Morse theory and nonlinear Schrödinger equations

Shibo Liu

Florida Institute of Technology

https://lausb.github.io

sliu@fit.edu

Contents

1. Morse theory, multiple solutions

Def1 (Critical groups, Chang(1993), Mawhin & Willem(1989), Chap 8). Let $f \in C^1(X)$, u be isolated critical point with $f(u) = c$, $f_c = \{f \le c\}$. Then

$$C_q(f, u) = H_q(f_c, f_c \backslash u, \mathbb{Q}), \qquad q \in \mathbb{N} = \{0, 1, \dots\}$$

1. Morse theory, multiple solutions

Def1 (Critical groups, Chang(1993), Mawhin & Willem(1989), Chap 8). Let $f \in C^1(X)$, u be isolated critical point with $f(u) = c$, $f_c = \{f \leq c\}$. Then

$$C_q(f, u) = H_q(f_c, f_c \backslash u, \mathbb{Q}), \qquad q \in \mathbb{N} = \{0, 1, \ldots\}$$

If f satisfies Cerami (C) and $\inf_{\mathscr{K}} f > -\infty$, then

$$C_q(f, \infty) = H_q(X, f_\alpha), \qquad \text{here } \alpha < \inf_{\mathscr{K}} f. \qquad \text{Bartsch \& Li(1997)}$$

1. Morse theory, multiple solutions

Def1 (Critical groups, Chang(1993), Mawhin & Willem(1989), Chap 8). Let $f \in C^1(X)$, u be isolated critical point with $f(u) = c$, $f_c = \{f \le c\}$. Then

$$C_q(f,u) = H_q(f_c, f_c \backslash u, \mathbb{Q}), \qquad q \in \mathbb{N} = \{0, 1, \ldots\}$$

If f satisfies Cerami (C) and $\inf_{\mathscr{K}} f > -\infty$, then

$$C_q(f,\infty) = H_q(X, f_\alpha), \qquad \text{here } \alpha < \inf_{\mathscr{K}} f.$$

Bartsch & Li(1997)

Thm1 (Morse inequalities). If $f \in C^1(X)$ satisfies (C) and $\#\mathscr{K} < \infty$, then

$$\sum_{q=0}^{\infty} (-1)^q M_q = \sum_{q=0}^{\infty} (-1)^q \beta_q,$$

1. Morse theory, multiple solutions

Def1 (Critical groups, Chang(1993), Mawhin & Willem(1989), Chap 8). Let $f \in C^1(X)$, u be isolated critical point with $f(u) = c$, $f_c = \{f \le c\}$. Then

$$C_q(f, u) = H_q(f_c, f_c \backslash u, \mathbb{Q}), \qquad q \in \mathbb{N} = \{0, 1, \ldots\}$$

If f satisfies Cerami (C) and $\inf_{\mathscr{K}} f > -\infty$, then

$$C_q(f, \infty) = H_q(X, f_\alpha), \qquad \text{here } \alpha < \inf_{\mathscr{K}} f.$$

Bartsch & Li(1997)

Thm1 (Morse inequalities). If $f \in C^1(X)$ satisfies (C) and $\#\mathscr{K} < \infty$, then

$$\sum_{q=0}^{\infty} (-1)^q M_q = \sum_{q=0}^{\infty} (-1)^q \beta_q,$$

where $\quad M_q = \sum_{u \in \mathscr{K}} \dim C_q(f, u), \quad \beta_q = \dim C_q(f, \infty).$

1. Morse theory, multiple solutions

Def1 (Critical groups, Chang(1993), Mawhin & Willem(1989), Chap 8). Let $f \in C^1(X)$, u be isolated critical point with $f(u) = c$, $f_c = \{f \le c\}$. Then

$$C_q(f,u) = H_q(f_c, f_c \backslash u, \mathbb{Q}), \qquad q \in \mathbb{N} = \{0, 1, \dots\}$$

If f satisfies Cerami (C) and $\inf_{\mathscr{K}} f > -\infty$, then

$$C_q(f,\infty) = H_q(X, f_\alpha), \qquad \text{here } \alpha < \inf_{\mathscr{K}} f. \qquad \text{Bartsch \& Li(1997)}$$

Thm1 (Morse inequalities). If $f \in C^1(X)$ satisfies (C) and $\#\mathscr{K} < \infty$, then

$$\sum_{q=0}^{\infty} (-1)^q M_q = \sum_{q=0}^{\infty} (-1)^q \beta_q,$$

where $\quad M_q = \sum_{u \in \mathscr{K}} \dim C_q(f,u), \quad \beta_q = \dim C_q(f,\infty).$

$$M_q - M_{q-1} + \cdots + (-1)^q M_0 \ge \beta_q - \beta_{q-1} + \cdots + (-1)^q \beta_0, \qquad q \in \mathbb{N}.$$

Exm1. (1) If u is a local maximizer of $f : \mathbb{R}^m \to \mathbb{R}$, then $C_q(f, u) = \delta_{qm}\mathbb{Q}$.

Exm1. (1) If u is a local maximizer of $f : \mathbb{R}^m \to \mathbb{R}$, then $C_q(f, u) = \delta_{qm}\mathbb{Q}$.

(2) If u is a local minimizer of $f : X \to \mathbb{R}$, then $C_q(f, u) = \delta_{q0}\mathbb{Q}$.

Exm1. (1) If u is a local maximizer of $f:\mathbb{R}^m\to\mathbb{R}$, then $C_q(f,u)=\delta_{qm}\mathbb{Q}$.

(2) If u is a local minimizer of $f:X\to\mathbb{R}$, then $C_q(f,u)=\delta_{q0}\mathbb{Q}$.

(3) If X is Hilbert and $f\in C^2(X)$, u is non-degenerate critical point of f with Morse index k,

Exm1. (1) If u is a local maximizer of $f : \mathbb{R}^m \to \mathbb{R}$, then $C_q(f, u) = \delta_{qm}\mathbb{Q}$.

(2) If u is a local minimizer of $f : X \to \mathbb{R}$, then $C_q(f, u) = \delta_{q0}\mathbb{Q}$.

(3) If X is Hilbert and $f \in C^2(X)$, u is non-degenerate critical point of f with Morse index k, Morse lemma yields

$$C_q(f, u) = \delta_{qk}\mathbb{Q}. \qquad \text{that is, } \dim C_q(f, u) = \delta_{qk}.$$

Therefore, critical groups are generalization of Morse index.

Exm1. (1) If u is a local maximizer of $f:\mathbb{R}^m\to\mathbb{R}$, then $C_q(f,u)=\delta_{qm}\mathbb{Q}$.

(2) If u is a local minimizer of $f:X\to\mathbb{R}$, then $C_q(f,u)=\delta_{q0}\mathbb{Q}$.

(3) If X is Hilbert and $f\in C^2(X)$, u is non-degenerate critical point of f with Morse index k, Morse lemma yields

$$C_q(f,u)=\delta_{qk}\mathbb{Q}. \qquad \text{that is, } \dim C_q(f,u)=\delta_{qk}.$$

Therefore, critical groups are generalization of Morse index.

Exm2. Perera(2003) constracted eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ for the p-Laplacian

Exm1. (1) If u is a local maximizer of $f:\mathbb{R}^m\to\mathbb{R}$, then $C_q(f,u)=\delta_{qm}\mathbb{Q}$.

(2) If u is a local minimizer of $f:X\to\mathbb{R}$, then $C_q(f,u)=\delta_{q0}\mathbb{Q}$.

(3) If X is Hilbert and $f\in C^2(X)$, u is non-degenerate critical point of f with Morse index k, Morse lemma yields

$$C_q(f,u)=\delta_{qk}\mathbb{Q}. \qquad \text{that is, } \dim C_q(f,u)=\delta_{qk}.$$

Therefore, critical groups are generalization of Morse index.

Exm2. Perera(2003) constracted eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ for the p-Laplacian such that, if $\lambda\in(\lambda_k,\lambda_{k+1})\setminus\sigma_p$, then $C_k(I,0)\neq 0$,

Exm1. (1) If u is a local maximizer of $f:\mathbb{R}^m\to\mathbb{R}$, then $C_q(f,u)=\delta_{qm}\mathbb{Q}$.

(2) If u is a local minimizer of $f:X\to\mathbb{R}$, then $C_q(f,u)=\delta_{q0}\mathbb{Q}$.

(3) If X is Hilbert and $f\in C^2(X)$, u is non-degenerate critical point of f with Morse index k, Morse lemma yields

$$C_q(f,u)=\delta_{qk}\mathbb{Q}. \qquad \text{that is, } \dim C_q(f,u)=\delta_{qk}.$$

Therefore, critical groups are generalization of Morse index.

Exm2. Perera(2003) constracted eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ for the p-Laplacian such that, if $\lambda\in(\lambda_k,\lambda_{k+1})\setminus\sigma_p$, then $C_k(I,0)\neq 0$, where $I:W_0^{1,p}\to\mathbb{R}$,

$$I(u)=\frac{1}{p}\int_\Omega\left(|\nabla u|^p-\lambda\,|u|^p\right).$$

Exm1. (1) If u is a local maximizer of $f:\mathbb{R}^m\to\mathbb{R}$, then $C_q(f,u)=\delta_{qm}\mathbb{Q}$.

(2) If u is a local minimizer of $f:X\to\mathbb{R}$, then $C_q(f,u)=\delta_{q0}\mathbb{Q}$.

(3) If X is Hilbert and $f\in C^2(X)$, u is non-degenerate critical point of f with Morse index k, Morse lemma yields

$$C_q(f,u)=\delta_{qk}\mathbb{Q}. \qquad \text{that is, } \dim C_q(f,u)=\delta_{qk}.$$

Therefore, critical groups are generalization of Morse index.

Exm2. Perera(2003) constracted eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ for the p-Laplacian such that, if $\lambda\in(\lambda_k,\lambda_{k+1})\setminus\sigma_p$, then $C_k(I,0)\neq 0$, where $I:W_0^{1,p}\to\mathbb{R}$,

$$I(u)=\frac{1}{p}\int_\Omega\left(|\nabla u|^p-\lambda\,|u|^p\right). \qquad -\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)=\lambda\,|u|^{p-2}\,u.$$

Exm1. (1) If u is a local maximizer of $f : \mathbb{R}^m \to \mathbb{R}$, then $C_q(f, u) = \delta_{qm}\mathbb{Q}$.

(2) If u is a local minimizer of $f : X \to \mathbb{R}$, then $C_q(f, u) = \delta_{q0}\mathbb{Q}$.

(3) If X is Hilbert and $f \in C^2(X)$, u is non-degenerate critical point of f with Morse index k, Morse lemma yields

$$C_q(f, u) = \delta_{qk}\mathbb{Q}. \qquad \text{that is, } \dim C_q(f, u) = \delta_{qk}.$$

Therefore, critical groups are generalization of Morse index.

Exm2. Perera(2003) constracted eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ for the p-Laplacian such that, if $\lambda \in (\lambda_k, \lambda_{k+1}) \setminus \sigma_p$, then $C_k(I, 0) \neq 0$, where $I : W_0^{1,p} \to \mathbb{R}$,

$$I(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p - \lambda\,|u|^p\right). \qquad -\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) = \lambda\,|u|^{p-2}\,u.$$

In Perera(2003) this result is applied to solve

$$-\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) = f(x, u) \quad \text{in } W_0^{1,p}(\Omega), \qquad \lim_{|t|\to 0}\frac{f(x,t)t}{|t|^p} = \lambda.$$

Exm1. (1) If u is a local maximizer of $f:\mathbb{R}^m\to\mathbb{R}$, then $C_q(f,u)=\delta_{qm}\mathbb{Q}$.

(2) If u is a local minimizer of $f:X\to\mathbb{R}$, then $C_q(f,u)=\delta_{q0}\mathbb{Q}$.

(3) If X is Hilbert and $f\in C^2(X)$, u is non-degenerate critical point of f with Morse index k, Morse lemma yields

$$C_q(f,u)=\delta_{qk}\mathbb{Q}. \qquad \text{that is, } \dim C_q(f,u)=\delta_{qk}.$$

Therefore, critical groups are generalization of Morse index.

Exm2. Perera(2003) constracted eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ for the p-Laplacian such that, if $\lambda\in(\lambda_k,\lambda_{k+1})\setminus\sigma_p$, then $C_k(I,0)\neq 0$, where $I:W_0^{1,p}\to\mathbb{R}$,

$$I(u)=\frac{1}{p}\int_\Omega\left(|\nabla u|^p-\lambda\,|u|^p\right). \qquad -\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)=\lambda\,|u|^{p-2}\,u.$$

In Perera(2003) this result is applied to solve

$$-\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)=f(x,u) \quad \text{in } W_0^{1,p}(\Omega), \qquad \lim_{|t|\to 0}\frac{f(x,t)t}{|t|^p}=\lambda.$$

If $p=2$ then 0 is non-degenerate and $C_q(I,0)=\delta_{qk}\mathbb{Q}$.

Pro1. If $f \in C^1(X)$ satisfies (C), $\exists \rho > 0$ and $\|v\| > \rho$ s.t.

$$\inf_{\|u\|=\rho} f(u) > f(0) \geq f(v),$$

Pro1. If $f \in C^1(X)$ satisfies (C), $\exists \rho > 0$ and $\|v\| > \rho$ s.t.

$$\inf_{\|u\|=\rho} f(u) > f(0) \geq f(v),$$

then f has critical point u, $C_1(f, u) \neq 0$.

Pro1. If $f \in C^1(X)$ satisfies (C), $\exists \rho > 0$ and $\|v\| > \rho$ s.t.

$$\inf_{\|u\|=\rho} f(u) > f(0) \geq f(v),$$

then f has critical point u, $C_1(f,u) \neq 0$. (in appls, $C_q(f,u) = \delta_{q,1}\mathbb{Q}$)

Pro1. If $f \in C^1(X)$ satisfies (C), $\exists \rho > 0$ and $\|v\| > \rho$ s.t.

$$\inf_{\|u\|=\rho} f(u) > f(0) \geq f(v),$$

then f has critical point u, $C_1(f, u) \neq 0$. (in appls, $C_q(f, u) = \delta_{q,1}\mathbb{Q}$)

Pro2 (Liu(1989), Thm 2.1). Suppose $f \in C^1(X, \mathbb{R})$ has a local linking at 0 wrt decomposition $X = Y \oplus Z$,

Pro1. If $f \in C^1(X)$ satisfies (C), $\exists \rho > 0$ and $\|v\| > \rho$ s.t.

$$\inf_{\|u\|=\rho} f(u) > f(0) \geq f(v),$$

then f has critical point u, $C_1(f, u) \neq 0$. (in appls, $C_q(f, u) = \delta_{q,1}\mathbb{Q}$)

Pro2 (Liu(1989), Thm 2.1). Suppose $f \in C^1(X, \mathbb{R})$ has a local linking at 0 wrt decomposition $X = Y \oplus Z$, i.e., $\exists \varepsilon > 0$,

$$\begin{aligned} f(u) &\leq 0 \quad \text{for } u \in Y \cap B_\varepsilon, \\ f(u) &> 0 \quad \text{for } u \in (Z\setminus\{0\}) \cap B_\varepsilon, \end{aligned}$$

$B_\varepsilon = \{u \in X \mid \|u\| \leq \varepsilon\}$.

Pro1. If $f \in C^1(X)$ satisfies (C), $\exists \rho > 0$ and $\|v\| > \rho$ s.t.

$$\inf_{\|u\|=\rho} f(u) > f(0) \geq f(v),$$

then f has critical point u, $C_1(f,u) \neq 0$. (in appls, $C_q(f,u) = \delta_{q,1}\mathbb{Q}$)

Pro2 (Liu(1989), Thm 2.1). Suppose $f \in C^1(X,\mathbb{R})$ has a local linking at 0 wrt decomposition $X = Y \oplus Z$, i.e., $\exists \varepsilon > 0$,

$$\begin{aligned} f(u) &\leq 0 \quad \text{for } u \in Y \cap B_\varepsilon, \\ f(u) &> 0 \quad \text{for } u \in (Z \setminus \{0\}) \cap B_\varepsilon, \end{aligned}$$

$B_\varepsilon = \{\, u \in X \mid \|u\| \leq \varepsilon\}$. If $\ell = \dim Y < \infty$, then $C_\ell(f,0) \neq 0$.

Pro1. If $f \in C^1(X)$ satisfies (C), $\exists \rho > 0$ and $\|v\| > \rho$ s.t.

$$\inf_{\|u\|=\rho} f(u) > f(0) \geq f(v),$$

then f has critical point u, $C_1(f, u) \neq 0$. (in appls, $C_q(f, u) = \delta_{q,1}\mathbb{Q}$)

Pro2 (Liu(1989), Thm 2.1). Suppose $f \in C^1(X, \mathbb{R})$ has a local linking at 0 wrt decomposition $X = Y \oplus Z$, i.e., $\exists \varepsilon > 0$,

$$\begin{aligned} f(u) &\leq 0 \quad \text{for } u \in Y \cap B_\varepsilon, \\ f(u) &> 0 \quad \text{for } u \in (Z \setminus \{0\}) \cap B_\varepsilon, \end{aligned}$$

$B_\varepsilon = \{ u \in X | \, \|u\| \leq \varepsilon \}$. If $\ell = \dim Y < \infty$, then $C_\ell(f, 0) \neq 0$.

Pro3 (Bartsch & Li(1997), Prop 3.8). Let $f \in C^1(X, \mathbb{R})$, sats (C).

Pro1. If $f \in C^1(X)$ satisfies (C), $\exists \rho > 0$ and $\|v\| > \rho$ s.t.

$$\inf_{\|u\|=\rho} f(u) > f(0) \geq f(v),$$

then f has critical point u, $C_1(f,u) \neq 0$. (in appls, $C_q(f,u) = \delta_{q,1}\mathbb{Q}$)

Pro2 (Liu(1989), Thm 2.1). Suppose $f \in C^1(X,\mathbb{R})$ has a local linking at 0 wrt decomposition $X = Y \oplus Z$, i.e., $\exists \varepsilon > 0$,

$$\begin{aligned} f(u) &\leq 0 \quad \text{for } u \in Y \cap B_\varepsilon, \\ f(u) &> 0 \quad \text{for } u \in (Z\backslash\{0\}) \cap B_\varepsilon, \end{aligned}$$

$B_\varepsilon = \{ u \in X | \ \|u\| \leq \varepsilon \}$. If $\ell = \dim Y < \infty$, then $C_\ell(f,0) \neq 0$.

Pro3 (Bartsch & Li(1997), Prop 3.8). Let $f \in C^1(X,\mathbb{R})$, sats (C). Assume $X = X^- \oplus X^+$, $\ell = \dim X^- < \infty$.

Pro1. If $f \in C^1(X)$ satisfies (C), $\exists \rho > 0$ and $\|v\| > \rho$ s.t.

$$\inf_{\|u\|=\rho} f(u) > f(0) \geq f(v),$$

then f has critical point u, $C_1(f,u) \neq 0$. (in appls, $C_q(f,u) = \delta_{q,1}\mathbb{Q}$)

Pro2 (Liu(1989), Thm 2.1). Suppose $f \in C^1(X,\mathbb{R})$ has a local linking at 0 wrt decomposition $X = Y \oplus Z$, i.e., $\exists \varepsilon > 0$,

$$\begin{aligned} f(u) &\leq 0 \quad \text{for } u \in Y \cap B_\varepsilon, \\ f(u) &> 0 \quad \text{for } u \in (Z\setminus\{0\}) \cap B_\varepsilon, \end{aligned}$$

$B_\varepsilon = \{u \in X \mid \|u\| \leq \varepsilon\}$. If $\ell = \dim Y < \infty$, then $C_\ell(f,0) \neq 0$.

Pro3 (Bartsch & Li(1997), Prop 3.8). Let $f \in C^1(X,\mathbb{R})$, sats (C). Assume $X = X^- \oplus X^+$, $\ell = \dim X^- < \infty$. If f is bounded from below on X^+ and

$$f(u) \to -\infty \quad \text{as } \|u\| \to \infty,\ u \in X^-,$$

Pro1. If $f \in C^1(X)$ satisfies (C), $\exists \rho > 0$ and $\|v\| > \rho$ s.t.

$$\inf_{\|u\|=\rho} f(u) > f(0) \geq f(v),$$

then f has critical point u, $C_1(f,u) \neq 0$. (in appls, $C_q(f,u) = \delta_{q,1}\mathbb{Q}$)

Pro2 (Liu(1989), Thm 2.1). Suppose $f \in C^1(X,\mathbb{R})$ has a local linking at 0 wrt decomposition $X = Y \oplus Z$, i.e., $\exists \varepsilon > 0$,

$$\begin{aligned} f(u) &\leq 0 \quad \text{for } u \in Y \cap B_\varepsilon, \\ f(u) &> 0 \quad \text{for } u \in (Z \setminus \{0\}) \cap B_\varepsilon, \end{aligned}$$

$B_\varepsilon = \{\, u \in X |\ \|u\| \leq \varepsilon\}$. If $\ell = \dim Y < \infty$, then $C_\ell(f,0) \neq 0$.

Pro3 (Bartsch & Li(1997), Prop 3.8). Let $f \in C^1(X,\mathbb{R})$, sats (C). Assume $X = X^- \oplus X^+$, $\ell = \dim X^- < \infty$. If f is bounded from below on X^+ and

$$f(u) \to -\infty \ \text{ as } \ \|u\| \to \infty,\ u \in X^-,$$

then $C_\ell(f,\infty) \neq 0$. (loc link at infinity)

Pro1. If $f \in C^1(X)$ satisfies (C), $\exists \rho > 0$ and $\|v\| > \rho$ s.t.

$$\inf_{\|u\|=\rho} f(u) > f(0) \geq f(v),$$

then f has critical point u, $C_1(f,u) \neq 0$. (in appls, $C_q(f,u) = \delta_{q,1}\mathbb{Q}$)

Pro2 (Liu(1989), Thm 2.1). Suppose $f \in C^1(X,\mathbb{R})$ has a local linking at 0 wrt decomposition $X = Y \oplus Z$, i.e., $\exists \varepsilon > 0$,

$$\begin{aligned} f(u) &\leq 0 \quad \text{for } u \in Y \cap B_\varepsilon, \\ f(u) &> 0 \quad \text{for } u \in (Z\backslash\{0\}) \cap B_\varepsilon, \end{aligned}$$

$B_\varepsilon = \{\, u \in X |\ \|u\| \leq \varepsilon\}$. If $\ell = \dim Y < \infty$, then $C_\ell(f,0) \neq 0$.

Pro3 (Bartsch & Li(1997), Prop 3.8). Let $f \in C^1(X,\mathbb{R})$, sats (C). Assume $X = X^- \oplus X^+$, $\ell = \dim X^- < \infty$. If f is bounded from below on X^+ and

$$f(u) \to -\infty \ \text{ as } \ \|u\| \to \infty,\ u \in X^-,$$

then $C_\ell(f,\infty) \neq 0$. (loc link at infinity)

Exm3 (Liu & Li(2003a)). If $f : \mathbb{R}^n \to \mathbb{R}$ is anti-coercive,

Pro1. If $f \in C^1(X)$ satisfies (C), $\exists \rho > 0$ and $\|v\| > \rho$ s.t.

$$\inf_{\|u\|=\rho} f(u) > f(0) \geq f(v),$$

then f has critical point u, $C_1(f, u) \neq 0$. (in appls, $C_q(f, u) = \delta_{q,1}\mathbb{Q}$)

Pro2 (Liu(1989), Thm 2.1). Suppose $f \in C^1(X, \mathbb{R})$ has a local linking at 0 wrt decomposition $X = Y \oplus Z$, i.e., $\exists \varepsilon > 0$,

$$\begin{aligned} f(u) &\leq 0 \quad \text{for } u \in Y \cap B_\varepsilon, \\ f(u) &> 0 \quad \text{for } u \in (Z \setminus \{0\}) \cap B_\varepsilon, \end{aligned}$$

$B_\varepsilon = \{ u \in X | \ \|u\| \leq \varepsilon\}$. If $\ell = \dim Y < \infty$, then $C_\ell(f, 0) \neq 0$.

Pro3 (Bartsch & Li(1997), Prop 3.8). Let $f \in C^1(X, \mathbb{R})$, sats (C). Assume $X = X^- \oplus X^+$, $\ell = \dim X^- < \infty$. If f is bounded from below on X^+ and

$$f(u) \to -\infty \ \text{ as } \ \|u\| \to \infty,\ u \in X^-,$$

then $C_\ell(f, \infty) \neq 0$. (loc link at infinity)

Exm3 (Liu & Li(2003a)). If $f : \mathbb{R}^n \to \mathbb{R}$ is anti-coercive, $C_q(f, \infty) = \delta_{q,n}\mathbb{Q}$.

Pro4 (Bartsch & Li(1997), Prop 3.6). If $f \in C^1(X, \mathbb{R})$, (C), for some $\ell \in \mathbb{N}$,

(1) if $C_\ell(f, \infty) \neq 0$, then f has a cri point u s.t. $C_\ell(f, u) \neq 0$.

Pro4 (Bartsch & Li(1997), Prop 3.6). If $f \in C^1(X,\mathbb{R})$, (C), for some $\ell \in \mathbb{N}$,

(1) if $C_\ell(f,\infty) \neq 0$, then f has a cri point u s.t. $C_\ell(f,u) \neq 0$.

(2) if $C_\ell(f,0) \neq C_\ell(f,\infty)$, then f has a nonzero cri point.

Pro4 (Bartsch & Li(1997), Prop 3.6). If $f \in C^1(X, \mathbb{R})$, (C), for some $\ell \in \mathbb{N}$,

(1) if $C_\ell(f, \infty) \neq 0$, then f has a cri point u s.t. $C_\ell(f, u) \neq 0$.

(2) if $C_\ell(f, 0) \neq C_\ell(f, \infty)$, then f has a nonzero cri point.

Thm2 (Poincare-Hopf). Let X be Hilbert, $f \in C^2(X)$ has an isolated critical point u, $\nabla f = 1_X - K$ being $K : X \to X$ compact,

Pro4 (Bartsch & Li(1997), Prop 3.6). If $f \in C^1(X,\mathbb{R})$, (C), for some $\ell \in \mathbb{N}$,

(1) if $C_\ell(f,\infty) \neq 0$, then f has a cri point u s.t. $C_\ell(f,u) \neq 0$.

(2) if $C_\ell(f,0) \neq C_\ell(f,\infty)$, then f has a nonzero cri point.

Thm2 (Poincare-Hopf). Let X be Hilbert, $f \in C^2(X)$ has an isolated critical point u, $\nabla f = 1_X - K$ being $K : X \to X$ compact, then

$$\operatorname{ind}(\nabla f, u) = \deg(\nabla f, B_\varepsilon(u), 0)$$

Pro4 (Bartsch & Li(1997), Prop 3.6). If $f \in C^1(X,\mathbb{R})$, (C), for some $\ell \in \mathbb{N}$,

(1) if $C_\ell(f,\infty) \neq 0$, then f has a cri point u s.t. $C_\ell(f,u) \neq 0$.

(2) if $C_\ell(f,0) \neq C_\ell(f,\infty)$, then f has a nonzero cri point.

Thm2 (Poincare-Hopf). Let X be Hilbert, $f \in C^2(X)$ has an isolated critical point u, $\nabla f = 1_X - K$ being $K : X \to X$ compact, then

$$\operatorname{ind}(\nabla f, u) = \deg(\nabla f, B_\varepsilon(u), 0) = \sum_{q=0}^{\infty} (-1)^q \dim C_q(f,u).$$

Pro4 (Bartsch & Li(1997), Prop 3.6). If $f \in C^1(X, \mathbb{R})$, (C), for some $\ell \in \mathbb{N}$,

(1) if $C_\ell(f, \infty) \neq 0$, then f has a cri point u s.t. $C_\ell(f, u) \neq 0$.

(2) if $C_\ell(f, 0) \neq C_\ell(f, \infty)$, then f has a nonzero cri point.

Thm2 (Poincare-Hopf). Let X be Hilbert, $f \in C^2(X)$ has an isolated critical point u, $\nabla f = 1_X - K$ being $K : X \to X$ compact, then

$$\operatorname{ind}(\nabla f, u) = \deg(\nabla f, B_\varepsilon(u), 0) = \sum_{q=0}^{\infty} (-1)^q \dim C_q(f, u).$$

Rek1. Critical group is better for describing local behavior.

2. Critical groups under SPR

$(E_\pm)$ Let $X = X^- \oplus X^+$ be a Hilbert space,

2. Critical groups under SPR

$(E_\pm)$ Let $X = X^- \oplus X^+$ be a Hilbert space, $f \in C^1(X)$, $\kappa > 0$ s.t.

$$\pm \langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2 ,$$

where $v \in X^-$ and $w_{1,2} \in X^+$.

2. Critical groups under SPR

$(E_\pm)$ Let $X = X^- \oplus X^+$ be a Hilbert space, $f \in C^1(X)$, $\kappa > 0$ s.t.

$$\pm \langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2 ,$$

where $v \in X^-$ and $w_{1,2} \in X^+$. some sort of monotonicity for $\nabla f(v + \cdot)$.

2. Critical groups under SPR

$(E_\pm)$ Let $X = X^- \oplus X^+$ be a Hilbert space, $f \in C^1(X)$, $\kappa > 0$ s.t.

$$\pm \langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2,$$

where $v \in X^-$ and $w_{1,2} \in X^+$. some sort of monotonicity for $\nabla f(v + \cdot)$.

Thm3 (Castro(1982)). Under (E_+) or (E_-), there is $\psi : X^- \to X^+$ s.t

$$\varphi : X^- \to \mathbb{R}, \qquad \varphi(v) = f(v + \psi(v))$$

is C^1. Moreover, v is a critical point of φ iff $v + \psi(v)$ is a critical point of f.

2. Critical groups under SPR

($E_\pm$) Let $X = X^- \oplus X^+$ be a Hilbert space, $f \in C^1(X)$, $\kappa > 0$ s.t.

$$\pm \langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2 ,$$

where $v \in X^-$ and $w_{1,2} \in X^+$. some sort of monotonicity for $\nabla f(v + \cdot)$.

Thm3 (Castro(1982)). Under (E_+) or (E_-), there is $\psi : X^- \to X^+$ s.t

$$\varphi : X^- \to \mathbb{R}, \qquad \varphi(v) = f(v + \psi(v))$$

is C^1. Moreover, v is a critical point of φ iff $v + \psi(v)$ is a critical point of f.

Rek2. Liu(2008), Cor 2.2 showed that

$$\nabla f = 1_X - \text{Compact} \quad \Longrightarrow \quad \nabla \varphi = 1_{X^-} - \text{Compact}.$$

2. Critical groups under SPR

$(E_\pm)$ Let $X = X^- \oplus X^+$ be a Hilbert space, $f \in C^1(X)$, $\kappa > 0$ s.t.

$$\pm \langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2 ,$$

where $v \in X^-$ and $w_{1,2} \in X^+$. some sort of monotonicity for $\nabla f(v + \cdot)$.

Thm3 (Castro(1982)). Under (E_+) or (E_-), there is $\psi : X^- \to X^+$ s.t

$$\varphi : X^- \to \mathbb{R}, \qquad \varphi(v) = f(v + \psi(v))$$

is C^1. Moreover, v is a critical point of φ iff $v + \psi(v)$ is a critical point of f.

Rek2. Liu(2008), Cor 2.2 showed that

$$\nabla f = 1_X - \text{Compact} \quad \Longrightarrow \quad \nabla\varphi = 1_{X^-} - \text{Compact}.$$

Thm4 (Liu & Li(2003a)). In case (E_+), if f sats (PS), $\inf_{\mathscr{K}} f > -\infty$,

2. Critical groups under SPR

$(E_\pm)$ Let $X = X^- \oplus X^+$ be a Hilbert space, $f \in C^1(X)$, $\kappa > 0$ s.t.

$$\pm \langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2 ,$$

where $v \in X^-$ and $w_{1,2} \in X^+$. some sort of monotonicity for $\nabla f(v + \cdot)$.

Thm3 (Castro(1982)). Under (E_+) or (E_-), there is $\psi : X^- \to X^+$ s.t

$$\varphi : X^- \to \mathbb{R}, \qquad \varphi(v) = f(v + \psi(v))$$

is C^1. Moreover, v is a critical point of φ iff $v + \psi(v)$ is a critical point of f.

Rek2. Liu(2008), Cor 2.2 showed that

$$\nabla f = 1_X - \text{Compact} \quad \Longrightarrow \quad \nabla \varphi = 1_{X^-} - \text{Compact}.$$

Thm4 (Liu & Li(2003a)). In case (E_+), if f sats (PS), $\inf_{\mathscr{K}} f > -\infty$, then

$$C_q(f, \infty) \cong C_q(\varphi, \infty), \qquad q \in \mathbb{N}.$$

2. Critical groups under SPR

$(E_\pm)$ Let $X = X^- \oplus X^+$ be a Hilbert space, $f \in C^1(X)$, $\kappa > 0$ s.t.

$$\pm \langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2 ,$$

where $v \in X^-$ and $w_{1,2} \in X^+$. some sort of monotonicity for $\nabla f(v + \cdot)$.

Thm3 (Castro(1982)). Under (E_+) or (E_-), there is $\psi : X^- \to X^+$ s.t

$$\varphi : X^- \to \mathbb{R}, \qquad \varphi(v) = f(v + \psi(v))$$

is C^1. Moreover, v is a critical point of φ iff $v + \psi(v)$ is a critical point of f.

Rek2. Liu(2008), Cor 2.2 showed that

$$\nabla f = 1_X - \text{Compact} \quad \Longrightarrow \quad \nabla\varphi = 1_{X^-} - \text{Compact}.$$

Thm4 (Liu & Li(2003a)). In case (E_+), if f sats (PS), $\inf_{\mathscr{K}} f > -\infty$, then

$$C_q(f, \infty) \cong C_q(\varphi, \infty), \qquad q \in \mathbb{N}.$$

Moreover, if $k = \dim X^- < \infty$ and $C_k(f, \infty) \neq 0$ (see Pro3),

2. Critical groups under SPR

$(E_\pm)$ Let $X = X^- \oplus X^+$ be a Hilbert space, $f \in C^1(X)$, $\kappa > 0$ s.t.

$$\pm \langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2 ,$$

where $v \in X^-$ and $w_{1,2} \in X^+$. some sort of monotonicity for $\nabla f(v + \cdot)$.

Thm3 (Castro(1982)). Under (E_+) or (E_-), there is $\psi : X^- \to X^+$ s.t

$$\varphi : X^- \to \mathbb{R}, \qquad \varphi(v) = f(v + \psi(v))$$

is C^1. Moreover, v is a critical point of φ iff $v + \psi(v)$ is a critical point of f.

Rek2. Liu(2008), Cor 2.2 showed that

$$\nabla f = 1_X - \text{Compact} \quad \Longrightarrow \quad \nabla \varphi = 1_{X^-} - \text{Compact}.$$

Thm4 (Liu & Li(2003a)). In case (E_+), if f sats (PS), $\inf_{\mathscr{K}} f > -\infty$, then

$$C_q(f, \infty) \cong C_q(\varphi, \infty), \qquad q \in \mathbb{N}.$$

Moreover, if $k = \dim X^- < \infty$ and $C_k(f, \infty) \neq 0$ (see Pro3), then

$$\dim C_q(f, \infty) = \delta_{qk}. \tag{1}$$

Rek3. To have (1), we prove: If $\varphi : \mathbb{R}^k \to \mathbb{R}$ sats (PS), then
$C_k(\varphi, \infty) \neq 0$

Rek3. To have (1), we prove: If $\varphi : \mathbb{R}^k \to \mathbb{R}$ sats (PS), then

$C_k(\varphi, \infty) \neq 0 \Longrightarrow C_q(\varphi, \infty) \cong \delta_{qk}\mathbb{Q}, \qquad q \in \mathbb{N}.$ Liu & Li(2003a), Lem 2.1

Rek3. To have (1), we prove: If $\varphi : \mathbb{R}^k \to \mathbb{R}$ sats (PS), then

$$C_k(\varphi,\infty) \neq 0 \Longrightarrow C_q(\varphi,\infty) \cong \delta_{qk}\mathbb{Q}, \qquad q \in \mathbb{N}.$$ Liu & Li(2003a), Lem 2.1

For $b < \inf_{\mathscr{K}} \varphi$, we need $C = \{\varphi \geq b\}$ to be connected.

Rek3. To have (1), we prove: If $\varphi : \mathbb{R}^k \to \mathbb{R}$ sats (PS), then

$$C_k(\varphi,\infty) \neq 0 \Longrightarrow C_q(\varphi,\infty) \cong \delta_{qk}\mathbb{Q}, \qquad q \in \mathbb{N}.$$ Liu & Li(2003a), Lem 2.1

For $b < \inf_{\mathscr{K}} \varphi$, we need $C = \{\varphi \geq b\}$ to be connected. It suffices to apply the following to $-\varphi$.

Pro5. If $\psi \in C^1(X)$ sats (C), $b > \sup_{\mathscr{K}} \psi$, then $\psi_b = \{\psi \leq b\}$ connected.

Rek3. To have (1), we prove: If $\varphi : \mathbb{R}^k \to \mathbb{R}$ sats (PS), then

$$C_k(\varphi,\infty) \neq 0 \Longrightarrow C_q(\varphi,\infty) \cong \delta_{qk}\mathbb{Q}, \qquad q \in \mathbb{N}.$$ Liu & Li(2003a), Lem 2.1

For $b < \inf_{\mathscr{K}} \varphi$, we need $C = \{\varphi \geq b\}$ to be connected. It suffices to apply the following to $-\varphi$.

Pro5. If $\psi \in C^1(X)$ sats (C), $b > \sup_{\mathscr{K}} \psi$, then $\psi_b = \{\psi \leq b\}$ connected.

Pf. ψ_b is a strong deformation retract of X, $\exists \eta : X \to \psi_b$ s.t. $\eta|_{\psi_b} = 1_{\psi_b}$.

Rek3. To have (1), we prove: If $\varphi : \mathbb{R}^k \to \mathbb{R}$ sats (PS), then

$$C_k(\varphi, \infty) \neq 0 \Longrightarrow C_q(\varphi, \infty) \cong \delta_{qk}\mathbb{Q}, \qquad q \in \mathbb{N}.$$ Liu & Li(2003a), Lem 2.1

For $b < \inf_{\mathscr{K}} \varphi$, we need $C = \{\varphi \geq b\}$ to be connected. It suffices to apply the following to $-\varphi$.

Pro5. If $\psi \in C^1(X)$ sats (C), $b > \sup_{\mathscr{K}} \psi$, then $\psi_b = \{\psi \leq b\}$ connected.

Pf. ψ_b is a strong deformation retract of X, $\exists \eta : X \to \psi_b$ s.t. $\eta|_{\psi_b} = 1_{\psi_b}$. For $a_\pm \in \psi_b$, $\exists \gamma : [-1, 1] \to X$ s.t. $\gamma(\pm 1) = a_\pm$.

Rek3. To have (1), we prove: If $\varphi:\mathbb{R}^k\to\mathbb{R}$ sats (PS), then

$$C_k(\varphi,\infty)\neq 0 \Longrightarrow C_q(\varphi,\infty)\cong\delta_{qk}\mathbb{Q},\qquad q\in\mathbb{N}.$$ Liu & Li(2003a), Lem 2.1

For $b<\inf_{\mathscr{K}}\varphi$, we need $C=\{\varphi\geq b\}$ to be connected. It suffices to apply the following to $-\varphi$.

Pro5. If $\psi\in C^1(X)$ sats (C), $b>\sup_{\mathscr{K}}\psi$, then $\psi_b=\{\psi\leq b\}$ connected.

Pf. ψ_b is a strong deformation retract of X, $\exists\eta:X\to\psi_b$ s.t. $\eta|_{\psi_b}=1_{\psi_b}$. For $a_\pm\in\psi_b$, $\exists\gamma:[-1,1]\to X$ s.t. $\gamma(\pm1)=a_\pm$. Now $\eta\circ\gamma:[-1,1]\to\psi_b$ is a curve in ψ_b connecting $a_\pm$.

Rek3. To have (1), we prove: If $\varphi : \mathbb{R}^k \to \mathbb{R}$ sats (PS), then

$$C_k(\varphi, \infty) \neq 0 \Longrightarrow C_q(\varphi, \infty) \cong \delta_{qk}\mathbb{Q}, \qquad q \in \mathbb{N}. \qquad \text{Liu \& Li(2003a), Lem 2.1}$$

For $b < \inf_{\mathscr{K}} \varphi$, we need $C = \{\varphi \geq b\}$ to be connected. It suffices to apply the following to $-\varphi$.

Pro5. If $\psi \in C^1(X)$ sats (C), $b > \sup_{\mathscr{K}} \psi$, then $\psi_b = \{\psi \leq b\}$ connected.

Pf. ψ_b is a strong deformation retract of X, $\exists \eta : X \to \psi_b$ s.t. $\eta|_{\psi_b} = 1_{\psi_b}$. For $a_\pm \in \psi_b$, $\exists \gamma : [-1, 1] \to X$ s.t. $\gamma(\pm 1) = a_\pm$. Now $\eta \circ \gamma : [-1, 1] \to \psi_b$ is a curve in ψ_b connecting $a_\pm$.

Since C closed connected, $C_k(\varphi, \infty) = H_k(\mathbb{R}^k, \mathbb{R}^k \backslash C) \neq 0$,

Rek3. To have (1), we prove: If $\varphi : \mathbb{R}^k \to \mathbb{R}$ sats (PS), then

$$C_k(\varphi,\infty) \neq 0 \Longrightarrow C_q(\varphi,\infty) \cong \delta_{qk}\mathbb{Q}, \qquad q \in \mathbb{N}.$$ Liu & Li(2003a), Lem 2.1

For $b < \inf_{\mathscr{K}} \varphi$, we need $C = \{\varphi \geq b\}$ to be connected. It suffices to apply the following to $-\varphi$.

Pro5. If $\psi \in C^1(X)$ sats (C), $b > \sup_{\mathscr{K}} \psi$, then $\psi_b = \{\psi \leq b\}$ connected.

Pf. ψ_b is a strong deformation retract of X, $\exists \eta : X \to \psi_b$ s.t. $\eta|_{\psi_b} = 1_{\psi_b}$. For $a_\pm \in \psi_b$, $\exists \gamma : [-1,1] \to X$ s.t. $\gamma(\pm 1) = a_\pm$. Now $\eta \circ \gamma : [-1,1] \to \psi_b$ is a curve in ψ_b connecting $a_\pm$.

Since C closed connected, $C_k(\varphi,\infty) = H_k(\mathbb{R}^k, \mathbb{R}^k \backslash C) \neq 0$, C is compact, φ bdd. from. above thus anti-coercive Li(1986).

Rek3. To have (1), we prove: If $\varphi : \mathbb{R}^k \to \mathbb{R}$ sats (PS), then

$$C_k(\varphi,\infty) \neq 0 \Longrightarrow C_q(\varphi,\infty) \cong \delta_{qk}\mathbb{Q}, \qquad q \in \mathbb{N}.$$ Liu & Li(2003a), Lem 2.1

For $b < \inf_{\mathscr{K}} \varphi$, we need $C = \{\varphi \geq b\}$ to be connected. It suffices to apply the following to $-\varphi$.

Pro5. If $\psi \in C^1(X)$ sats (C), $b > \sup_{\mathscr{K}} \psi$, then $\psi_b = \{\psi \leq b\}$ connected.

Pf. ψ_b is a strong deformation retract of X, $\exists \eta : X \to \psi_b$ s.t. $\eta|_{\psi_b} = 1_{\psi_b}$. For $a_\pm \in \psi_b$, $\exists \gamma : [-1,1] \to X$ s.t. $\gamma(\pm 1) = a_\pm$. Now $\eta \circ \gamma : [-1,1] \to \psi_b$ is a curve in ψ_b connecting $a_\pm$.

Since C closed connected, $C_k(\varphi,\infty) = H_k(\mathbb{R}^k, \mathbb{R}^k \backslash C) \neq 0$, C is compact, φ bdd. from. above thus anti-coercive Li(1986). Now apply Exm3.

Rek3. To have (1), we prove: If $\varphi : \mathbb{R}^k \to \mathbb{R}$ sats (PS), then

$$C_k(\varphi,\infty) \neq 0 \implies C_q(\varphi,\infty) \cong \delta_{qk}\mathbb{Q}, \qquad q \in \mathbb{N}.$$ Liu & Li(2003a), Lem 2.1

For $b < \inf_{\mathscr{K}} \varphi$, we need $C = \{\varphi \geq b\}$ to be connected. It suffices to apply the following to $-\varphi$.

Pro5. If $\psi \in C^1(X)$ sats (C), $b > \sup_{\mathscr{K}} \psi$, then $\psi_b = \{\psi \leq b\}$ connected.

Pf. ψ_b is a strong deformation retract of X, $\exists \eta : X \to \psi_b$ s.t. $\eta|_{\psi_b} = 1_{\psi_b}$. For $a_\pm \in \psi_b$, $\exists \gamma : [-1,1] \to X$ s.t. $\gamma(\pm 1) = a_\pm$. Now $\eta \circ \gamma : [-1,1] \to \psi_b$ is a curve in ψ_b connecting $a_\pm$.

Since C closed connected, $C_k(\varphi,\infty) = H_k(\mathbb{R}^k, \mathbb{R}^k \backslash C) \neq 0$, C is compact, φ bdd. from. above thus anti-coercive Li(1986). Now apply Exm3.

Thm5 (Liu & Li(2003a)). In case (E_-), if f satisfies (PS), $\inf_{\mathscr{K}} f > -\infty$ and $j = \dim X^+ < \infty$,

Rek3. To have (1), we prove: If $\varphi : \mathbb{R}^k \to \mathbb{R}$ sats (PS), then

$$C_k(\varphi,\infty) \neq 0 \Longrightarrow C_q(\varphi,\infty) \cong \delta_{qk}\mathbb{Q}, \qquad q \in \mathbb{N}. \qquad \text{Liu \& Li(2003a), Lem 2.1}$$

For $b < \inf_{\mathscr{K}} \varphi$, we need $C = \{\varphi \geq b\}$ to be connected. It suffices to apply the following to $-\varphi$.

Pro5. If $\psi \in C^1(X)$ sats (C), $b > \sup_{\mathscr{K}} \psi$, then $\psi_b = \{\psi \leq b\}$ connected.

Pf. ψ_b is a strong deformation retract of X, $\exists \eta : X \to \psi_b$ s.t. $\eta|_{\psi_b} = 1_{\psi_b}$. For $a_\pm \in \psi_b$, $\exists \gamma : [-1,1] \to X$ s.t. $\gamma(\pm 1) = a_\pm$. Now $\eta \circ \gamma : [-1,1] \to \psi_b$ is a curve in ψ_b connecting $a_\pm$.

Since C closed connected, $C_k(\varphi,\infty) = H_k(\mathbb{R}^k, \mathbb{R}^k \backslash C) \neq 0$, C is compact, φ bdd. from. above thus anti-coercive Li(1986). Now apply Exm3.

Thm5 (Liu & Li(2003a)). In case (E_-), if f satisfies (PS), $\inf_{\mathscr{K}} f > -\infty$ and $j = \dim X^+ < \infty$, then

$$C_q(f,\infty) \cong C_{q-j}(\varphi,\infty), \qquad q \in \mathbb{N}.$$

Rek3. To have (1), we prove: If $\varphi : \mathbb{R}^k \to \mathbb{R}$ sats (PS), then

$$C_k(\varphi,\infty) \neq 0 \Longrightarrow C_q(\varphi,\infty) \cong \delta_{qk}\mathbb{Q}, \qquad q \in \mathbb{N}.$$ Liu & Li(2003a), Lem 2.1

For $b < \inf_{\mathscr{K}} \varphi$, we need $C = \{\varphi \geq b\}$ to be connected. It suffices to apply the following to $-\varphi$.

Pro5. If $\psi \in C^1(X)$ sats (C), $b > \sup_{\mathscr{K}} \psi$, then $\psi_b = \{\psi \leq b\}$ connected.

Pf. ψ_b is a strong deformation retract of X, $\exists \eta : X \to \psi_b$ s.t. $\eta|_{\psi_b} = 1_{\psi_b}$. For $a_\pm \in \psi_b$, $\exists \gamma : [-1,1] \to X$ s.t. $\gamma(\pm 1) = a_\pm$. Now $\eta \circ \gamma : [-1,1] \to \psi_b$ is a curve in ψ_b connecting $a_\pm$.

Since C closed connected, $C_k(\varphi,\infty) = H_k(\mathbb{R}^k, \mathbb{R}^k \backslash C) \neq 0$, C is compact, φ bdd. from. above thus anti-coercive Li(1986). Now apply Exm3.

Thm5 (Liu & Li(2003a)). In case (E_-), if f satisfies (PS), $\inf_{\mathscr{K}} f > -\infty$ and $j = \dim X^+ < \infty$, then

$$C_q(f,\infty) \cong C_{q-j}(\varphi,\infty), \qquad q \in \mathbb{N}. \tag{2}$$

Moreover, if $C_j(f,\infty) \neq 0$, then $\dim C_q(f,\infty) = \delta_{qj}$.

Rek4. The shift of index in (2) is due to Künneth formula.

Thm6 (Liu(2007)). In case (E_+), if u is a critical point of f and $v = P_{X^-}u$,

Thm6 (Liu(2007)). In case (E_+), if u is a critical point of f and $v = P_{X^-}u$, then $C_q(f,u) \cong C_q(\varphi, v)$, $q \in \mathbb{N}$.

Thm6 (Liu(2007)). In case (E_+), if u is a critical point of f and $v = P_{X^-}u$, then $C_q(f,u) \cong C_q(\varphi, v)$, $q \in \mathbb{N}$.

Thm7 (Li & Liu(2013)). In case (E_-), if u is a critical point of f and $v = P_{X^-}u$, φ satisfies (PS), $j = \dim X^+ < \infty$,

Thm6 (Liu(2007)). In case (E_+), if u is a critical point of f and $v = P_{X^-}u$, then $C_q(f, u) \cong C_q(\varphi, v)$, $q \in \mathbb{N}$.

Thm7 (Li & Liu(2013)). In case (E_-), if u is a critical point of f and $v = P_{X^-}u$, φ satisfies (PS), $j = \dim X^+ < \infty$, then

$$C_q(f, u) \cong C_{q-j}(\varphi, v), \qquad q \in \mathbb{N}.$$

Künneth again

Thm6 (Liu(2007)). In case (E_+), if u is a critical point of f and $v = P_{X^-}u$, then $C_q(f,u) \cong C_q(\varphi, v)$, $q \in \mathbb{N}$.

Thm7 (Li & Liu(2013)). In case (E_-), if u is a critical point of f and $v = P_{X^-}u$, φ satisfies (PS), $j = \dim X^+ < \infty$, then

$$C_q(f,u) \cong C_{q-j}(\varphi, v), \qquad q \in \mathbb{N}.$$

Künneth again

Rek5. Based on Thm4,

- Alexander Duality Theorem,

Thm6 (Liu(2007)). In case (E_+), if u is a critical point of f and $v = P_{X^-}u$, then $C_q(f,u) \cong C_q(\varphi, v)$, $q \in \mathbb{N}$.

Thm7 (Li & Liu(2013)). In case (E_-), if u is a critical point of f and $v = P_{X^-}u$, φ satisfies (PS), $j = \dim X^+ < \infty$, then

$$C_q(f,u) \cong C_{q-j}(\varphi, v), \qquad q \in \mathbb{N}.$$

Künneth again

Rek5. Based on Thm 4,

- Alexander Duality Theorem,
- Homotopy Invariance Theorem of Li, Perera & Su.(2001),

Liu(2009) proposed a universal method for computing $C_*(f,\infty)$ for f arising in elliptic resonant problems.

3. Multiple solutions for resonant problem

Let $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ be eigenvalues of $\left(-\Delta, H_0^1(\Omega)\right)$. Consider

$$-\Delta u = p(u), \qquad u \in H_0^1(\Omega).$$

3. Multiple solutions for resonant problem

Let $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ be eigenvalues of $\left(-\Delta, H_0^1(\Omega)\right)$. Consider

$$-\Delta u = p(u), \qquad u \in H_0^1(\Omega). \tag{3}$$

(p_1) $p \in C^1(\mathbb{R})$, $p(0) = 0$, $p'(0) < \lambda_1 < p_\infty = \lambda_m$, where

$$p_\infty = \lim_{|t|\to\infty} \frac{p(t)}{t}.$$

(asy lin, has solutions if cross eigenvalue)

3. Multiple solutions for resonant problem

Let $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ be eigenvalues of $\left(-\Delta, H_0^1(\Omega)\right)$. Consider

$$-\Delta u = p(u), \qquad u \in H_0^1(\Omega). \tag{3}$$

(p_1) $p \in C^1(\mathbb{R})$, $p(0) = 0$, $p'(0) < \lambda_1 < p_\infty = \lambda_m$, where

$$p_\infty = \lim_{|t|\to\infty} \frac{p(t)}{t}.$$

(asy lin, has solutions if cross eigenvalue)

(p_2) for some $\gamma \in \mathbb{R}$, $p'(t) \le \gamma < \lambda_{m+1}$.

Rek6. If $p_\infty \in (\lambda_m, \lambda_{m+1})$, Castro & Cossio(1994) obtained 4 nontrivial solutions for (3).

3. Multiple solutions for resonant problem

Let $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ be eigenvalues of $\left(-\Delta, H_0^1(\Omega)\right)$. Consider

$$-\Delta u = p(u), \qquad u \in H_0^1(\Omega). \tag{3}$$

(p_1) $p \in C^1(\mathbb{R})$, $p(0) = 0$, $p'(0) < \lambda_1 < p_\infty = \lambda_m$, where

$$p_\infty = \lim_{|t|\to\infty} \frac{p(t)}{t}. \qquad \text{(asy lin, has solutions if cross eigenvalue)}$$

(p_2) for some $\gamma \in \mathbb{R}$, $p'(t) \le \gamma < \lambda_{m+1}$.

Rek6. If $p_\infty \in (\lambda_m, \lambda_{m+1})$, Castro & Cossio(1994) obtained 4 nontrivial solutions for (3). Li & Zhang(1999) extended to $p_\infty = \lambda_m$ provided

(p_3) $\exists \alpha \in [0,1)$, $c > 0$ s.t. $|p(t) - \lambda_m t| \le c\left(1 + |t|^\alpha\right)$,

3. Multiple solutions for resonant problem

Let $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ be eigenvalues of $\left(-\Delta, H_0^1(\Omega)\right)$. Consider

$$-\Delta u = p(u), \qquad u \in H_0^1(\Omega). \tag{3}$$

(p_1) $p \in C^1(\mathbb{R})$, $p(0) = 0$, $p'(0) < \lambda_1 < p_\infty = \lambda_m$, where

$$p_\infty = \lim_{|t|\to\infty} \frac{p(t)}{t}.$$

(asy lin, has solutions if cross eigenvalue)

(p_2) for some $\gamma \in \mathbb{R}$, $p'(t) \le \gamma < \lambda_{m+1}$.

Rek6. If $p_\infty \in (\lambda_m, \lambda_{m+1})$, Castro & Cossio(1994) obtained 4 nontrivial solutions for (3). Li & Zhang(1999) extended to $p_\infty = \lambda_m$ provided

(p_3) $\exists \alpha \in [0,1)$, $c > 0$ s.t. $|p(t) - \lambda_m t| \le c\left(1 + |t|^\alpha\right)$,

$$\frac{1}{|t|^{2\alpha}}\left(P(t) - \frac{1}{2}\lambda_m t^2\right) \to +\infty, \qquad \text{as } |t| \to \infty.$$

Thm8 (Liu(2007)). Assume (p_1), (p_2)

$$(p_4)\ \ P(t) - \frac{1}{2}\lambda_m t^2 \to +\infty \text{ as } |t| \to \infty,$$

much weaker than (p_3)

Thm8 (Liu(2007)). Assume (p_1), (p_2)

(p_4) $P(t) - \frac{1}{2}\lambda_m t^2 \to +\infty$ as $|t| \to \infty$, much weaker than (p_3)

then (3) has 4 nontrivial solutions.

Thm8 (Liu(2007)). Assume (p_1), (p_2)

(p_4) $P(t) - \frac{1}{2}\lambda_m t^2 \to +\infty$ as $|t| \to \infty$, much weaker than (p_3)

then (3) has 4 nontrivial solutions.

Solutions of (3) are critical points of $f : H_0^1(\Omega) \to \mathbb{R}$,

$$f(u) = \frac{1}{2}\int |\nabla u|^2 - \int P(u).$$

Thm8 (Liu(2007)). Assume (p_1), (p_2)

(p_4) $P(t) - \frac{1}{2}\lambda_m t^2 \to +\infty$ as $|t| \to \infty$, much weaker than (p_3)

then (3) has 4 nontrivial solutions.

Solutions of (3) are critical points of $f : H_0^1(\Omega) \to \mathbb{R}$,

$$f(u) = \frac{1}{2}\int |\nabla u|^2 - \int P(u)..$$

Rek7. (1) In Castro & Cossio(1994), Li & Zhang(1999), f satisfies (PS), while under our conditions, f may not satisfy (PS).

Thm8 (Liu(2007)). Assume (p_1), (p_2)

(p_4) $P(t) - \frac{1}{2}\lambda_m t^2 \to +\infty$ as $|t| \to \infty$, much weaker than (p_3)

then (3) has 4 nontrivial solutions.

Solutions of (3) are critical points of $f : H_0^1(\Omega) \to \mathbb{R}$,

$$f(u) = \frac{1}{2}\int |\nabla u|^2 - \int P(u)..$$

Rek7. (1) In Castro & Cossio(1994), Li & Zhang(1999), f satisfies (PS), while under our conditions, f may not satisfy (PS).

(2) Condition (p_2) enables us to reduce f to the subspace

$$X^- = \mathrm{span}\{\phi_1, \ldots, \phi_m\}$$

and consider $\varphi : X^- \to \mathbb{R}$.

Thm8 (Liu(2007)). Assume (p_1), (p_2)

(p_4) $P(t) - \frac{1}{2}\lambda_m t^2 \to +\infty$ as $|t| \to \infty$, much weaker than (p_3)

then (3) has 4 nontrivial solutions.

Solutions of (3) are critical points of $f : H_0^1(\Omega) \to \mathbb{R}$,

$$f(u) = \frac{1}{2}\int |\nabla u|^2 - \int P(u)..$$

Rek7. (1) In Castro & Cossio(1994), Li & Zhang(1999), f satisfies (PS), while under our conditions, f may not satisfy (PS).

(2) Condition (p_2) enables us to reduce f to the subspace

$$X^- = \operatorname{span}\{\phi_1, \ldots, \phi_m\}$$

and consider $\varphi : X^- \to \mathbb{R}$. It turns out that φ is anti–coercive:

$$\varphi(v) \to -\infty \qquad \text{as } \|v\| \to \infty$$

under (p_4), noting $\dim X^- < \infty$.

Thm8 (Liu(2007)). Assume (p_1), (p_2)

(p_4) $P(t) - \frac{1}{2}\lambda_m t^2 \to +\infty$ as $|t| \to \infty$, much weaker than (p_3)

then (3) has 4 nontrivial solutions.

Solutions of (3) are critical points of $f : H_0^1(\Omega) \to \mathbb{R}$,

$$f(u) = \frac{1}{2}\int |\nabla u|^2 - \int P(u)..$$

Rek7. (1) In Castro & Cossio(1994), Li & Zhang(1999), f satisfies (PS), while under our conditions, f may not satisfy (PS).

(2) Condition (p_2) enables us to reduce f to the subspace

$$X^- = \mathrm{span}\,\{\phi_1, \ldots, \phi_m\}$$

and consider $\varphi : X^- \to \mathbb{R}$. It turns out that φ is anti–coercive:

$$\varphi(v) \to -\infty \qquad \text{as } \|v\| \to \infty$$

under (p_4), noting $\dim X^- < \infty$. For $\dim X^- = \infty$, see Liu(2008).

Pf. It is known that $u = 0$ is loc. min. of f. Thus

$$\dim C_q(f, 0) = \delta_{q0}.$$

Pf. It is known that $u = 0$ is loc. min. of f. Thus

$$\dim C_q(f, 0) = \delta_{q0}.$$

It is also known that $f_\pm$ satisfy (PS), therefore we can obtain two solutions $u_\pm$ via Mountain Pass Theorem such that

$$\dim C_q(f, u_\pm) = \delta_{q1}.$$

Pf. It is known that $u = 0$ is loc. min. of f. Thus

$$\dim C_q(f, 0) = \delta_{q0}.$$

It is also known that $f_\pm$ satisfy (PS), therefore we can obtain two solutions $u_\pm$ via Mountain Pass Theorem such that

$$\dim C_q(f, u_\pm) = \delta_{q1}.$$

By Thm 6, $(v_\pm = P_{X^-} u_\pm)$

$$\dim C_q(\varphi, 0) = \dim C_q(f, 0) = \delta_{q0},$$

$$\dim C_q(\varphi, v_\pm) = \dim C_q(f, u_\pm) = \delta_{q1}.$$

Pf. It is known that $u = 0$ is loc. min. of f. Thus

$$\dim C_q(f, 0) = \delta_{q0}.$$

It is also known that $f_\pm$ satisfy (PS), therefore we can obtain two solutions $u_\pm$ via Mountain Pass Theorem such that

$$\dim C_q(f, u_\pm) = \delta_{q1}.$$

By Thm 6, $(v_\pm = P_{X^-} u_\pm)$

$$\dim C_q(\varphi, 0) = \dim C_q(f, 0) = \delta_{q0},$$

$$\dim C_q(\varphi, v_\pm) = \dim C_q(f, u_\pm) = \delta_{q1}.$$

Since φ is anti-coercive, φ has a global max $v \in X^-$, with

$$\dim C_q(\varphi, v) = \delta_{qm}.$$

Pf. It is known that $u = 0$ is loc. min. of f. Thus

$$\dim C_q(f, 0) = \delta_{q0}.$$

It is also known that $f_\pm$ satisfy (PS), therefore we can obtain two solutions $u_\pm$ via Mountain Pass Theorem such that

$$\dim C_q(f, u_\pm) = \delta_{q1}.$$

By Thm 6, ($v_\pm = P_{X^-} u_\pm$)

$$\dim C_q(\varphi, 0) = \dim C_q(f, 0) = \delta_{q0},$$

$$\dim C_q(\varphi, v_\pm) = \dim C_q(f, u_\pm) = \delta_{q1}.$$

Since φ is anti-coercive, φ has a global max $v \in X^-$, with

$$\dim C_q(\varphi, v) = \delta_{qm}.$$

We also know that $\beta_q = \dim C_q(\varphi, \infty) = \delta_{qm}$, see Exm 3.

Pf. It is known that $u=0$ is loc. min. of f. Thus

$$\dim C_q(f,0)=\delta_{q0}.$$

It is also known that $f_\pm$ satisfy (PS), therefore we can obtain two solutions $u_\pm$ via Mountain Pass Theorem such that

$$\dim C_q(f,u_\pm)=\delta_{q1}.$$

By Thm 6, $(v_\pm = P_{X^-}u_\pm)$

$$\dim C_q(\varphi,0)=\dim C_q(f,0)=\delta_{q0},$$

$$\dim C_q(\varphi,v_\pm)=\dim C_q(f,u_\pm)=\delta_{q1}.$$

Since φ is anti-coercive, φ has a global max $v\in X^-$, with

$$\dim C_q(\varphi,v)=\delta_{qm}.$$

We also know that $\beta_q=\dim C_q(\varphi,\infty)=\delta_{qm}$, see Exm 3.
If $0, v_\pm$ and v were the only critical points of φ, then

$$M_0=1,\quad M_1=2,\quad M_m=1,\quad M_q=0 \text{ for } q\neq 0,1,m.$$

Pf. It is known that $u=0$ is loc. min. of f. Thus
$$\dim C_q(f,0)=\delta_{q0}.$$
It is also known that $f_\pm$ satisfy (PS), therefore we can obtain two solutions $u_\pm$ via Mountain Pass Theorem such that
$$\dim C_q(f,u_\pm)=\delta_{q1}.$$
By Thm 6, $(v_\pm=P_{X^-}u_\pm)$
$$\dim C_q(\varphi,0)=\dim C_q(f,0)=\delta_{q0},$$
$$\dim C_q(\varphi,v_\pm)=\dim C_q(f,u_\pm)=\delta_{q1}.$$
Since φ is anti-coercive, φ has a global max $v\in X^-$, with
$$\dim C_q(\varphi,v)=\delta_{qm}.$$
We also know that $\beta_q=\dim C_q(\varphi,\infty)=\delta_{qm}$, see Exm 3.
If 0, $v_\pm$ and v were the only critical points of φ, then
$$M_0=1,\quad M_1=2,\quad M_m=1,\quad M_q=0 \text{ for } q\neq 0,1,m.$$
The Morse relation $$\sum(-1)^q M_q=\sum(-1)^q\beta_q$$
becomes $1-2+(-1)^m=(-1)^m$, a contradition.

4. Stationary Schrödinger equations

Looking for standing waves $\psi(t,x) = e^{-i\omega t}u(x)$ for NSE

$$i\psi_t = -\Delta\psi + U(x)\psi - \tilde{g}(|\psi|)\psi$$

leads to

$$-\Delta u + V(x)u = g(u), \qquad u \in H^1(\mathbb{R}^N).$$

4. Stationary Schrödinger equations

Looking for standing waves $\psi(t,x) = e^{-i\omega t}u(x)$ for NSE

$$i\psi_t = -\Delta\psi + U(x)\psi - \tilde{g}(|\psi|)\psi$$

leads to

$$-\Delta u + V(x)u = g(u), \qquad u \in H^1(\mathbb{R}^N). \tag{4}$$

Similar Schrödinger type equations require solving

$$\begin{cases} -\Delta u + V(x) + \phi u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

4. Stationary Schrödinger equations

Looking for standing waves $\psi(t,x) = e^{-i\omega t}u(x)$ for NSE

$$i\psi_t = -\Delta\psi + U(x)\psi - \tilde{g}(|\psi|)\psi$$

leads to

$$-\Delta u + V(x)u = g(u), \qquad u \in H^1(\mathbb{R}^N). \tag{4}$$

Similar Schrödinger type equations require solving

$$\begin{cases} -\Delta u + V(x) + \phi u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{5}$$

and

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad u \in H^1(\mathbb{R}^N).$$

4. Stationary Schrödinger equations

Looking for standing waves $\psi(t,x) = e^{-i\omega t}u(x)$ for NSE

$$i\psi_t = -\Delta\psi + U(x)\psi - \tilde{g}(|\psi|)\psi$$

leads to

$$-\Delta u + V(x)u = g(u), \qquad u \in H^1(\mathbb{R}^N). \tag{4}$$

Similar Schrödinger type equations require solving

$$\begin{cases} -\Delta u + V(x) + \phi u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{5}$$

and

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad u \in H^1(\mathbb{R}^N). \tag{6}$$

(4), (5), (6) are called Schrödinger equations, Schrödinger-Poisson systems and quasilinear Schrödinger equations.

4. Stationary Schrödinger equations

Looking for standing waves $\psi(t,x) = e^{-i\omega t}u(x)$ for NSE

$$i\psi_t = -\Delta\psi + U(x)\psi - \tilde{g}(|\psi|)\psi$$

leads to

$$-\Delta u + V(x)u = g(u), \qquad u \in H^1(\mathbb{R}^N). \tag{4}$$

Similar Schrödinger type equations require solving

$$\begin{cases} -\Delta u + V(x) + \phi u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{5}$$

and

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad u \in H^1(\mathbb{R}^N). \tag{6}$$

(4), (5), (6) are called Schrödinger equations, Schrödinger-Poisson systems and quasilinear Schrödinger equations. Here

$$V(x) = U(x) - \omega, \qquad g(u) = \tilde{g}(|u|)u.$$

From now on, all integrals are over $\mathbb{R}^N$.

Most studies on (4) in the last decades require

$$\inf_{\mathbb{R}^N} V > 0, \qquad \text{so that} \qquad \mathscr{B}(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right)$$

is positive definite.

Most studies on (4) in the last decades require

$$\inf_{\mathbb{R}^N} V > 0, \qquad \text{so that} \qquad \mathscr{B}(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right)$$

is positive definite. Then, $u = 0$ is loc min of

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) - \int G(u), \qquad \int G(u) = o\left(\|u\|^2\right) \text{ as } u \to 0.$$

Most studies on (4) in the last decades require

$$\inf_{\mathbb{R}^N} V > 0, \qquad \text{so that} \qquad \mathscr{B}(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right)$$

is positive definite. Then, $u = 0$ is loc min of

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) - \int G(u), \qquad \int G(u) = o\left(\|u\|^2\right) \text{ as } u \to 0.$$

and MPT applies. The same is true for Eqs (5) and (6).

Most studies on (4) in the last decades require

$$\inf_{\mathbb{R}^N} V > 0, \qquad \text{so that} \qquad \mathscr{B}(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right)$$

is positive definite. Then, $u = 0$ is loc min of

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) - \int G(u), \qquad \int G(u) = o\left(\|u\|^2\right) \text{ as } u \to 0.$$

and MPT applies. The same is true for Eqs (5) and (6).

Prb1. If $\omega \gg 1$, $\mathscr{B}$ is indefinite ($V = U - \omega$), MPT is no longer applicable. We will focus on this situation.

Most studies on (4) in the last decades require

$$\inf_{\mathbb{R}^N} V > 0, \qquad \text{so that} \qquad \mathscr{B}(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right)$$

is positive definite. Then, $u = 0$ is loc min of

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) - \int G(u), \qquad \int G(u) = o\left(\|u\|^2\right) \text{ as } u \to 0.$$

and MPT applies. The same is true for Eqs (5) and (6).

Prb1. If $\omega \gg 1$, $\mathscr{B}$ is indefinite ($V = U - \omega$), MPT is no longer applicable. We will focus on this situation.

For (4), the most interesting situation is

(V) $V \in C(\mathbb{R}^N)$ is $\mathbb{Z}^N$-per, 0 lies in a spectral gap of $-\Delta + V$.

Then both $\pm$-spaces of $\mathscr{B}$ are infinite dim, strongly indefinite.

Most studies on (4) in the last decades require

$$\inf_{\mathbb{R}^N} V > 0, \qquad \text{so that} \qquad \mathscr{B}(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right)$$

is positive definite. Then, $u = 0$ is loc min of

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) - \int G(u), \qquad \int G(u) = o\left(\|u\|^2\right) \text{ as } u \to 0.$$

and MPT applies. The same is true for Eqs (5) and (6).

Prb1. If $\omega \gg 1$, $\mathscr{B}$ is indefinite ($V = U - \omega$), MPT is no longer applicable. We will focus on this situation.

For (4), the most interesting situation is

(V) $V \in C(\mathbb{R}^N)$ is $\mathbb{Z}^N$-per, 0 lies in a spectral gap of $-\Delta + V$.

Then both $\pm$-spaces of $\mathscr{B}$ are infinite dim, strongly indefinite.
Kryszewski & Szulkin(1998) obtain nonzero solution, provided g satisfies

(g_0) g is subcritical, $g(t) = o(t)$ as $t \to 0$,

(g_1) $\exists \mu > 2$ s.t. $0 < \mu G(t) \le t g(t)$ for $t \ne 0$.

Most studies on (4) in the last decades require

$$\inf_{\mathbb{R}^N} V > 0, \qquad \text{so that} \qquad \mathscr{B}(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right)$$

is positive definite. Then, $u = 0$ is loc min of

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) - \int G(u), \qquad \int G(u) = o\left(\|u\|^2\right) \text{ as } u \to 0.$$

and MPT applies. The same is true for Eqs (5) and (6).

Prb1. If $\omega \gg 1$, $\mathscr{B}$ is indefinite ($V = U - \omega$), MPT is no longer applicable. We will focus on this situation.

For (4), the most interesting situation is

(V) $V \in C(\mathbb{R}^N)$ is $\mathbb{Z}^N$-per, 0 lies in a spectral gap of $-\Delta + V$.

Then both $\pm$-spaces of $\mathscr{B}$ are infinite dim, strongly indefinite.
Kryszewski & Szulkin(1998) obtain nonzero solution, provided g satisfies

(g_0) g is subcritical, $g(t) = o(t)$ as $t \to 0$,

(g_1) $\exists \mu > 2$ s.t. $0 < \mu G(t) \leq tg(t)$ for $t \neq 0$. $\quad \Rightarrow |G(t)| \geq c_1 |t|^\mu - c_2$ suplin

Superlinear functions like

$$g(t) = t \log(1 + |t|)$$

violates (g_1). Without (g_1) unable to get bdness of (PS) seqs.

Superlinear functions like

$$g(t) = t\log(1+|t|)$$

violates (g_1). Without (g_1) unable to get bdness of (PS) seqs.
Szulkin & Weth(2009) solved (4) via Pankov manifold, replacing (g_1) by

(g_2) $t \mapsto \dfrac{g(t)}{|t|}$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$,

$$\lim_{|t|\to\infty} \frac{G(t)}{t^2} = +\infty.$$

Superlinear functions like
$$g(t) = t \log(1 + |t|)$$
violates (g_1). Without (g_1) unable to get bdness of (PS) seqs.
Szulkin & Weth(2009) solved (4) via Pankov manifold, replacing (g_1) by

(g_2) $t \mapsto \dfrac{g(t)}{|t|}$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$,
$$\lim_{|t|\to\infty} \frac{G(t)}{t^2} = +\infty. \tag{7}$$

Thm9 (Liu(2012)). Assume (V), (g_0), (7) and

(g_2') $t \mapsto \dfrac{g(t)}{|t|}$ is increasing on $(-\infty, 0)$ and $(0, \infty)$,

then (4) has a ground state solution.

Superlinear functions like

$$g(t) = t\log(1+|t|)$$

violates (g_1). Without (g_1) unable to get bdness of (PS) seqs.
Szulkin & Weth(2009) solved (4) via Pankov manifold, replacing (g_1) by

(g_2) $t \mapsto \dfrac{g(t)}{|t|}$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$,

$$\lim_{|t|\to\infty} \frac{G(t)}{t^2} = +\infty. \tag{7}$$

Thm9 (Liu(2012)). Assume (V), (g_0), (7) and

(g_2') $t \mapsto \dfrac{g(t)}{|t|}$ is increasing on $(-\infty, 0)$ and $(0, \infty)$,

then (4) has a ground state solution.

Pf. For $N = \left\{ u \in X^+ \middle| \, \|u\| = \rho \right\}$, $M = \left\{ u \in X^- \oplus \mathbb{R}^+ h \middle| \, \|u\| \le R \right\}$,

$$\inf_N \Phi > \sup_{\partial M} \Phi,$$

Using Li & Szulkin(2002) to get $(C)_c$ sequence, then show its bdd.

5. Schrödinger-Poisson systems

To solve the S-P system

$$\begin{cases} -\Delta u + V(x) + \phi u = g(u), \\ -\Delta \phi = u^2, \qquad (u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3), \end{cases}$$

5. Schrödinger-Poisson systems

To solve the S-P system

$$\begin{cases} -\Delta u + V(x) + \phi u = g(u), \\ -\Delta \phi = u^2, \qquad (u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3), \end{cases} \tag{8}$$

by Benci & Fortunato(1998), it suffices to find critical pt of $\Phi : H^1 \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int \phi_u u^2 - \int G(u),$$

5. Schrödinger-Poisson systems

To solve the S-P system

$$\begin{cases} -\Delta u + V(x) + \phi u = g(u), \\ -\Delta \phi = u^2, \qquad (u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3), \end{cases} \tag{8}$$

by Benci & Fortunato(1998), it suffices to find critical pt of $\Phi : H^1 \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2} \int \left(|\nabla u|^2 + V(x) u^2 \right) + \frac{1}{4} \int \phi_u u^2 - \int G(u),$$

where ϕ_u is solution of $-\Delta \phi = u^2$. If $\inf_{\mathbb{R}^N} V > 0$, there are many results.

5. Schrödinger-Poisson systems

To solve the S-P system

$$\begin{cases} -\Delta u + V(x) + \phi u = g(u), \\ -\Delta \phi = u^2, \qquad (u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3), \end{cases} \tag{8}$$

by Benci & Fortunato(1998), it suffices to find critical pt of $\Phi : H^1 \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2} \int \left(|\nabla u|^2 + V(x) u^2 \right) + \frac{1}{4} \int \phi_u u^2 - \int G(u),$$

where ϕ_u is solution of $-\Delta \phi = u^2$. If $\inf_{\mathbb{R}^N} V > 0$, there are many results.

If $\inf_{\mathbb{R}^N} V < 0$, due to the term involving ϕ_u, Φ does not have a linking

$$\inf_N \Phi > \sup_{\partial M} \Phi,$$

we no longer have $\Phi \le 0$ on X^-

5. Schrödinger-Poisson systems

To solve the S-P system

$$\begin{cases} -\Delta u + V(x) + \phi u = g(u), \\ -\Delta \phi = u^2, \qquad (u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3), \end{cases} \tag{8}$$

by Benci & Fortunato(1998), it suffices to find critical pt of $\Phi : H^1 \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int \phi_u u^2 - \int G(u),$$

where ϕ_u is solution of $-\Delta\phi = u^2$. If $\inf_{\mathbb{R}^N} V > 0$, there are many results. If $\inf_{\mathbb{R}^N} V < 0$, due to the term involving ϕ_u, Φ does not have a linking

$$\inf_N \Phi > \sup_{\partial M} \Phi,$$

we no longer have $\Phi \le 0$ on X^-

where

$$N = \left\{ u \in X^+ \,\middle|\, \|u\| = \rho \right\},$$

$$M = \left\{ u \in X^- \oplus \mathbb{R}^+ h \,\middle|\, \|u\| \le R \right\}.$$

Thus, unlike (4), we can not get solution via linking theorem!

Thm10 (Chen & Liu(2015)). Assume (V_0) $V \in C(\mathbb{R}^3)$, $\lim_{|x|\to\infty} V(x) = +\infty$.

(g_0) g is subcritical, $g(t) = o(t)$ as $t \to 0$.

(g_1) $\exists b > 0$ s.t. $4G(t) \le tg(t) + bt^2$, $\lim_{|t|\to\infty} \frac{g(t)}{t^3} = +\infty$.

Thm10 (Chen & Liu(2015)). Assume (V_0) $V \in C(\mathbb{R}^3)$, $\lim\limits_{|x|\to\infty} V(x) = +\infty$.

(g_0) g is subcritical, $g(t) = o(t)$ as $t \to 0$.

(g_1) $\exists b > 0$ s.t. $4G(t) \le tg(t) + bt^2$, $\lim\limits_{|t|\to\infty} \frac{g(t)}{t^3} = +\infty$.

If 0 is not eigen value of $-\Delta u + V(x)u = \lambda u$, then (8) has a nonzero solution.

Thm10 (Chen & Liu(2015)). Assume (V_0) $V \in C(\mathbb{R}^3)$, $\lim_{|x|\to\infty} V(x) = +\infty$.

(g_0) g is subcritical, $g(t) = o(t)$ as $t \to 0$.

(g_1) $\exists b > 0$ s.t. $4G(t) \leq tg(t) + bt^2$, $\lim_{|t|\to\infty} \frac{g(t)}{t^3} = +\infty$.

If 0 is not eigen value of $-\Delta u + V(x)u = \lambda u$, then (8) has a nonzero solution.

Pf. Ingredients of the proof ($\ell = \dim X^-$):

Thm10 (Chen & Liu(2015)). Assume (V_0) $V \in C(\mathbb{R}^3)$, $\lim_{|x|\to\infty} V(x) = +\infty$.

(g_0) g is subcritical, $g(t) = o(t)$ as $t \to 0$.

(g_1) $\exists b > 0$ s.t. $4G(t) \le tg(t) + bt^2$, $\lim_{|t|\to\infty} \frac{g(t)}{t^3} = +\infty$.

If 0 is not eigen value of $-\Delta u + V(x)u = \lambda u$, then (8) has a nonzero solution.

Pf. Ingredients of the proof ($\ell = \dim X^-$):

(1) (V_0) allows working on subspace $X \subset H^1(\mathbb{R}^3)$, s.t. $X \hookrightarrow L^2(\mathbb{R}^3)$ comptly.

Thm10 (Chen & Liu(2015)). Assume (V_0) $V \in C(\mathbb{R}^3)$, $\lim_{|x|\to\infty} V(x) = +\infty$.

(g_0) g is subcritical, $g(t) = o(t)$ as $t \to 0$.

(g_1) $\exists b > 0$ s.t. $4G(t) \le tg(t) + bt^2$, $\lim_{|t|\to\infty} \frac{g(t)}{t^3} = +\infty$.

If 0 is not eigen value of $-\Delta u + V(x)u = \lambda u$, then (8) has a nonzero solution.

Pf. Ingredients of the proof ($\ell = \dim X^-$):

(1) (V_0) allows working on subspace $X \subset H^1(\mathbb{R}^3)$, s.t. $X \hookrightarrow L^2(\mathbb{R}^3)$ comptly.

(2) (g_0) and $\left|\frac{1}{4}\int \phi_u u^2\right| \le c\,\|u\|^4$ give $C_q(\Phi, 0) = \delta_{q,0}\mathbb{Q}$?

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int \phi_u u^2 - \int G(u)$$

Thm10 (Chen & Liu(2015)). Assume (V_0) $V \in C(\mathbb{R}^3)$, $\lim_{|x|\to\infty} V(x) = +\infty$.

(g_0) g is subcritical, $g(t) = o(t)$ as $t \to 0$.

(g_1) $\exists b > 0$ s.t. $4G(t) \le tg(t) + bt^2$, $\lim_{|t|\to\infty} \frac{g(t)}{t^3} = +\infty$.

If 0 is not eigen value of $-\Delta u + V(x)u = \lambda u$, then (8) has a nonzero solution.

Pf. Ingredients of the proof ($\ell = \dim X^-$):

(1) (V_0) allows working on subspace $X \subset H^1(\mathbb{R}^3)$, s.t. $X \hookrightarrow L^2(\mathbb{R}^3)$ comptly.

(2) (g_0) and $\left|\frac{1}{4}\int \phi_u u^2\right| \le c\,\|u\|^4$ give $C_q(\Phi, 0) = \delta_{q,0}\mathbb{Q}$?

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int \phi_u u^2 - \int G(u)$$

$$= \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + o\left(\|u\|^2\right), \qquad \text{as } u \to 0.$$

Thm10 (Chen & Liu(2015)). Assume (V_0) $V \in C(\mathbb{R}^3)$, $\lim\limits_{|x|\to\infty} V(x) = +\infty$.

(g_0) g is subcritical, $g(t) = o(t)$ as $t \to 0$.

(g_1) $\exists b > 0$ s.t. $4G(t) \le tg(t) + bt^2$, $\lim\limits_{|t|\to\infty} \frac{g(t)}{t^3} = +\infty$.

If 0 is not eigen value of $-\Delta u + V(x)u = \lambda u$, then (8) has a nonzero solution.

Pf. Ingredients of the proof ($\ell = \dim X^-$):

(1) (V_0) allows working on subspace $X \subset H^1(\mathbb{R}^3)$, s.t. $X \hookrightarrow L^2(\mathbb{R}^3)$ comptly.

(2) (g_0) and $\left|\frac{1}{4}\int \phi_u u^2\right| \le c\,\|u\|^4$ give $\qquad C_q(\Phi, 0) = \delta_{q,0}\mathbb{Q}$?

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int \phi_u u^2 - \int G(u)$$

$$= \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + o\left(\|u\|^2\right), \qquad \text{as } u \to 0.$$

So Φ has a loc link at 0 w.r.p. to $X = X^- \oplus X^+$. $C_\ell(\Phi, 0) \ne 0$ (see Pro 2).

Thm10 (Chen & Liu(2015)). Assume (V_0) $V \in C(\mathbb{R}^3)$, $\lim_{|x|\to\infty} V(x) = +\infty$.

(g_0) g is subcritical, $g(t) = o(t)$ as $t \to 0$.

(g_1) $\exists b > 0$ s.t. $4G(t) \le tg(t) + bt^2$, $\lim_{|t|\to\infty} \frac{g(t)}{t^3} = +\infty$.

If 0 is not eigen value of $-\Delta u + V(x)u = \lambda u$, then (8) has a nonzero solution.

Pf. Ingredients of the proof ($\ell = \dim X^-$):

(1) (V_0) allows working on subspace $X \subset H^1(\mathbb{R}^3)$, s.t. $X \hookrightarrow L^2(\mathbb{R}^3)$ comptly.

(2) (g_0) and $\left|\frac{1}{4}\int \phi_u u^2\right| \le c\,\|u\|^4$ give $\qquad C_q(\Phi, 0) = \delta_{q,0}\mathbb{Q}$?

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int \phi_u u^2 - \int G(u)$$
$$= \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u^2\right) + o\left(\|u\|^2\right), \qquad \text{as } u \to 0.$$

So Φ has a loc link at 0 w.r.p. to $X = X^- \oplus X^+$. $C_\ell(\Phi, 0) \ne 0$ (see Pro 2).

(3) Since $C_q(\Phi, \infty) = 0$ for all q, Pro 4 yields a critical point $u \ne 0$.

Rek8. Based on Pohozaev identity (need $V \in C^1$) and Ruiz(2006)

$$\int |u|^3 \le \frac{1}{2}\int \left(|\nabla u|^2 + \phi_u u^2\right),$$

Rek8. Based on Pohozaev identity (need $V \in C^1$) and Ruiz(2006)

$$\int |u|^3 \leq \frac{1}{2} \int \left(|\nabla u|^2 + \phi_u u^2 \right),$$

Liu & Mosconi(2020) studied the case that $g(t) \approx |t|^{\mu-2}\, t$ and obtained

- two nontrivial solutions if $\mu \in (2, 3)$,

Rek8. Based on Pohozaev identity (need $V \in C^1$) and Ruiz(2006)

$$\int |u|^3 \leq \frac{1}{2} \int \left(|\nabla u|^2 + \phi_u u^2 \right),$$

Liu & Mosconi(2020) studied the case that $g(t) \approx |t|^{\mu-2} t$ and obtained

- two nontrivial solutions if $\mu \in (2, 3)$,
- one nontrivial solution if $\mu \in (3, 4]$.

Rek8. Based on Pohozaev identity (need $V \in C^1$) and Ruiz(2006)

$$\int |u|^3 \leq \frac{1}{2} \int \left(|\nabla u|^2 + \phi_u u^2 \right),$$

Liu & Mosconi(2020) studied the case that $g(t) \approx |t|^{\mu-2} t$ and obtained

- two nontrivial solutions if $\mu \in (2, 3)$,
- one nontrivial solution if $\mu \in (3, 4]$.

Rek9. Motivated by Liu & Mosconi(2020), Jiang & Liu(2022) got two nontrivial solutions for

$$-\left(1 + \int |\nabla u|^2\right) \Delta u + V(x) = g(u), \qquad u \in H^1(\mathbb{R}^3),$$

Rek8. Based on Pohozaev identity (need $V \in C^1$) and Ruiz(2006)

$$\int |u|^3 \le \frac{1}{2} \int \left(|\nabla u|^2 + \phi_u u^2 \right),$$

Liu & Mosconi(2020) studied the case that $g(t) \approx |t|^{\mu-2}\, t$ and obtained

- two nontrivial solutions if $\mu \in (2,3)$,
- one nontrivial solution if $\mu \in (3,4]$.

Rek9. Motivated by Liu & Mosconi(2020), Jiang & Liu(2022) got two nontrivial solutions for

$$-\left(1 + \int |\nabla u|^2\right) \Delta u + V(x) = g(u), \qquad u \in H^1(\mathbb{R}^3),$$

where

$$\lim_{t \to 0} \frac{g(t)}{t} = 0, \qquad \lim_{|t| \to \infty} \frac{g(t)}{t^2} = 0;$$

Rek8. Based on Pohozaev identity (need $V \in C^1$) and Ruiz(2006)

$$\int |u|^3 \le \frac{1}{2} \int \left(|\nabla u|^2 + \phi_u u^2\right),$$

Liu & Mosconi(2020) studied the case that $g(t) \approx |t|^{\mu-2}\, t$ and obtained

- two nontrivial solutions if $\mu \in (2, 3)$,
- one nontrivial solution if $\mu \in (3, 4]$.

Rek9. Motivated by Liu & Mosconi(2020), Jiang & Liu(2022) got two non-trivial solutions for

$$-\left(1 + \int |\nabla u|^2\right) \Delta u + V(x) = g(u), \qquad u \in H^1(\mathbb{R}^3),$$

where

$$\lim_{t\to 0} \frac{g(t)}{t} = 0, \qquad \lim_{|t|\to\infty} \frac{g(t)}{t^2} = 0;$$

and for some $\gamma > 1$, $V(x) \ge \alpha\, |x|^\gamma$ for $|x| \gg 1$,

Rek8. Based on Pohozaev identity (need $V \in C^1$) and Ruiz(2006)

$$\int |u|^3 \le \frac{1}{2} \int \left(|\nabla u|^2 + \phi_u u^2\right),$$

Liu & Mosconi(2020) studied the case that $g(t) \approx |t|^{\mu-2}\, t$ and obtained

- two nontrivial solutions if $\mu \in (2,3)$,
- one nontrivial solution if $\mu \in (3,4]$.

Rek9. Motivated by Liu & Mosconi(2020), Jiang & Liu(2022) got two nontrivial solutions for

$$-\left(1 + \int |\nabla u|^2\right) \Delta u + V(x) = g(u), \qquad u \in H^1(\mathbb{R}^3),$$

where

$$\lim_{t\to 0} \frac{g(t)}{t} = 0, \qquad \lim_{|t|\to\infty} \frac{g(t)}{t^2} = 0;$$

and for some $\gamma > 1$, $V(x) \ge \alpha\, |x|^\gamma$ for $|x| \gg 1$, so that $X \hookrightarrow L^{3/2}$ and

$$c \int |u|^3 \le \left(\int |\nabla u|^2\right)^2 + \|u\|^2.$$

6. Quasilinear Schrödinger equations

The quasilinear Schrödinger equation

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad \text{in } H^1(\mathbb{R}^N).$$

6. Quasilinear Schrödinger equations

The quasilinear Schrödinger equation

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad \text{in } H^1(\mathbb{R}^N). \tag{9}$$

is the Euler-Lagrange eqn for

$$J(u) = \frac{1}{2}\int \left(1+2u^2\right)|\nabla u|^2 + \frac{1}{2}\int V(x)u^2 - \int G(u),$$

but J could not be defined on all of $H^1(\mathbb{R}^N)$.

6. Quasilinear Schrödinger equations

The quasilinear Schrödinger equation

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad \text{in } H^1(\mathbb{R}^N). \tag{9}$$

is the Euler-Lagrange eqn for

$$J(u) = \frac{1}{2}\int \left(1+2u^2\right)|\nabla u|^2 + \frac{1}{2}\int V(x)u^2 - \int G(u),$$

but J could not be defined on all of $H^1(\mathbb{R}^N)$.

Liu et al.(2003), Colin & Jeanjean(2004) introduced transformation f (11) s.t. if $v \in H^1(\mathbb{R}^N)$ is critical for $\Phi : H^1 \to \mathbb{R}$,

$$\Phi(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x)f^2(v)\right) - \int G(f(v)),$$

then $u = f(v)$ is solution for (9).

6. Quasilinear Schrödinger equations

The quasilinear Schrödinger equation

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad \text{in } H^1(\mathbb{R}^N). \tag{9}$$

is the Euler-Lagrange eqn for

$$J(u) = \frac{1}{2}\int \left(1+2u^2\right)|\nabla u|^2 + \frac{1}{2}\int V(x)u^2 - \int G(u),$$

but J could not be defined on all of $H^1(\mathbb{R}^N)$.
Liu et al.(2003), Colin & Jeanjean(2004) introduced transformation f (11) s.t. if $v \in H^1(\mathbb{R}^N)$ is critical for $\Phi : H^1 \to \mathbb{R}$,

$$\Phi(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x)f^2(v)\right) - \int G(f(v)),$$

then $u = f(v)$ is solution for (9).
Since then many results appear,

- all require $v = 0$ is loc min for Φ (e.g., $\inf_{\mathbb{R}^N} V > 0$),

6. Quasilinear Schrödinger equations

The quasilinear Schrödinger equation

$$-\Delta u + V(x)u - u\Delta(u^2) = g(u), \qquad \text{in } H^1(\mathbb{R}^N). \tag{9}$$

is the Euler-Lagrange eqn for

$$J(u) = \frac{1}{2}\int \left(1 + 2u^2\right)|\nabla u|^2 + \frac{1}{2}\int V(x)u^2 - \int G(u),$$

but J could not be defined on all of $H^1(\mathbb{R}^N)$.

Liu et al.(2003), Colin & Jeanjean(2004) introduced transformation f (11) s.t. if $v \in H^1(\mathbb{R}^N)$ is critical for $\Phi : H^1 \to \mathbb{R}$,

$$\Phi(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x)f^2(v)\right) - \int G(f(v)),$$

then $u = f(v)$ is solution for (9).

Since then many results appear,

- all require $v = 0$ is loc min for Φ (e.g., $\inf_{\mathbb{R}^N} V > 0$),

Then MPT applies.

We consider the case that $v = 0$ **fails to be** a loc min of Φ.

Unlike in semilinear problems (4), the principle part of Φ,

$$Q(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x)f^2(v)\right)$$

is not a quadratic form on v, linking theorem is not applicable:

Unlike in semilinear problems (4), the principle part of Φ,

$$Q(v) = \frac{1}{2} \int \left(|\nabla v|^2 + V(x) f^2(v) \right)$$

is not a quadratic form on v, linking theorem is not applicable: We don't know how to decompose space.

Unlike in semilinear problems (4), the principle part of Φ,

$$Q(v) = \frac{1}{2} \int \left(|\nabla v|^2 + V(x) f^2(v) \right)$$

is not a quadratic form on v, linking theorem is not applicable: We don't know how to decompose space.

Our crucial observation is that Φ still has a loc link at $v = 0$.

Unlike in semilinear problems (4), the principle part of Φ,

$$Q(v) = \frac{1}{2} \int \left(|\nabla v|^2 + V(x) f^2(v) \right)$$

is not a quadratic form on v, linking theorem is not applicable: We don't know how to decompose space.

Our crucial observation is that Φ still has a loc link at $v = 0$.

Thm11 (Liu & Zhou(2018)). Assume

$$(V) \quad V \in C(\mathbb{R}^3), \lim_{|x| \to \infty} V(x) = +\infty.$$

$$(g_0) \quad g \in C(\mathbb{R}^N),\ g(t) = o(t) \text{ as } t \to 0,\ \exists p \in (4, 2 \cdot 2^*),$$
$$|g(t)| \leq C \left(|t| + |t|^{p-1} \right).$$

Unlike in semilinear problems (4), the principle part of Φ,

$$Q(v) = \frac{1}{2} \int \left(|\nabla v|^2 + V(x) f^2(v) \right)$$

is not a quadratic form on v, linking theorem is not applicable: We don't know how to decompose space.

Our crucial observation is that Φ still has a loc link at $v = 0$.

Thm11 (Liu & Zhou(2018)). Assume

(V) $V \in C(\mathbb{R}^3)$, $\lim_{|x| \to \infty} V(x) = +\infty$.

(g_0) $g \in C(\mathbb{R}^N)$, $g(t) = o(t)$ as $t \to 0$, $\exists p \in (4, 2 \cdot 2^*)$,

$$|g(t)| \le C \left(|t| + |t|^{p-1} \right).$$

(g_1) $\exists \mu > 4$, s.t. $0 < \mu G(t) \le t g(t)$ for $t \ne 0$. (roughly $g(t) \approx |t|^{\mu-2} t$)

Unlike in semilinear problems (4), the principle part of Φ,

$$Q(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x) f^2(v)\right)$$

is not a quadratic form on v, linking theorem is not applicable: We don't know how to decompose space.

Our crucial observation is that Φ still has a loc link at $v = 0$.

Thm11 (Liu & Zhou(2018)). Assume

(V) $V \in C(\mathbb{R}^3)$, $\lim_{|x|\to\infty} V(x) = +\infty$.

(g_0) $g \in C(\mathbb{R}^N)$, $g(t) = o(t)$ as $t \to 0$, $\exists p \in (4, 2\cdot 2^*)$,

$$|g(t)| \le C\left(|t| + |t|^{p-1}\right).$$

(g_1) $\exists \mu > 4$, s.t. $0 < \mu G(t) \le t g(t)$ for $t \ne 0$. (roughly $g(t) \approx |t|^{\mu-2}t$)

If 0 is not eigenvalue for $-\Delta u + V(x)u = \lambda u$, then (9) has a nontrivial solution.

Unlike in semilinear problems (4), the principle part of Φ,

$$Q(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x) f^2(v)\right)$$

is not a quadratic form on v, linking theorem is not applicable: We don't know how to decompose space.

Our crucial observation is that Φ still has a loc link at $v = 0$.

Thm11 (Liu & Zhou(2018)). Assume

(V) $V \in C(\mathbb{R}^3)$, $\lim_{|x|\to\infty} V(x) = +\infty$.

(g_0) $g \in C(\mathbb{R}^N)$, $g(t) = o(t)$ as $t \to 0$, $\exists p \in (4, 2\cdot 2^*)$,

$$|g(t)| \le C\left(|t| + |t|^{p-1}\right).$$

(g_1) $\exists \mu > 4$, s.t. $0 < \mu G(t) \le t g(t)$ for $t \ne 0$. (roughly $g(t) \approx |t|^{\mu-2}\, t$)

If 0 is not eigenvalue for $-\Delta u + V(x) u = \lambda u$, then (9) has a nontrivial solution.

Pf. The idea is similar to Thm10. We only verify local linking here.

Because f'' is bounded, $f(0)=0$, $f'(0)=1$, we have $Q \in C^2(X)$, $Q'(0)=0$,

$$\langle Q''(0)\phi, \psi\rangle = \int (\nabla\phi \cdot \nabla\psi + V(x)\phi\psi)\,. \qquad Q(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x) f^2(v)\right)$$

Because f'' is bounded, $f(0)=0$, $f'(0)=1$, we have $Q \in C^2(X)$, $Q'(0)=0$,

$$\langle Q''(0)\phi, \psi\rangle = \int (\nabla\phi \cdot \nabla\psi + V(x)\phi\psi)\,. \qquad Q(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x) f^2(v)\right)$$

Hence by Taylor, as $\|v\| \to 0$,

$$Q(v) = Q(0) + Q'(0)v + \tfrac{1}{2}\langle Q''(0)v, v\rangle + o\left(\|v\|^2\right)$$

Because f'' is bounded, $f(0)=0$, $f'(0)=1$, we have $Q \in C^2(X)$, $Q'(0)=0$,

$$\langle Q''(0)\phi, \psi\rangle = \int (\nabla\phi\cdot\nabla\psi + V(x)\phi\psi). \qquad Q(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x)f^2(v)\right)$$

Hence by Taylor, as $\|v\| \to 0$,

$$Q(v) = Q(0) + Q'(0)v + \tfrac{1}{2}\langle Q''(0)v, v\rangle + o\left(\|v\|^2\right)$$
$$= \frac{1}{2}\int \left(|\nabla v|^2 + V(x)v^2\right) + o\left(\|v\|^2\right),$$

Because f'' is bounded, $f(0)=0$, $f'(0)=1$, we have $Q \in C^2(X)$, $Q'(0)=0$,

$$\langle Q''(0)\phi, \psi \rangle = \int (\nabla\phi \cdot \nabla\psi + V(x)\phi\psi). \qquad Q(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x) f^2(v)\right)$$

Hence by Taylor, as $\|v\| \to 0$,

$$\begin{aligned} Q(v) &= Q(0) + Q'(0)v + \tfrac{1}{2}\langle Q''(0)v, v\rangle + o\left(\|v\|^2\right) \\ &= \frac{1}{2}\int \left(|\nabla v|^2 + V(x)v^2\right) + o\left(\|v\|^2\right), \end{aligned}$$

$$\Phi(v) = Q(v) - \int G(f(v)) = Q(v) + o\left(\|v\|^2\right)$$

Because f'' is bounded, $f(0)=0$, $f'(0)=1$, we have $Q \in C^2(X)$, $Q'(0)=0$,

$$\langle Q''(0)\phi, \psi\rangle = \int (\nabla\phi\cdot\nabla\psi + V(x)\phi\psi). \qquad Q(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V(x)f^2(v)\right)$$

Hence by Taylor, as $\|v\| \to 0$,

$$\begin{aligned} Q(v) &= Q(0) + Q'(0)v + \tfrac{1}{2}\left\langle Q''(0)v, v\right\rangle + o\left(\|v\|^2\right) \\ &= \frac{1}{2}\int \left(|\nabla v|^2 + V(x)v^2\right) + o\left(\|v\|^2\right), \\ \Phi(v) &= Q(v) - \int G(f(v)) = Q(v) + o\left(\|v\|^2\right) \\ &= \frac{1}{2}\int \left(|\nabla v|^2 + V(x)v^2\right) + o\left(\|v\|^2\right). \end{aligned}$$

Because f'' is bounded, $f(0)=0$, $f'(0)=1$, we have $Q\in C^2(X)$, $Q'(0)=0$,

$$\langle Q''(0)\phi,\psi\rangle = \int (\nabla\phi\cdot\nabla\psi + V(x)\phi\psi)\,. \qquad Q(v)=\frac{1}{2}\int\left(|\nabla v|^2+V(x)f^2(v)\right)$$

Hence by Taylor, as $\|v\|\to 0$,

$$\begin{aligned}
Q(v) &= Q(0)+Q'(0)v+\tfrac{1}{2}\left\langle Q''(0)v,v\right\rangle + o\left(\|v\|^2\right)\\
&= \frac{1}{2}\int\left(|\nabla v|^2+V(x)v^2\right)+o\left(\|v\|^2\right),\\
\Phi(v) &= Q(v)-\int G(f(v)) = Q(v)+o\left(\|v\|^2\right)\\
&= \frac{1}{2}\int\left(|\nabla v|^2+V(x)v^2\right)+o\left(\|v\|^2\right).
\end{aligned}$$

Thus Φ as a loc link w.r.p. $X=X^-\oplus X^+$ and $C_\ell(\Phi,0)\neq 0$.

Because f'' is bounded, $f(0)=0$, $f'(0)=1$, we have $Q\in C^2(X)$, $Q'(0)=0$,

$$\langle Q''(0)\phi,\psi\rangle=\int(\nabla\phi\cdot\nabla\psi+V(x)\phi\psi). \qquad Q(v)=\frac{1}{2}\int\left(|\nabla v|^2+V(x)f^2(v)\right)$$

Hence by Taylor, as $\|v\|\to 0$,

$$\begin{aligned}
Q(v)&=Q(0)+Q'(0)v+\tfrac{1}{2}\langle Q''(0)v,v\rangle+o\left(\|v\|^2\right)\\
&=\frac{1}{2}\int\left(|\nabla v|^2+V(x)v^2\right)+o\left(\|v\|^2\right),\\
\Phi(v)&=Q(v)-\int G(f(v))=Q(v)+o\left(\|v\|^2\right)\\
&=\frac{1}{2}\int\left(|\nabla v|^2+V(x)v^2\right)+o\left(\|v\|^2\right).
\end{aligned}$$

Thus Φ as a loc link w.r.p. $X=X^-\oplus X^+$ and $C_\ell(\Phi,0)\neq 0$.

Rek10. Yin & Liu(2023) got related results for

$$-\Delta u+V(x)u-\frac{u}{2\sqrt{1+u^2}}\Delta\sqrt{1+u^2}=g(u), \qquad u\in H^1(\mathbb{R}^N),$$

where $0\le 4G(t)\le tg(t)$, thus $g(t)=t^3\ln(1+|t|)$ is allowed. Thm11

7. QSE: ∞ solutions in $\mathbb{R}^N$

Let $1 < q < 2 < s < \infty$, consider

$$\begin{cases} -\Delta u - u\Delta(u^2) = k(x)\,|u|^{q-2}\,u - h(x)\,|u|^{s-2}\,u, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases}$$

7. QSE: ∞ solutions in $\mathbb{R}^N$

Let $1 < q < 2 < s < \infty$, consider

$$\begin{cases} -\Delta u - u\Delta(u^2) = k(x)\,|u|^{q-2}\,u - h(x)\,|u|^{s-2}\,u, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{10}$$

As the last section, we should find critical points of

$$\Phi(v) = J(f(v)) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{q}\int k\,|f(v)|^q + \frac{1}{s}\int h\,|f(v)|^s\,.$$

7. QSE: ∞ solutions in $\mathbb{R}^N$

Let $1 < q < 2 < s < \infty$, consider

$$\begin{cases} -\Delta u - u\Delta(u^2) = k(x)\,|u|^{q-2}\,u - h(x)\,|u|^{s-2}\,u, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{10}$$

As the last section, we should find critical points of

$$\Phi(v) = J(f(v)) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{q}\int k\,|f(v)|^q + \frac{1}{s}\int h\,|f(v)|^s\,.$$

Elliptic problems involving concave and convex terms

- great attention since Ambrosetti et al.(1994) and Bartsch & Willem(1995) on semilinear problems on bounded domain.

7. QSE: ∞ solutions in $\mathbb{R}^N$

Let $1 < q < 2 < s < \infty$, consider

$$\begin{cases} -\Delta u - u\Delta(u^2) = k(x)\,|u|^{q-2}\,u - h(x)\,|u|^{s-2}\,u, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{10}$$

As the last section, we should find critical points of

$$\Phi(v) = J(f(v)) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{q}\int k\,|f(v)|^q + \frac{1}{s}\int h\,|f(v)|^s.$$

Elliptic problems involving concave and convex terms

- great attention since Ambrosetti et al.(1994) and Bartsch & Willem(1995) on semilinear problems on bounded domain.
- relatively less for quasilinear Schrödinger equations, do Ó & Severo(2009) is the first in this direction (Santos & Santos Júnior(2019) more recent).

7. QSE: ∞ solutions in $\mathbb{R}^N$

Let $1 < q < 2 < s < \infty$, consider

$$\begin{cases} -\Delta u - u\Delta(u^2) = k(x)\,|u|^{q-2}\,u - h(x)\,|u|^{s-2}\,u, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{10}$$

As the last section, we should find critical points of

$$\Phi(v) = J(f(v)) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{q}\int k\,|f(v)|^q + \frac{1}{s}\int h\,|f(v)|^s\,.$$

Elliptic problems involving concave and convex terms

- great attention since Ambrosetti et al.(1994) and Bartsch & Willem(1995) on semilinear problems on bounded domain.
- relatively less for quasilinear Schrödinger equations, do Ó & Severo(2009) is the first in this direction (Santos & Santos Júnior(2019) more recent).

These and most papers on quasilinear Schrödinger equations require

$$|g(x,u)| \le C\left(1 + |u|^{2\cdot 2^*-1}\right), \qquad \text{(under critical)}$$

here $2^* = 2N/(N-2)$. We allow supercritical.

For supercritical problems (Figueiredo et al.(2015), Liu(2016)),

- modify $g(x, u)$ subcritically for $|u|$ large,

For supercritical problems (Figueiredo et al.(2015), Liu(2016)),

- modify $g(x,u)$ subcritically for $|u|$ large,
- get solutions for the problem obtained via variational methods,

For supercritical problems (Figueiredo et al.(2015), Liu(2016)),

- modify $g(x, u)$ subcritically for $|u|$ large,
- get solutions for the problem obtained via variational methods,
- solutions of truncated problem have small L^∞-norm, thus are solutions of the original problem.

For supercritical problems (Figueiredo et al.(2015), Liu(2016)),

- modify $g(x,u)$ subcritically for $|u|$ large,
- get solutions for the problem obtained via variational methods,
- solutions of truncated problem have small L^∞-norm, thus are solutions of the original problem.

Our approach (motivated by Liu & Li(2003b)) does not require truncation and L^∞-estimate, geometric properties of f play essential role. Let

For supercritical problems (Figueiredo et al.(2015), Liu(2016)),

- modify $g(x,u)$ subcritically for $|u|$ large,
- get solutions for the problem obtained via variational methods,
- solutions of truncated problem have small L^∞-norm, thus are solutions of the original problem.

Our approach (motivated by Liu & Li(2003b)) does not require truncation and L^∞-estimate, geometric properties of f play essential role. Let

$$p_0 = \frac{2N}{2N - p(N-2)}, \qquad p' = \frac{p}{p-1}.$$

For supercritical problems (Figueiredo et al.(2015), Liu(2016)),

- modify $g(x,u)$ subcritically for $|u|$ large,
- get solutions for the problem obtained via variational methods,
- solutions of truncated problem have small L^∞-norm, thus are solutions of the original problem.

Our approach (motivated by Liu & Li(2003b)) does not require truncation and L^∞-estimate, geometric properties of f play essential role. Let

$$p_0 = \frac{2N}{2N - p(N-2)}, \qquad p' = \frac{p}{p-1}.$$

Thm12 (Liu & Yin(2023)). Assume

(k) $k \in L^{q_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $k \geq 0$, $k \not\equiv 0$,

For supercritical problems (Figueiredo et al.(2015), Liu(2016)),

- modify $g(x,u)$ subcritically for $|u|$ large,
- get solutions for the problem obtained via variational methods,
- solutions of truncated problem have small L^∞-norm, thus are solutions of the original problem.

Our approach (motivated by Liu & Li(2003b)) does not require truncation and L^∞-estimate, geometric properties of f play essential role. Let

$$p_0 = \frac{2N}{2N - p(N-2)}, \qquad p' = \frac{p}{p-1}.$$

Thm12 (Liu & Yin(2023)). Assume

(k) $k \in L^{q_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $k \geq 0$, $k \not\equiv 0$,

(h) $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $h \geq 0$,

then then (10) has solutions u_n s.t. $J(u_n) < 0$ and $J(u_n) \to 0$ as $n \to \infty$.

Thm13 (Liu & Yin(2023)). Assume (k) and (h), then (10) has a nonegative solution u s.t. $J(u) < 0$.

Thm13 (Liu & Yin(2023)). Assume (k) and (h), then (10) has a nonegative solution u s.t. $J(u) < 0$.

Rek11. Thm 13 is closely related to Miyagaki & Moreira(2015), where for $4 \le q < s < \infty$, problem

$$-\Delta u - \Delta(u^2) = \lambda u + k(x)\,|u|^{q-2}\,u - h(x)\,|u|^{s-2}\,u, \qquad u \in H_0^1(\Omega)$$

on a bounded domain Ω is considered.

Thm13 (Liu & Yin(2023)). Assume (k) and (h), then (10) has a nonegative solution u s.t. $J(u) < 0$.

Rek11. Thm 13 is closely related to Miyagaki & Moreira(2015), where for $4 \le q < s < \infty$, problem

$$-\Delta u - \Delta(u^2) = \lambda u + k(x)\,|u|^{q-2}\,u - h(x)\,|u|^{s-2}\,u, \qquad u \in H_0^1(\Omega)$$

on a bounded domain Ω is considered.

Following Colin & Jeanjean(2004), Liu et al.(2003), let f be the odd function defined by

$$f'(t) = \frac{1}{\sqrt{1+2f^2(t)}}, \qquad f(0) = 0 \tag{11}$$

on $[0, +\infty)$.

Thm13 (Liu & Yin(2023)). Assume (k) and (h), then (10) has a nonegative solution u s.t. $J(u) < 0$.

Rek11. Thm 13 is closely related to Miyagaki & Moreira(2015), where for $4 \le q < s < \infty$, problem

$$-\Delta u - \Delta(u^2) = \lambda u + k(x)\,|u|^{q-2}\,u - h(x)\,|u|^{s-2}\,u, \qquad u \in H_0^1(\Omega)$$

on a bounded domain Ω is considered.

Following Colin & Jeanjean(2004), Liu et al.(2003), let f be the odd function defined by

$$f'(t) = \frac{1}{\sqrt{1+2f^2(t)}}, \qquad f(0) = 0 \tag{11}$$

on $[0, +\infty)$.

Pro6. The function f possesses the following properties:

(1) $f \in C^\infty(\mathbb{R})$ is strictly increasing, therefore is invertible.

(2) $|f(t)| \le |t|$, $f'(0) = 1$, $\left|f'(t)\right| \le 1$ for all $t \in \mathbb{R}$.

(3) $\left|f(t)f'(t)\right| \le 1$, $|f(t)| \le 2^{1/4}|t|^{1/2}$.

(4) There exists a positive constant μ such that

$$|f(t)| \ge \mu|t| \quad \text{for } |t| \le 1, \qquad |f(t)| \ge \mu|t|^{1/2} \quad \text{for } |t| \ge 1. \tag{12}$$

(5) For all $t \in \mathbb{R}$ we have $f^2(t) \ge f(t)f'(t)t \ge \frac{1}{2}f^2(t)$.

(3) $\left|f(t)f'(t)\right| \le 1$, $|f(t)| \le 2^{1/4}|t|^{1/2}$.

(4) There exists a positive constant μ such that

$$|f(t)| \ge \mu|t| \quad \text{for } |t| \le 1, \qquad |f(t)| \ge \mu|t|^{1/2} \quad \text{for } |t| \ge 1. \tag{12}$$

(5) For all $t \in \mathbb{R}$ we have $f^2(t) \ge f(t)f'(t)t \ge \frac{1}{2}f^2(t)$.

Let E be the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\begin{aligned} \|v\| &= \|v\|_D + |h^{2/s}v|_{s/2} \\ &= \left(\int |\nabla v|^2\right)^{1/2} + \left(\int h|v|^{s/2}\right)^{2/s}, \end{aligned} \tag{13}$$

where $\|\cdot\|_D$ and $|\cdot|_p$ are the standard $D^{1,2}$-norm and L^p-norm ($p \in [1,\infty]$).

(3) $\left|f(t)f'(t)\right| \leq 1$, $|f(t)| \leq 2^{1/4}|t|^{1/2}$.

(4) There exists a positive constant μ such that

$$|f(t)| \geq \mu|t| \quad \text{for } |t| \leq 1, \qquad |f(t)| \geq \mu|t|^{1/2} \quad \text{for } |t| \geq 1. \tag{12}$$

(5) For all $t \in \mathbb{R}$ we have $f^2(t) \geq f(t)f'(t)t \geq \frac{1}{2}f^2(t)$.

Let E be the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\begin{aligned} \|v\| &= \|v\|_D + |h^{2/s}v|_{s/2} \\ &= \left(\int |\nabla v|^2\right)^{1/2} + \left(\int h|v|^{s/2}\right)^{2/s}, \end{aligned} \tag{13}$$

where $\|\cdot\|_D$ and $|\cdot|_p$ are the standard $D^{1,2}$-norm and L^p-norm ($p \in [1, \infty]$).

Lem1. If $v \in E$ and $\phi \in C_0^\infty(\mathbb{R}^N)$, then

$$\xi = \frac{\phi}{f'(v)} = \sqrt{1 + 2f^2(v)}\phi \tag{14}$$

belongs to E.

Under our assumptions on k and h, the functional $\Phi : E \to \mathbb{R}$

$$\Phi(v) = J(f(v)) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{q}\int k\,|f(v)|^q + \frac{1}{s}\int h\,|f(v)|^s$$

is C^1.

Under our assumptions on k and h, the functional $\Phi : E \to \mathbb{R}$

$$\Phi(v) = J(f(v)) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{q}\int k\,|f(v)|^q + \frac{1}{s}\int h\,|f(v)|^s$$

is C^1. If $\Phi'(v) = 0$, by Lem 1 for $\phi \in C_0^\infty(\mathbb{R}^N)$ we have $\xi = \phi/f'(v) \in E$,

Under our assumptions on k and h, the functional $\Phi : E \to \mathbb{R}$

$$\Phi(v) = J(f(v)) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{q}\int k\,|f(v)|^q + \frac{1}{s}\int h\,|f(v)|^s$$

is C^1. If $\Phi'(v) = 0$, by Lem 1 for $\phi \in C_0^\infty(\mathbb{R}^N)$ we have $\xi = \phi/f'(v) \in E$, hence $\langle \Phi'(v), \xi \rangle = 0$.

Under our assumptions on k and h, the functional $\Phi : E \to \mathbb{R}$

$$\Phi(v) = J(f(v)) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{q}\int k\,|f(v)|^q + \frac{1}{s}\int h\,|f(v)|^s$$

is C^1. If $\Phi'(v) = 0$, by Lem 1 for $\phi \in C_0^\infty(\mathbb{R}^N)$ we have $\xi = \phi/f'(v) \in E$, hence $\langle \Phi'(v), \xi \rangle = 0$. Let $u = f(v)$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} J(u + t\phi), \qquad u \text{ is a weak solution of the problem (10).}$$

Under our assumptions on k and h, the functional $\Phi : E \to \mathbb{R}$

$$\Phi(v) = J(f(v)) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{q}\int k\,|f(v)|^q + \frac{1}{s}\int h\,|f(v)|^s$$

is C^1. If $\Phi'(v) = 0$, by Lem 1 for $\phi \in C_0^\infty(\mathbb{R}^N)$ we have $\xi = \phi/f'(v) \in E$, hence $\langle \Phi'(v), \xi \rangle = 0$. Let $u = f(v)$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} J(u + t\phi), \qquad u \text{ is a weak solution of the problem (10).}$$

Pro7. Suppose $s \geq 4$. If $v \in E$ is a critical point of Φ, then $\Phi(v) \leq 0$. Thm 12

Under our assumptions on k and h, the functional $\Phi : E \to \mathbb{R}$

$$\Phi(v) = J(f(v)) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{q}\int k\,|f(v)|^q + \frac{1}{s}\int h\,|f(v)|^s$$

is C^1. If $\Phi'(v) = 0$, by Lem 1 for $\phi \in C_0^\infty(\mathbb{R}^N)$ we have $\xi = \phi/f'(v) \in E$, hence $\langle \Phi'(v), \xi \rangle = 0$. Let $u = f(v)$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} J(u + t\phi), \qquad u \text{ is a weak solution of the problem (10).}$$

Pro7. Suppose $s \geq 4$. If $v \in E$ is a critical point of Φ, then $\Phi(v) \leq 0$. Thm 12

Lem2. $\Phi : E \to \mathbb{R}$ is coercive.

Under our assumptions on k and h, the functional $\Phi : E \to \mathbb{R}$

$$\Phi(v) = J(f(v)) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{q}\int k\,|f(v)|^q + \frac{1}{s}\int h\,|f(v)|^s$$

is C^1. If $\Phi'(v) = 0$, by Lem 1 for $\phi \in C_0^\infty(\mathbb{R}^N)$ we have $\xi = \phi / f'(v) \in E$, hence $\langle \Phi'(v), \xi \rangle = 0$. Let $u = f(v)$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} J(u + t\phi), \qquad u \text{ is a weak solution of the problem (10).}$$

Pro7. Suppose $s \geq 4$. If $v \in E$ is a critical point of Φ, then $\Phi(v) \leq 0$. Thm 12

Lem2. $\Phi : E \to \mathbb{R}$ is coercive.

Pf. Let ℓ be the norm of $D^{1,2} \hookrightarrow L^{2^*}$,

$$\|v_n\| = \|v_n\|_D + |h^{2/s} v_n|_{s/2} \to +\infty.$$

Under our assumptions on k and h, the functional $\Phi : E \to \mathbb{R}$

$$\Phi(v) = J(f(v)) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{q}\int k\,|f(v)|^q + \frac{1}{s}\int h\,|f(v)|^s$$

is C^1. If $\Phi'(v) = 0$, by Lem 1 for $\phi \in C_0^\infty(\mathbb{R}^N)$ we have $\xi = \phi/f'(v) \in E$, hence $\langle \Phi'(v), \xi \rangle = 0$. Let $u = f(v)$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} J(u + t\phi), \qquad u \text{ is a weak solution of the problem (10).}$$

Pro7. Suppose $s \geq 4$. If $v \in E$ is a critical point of Φ, then $\Phi(v) \leq 0$. Thm 12

Lem2. $\Phi : E \to \mathbb{R}$ is coercive.

Pf. Let ℓ be the norm of $D^{1,2} \hookrightarrow L^{2^*}$,

$$\|v_n\| = \|v_n\|_D + |h^{2/s} v_n|_{s/2} \to +\infty.$$

If $\|v_n\|_D \to \infty$, noting $q < 2$ we easy have

$$\Phi(v_n) = \tfrac{1}{2}\int |\nabla v_n|^2 - \tfrac{1}{q}\int k|v_n|^q + \tfrac{1}{s}\int h|f(v_n)|^s$$

Under our assumptions on k and h, the functional $\Phi : E \to \mathbb{R}$

$$\Phi(v) = J(f(v)) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{q}\int k\,|f(v)|^q + \frac{1}{s}\int h\,|f(v)|^s$$

is C^1. If $\Phi'(v) = 0$, by Lem 1 for $\phi \in C_0^\infty(\mathbb{R}^N)$ we have $\xi = \phi/f'(v) \in E$, hence $\langle \Phi'(v), \xi \rangle = 0$. Let $u = f(v)$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} J(u + t\phi),$$

u is a weak solution of the problem (10).

Pro7. Suppose $s \geq 4$. If $v \in E$ is a critical point of Φ, then $\Phi(v) \leq 0$. Thm 12

Lem2. $\Phi : E \to \mathbb{R}$ is coercive.

Pf. Let ℓ be the norm of $D^{1,2} \hookrightarrow L^{2^*}$,

$$\|v_n\| = \|v_n\|_D + |h^{2/s} v_n|_{s/2} \to +\infty.$$

If $\|v_n\|_D \to \infty$, noting $q < 2$ we easy have

$$\Phi(v_n) = \frac{1}{2}\int |\nabla v_n|^2 - \frac{1}{q}\int k|v_n|^q + \frac{1}{s}\int h|f(v_n)|^s$$

$$\geq \frac{1}{2}\|v_n\|_D^2 - \frac{1}{q}\ell^q\,|k|_{q_0}\,\|v_n\|_D^q \to +\infty,$$

If $\|v_n\|_D \ll \infty$, using (12) and $h \in L^1$ we get

$$\int h\,|f(v_n)|^s = \int_{|v_n|\le 1} h\,|f(v_n)|^s + \int_{v_n>1} h\,|f(v_n)|^s$$

If $\|v_n\|_D \ll \infty$, using (12) and $h \in L^1$ we get

$$\int h\,|f(v_n)|^s = \int_{|v_n|\leq 1} h\,|f(v_n)|^s + \int_{v_n>1} h\,|f(v_n)|^s$$

$$\geq \mu \int_{v_n>1} h\,|v_n|^{s/2}$$

If $\|v_n\|_D \ll \infty$, using (12) and $h \in L^1$ we get

$$\int h\,|f(v_n)|^s = \int_{|v_n|\le 1} h\,|f(v_n)|^s + \int_{v_n>1} h\,|f(v_n)|^s$$

$$\ge \mu \int_{v_n>1} h\,|v_n|^{s/2} = \mu \int h\,|v_n|^{2/s} - \mu \int_{|v_n|\le 1} h\,|v_n|^{2/s}$$

If $\|v_n\|_D \ll \infty$, using (12) and $h \in L^1$ we get

$$\int h\,|f(v_n)|^s = \int_{|v_n|\le 1} h\,|f(v_n)|^s + \int_{v_n>1} h\,|f(v_n)|^s$$

$$\ge \mu \int_{v_n>1} h\,|v_n|^{s/2} = \mu \int h\,|v_n|^{2/s} - \mu \int_{|v_n|\le 1} h\,|v_n|^{2/s}$$

$$\ge \mu \int h\,|v_n|^{2/s} - \mu\,|h|_1 \to +\infty.$$

If $\|v_n\|_D \ll \infty$, using (12) and $h \in L^1$ we get

$$\begin{aligned}\int h\,|f(v_n)|^s &= \int_{|v_n|\le 1} h\,|f(v_n)|^s + \int_{v_n>1} h\,|f(v_n)|^s \\ &\ge \mu \int_{v_n>1} h\,|v_n|^{s/2} = \mu \int h\,|v_n|^{2/s} - \mu \int_{|v_n|\le 1} h\,|v_n|^{2/s} \\ &\ge \mu \int h\,|v_n|^{2/s} - \mu\,|h|_1 \to +\infty.\end{aligned}$$

Thus

$$\begin{aligned}\Phi(v_n) \ge \frac{1}{2}\|v_n\|_D^2 &- \frac{1}{q}\ell^q\,|k|_{q_0}\,\|v_n\|_D^q \\ &+ \frac{1}{s}\int h\,|f(v_n)|^s \to +\infty.\end{aligned}$$

If $\|v_n\|_D \ll \infty$, using (12) and $h \in L^1$ we get

$$\begin{aligned}\int h\,|f(v_n)|^s &= \int_{|v_n|\le 1} h\,|f(v_n)|^s + \int_{v_n>1} h\,|f(v_n)|^s \\ &\ge \mu \int_{v_n>1} h\,|v_n|^{s/2} = \mu \int h\,|v_n|^{2/s} - \mu \int_{|v_n|\le 1} h\,|v_n|^{2/s} \\ &\ge \mu \int h\,|v_n|^{2/s} - \mu\,|h|_1 \to +\infty.\end{aligned}$$

Thus

$$\begin{aligned}\Phi(v_n) \ge \frac{1}{2}\|v_n\|_D^2 - \frac{1}{q}\ell^q\,|k|_{q_0}\,\|v_n\|_D^q \\ + \frac{1}{s}\int h\,|f(v_n)|^s \to +\infty.\end{aligned}$$

Lem3. Given $\alpha \in \mathbb{R}$, the function $\eta : \mathbb{R} \to \mathbb{R}$, $\eta(t) = |f(t)|^s$, is convex. Hence for $\alpha, \beta \in \mathbb{R}$ we have

$$|f(\alpha)|^s \le |f(\beta)|^s + s\,|f(\alpha)|^{s-2} f(\alpha) f'(\alpha)\,(\alpha - \beta)\,. \tag{15}$$

Lem4. Φ satisfies the Palais-Smale condition.

Lem4. Φ satisfies the Palais-Smale condition.

Pf. Any (PS) sequence $\{v_n\} \subset E$ is bounded. Up to a subsequence

$$v_n \rightharpoonup v \quad \text{in } D^{1,2}(\mathbb{R}^N), \qquad h^{2/s} v_n \rightharpoonup h^{2/s} v \quad \text{in } L^{s/2}(\mathbb{R}^N).$$

Lem4. Φ satisfies the Palais-Smale condition.

Pf. Any (PS) sequence $\{v_n\} \subset E$ is bounded. Up to a subsequence

$$v_n \rightharpoonup v \quad \text{in } D^{1,2}(\mathbb{R}^N), \qquad h^{2/s} v_n \rightharpoonup h^{2/s} v \quad \text{in } L^{s/2}(\mathbb{R}^N). \tag{16}$$

By do Ó(1997), $\psi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$,

$$\psi(v) = \int k\,|v|^q \quad \text{is weakly continuous,} \ \int k\,|v_n - v|^q \to 0.$$

Lem4. Φ satisfies the Palais-Smale condition.

Pf. Any (PS) sequence $\{v_n\} \subset E$ is bounded. Up to a subsequence

$$v_n \rightharpoonup v \quad \text{in } D^{1,2}(\mathbb{R}^N), \qquad h^{2/s} v_n \rightharpoonup h^{2/s} v \quad \text{in } L^{s/2}(\mathbb{R}^N). \tag{16}$$

By do Ó(1997), $\psi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$,

$$\psi(v) = \int k\,|v|^q \quad \text{is weakly continuous,} \int k\,|v_n - v|^q \to 0. \tag{17}$$

Is E reflexive? We could not get $v_n \rightharpoonup v$ in E and deduce

$$\langle \Phi'(v), v_n - v \rangle \to 0.$$

Lem4. Φ satisfies the Palais-Smale condition.

Pf. Any (PS) sequence $\{v_n\} \subset E$ is bounded. Up to a subsequence

$$v_n \rightharpoonup v \quad \text{in } D^{1,2}(\mathbb{R}^N), \qquad h^{2/s} v_n \rightharpoonup h^{2/s} v \quad \text{in } L^{s/2}(\mathbb{R}^N). \tag{16}$$

By do Ó(1997), $\psi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$,

$$\psi(v) = \int k\,|v|^q \quad \text{is weakly continuous,} \ \int k\,|v_n - v|^q \to 0. \tag{17}$$

Is E reflexive? We could not get $v_n \rightharpoonup v$ in E and deduce

$$\langle \Phi'(v), v_n - v \rangle \to 0. \tag{18}$$

By Hölder and (17),

$$\int k\,|f(v)|^{q-2} f(v) f'(v)\,(v_n - v) \to 0.$$

Lem4. Φ satisfies the Palais-Smale condition.

Pf. Any (PS) sequence $\{v_n\} \subset E$ is bounded. Up to a subsequence

$$v_n \rightharpoonup v \quad \text{in } D^{1,2}(\mathbb{R}^N), \qquad h^{2/s} v_n \rightharpoonup h^{2/s} v \quad \text{in } L^{s/2}(\mathbb{R}^N). \tag{16}$$

By do Ó(1997), $\psi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$,

$$\psi(v) = \int k\,|v|^q \quad \text{is weakly continuous,} \quad \int k\,|v_n - v|^q \to 0. \tag{17}$$

Is E reflexive? We could not get $v_n \rightharpoonup v$ in E and deduce

$$\langle \Phi'(v), v_n - v \rangle \to 0. \tag{18}$$

By Hölder and (17),

$$\int k\,|f(v)|^{q-2} f(v) f'(v)\,(v_n - v) \to 0. \tag{19}$$

By $h^{1-2/s}\,|f(v)|^{s-2} f(v) f'(v) \in L^{(s/2)'}(\mathbb{R}^N)$,

Lem4. Φ satisfies the Palais-Smale condition.

Pf. Any (PS) sequence $\{v_n\} \subset E$ is bounded. Up to a subsequence

$$v_n \rightharpoonup v \quad \text{in } D^{1,2}(\mathbb{R}^N), \qquad h^{2/s} v_n \rightharpoonup h^{2/s} v \quad \text{in } L^{s/2}(\mathbb{R}^N). \tag{16}$$

By do Ó(1997), $\psi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$,

$$\psi(v) = \int k\, |v|^q \quad \text{is weakly continuous,} \int k\, |v_n - v|^q \to 0. \tag{17}$$

Is E reflexive? We could not get $v_n \rightharpoonup v$ in E and deduce

$$\langle \Phi'(v), v_n - v \rangle \to 0. \tag{18}$$

By Hölder and (17),

$$\int k\, |f(v)|^{q-2} f(v) f'(v)\, (v_n - v) \to 0. \tag{19}$$

By $h^{1-2/s}\, |f(v)|^{s-2} f(v) f'(v) \in L^{(s/2)'}(\mathbb{R}^N)$, $h^{2/s} v_n \rightharpoonup h^{2/s} v$ in $L^{s/2}(\mathbb{R}^N)$ we get

$$\int h\, |f(v)|^{s-2} f(v) f'(v)\, (v_n - v) \to 0.$$

Lem4. Φ satisfies the Palais-Smale condition.

Pf. Any (PS) sequence $\{v_n\} \subset E$ is bounded. Up to a subsequence

$$v_n \rightharpoonup v \quad \text{in } D^{1,2}(\mathbb{R}^N), \qquad h^{2/s} v_n \rightharpoonup h^{2/s} v \quad \text{in } L^{s/2}(\mathbb{R}^N). \tag{16}$$

By do Ó(1997), $\psi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$,

$$\psi(v) = \int k\,|v|^q \quad \text{is weakly continuous,} \int k\,|v_n - v|^q \to 0. \tag{17}$$

Is E reflexive? We could not get $v_n \rightharpoonup v$ in E and deduce

$$\langle \Phi'(v), v_n - v \rangle \to 0. \tag{18}$$

By Hölder and (17),

$$\int k\,|f(v)|^{q-2} f(v) f'(v)\,(v_n - v) \to 0. \tag{19}$$

By $h^{1-2/s}\,|f(v)|^{s-2} f(v) f'(v) \in L^{(s/2)'}(\mathbb{R}^N)$, $h^{2/s} v_n \rightharpoonup h^{2/s} v$ in $L^{s/2}(\mathbb{R}^N)$ we get

$$\int h\,|f(v)|^{s-2} f(v) f'(v)\,(v_n - v) \to 0. \tag{20}$$

Claim (18) follows from (20), (16) and (19).

We also have

$$\int k\left(|f(v_n)|^{q-2}f(v_n)f'(v_n)-|f(v)|^{q-2}f(v)f'(v)\right)(v_n-v)\to 0,$$

We also have

$$\int k\left(|f(v_n)|^{q-2}f(v_n)f'(v_n)-|f(v)|^{q-2}f(v)f'(v)\right)(v_n-v)\to 0, \tag{21}$$

$$H_n := \int h\left(|f(v_n)|^{s-2}f(v_n)f'(v_n)-|f(v)|^{s-2}f(v)f'(v)\right)(v_n-v)\geq 0$$

because $t\mapsto s\,|f(t)|^{s-2}f(t)f'(t)$ is increasing.

We also have

$$\int k\left(|f(v_n)|^{q-2}f(v_n)f'(v_n)-|f(v)|^{q-2}f(v)f'(v)\right)(v_n-v)\to 0, \tag{21}$$

$$H_n:=\int h\left(|f(v_n)|^{s-2}f(v_n)f'(v_n)-|f(v)|^{s-2}f(v)f'(v)\right)(v_n-v)\geq 0$$

because $t\mapsto s\,|f(t)|^{s-2}f(t)f'(t)$ is increasing. By (18) and (21),

$$o(1)=\langle\Phi'(v_n)-\Phi'(v),v_n-v\rangle$$

We also have

$$\int k\left(|f(v_n)|^{q-2} f(v_n)f'(v_n) - |f(v)|^{q-2} f(v)f'(v)\right)(v_n - v) \to 0, \qquad (21)$$

$$H_n := \int h\left(|f(v_n)|^{s-2} f(v_n)f'(v_n) - |f(v)|^{s-2} f(v)f'(v)\right)(v_n - v) \geq 0$$

because $t \mapsto s\,|f(t)|^{s-2} f(t)f'(t)$ is increasing. By (18) and (21),

$$\begin{aligned} o(1) &= \langle \Phi'(v_n) - \Phi'(v), v_n - v\rangle \\ &= \int |\nabla(v_n - v)|^2 \\ &\quad - \int k\left(|f(v_n)|^{q-2} f(v_n)f'(v_n) - |f(v)|^{q-2} f(v)f'(v)\right)(v_n - v) \\ &\quad + \int h\left(|f(v_n)|^{s-2} f(v_n)f'(v_n) - |f(v)|^{s-2} f(v)f'(v)\right)(v_n - v) \end{aligned}$$

We also have

$$\int k\left(|f(v_n)|^{q-2}f(v_n)f'(v_n)-|f(v)|^{q-2}f(v)f'(v)\right)(v_n-v)\to 0, \qquad (21)$$

$$H_n:=\int h\left(|f(v_n)|^{s-2}f(v_n)f'(v_n)-|f(v)|^{s-2}f(v)f'(v)\right)(v_n-v)\geq 0$$

because $t\mapsto s\,|f(t)|^{s-2}f(t)f'(t)$ is increasing. By (18) and (21),

$$\begin{aligned}
o(1)&=\langle\Phi'(v_n)-\Phi'(v),v_n-v\rangle\\
&=\int|\nabla(v_n-v)|^2\\
&\qquad-\int k\left(|f(v_n)|^{q-2}f(v_n)f'(v_n)-|f(v)|^{q-2}f(v)f'(v)\right)(v_n-v)\\
&\qquad+\int h\left(|f(v_n)|^{s-2}f(v_n)f'(v_n)-|f(v)|^{s-2}f(v)f'(v)\right)(v_n-v)\\
&=\int|\nabla(v_n-v)|^2+H_n+o(1).
\end{aligned}$$

We also have

$$\int k\left(|f(v_n)|^{q-2}f(v_n)f'(v_n)-|f(v)|^{q-2}f(v)f'(v)\right)(v_n-v)\to 0, \qquad (21)$$

$$H_n:=\int h\left(|f(v_n)|^{s-2}f(v_n)f'(v_n)-|f(v)|^{s-2}f(v)f'(v)\right)(v_n-v)\geq 0$$

because $t\mapsto s|f(t)|^{s-2}f(t)f'(t)$ is increasing. By (18) and (21),

$$\begin{aligned} o(1) &= \langle \Phi'(v_n)-\Phi'(v), v_n-v\rangle \\ &= \int |\nabla(v_n-v)|^2 \\ &\quad -\int k\left(|f(v_n)|^{q-2}f(v_n)f'(v_n)-|f(v)|^{q-2}f(v)f'(v)\right)(v_n-v) \\ &\quad +\int h\left(|f(v_n)|^{s-2}f(v_n)f'(v_n)-|f(v)|^{s-2}f(v)f'(v)\right)(v_n-v) \\ &= \int |\nabla(v_n-v)|^2+H_n+o(1). \end{aligned} \qquad (22)$$

Consequently, noting $H_n\geq 0$ we deduce

$$v_n\to v \quad \text{in } D^{1,2}(\mathbb{R}^N), \qquad H_n\to 0. \qquad (23)$$

Since $H_n \to 0$, from (20) we have

$$\int h |f(v_n)|^{s-2} f(v_n) f'(v_n) (v_n - v) \to 0.$$

Since $H_n \to 0$, from (20) we have

$$\int h\,|f(v_n)|^{s-2} f(v_n) f'(v_n)\,(v_n - v) \to 0.$$

Replacing α and β in (15) with v_n and v respectively, we get

$$\overline{\lim_{n\to\infty}} \int h\,|f(v_n)|^s \le \int h\,|f(v)|^s + s \lim_{n\to\infty} \int h\,|f(v_n)|^{s-2} f(v_n) f'(v_n)\,(v_n - v)$$

Since $H_n \to 0$, from (20) we have

$$\int h\,|f(v_n)|^{s-2} f(v_n) f'(v_n)\,(v_n - v) \to 0.$$

Replacing α and β in (15) with v_n and v respectively, we get

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int h\,|f(v_n)|^s &\le \int h\,|f(v)|^s + s \lim_{n\to\infty} \int h\,|f(v_n)|^{s-2} f(v_n) f'(v_n)\,(v_n - v)\\ &= \int h\,|f(v)|^s\end{aligned}$$

Since $H_n \to 0$, from (20) we have

$$\int h\,|f(v_n)|^{s-2}\,f(v_n)f'(v_n)\,(v_n - v) \to 0.$$

Replacing α and β in (15) with v_n and v respectively, we get

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int h\,|f(v_n)|^s &\le \int h\,|f(v)|^s + s\lim_{n\to\infty}\int h\,|f(v_n)|^{s-2}\,f(v_n)f'(v_n)\,(v_n - v)\\ &= \int h\,|f(v)|^s \le \underline{\lim_{n\to\infty}} \int h\,|f(v_n)|^s\,.\end{aligned}$$

Since $H_n \to 0$, from (20) we have

$$\int h\,|f(v_n)|^{s-2} f(v_n) f'(v_n)\,(v_n - v) \to 0.$$

Replacing α and β in (15) with v_n and v respectively, we get

$$\varlimsup_{n\to\infty} \int h\,|f(v_n)|^s \le \int h\,|f(v)|^s + s \lim_{n\to\infty} \int h\,|f(v_n)|^{s-2} f(v_n) f'(v_n)\,(v_n - v)$$
$$= \int h\,|f(v)|^s \le \varliminf_{n\to\infty} \int h\,|f(v_n)|^s\,.$$

Hence
$$\int h\,|f(v_n)|^s \to \int h\,|f(v)|^s\,.$$

Since $H_n \to 0$, from (20) we have

$$\int h\,|f(v_n)|^{s-2} f(v_n) f'(v_n)\,(v_n - v) \to 0.$$

Replacing α and β in (15) with v_n and v respectively, we get

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int h\,|f(v_n)|^s &\le \int h\,|f(v)|^s + s \lim_{n\to\infty} \int h\,|f(v_n)|^{s-2} f(v_n) f'(v_n)\,(v_n - v) \\ &= \int h\,|f(v)|^s \le \underline{\lim_{n\to\infty}} \int h\,|f(v_n)|^s .\end{aligned}$$

Hence
$$\int h\,|f(v_n)|^s \to \int h\,|f(v)|^s . \tag{24}$$

Now, by the grow of f (see (12)),

$$h\,|v_n|^{s/2} \le h + \frac{1}{\mu^s} h\,|f(v_n)|^s$$

and $h\,|v_n|^{s/2} \to h\,|v|^{s/2}$ a.e. in $\mathbb{R}^N$,

Since $H_n \to 0$, from (20) we have

$$\int h\,|f(v_n)|^{s-2} f(v_n) f'(v_n)\,(v_n - v) \to 0.$$

Replacing α and β in (15) with v_n and v respectively, we get

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int h\,|f(v_n)|^s &\le \int h\,|f(v)|^s + s \lim_{n\to\infty} \int h\,|f(v_n)|^{s-2} f(v_n) f'(v_n)\,(v_n - v)\\ &= \int h\,|f(v)|^s \le \underline{\lim_{n\to\infty}} \int h\,|f(v_n)|^s .\end{aligned}$$

Hence

$$\int h\,|f(v_n)|^s \to \int h\,|f(v)|^s . \tag{24}$$

Now, by the grow of f (see (12)),

$$h\,|v_n|^{s/2} \le h + \frac{1}{\mu^s} h\,|f(v_n)|^s$$

and $h\,|v_n|^{s/2} \to h\,|v|^{s/2}$ a.e. in $\mathbb{R}^N$, by Pro 8 and (24) we get

$$\int h\,|v_n|^{s/2} \to \int h\,|v|^{s/2} , \qquad \text{i.e., } h^{2/s} v_n|_{s/2} \to |h^{2/s} v|_{s/2}.$$

Since $H_n \to 0$, from (20) we have

$$\int h\,|f(v_n)|^{s-2} f(v_n) f'(v_n)\,(v_n - v) \to 0.$$

Replacing α and β in (15) with v_n and v respectively, we get

$$\overline{\lim_{n\to\infty}} \int h\,|f(v_n)|^s \le \int h\,|f(v)|^s + s \lim_{n\to\infty} \int h\,|f(v_n)|^{s-2} f(v_n) f'(v_n)\,(v_n - v)$$
$$= \int h\,|f(v)|^s \le \underline{\lim_{n\to\infty}} \int h\,|f(v_n)|^s\,.$$

Hence
$$\int h\,|f(v_n)|^s \to \int h\,|f(v)|^s\,. \tag{24}$$

Now, by the grow of f (see (12)),

$$h\,|v_n|^{s/2} \le h + \frac{1}{\mu^s} h\,|f(v_n)|^s$$

and $h\,|v_n|^{s/2} \to h\,|v|^{s/2}$ a.e. in $\mathbb{R}^N$, by Pro 8 and (24) we get

$$\int h\,|v_n|^{s/2} \to \int h\,|v|^{s/2}\,, \qquad \text{i.e., } h^{2/s} v_n|_{s/2} \to |h^{2/s} v|_{s/2}.$$

But $h^{2/s} v_n \rightharpoonup h^{2/s} v$ in $L^{s/2}(\mathbb{R}^N)$, we deduce $h^{2/s} v_n \to h^{2/s} v$ in $L^{s/2}(\mathbb{R}^N)$.

Combining this with (23) we get

$$\|v_n - v\| = \|v_n - v\|_D + \left| h^{2/s} v_n - h^{2/s} v \right|_{s/2} \to 0.$$

Combining this with (23) we get

$$\|v_n - v\| = \|v_n - v\|_D + \left| h^{2/s} v_n - h^{2/s} v \right|_{s/2} \to 0.$$

Pro8. Let $f_n, g_n : \Omega \to \mathbb{R}$ be measurable functions over the measurable set Ω, $f_n \to f$ a.e. in Ω, $g_n \to g$ a.e. in Ω, $|f_n| \leq g_n$.

Combining this with (23) we get

$$\|v_n - v\| = \|v_n - v\|_D + \left|h^{2/s}v_n - h^{2/s}v\right|_{s/2} \to 0.$$

Pro8. Let $f_n, g_n : \Omega \to \mathbb{R}$ be measurable functions over the measurable set Ω, $f_n \to f$ a.e. in Ω, $g_n \to g$ a.e. in Ω, $|f_n| \leq g_n$. Then

$$\int_\Omega |f_n - f| \to 0$$

provided $\int_\Omega g_n \to \int_\Omega g$ and $\int_\Omega g < +\infty$.

Combining this with (23) we get

$$\|v_n - v\| = \|v_n - v\|_D + \left|h^{2/s}v_n - h^{2/s}v\right|_{s/2} \to 0.$$

Pro8. Let $f_n, g_n : \Omega \to \mathbb{R}$ be measurable functions over the measurable set Ω, $f_n \to f$ a.e. in Ω, $g_n \to g$ a.e. in Ω, $|f_n| \le g_n$. Then

$$\int_\Omega |f_n - f| \to 0$$

provided $\int_\Omega g_n \to \int_\Omega g$ and $\int_\Omega g < +\infty$.

Rek12. When $g_n \equiv g$ for all n, Pro 8 reduces to the usual Lebesgue dominating theorem.

Combining this with (23) we get

$$\|v_n - v\| = \|v_n - v\|_D + \left|h^{2/s} v_n - h^{2/s} v\right|_{s/2} \to 0.$$

Pro8. Let $f_n, g_n : \Omega \to \mathbb{R}$ be measurable functions over the measurable set Ω, $f_n \to f$ a.e. in Ω, $g_n \to g$ a.e. in Ω, $|f_n| \le g_n$. Then

$$\int_\Omega |f_n - f| \to 0$$

provided $\int_\Omega g_n \to \int_\Omega g$ and $\int_\Omega g < +\infty$.

Rek12. When $g_n \equiv g$ for all n, Pro 8 reduces to the usual Lebesgue dominating theorem.

Pro9 (Wang(2001), Lemma 2.4). Let E be a Banach space and $\Phi \in C^1(E, \mathbb{R})$ be an even coercive functional satisfying the (PS) and $\Phi(0) = 0$.

Combining this with (23) we get

$$\|v_n - v\| = \|v_n - v\|_D + \left|h^{2/s}v_n - h^{2/s}v\right|_{s/2} \to 0.$$

Pro8. Let $f_n, g_n : \Omega \to \mathbb{R}$ be measurable functions over the measurable set Ω, $f_n \to f$ a.e. in Ω, $g_n \to g$ a.e. in Ω, $|f_n| \le g_n$. Then

$$\int_\Omega |f_n - f| \to 0$$

provided $\int_\Omega g_n \to \int_\Omega g$ and $\int_\Omega g < +\infty$.

Rek12. When $g_n \equiv g$ for all n, Pro 8 reduces to the usual Lebesgue dominating theorem.

Pro9 (Wang(2001), Lemma 2.4). Let E be a Banach space and $\Phi \in C^1(E,\mathbb{R})$ be an even coercive functional satisfying the (PS) and $\Phi(0) = 0$. If for any $n \in \mathbb{N}$, there is an n-dimensional subspace X_n and $\rho_n > 0$ such that

$$\sup_{X_n \cap S_{\rho_n}} \Phi < 0,$$

where $S_r = \{u \in E | \ \|u\| = r\}$,

Combining this with (23) we get

$$\|v_n - v\| = \|v_n - v\|_D + \left|h^{2/s}v_n - h^{2/s}v\right|_{s/2} \to 0.$$

Pro8. Let $f_n, g_n : \Omega \to \mathbb{R}$ be measurable functions over the measurable set Ω, $f_n \to f$ a.e. in Ω, $g_n \to g$ a.e. in Ω, $|f_n| \le g_n$. Then

$$\int_\Omega |f_n - f| \to 0$$

provided $\int_\Omega g_n \to \int_\Omega g$ and $\int_\Omega g < +\infty$.

Rek12. When $g_n \equiv g$ for all n, Pro 8 reduces to the usual Lebesgue dominating theorem.

Pro9 (Wang(2001), Lemma 2.4). Let E be a Banach space and $\Phi \in C^1(E, \mathbb{R})$ be an even coercive functional satisfying the (PS) and $\Phi(0) = 0$. If for any $n \in \mathbb{N}$, there is an n-dimensional subspace X_n and $\rho_n > 0$ such that

$$\sup_{X_n \cap S_{\rho_n}} \Phi < 0,$$

where $S_r = \{u \in E|\ \|u\| = r\}$, then Φ has a sequence of critical values $c_n \uparrow 0$.

8. QSE: ∞ solutions in bdd Ω

He & Wu(2020) studied the following elliptic boundary value problem

$$-\Delta u + V(x)u = f(x,u), \qquad u \in H_0^1(\Omega)$$

with indefinite linear part $-\Delta + V$,

8. QSE: ∞ solutions in bdd Ω

He & Wu(2020) studied the following elliptic boundary value problem

$$-\Delta u + V(x)u = f(x,u), \qquad u \in H_0^1(\Omega)$$

with indefinite linear part $-\Delta + V$, where

(1) $\Omega \subset \mathbb{R}^N$ is bounded;

(2) the odd nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is sublinear at zero:

$$\lim_{|t|\to 0} \frac{1}{t^2} \int_0^t f(x,s)\,\mathrm{d}s = +\infty.$$

8. QSE: ∞ solutions in bdd Ω

He & Wu(2020) studied the following elliptic boundary value problem

$$-\Delta u + V(x)u = f(x,u), \qquad u \in H_0^1(\Omega)$$

with indefinite linear part $-\Delta + V$, where

(1) $\Omega \subset \mathbb{R}^N$ is bounded;

(2) the odd nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is sublinear at zero:

$$\lim_{|t|\to 0} \frac{1}{t^2} \int_0^t f(x,s)\,\mathrm{d}s = +\infty.$$

Using truncating technique and Liu–Wang's variant of Clark's theorem Liu & Wang(2015), Theorem 1.1, they obtained a sequence of solutions conversing to zero in $H_0^1(\Omega)$.

8. QSE: ∞ solutions in bdd Ω

He & Wu(2020) studied the following elliptic boundary value problem

$$-\Delta u + V(x)u = f(x,u), \qquad u \in H_0^1(\Omega)$$

with indefinite linear part $-\Delta + V$, where

(1) $\Omega \subset \mathbb{R}^N$ is bounded;

(2) the odd nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is sublinear at zero:

$$\lim_{|t|\to 0} \frac{1}{t^2} \int_0^t f(x,s)\,\mathrm{d}s = +\infty.$$

Using truncating technique and Liu–Wang's variant of Clark's theorem Liu & Wang(2015), Theorem 1.1, they obtained a sequence of solutions conversing to zero in $H_0^1(\Omega)$.

Motivated by He & Wu(2020), we consider Kirchhoff equation on bdd $\Omega \subset \mathbb{R}^N$,

$$-\left(1 + \int_\Omega |\nabla u|^2\right) \Delta u + V(x)u = f(x,u), \qquad u \in H_0^1(\Omega). \tag{25}$$

Assume

Rem 9

(V) $V \in C(\Omega)$ is bounded;

(f_1) $f \in C(\Omega \times \mathbb{R})$ is subcritical, that is

$$\lim_{|t|\to\infty} \frac{f(x,t)t}{|t|^{2^*}} = 0, \quad \text{where } 2^* = \frac{2N}{N-2} \text{ is the critical exponent;}$$

(V) $V \in C(\Omega)$ is bounded;

(f_1) $f \in C(\Omega \times \mathbb{R})$ is subcritical, that is

$$\lim_{|t|\to\infty} \frac{f(x,t)t}{|t|^{2^*}} = 0, \quad \text{where } 2^* = \frac{2N}{N-2} \text{ is the critical exponent;}$$

(f_2) $f(x,\cdot)$ is odd for all $x \in \Omega$, $f(x,0) = 0$, and is sublinear at zero:

$$\lim_{|t|\to 0} \frac{F(x,t)}{t^2} = +\infty, \qquad \text{where } F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s.$$

(V) $V \in C(\Omega)$ is bounded;

(f_1) $f \in C(\Omega \times \mathbb{R})$ is subcritical, that is

$$\lim_{|t|\to\infty} \frac{f(x,t)t}{|t|^{2^*}} = 0, \quad \text{where } 2^* = \frac{2N}{N-2} \text{ is the critical exponent;}$$

(f_2) $f(x,\cdot)$ is odd for all $x \in \Omega$, $f(x,0) = 0$, and is sublinear at zero:

$$\lim_{|t|\to 0} \frac{F(x,t)}{t^2} = +\infty, \qquad \text{where } F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s. \tag{26}$$

Thm14. Suppose (V), (f_1) and (f_2) hold, then the problem (25) possesses a sequence of nontrivial solutions converging to zero.

(V) $V \in C(\Omega)$ is bounded;

(f_1) $f \in C(\Omega \times \mathbb{R})$ is subcritical, that is

$$\lim_{|t| \to \infty} \frac{f(x,t)t}{|t|^{2^*}} = 0, \quad \text{where } 2^* = \frac{2N}{N-2} \text{ is the critical exponent;}$$

(f_2) $f(x,\cdot)$ is odd for all $x \in \Omega$, $f(x,0) = 0$, and is sublinear at zero:

$$\lim_{|t| \to 0} \frac{F(x,t)}{t^2} = +\infty, \qquad \text{where } F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s. \tag{26}$$

Thm14. Suppose (V), (f_1) and (f_2) hold, then the problem (25) possesses a sequence of nontrivial solutions converging to zero.

BVPs of the form (25) are related to the Kirchhoff wave equation

$$\psi_{tt} - \left(a + b\int_\Omega |\nabla\psi|^2\right)\Delta\psi = g(x,\psi). \qquad \text{(vibrating string, changing length)}$$

(V) $V \in C(\Omega)$ is bounded;

(f_1) $f \in C(\Omega \times \mathbb{R})$ is subcritical, that is

$$\lim_{|t|\to\infty} \frac{f(x,t)t}{|t|^{2^*}} = 0, \quad \text{where } 2^* = \frac{2N}{N-2} \text{ is the critical exponent;}$$

(f_2) $f(x,\cdot)$ is odd for all $x \in \Omega$, $f(x,0)=0$, and is sublinear at zero:

$$\lim_{|t|\to 0} \frac{F(x,t)}{t^2} = +\infty, \qquad \text{where } F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s. \tag{26}$$

Thm14. Suppose (V), (f_1) and (f_2) hold, then the problem (25) possesses a sequence of nontrivial solutions converging to zero.

BVPs of the form (25) are related to the Kirchhoff wave equation

$$\psi_{tt} - \left(a + b\int_\Omega |\nabla\psi|^2\right)\Delta\psi = g(x,\psi). \qquad \text{(vibrating string, changing length)}$$

Variational approach is developed to solve (25) in Alves et al.(2005), Perera & Zhang(2006), Sun & Liu(2012).

Cheng et al.(2012) considered the case $V(x) = 0$ and

$$f(x,t) = \alpha(x)\,|t|^{q-2}\,t + g(x,t), \tag{27}$$

where $q \in (1,2)$, $N \leq 3$ (they need $H_0^1 \hookrightarrow L^{r>4}$),

$$\lim_{t\to 0} \frac{g(x,t)}{t} = 0, \qquad \lim_{|t|\to\infty} \frac{g(x,t)t}{t^4} = +\infty.$$

Cheng et al.(2012) considered the case $V(x) = 0$ and

$$f(x,t) = \alpha(x)\,|t|^{q-2}\,t + g(x,t), \tag{27}$$

where $q \in (1,2)$, $N \le 3$ (they need $H_0^1 \hookrightarrow L^{r>4}$),

$$\lim_{t\to 0} \frac{g(x,t)}{t} = 0, \qquad \lim_{|t|\to\infty} \frac{g(x,t)t}{t^4} = +\infty.$$

Furtado & Zanata(2017) also considered (25) with $V(x) = 0$ and f as in (27); but they only imposed local conditions to $g(x,t)$ for $|t|$ small.

Cheng et al.(2012) considered the case $V(x)=0$ and

$$f(x,t)=\alpha(x)\,|t|^{q-2}\,t+g(x,t), \tag{27}$$

where $q\in(1,2)$, $N\le 3$ (they need $H_0^1\hookrightarrow L^{r>4}$),

$$\lim_{t\to 0}\frac{g(x,t)}{t}=0, \qquad \lim_{|t|\to\infty}\frac{g(x,t)t}{t^4}=+\infty.$$

Furtado & Zanata(2017) also considered (25) with $V(x)=0$ and f as in (27); but they only imposed local conditions to $g(x,t)$ for $|t|$ small. Here we coinsider indefinite case.

Cheng et al.(2012) considered the case $V(x) = 0$ and

$$f(x,t) = \alpha(x)\,|t|^{q-2}\,t + g(x,t), \tag{27}$$

where $q \in (1,2)$, $N \leq 3$ (they need $H_0^1 \hookrightarrow L^{r>4}$),

$$\lim_{t\to 0} \frac{g(x,t)}{t} = 0, \qquad \lim_{|t|\to\infty} \frac{g(x,t)t}{t^4} = +\infty.$$

Furtado & Zanata(2017) also considered (25) with $V(x) = 0$ and f as in (27); but they only imposed local conditions to $g(x,t)$ for $|t|$ small. Here we coinsider indefinite case. For Schrödinger-Poisson system on a bounded smooth domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) & \text{in } \Omega, \\ -\Delta\phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

we have similar result.

Cheng et al.(2012) considered the case $V(x)=0$ and

$$f(x,t)=\alpha(x)\,|t|^{q-2}\,t+g(x,t), \tag{27}$$

where $q\in(1,2)$, $N\le 3$ (they need $H_0^1\hookrightarrow L^{r>4}$),

$$\lim_{t\to 0}\frac{g(x,t)}{t}=0,\qquad \lim_{|t|\to\infty}\frac{g(x,t)t}{t^4}=+\infty.$$

Furtado & Zanata(2017) also considered (25) with $V(x)=0$ and f as in (27); but they only imposed local conditions to $g(x,t)$ for $|t|$ small. Here we coinsider indefinite case. For Schrödinger-Poisson system on a bounded smooth domain $\Omega\subset\mathbb{R}^3$

$$\begin{cases} -\Delta u+V(x)u+\phi u=f(x,u) & \text{in } \Omega,\\ -\Delta\phi=u^2 & \text{in } \Omega,\\ u=\phi=0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

we have similar result.

Thm15. Suppose (V), (f_1) and (f_2) hold, then the problem (28) possesses a sequence of nontrivial solutions $(u_n,\phi_n)\to(0,0)$ in $H_0^1(\Omega)\times H_0^1(\Omega)$.

Weak solutions of (25) are critical points of the C^1-functional $\Phi : H_0^1(\Omega) \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u\right) + \frac{1}{4}\left(\int |\nabla u|^2\right)^2 - \int F(u), \qquad \int = \int_\Omega .$$

Weak solutions of (25) are critical points of the C^1-functional $\Phi : H_0^1(\Omega) \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u\right) + \frac{1}{4}\left(\int |\nabla u|^2\right)^2 - \int F(u), \qquad \int = \int_\Omega . \tag{29}$$

Let $E^\pm$, E^0 be the $\pm$ and null spaces of the quadratic form. For $u \in E := H_0^1(\Omega)$, $u^\pm$ and u^0 are orthogonal projections on $E^\pm$ and E^0.

Weak solutions of (25) are critical points of the C^1-functional $\Phi : H_0^1(\Omega) \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u\right) + \frac{1}{4}\left(\int |\nabla u|^2\right)^2 - \int F(u), \qquad \int = \int_\Omega . \tag{29}$$

Let $E^\pm$, E^0 be the $\pm$ and null spaces of the quadratic form. For $u \in E := H_0^1(\Omega)$, $u^\pm$ and u^0 are orthogonal projections on $E^\pm$ and E^0. There is an equivalent norm $\|\cdot\|$ on E s.t.

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + \frac{1}{4}\left(\int |\nabla u|^2\right)^2 - \int F(u). \tag{30}$$

We denote by $(\cdot,\cdot)$ the corresponding inner product.

Weak solutions of (25) are critical points of the C^1-functional $\Phi : H_0^1(\Omega) \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u\right) + \frac{1}{4}\left(\int |\nabla u|^2\right)^2 - \int F(u), \qquad \int = \int_\Omega. \tag{29}$$

Let $E^\pm$, E^0 be the $\pm$ and null spaces of the quadratic form. For $u \in E := H_0^1(\Omega)$, $u^\pm$ and u^0 are orthogonal projections on $E^\pm$ and E^0. There is an equivalent norm $\|\cdot\|$ on E s.t.

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + \frac{1}{4}\left(\int |\nabla u|^2\right)^2 - \int F(u). \tag{30}$$

We denote by $(\cdot,\cdot)$ the corresponding inner product. ECHO is on. We need the following variant of the Clark's theorem.

Thm16 (Liu & Wang(2015), Theorem 1.1). Let E is Banach space, $\Phi \in C^1(E,\mathbb{R})$ be even and coercive, satisfying $(PS)_{c\le 0}$ and $\Phi(0) = 0$.

Weak solutions of (25) are critical points of the C^1-functional $\Phi : H_0^1(\Omega) \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u\right) + \frac{1}{4}\left(\int |\nabla u|^2\right)^2 - \int F(u), \qquad \int = \int_\Omega . \tag{29}$$

Let $E^\pm$, E^0 be the $\pm$ and null spaces of the quadratic form. For $u \in E := H_0^1(\Omega)$, $u^\pm$ and u^0 are orthogonal projections on $E^\pm$ and E^0. There is an equivalent norm $\|\cdot\|$ on E s.t.

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + \frac{1}{4}\left(\int |\nabla u|^2\right)^2 - \int F(u). \tag{30}$$

We denote by $(\cdot,\cdot)$ the corresponding inner product. ECHO is on. We need the following variant of the Clark's theorem.

Thm16 (Liu & Wang(2015), Theorem 1.1). Let E is Banach space, $\Phi \in C^1(E,\mathbb{R})$ be even and coercive, satisfying $(PS)_{c\le 0}$ and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there is a k-dimensional subspace X_k and $\rho_k > 0$ such that

$$\sup_{X_k \cap S_{\rho_k}} \Phi < 0,$$

Weak solutions of (25) are critical points of the C^1-functional $\Phi : H_0^1(\Omega) \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 + V(x)u\right) + \frac{1}{4}\left(\int |\nabla u|^2\right)^2 - \int F(u), \qquad \int = \int_\Omega . \tag{29}$$

Let $E^\pm$, E^0 be the $\pm$ and null spaces of the quadratic form. For $u \in E := H_0^1(\Omega)$, $u^\pm$ and u^0 are orthogonal projections on $E^\pm$ and E^0. There is an equivalent norm $\|\cdot\|$ on E s.t.

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + \frac{1}{4}\left(\int |\nabla u|^2\right)^2 - \int F(u). \tag{30}$$

We denote by $(\cdot,\cdot)$ the corresponding inner product. ECHO is on. We need the following variant of the Clark's theorem.

Thm16 (Liu & Wang(2015), Theorem 1.1). Let E is Banach space, $\Phi \in C^1(E,\mathbb{R})$ be even and coercive, satisfying $(PS)_{c\le 0}$ and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there is a k-dimensional subspace X_k and $\rho_k > 0$ such that

$$\sup_{X_k \cap S_{\rho_k}} \Phi < 0, \tag{31}$$

where $S_r = \{u \in E \mid \|u\| = r\}$, then Φ has critical points $u_k \neq 0$ such that $\Phi(u_k) \le 0$, $u_k \to 0$. (Compare Pro 9)

To verify $(PS)_{c\leq 0}$ we need

Lem5. If $u_n \rightharpoonup u$ in E,

To verify $(PS)_{c\le 0}$ we need

Lem5. If $u_n \rightharpoonup u$ in E, then

$$\varliminf_{n\to\infty}\left[\left(\int |\nabla u_n|^2\right)\int \nabla u_n\cdot\nabla(u_n-u)-\left(\int|\nabla u|^2\right)\int\nabla u\cdot\nabla(u_n-u)\right]\ge 0.$$

To verify $(PS)_{c\le 0}$ we need

Lem5. If $u_n \rightharpoonup u$ in E, then

$$\varliminf_{n\to\infty}\left[\left(\int|\nabla u_n|^2\right)\int\nabla u_n\cdot\nabla(u_n-u)-\left(\int|\nabla u|^2\right)\int\nabla u\cdot\nabla(u_n-u)\right]\ge 0. \tag{32}$$

Pf of Thm14. Let $\phi:[0,\infty)\to\mathbb{R}$ be a decreasing, $\left|\phi'(t)\right|\le 2$,

$$\phi(t)=1\quad\text{for } t\in[0,1]\,,\qquad \phi(t)=0\quad\text{for } t\ge 2.$$

To verify $(PS)_{c\le 0}$ we need

Lem5. If $u_n \to u$ in E, then

$$\varliminf_{n\to\infty}\left[\left(\int|\nabla u_n|^2\right)\int\nabla u_n\cdot\nabla(u_n-u)-\left(\int|\nabla u|^2\right)\int\nabla u\cdot\nabla(u_n-u)\right]\ge 0. \tag{32}$$

Pf of Thm14. Let $\phi:[0,\infty)\to\mathbb{R}$ be a decreasing, $\left|\phi'(t)\right|\le 2$,

$$\phi(t)=1 \quad \text{for } t\in[0,1], \qquad \phi(t)=0 \quad \text{for } t\ge 2.$$

Consider truncated functional $I:E\to\mathbb{R}$, $\quad I(u)=\Phi(u)$ if $\|u\|\le 1$, see (30)

$$I(u)=\frac{1}{2}\|u\|^2-\frac{1}{2}\left(\|u^*\|^2+2\int F(u)\right)\phi(\|u\|^2)+\frac{1}{4}\left(\int|\nabla u|^2\right)^2, \tag{33}$$

where $u^*=u^-+u^0\in E^-\oplus E^0$. $\|u^+\|^2-\|u^-\|^2=\|u\|^2-\|u^*\|^2$

To verify $(PS)_{c\le 0}$ we need

Lem5. If $u_n \to u$ in E, then

$$\lim_{n\to\infty}\left[\left(\int|\nabla u_n|^2\right)\int\nabla u_n\cdot\nabla(u_n-u)-\left(\int|\nabla u|^2\right)\int\nabla u\cdot\nabla(u_n-u)\right]\ge 0. \tag{32}$$

Pf of Thm14. Let $\phi:[0,\infty)\to\mathbb{R}$ be a decreasing, $\left|\phi'(t)\right|\le 2$,

$$\phi(t)=1 \quad \text{for } t\in[0,1], \qquad \phi(t)=0 \quad \text{for } t\ge 2.$$

Consider truncated functional $I:E\to\mathbb{R}$, $\qquad I(u)=\Phi(u)$ if $\|u\|\le 1$, see (30)

$$I(u)=\frac{1}{2}\|u\|^2-\frac{1}{2}\left(\|u^*\|^2+2\int F(u)\right)\phi(\|u\|^2)+\frac{1}{4}\left(\int|\nabla u|^2\right)^2, \tag{33}$$

where $u^*=u^-+u^0\in E^-\oplus E^0$. $\|u^+\|^2-\|u^-\|^2=\|u\|^2-\|u^*\|^2$ Obviously I is even. If $\|u\|\ge 2$, then $\phi(\|u\|^2)=0$. Hence

$$I(u)=\frac{1}{2}\|u\|^2+\frac{1}{4}\left(\int|\nabla u|^2\right)^2$$

To verify $(PS)_{c\le 0}$ we need

Lem5. If $u_n \to u$ in E, then

$$\lim_{n\to\infty}\left[\left(\int|\nabla u_n|^2\right)\int\nabla u_n\cdot\nabla(u_n-u)-\left(\int|\nabla u|^2\right)\int\nabla u\cdot\nabla(u_n-u)\right]\ge 0. \tag{32}$$

Pf of Thm14. Let $\phi:[0,\infty)\to\mathbb{R}$ be a decreasing, $\left|\phi'(t)\right|\le 2$,

$$\phi(t)=1 \quad \text{for } t\in[0,1]\,, \qquad \phi(t)=0 \quad \text{for } t\ge 2.$$

Consider truncated functional $I:E\to\mathbb{R}$, $\qquad I(u)=\Phi(u)$ if $\|u\|\le 1$, see (30)

$$I(u)=\frac{1}{2}\|u\|^2-\frac{1}{2}\left(\|u^*\|^2+2\int F(u)\right)\phi(\|u\|^2)+\frac{1}{4}\left(\int|\nabla u|^2\right)^2, \tag{33}$$

where $u^*=u^-+u^0\in E^-\oplus E^0$. $\|u^+\|^2-\|u^-\|^2=\|u\|^2-\|u^*\|^2$ Obviously I is even. If $\|u\|\ge 2$, then $\phi(\|u\|^2)=0$. Hence

$$\begin{aligned} I(u)&=\frac{1}{2}\|u\|^2+\frac{1}{4}\left(\int|\nabla u|^2\right)^2\\ &\ge\frac{1}{2}\|u\|^2\to+\infty, \qquad \text{as } \|u\|\to\infty. \qquad (I \text{ is coercive})\end{aligned}$$

To verify $(PS)_{c\leq 0}$, let $\{u_n\} \subset E$ be such that

$$I(u_n) \to c \leq 0, \qquad I'(u_n) \to 0.$$

To verify $(PS)_{c\leq 0}$, let $\{u_n\} \subset E$ be such that

$$I(u_n) \to c \leq 0, \qquad I'(u_n) \to 0.$$

Then $\{u_n\}$ is bounded in E and we assume that $u_n \rightharpoonup u$ in E.

To verify $(PS)_{c\leq 0}$, let $\{u_n\}\subset E$ be such that

$$I(u_n)\to c\leq 0, \qquad I'(u_n)\to 0.$$

Then $\{u_n\}$ is bounded in E and we assume that $u_n \rightharpoonup u$ in E. Then

$$-\left(\|u_n^*\|^2+2\int F(u_n)\right)\phi(\|u_n\|^2)=2I(u_n)-\|u_n\|^2-\frac{1}{2}\left(\int|\nabla u_n|^2\right)^2\leq 0.$$

To verify $(PS)_{c\leq 0}$, let $\{u_n\}\subset E$ be such that

$$I(u_n)\to c\leq 0, \qquad I'(u_n)\to 0.$$

Then $\{u_n\}$ is bounded in E and we assume that $u_n \rightharpoonup u$ in E. Then

$$-\left(\|u_n^*\|^2+2\int F(u_n)\right)\phi(\|u_n\|^2)=2I(u_n)-\|u_n\|^2-\frac{1}{2}\left(\int|\nabla u_n|^2\right)^2\leq 0.$$

Hence

$$\|u_n^*\|^2+2\int F(u_n)\geq 0.$$

To verify $(PS)_{c\le 0}$, let $\{u_n\}\subset E$ be such that

$$I(u_n)\to c\le 0, \qquad I'(u_n)\to 0.$$

Then $\{u_n\}$ is bounded in E and we assume that $u_n \rightharpoonup u$ in E. Then

$$-\left(\|u_n^*\|^2+2\int F(u_n)\right)\phi(\|u_n\|^2)=2I(u_n)-\|u_n\|^2-\frac{1}{2}\left(\int|\nabla u_n|^2\right)^2\le 0.$$

Hence

$$\|u_n^*\|^2+2\int F(u_n)\ge 0. \tag{34}$$

Because $\phi'(\|u_n\|^2)\le 0$ and

$$\varliminf_{n\to\infty}(u_n,u_n-u)=\varliminf_{n\to\infty}\|u_n\|^2-\|u\|^2\ge 0,$$

To verify $(PS)_{c\le 0}$, let $\{u_n\}\subset E$ be such that

$$I(u_n)\to c\le 0,\qquad I'(u_n)\to 0.$$

Then $\{u_n\}$ is bounded in E and we assume that $u_n\rightharpoonup u$ in E. Then

$$-\left(\|u_n^*\|^2+2\int F(u_n)\right)\phi(\|u_n\|^2)=2I(u_n)-\|u_n\|^2-\frac{1}{2}\left(\int|\nabla u_n|^2\right)^2\le 0.$$

Hence

$$\|u_n^*\|^2+2\int F(u_n)\ge 0. \tag{34}$$

Because $\phi'(\|u_n\|^2)\le 0$ and

$$\varliminf_{n\to\infty}(u_n,u_n-u)=\varliminf_{n\to\infty}\|u_n\|^2-\|u\|^2\ge 0,$$

up to a further subsequence we may assume

$$\left(\|u_n^*\|^2+2\int F(u_n)\right)\phi'(\|u_n\|^2)(u_n,u_n-u)\longrightarrow\alpha\le 0,$$

To verify $(PS)_{c\le 0}$, let $\{u_n\} \subset E$ be such that

$$I(u_n) \to c \le 0, \qquad I'(u_n) \to 0.$$

Then $\{u_n\}$ is bounded in E and we assume that $u_n \rightharpoonup u$ in E. Then

$$-\left(\|u_n^*\|^2 + 2\int F(u_n)\right)\phi(\|u_n\|^2) = 2I(u_n) - \|u_n\|^2 - \frac{1}{2}\left(\int |\nabla u_n|^2\right)^2 \le 0.$$

Hence

$$\|u_n^*\|^2 + 2\int F(u_n) \ge 0. \tag{34}$$

Because $\phi'(\|u_n\|^2) \le 0$ and

$$\varliminf_{n\to\infty} (u_n, u_n - u) = \varliminf_{n\to\infty} \|u_n\|^2 - \|u\|^2 \ge 0,$$

up to a further subsequence we may assume

$$\left(\|u_n^*\|^2 + 2\int F(u_n)\right)\phi'(\|u_n\|^2)(u_n, u_n - u) \longrightarrow \alpha \le 0, \tag{35}$$

note since $\{u_n\}$ bdd, the coefficient of $(u_n, u_n - u)$ is bounded. By Lem 5,

we may assume

$$\left(\int |\nabla u_n|^2\right)\int \nabla u_n \cdot \nabla(u_n - u) - \left(\int |\nabla u|^2\right)\int \nabla u \cdot \nabla(u_n - u) \longrightarrow \beta \geq 0.$$

we may assume

$$\left(\int |\nabla u_n|^2\right) \int \nabla u_n \cdot \nabla(u_n - u) - \left(\int |\nabla u|^2\right) \int \nabla u \cdot \nabla(u_n - u) \longrightarrow \beta \geq 0. \tag{36}$$

By subcritical assumption (f_1) and compact embedding $E \hookrightarrow L^2(\Omega)$,

$$\int f(u_n)(u_n - u) \to 0, \qquad \int f(u)(u_n - u) \to 0.$$

we may assume

$$\left(\int |\nabla u_n|^2\right) \int \nabla u_n \cdot \nabla(u_n - u) - \left(\int |\nabla u|^2\right) \int \nabla u \cdot \nabla(u_n - u) \longrightarrow \beta \geq 0. \quad (36)$$

By subcritical assumption (f_1) and compact embedding $E \hookrightarrow L^2(\Omega)$,

$$\int f(u_n)(u_n - u) \to 0, \qquad \int f(u)(u_n - u) \to 0. \quad (37)$$

Since $\dim(E^- \oplus E^0) < \infty$, $u_n^* \to u^*$, so

$$\left(u_n^*, u_n^* - u^*\right) \to 0, \qquad \left(u^*, u_n^* - u^*\right) \to 0.$$

we may assume

$$\left(\int |\nabla u_n|^2\right) \int \nabla u_n \cdot \nabla(u_n - u) - \left(\int |\nabla u|^2\right) \int \nabla u \cdot \nabla(u_n - u) \longrightarrow \beta \geq 0. \tag{36}$$

By subcritical assumption (f_1) and compact embedding $E \hookrightarrow L^2(\Omega)$,

$$\int f(u_n)(u_n - u) \to 0, \qquad \int f(u)(u_n - u) \to 0. \tag{37}$$

Since $\dim(E^- \oplus E^0) < \infty$, $u_n^* \to u^*$, so

$$\left(u_n^*, u_n^* - u^*\right) \to 0, \qquad \left(u^*, u_n^* - u^*\right) \to 0. \tag{38}$$

We deduce from (35), (36), (37) and (38) that

$$\|u_n - u\|^2 = \langle I'(u_n) - I'(u), u_n - u\rangle + \text{6 terms}$$

we may assume

$$\left(\int |\nabla u_n|^2\right)\int \nabla u_n\cdot\nabla(u_n-u)-\left(\int|\nabla u|^2\right)\int\nabla u\cdot\nabla(u_n-u)\longrightarrow\beta\geq 0. \tag{36}$$

By subcritical assumption (f_1) and compact embedding $E\hookrightarrow L^2(\Omega)$,

$$\int f(u_n)(u_n-u)\to 0,\qquad \int f(u)(u_n-u)\to 0. \tag{37}$$

Since $\dim(E^-\oplus E^0)<\infty$, $u_n^*\to u^*$, so

$$\left(u_n^*,u_n^*-u^*\right)\to 0,\qquad \left(u^*,u_n^*-u^*\right)\to 0. \tag{38}$$

We deduce from (35), (36), (37) and (38) that

$$\begin{aligned}\|u_n-u\|^2&=\langle I'(u_n)-I'(u),u_n-u\rangle+6\text{ terms}\\&=[o(1)+\alpha-\beta]\to(\alpha-\beta)\leq 0.\end{aligned} \tag{39}$$

It follows that $u_n\to u$ in E and I satisfies $(PS)_{c\leq 0}$ for $c\leq 0$.

we may assume

$$\left(\int |\nabla u_n|^2\right) \int \nabla u_n \cdot \nabla (u_n - u) - \left(\int |\nabla u|^2\right) \int \nabla u \cdot \nabla (u_n - u) \longrightarrow \beta \geq 0. \quad (36)$$

By subcritical assumption (f_1) and compact embedding $E \hookrightarrow L^2(\Omega)$,

$$\int f(u_n)(u_n - u) \to 0, \qquad \int f(u)(u_n - u) \to 0. \quad (37)$$

Since $\dim(E^- \oplus E^0) < \infty$, $u_n^* \to u^*$, so

$$\left(u_n^*, u_n^* - u^*\right) \to 0, \qquad \left(u^*, u_n^* - u^*\right) \to 0. \quad (38)$$

We deduce from (35), (36), (37) and (38) that

$$\begin{aligned} \|u_n - u\|^2 &= \langle I'(u_n) - I'(u), u_n - u\rangle + 6 \text{ terms} \\ &= [o(1) + \alpha - \beta] \to (\alpha - \beta) \leq 0. \end{aligned} \quad (39)$$

It follows that $u_n \to u$ in E and I satisfies $(PS)_{c\leq 0}$ for $c \leq 0$. For $k \in \mathbb{N}$, let X_k be k-dim subspace of E. Using (f_2), we can find $\rho_k > 0$ s.t.

$$\sup_{X_k \cap S_{\rho_k}} I < 0.$$

we may assume

$$\left(\int |\nabla u_n|^2\right)\int \nabla u_n\cdot\nabla(u_n-u)-\left(\int |\nabla u|^2\right)\int \nabla u\cdot\nabla(u_n-u)\longrightarrow \beta\geq 0. \quad (36)$$

By subcritical assumption (f_1) and compact embedding $E\hookrightarrow L^2(\Omega)$,

$$\int f(u_n)(u_n-u)\to 0,\qquad \int f(u)(u_n-u)\to 0. \quad (37)$$

Since $\dim(E^-\oplus E^0)<\infty$, $u_n^*\to u^*$, so

$$\left(u_n^*,u_n^*-u^*\right)\to 0,\qquad \left(u^*,u_n^*-u^*\right)\to 0. \quad (38)$$

We deduce from (35), (36), (37) and (38) that

$$\begin{aligned}\|u_n-u\|^2 &= \langle I'(u_n)-I'(u),u_n-u\rangle+6\text{ terms}\\ &= [o(1)+\alpha-\beta]\to(\alpha-\beta)\leq 0. \quad (39)\end{aligned}$$

It follows that $u_n\to u$ in E and I satisfies $(PS)_{c\leq 0}$ for $c\leq 0$. For $k\in\mathbb{N}$, let X_k be k-dim subspace of E. Using (f_2), we can find $\rho_k>0$ s.t.

$$\sup_{X_k\cap S_{\rho_k}} I<0.$$

By Thm 16, I has a sequence of critical points $\{u_k\}$ such that $u_k\to 0$ in E.

Pf of Thm 15. Given $u \in H_0^1(\Omega)$, let $\phi_u \in H_0^1(\Omega)$ be solution of $-\Delta\phi = u^2$.

Pf of Thm 15. Given $u \in H_0^1(\Omega)$, let $\phi_u \in H_0^1(\Omega)$ be solution of $-\Delta\phi = u^2$. To verify $(PS)_{c\leq 0}$ we need analogue of Lem 5.

Lem6. If $u_n \rightharpoonup u$ in $E = H_0^1(\Omega)$,

Pf of Thm 15. Given $u \in H_0^1(\Omega)$, let $\phi_u \in H_0^1(\Omega)$ be solution of $-\Delta\phi = u^2$. To verify $(PS)_{c\leq 0}$ we need analogue of Lem 5.

Lem6. If $u_n \rightharpoonup u$ in $E = H_0^1(\Omega)$, then

$$\lim_{n\to\infty}\left(\int \phi_{u_n} u_n (u_n - u) - \int \phi_u u (u_n - u)\right) = 0.$$

Pf of Thm 15. Given $u \in H_0^1(\Omega)$, let $\phi_u \in H_0^1(\Omega)$ be solution of $-\Delta\phi = u^2$. To verify $(PS)_{c\leq 0}$ we need analogue of Lem 5.

Lem6. If $u_n \rightharpoonup u$ in $E = H_0^1(\Omega)$, then

$$\lim_{n\to\infty}\left(\int \phi_{u_n} u_n (u_n - u) - \int \phi_u u (u_n - u)\right) = 0. \tag{40}$$

Pf. ϕ_u is obtained by applying Riesz lemma to $\ell_u : v \mapsto \int u^2 v$ on E.

Pf of Thm 15. Given $u \in H_0^1(\Omega)$, let $\phi_u \in H_0^1(\Omega)$ be solution of $-\Delta\phi = u^2$. To verify $(PS)_{c\le 0}$ we need analogue of Lem 5.

Lem6. If $u_n \rightharpoonup u$ in $E = H_0^1(\Omega)$, then

$$\lim_{n\to\infty}\left(\int \phi_{u_n} u_n (u_n - u) - \int \phi_u u (u_n - u)\right) = 0. \tag{40}$$

Pf. ϕ_u is obtained by applying Riesz lemma to $\ell_u : v \mapsto \int u^2 v$ on E. Thus

$$\|\phi_u\| = \|\ell_u\|$$

Pf of Thm 15. Given $u \in H_0^1(\Omega)$, let $\phi_u \in H_0^1(\Omega)$ be solution of $-\Delta\phi = u^2$. To verify $(PS)_{c\le 0}$ we need analogue of Lem 5.

Lem6. If $u_n \rightharpoonup u$ in $E = H_0^1(\Omega)$, then

$$\lim_{n\to\infty}\left(\int \phi_{u_n} u_n (u_n - u) - \int \phi_u u (u_n - u)\right) = 0. \tag{40}$$

Pf. ϕ_u is obtained by applying Riesz lemma to $\ell_u : v \mapsto \int u^2 v$ on E. Thus

$$\|\phi_u\| = \|\ell_u\| = \sup_{\|v\|=1}\left|\int u^2 v\right|$$

Pf of Thm 15. Given $u \in H_0^1(\Omega)$, let $\phi_u \in H_0^1(\Omega)$ be solution of $-\Delta\phi = u^2$. To verify $(PS)_{c\le 0}$ we need analogue of Lem 5.

Lem6. If $u_n \rightharpoonup u$ in $E = H_0^1(\Omega)$, then

$$\lim_{n\to\infty}\left(\int \phi_{u_n} u_n (u_n - u) - \int \phi_u u (u_n - u)\right) = 0. \tag{40}$$

Pf. ϕ_u is obtained by applying Riesz lemma to $\ell_u : v \mapsto \int u^2 v$ on E. Thus

$$\|\phi_u\| = \|\ell_u\| = \sup_{\|v\|=1}\left|\int u^2 v\right|$$

$$\le \sup_{\|v\|=1}\left(|u^2|_3\,|v|_{3/2}\right) = |u|_6^2 \sup_{\|v\|=1}|v|_{3/2} \le C\,\|u\|^2 .$$

Pf of Thm 15. Given $u \in H_0^1(\Omega)$, let $\phi_u \in H_0^1(\Omega)$ be solution of $-\Delta\phi = u^2$. To verify $(PS)_{c\le 0}$ we need analogue of Lem 5.

Lem6. If $u_n \rightharpoonup u$ in $E = H_0^1(\Omega)$, then

$$\lim_{n\to\infty}\left(\int \phi_{u_n} u_n (u_n - u) - \int \phi_u u (u_n - u)\right) = 0. \tag{40}$$

Pf. ϕ_u is obtained by applying Riesz lemma to $\ell_u : v \mapsto \int u^2 v$ on E. Thus

$$\begin{aligned} \|\phi_u\| = \|\ell_u\| &= \sup_{\|v\|=1}\left|\int u^2 v\right| \\ &\le \sup_{\|v\|=1}\left(|u^2|_3\,|v|_{3/2}\right) = |u|_6^2 \sup_{\|v\|=1}|v|_{3/2} \le C\,\|u\|^2 . \end{aligned} \tag{41}$$

Since $\{u_n\}$ is bdd, $\{\phi_{u_n}\}$ is also bdd in $H_0^1(\Omega)$.

Pf of Thm 15. Given $u \in H_0^1(\Omega)$, let $\phi_u \in H_0^1(\Omega)$ be solution of $-\Delta\phi = u^2$. To verify $(PS)_{c\le 0}$ we need analogue of Lem 5.

Lem6. If $u_n \rightharpoonup u$ in $E = H_0^1(\Omega)$, then

$$\lim_{n\to\infty}\left(\int \phi_{u_n} u_n (u_n - u) - \int \phi_u u (u_n - u)\right) = 0. \tag{40}$$

Pf. ϕ_u is obtained by applying Riesz lemma to $\ell_u : v \mapsto \int u^2 v$ on E. Thus

$$\begin{aligned} \|\phi_u\| = \|\ell_u\| &= \sup_{\|v\|=1}\left|\int u^2 v\right| \\ &\le \sup_{\|v\|=1}\left(|u^2|_3\,|v|_{3/2}\right) = |u|_6^2 \sup_{\|v\|=1}|v|_{3/2} \le C\,\|u\|^2. \end{aligned} \tag{41}$$

Since $\{u_n\}$ is bdd, $\{\phi_{u_n}\}$ is also bdd in $H_0^1(\Omega)$. But $E \hookrightarrow L^{12/5}(\Omega)$ is compact, may assume $u_n \to u$ in $L^{12/5}(\Omega)$.

Pf of Thm 15. Given $u \in H_0^1(\Omega)$, let $\phi_u \in H_0^1(\Omega)$ be solution of $-\Delta\phi = u^2$. To verify $(PS)_{c\le 0}$ we need analogue of Lem 5.

Lem6. If $u_n \rightharpoonup u$ in $E = H_0^1(\Omega)$, then

$$\lim_{n\to\infty}\left(\int \phi_{u_n} u_n (u_n - u) - \int \phi_u u (u_n - u)\right) = 0. \tag{40}$$

Pf. ϕ_u is obtained by applying Riesz lemma to $\ell_u : v \mapsto \int u^2 v$ on E. Thus

$$\begin{aligned}\|\phi_u\| = \|\ell_u\| &= \sup_{\|v\|=1}\left|\int u^2 v\right| \\ &\le \sup_{\|v\|=1}\left(|u^2|_3\,|v|_{3/2}\right) = |u|_6^2 \sup_{\|v\|=1}|v|_{3/2} \le C\,\|u\|^2.\end{aligned} \tag{41}$$

Since $\{u_n\}$ is bdd, $\{\phi_{u_n}\}$ is also bdd in $H_0^1(\Omega)$. But $E \hookrightarrow L^{12/5}(\Omega)$ is compact, may assume $u_n \to u$ in $L^{12/5}(\Omega)$. By Hölder,

$$\left|\int \phi_{u_n} u_n (u_n - u)\right| \le \left|\phi_{u_n}\right|_6 |u_n|_{12/5}\,|u_n - u|_{12/5}$$

Pf of Thm 15. Given $u \in H_0^1(\Omega)$, let $\phi_u \in H_0^1(\Omega)$ be solution of $-\Delta\phi = u^2$. To verify $(PS)_{c\le 0}$ we need analogue of Lem 5.

Lem6. If $u_n \rightharpoonup u$ in $E = H_0^1(\Omega)$, then

$$\lim_{n\to\infty}\left(\int \phi_{u_n} u_n (u_n - u) - \int \phi_u u (u_n - u)\right) = 0. \tag{40}$$

Pf. ϕ_u is obtained by applying Riesz lemma to $\ell_u : v \mapsto \int u^2 v$ on E. Thus

$$\begin{aligned} \|\phi_u\| = \|\ell_u\| &= \sup_{\|v\|=1}\left|\int u^2 v\right| \\ &\le \sup_{\|v\|=1}\left(|u^2|_3\,|v|_{3/2}\right) = |u|_6^2 \sup_{\|v\|=1}|v|_{3/2} \le C\,\|u\|^2. \end{aligned} \tag{41}$$

Since $\{u_n\}$ is bdd, $\{\phi_{u_n}\}$ is also bdd in $H_0^1(\Omega)$. But $E \hookrightarrow L^{12/5}(\Omega)$ is compact, may assume $u_n \to u$ in $L^{12/5}(\Omega)$. By Hölder,

$$\left|\int \phi_{u_n} u_n (u_n - u)\right| \le \left|\phi_{u_n}\right|_6 |u_n|_{12/5}\,|u_n - u|_{12/5} \to 0,$$

Similarly, the second integral in (40) vanishes as $n \to \infty$.

References

Alves et al.(2005) Alves CO, Corrêa FJSA, Ma TF. Positive solutions for a quasilinear elliptic equation of Kirchhoff type. Comput. Math. Appl., 49(2005)(1) 85–93.

Ambrosetti et al.(1994) Ambrosetti A, Brezis H, Cerami G. Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal., 122(1994)(2) 519–543.

Bartsch & Li(1997) Bartsch T, Li S. Critical point theory for asymptotically quadratic functionals and applications to problems with resonance. Nonlinear Anal., 28(1997)(3) 419–441.

Bartsch & Willem(1995) Bartsch T, Willem M. On an elliptic equation with concave and convex nonlinearities. Proc. Amer. Math. Soc., 123(1995)(11) 3555–3561.

Benci & Fortunato(1998) Benci V, Fortunato D. An eigenvalue problem for the Schrödinger-Maxwell equations. Topol. Methods Nonlinear Anal., 11(1998)(2) 283–293.

Castro(1982) Castro A. Reduction methods via minimax. In Differential equations (S ao Paulo, 1981), vol. 957 of Lecture Notes in Math., pp. 1–20. Springer, Berlin (1982).

Castro & Cossio(1994) Castro A, Cossio J. Multiple solutions for a nonlinear Dirichlet problem. SIAM J. Math. Anal., 25(1994)(6) 1554–1561.

Chang(1993) Chang Kc. Infinite-dimensional Morse theory and multiple solution problems. Progress in Nonlinear Differential Equations and their Applications, 6. Birkhäuser Boston, Inc., Boston, MA (1993).

Chen & Liu(2015) Chen H, Liu S. Standing waves with large frequency for 4-superlinear Schrödinger-Poisson systems. Ann. Mat. Pura Appl. (4), 194(2015)(1) 43–53.

Cheng et al.(2012) Cheng B, Wu X, Liu J. Multiple solutions for a class of Kirchhoff type problems with concave nonlinearity. NoDEA Nonlinear Differential Equations Appl., 19(2012)(5) 521–537.

Colin & Jeanjean(2004) Colin M, Jeanjean L. Solutions for a quasilinear Schrödinger equation: a dual approach. Nonlinear Anal., 56(2004)(2) 213–226.

do Ó & Severo(2009) do Ó JaM, Severo U. Quasilinear Schrödinger equations involving concave and convex nonlinearities. Commun. Pure Appl. Anal., 8(2009)(2) 621–644.

do Ó(1997) do Ó JaMB. Solutions to perturbed eigenvalue problems of the p-Laplacian in $\mathbf{R}^N$. Electron. J. Differential Equations, (1997) No. 11, 15.

Figueiredo et al.(2015) Figueiredo GM, Miyagaki OH, Moreira SI. Nonlinear perturbations of a periodic Schrödinger equation with supercritical growth. Z. Angew. Math. Phys., 66(2015)(5) 2379–2394.

Furtado & Zanata(2017) Furtado MF, Zanata HR. Multiple solutions for a Kirchhoff equation with nonlinearity having arbitrary growth. Bull. Aust. Math. Soc., 96(2017)(1) 98–109.

He & Wu(2020) He W, Wu Q. Multiplicity results for sublinear elliptic equations with sign-changing potential and general nonlinearity. Bound. Value Probl., (2020) Paper No. 159, 9.

Jiang & Liu(2022) Jiang S, Liu S. Multiple solutions for Schrödinger-Kirchhoff equations with indefinite potential. Appl. Math. Lett., 124(2022) Paper No. 107672, 9.

Kryszewski & Szulkin(1998) Kryszewski W, Szulkin A. Generalized linking theorem with an application to a semilinear Schrödinger equation. Adv. Differential Equations, 3(1998)(3) 441–472.

Li & Liu(2013) Li C, Liu S. Homology of saddle point reduction and applications to resonant elliptic systems. Nonlinear Anal., 81(2013) 236–246.

Li & Szulkin(2002) Li G, Szulkin A. An asymptotically periodic Schrödinger equation with indefinite linear part. Commun. Contemp. Math., 4(2002)(4) 763–776.

Li(1986) Li S. Some existence theorems of critical points and applications. Tech. Rep. IC/86/90, ICTP, Trieste (1986).

Li, Perera & Su.(2001) Li S, Perera K, Su J. Computation of critical groups in elliptic boundary-value problems where the asymptotic limits may not exist. Proc. Roy. Soc. Edinburgh Sect. A, 131(2001)(3) 721–732.

Li & Zhang(1999) Li S, Zhang Z. Multiple solutions theorems for semilinear elliptic boundary value problems with resonance at infinity. Discrete Contin. Dynam. Systems, 5(1999)(3) 489–493.

Liu(2016) Liu H. Positive solution for a quasilinear elliptic equation involving critical or supercritical exponent. J. Math. Phys., 57(2016)(4) 041506, 11.

Liu(1989) Liu JQ. The Morse index of a saddle point. Systems Sci. Math. Sci., 2(1989)(1) 32–39.

Liu et al.(2003) Liu Jq, Wang Yq, Wang ZQ. Soliton solutions for quasilinear Schrödinger equations. II. J. Differential Equations, 187(2003)(2) 473–493.

Liu(2007) Liu S. Remarks on multiple solutions for elliptic resonant problems. J. Math. Anal. Appl., 336(2007)(1) 498–505.

Liu(2008) Liu S. Multiple solutions for elliptic resonant problems. Proc. Roy. Soc. Edinburgh Sect. A, 138(2008)(6) 1281–1289.

Liu(2009) Liu S. Nontrivial solutions for elliptic resonant problems. Nonlinear Anal., 70(2009)(5) 1965–1974.

Liu(2012) Liu S. On superlinear Schrödinger equations with periodic potential. Calc. Var. Partial Differential Equations, 45(2012)(1-2) 1–9.

Liu & Li(2003a) Liu S, Li S. Critical groups at infinity, saddle point reduction and elliptic resonant problems. Commun. Contemp. Math., 5(2003a)(5) 761–773.

Liu & Li(2003b) Liu S, Li S. An elliptic equation with concave and convex nonlinearities. Nonlinear Anal., 53(2003b)(6) 723–731.

Liu & Mosconi(2020) Liu S, Mosconi S. On the Schrödinger-Poisson system with indefinite potential and 3-sublinear nonlinearity. J. Differential Equations, 269(2020)(1) 689–712.

Liu & Yin(2023) Liu S, Yin LF. Quasilinear schrödinger equations with concave and convex nonlinearities. Calc. Var. Partial Differential Equations, in press(2023).

Liu & Zhou(2018) Liu S, Zhou J. Standing waves for quasilinear Schrödinger equations with indefinite potentials. J. Differential Equations, 265(2018)(9) 3970–3987.

Liu & Wang(2015) Liu Z, Wang ZQ. On Clark's theorem and its applications to partially sublinear problems. Ann. Inst. H. Poincaré Anal. Non Linéaire, 32(2015)(5) 1015–1037.

Mawhin & Willem(1989) Mawhin J, Willem M. Critical point theory and Hamiltonian systems, vol. 74 of Applied Mathematical Sciences. Springer-Verlag, New York (1989).

Miyagaki & Moreira(2015) Miyagaki OH, Moreira SI. Nonnegative solution for quasilinear Schrödinger equations that include supercritical exponents with nonlinearities that are indefinite in sign. J. Math. Anal. Appl., 421(2015)(1) 643–655.

Perera(2003) Perera K. Nontrivial critical groups in p-Laplacian problems via the Yang index. Topol. Methods Nonlinear Anal., 21(2003)(2) 301–309.

Perera & Zhang(2006) Perera K, Zhang Z. Nontrivial solutions of Kirchhoff-type problems via the Yang index. J. Differential Equations, 221(2006)(1) 246–255.

Ruiz(2006) Ruiz D. The Schrödinger-Poisson equation under the effect of a nonlinear local term. J. Funct. Anal., 237(2006)(2) 655–674.

Santos & Santos Júnior(2019) Santos AV, Santos Júnior JaR. Multiple solutions for a generalised Schrödinger problem with "concave-convex" nonlinearities. Z. Angew. Math. Phys., 70(2019)(5) Paper No. 158, 19.

Sun & Liu(2012) Sun J, Liu S. Nontrivial solutions of Kirchhoff type problems. Appl. Math. Lett., 25(2012)(3) 500–504.

Szulkin & Weth(2009) Szulkin A, Weth T. Ground state solutions for some indefinite variational problems. J. Funct. Anal., 257(2009)(12) 3802–3822.

Wang(2001) Wang ZQ. Nonlinear boundary value problems with concave nonlinearities near the origin. NoDEA Nonlinear Differential Equations Appl., 8(2001)(1) 15–33.

Yin & Liu(2023) Yin LF, Liu S. Solutions for quasilinear Schr"odinger equation on $\mathbb{R}^n$ involving indefinite potentials. Complex Var. Elliptic Equ., in press(2023).