

ON DENSITY, MASS AND MULTIPLE INTEGRALS

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1. MASS AND DOUBLE INTEGRALS

Let's recall the 1-D case. Consider a thin rod occupying $I = [0, a]$ on the x -axis with 1-D density $f(x)$ (*mass per unit length*). To find the mass of I , we consider an infinitesimal segment $[x, x + dx]$ at $x \in I$, whose mass is approximately $dM \approx f(x)dx$. Therefore the mass of the rod I is

$$M = \int dM = \int_0^a f(x)dx.$$

We get the mass of the whole rod by integrating the *mass element* dM . This is what we have learned as application of definite integral in Calculus 1.

Similar idea applies to higher dimensional problems. For simplicity, it is standard to denote a rectangle

$$R : \quad a \leq x \leq b, c \leq y \leq d$$

by product of intervals $R = [a, b] \times [c, d]$. Similarly, $[a, b] \times [c, d] \times [e, f]$ is a box in 3-D space.

Let $Q \subset \mathbb{R}^2$ be a thin sheet on the xy -plane with 2-D density $f(x, y)$ (*mass per unit area*). Then the mass of the infinitesimal rectangle

$$dA = [x, x + dx] \times [y, y + dy]$$

based at $(x, y) \in Q$ is approximately

$$dM \approx f(x, y)dA,$$

here we use the same notation dA to denote the area of the small rectangle. Thus the mass of Q is

$$M = \iint dM = \iint_Q f(x, y) dA. \quad (1.1)$$

This is the simplest interpretation of double integrals. Since $dA = dx dy$, we also write $\iint_Q f dx dy$ for $\iint_Q f dA$.

2. DOUBLE INTEGRALS AND ITERATED INTEGRALS

Now we assume that the thin sheet Q is bounded by the curves $y = g(x)$ and $y = h(x)$, $x \in [0, a]$, i.e.,

$$Q = \{(x, y) \mid g(x) \leq y \leq h(x), x \in [0, a]\}.$$

At $x \in [0, a]$ we consider infinitesimal increment dx . The part of Q above $[x, x + dx]$ is approximately the thin strip (*the vertical green region in the picture*)

$$dY = [x, x + dx] \times [g(x), h(x)].$$

Since dx is very small, dY can be viewed as a vertical line with 1-D density⁽¹⁾ $f(x, y)dx$ at $y \in [g(x), h(x)]$. Thus its mass is

$$dM = \int_{g(x)}^{h(x)} [f(x, y)dx] dy = \left(\int_{g(x)}^{h(x)} f(x, y) dy \right) dx,$$

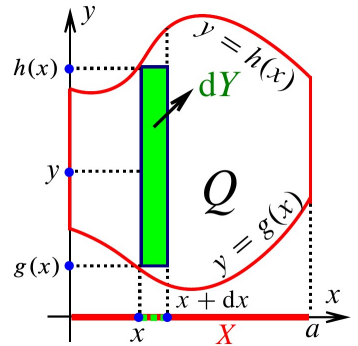
here we are integrating with respect to y , so dx is constant and can be moved out of the integral sign.

At this moment, we can already integrate dM against $x \in [0, a]$ to get $\iint_Q f dA$, the mass of Q , thus deduce the equality

$$\int_0^a \left(\int_{g(x)}^{h(x)} f(x, y) dy \right) dx = \iint_Q f(x, y) dA. \quad (2.1)$$

If this does not convince you, read the next paragraph for more details. That paragraph plays the same role explaining *why integrating the area of cross section yields the volume* in the volume interpretation of the iterated integral formula (2.1).

Imaging that we squeeze Q vertically to the x -axis. Then the squeezed Q becomes the 1-D segment $X = [0, a]$ on the x -axis (*the thick red segment lying on the x -axis*



⁽¹⁾Multiplying the horizontal width dx to the 2-D density $f(x, y)$, we get the 1-D density in the vertical direction.

in the picture), with **the same mass**. The mass of the small segment (*the dotted green segment on X*) $[x, x + dx]$ on X equals the mass of dY we obtained above:

$$dM = \left(\int_{g(x)}^{h(x)} f(x, y) dy \right) dx,$$

Thus, the 1-D density of X at $x \in [0, a]$ is

$$\rho(x) = \frac{dM}{dx} = \int_{g(x)}^{h(x)} f(x, y) dy. \quad (2.2)$$

Hence the mass of the squeezed Q (that is the segment X) is

$$\int_0^a \left(\int_{g(x)}^{h(x)} f(x, y) dy \right) dx = \iint_Q f(x, y) dA.$$

The equality here is because: LHS is the mass of X obtained by integrating its 1-D density given in (2.2); RHS is the mass of Q , which equals that of X (the squeezed Q).

3. TRIPLE INTEGRALS AND ITERATED INTEGRALS

Let $D \subset \mathbb{R}^3$ be a solid body in 3-D space, $f(x, y, z)$ be the 3-D density (*mass per unit volume*) at point $(x, y, z) \in D$. Then the mass of the infinitesimal box

$$dV = [x, x + dx] \times [y, y + dy] \times [z, z + dz]$$

based at $(x, y, z) \in D$ is approximately

$$dM \approx f(x, y, z) dV,$$

here we use the same notation dV to denote the volume of the small box. Thus the mass of D is

$$M = \iiint dM = \iiint_D f(x, y, z) dV.$$

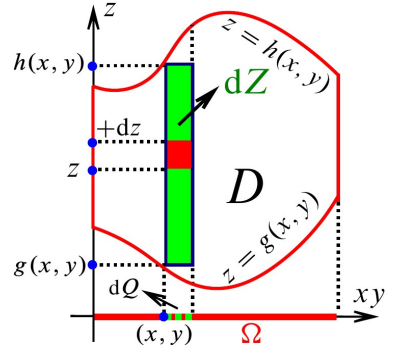
Since $dV = dx dy dz$, we also write $\iiint_D f dx dy dz$ for the triple integral $\iiint_D f dV$.

Assume that D is bounded by the surfaces $z = g(x, y)$ and $z = h(x, y)$, $(x, y) \in \Omega$, i.e.,

$$D = \{(x, y, z) \mid g(x, y) \leq z \leq h(x, y), (x, y) \in \Omega\},$$

see the picture, where the xy -plane is *represented as the horizontal axis* to avoid a messy 3-D illustration (this trick can be applied to the volume interpretation of (2.1)).

At $(x, y) \in \Omega$ we consider infinitesimal increments dx and dy . The thin vertical bar (*the vertical green region in the picture*) dZ with base (*the dotted green segment on Ω*) $dQ = [x, x + dx] \times [y, y + dy]$ can be view as a vertical line bounded by $z = g(x, y)$ and $z = h(x, y)$, with 1-D density $f(x, y, z)dx dy$; because the portion (*the small solid red rectangle*) between z and $z + dz$ has mass $dm \approx f(x, y, z)dx dy dz$, dividing the infinitesimal length dz we get the desired 1-D density $f(x, y, z)dx dy$.



Thus the mass of dZ is

$$dM = \int_{g(x,y)}^{h(x,y)} [f(x, y, z)dx dy] dz = dx dy \int_{g(x,y)}^{h(x,y)} f(x, y, z)dz. \quad (3.1)$$

At this moment, we can already integrate dM against $(x, y) \in \Omega$ to get $\iiint_D f dV$, the mass of D , thus deduce

$$\iiint_D f(x, y, z)dx dy dz = \iint_{\Omega} dx dy \int_{g(x,y)}^{h(x,y)} f(x, y, z)dz. \quad (3.2)$$

If this does not convince you, read the next paragraph for more details.

We compress D vertically onto Ω , the mass M does not change. Now Ω is a planar region on xy -plane with 2-D density at $(x, y) \in \Omega$

$$\rho(x, y) = \frac{dM}{dx dy} = \int_{g(x,y)}^{h(x,y)} f(x, y, z)dz, \quad (3.3)$$

because the mass of the infinitesimal rectangle (*the dotted green segment on Ω*) dQ is the dM given in (3.1) and the area is $dx dy$.

Therefore we deduce

$$\begin{aligned} \iiint_D f(x, y, z)dV &= \iint_{\Omega} \rho(x, y)dx dy \\ &= \iint_{\Omega} \left(\int_{g(x,y)}^{h(x,y)} f(x, y, z)dz \right) dx dy \\ &= \iint_{\Omega} dx dy \int_{g(x,y)}^{h(x,y)} f(x, y, z)dz. \end{aligned}$$

Here, the first equality is because: the LHS is the mass of D , the RHS is the mass of Ω obtained by integrating its 2-D density given in (3.3); Ω , being the vertical compression of D on the xy -plane, has the same mass as D .

After deducing (3.2), we may further evaluate the outer integral on Ω using iterated integral: Assuming Ω is bounded by two curves $y = \varphi_{\pm}(x)$, $x \in [a, b]$, then

$$\begin{aligned} \iiint_D f(x, y, z) dx dy dz &= \iint_{\Omega} dx dy \int_{g(x, y)}^{h(x, y)} f(x, y, z) dz \\ &= \int_a^b dx \int_{\varphi_-(x)}^{\varphi_+(x)} dy \int_{g(x, y)}^{h(x, y)} f(x, y, z) dz. \end{aligned} \quad (3.4)$$

4. ANOTHER ITERATED TRIPLE INTEGRAL FORMULA

If D is bounded by $z = c$ and $z = d$, D_z (the thick blue segment) is the intersection of D and the horizontal plane π_z at level $z \in [c, d]$. Given infinitesimal increment dz , the part dZ (the green region in the picture) of D between π_z and π_{z+dz} can be view as planar region with 2-D density⁽²⁾

$$\rho(x, y) = f(x, y, z) dz, \quad (x, y) \in D(z), \quad (4.1)$$

where $D(z)$ (the thick green segment on the horizontal axis) is the projection of D_z on the xy -plane.

Therefor the mass of dZ is

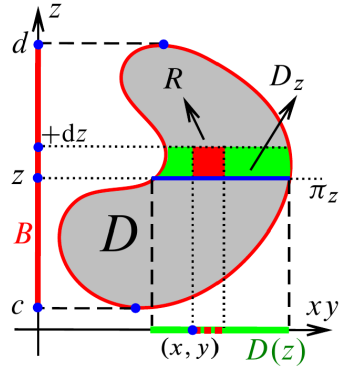
$$dM = \iint_{D(z)} [f(x, y, z) dz] dx dy = dz \iint_{D(z)} f(x, y, z) dx dy.$$

Imaging that we compress D horizontally onto the z -axis. Then D becomes a thin bar B (the red vertical segment in the picture) on the z -axis with the same mass. The 1-D density of B at z is

$$\varrho(z) = \frac{dM}{dz} = \iint_{D(z)} f(x, y, z) dx dy.$$

Thus, the mass of B (which equals $\iiint_D f dV$, the mass of D), is

$$\iiint_D f(x, y, z) dV = \int_c^d \varrho(z) dz = \int_c^d dz \iint_{D(z)} f(x, y, z) dx dy. \quad (4.2)$$



⁽²⁾To get (4.1) we take infinitesimal increments dx and dy at $(x, y) \in D_z$. Then the 2-D density

$$\rho(x, y) = \frac{dm}{dxdy} = f(x, y, z) dz, \quad (x, y) \in D(z),$$

where $dm = f(x, y, z) dx dy dz$ is the mass of the small box (the red solid rectangle)

$$R = [x, x + dx] \times [y, y + dy] \times [z, z + dz] \quad \text{based at } (x, y, z).$$