Gagliardo-Nirenberg-Sobolev inequality: an induction proof

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1. Introduction

Let $p \in [1, n)$, then there is a constant $C_{n,p} > 0$ such that

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\right)^{1/p^*} \le C_{n,p}\left(\int_{\mathbb{R}^n} |\nabla u|^p\right)^{1/p} \qquad \text{for all } u \in C_0^1(\mathbb{R}^n), \tag{1.1}$$ ee

where $p^* = np/(n-p)$ is the critical Sobolev exponent.

This is the Gagliardo-Nirenberg-Sobolev inequality, which is very fundamental in the theory of Sobolev spaces and partial differential equations. It is well known that (1.1) follows from applying the result for the case $p = 1$ to $|u|^\gamma$ with some suitable exponent γ. Traditionally (see e.g. [**2**, Page 277–278]), the inequality (1.1) for $p = 1$ is proved by integrating

$$|u(x)|^{n/(n-1)} \le \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty} |\partial_i u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)|\,\mathrm{d}y_i\right)^{1/(n-1)}$$

with respect to x_1 and applying the extended Hölder inequality, then repeat this procedure with respect to x_2, x_3, ..., x_n successively. In every step the extended Hölder inequality is applied. This tedious procedure is not very transparent, and is not easy to follow (especially for the beginning graduate students).

Observe that the inequality (for $p = 1$) is a proposition about the dimension n. Therefore it should be natural to prove the result by induction on n. Surprisingly, we could not find such a proof in the literature. The purpose of this note is to present the induction proof, which is very simple and transparent.

t1 **Theorem 1.1.** *Let $n > 1$, $u \in C_0^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |u|^{n/(n-1)} \le \left(\int_{\mathbb{R}^n} |\nabla u|\right)^{n/(n-1)}. \tag{1.2}$$ e0

Proof. Our induction proof starts with $n = 2$, which can be found in [**1**, Page 27]. We present it here for the reader's convenience. Since $u \in C_0^1(\mathbb{R}^n)$, we have

$$|u(x)| = \left| \int_{-\infty}^{x_1} \partial_1 u(s, x_2)\, \mathrm{d}s \right| \le \int_{-\infty}^{\infty} |\nabla u(s, x_2)|\, \mathrm{d}s.$$

Replacing the right hand side by $\int_{-\infty}^{\infty} |\nabla u(x_1, s)|\, \mathrm{d}s$, the inequality is also true. Thus

$$|u(x)|^2 \le \left(\int_{-\infty}^{\infty} |\nabla u(s, x_2)|\, \mathrm{d}s \right) \left(\int_{-\infty}^{\infty} |\nabla u(x_1, s)|\, \mathrm{d}s \right).$$

Integrating both sides over $\mathbb{R}^2$ gives (1.2) for $n = 2$.

Assume that the result is true for $n = m$, namely

$$\int_{\mathbb{R}^m} |u|^{m/(m-1)} \le \left(\int_{\mathbb{R}^m} |\nabla u| \right)^{m/(m-1)} \qquad \text{for } u \in C_0^1(\mathbb{R}^m). \tag{1.3}$$

$\boxed{\text{e}}$

To consider the case that $n = m + 1$, we write points in $\mathbb{R}^{m+1}$ by (x, t), being $x \in \mathbb{R}^m$ and $t \in \mathbb{R}$. Then for $u \in C_0^1(\mathbb{R}^{m+1})$,

$$|u(x, t)| = \left| \int_{-\infty}^{t} \partial_{m+1} u(x, s)\, \mathrm{d}s \right| \le \int_{-\infty}^{\infty} |\nabla u(x, s)|\, \mathrm{d}s,$$

and

$$\int_{\mathbb{R}^m} |u(x, t)|\, \mathrm{d}x \le \int_{\mathbb{R}^m} \mathrm{d}x \int_{-\infty}^{\infty} |\nabla u(x, s)|\, \mathrm{d}s = \int_{\mathbb{R}^{m+1}} |\nabla u|. \tag{1.4}$$

$\boxed{\text{e1}}$

Here we write the differentials like $\mathrm{d}s$ to indicate with respect to which variable we are integrating; if we are integrating with respect to all variables, we omit the differentials (for example, $\mathrm{d}x\mathrm{d}s$ is omitted in the final integral in (1.4)).

Using the Hölder inequality, (1.3) with u replaced by $u(\cdot, t) \in C_0^1(\mathbb{R}^m)$, and (1.4), we have

$$\begin{aligned} \int_{\mathbb{R}^m} |u(x, t)|^{(m+1)/m}\, \mathrm{d}x &= \int_{\mathbb{R}^m} |u|^{1/m} |u|\, \mathrm{d}x \\ &\le \left(\int_{\mathbb{R}^m} |u|\, \mathrm{d}x \right)^{1/m} \left(\int_{\mathbb{R}^m} |u|^{m/(m-1)}\, \mathrm{d}x \right)^{(m-1)/m} \\ &\le \left(\int_{\mathbb{R}^{m+1}} |\nabla u| \right)^{1/m} \left(\int_{\mathbb{R}^m} |\nabla_x u(x, t)|\, \mathrm{d}x \right), \end{aligned} \tag{1.5}$$

$\boxed{\text{x}}$

where $\nabla_x u(x,t)$ is the gradient of $u(\cdot,t)$ at x. Obviously $|\nabla_x u(x,t)| \le |\nabla u(x,t)|$, thus integrating both sides with respect to t over $\mathbb{R}$ yields

$$\begin{aligned}\int_{\mathbb{R}^{m+1}} |u|^{(m+1)/m} &= \int_{-\infty}^{\infty} \mathrm{d}t \int_{\mathbb{R}^m} |u(x,t)|^{(m+1)/m}\, \mathrm{d}x \\ &\le \left(\int_{\mathbb{R}^{m+1}} |\nabla u|\right)^{1/m} \left[\int_{-\infty}^{\infty} \mathrm{d}t \int_{\mathbb{R}^m} |\nabla u(x,t)|\, \mathrm{d}x\right] \\ &= \left(\int_{\mathbb{R}^{m+1}} |\nabla u|\right)^{(m+1)/m},\end{aligned}$$

we have obtained (1.2) for $n = m+1$. So (1.2) is true for all $n \ge 2$.

Remark 1.2. Using the same idea, a stronger inequality

$$\int_{\mathbb{R}^n} |u|^{n/(n-1)} \le \left(\prod_{i=1}^{n} \int_{\mathbb{R}^n} |\partial_i u|\right)^{1/(n-1)} \tag{S_n}$$

can be proved. To derive (S_{m+1}) from (S_m), we replace all the ∇u before (1.5) by $\partial_{m+1} u$, then apply the induction assumption (S_m) to $\int |u|^{m/(m-1)}$ in (1.5). When integrating with respect to t we need the extended Hölder inequality, which is not needed in our proof of Theorem 1.1.

In [**3**], combining a simple iteration scheme with this stronger version of the Gagliardo-Nirenberg-Sobolev inequality, the author gives a new proof of the L^∞-embedding

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le C_p \left(\|u\|_{L^p(\mathbb{R}^n)} + \|\nabla u\|_{L^p(\mathbb{R}^n)}\right),$$

where $p > n$, $u \in C_0^1(\mathbb{R}^n)$.

2. ODE

Consider

$$y'' + py' + qy = e^{\lambda x} p_m(x).$$

Let $y = e^{\lambda x} z$, then

$$\begin{aligned} y' &= e^{\lambda x} z' + \lambda e^{\lambda x} z, \\ y'' &= e^{\lambda x} z'' + 2\lambda e^{\lambda x} z' + \lambda^2 e^{\lambda x} z.\end{aligned}$$

The ODE becomes

$$\begin{aligned} e^{\lambda x}\left(z'' + (2\lambda + p) z' + \left(\lambda^2 + p\lambda + q\right) z\right) &= e^{\lambda x} p_m(x), \\ z'' + (2\lambda + p) z' + \left(\lambda^2 + p\lambda + q\right) z &= p_m(x),\end{aligned}$$

which can be solved by letting $z = q_m(x)$.

Example 2.1. $y'' + 3y' + 2y = e^{-2x}\left(x^2 + 2x - 1\right)$.

Proof. Let $y = e^{-2x}z$, then $y' = e^{-2x}z' - 2e^{-2x}z$,

$$y'' = e^{-2x}z'' - 4e^{-2x}z' + 4e^{-2x}z.$$

The ODE hecomes

$$z'' - z' = x^2 + 2x - 1.$$

Asusme $z = \sum_{i=1}^{3} a_i x^i$, then

$$-3a_3x^2 + (6a_3 - 2a_2)x + (2a_2 - a_1) = x^2 + 2x - 1.$$

Thus

$$-3a_3 = 1, \qquad 6a_3 - 2a_2 = 2, \qquad 2a_2 - a_1 = -1.$$

Hence

$$a_3 = -\frac{1}{3}, \qquad a_2 = -2, \qquad a_1 = -3,$$

$$z = -\frac{x^3}{3} - 2x^2 - 3x.$$

A solution of our ODE is

$$y = e^{-2x}\left(-\frac{x^3}{3} - 2x^2 - 3x\right).$$

Example 2.2. $y'' - 6y' + 13y = xe^{3x}\sin 2x$.

Proof. Let $y = e^{3x}z$, then $y' = e^{3x}z' + 3e^{3x}z$,

$$y'' = e^{3x}z'' + 6e^{3x}z' + 9e^{3x}z.$$

The ODE becomes

$$z'' + 4z = x\sin 2x.$$

Substituting (since $2i$ is a root of the characteristic equation, we multiply x)

$$\begin{aligned} z &= x\left[(ax+b)\cos 2x + (cx+d)\sin 2x\right] \\ &= \left(ax^2 + bx\right)\cos 2x + \left(cx^2 + dx\right)\sin 2x \end{aligned}$$

into the ODE, we get

$$(2a + 4d + 8cx)\cos 2x + (2c - 4b - 8ax)\sin 2x = x\sin 2x.$$

So

$$\begin{aligned} 2a + 4d + 8cx &= 0, \\ 2c - 4b - 8ax &= x. \end{aligned}$$

We get $(c, a, d, b) = \left(0, -\frac{1}{8}, \frac{1}{16}, 0\right)$,

$$z = -\frac{x^2}{8}\cos 2x + \frac{x\sin 2x}{16},$$

$$y = e^{3x}\left(-\frac{x^2}{8}\cos 2x + \frac{x\sin 2x}{16}\right).$$

References

[1] W. Chen, C. Li, Methods on nonlinear elliptic equations, *AIMS Series on Differential Equations & Dynamical Systems*, vol. 4, American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2010.

[2] L. C. Evans, Partial differential equations, *Graduate Studies in Mathematics*, vol. 19, American Mathematical Society, Providence, RI, 2nd ed., 2010.

[3] A. Porretta, A note on the Sobolev and Gagliardo-Nirenberg inequality when $p > N$, Adv. Nonlinear Stud. 20 (2020) 361–371.