Gradient Flow, Coarea Formula, Mountain Pass and Global Homeomorphism Thms

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Dedicated to Shibo Liu on his 50^{th} *birthday*

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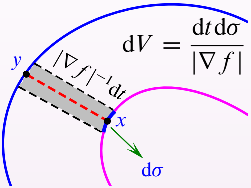
Contents

0.1. Physical derivation of coarea formula

Fundmal result in geometric measure theory Federer (1969).
Given $f : \mathbb{R}^n \to \mathbb{R}$, $\Omega = f^{-1}[a,b]$, density $g(x)$, Fig 1. The mass of Ω is

$$M = \int_\Omega g(x)\mathrm{d}x. \tag{1}$$



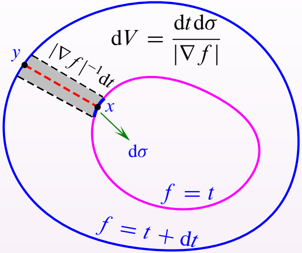
For $t \in [a,b]$, take $\mathrm{d}\sigma$ at $x \in f^{-1}(t)$, $\{y\} = f^{-1}(t+\mathrm{d}t) \cap \mathrm{N}_x f^{-1}(t)$.
By the first order Taylor expansion

$$\mathrm{d}t = f(y) - f(x) \approx \nabla f(x) \cdot (y - x). \tag{2}$$

Since $\nabla f(x) \parallel (y - x)$,

$$|y - x| = \frac{\mathrm{d}t}{|\nabla f(x)|}. \tag{3}$$

V and m of the small cylinder base $\mathrm{d}\sigma$ between $f^{-1}(t)$ and $f^{-1}(t+\mathrm{d}t)$ are



$$\mathrm{d}V = \frac{1}{|\nabla f(x)|}\mathrm{d}t\mathrm{d}\sigma, \qquad \mathrm{d}m = \frac{g(x)}{|\nabla f(x)|}\mathrm{d}t\mathrm{d}\sigma.$$

Mass of the thin region $f^{-1}[t, t+\mathrm{d}t]$:

$$\mathrm{d}M = \int_{f^{-1}(t)} \mathrm{d}m = \int_{f^{-1}(t)} \frac{g(x)}{|\nabla f(x)|}\mathrm{d}t\mathrm{d}\sigma$$
$$= \mathrm{d}t \int_{f^{-1}(t)} \frac{g(x)}{|\nabla f(x)|}\mathrm{d}\sigma.$$

Total mass of $\Omega = f^{-1}[a,b]$ is

$$\int_{f^{-1}[a,b]} g(x)\mathrm{d}x = \int_a^b \mathrm{d}M = \int_a^b \mathrm{d}t \int_{f^{-1}(t)} \frac{g(x)}{|\nabla f(x)|}\mathrm{d}\sigma. \tag{4}$$

0.2. Gradient flow and proof of coarea

Thm 1 (Coarea formula). Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^2,

(f) $\Omega = f^{-1}[a,b]$ is compact, $|\nabla f| \neq 0$ on Ω.

Then (4) holds for any continuous $g : \Omega \to \mathbb{R}$.

Pf. By (f), $|\nabla f| \geq \varepsilon$ on W for an bopen $W \supset \Omega$. Set $\phi : W \to \mathbb{R}^n$ via

$$\phi(x) = |\nabla f(x)|^{-2} \nabla f(x).$$

There are maximal open $G \subset \mathbb{R} \times W$, and flow $\eta : G \to \mathbb{R}^n$

$$\partial_t \eta(t,q) = \phi(\eta(t,q)), \quad \eta(a,q) = q; \qquad \text{for } (t,q) \in G. \tag{5}$$

For $p \in f^{-1}(a)$ write $x_p = \eta(\cdot, p)$, i.e., $x_p' = \phi(x_p)$, $x_p(a) = p$.

Let $\beta \in \mathbb{R}$ be maximal such that x_p is defined on $[a, \beta)$.

For $t \in [a, \beta)$,

$$(f \circ x_p)'(t) = \nabla f(x_p(t)) \cdot x_p'(t) = \nabla f(x_p(t)) \cdot \frac{\nabla f(x_p(t))}{|\nabla f(x_p(t))|^2} = 1.$$

Claim: $\boldsymbol{\beta > b}$. If $\beta \le b$, for $t \in [a, \beta)$

$$f(x_p(t)) = f(p) + \int_a^t (f \circ x_p)'(s)\,\mathrm{d}s = t, \quad \text{i.e., } x_p(t) \in W. \tag{6}$$

Let $t \to \beta^-$, $f(x_p(\beta)) = \beta \le b$; $x_p(\beta) \in W$, max $\ne \beta$. Thus $\beta > b$.
$\exists\, \varepsilon(p) > 0$, for $q \in B_{\varepsilon(p)}(p)$, $x_q = \eta(\cdot, q)$ is defined on $[a, b]$.
Then

$$A = W \cap \bigcup_{p \in f^{-1}(a)} B_{\varepsilon(p)}(p)$$

is open nbhd of $f^{-1}(a)$, s.t. $[a, b] \times A \subset G$.
$(t, q) \in G$ iff η is defined at (t, q), i.e., x_q is defined at t.

The restriction $\eta : [a,b] \times A \to \mathbb{R}^n$ is C^1,

(a) $\eta : [a,b] \times f^{-1}(a) \to f^{-1}[a,b]$ is bijective;

(b) for $(t,q) \in [a,b] \times f^{-1}(a)$, $\eta(t,\cdot)$ is a local diffeom at q, thus $\partial_x \eta = \left(\partial_{x^j} \eta^i\right)_{i,j \in \overline{n}}$ at (t,q) is invertible.

Let $f^{-1}(a)$ be image of $\varphi : U \to \mathbb{R}^n$, $U \subset \mathbb{R}^{n-1}$ (see Fig 1), $\varphi|_{U^\circ}$ is injective, $\operatorname{rank} \varphi'(u) = n-1$ for $u \in U^\circ$. Define

$$\Phi : [a,b] \times U \to \mathbb{R}^n, \qquad \Phi(t,u) = \eta(t, \varphi(u)). \tag{7}$$

Then $\Phi([a,b] \times U) = f^{-1}[a,b]$, Φ maps $(a,b) \times U^\circ$ injectively *into* $f^{-1}[a,b]$. Using (b) and $\operatorname{rank} \varphi'(u) = n-1$,

$$\operatorname{rank} \partial_u \Phi(t,u) = \operatorname{rank}[\partial_x \eta(t, \varphi(u)) \varphi'(u)] = n-1. \tag{8}$$

So $\Phi(t,\cdot) : U \to \mathbb{R}^n$ is a parametrization of $f^{-1}(t)$ for $t \in [a,b]$, which determines normal vector of $f^{-1}(t)$ at $\Phi(t,u)$:

$$N_t(u) = \left(\frac{\partial(\Phi^2,\dots,\Phi^n)}{\partial(u^1,\dots,u^{n-1})}, \dots, (-1)^{n+1}\frac{\partial(\Phi^1,\dots,\Phi^{n-1})}{\partial(u^1,\dots,u^{n-1})} \right).$$

For $(t,u) \in (a,b) \times U^\circ$,

$$\Phi'(t,u) = (\partial_t \Phi, \partial_u \Phi) = \begin{pmatrix} \partial_t \Phi^1 & \partial_{u^1}\Phi^1 & \cdots & \partial_{u^{n-1}}\Phi^1 \\ \partial_t \Phi^2 & \partial_{u^1}\Phi^2 & \cdots & \partial_{u^{n-1}}\Phi^2 \\ \vdots & \vdots & & \vdots \\ \partial_t \Phi^n & \partial_{u^1}\Phi^n & \cdots & \partial_{u^{n-1}}\Phi^n \end{pmatrix}.$$

$\partial_t \Phi = \partial_t \eta$, for $i \in \overline{n}$ $\mathrm{cof}[\partial_t \Phi^i] = N_t^i(u)$. Expanding $\det \Phi'(t,u)$ via the 1st col and using $\partial_t \eta = |\nabla f(\eta)|^{-2} \nabla f(\eta)$ in (5), (Fig 1)

$$\begin{aligned} |\det \Phi'(t,u)| &= |\partial_t \Phi(t,u) \cdot N_t(u)| = |\partial_t \eta(t,\varphi(u)) \cdot N_t(u)| \\ &= \left| \frac{\nabla f(\Phi(t,u))}{|\nabla f(\Phi(t,u))|^2} \cdot N_t(u) \right| = \frac{|N_t(u)|}{|\nabla f(\Phi(t,u))|} \neq 0. \end{aligned}$$

Hence the bijection (Fig 1)

$$\Phi : (a,b)\times U^\circ \to \Phi((a,b)\times U^\circ) = (f^{-1}[a,b])^\circ$$

is a diffeosm. Changing variable $x = \Phi(t,u)$ and Fubini,

$$\begin{aligned}
\int_{f^{-1}[a,b]} g(x)\mathrm{d}x &= \int_{\Phi([a,b]\times U)} g(x)\mathrm{d}x \\
&= \int_{[a,b]\times U} g(\Phi(t,u))|\det \Phi'(t,u)|\mathrm{d}u\mathrm{d}t \\
&= \int_a^b \mathrm{d}t \int_U g(\Phi(t,u))\frac{|N_t(u)|}{|\nabla f(\Phi(t,u))|}\mathrm{d}u \\
&= \int_a^b \mathrm{d}t \int_{f^{-1}(t)} \frac{g(x)}{|\nabla f(x)|}\mathrm{d}\sigma, \qquad (9)
\end{aligned}$$

the last equality is because $u \mapsto \Phi(t,u)$ is paramation of $f^{-1}(t)$.

Rek 1. For $n=2$ see Ding (2024) (cross product in $\mathbb{R}^3$ is used). A novelty of our proof is the clever computation of $\det\Phi'(t,u)$.

Rek 2. Flows (GF, Ricci flow $\gg$ Poincaré Conjecture) are used in many areas. GF is used to prove the Deformation Lemma in critical point theory (Willem (1996)). For $c\in\mathbb{R}$, $f^c=f^{-1}(-\infty,c]$.

Cor 1. Under Thm 1, there is $h:f^b\to f^a$ conts s.t. $h|_{f^a}=\mathbb{1}_{f^a}$.

Pf. For $x\in f^b\backslash f^a$, $f(x)=t$. $\exists 1\, y_x\in f^{-1}(a)$ s.t. $\eta_t(y_x)=x$. Set

$$h:f^b\to f^a,\qquad h(x)=\begin{cases} y_x & \text{if } f(x)>a,\\ x & \text{if } f(x)\le a.\end{cases}$$

Lem 1. If $\Phi\in C^1(X)$ satfes Palais–Smale, $\Phi^{-1}[a,b]\cap\mathscr{K}=\emptyset$, then there is $\eta:\Phi^b\to\Phi^a$ cont s.t. $\eta|_{\Phi^a}=\mathbb{1}_{\Phi^a}$.

Palais–Smale: $\{u_n\}$ is precpt if $|\Phi(u_n)|\ll\infty$, $\Phi'(u_n)\to 0$.

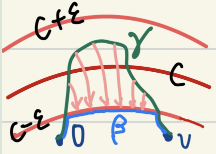
0.3. Deformation lemma and mountain pass

Thm 2 (Ambrosetti & Rabinowitz (1973)). Let $\Phi \in C^1(X)$ satfes (PS), $\|v\| > r$, Willem (1996)

$$b = \inf_{\partial \mathscr{B}_r} \Phi > \Phi(0) \geq \Phi(v). \qquad \Gamma = \{\text{paths joining } 0 \text{ and } v\},$$

then Φ has a critical value $c \geq b$ given by $c = \inf_{\gamma \in \Gamma} \max_{\gamma} \Phi$.

Pf. If c is not critical, $\Phi^{-1}[c-\varepsilon, c+\varepsilon] \cap \mathscr{K} = \emptyset$ if $\varepsilon \ll 1$.



(1) $\exists \gamma \in \Gamma$ s.t. $\max_{\gamma} \Phi \leq c + \varepsilon$, i.e. $\gamma \subset \Phi^{c+\varepsilon}$.

(2) By Lem 1, $\exists \eta : \Phi^{c+\varepsilon} \to \Phi^{c-\varepsilon}$, $\eta|_{\Phi^{c-\varepsilon}} = \mathbb{1}_{\Phi^{c-\varepsilon}}$.

(3) $\{0, v\} \subset \Phi^{c-\varepsilon}$, $\beta = \eta(\gamma) \in \Gamma$,

so $\beta \subset \Phi^{c-\varepsilon}$, $\max_{\beta} \Phi \leq c - \varepsilon$, controdicn.

Thm 3. If non-empty $A \subset \mathbb{R}^n$ is clopen. Then $A = \mathbb{R}^n$.

Pf. Take $a \in \mathbb{R}^n \setminus A$. Since A is closed, $\exists x \in A$ such that

$$|x - a| = d := \inf_{y \in A} |y - a|. \tag{10}$$

But A is open, $\exists r \in (0, d)$ such that $B_{2r}(x) \subset A$. Let

$$x' = x - \frac{r}{d}(x - a),$$

then $x' \in B_r(x) \subset A$; but $|x' - a| < |x - a|$, violating (10).

0.4. Global homeomorphism theorem

Thm 4 (Inv. Fun). Let $f : B_r^n(a) \to \mathbb{R}^n$ be C^1, $\det f'(a) \neq 0$. Then $\exists \varrho \in (0, r)$ s.t. $V = f(B_\varrho(a))$ is open and $f : B_\varrho(a) \to V$ is diffeom.

Cor 2 (Open map). Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1. If $\det f'(x) \neq 0$ for all $x \in \mathbb{R}^n$, then $f(\mathbb{R}^n)$ is open.

Thm 5. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1, $\det f'(x) \neq 0$ for all x. If coercive:

$$\lim_{|x| \to \infty} |f(x)| = +\infty, \tag{11}$$

then $f(\mathbb{R}^n) = \mathbb{R}^n$. $f(x) = b$ is solvable for all $b \in \mathbb{R}^n$

Pf. By Cor2, $f(\mathbb{R}^n)$ is open. By (11), $f(\mathbb{R}^n)$ is closed.
Thus $f(\mathbb{R}^n) = \mathbb{R}^n$ by Thm 3.

Alt. Proof without IFT: consider min of $\varphi(x) = |f(x) - b|^2$.

Thm 6 (Liu & Liu (2018)). Let $n \geq 2$, $f : \mathbb{R}^m \to \mathbb{R}^n$ be C^1. If $f(\mathbb{R}^m)$ is closed and $\#\{x \in \mathbb{R}^m \mid \operatorname{rank} f'(x) < n\} < \infty$, then $f(\mathbb{R}^m) = \mathbb{R}^n$.

Generalizes Thm 5 and yields FTA. (vector fun on cpt manfd)

Thm 7 (Hadamard). In Thm 5, $f : \mathbb{R}^n \to \mathbb{R}^n$ is injective, thus a diffeom.

Pf(Katriel (1994)). If not, $f(0) = f(a)$ for some $a \neq 0$. Let

$$\Phi : \mathbb{R}^n \to \mathbb{R}, \qquad \Phi(x) = \frac{1}{2}|f(x) - f(0)|^2.$$

Since $\det f'(0) \neq 0$, $f : B_r \to \mathbb{R}^n$ is inject for an $r \in (0, |a|)$. Thus

$$b = \inf_{\partial B_{r/2}} \Phi > \Phi(0) = \Phi(a) = 0, \quad \text{because} \ \ x \in \partial B_{r/2} \Rightarrow f(x) \neq f(0).$$

Thm 2 (MPT) gives $u \in \mathbb{R}^n$ s.t. $\Phi(u) \geq b$ so $f(u) \neq f(0)$, and

$$0 = \nabla\Phi(u) = f'(u)(f(u) - f(0)) \quad \Rightarrow \quad \det f'(u) = 0. \qquad \otimes$$

0.5. Figures

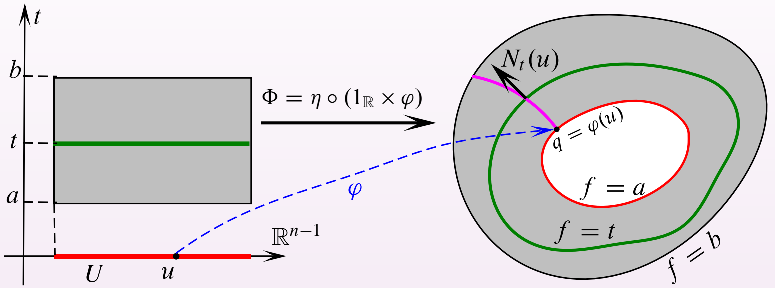


Fig 1: $\Phi(t,u) = \eta(t, \varphi(u))$

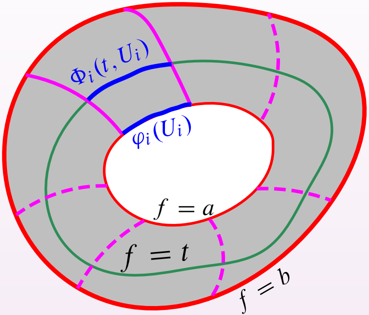


Fig 2: Decomposition of $f^{-1}[a, b]$ and $f^{-1}(t)$

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The Poincaré conjecture, proposed by Henri Poincaré in 1904, is a theorem in topology stating that any simply connected, closed, three-dimensional manifold is topologically equivalent to a three-sphere

The Poincaré conjecture, proposed by Henri Poincaré in 1904, is a theorem in topology stating that any simply connected, closed, three-dimensional manifold is topologically equivalent to a three-sphere. In simpler terms, it suggests that if you have a bounded, three-dimensional object with no "holes" and you can shrink any loop on its surface to a single point.

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