Gagliardo-Nirenberg-Sobolev Inequality: An Induction Proof

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Abstract. This paper presents an induction proof of the G-N-S inequality when $p = 1$, which the author believes is much simpler and more transparent than other proofs found in the literature.

Let $p \in [1, n)$, then there is a constant $C_{n,p} > 0$ such that

$$\left(\int_{\mathbb{R}^n} |u(x)|^{p^*} \, \mathrm{d}x\right)^{1/p^*} \leq C_{n,p} \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p \, \mathrm{d}x\right)^{1/p}, \quad \text{for all } u \in C_0^1(\mathbb{R}^n), \tag{1}$$

where $C_0^1(\mathbb{R}^n)$ is the space of continuously differentiable functions on $\mathbb{R}^n$ with compact support, and $p^* = np/(n-p)$ is the critical Sobolev exponent.

This is the Gagliardo-Nirenberg-Sobolev inequality, which is very fundamental in the theory of Sobolev spaces and partial differential equations. It is well known that (1) follows from applying the result for the case $p = 1$ to $|u|^\gamma$ with some suitable exponent γ. Traditionally (see e.g., [**2**, pp. 277–278]), the inequality (1) for $p = 1$ is proved by integrating

$$|u(x)|^{n/(n-1)} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |\partial_i u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)| \, \mathrm{d}y_i\right)^{1/(n-1)}$$

with respect to x_1 and applying the extended Hölder inequality, then repeating this procedure with respect to $x_2, x_3, \ldots, x_n$ successively. In every step the extended Hölder inequality is applied. This tedious procedure is not very transparent, and is not easy to follow (especially for beginning graduate students).

Observe that the inequality (for $p = 1$) is a proposition about the dimension n. Therefore it should be natural to prove the result by induction on n. Surprisingly, we could not find such a proof in the literature. Although the case $1 < p < n$ still depends on an application of Hölder's inequality, the purpose of this note is to present the induction proof that is simple and transparent in the case $p = 1$.

Theorem 1. *Let $n > 1$, $u \in C_0^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |u(x)|^{n/(n-1)} \, \mathrm{d}x \leq \left(\int_{\mathbb{R}^n} |\nabla u(x)| \, \mathrm{d}x\right)^{n/(n-1)}. \tag{2}$$

Proof. Our induction proof starts with $n = 2$, which can be found in [**1**, Page 27]. We present it here for the reader's convenience. Since $u \in C_0^1(\mathbb{R}^n)$, we have

$$u(x) = \int_{-\infty}^{x_1} \partial_1 u(s, x_2) \, \mathrm{d}s = \int_{-\infty}^{x_2} \partial_2 u(x_1, s) \, \mathrm{d}s,$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, where ∂_1 and ∂_2 denote a shorthand for $\partial/\partial x_1$ and $\partial/\partial x_2$, respectively. Thus

$$|u(x)|^2 \leq \left(\int_{-\infty}^{\infty} |\partial_1 u(s, x_2)|\, \mathrm{d}s\right)\left(\int_{-\infty}^{\infty} |\partial_2 u(x_1, s)|\, \mathrm{d}s\right).$$

Integrating both sides over $\mathbb{R}^2$, then applying the elementary pointwise bound $|\partial_j u(x)| \leq |\nabla u(x)|$, for all $x \in \mathbb{R}^2$ and $j = 1, 2$, yields

$$\begin{aligned} \int_{\mathbb{R}^2} |u(x)|^2\, \mathrm{d}x &\leq \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_1 u(s, x_2)|\, \mathrm{d}s\mathrm{d}x_2\right)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_2 u(x_1, s)|\, \mathrm{d}s\mathrm{d}x_1\right) \\ &\leq \left(\int_{\mathbb{R}^2} |\nabla u(x)|\, \mathrm{d}x\right)^2. \end{aligned}$$

This completes the proof of (2) for $n = 2$.

Assume that the result is true for $n = m$, namely

$$\int_{\mathbb{R}^m} |u(x)|^{m/(m-1)}\, \mathrm{d}x \leq \left(\int_{\mathbb{R}^m} |\nabla u(x)|\, \mathrm{d}x\right)^{m/(m-1)}, \qquad \text{for } u \in C_0^1(\mathbb{R}^m). \tag{3}$$

To consider the case that $n = m + 1$, we write points in $\mathbb{R}^{m+1}$ by (x, t), where $x \in \mathbb{R}^m$ and $t \in \mathbb{R}$. Then for $u \in C_0^1(\mathbb{R}^{m+1})$,

$$|u(x, t)| = \left|\int_{-\infty}^{t} \partial_{m+1} u(x, s)\, \mathrm{d}s\right| \leq \int_{-\infty}^{\infty} |\nabla u(x, s)|\, \mathrm{d}s,$$

and

$$\int_{\mathbb{R}^m} |u(x, t)|\, \mathrm{d}x \leq \int_{\mathbb{R}^m} \mathrm{d}x \int_{-\infty}^{\infty} |\nabla u(x, s)|\, \mathrm{d}s = \int_{\mathbb{R}^{m+1}} |\nabla u(y)|\, \mathrm{d}y. \tag{4}$$

Here, to avoid confusion we use y to denote points in $\mathbb{R}^{m+1}$.

Using the Hölder inequality, (3) with u replaced by $u(\cdot, t) \in C_0^1(\mathbb{R}^m)$, and (4), we have

$$\begin{aligned} \int_{\mathbb{R}^m} |u(x, t)|^{(m+1)/m}\, \mathrm{d}x &= \int_{\mathbb{R}^m} |u(x, t)|^{1/m}\, |u(x, t)|\, \mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^m} |u(x, t)|\, \mathrm{d}x\right)^{1/m}\left(\int_{\mathbb{R}^m} |u(x, t)|^{m/(m-1)}\, \mathrm{d}x\right)^{(m-1)/m} \\ &\leq \left(\int_{\mathbb{R}^{m+1}} |\nabla u(y)|\, \mathrm{d}y\right)^{1/m}\left(\int_{\mathbb{R}^m} |\tilde{\nabla} u(x, t)| \mathrm{d}x\right), \end{aligned} \tag{5}$$

where $\tilde{\nabla} u(x, t)$ is the gradient of $u(\cdot, t)$ at x. Obviously $|\tilde{\nabla} u(x, t)| \leq |\nabla u(x, t)|$, thus integrating both sides with respect to t over $\mathbb{R}$ yields

$$\begin{aligned} \int_{\mathbb{R}^{m+1}} |u(y)|^{(m+1)/m}\, \mathrm{d}y &= \int_{-\infty}^{\infty} \mathrm{d}t \int_{\mathbb{R}^m} |u(x, t)|^{(m+1)/m}\, \mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^{m+1}} |\nabla u(y)|\, \mathrm{d}y\right)^{1/m}\left[\int_{-\infty}^{\infty} \mathrm{d}t \int_{\mathbb{R}^m} |\tilde{\nabla} u(x, t)| \mathrm{d}x\right] \end{aligned}$$

$$\leq \left(\int_{\mathbb{R}^{m+1}} |\nabla u(y)|\, \mathrm{d}y\right)^{1/m} \left[\int_{-\infty}^{\infty} \mathrm{d}t \int_{\mathbb{R}^m} |\nabla u(x,t)|\, \mathrm{d}x\right]$$
$$= \left(\int_{\mathbb{R}^{m+1}} |\nabla u(y)|\, \mathrm{d}y\right)^{(m+1)/m},$$

and we have obtained (2) for $n = m + 1$. So (2) is true for all $n \geq 2$. ∎

Remark 2. Using the same idea, a stronger Gagliardo-Nirenberg-Sobolev inequality

$$\int_{\mathbb{R}^n} |u(x)|^{n/(n-1)}\, \mathrm{d}x \leq \left(\prod_{i=1}^{n} \int_{\mathbb{R}^n} |\partial_i u(x)|\, \mathrm{d}x\right)^{1/(n-1)} \tag{S_n}$$

can be proved. To derive (S_{m+1}) from (S_m), we replace all the ∇u before (5) by $\partial_{m+1} u$, then apply the induction assumption (S_m) to $\int |u|^{m/(m-1)}$ in (5). When integrating with respect to t we need the extended Hölder inequality, which is not needed in our proof of Theorem 1.

Remark 3. In [**3**], combining a simple iteration scheme with this stronger version of the Gagliardo-Nirenberg-Sobolev inequality, the author gives a new proof of the L^∞-embedding

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C_p \left(\|u\|_{L^p(\mathbb{R}^n)} + \|\nabla u\|_{L^p(\mathbb{R}^n)}\right),$$

where $p > n$, $u \in C_0^1(\mathbb{R}^n)$.

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