Research of Dr. Shibo Liu

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According to MathSciNet of the American Mathematical Society, I have published 44 research papers, which were cited 929 times by 708 authors.

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1. Morse theory and elliptic resonant problems

In Morse theory, the critical group[(1)] $C_*(f,u)$ of $f : X \to \mathbb{R}$ at an isolated critical point u describes the local behavior of f near u (thus can be used to distinguish critical points); the critical group $C_*(f,\infty)$ of f at infinity describes the global property of f. Their relation is demonstrated by the Morse inequalities. Morse theory is very useful in the study of multiple solutions problems.

Saddle point reduction is a useful technique in critical point theory. Let $X = Y \oplus Z$ be a separable Hilbert space. If there is $m > 0$ such that for $v \in Y$, $w_1, w_2 \in Z$ there holds

$$\pm \langle \nabla f(v+w_1) - \nabla f(v+w_2), w_1 - w_2 \rangle \geq m \left\| w_1 - w_2 \right\|^2, \tag{S$_\pm$}$$

then there is a reduced functional $\varphi : Y \to \mathbb{R}$ such that:

- u is critical point of f if and only if $v = Pu$ is critical point of φ, where $P : X \to Y$ is the orthogonal projection.

[(1)]Critical groups are defined using singular homology. For simplicity we use $\mathbb{Q}$ as coefficient group, so that the resulting critical groups are vector spaces.

Thus, finding critical points of f is converted to finding critical points of φ, which is simpler because φ is defined on a (usually finite dimensional) subspace Y.

To combine Morse theory and saddle point reduction, one needs to know the relations between the critical groups of f and that of φ. We systematically investigated this problem in [LL03, Liu07, LL13]. Let $\mathbb{N}$ be the set of nonnegative integers, $\mathcal{K}$ be the set of critical points of f.

Theorem 1.1. *Under the condition* (S_+),

(1) *if u is an isolated critical point of f and $v = Pu$, then $C_q(f, u) \cong C_q(\varphi, v)$ for all $q \in \mathbb{N}$.* [Liu07]

(2) *if f satisfies (PS) and $\inf_{\mathcal{K}} f > -\infty$, then $C_q(f, \infty) \cong C_q(\varphi, \infty)$ for all $q \in \mathbb{N}$. If $\dim Y = k$ and $C_k(f, \infty) \neq 0$, then $\dim C_q(f, \infty) \cong \delta_{qk}$.* [LL03]

Theorem 1.2. *Assume the condition* (S_-) *and* $\dim Z = j$,

(1) *if u is an isolated critical point of f and $v = Pu$, φ satisfies (PS), then $C_q(f, u) \cong C_{q-j}(\varphi, v)$ for all $q \in \mathbb{N}$.* [LL13]

(2) *if f satisfies (PS) and $\inf_{\mathcal{K}} f > -\infty$, then $C_q(f, \infty) \cong C_{q-j}(\varphi, \varphi)$ for all $q \in \mathbb{N}$. Moreover, if $C_j(f, \infty) \neq 0$, then $\dim C_q(f, \infty) \cong \delta_{qj}$.* [LL03]

We applied these abstract results to various elliptic resonant problems. For example, consider the following Dirichlet problem on a bounded domain $\Omega \subset \mathbb{R}^N$:

$$\Delta u + p(u) = 0, \qquad u \in H_0^1(\Omega). \tag{1.1}$$

Theorem 1.3 ([Liu07])**.** *Assume*

(p_1) *$p \in C^1(\mathbb{R})$, $p(0) = 0$, $p'(0) < \lambda_1 < p_\infty = \lambda_m$, where*

$$p_\infty = \lim_{|t|\to\infty} \frac{p(t)}{t},$$

(p_2) *$p'(t) \le \gamma < \lambda_{m+1}$ for some $\gamma \in \mathbb{R}$,*

(p_3) *$P(t) - \frac{1}{2}\lambda_m t^2 \to +\infty$ as $|t| \to \infty$,*

then (1.1) has at least four nontrivial solutions.

Remark 1.4. Castro and Cossio [CC94] studied the nonresonant case that p_∞ is not an eigenvalue. Li and Zhang [LZ99] studied the resonant case $p_\infty = \lambda_m$ assuming (p_1), (p_2) and the much stronger condition

(p_3^*) there exist $\alpha \in [0, 1)$ and $c > 0$ such that

$$|p(t) - \lambda_m t| \le c\left(1 + |t|^\alpha\right), \qquad \lim_{|t|\to\infty} \frac{1}{|t|^{2\alpha}} \left(P(t) - \frac{1}{2}\lambda_m t^2\right) = +\infty.$$

This condition ensures that the energy functional f satisfies (PS) condition. In our Theorem 1.3 we don't know whether f satisfies (PS); but we observed that the saddle point reduced functional φ satisfies (PS), which enables us to get four nonzero critical points for φ via Morse theory (for this, Theorem 1.1 (1) is needed).

We have also studied the case that $p'(0) > \lambda_1$.

Theorem 1.5 (Special case of [Liu08, Thm 1.1])**.** *Assume $|p(t)| \le \Lambda |t|$ for some $\Lambda \in \mathbb{R}$, and*

(p_1') *$p \in C^1(\mathbb{R})$, $p(0) = 0$, $p'(0) \in (\lambda_k, \lambda_{k+1})$,*
(p_2') *$p'(t) \geq \gamma > \lambda_{m-1}$ for some $\gamma \in \mathbb{R}$,*
(p_3') *$\frac{1}{2}\lambda_m t^2 - P(t) \to +\infty$ as $|t| \to \infty$,*

then (1.1) has at least two nontrivial solutions.

Theorem 1.5 improves related results of Li and Willem [LW98], where to ensures that the energy functional f satisfies (PS) condition, instead of (p_3') they assume the stronger condition

$(p_3^\#)$ there exist $\alpha \in [0, 1)$ and $c > 0$ such that

$$|p(t) - \lambda_m t| \leq c\left(1 + |t|^\alpha\right), \qquad \lim_{|t|\to\infty} \frac{1}{|t|^{2\alpha}}\left(\frac{1}{2}\lambda_m t^2 - P(t)\right) = +\infty.$$

2. p-Laplacian equations

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For the p-Laplacian equation

$$-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = g(u), \qquad u \in W_0^{1,p}(\Omega) \tag{2.1}$$

with p-superlinear nonlinearity

$$\lim_{|t|\to\infty} \frac{g(t)}{|t|^{p-2}t} = +\infty,$$

all existence results (except [FL00]) require

$$\overline{\lim_{|t|\to 0}} \frac{pG(t)}{|t|^p} < \lambda_1, \qquad \text{here and below } G(t) = \int_0^t g,$$

so that $u = 0$ is a local minimizer of the energy functional Φ and mountain pass theorem applies. By constructing a global linking, Fan and Li [FL00] obtained nontrivial solution of (2.1) assuming g satisfies the Ambrosetti–Rabinowitz condition (AR) and for some $\lambda \in [\lambda_1, \lambda_2)$ there hold

$$G(t) \geq \frac{\lambda}{p}|t|^p \quad \text{for all } t \in \mathbb{R}, \qquad \lim_{|t|\to 0} \frac{pG(t)}{|t|^p} = \lambda. \tag{2.2}$$

This is the first result for (2.1) when Φ is unbounded below and $u = 0$ is not a local minimizer of Φ.

In [Liu01], by using Morse theory and local linking, we improved the above result of [FL00] by weakening the global condition (2.2) to the following local condition:

(g_0) there exist $r > 0$ and $\bar{\lambda} \in (\lambda_1, \lambda_2)$ such that

$$\lambda_1 |t|^p \leq pG(t) \leq \bar{\lambda}\,|t|^p \qquad \text{for } |t| \leq r.$$

Using the Yang index, Perera [Per03] dealt with the case that g interacts with higher eigenvalues:

$$\lim_{|t|\to 0} \frac{g(t)}{|t|^{p-2}t} = \lambda \notin \sigma(-\Delta_p).$$

By an argument dual to [Per03], we [LL04] obtained nontrivial solution for (2.1) when g is sublinear at 0 and asymptotically p-linear at infinity

$$\lim_{|t|\to\infty} \frac{g(t)}{|t|^{p-2}\,t} = \lambda \notin \sigma(-\Delta_p).$$

Back to the p-superlinear case, both [Liu01] and [Per03] require the (AR) condition. In [Liu10a] we got the same existence result with (AR) replacing by

- for some $R > 0$, $\eta : t \mapsto |t|^{1-p}\, g(t)$ is increasing on $(-\infty, -R)$ and (R, ∞), and
$$\lim_{|t|\to\infty} \frac{G(t)}{|t|^p} = +\infty.$$

The point is that, although we could not get the boundedness of (PS) sequences without (AR), the Cerami sequences are bounded. We also applied this idea and technique to more difficult problems, such as $p(x)$-Laplacian equations [AL10] and strongly indefinite Schrödinger equations (see §3.1).

3. Indefinite Schrödinger equations

3.1. Strongly indefinite Schrödinger equations. Let $V \in C(\mathbb{R}^N)$ be $\mathbb{Z}^N$-periodic, we consider the stationary Schrödinger equation

$$-\Delta u + V(x)u = g(u), \qquad u \in H^1(\mathbb{R}^N). \tag{3.1}$$

If g satisfies the Ambrosetti–Rabinowitz (AR) condition, the problem has been studied by Coti Zelati and Rabinowitz [CZR92] for the definite case $0 < \inf \sigma(-\Delta + V)$ and by Kryszewski and Szulkin [KS98] for the indefinite case

$$0 > \inf \sigma(-\Delta + V), \qquad 0 \notin \sigma(-\Delta + V).$$

Since there are many superlinear functions, such as $g(t) = t\ln(1 + |t|)$, violating (AR), it is interesting to replace (AR) by other superlinear conditions. In [Liu12] we obtained the following theorem.

Theorem 3.1. *Let $V \in C(\mathbb{R}^N)$ be $\mathbb{Z}^N$-periodic, $0 \notin \sigma(-\Delta + V)$, $g \in C(\mathbb{R})$ be subcritical and superlinear at* 0*. If*

- *$\eta : t \mapsto g(t)/|t|$ is increasing on $(-\infty, 0)$ and $(0, \infty)$,*
$$\lim_{|t|\to\infty} \frac{G(t)}{t^2} = +\infty,$$

then (3.1) has a ground state solution.

This improves the result of Szulkin and Weth [SW09], because they required η to be *strictly* increasing. Although the energy functional Φ verifies the linking geometry

$$b = \inf_N \Phi > \sup_{\partial M} \Phi, \tag{3.2}$$

without (AR) condition one can not get the boundedness of (PS) sequences. This prevents them to apply the linking theorem of [KS98]. To overcome this difficulty they applied the generalized Nehari manifold technique.

Our key observation is that, (even for the strongly indefinite case) the Cerami sequences of Φ are bounded. This enables us to get the desired result via standard variational method and the linking theorem of [LS02]. This approach is much simpler.

3.2. Schrödinger-Poisson systems. To solve the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = g(u), \\ -\Delta\phi = u^2, \\ (u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3), \end{cases} \tag{3.3}$$

Benci and Fortunato [BF98] introduced the reduction method: let ϕ_u be the solution of the second equation, if u is a critical point of $\Phi : H^1(\mathbb{R}^3) \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int \phi_u u^2 - \int G(u), \tag{3.4}$$

then (u, ϕ_u) solves (3.3). Using this idea, many results on (3.3) appear, all of them require the Schrödinger operator $S = -\Delta + V$ to be positive definite, so that $u = 0$ is a local minimizer of Φ and Φ satisfies the mountain pass geometry.

If the Schrödinger operator S is indefinite, $u = 0$ will not be a local minimizer of Φ, the mountain pass theorem (and Nehari technique) could not be applied any more. One may think of using the linking theorem instead.

Let Y be the negative space of S. We observed that because $\int \phi_u u^2 > 0$ for $u \neq 0$, we could not have $\Phi|_Y \le 0$, which is needed for verifying the linking geometry (3.2). Therefore, unlike for the semilinear Schrödinger equation (3.1), linking theorem could not be used to solve the indefinite Schrödinger-Poisson system (3.3).

We discovered that the concept of local linking originated by Liu and Li [LL84] is quite suitable for treating this kind of problems. Using local linking theory, in [CL15] we obtained the following result.

Theorem 3.2. *Let $V \in C(\mathbb{R}^N)$ be coercive, $g \in C(\mathbb{R})$ be subcritical, superlinear at* 0, *and satisfy*

- $4G(t) \le tg(t) + bt^2$ *for some* $b > 0$ *and*
$$\lim_{|t|\to\infty} \frac{g(t)}{t^3} = +\infty.$$

Then, if $0 \notin \sigma(-\Delta + V)$, *the problem (3.3) has a nontrivial solution.*

This is the first result on the Schrödinger-Poisson system when the Schrödinger operator $S = -\Delta + V$ is indefinite. In [LW17, LM20] we obtained more results in this direction.

3.3. Quasilinear Schrödinger equations. The quasilinear Schrödinger equation

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad u \in H^1(\mathbb{R}^N) \tag{3.5}$$

is the Euler-Lagrange equation of the functional

$$J(u) = \frac{1}{2}\int \left(1 + 2u^2\right)|\nabla u|^2 + \frac{1}{2}\int V(x)u^2 - \int G(u).$$

But J is not defined at some $u \in H^1(\mathbb{R}^N)$, so we could not get solutions of (3.5) by applying critical point theory to J.

Liu *et al.* [LWW03] and Colin and Jeanjean [CJ04] introduced a nonlinear transformation f, so that the quasilinear equation (3.5) is converted into a semilinear one, which is the Euler-Lagrange equation of $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$,

$$\Phi(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x)f^2(v)\right) - \int G(f(v)).$$

If v is a critical point of Φ, then $u = f(v)$ is solution of (3.5).

Since then, many results on (3.5) appear. Again, all of them require the Schrödinger operator $S = -\Delta + V$ be positive definite, so that $v = 0$ is a local minimizer of Φ and the mountain pass theorem applies.

In [LZ18] we initiated the study on the case that S is indefinite and $v = 0$ fails to be a local minimizer of Φ. Then, mountain pass theorem is not applicable. Since the principle part

$$Q(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x)f^2(v)\right)$$

of Φ, is not a quadratic form, it is also not clear how to decompose the working space and verify linking or local linking geometry.

We observed that Q is of class C^2 (due to the nice properties of f). Since

$$\int G(f(v)) = o(\|v\|^2) \qquad \text{as } \|v\| \to 0,$$

by applying Taylor expansion to Q we deduce that as $\|v\| \to 0$,

$$\Phi(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x)v^2\right) + o(\|v\|^2) \qquad \text{as } \|v\| \to 0.$$

Now it is clear Φ has a local linking at $v = 0$ with respect to the decomposition of the space via the quadratic form above. This enables us to establish the following theorem.

Theorem 3.3. *Let $V \in C(\mathbb{R}^N)$ be coercive, $g \in C(\mathbb{R})$ be superlinear at* 0*, and satisfy*

(g_1) *there exist $C > 0$ and $p \in (4, 2\cdot 2^*)$ such that*

$$|g(t)| \le C\left(1 + |t|^{p-1}\right),$$

(g_2) *there exists $\mu > 4$ such that*

$$0 < \mu G(t) \le tg(t) \qquad \text{for } t \neq 0,$$

Then, if $0 \notin \sigma(-\Delta + V)$, the problem (3.5) has a nontrivial solution.

This is the first result on the quasilinear Schrödinger equations when the Schrödinger operater $S = -\Delta + V$ is indefinite.

In [LZ19] we considered a related indefinite problem ($N \le 6$)

$$\Delta^2 u - \Delta u + V(x)u - \tfrac{1}{2}u\Delta(u^2) = g(x,u), \qquad u \in H^2(\mathbb{R}^N).$$

Since H^2 is more regular than H^1, the natural energy functional is well defined on H^2. Therefore, unlike in the study of (3.5), we don't need to apply nonlinear transformation.

This enables us to get better results, for example, we can deal with the case that $V \in L^\infty(\mathbb{R}^N)$.

In [LY23], we studied the super-critical problem

$$-\Delta u - u\Delta(u^2) = k(x)\,|u|^{q-2}\,u - h(x)\,|u|^{s-2}\,u, \qquad u \in D^{1,2}(\mathbb{R}^N), \tag{3.6}$$

where $1 < q < 2 < s < \infty$. Under our assumptions on k and h, it turns out that the suitable working space E is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\| = \|u\|_{D^{1,2}} + |h^{s/2}u|_{s/2}.$$

Although we don't know whether E is reflexive, we still verified the (PS) condition and got a sequence of solutions for the problem (3.6).

4. Elementary topics

In addition to frontier research described above, because of my teaching I also revisited some classical (thus elementary, undergraduate level) topics.

We say that $A \in \mathcal{L}(X, Y)$ is regular if A is bijective and $A^{-1} \in \mathcal{L}(Y, X)$. It is well known that if A is regular then operators near A are also regular, thus the set of regular operators from X to Y is open in $\mathcal{L}(X, Y)$. Traditionaly this is proven using Neumann series. In [Liu10b] we provided a simple geometric proof using the Riesz lemma (the one used in proving that infinite dimensional spheres are not compact).

Motivated by an excerise in [dC76] (where two dimensional case is considered), by induction argument and linear algebraic techniques (such as the Cauchy-Binet formula and the Laplace expansion of determinants), in [LZ17] we presented an easy proof of the m-dimsneional change of variables formula

$$\int_{\varphi(\Omega)} f(y)\,\mathrm{d}y = \int_\Omega f(\varphi(x))\left|\det \varphi'(x)\right|\,\mathrm{d}x.$$

Since for simple Ω (such as balls in $\mathbb{R}^m$) we only required that φ is a diffeomorphism on $\partial\Omega$, as an easy corollary we obtained the m-dimensional Brouwer fixed point theorem.

A classical result in multivariable calculus says that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is coercive and $\det f'(x) \neq 0$ for all $x \in \mathbb{R}^n$, then $f(\mathbb{R}^n) = \mathbb{R}^n$. To produce a more elementary proof of the fundamental theorem of algebra, in [LL18] we extended the above classical result as follow:

- Suppose $n \geq 2$ and $f : \mathbb{R}^m \to \mathbb{R}^n$ has at most finitely many critical points. If $f(\mathbb{R}^m)$ is closed in $\mathbb{R}^n$, then $f(\mathbb{R}^m) = \mathbb{R}^n$.

As a corollary, we deduced that *any vector-valued function over a compact manifold has infinitely many critical points.*

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