c\in\mathbb{R}$.

**Pf.** Under $(V_1)$, $(f_1)$ and the following condition weaker than $(f_3^*)$:

$(f_3)$ There exists $h\in(0,\lambda_\infty)$ such that $F(x,t)\le\frac{1}{2}ht^2$ for all $(x,t)\in\mathbb{R}\times\mathbb{R}$,

it has been shown by Wang & Yang (2015) that $\Phi$ is coercive.
Let $\{u_n\}$ be a $(PS)_c$ sequence of $\Phi$, that is

$$\Phi(u_n)\to c,\qquad \Phi'(u_n)\to 0.$$

By the coerciveness of $\Phi$, $\{u_n\}$ is bounded in $X$.
We may assume $u_n\rightharpoonup u$ in $X$. Since $\dim E^-<\infty$, we have $u_n^-\to u^-$ and $\|u_n^-\|\to\|u^-\|$.

Since $\Phi'(u_n) \to 0$, we have

$$\begin{aligned} o(1) = o(\|u_n\|) &= \langle \Phi'(u_n), u_n \rangle \\ &= \int \left( \dot{u}_n^2 + V(x)u_n^2 - hu_n^2 \right) + \langle N'(u_n), u_n \rangle - \int \left( f(x,u_n)u_n - hu_n^2 \right) \\ &= \|u_n^+\|^2 - \|u_n^-\|^2 + 4N(u_n) + \int \left( hu_n^2 - f(x,u_n)u_n \right). \end{aligned} \tag{8}$$

Applying Lem 2 to $g : (x,t) \mapsto ht - f(x,t)$ yields

$$\int (hu_n u - f(x,u_n)u) = \int \left( hu^2 - f(x,u)u \right) + o(1).$$

Therefore

$$\begin{aligned} o(1) &= \langle \Phi'(u_n), u \rangle \\ &= \|u^+\|^2 - \|u^-\|^2 + \langle N'(u_n), u \rangle + \int \left( hu^2 - f(x,u)u \right) + o(1). \end{aligned} \tag{9}$$

From (8), (9), Lem 1, and

$$\int \left( hu^2 - f(x,u)u \right) \le \varliminf_{n\to\infty} \int \left( hu_n^2 - f(x,u_n)u_n \right), \qquad ((f_3^*) \text{ \& Fatou})$$

as well as $\|u_n^-\| \to \|u^-\|$,

we deduce

$$\begin{aligned}\varlimsup_{n\to\infty}\|u_n^+\|^2 &= \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) \\ &\quad + \varlimsup_{n\to\infty}\left(\left[\langle N'(u_n),u\rangle - 4N(u_n)\right] - \int (hu_n u - f(x,u_n)u)\right) \\ &\le \|u^+\|^2 + \int \left(hu^2 - f(x,u)u\right) - \varliminf_{n\to\infty}\int \left(hu_n^2 - f(x,u_n)u_n\right) \\ &\le \|u^+\|^2.\end{aligned}$$

This and the weak lower semi-continuity of norm functional $u \mapsto \|u\|$, i.e.,

$$\|u^+\|^2 \le \varliminf_{n\to\infty}\|u_n^+\|^2, \tag{10}$$

yields $\|u_n^+\| \to \|u^+\|$. Therefore $\|u_n\| \to \|u\|$ and $u_n \to u$ in $X$.

**Pf of Thm 1.** Let $X^\pm$ be $\pm$-spaces of $S$. As $u \to 0$, Clark (1972/73)

$$\Phi(u) = \frac{1}{2}\int \left(\dot u^2 + V(x)u^2\right) + \int \dot u^2 u^2 - \int F(x,u) = \frac{1}{2}\int \left(\dot u^2 + V(x)u^2\right) + o(\|u\|^2),$$

So $\Phi$ has a loc link at $u = 0$, hence has **3** critical points (Liu (1989)).

# 3. Proof of Thm 2

Let $X^-$ be the negative subspace of $S$ and $X^+ = (X^-)^\perp$. Then $\dim X^- < \infty$ and there is an equivalent norm $\|\cdot\|$ on $X$ such that

$$\int \left(\dot{u}^2 + V(x)u^2\right) = \|u^+\|^2 - \|u^-\|^2,$$

where $u^\pm$ are the orthogonal projections of $u$ on $X^\pm$. Hence

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + N(u) - \int F(x,u).$$

**Lem 4.** Under the conditions $(V_1)$, $(f_1)$, $(f_4)$ and $(f_5)$, $\Phi$ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.

**Pf.** Let $\{u_n\}$ be a $(PS)_c$ sequence of $\Phi$.
Step 1. If $\|u_n\| \to \infty$. Let $v_n = \|u_n\|^{-1}u_n$, then $v_n \rightharpoonup v$ in $X$,

$$v_n^- \to v^- \quad \text{in } X$$

because $\dim X^- < \infty$.

If $v=0$, then $v^-=0$, $\|v_n^+\|^2-\|v_n^-\|^2\geq\frac{1}{2}$ for large $n$. By $(f_4)$

$$
\begin{aligned}
1+c+\|u_n\| &\geq \Phi(u_n)-\frac{1}{4}\langle\Phi'(u_n),u_n\rangle\\
&=\frac{1}{4}\|u_n\|^2\left(\|v_n^+\|^2-\|v_n^-\|^2\right)+\int\left(\frac{1}{4}f(x,u_n)u_n-F(x,u_n)\right)\\
&\geq\frac{1}{8}\|u_n\|^2,
\end{aligned}
$$

contradicting $\|u_n\|\to\infty$.

If $v\neq 0$, by Fatou's lemma (5) implies

$$\int\frac{F(x,u_n)}{\|u_n\|^4}\geq\int_{v\neq 0}\frac{F(x,u_n)}{\|u_n\|^4}v_n^4\to+\infty. \tag{11}$$

Using $|N(u)|\leq B\|u\|^4$ we get a contradiction:

$$\frac{c-1}{\|u_n\|^4}\leq\frac{\Phi(u_n)}{\|u_n\|^4}\leq\frac{\|u_n^+\|^2-\|u_n^-\|^2}{2\|u_n\|^4}+B-\int\frac{F(x,u_n)}{\|u_n\|^4}\to-\infty.$$

Step 2. Assume $u_n \rightharpoonup u$. We show that $\|u_n\| \to \|u\|$. Since $\dim X^- < \infty$, we have $\|u_n^-\| \to \|u^-\|$. Noting

$$\langle N'(u_n), u_n - u\rangle = 4N(u_n) - \langle N'(u_n), u\rangle,$$

by direct computation we deduce

$$\begin{aligned} 0 &= \langle \Phi'(u_n), u_n - u\rangle + o(1) \\ &= \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) - \left(\|u^+\|^2 - \|u^-\|^2\right) \\ &\qquad + 4N(u_n) - \langle N'(u_n), u\rangle - \int f(x, u_n)(u_n - u). \end{aligned}$$

Now, applying Lem 1, using $\|u_n^-\| \to \|u^-\|$ and condition $(f_5)$ we get

$$\begin{aligned} \overline{\lim_{n\to\infty}} \|u_n^+\|^2 &= \|u^+\|^2 + \overline{\lim_{n\to\infty}} \left( \left[\langle N'(u_n), u\rangle - 4N(u_n)\right] + \int f(x, u_n)(u_n - u)\right) \\ &\le \|u^+\|^2. \end{aligned}$$

Hence $\|u_n^+\| \to \|u^+\|$. Noting $\|u_n^-\| \to \|u^-\|$, we conclude $\|u_n\| \to \|u\|$.

**Lem 5.** Assume $(V_1)$, $(f_1)$, $(f_4)$, there exists $A < \inf_{B_2} \Phi$, s.t. if $\Phi(u) \le A$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu) < 0.$$

**Pf.** Otherwise, there is a sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \le -n$ but

$$\langle \Phi'(u_n), u_n \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi(tu_n) \ge 0. \tag{12}$$

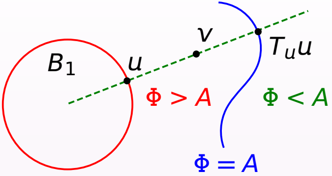
Using $(f_4)$, we deduce

$$\begin{aligned} \|u_n^+\|^2 - \|u_n^-\|^2 &\le \left(\|u_n^+\|^2 - \|u_n^-\|^2\right) + \int [f(x,u_n)u_n - 4F(x,u_n)] \\ &= 4\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle \le -4n. \end{aligned} \tag{13}$$

Let $v_n = \|u_n\|^{-1} u_n$, then $v_n^- \to v^-$ in $X$ ($\dim X^- < \infty$), $v^- \neq 0$. Hence

$$\int \frac{f(x,u_n)u_n}{\|u_n\|^4} \ge 4 \int \frac{F(x,u_n)}{\|u_n\|^4} \to +\infty. \qquad \text{(see (11))}$$

$$0 \le \frac{\langle \Phi'(u_n), u_n \rangle}{\|u_n\|^4} \le \frac{\|u_n^+\|^2 - \|u_n^-\|^2}{\|u_n\|^4} + \frac{\langle N'(u_n), u_n \rangle}{\|u_n\|^4} - \int \frac{f(x,u_n)u_n}{\|u_n\|^4} \to -\infty.$$




$$\varphi : X \backslash B_1 \to \Phi_A,$$

$$\varphi(v) = \begin{cases} T_{v/\|v\|} \dfrac{v}{\|v\|} & \text{if } \Phi(v) > A \\ v & \text{if } \Phi(v) \le A \end{cases}$$

**Pf of Thm 2.** For $u \in \partial B_1$, it is clear that

$$\Phi(tu) \to -\infty \qquad \text{as } t \to +\infty.$$

Thus $\exists 1\, T_u > 1$ s.t. $\Phi(T_u u) = A$. By Lem 5

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=T_u} \Phi(su) = \frac{1}{T_u} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \Phi(t \cdot T_u u) < 0.$$

Following Wang (1991), by IFT $u \mapsto T_u$ is continuous, and we construct a deformation $\varphi : X \backslash B_1 \to \Phi_A$, and deduce (critical groups)

$$C_i(\Phi, \infty) = \mathrm{H}_i(X, \Phi_A) = \mathrm{H}_i(X, X \backslash B_1) = 0 \qquad \text{for } i \in \mathbb{N}_0.$$

Since $\Phi$ has a loc link at $u = 0$, by Liu (1989) (for $\ell = \dim X^-$)

$$C_\ell(\Phi, 0) \ne 0. \qquad \text{Hence } C_\ell(\Phi, 0) \ne C_\ell(\Phi, \infty),$$

Applying Bartsch & Li (1997), $\Phi$ has a crt pt $u \ne 0$.

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# Thank you!

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