On the sign of Jacobian and orientation of parametrized surfaces

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Let $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ be a local diffeomorphism at the point $a \in \mathbb{R}^m$. Then the tangent map $\varphi_* : T_a(\mathbb{R}^m) \to T_b(\mathbb{R}^m)$ is a linear isomorphism given by

$$\varphi_*(a, v) = (\varphi(a), \varphi'(a)v),$$

here we use (a, v) to denote a tangent vector based at a and in the direction $v \in \mathbb{R}^m$. Taking determinent on both sides of the following matrix identity

$$\left(\varphi'(a)v_1, \dots, \varphi'(a)v_m\right) = \varphi'(a)\left(v_1, \dots, v_m\right),$$

it follows that if $\det \varphi'(a) > 0$ and $\{e_i\}_{i=1}^m$ being $e_i = (a, v_i)$ is a positive base of $T_a(\mathbb{R}^m)$, meaning that the determinent with columns $v_1, \dots, v_m$ is positive, then $\{\varphi_* e_i\}_{i=1}^m$ is a positive base of $T_b(\mathbb{R}^m)$. We say that diffeomorphisms with positive Jacobian preserve orientation.

Suppose that $\varphi : \bar{\Omega} \to \bar{D}$ is a C^1-diffeomorphism between two smooth domains in $\mathbb{R}^m$, $a \in \partial\Omega$. It is well known that the submanifold $\partial\Omega$ is orientable and its orientation can be interprited by a nozero normal vector N at a. In many applications, N is given by a local parametrization $x : U \to \partial\Omega$ near a via

$$N = \left(\frac{\partial\left(x^2, \dots, x^m\right)}{\partial\left(u^1, \dots, u^{m-1}\right)}, -\frac{\partial\left(x^1, x^3, \dots, x^m\right)}{\partial\left(u^1, \dots, u^{m-1}\right)}, \dots, (-1)^{m+1}\frac{\partial\left(x^1, \dots, x^{m-1}\right)}{\partial\left(u^1, \dots, u^{m-1}\right)}\right)_{u_0},$$

being U an open subset of $\mathbb{R}^{m-1}$ and $u_0 \in U$ such that $a = x(u_0)$. Because φ is a diffeomorphism between $\partial\Omega$ and ∂D, we automatically get a local parametrization $y = \varphi \circ x$ of ∂D near $b = \varphi(a)$, which gives raise to a normal vector

$$\tilde{N} = \left(\frac{\partial\left(y^2, \dots, y^m\right)}{\partial\left(u^1, \dots, u^{m-1}\right)}, -\frac{\partial\left(y^1, y^3, \dots, y^m\right)}{\partial\left(u^1, \dots, u^{m-1}\right)}, \dots, (-1)^{m+1}\frac{\partial\left(y^1, \dots, y^{m-1}\right)}{\partial\left(u^1, \dots, u^{m-1}\right)}\right)_{u_0}$$

of ∂D at b. The main result of this note is the following theorem.

Theorem 0.1. *Let $\varphi : \bar{\Omega} \to \bar{D}$ be a C^1-diffeomorphism, $a \in \partial\Omega$, $J_\varphi(a) > 0$. If N is an outward normal vector of $\partial\Omega$ at a, then $\tilde{N}$ is an outward normal vector of ∂D at b.*

Firstly we explain what an outward normal vector means. For $x \in \mathbb{R}^m$, let $\mathcal{N}(x)$ denotes the set of all open neighborhoods of x in $\mathbb{R}^m$. Let Ω be an open subset of $\mathbb{R}^m$. We say that $\partial\Omega$ is of class C^k, if for every $a \in \partial\Omega$, there are $U \in \mathcal{N}(a)$ and a C^k-function $g : U \to \mathbb{R}$

Let Ω be an open subset of $\mathbb{R}^m$. We say that $\partial\Omega$ is of class C^k, if for every $a \in \partial\Omega$, there are $U \in \mathcal{N}(a)$, $V \in \mathcal{N}(0)$ and a C^k-diffeomorphism $\phi : U \to V$

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such that $\phi(a) = 0$ and

$$\phi(U \cap \Omega) = V \cap \{y^m > 0\}, \qquad \phi(U \cap \partial\Omega) = V \cap \{y^m = 0\}. \tag{0.1}$$

Obviously $\partial\Omega \cap U = \{\phi^m = 0\}$ and since ϕ is a C^1-diffeomprphism, the Jacobi matrix

$$\phi'(a) = \begin{pmatrix} \partial_1\phi^1 & \cdots & \partial_m\phi^1 \\ \vdots & & \vdots \\ \partial_1\phi^m & \cdots & \partial_m\phi^m \end{pmatrix}_a = \begin{pmatrix} \nabla\phi^1(a) \\ \vdots \\ \nabla\phi^m(a) \end{pmatrix}$$

is invertible, which implies that $\nabla\phi^m(a) \neq 0$. Actually, $\nabla\phi^m(a)$ is a normal vector of $\partial\Omega$ at a.

Suppose N is a nonzero normal vector of $\partial\Omega$ at a. Since a is an interior point of U, for some $\delta > 0$ we can define a C^1-function $\eta : (-\delta, \delta) \to \mathbb{R}$ by $\eta(t) = \phi^m(a + tN)$. Obviously,

$$\eta(0) = \phi^m(a) = 0, \qquad \dot\eta(0) = \nabla\phi^m(a) \cdot N \neq 0$$

because N is parallel to $\nabla\phi^m(a)$. Assume $\dot\eta(0) < 0$, then there exists $\varepsilon > 0$ such that

$$\phi^m(a + tN) = \eta(t) > \eta(0) = 0$$

for $t \in (-\varepsilon, 0)$. From (0.1) we get $a + tN \in \Omega$. If $\dot\eta(0) > 0$ we will get $a - tN \in \Omega$ for $t \in (-\varepsilon, 0)$. The above discussion justifies the following definition.

Definition 0.2. Let Ω be an open subset of $\mathbb{R}^m$ with C^1-boundary $\partial\Omega$, $a \in \partial\Omega$ and N is a normal vector of $\partial\Omega$ at a. If there exists $\varepsilon > 0$ such that $a + tN \in \Omega$ for $t \in (-\varepsilon, 0)$, then we say that N is an *outward normal vector* of $\partial\Omega$ at a.

Proof (Proof of Theorem 0.1). By the chain role,

$$\begin{pmatrix} \dfrac{\partial y^1}{\partial u^1} & \dfrac{\partial y^1}{\partial u^2} & \cdots & \dfrac{\partial y^1}{\partial u^{m-1}} \\ \dfrac{\partial y^2}{\partial u^1} & \dfrac{\partial y^2}{\partial u^2} & \cdots & \dfrac{\partial y^2}{\partial u^{m-1}} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial y^m}{\partial u^1} & \dfrac{\partial y^m}{\partial u^2} & \cdots & \dfrac{\partial y^m}{\partial u^{m-1}} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial y^1}{\partial x^1} & \dfrac{\partial y^1}{\partial x^2} & \cdots & \dfrac{\partial y^1}{\partial x^m} \\ \dfrac{\partial y^2}{\partial x^1} & \dfrac{\partial y^2}{\partial x^2} & \cdots & \dfrac{\partial y^2}{\partial x^m} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial y^m}{\partial x^1} & \dfrac{\partial y^m}{\partial x^2} & \cdots & \dfrac{\partial y^m}{\partial x^m} \end{pmatrix} \begin{pmatrix} \dfrac{\partial x^1}{\partial u^1} & \dfrac{\partial x^1}{\partial u^2} & \cdots & \dfrac{\partial x^1}{\partial u^{m-1}} \\ \dfrac{\partial x^2}{\partial u^1} & \dfrac{\partial x^2}{\partial u^2} & \cdots & \dfrac{\partial x^2}{\partial u^{m-1}} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial x^m}{\partial u^1} & \dfrac{\partial x^m}{\partial u^2} & \cdots & \dfrac{\partial x^m}{\partial u^{m-1}} \end{pmatrix},$$

here and in what follows, $\partial_{u^i} y^j$, $\partial_{x^i} y^j$ and $\partial_{u^i} x^j$ are evaluated at u_0, a and u_0, respectively. Let A_i^j be the cofactor of $\partial_{x^i} y^j$ in $A = \varphi'(a)$, A^* be the adjugate matrix of A. By the Cauchy-Binet formula,

$$\begin{aligned} \tilde N^j &= (-1)^{j+1} \frac{\partial(y^1, \ldots, \hat y^j, \ldots, y^m)}{\partial(u^1, \ldots, u^{m-1})} \\ &= (-1)^{j+1} \sum_{i=1}^m \frac{\partial\left(y^1, \ldots, \hat y^j, \ldots, y^m\right)}{\partial\left(x^1, \ldots, \hat x^i, \ldots, x^m\right)} \frac{\partial\left(x^1, \ldots, \hat x^i, \ldots, x^m\right)}{\partial\left(u^1, \ldots, u^{m-1}\right)} \\ &= \sum_{i=1}^m \left\{(-1)^{i+j} \frac{\partial\left(y^1, \ldots, \hat y^j, \ldots, y^m\right)}{\partial\left(x^1, \ldots, \hat x^i, \ldots, x^m\right)}\right\} \left\{(-1)^{i+1} \frac{\partial\left(x^1, \ldots, \hat x^i, \ldots, x^m\right)}{\partial\left(u^1, \ldots, u^{m-1}\right)}\right\} \\ &= \sum_{i=1}^m A_i^j N^i. \end{aligned}$$

Therefore

$$\tilde{N} = \begin{pmatrix} A_1^1 & \cdots & A_m^1 \\ \vdots & & \vdots \\ A_1^m & \cdots & A_m^m \end{pmatrix} N = (A^*)^{\mathrm{T}} N. \tag{0.2}$$

Since $a \in \partial\Omega$ and N is an outward normal vector of $\partial\Omega$ at a, there is $\varepsilon > 0$, such that for $t \in (-\varepsilon, 0)$ we have $a + tN \in \Omega$ and consequently $\varphi(a + tN) \in D$. Thus,

$$\gamma : t \mapsto \varphi(a + tN), \qquad t \in (-\varepsilon, 0]$$

is a smooth curve from the interior of D to $b \in \partial D$, whose velocity vector at $b = \varphi(a)$ is

$$v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi(a + tN) = \varphi'(a)N = AN$$

Because $A^*A = J_\varphi(a)\mathrm{I}_m$, where I_m is the $m \times m$ identity matrix, we deduce from (0.2) that

$$\begin{aligned} v \cdot \tilde{N} &= (AN)^{\mathrm{T}} (A^*)^{\mathrm{T}} N \\ &= N^{\mathrm{T}} A^{\mathrm{T}} (A^*)^{\mathrm{T}} N \\ &= N^{\mathrm{T}}(A^*A)^{\mathrm{T}} N = J_\varphi(a)\,|N|^2 > 0. \end{aligned}$$

On the other hand, since ∂D is of class C^1, we can choose $U \in \mathcal{N}(b)$ and a C^1-function $g : U \to \mathbb{R}$ such that $g(b) = 0$,

$$U \cap D = \{g > 0\}, \qquad U \cap \partial D = \{g = 0\}.$$

Because $\nabla g(b)$ is a normal vector of ∂D at b, there exists a constant $k \neq 0$ such that $\nabla g(b) = k\tilde{N}$.

We claim that $k < 0$. In fact, take $r > 0$ such that $\gamma(t) \in U$ for $t \in (-r, 0)$, then consider the function $f : (-r, 0] \to \mathbb{R}$ defined by $f(t) = g(\gamma(t))$. We have $f(0) = g(b) = 0$,

$$\begin{aligned} k\tilde{N} \cdot v &= \nabla g(b) \cdot \dot{\gamma}(0) \\ &= \dot{f}(0) = \lim_{t \to 0^-} \frac{g(\gamma(t))}{t} \leq 0, \end{aligned}$$

because $\gamma(t) \in D$ for $t \in (-r, 0)$. Since $v \cdot \tilde{N} > 0$ and $k \neq 0$, we deduce that $k < 0$.

To conclude the proof, we need to find a $\delta > 0$ such that $b + t\tilde{N} \in D$ for $t \in (-\delta, 0)$. For this purpose, consider the C^1-function $\eta : (-s, 0] \to \mathbb{R}$ defined by $\eta(t) = g(b + t\tilde{N})$ for some $s > 0$ small enough. Then $\eta(0) = g(b) = 0$,

$$\dot{\eta}(0) = \nabla g(b) \cdot \tilde{N} = k\tilde{N} \cdot \tilde{N} = k|\tilde{N}|^2 < 0.$$

Hence, there exists $\delta > 0$ such that for $t \in (-\delta, 0)$ we have

$$g(b + t\tilde{N}) = \eta(t) > 0,$$

that is $b + t\tilde{N} \in D$. Consequently, $\tilde{N}$ is an outward normal vector of ∂D at b.