Multiple solutions for Schrödinger–Poisson–Slater equations with critical growth

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†***Dedicated to Professor Shujie Li on the Occasion of his 85th Birthday***

Abstract

We obtain multiple solutions for the zero mass Schrödinger–Poisson–Slater equation

$$-\Delta u + \left(\frac{1}{4\pi|x|} * u^2\right) u = \lambda g(x)|u|^{p-2}u + |u|^{6-2}u, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$$

for $\lambda \gg 1$, where $p \in (4, 6)$ and $g \in L^{6/(6-p)}(\mathbb{R}^3)$. The crucial $(\mathrm{PS})_c$ condition is verified using a simpler method. Similar multiplicity result is also obtained for related equation with an external potential.

Keywords: Schrödinger–Poisson–Slater equation, Coulomb–Sobolev space, Variational methods, Critical growth

MSC Classification: 35J91 , 35J20 , 47J30

1 Introduction

We consider the following zero mass Schrödinger–Poisson–Slater equation

$$-\Delta u + \left(\frac{1}{4\pi|x|} * u^2\right) u = \lambda g(x)|u|^{p-2}u + |u|^{6-2}u, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^3). \tag{1.1}$$

Note that the exponent 6 in the last term is the critical exponent for the Sobolev embedding.

Nonlocal elliptic equations like (1.1) and its counterpart (4.1) arise from finding standing waves $\psi(t,x) = \mathrm{e}^{-i\omega t}u(x)$ for the following nonlocal Schrödinger equation

$$i\partial_t\psi = -\Delta\psi + U(x)\psi + \left(\frac{1}{4\pi|x|} * |\psi|^2\right)\psi - f(x,|\psi|)\psi, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3,$$

which comes from an approximation of the Hartree–Fock model of a quantum many-body system of electrons, in which $|\psi|^2$ is the density of electrons and the nonlocal convolution term represents the Coulombic repulsion between the electrons.

Ruiz [14] and Ianni & Ruiz [7] studied (1.1) for the case that the nonlinearity (the right hand side) is a pure power function $|u|^{q-2}u$ with $q \in (2,6)$. Liu *et al.* [10] studied (1.1) for the case that $g(x) \equiv 1$, they obtained ground state solution for $p \in (3,6)$ and positive radial solution for $p \in \left(\frac{18}{7},3\right)$, see also [6].

More recently, for $g(x) \equiv 1$ and $p \in (3,6)$, Mercuri & Perera [12, Theorem 1.28] obtained arbitrarily many solutions for the equation (1.1), provided λ is large enough. Since $g(x)$ is constant, they can work with the radial subspace $E_r(\mathbb{R}^3)$ of the Coulomb–Sobolev space $E(\mathbb{R}^3)$. The variational functional enjoys nice 3-scaling property, which enable them to apply their critical point theorem for scaled functionals ([12, Corollary 2.34]).

In this paper we study the case that g is not a constant, not even radially symmetric. Thus, we have to work on the general Coulomb–Sobolev space $E(\mathbb{R}^3)$ and the crucial 3-scaling property is lost. Nevertheless, we still obtain similar multiplicity result with narrower range of p. We assume that g satisfies

(g) there is $p \in (4,6)$ such that $g \in L^{6/(6-p)}(\mathbb{R}^3)$, $g \geqslant 0$, $\Omega := \{g > 0\}$ is nonempty open subset of $\mathbb{R}^3$.

Then, we have the following theorem.

Theorem 1.1 *Let g satisfy the condition (g). Given $m \in \mathbb{N}$, there is $\lambda_m > 0$ such that (1.1) has m pairs of solutions with positive energy for all $\lambda \geqslant \lambda_m$.*

Our proof of this theorem is based on a critical point theorem of Perera [13, Theorem 2.1]. Like almost all critical point theorem, Palais–Smale (PS) condition is crucial for applying this theorem. Since (1.1) is of critical growth, the most we can expect is local (PS) condition, that is $(\mathrm{PS})_c$ for all $c \in (0,c^*)$ for some $c^* > 0$. In Mercuri & Perera [12] the proof of $(\mathrm{PS})_c$ for the case $g \equiv 1$ depends on the Pohozaev identity [3, Lemma 2.4]. For our case that g is not constant, we will give a simpler proof in Section 3. In Section 2 we first recall the Coulomb–Sobolev space $E(\mathbb{R}^3)$ introduced by Ruiz [14], then present the proof of Theorem 1.1. In Section 4, we present similar result for Schrödinger–Poisson–Slater equation with an external potential (see Eq. (4.1)). Finally, in Section 5 we present some variants of the results we have obtained so far.

2 Variational setting and proof of Theorem 1.1

Instead of the standard Sobolev space $H^1(\mathbb{R}^3)$, the correct functional space for studying the zero mass problem (1.1) is the Coulomb–Sobolev space $E(\mathbb{R}^3)$ introduced by Ruiz [14], where

$E(\mathbb{R}^3)$ is the vector space

$$E = E(\mathbb{R}^3) = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \,\middle|\, \iint \frac{u^2(x)u^2(y)}{|x-y|} < \infty \right\}$$

equipped with the norm

$$\|u\| = \left[\int |\nabla u|^2 + \left(\iint \frac{u^2(x)u^2(y)}{|x-y|} \right)^{1/2} \right]^{1/2}.$$

Here and in what follows, unless stated explicitly, all integrals are taken over $\mathbb{R}^3$, all double integrals are taken with respect to (x, y) over $\mathbb{R}^3 \times \mathbb{R}^3$.

It has been proved in [14, Theorem 1.5] that $(E, \|\cdot\|)$ is a uniformly convex Banach space which is embedded in $L^q(\mathbb{R}^3)$ continuously for $q \in [3, 6]$.

We consider the functional $\Phi : E \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2} \int |\nabla u|^2 + \frac{1}{16\pi} \iint \frac{u^2(x)u^2(y)}{|x-y|} - \frac{\lambda}{p} \int g|u|^p - \frac{1}{6} \int |u|^6.$$

Then, it is well known that $\Phi \in C^1(E)$ with derivative given by

$$\langle \Phi'(u), v \rangle = \int \nabla u \cdot \nabla v + \frac{1}{4\pi} \iint \frac{u^2(x)u(y)v(y)}{|x-y|} - \frac{\lambda}{p} \int g|u|^{p-2}uv - \int |u|^{6-2}uv.$$

Hence, critical points of Φ are weak solutions of the problem (1.1). By regularity results, weak solutions are classical solutions.

Therefore, we will focus on finding multiple critical points of Φ. For this purpose, we need the following critical point theorem of Perera [13]. For a symmetric subset A of $E\backslash\{0\}$, we denote by $i(A)$ the cohomological index of A, which was introduced by Fadell & Rabinowitz [4]. If A is homeomorphic to the unit sphere S^{m-1} in $\mathbb{R}^m$, then $i(A) = m$.

Proposition 2.1 ([13, Theorem 2.1]) *Let E be a Banach space, $\Phi : E \to \mathbb{R}$ be an even C^1-functional satisfying $(\mathrm{PS})_c$ for $c \in (0, c^*)$ being c^* some positive constant. If 0 is a strict local minimizer of Φ and there are $R > 0$ and a compact symmetric set $A \subset \partial\mathfrak{B}_R$, where $\mathfrak{B}_R$ is the R-ball in E, such that $i(A) = m$,*

$$\max_A \Phi \leqslant 0, \qquad \max_B \Phi < c^*, \tag{2.1}$$

where $B = \{tu \mid t \in [0, 1], u \in A\}$, then Φ has m pairs of nonzero critical points with positive critical values.

Proof of Theorem 1.1

As has been pointed out by Ianni & Ruiz [7], it is clear that

$$\frac{1}{2}\|u\|^4 \leqslant \int |\nabla u|^2 + \iint \frac{u^2(x)u^2(y)}{|x-y|} \qquad \text{if } \|u\| \leqslant 1.$$

Since $p > 4$, using the above inequality it is clear that $u = 0$ is a strict local minimizer of Φ.

Let

$$S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{|\nabla u|_2^2}{|u|_6^2}, \qquad c^* = \frac{1}{3} S^{3/2}.$$

In Section 3 we will show that Φ satisfies $(\mathrm{PS})_c$ for $c \in (0, c^*)$. To conclude the proof of Theorem 1.1, it suffices to find the subsets A and B satisfying the geometric assumption (2.1) for any given $m \in \mathbb{N}$. We will adapt the argument used in [9], where a (p, q)-Laplacian equation

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda h(x) |u|^{r-2} u + g(x) |u|^{p^*-2} u, \\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \cap \mathcal{D}^{1,q}(\mathbb{R}^N) \end{cases}$$

is considered.

Given $m \in \mathbb{N}$, let

$$Z = \{u \in E \mid \operatorname{supp} u \subset \Omega\},$$

Z_m be an m-dimensional subspace of Z. Since $g > 0$ on Ω,

$$[u]_g = \left(\frac{1}{p} \int g |u|^p \right)^{1/p}$$

is a norm on Z and Z_m. For $u \in Z_m$ we have

$$\begin{aligned} \Phi(u) &\leqslant \frac{1}{2} \|u\|^2 + \frac{1}{16\pi} \|u\|^4 - \lambda [u]_g^p - \frac{1}{6} |u|_6^6 \\ &\leqslant \frac{1}{2} \|u\|^2 + \frac{1}{16\pi} \|u\|^4 - \lambda a_1 \|u\|^p - a_2 \|u\|^6 \end{aligned} \tag{2.2}$$

because all norms on Z_m are equivalent. Take $R > 1$ such that

$$f(R) := \frac{1}{2} R^2 + \frac{1}{16\pi} R^4 - a_2 R^6 < 0. \tag{2.3}$$

Let $A = Z_m \cap \partial \mathfrak{B}_R$, then $i(A) = m$. If $\lambda > 0$, then for any $u \in A$, from (2.2) we have $\Phi(u) \leqslant f(R)$. Thus

$$\max_A \Phi < 0.$$

For the function f defined in (2.3), there is $\delta \in (0, R)$ such that $f(s) < c^*$ for all $s \in [0, \delta]$. Set

$$\lambda_m = 1 + \max_{s \in [\delta, R]} \left| \frac{f(s) - c^*}{a_1 s^p} \right|.$$

Then if $\lambda \geqslant \lambda_m$ we have

$$f(s) - \lambda a_1 s^p < c^* \qquad \text{for } s \in [\delta, R].$$

Therefore, for $u \in A$,

1. if $t \in \left[\frac{\delta}{R}, 1\right]$, then $\|tu\| \in [\delta, R]$,

$$\Phi(tu) \leqslant f(\|tu\|) - \lambda a_1 \|tu\|^p < c^*;$$

2. if $t \in \left[0, \frac{\delta}{R}\right]$, then $\|tu\| \leqslant \delta$ and $\Phi(tu) \leqslant f(\|tu\|) < c^*$.

From this, we deduce that for $B = \{tu \mid t \in [0,1], u \in A\}$ there holds

$$\max_B \Phi < c^*.$$

By Proposition 2.1, Φ has m pairs of nonzero critical points, and (1.1) has m-pairs of nontrivial solutions.

3 $(\mathrm{PS})_c$ condition

In this section we show that Φ satisfies $(\mathrm{PS})_c$ condition for all $c \in (0, c^*)$. Let $\{u_n\}$ be a $(\mathrm{PS})_c$ sequence with $c \in (0, c^*)$, that is

$$\begin{aligned}
\Phi(u_n) &= \frac{1}{2}\int |\nabla u_n|^2 + \frac{1}{16\pi}\iint \frac{u_n^2(x)u_n^2(y)}{|x-y|} - \frac{\lambda}{p}\int g|u_n|^p - \frac{1}{6}\int |u_n|^6 \to c,\\
\langle \Phi'(u_n), v\rangle &= \int \nabla u_n \cdot \nabla v + \frac{1}{4\pi}\iint \frac{u_n^2(x)u_n(y)v(y)}{|x-y|}\\
&\quad - \lambda \int g|u_n|^{p-2}u_n v - \int |u_n|^{6-2}u_n v = o(\|v\|). \qquad (3.1)
\end{aligned}$$

Since $p \in (4,6)$, we may take $\mu \in (4,p)$. Then for $n \gg 1$ we have

$$\begin{aligned}
c + 1 &\geqslant \Phi(u_n) - \frac{1}{\mu}\langle \Phi'(u_n), u_n\rangle\\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right)\int |\nabla u_n|^2 + \left(\frac{1}{16\pi} - \frac{1}{4\mu\pi}\right)\iint \frac{u_n^2(x)u_n^2(y)}{|x-y|}\\
&\quad + \left(\frac{\lambda}{\mu} - \frac{\lambda}{p}\right)\int g|u_n|^p + \left(\frac{1}{\mu} - \frac{1}{6}\right)\int |u_n|^6\\
&\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\int |\nabla u_n|^2 + \left(\frac{1}{16\pi} - \frac{1}{4\mu\pi}\right)\iint \frac{u_n^2(x)u_n^2(y)}{|x-y|}. \qquad (3.2)
\end{aligned}$$

Since $\mu > 4$, the coefficients of the integrals at the end are positive. It follows that $\{u_n\}$ is bounded.

Up to a subsequence we may assume $u_n \rightharpoonup u$ in E, and

$$u_n \rightharpoonup u \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^3), \qquad u_n \to u \quad \text{a.e. on } \mathbb{R}^3.$$

Since $\{|u_n|^{6-2}u_n\}$ is bounded in $L^{6/5}(\mathbb{R}^3)$ and $|u_n|^{6-2}u_n \to |u|^{6-2}u$ a.e. on $\mathbb{R}^3$, using [2, Page 487] we have $|u_n|^{6-2}u_n \rightharpoonup |u|^{6-2}u$ in $L^{6/5}(\mathbb{R}^3)$. Therefore

$$\int |u_n|^{6-2}u_n v \to \int |u|^{6-2}uv \qquad \text{for } v \in E. \tag{3.3}$$

Similarly, since $\{|u_n|^p\}$ and $\{|u_n|^{p-2}u_n v\}$ are bounded in $L^{6/p}(\mathbb{R}^3)$, using [2, Page 487] again we have

$$|u_n|^p \rightharpoonup |u|^p \quad \text{in } L^{6/p}(\mathbb{R}^3), \qquad |u_n|^{p-2}u_n v \rightharpoonup |u|^{p-2}uv \quad \text{in } L^{6/p}(\mathbb{R}^3).$$

Because $g \in L^{6/(6-p)}(\mathbb{R}^3)$, the dual of $L^{6/p}(\mathbb{R}^3)$ we deduce

$$\int g|u_n|^p \to \int g|u|^p, \qquad \int g|u_n|^{p-2}u_n v \to \int g|u|^{p-2}uv. \tag{3.4}$$

Moreover, since $u_n \rightharpoonup u$ in E, using [7, Lemma 2.3] we have

$$\iint \frac{u_n^2(x)u_n(y)v(y)}{|x-y|} \to \iint \frac{u^2(x)u(y)v(y)}{|x-y|}. \tag{3.5}$$

Using (3.3), (3.4) and (3.5) we deduce

$$\begin{aligned} 0 &= \lim_{n\to\infty} \langle \Phi'(u_n), v\rangle \\ &= \int \nabla u \cdot \nabla v + \frac{1}{4\pi} \iint \frac{u^2(x)u(y)v(y)}{|x-y|} - \lambda \int g|u|^{p-2}uv - \int |u|^{6-2}uv \\ &= \langle \Phi'(u), v\rangle. \end{aligned} \tag{3.6}$$

So $\Phi'(u) = 0$. We also have $\Phi(u) \geqslant 0$ because

$$\begin{aligned} 4\Phi(u) &= 4\Phi(u) - \langle \Phi'(u), u\rangle \\ &= \int |\nabla u|^2 + \lambda \left(1 - \frac{4}{p}\right) \int g|u|^p + \left(1 - \frac{4}{6}\right) \int |u|^6. \end{aligned}$$

Let $v_n = u_n - u$. By Brezis–Lieb lemma [2, Theorem 1] (see also [16, Lemma 1.32],

$$\int |u_n|^6 = \int |u|^6 + \int |v_n|^6 + o(1). \tag{3.7}$$

Using this and

$$\int |\nabla u_n|^2 = \int |\nabla u|^2 + \int |\nabla v_n|^2 + o(1), \tag{3.8}$$

$$\int g|u_n|^p = \int g|u|^p + o(1), \tag{3.9}$$

as well as $\Phi(u_n) \to c$, we deduce

$$\begin{aligned}
&4\Phi(u) + 2\int |\nabla v_n|^2 - \frac{2}{3}\int |v_n|^6 \\
&\quad + \frac{1}{4\pi}\iint \frac{u_n^2(x)u_n^2(y)}{|x-y|} - \frac{1}{4\pi}\iint \frac{u^2(x)u^2(y)}{|x-y|} \to 4c.
\end{aligned} \tag{3.10}$$

On the other hand, it follows from $\langle \Phi'(u_n), u_n\rangle \to 0$ and $\Phi'(u) = 0$ that

$$\begin{aligned}
0 &= \langle \Phi'(u_n), u_n\rangle + o(1) \\
&= \int |\nabla u_n|^2 + \frac{1}{4\pi}\iint \frac{u_n^2(x)u_n^2(y)}{|x-y|} - \lambda\int g|u_n|^p - \int |u_n|^6 + o(1) \\
&= \int |\nabla u|^2 + \int |\nabla v_n|^2 + \frac{1}{4\pi}\iint \frac{u_n^2(x)u_n^2(y)}{|x-y|} \\
&\qquad - \lambda\int g|u|^p - \int |u|^6 - \int |v_n|^6 + o(1) \\
&= \langle \Phi'(u), u\rangle + \int |\nabla v_n|^2 - \int |v_n|^6 \\
&\qquad + \frac{1}{4\pi}\iint \frac{u_n^2(x)u_n^2(y)}{|x-y|} - \frac{1}{4\pi}\iint \frac{u^2(x)u^2(y)}{|x-y|} + o(1) \\
&= \int |\nabla v_n|^2 - \int |v_n|^6 + \frac{1}{4\pi}\iint \frac{u_n^2(x)u_n^2(y)}{|x-y|} - \frac{1}{4\pi}\iint \frac{u^2(x)u^2(y)}{|x-y|} + o(1).
\end{aligned} \tag{3.11}$$

Using (3.10), (3.11) we get (the double integral terms cancel)

$$4\Phi(u) + |\nabla v_n|_2^2 + \frac{1}{3}|v_n|_6^6 \to 4c. \tag{3.12}$$

Since $\{|\nabla v_n|_2^2\}$ and $\{|v_n|_6^6\}$ are bounded, up to a subsequence we may assume that

$$|\nabla v_n|_2^2 \to a, \qquad |v_n|_6^6 \to b.$$

Since $u_n \to u$ a.e. on $\mathbb{R}^3$, by Fatou lemma we have

$$\varliminf_{n\to\infty} \iint \frac{u_n^2(x)u_n^2(y)}{|x-y|} \geqslant \iint \frac{u^2(x)u^2(y)}{|x-y|}. \tag{3.13}$$

From this and (3.11) we see that $b \geqslant a$.

By the definition of S, $|\nabla v_n|_2^2 \geqslant S|v_n|_6^2$, which implies

$$a \geqslant Sb^{2/6} \geqslant Sa^{2/6}.$$

So either $a = 0$ or $a \geqslant S^{3/2}$. If $a \geqslant S^{3/2}$, we deduce from (3.12), $\Phi(u) \geqslant 0$ and $b \geqslant a$ that

$$c \geqslant \frac{1}{4}a + \frac{1}{12}b \geqslant \frac{1}{3}a \geqslant \frac{1}{3}S^{3/2} = c^*,$$

which contradicts our assumption that $c < c^*$. Therefore, $a = 0$. That is, $v_n \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.

Now we follow the argument in [12, Page 58]. By the nonlocal Brezis–Lieb lemma [11, Proposition 4.1], we have

$$\iint \frac{u_n^2(x)u_n^2(y)}{|x-y|} \geqslant \iint \frac{u^2(x)u^2(y)}{|x-y|} + \iint \frac{v_n^2(x)v_n^2(y)}{|x-y|} + o(1). \tag{3.14}$$

Subtracting (3.6) with $v = u$, that is

$$\int |\nabla u|^2 + \frac{1}{4\pi} \iint \frac{u^2(x)u^2(y)}{|x-y|} - \lambda \int g|u|^p - \int |u|^6 = 0,$$

from (3.1) with $v = u_n$, that is

$$\int |\nabla u_n|^2 + \frac{1}{4\pi} \iint \frac{u_n^2(x)u_n^2(y)}{|x-y|} - \lambda \int g|u_n|^p - \int |u_n|^6 = \langle \Phi'(u_n), u_n \rangle = o(1),$$

then using (3.7), (3.8), (3.9) and (3.14), as well as $v_n \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, we get

$$\begin{aligned} \int |\nabla v_n|^2 + \iint \frac{v_n^2(x)v_n^2(y)}{|x-y|} &\leqslant \int |v_n|^6 + o(1) \\ &\leqslant S^{-3}\left(\int |\nabla v_n|^2\right)^3 + o(1) \to 0. \end{aligned} \tag{3.15}$$

This means $v_n \to 0$ in E, that is $u_n \to u$ in E.

Remark 1 Unlike the proof of Mercuri & Perera [12, Lemma 3.6] for the case $g \equiv 1$, our proof here does not depend on the Pohozaev identity [3, Lemma 2.4], therefore is somewhat simpler. If $g \equiv 1$ then as in [12] instead of E we should work on E_r. Thanks to the compact embedding $E_r \hookrightarrow L^p(\mathbb{R}^3)$, (3.4) is still valid and our argument works as well.

4 Potential case

The argument above can be applied to similar equations with an external potential

$$-\Delta u + V(x)u + \left(\frac{1}{4\pi|x|} * u^2\right)u = \lambda g(x)|u|^{p-2}u + |u|^{q-2}u, \qquad u \in H^1(\mathbb{R}^3), \tag{4.1}$$

where $4 < p < q \leqslant 6$, $V : \mathbb{R}^3 \to (0, \infty)$ is any external potential such that

$$\|u\|_V = \left(\int (|\nabla u|^2 + Vu^2) \right)^{1/2}$$

is equivalent to the standard H^1-norm. The equation (4.1) is equivalent to the nonlinear Schrödinger-Poisson systems with critical or subcritical growth:

$$\begin{cases} -\Delta u + V(x)u + \phi u = \lambda g(x)|u|^{p-2}u + |u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

for which, there are many results, see [5, 8, 15, 17–19]. We have the following multiplicity result.

Theorem 4.1 *Let g satisfy the condition (g). Given $m \in \mathbb{N}$, there is $\lambda_m > 0$ such that (4.1) has m pairs of solutions with positive energy for all $\lambda \geqslant \lambda_m$.*

For the proof, let X be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\| = \left[\int (|\nabla u|^2 + Vu^2) + \left(\iint \frac{u^2(x)u^2(y)}{|x-y|} \right)^{1/2} \right]^{1/2}. \tag{4.2}$$

Clearly we have continuous embeddings $X \hookrightarrow E$ and $X \hookrightarrow H^1(\mathbb{R}^3)$. Therefore we can define $\Phi : X \to \mathbb{R}$ via

$$\Phi(u) = \frac{1}{2} \int (|\nabla u|^2 + Vu^2) + \frac{1}{16\pi} \iint \frac{u^2(x)u^2(y)}{|x-y|} - \frac{\lambda}{p} \int g|u|^p - \frac{1}{q} \int |u|^q.$$

Then $\Phi \in C^1(X)$ and critical points of Φ are solutions of (4.1). Let

$$S_q = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \frac{\|u\|_V^2}{|u|_q^2}, \qquad c^* = \frac{1}{3} S_q^{3/2}. \tag{4.3}$$

Similar to the proof of Theorem 1.1, for $m \in \mathbb{N}$ let

$$Z = \{u \in X \mid \operatorname{supp} u \subset \Omega\}$$

and Z_m be an m-dimensional subspace of Z. As in the proof of Theorem 1.1, there is $R > 0$ such that (2.1) holds for $A = Z_m \cap \partial \mathfrak{B}_R$ and

$$B = \{tu \mid t \in [0, 1], u \in A\}.$$

Since $i(A) = m$, we see that the geometric conditions of Proposition 2.1 hold. To get m-pairs of critical points for Φ, it suffices to verify $(\mathrm{PS})_c$ for $c \in (0, c^*)$ with c^* now given in (4.3).

Thus, let $\{u_n\}$ be a $(\mathrm{PS})_c$ sequence with $c \in (0, c^*)$. Similar to (3.2), we have

$$c + 1 \geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u\|_V^2 + \left(\frac{1}{16\pi} - \frac{1}{4\mu\pi}\right) \iint \frac{u_n^2(x) u_n^2(y)}{|x-y|}.$$

It follows that u_n is bounded in X.

Thanks to the continuous embeddings $X \hookrightarrow E$ and $X \hookrightarrow H^1(\mathbb{R}^3)$, up to a subsequence we have

$$u_n \rightharpoonup u \quad \text{in } E, \qquad \text{and} \qquad u_n \rightharpoonup u \quad \text{in } H^1(\mathbb{R}^3).$$

Hence, replacing the exponent 6 by q, the argument between (3.3) through (3.12) remains valid except the $\mathcal{D}^{1,2}$-norm needs to be replaced by $\|\cdot\|_V$, the equivalent H^1-norm. In particular, (3.12) now reads

$$4\Phi(u) + \|v_n\|_V^2 + \frac{1}{3}|v_n|_q^q \to 4c, \tag{4.4}$$

being $v_n = u_n - u$. Note that u is a critical point of $\Phi : X \to \mathbb{R}$ with $\Phi(u) \geqslant 0$.

As before, assuming

$$\|v_n\|_V^2 \to a, \qquad |v_n|_q^q \to b.$$

Then $b \geqslant a$. Combining $\|v_n\|_V^2 \geqslant S_q |v_n|_q^2$, a consequence of (4.3), we get $a \geqslant S_q a^{1/3}$. If $a \geqslant S_q^{3/2}$ we get from (4.4) and $b \geqslant a$ that

$$c \geqslant \frac{1}{4}a + \frac{1}{12}b \geqslant \frac{1}{3}a \geqslant \frac{1}{3}S_q^{3/2} = c^*,$$

contradicting $c \in (0, c^*)$. Thus $a = 0$ and $v_n \to 0$ in $H^1(\mathbb{R}^3)$. The estimate (3.15) now reads

$$\|v_n\|_V^2 + \iint \frac{v_n^2(x) v_n^2(y)}{|x-y|} \leqslant S_q^{-3} \|v_n\|_V^3 + o(1) \to 0,$$

which means $u_n \to u$ in X.

5 Variants of Theorem 4.1

Checking the proofs of Theorems 1.1 and 4.1, we see that the condition that $g \in L^{6/(6-p)}(\mathbb{R}^3)$ is only used to ensure

$$\int g|u_n|^p \to \int g|u|^p, \qquad \int g|u_n|^{p-2}u_n v \to \int g|u|^{p-2}uv \tag{5.1}$$

for $u_n \rightharpoonup u$ in E or X, see (3.4). Therefore, we can replace this conditions by other conditions ensuring (5.1). For example, it is well known that (5.1) holds provided

$$\lim_{|x|\to\infty} g(x) = 0. \tag{5.2}$$

So we have the following variant of Theorems 1.1 and 4.1.

Theorem 5.1 *Assume that the continuous function* $g : \mathbb{R}^3 \to (0, \infty)$ *satisfy* (5.2), $p \in (4, 6)$. *Given* $m \in \mathbb{N}$, *there is* $\lambda_m > 0$ *such that both* (1.1) *and* (4.1) *have* m *pairs of solutions with positive energy for all* $\lambda \geqslant \lambda_m$.

On the other hand, if the potential V is coercive

$$\lim_{|x|\to\infty} V(x) = +\infty, \tag{5.3}$$

then by Bartsch & Wang [1] we have a compact embedding $H_V \hookrightarrow L^2(\mathbb{R}^3)$. As a consequence the embedding $X \hookrightarrow L^2(\mathbb{R}^3)$ is also compact and (5.1) is valid provided $g \in L^\infty(\mathbb{R}^3)$. Hence, for V satisfying (5.3) the same multiplicity result is true assuming $g \in L^\infty(\mathbb{R}^3)$ and $p \in (4, 6)$.

Data availability

This manuscript has no associated data.

Declarations

Conflict of interest

The author declare that he has no conflict of interest.

References

[1] Bartsch, T., Wang, Z.Q.: Existence and multiplicity results for some superlinear elliptic problems on $\mathbf{R}^N$. Comm. Partial Differential Equations **20**, 1725–1741 (1995)

[2] Brézis, H., Lieb, E.: A relation between pointwise convergence of functions and convergence of functionals. Proc. Amer. Math. Soc. **88**, 486–490 (1983)

[3] Dutko, T., Mercuri, C., Tyler, T.M.: Groundstates and infinitely many high energy solutions to a class of nonlinear Schrödinger-Poisson systems. Calc. Var. Partial Differential Equations **60**, Paper No. 174, 46 (2021)

[4] Fadell, E.R., Rabinowitz, P.H.: Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. Invent. Math. **45**, 139–174 (1978)

[5] Furtado, M.F., Wang, Y., Zhang, Z.: Positive and nodal ground state solutions for a critical Schrödinger-Poisson system with indefinite potentials. J. Math. Anal. Appl. **526**, Paper No. 127252, 23 (2023)

[6] Gu, Y., Liao, F.: Ground state solutions of Nehari-Pohozaev type for Schrödinger-Poisson-Slater equation with zero mass and critical growth. J. Geom. Anal. **34**, Paper No. 221, 19 (2024)

[7] Ianni, I., Ruiz, D.: Ground and bound states for a static Schrödinger-Poisson-Slater problem. Commun. Contemp. Math. **14**, 1250003, 22 (2012)

[8] Kang, J.C., Liu, X.Q., Tang, C.L.: Ground state sign-changing solutions for critical Schrödinger-Poisson system with steep potential well. J. Geom. Anal. **33**, Paper No. 59, 24 (2023)

[9] Liu, S., Perera, K.: Multiple solutions for (p, q)-Laplacian equations in $\mathbb{R}^n$ with critical or subcritical exponents. Calc. Var. Partial Differential Equations **63**, Paper No. 199, 15 (2024)

[10] Liu, Z., Zhang, Z., Huang, S.: Existence and nonexistence of positive solutions for a static Schrödinger-Poisson-Slater equation. J. Differential Equations **266**, 5912–5941 (2019)

[11] Mercuri, C., Moroz, V., Van Schaftingen, J.: Groundstates and radial solutions to nonlinear Schrödinger-Poisson-Slater equations at the critical frequency. Calc. Var. Partial Differential Equations **55**, Art. 146, 58 (2016)

[12] Mercuri, C., Perera, K.: Variational methods for scaled functionals with applications to the schrödinger-poisson-slater equation (2024), arXiv:2411.15887

[13] Perera, K.: Abstract multiplicity theorems and applications to critical growth problems (2024), arXiv:2308.07901

[14] Ruiz, D.: On the Schrödinger-Poisson-Slater system: behavior of minimizers, radial and nonradial cases. Arch. Ration. Mech. Anal. **198**, 349–368 (2010)

[15] Wang, Y., Yuan, R.: Nonexistence of ground state sign-changing solutions for autonomous Schrödinger-Poisson system with critical growth. Appl. Anal. **102**, 4652–4658 (2023)

[16] Willem, M.: Minimax theorems. Progress in Nonlinear Differential Equations and their Applications, 24, Birkhäuser Boston Inc., Boston, MA, 1996

[17] Yao, X., Li, X., Zhang, F., Mu, C.: Infinitely many solutions of Schrödinger-Poisson equations with critical and sublinear terms. Adv. Math. Phys. Art. ID 8453176, 9 (2019)

[18] Zhang, J.: Multi-bump solutions to Schrödinger-Poisson equations with critical growth in $\mathbb{R}^3$. Calc. Var. Partial Differential Equations **64**, Paper No. 62, 28 (2025)

[19] Zhao, L., Zhao, F.: Positive solutions for Schrödinger-Poisson equations with a critical exponent. Nonlinear Anal. **70**, 2150–2164 (2009)