

On the integral formula of the Jacobian determinant

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ABSTRACT. It is known that the integral of the Jacobian determinant of a smooth map $f : \bar{\Omega} \rightarrow \mathbb{R}^n$ depends only on $f|_{\partial\Omega}$ and this result leads to an analytic proof of the Brouwer fixed point theorem. In this note we provide two new proofs of this result, one by classical analysis and one by differential forms and Stokes formula.

1. Introduction

In a recent paper [Krylov \(2024\)](#), to respond inquiry from some readers about the solution of Exercise 1.2.1 in his book [Krylov \(2008\)](#), Krylov gives a proof of the following theorem.

Theorem 1.1 ([Krylov \(2024, Lemma 1\)](#)). *Let Ω be a bounded domain in \mathbb{R}^n , $f_{\pm} : \bar{\Omega} \rightarrow \mathbb{R}^n$ be C^2 -maps such that $f_+|_{\partial\Omega} = f_-|_{\partial\Omega}$. then*

$$\int_{\Omega} \det f'_+(x) dx = \int_{\Omega} \det f'_-(x) dx. \quad (1.1)$$

Using this theorem, the *no-retraction theorem* ([Krylov \(2024, Corollary 1\)](#)) follows immediately. Let B be the closed unit ball in \mathbb{R}^n . If there was a smooth retraction $R : B \rightarrow \partial B$, since $R|_{\partial B} = \text{id}|_{\partial B}$, Theorem 1.1 with $f_+ = R$, $f_- = \text{id}$ yields a contradiction

$$\int_B \det R'(x) dx = \int_B dx = \text{Vol}(B) > 0, \quad (1.2)$$

because $\det R'(x) = 0$, a consequence of $R(B) \subset \partial B$. That the two integrals in (1.2) are equal also follows from a version of the changing variable formula given in [Liu & Zhang \(2017, Theorem 3.1\)](#): *For smooth closed bounded domain D in \mathbb{R}^n and smooth map $\varphi : B \rightarrow D$, if $\varphi : \partial B \rightarrow \partial D$ is a diffeomorphism, then for continuous $f : D \rightarrow \mathbb{R}$ there holds*

$$\int_D f(y) dy = \pm \int_B f(\varphi(x)) \det \varphi'(x) dx.$$

The first equality in (1.2) follows by letting $f = 1$, $D = B$ and $\varphi = R$ in this formula. It is well known that the no-retraction theorem is equivalent to the famous Brouwer fixed point theorem.

Krylov's proof of Theorem 1.1 in [Krylov \(2024\)](#) is based on the observation: for small t , $f_{\pm}^t = \text{id} + tf_{\pm}$ are diffeomorphisms and $f_+^t(\Omega) = f_-^t(\Omega)$ because $f_+|_{\partial\Omega} = f_-|_{\partial\Omega}$, thus by the changing variable formula

$$\int_{\Omega} \det(I + tf'_+(x)) dx = \text{Vol}(f_+^t(\Omega)) = \text{Vol}(f_-^t(\Omega)) = \int_{\Omega} \det(I + tf'_-(x)) dx.$$

Since the two sides are polynomials in t , comparing the coefficients of t^n gives (1.1). To make the argument rigorous some issues including why $f_{\pm}^t(\partial\Omega) = \partial f_{\pm}^t(\Omega)$ and

why $\partial f_+^t(\Omega) = \partial f_-^t(\Omega)$ implies $f_+^t(\Omega) = f_-^t(\Omega)$ need to be handled. These are clarified via a careful point set analysis in a long remark following the proof. One of the advantages of this argument is that the maps f_\pm only need to be C^1 .

Theorem 1.1 also appears in Evans (2010); Kulpa (1989). In Kulpa (1989), Kulpa defines

$$I(h^1, \dots, h^n) = \int_{\Omega} \det h'(x) dx$$

for smooth $h : \bar{\Omega} \rightarrow \mathbb{R}^n$ and manages to show

$$I(f_+^1, f_+^2, \dots, f_+^n) = I(f_-^1, f_+^2, \dots, f_+^n) \quad (1.3)$$

using Fubini theorem and projections $\Pi_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ given by deleting the i -th component. Then (1.1) follows by applying (1.3) successively to replace the remaining columns of $\det f_+^t(x)$ by those of $\det f_-^t(x)$. While in Evans (2010, pp. 464), (1.1) is proved using the facts that $L(P) = \det P$ is a null Lagrangian and the energy $\int_{\Omega} L(w'(x), w(x), x) dx$ of null Lagrangian $L : \mathbb{M}^{m \times n} \times \mathbb{R}^m \times \bar{\Omega} \rightarrow \mathbb{R}$ depends only on boundary conditions.

The purpose of this note is to present another two proofs of Theorem 1.1. The first one is performed by playing classical analysis, which is written for readers not familiar with differential forms and in our opinion is very elementary and transparent. The second one is quite short, which depends on differential form and Stokes formula on manifolds. As demonstrated in the paragraph after Theorem 1.1, our note is useful for understanding the Brouwer fixed point theorem from the analytic point of view. In addition to Evans (2010); Krylov (2024); Kulpa (1989); Liu & Zhang (2017) mentioned above, other analytic proofs of the Brouwer fixed point theorem can be found in Kannai (1981); Lax (1999); Milnor (1978); Rogers (1980).

2. Proof of Theorem 1.1

Let f_{\pm}^i be the i -th component of f_{\pm} and $\Phi_{\pm} = (f_{\pm}^2, \dots, f_{\pm}^n)$. Then $\Phi_+ = \Phi_-$ on $\partial\Omega$. Let

$$A_{\pm} = \left(\frac{\partial(f_{\pm}^2, \dots, f_{\pm}^n)}{\partial(x^2, \dots, x^n)}, -\frac{\partial(f_{\pm}^2, f_{\pm}^3, \dots, f_{\pm}^n)}{\partial(x^1, x^3, \dots, x^n)}, \dots, (-1)^{n+1} \frac{\partial(f_{\pm}^2, \dots, f_{\pm}^n)}{\partial(x^1, \dots, x^{n-1})} \right),$$

then the i -th component of A_{\pm} is just the cofactor of $\partial_{x^i} f_{\pm}^1$ in $\det f'_{\pm}(x)$. By Jacobi identity (see e.g. Kannai (1981, Eq. (5)) or Brezis *et al.* (2024, Eq. (0.3))), we have $\operatorname{div} A_{\pm} = 0$. Thus

$$\operatorname{div}(f_{\pm}^1 A_{\pm}) = \nabla f_{\pm}^1 \cdot A_{\pm} + f_{\pm}^1 \operatorname{div} A_{\pm} = \nabla f_{\pm}^1 \cdot A_{\pm} = \det f'_{\pm}(x). \quad (2.1)$$

For $a \in \partial\Omega$, let $\eta : (u^1, \dots, u^{n-1}) \mapsto x$ be a local parametrization of $\partial\Omega$ at $a = \eta(\alpha)$ such that

$$N(a) = \left(\frac{\partial(x^2, \dots, x^n)}{\partial(u^1, \dots, u^{n-1})}, -\frac{\partial(x^1, x^3, \dots, x^n)}{\partial(u^1, \dots, u^{n-1})}, \dots, (-1)^{n+1} \frac{\partial(x^1, \dots, x^{n-1})}{\partial(u^1, \dots, u^{n-1})} \right)_{\alpha}$$

is an outward normal vector of $\partial\Omega$ at a . Applying the chain rule to the composition $y_{\pm} = \Phi_{\pm}(\eta(u))$, we see that

$$\begin{pmatrix} \frac{\partial y_{\pm}^2}{\partial u^1} & \cdots & \frac{\partial y_{\pm}^2}{\partial u^{n-1}} \\ \frac{\partial y_{\pm}^n}{\partial u^1} & \cdots & \frac{\partial y_{\pm}^n}{\partial u^{n-1}} \end{pmatrix}_{\alpha} = \begin{pmatrix} \frac{\partial y_{\pm}^2}{\partial x^1} & \cdots & \frac{\partial y_{\pm}^2}{\partial x^n} \\ \frac{\partial y_{\pm}^n}{\partial x^1} & \cdots & \frac{\partial y_{\pm}^n}{\partial x^n} \end{pmatrix}_a \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^1}{\partial u^{n-1}} \\ \frac{\partial x^n}{\partial u^1} & \cdots & \frac{\partial x^n}{\partial u^{n-1}} \end{pmatrix}_{\alpha},$$

where the matrix on the left is the Jacobian matrix $(\Phi_{\pm} \circ \eta)'(\alpha)$.

Because $\Phi_+ = \Phi_-$ on $\partial\Omega$ and $\eta(u) \in \partial\Omega$ for u near α , we have $\Phi_+ \circ \eta = \Phi_- \circ \eta$ near α . Using the Cauchy-Binet formula we deduce

$$\begin{aligned} (A_+ \cdot N)(a) &= \sum_{i=1}^n \frac{\partial(f_+^2, \dots, f_+^n)}{\partial(x^1, \dots, \hat{x}^i, \dots, x^n)} \Big|_a \cdot \frac{\partial(x^1, \dots, \hat{x}^i, \dots, x^n)}{\partial(u^1, \dots, u^{n-1})} \Big|_{\alpha} \\ &= \det(\Phi_+ \circ \eta)'(\alpha) = \det(\Phi_- \circ \eta)'(\alpha) = (A_- \cdot N)(a). \end{aligned}$$

Therefore

$$A_+ \cdot \nu = \frac{A_+ \cdot N}{|N|} = \frac{A_- \cdot N}{|N|} = A_- \cdot \nu \quad (2.2)$$

on $\partial\Omega$, where $\nu = N/|N|$ is outward unit normal vector field on $\partial\Omega$.

Since $f_+^1 = f_-^1$ on $\partial\Omega$, using the divergence theorem we deduce from (2.1) and (2.2)

$$\begin{aligned} \int_{\Omega} \det f'_+(x) dx &= \int_{\Omega} \operatorname{div}(f_+^1 A_+) dx = \int_{\partial\Omega} f_+^1 A_+ \cdot \nu d\sigma \\ &= \int_{\partial\Omega} f_-^1 A_- \cdot \nu d\sigma = \int_{\Omega} \det f'_-(x) dx. \end{aligned}$$

Example 2.1. Let A be an $n \times n$ matrix, $\varphi \in C_0^\infty(\Omega, \mathbb{R}^n)$. Set $f_+(x) = Ax + \varphi(x)$, $f_-(x) = Ax$, we get the formula stated in [Ambrosio *et al.* \(2018, pp. 21\)](#):

$$\int_{\Omega} \det(A + \varphi'(x)) dx = \det(A) \cdot m(\Omega).$$

3. Another proof using differential forms

To conclude this note, we present another proof using differential forms. Let f^i be the components of $f : \bar{\Omega} \rightarrow \mathbb{R}^n$, then it is well known that

$$\begin{aligned} \det f'(x) dx^1 \wedge \dots \wedge dx^n &= df^1 \wedge df^2 \wedge \dots \wedge df^n. \\ &= d(f^1 df^2 \wedge \dots \wedge df^n), \end{aligned}$$

where in the second equality we used $d^2 = 0$. Recall the Stokes formula

$$\int_{\Omega} d\omega = \int_{\partial\Omega} i^* \omega,$$

where ω is an $(n-1)$ -form on Ω and i^* is the pull back induced by the embedding $i : \partial\Omega \rightarrow \Omega$. Because $i^* \circ d = d \circ i^*$, we conclude

$$\begin{aligned} \int_{\Omega} \det f'(x) dx &= \int_{\Omega} \det f'(x) dx^1 \wedge \dots \wedge dx^n \\ &= \int_{\Omega} d(f^1 df^2 \wedge \dots \wedge df^n) = \int_{\partial\Omega} i^*(f^1 df^2 \wedge \dots \wedge df^n) \\ &= \int_{\partial\Omega} (f^1 \circ i) d(f^2 \circ i) \wedge \dots \wedge d(f^n \circ i). \end{aligned}$$

Clearly the right hand side depends only on $f \circ i = f|_{\partial\Omega}$.

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