Surjections between Euclidean spaces, changing variable formula and Brouwer fixed point theorem

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Contents

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$$F'(a)=\begin{pmatrix}\left(\partial_i f^j\right)_{i,j=1,\ldots,n} & \left(\partial_i f^j\right)_{i>n}\\ 0 & I_{m-n}\end{pmatrix}$$

is invertible. We apply the Inverse Function Theorem to F.

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Thm3 (FTA). Let $a_i \in \mathbb{C}$, $n \geq 1$,

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- Our motivation was to avoid subspace topology.

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From Cauchy-Riemann equation we know

$$p'(z) = 0 \iff \det p'(x,y) = 0, \quad (x,y) \text{ is critical point of } p : \mathbb{R}^2 \to \mathbb{R}^2$$

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(2) (3) means that $f(\mathbb{R}^n)$ is closed in $\mathbb{R}^n$. This motivates our Thm5.

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Thm4 (Local surjection). Let $\Omega \subset \mathbb{R}^m$, $f : \Omega \to \mathbb{R}^n$ be C^1, $a \in \Omega^\circ$. If $Df(a) : \mathbb{R}^m \to \mathbb{R}^n$ is surjective (i.e. $\operatorname{rank} Df(a) = n$), then $f(a) \in [f(\Omega)]^\circ$.

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- Noting that $f(\mathbb{R}^m)=\overline{f(\mathbb{R}^m)}\supset\overline{A}$, it suffices to prove the intuitive result:

  if the union of open set A and a finite set is closed, then $\overline{A}=\mathbb{R}^n$.

Lem1. Let $n \geq 2$, A be nonempty open set in $\mathbb{R}^n$. If there are k points $p_i \in \mathbb{R}^n$ s.t. $A \cup \{p_i\}_{i=1}^k$ is closed, then $\overline{A} = \mathbb{R}^n$.

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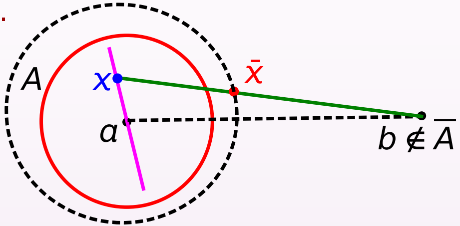

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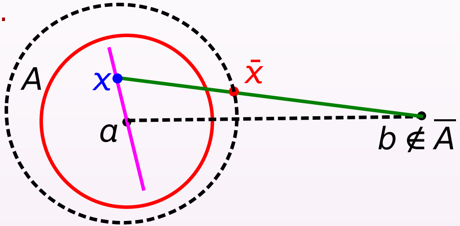

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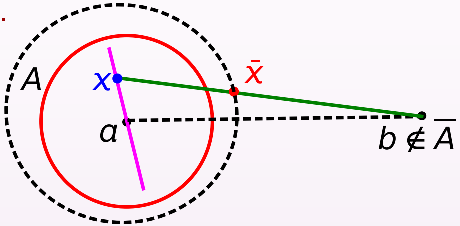

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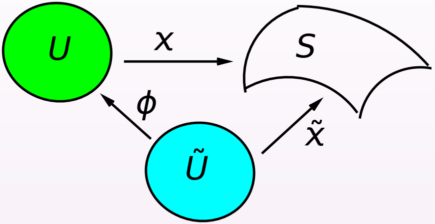
Cor3. Let $n \geq 2$, M be m-dimensional compact manifold without boundary. If $f : M \to \mathbb{R}^n$ is C^1-map, then f has infinitely many critical points.

3. Changing variable for multiple integrals

Motivated by [dC76] (CVF for double integral via Green Theorem), we assume CVF for $(m-1)$-integrals, define **surface integral** in $\mathbb{R}^m$ and prove the **Divergence Theorem**, then prove CVF for m-integrals.



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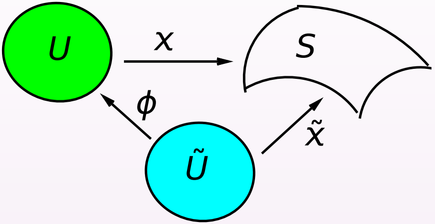

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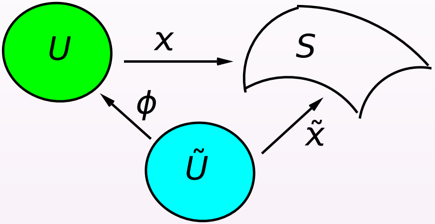

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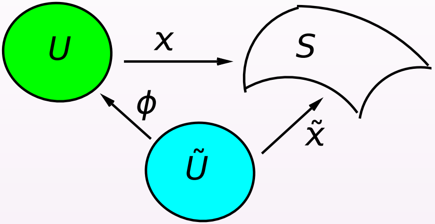

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  Then $\tilde{x}(\tilde{U}) = x(U)$, we can identify the equivalent class $[x]$ with $x(U)$, called **smooth surface**, and denoted by $S = x(U)$ or $S = [x : U \to \mathbb{R}^m]$.

Using Cramer we know that a **normal vector** of $S = [x : U \to \mathbb{R}^m]$ at $x(u)$ is

$$N(u) = \left(\frac{\partial(x^2, \ldots, x^m)}{\partial(u^1, \ldots, u^{m-1})}, \ldots, (-1)^{m+1} \frac{\partial(x^1, \ldots, x^{m-1})}{\partial(u^1, \ldots, u^{m-1})} \right).$$

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Thm6 (Divergence Theorem). Let $D \subset \mathbb{R}^m$ be bounded closed domain, ∂D piece-wise smooth, $F \in C^1(D, \mathbb{R}^m)$, n is unit outer normal of ∂D, then

$$\int_D \operatorname{div} F \, \mathrm{d}x = \int_{\partial D} F \cdot n \, \mathrm{d}\sigma.$$

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$$N(u)=\left(\frac{\partial(x^2,\ldots,x^m)}{\partial(u^1,\ldots,u^{m-1})},\ldots,(-1)^{m+1}\frac{\partial(x^1,\ldots,x^{m-1})}{\partial(u^1,\ldots,u^{m-1})}\right).$$

The **surface integral** of cont $f:S\to\mathbb{R}$ on $S=[x:U\to\mathbb{R}^m]$ is defined by

$$\int_S f(x)\mathrm{d}\sigma=\int_U f(x(u))\,|N(u)|\,\mathrm{d}u. \tag{4}$$

By $(m-1)$-dim CVF, the RHS is **indept** with the parametrization of S.

For piece-wise smooth surface $\Sigma=\bigcup_{i=1}^{\ell}S_i$, where $S_i=x_i(U_i)$ interiorly-disjoint, set

$$\int_\Sigma f\mathrm{d}\sigma=\sum_{i=1}^{\ell}\int_{S_i}f\mathrm{d}\sigma. \qquad x_i(U_i^\circ)\cap x_j(U_j^\circ)=\varnothing.$$

Thm6 (Divergence Theorem). Let $D\subset\mathbb{R}^m$ be bounded closed domain, ∂D piece-wise smooth, $F\in C^1(D,\mathbb{R}^m)$, n is unit outer normal of ∂D, then

$$\int_D \operatorname{div}F\,\mathrm{d}x=\int_{\partial D}F\cdot n\,\mathrm{d}\sigma. \qquad \int_D\partial_i f\,\mathrm{d}x=\int_{\partial D}fn^i\,\mathrm{d}\sigma \quad \text{for } f\in C^1(D).$$

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A bdd domain Ω is **simple**, if there is $(m-1)$-dim C^1–parametrized surface $x : U \to \mathbb{R}^m$ s.t. $\partial\Omega = x(U)$. Note that U is closed, x is injective in U°.

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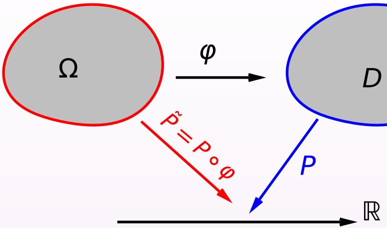
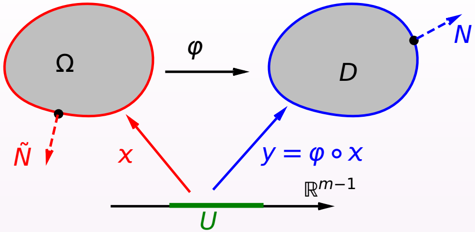
Lax 99: $f \in C_0(\mathbb{R}^m)$
φ is identity outside some ball

$$\int_{\mathbb{R}^m} f(y)\mathrm{d}y = \int_{\mathbb{R}^m} f(\varphi(x))\det\left(\frac{\partial y}{\partial x}\right)\mathrm{d}x.$$





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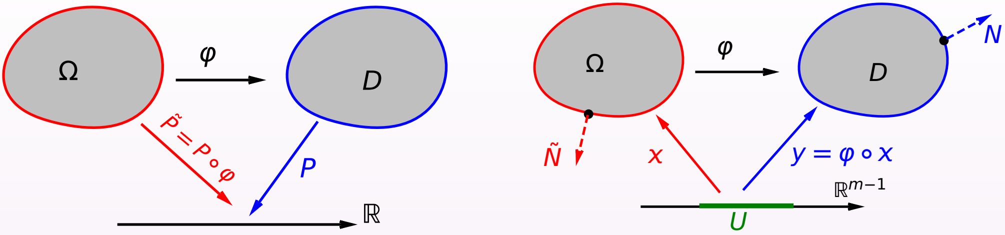

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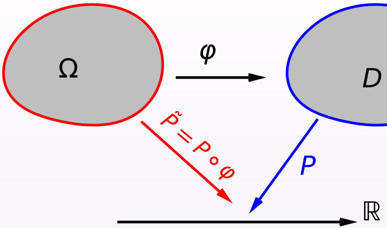

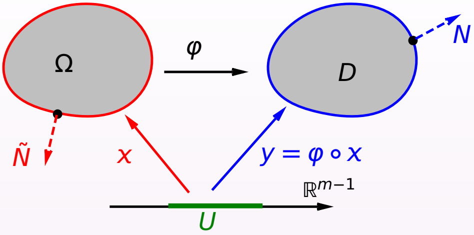

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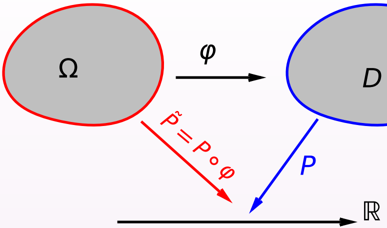

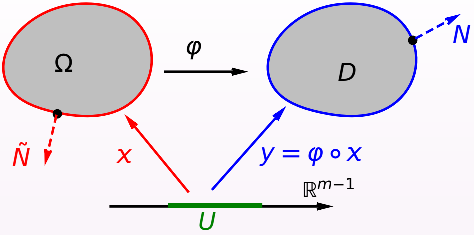

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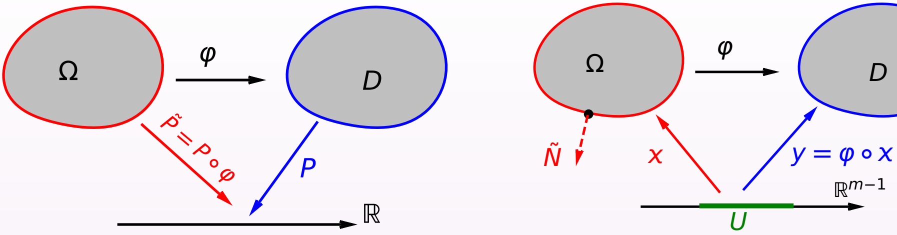

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To compute $\operatorname{div}(\tilde{P}A)$, let $y_i^k = \partial y^k / \partial x^i$, then A_i is algebraic cofactor of y_i^1 in

$$\frac{\partial(y^1,\ldots,y^m)}{\partial(x^1,\ldots,x^m)} = \det \begin{pmatrix} y_1^1 & y_2^1 & \cdots & y_i^1 & \cdots & y_m^1 \\ y_1^2 & y_2^2 & \cdots & y_i^2 & \cdots & y_m^2 \\ \vdots & \vdots & & \vdots & & \vdots \\ y_1^m & y_2^m & \cdots & y_i^m & \cdots & y_m^m \end{pmatrix}.$$

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From (5) we get

$$\int_D f(y)\mathrm{d}y = \pm\int_\Omega f(\varphi(x)) J_\varphi(x)\mathrm{d}x.$$

Cor4. Under assumptions of Thm 7 , if J_φ does not change sign on $\overline{\Omega}$, then

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Thm8. Let D and Ω be Jordan measurable bounded open domains in $\mathbb{R}^m$, $f \in C(\overline{D})$, $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^m)$, $\varphi : \Omega \to D$ is diffeomorphism, then

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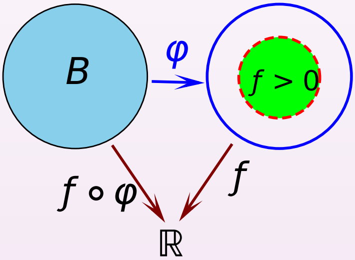

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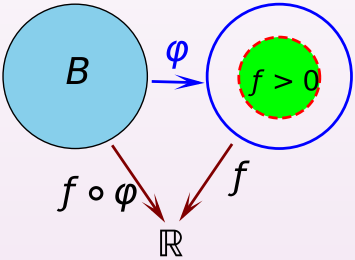

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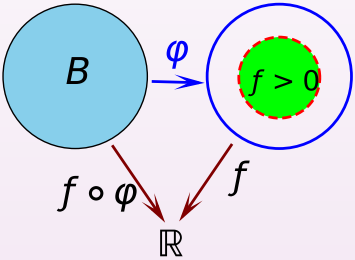

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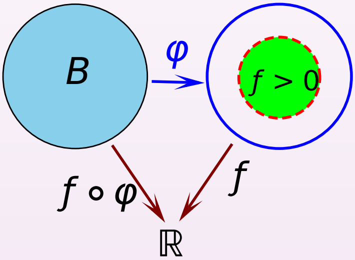

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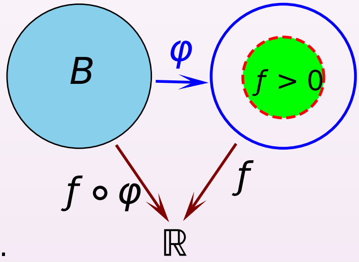

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5. Coarea formula and applications

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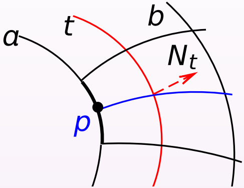

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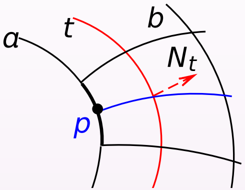

Thm10. Let $G \subset \mathbb{R}^m$ be bdd open, $f \in C^2(G)$, $\nabla f(x) \neq 0$ for $\forall x \in G$. $\Omega = f^{-1}[a,b] \subset G$.

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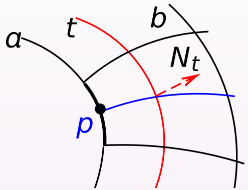

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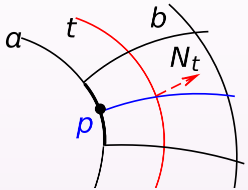

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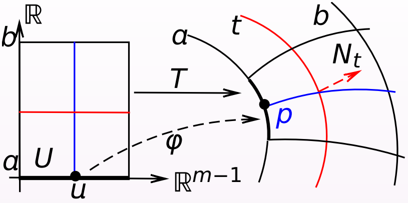
Let $\varphi : U \to \mathbb{R}^m$ be par of $f^{-1}(a)$, then the C^1-map $T : U \times [a,b] \to \mathbb{R}^m$,

$$T(u,t) = x(t-a, \varphi(u)) \qquad \text{is interiorly injective,}$$

$T(\cdot, t) : U \to \mathbb{R}^m$ is par of $f^{-1}(t)$, with normal $N_t(u)$.



Expanding $\det T'(u,t)$ and using (6) yield

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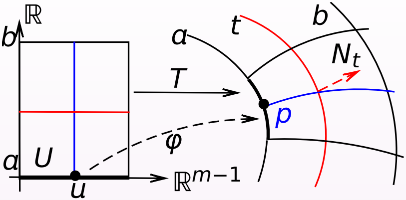

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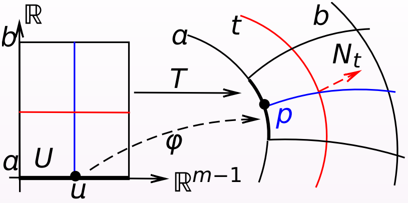

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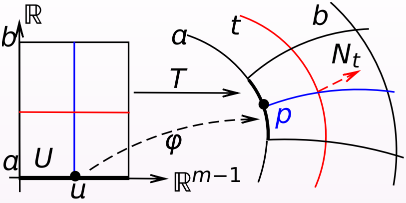

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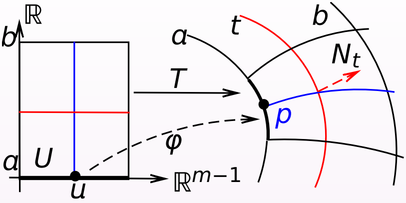

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Exm3. Let B be unit ball in $\mathbb{R}^m$, $f \in C^1(B)$, $f|_{\partial B} = 0$. Find

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$$= \int_{|y|=1} [-f(\varepsilon y)]\, \mathrm{d}\sigma \to -f(0)\omega_m.$$

Rek8. This can also be solved using divergence theorem

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Appendix

[*] Pf of L-multipliers,

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$$\begin{aligned} g^1(x) &= 0 \\ &\vdots \\ g^n(x) &= 0 \end{aligned}$$

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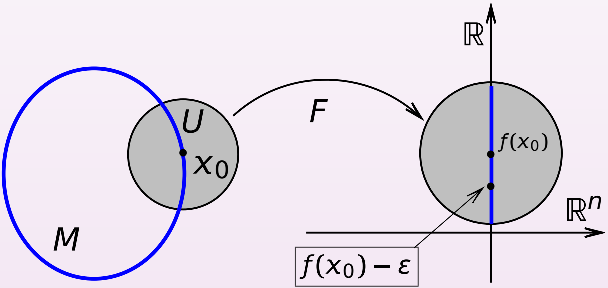

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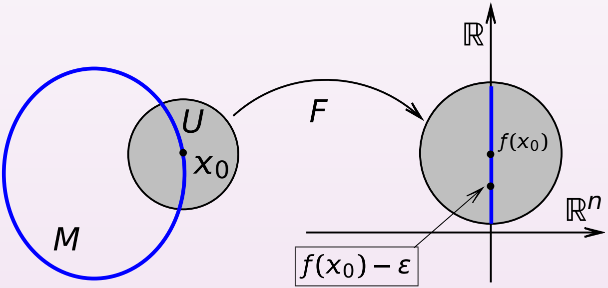

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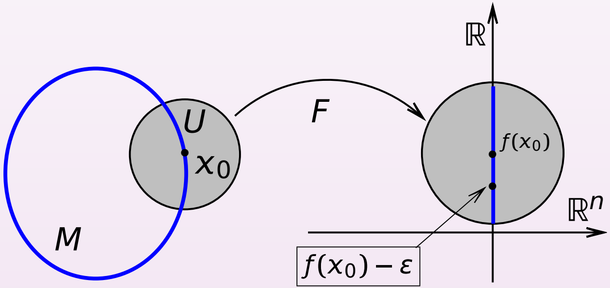

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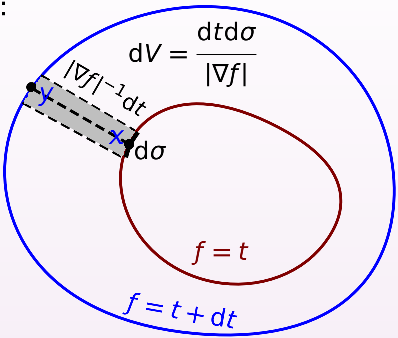

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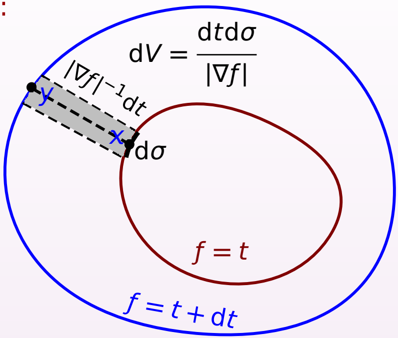

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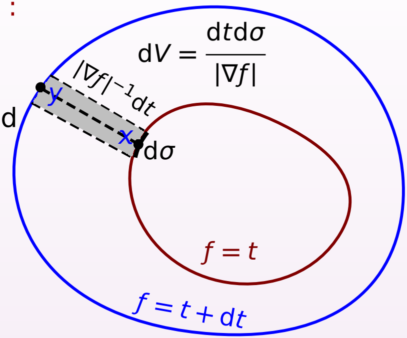

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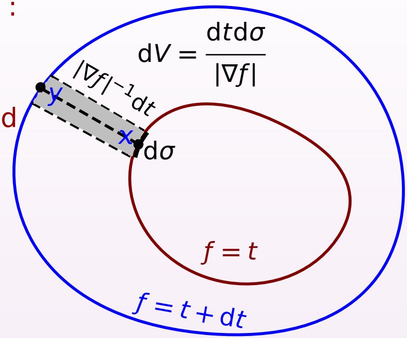

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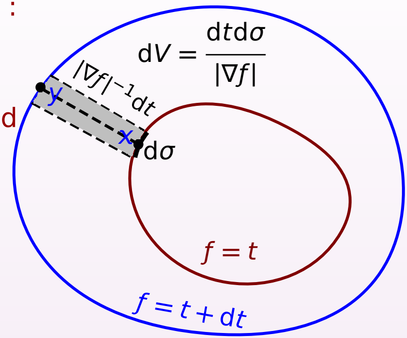

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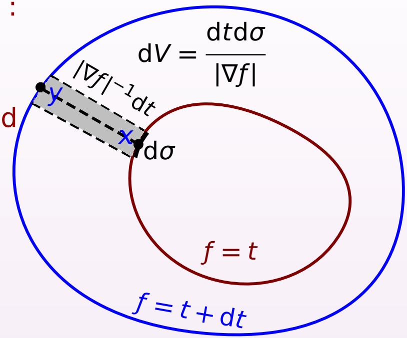

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Volume of the gray column with base $\mathrm{d}\sigma$ and high $|y - x|$ is

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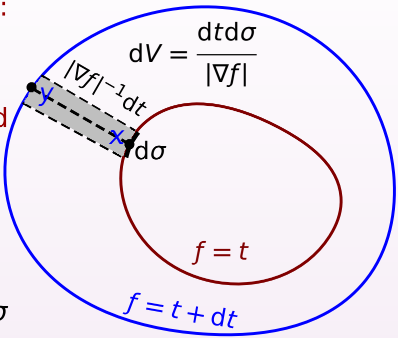

[✱] Coarea and method of element

Let $G \subset \mathbb{R}^m$, $f : G \to \mathbb{R}$. $\Omega = f^{-1}[a, b]$, $g : \Omega \to \mathbb{R}$.

Take surface element $\mathrm{d}\sigma$ at $x \in f^{-1}(t)$. Let y be the intersection of $f^{-1}(t + \mathrm{d}t)$ and normal line of $f^{-1}(t)$ at x. Then

$$\mathrm{d}t = f(y) - f(x) \approx \nabla f(x) \cdot (y - x),$$

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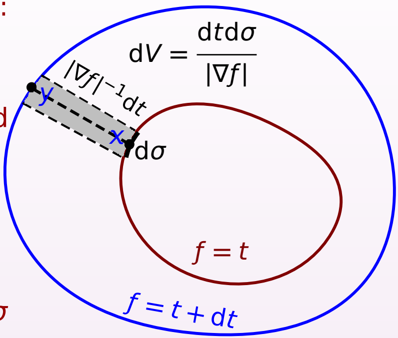

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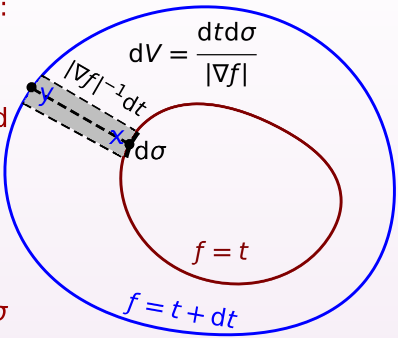

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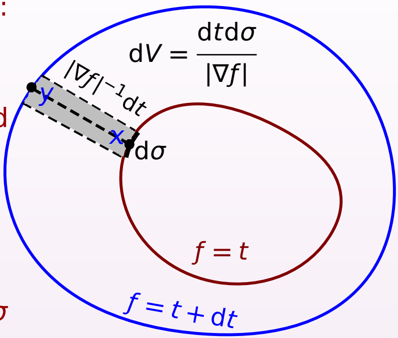

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Hence mass of Ω is

$$\int_\Omega g(x)\mathrm{d}x = \int_a^b \mathrm{d}t \int_{f=t} \frac{g(x)}{|\nabla f(x)|}\mathrm{d}\sigma.$$

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$$\begin{aligned}\frac{d}{dr}\int_{B_r} g(x,r)\,dx &= \int_{|x|=r} g(x,r)\,d\sigma + \int_0^r dt \int_{|x|=t} \partial_r g(x,r)\,d\sigma \\ &= \int_{\partial B_r} g(x,r)\,d\sigma + \int_{B_r} \partial_r g(x,r)\,dx.\end{aligned}$$

[✱] Chain rule and Cauchy-Binet

$$
\begin{pmatrix}
\frac{\partial y^2}{\partial u^1} & \cdots & \frac{\partial y^2}{\partial u^{m-1}} \\
\vdots & & \vdots \\
\frac{\partial y^m}{\partial u^1} & \cdots & \frac{\partial y^m}{\partial u^{m-1}}
\end{pmatrix}
=
\begin{pmatrix}
\frac{\partial y^2}{\partial x^1} & \cdots & \color{red}{\frac{\partial y^2}{\partial x^i}} & \cdots & \frac{\partial y^2}{\partial x^m} \\
\vdots & & \color{red}{\vdots} & & \vdots \\
\frac{\partial y^m}{\partial x^1} & \cdots & \color{red}{\frac{\partial y^m}{\partial x^i}} & \cdots & \frac{\partial y^m}{\partial x^m}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^1}{\partial u^{m-1}} \\
\vdots & & \vdots \\
\color{red}{\frac{\partial x^i}{\partial u^1}} & \color{red}{\cdots} & \color{red}{\frac{\partial x^i}{\partial u^{m-1}}} \\
\vdots & & \vdots \\
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\end{pmatrix}
$$

$(m-1)\times(m-1)$ $\qquad$ $(m-1)\times m$ $\qquad$ $m\times(m-1)$

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$$\begin{pmatrix} \frac{\partial y^2}{\partial u^1} & \cdots & \frac{\partial y^2}{\partial u^{m-1}} \\ \vdots & & \vdots \\ \frac{\partial y^m}{\partial u^1} & \cdots & \frac{\partial y^m}{\partial u^{m-1}} \end{pmatrix} = \begin{pmatrix} \frac{\partial y^2}{\partial x^1} & \cdots & \frac{\partial y^2}{\partial x^i} & \cdots & \frac{\partial y^2}{\partial x^m} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial y^m}{\partial x^1} & \cdots & \frac{\partial y^m}{\partial x^i} & \cdots & \frac{\partial y^m}{\partial x^m} \end{pmatrix} \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^1}{\partial u^{m-1}} \\ \vdots & & \vdots \\ \frac{\partial x^i}{\partial u^1} & \cdots & \frac{\partial x^i}{\partial u^{m-1}} \\ \vdots & & \vdots \\ \frac{\partial x^m}{\partial u^1} & \cdots & \frac{\partial x^m}{\partial u^{m-1}} \end{pmatrix}$$

$(m-1)\times(m-1) \qquad (m-1)\times m \qquad m\times(m-1)$

Cauchy-Binet yields

$$\frac{\partial(y^2,\ldots,y^m)}{\partial(u^1,\ldots,u^{m-1})} = \sum_{i=1}^{m} \frac{\partial(y^2,\ldots,y^m)}{\partial(x^1,\ldots,\hat{x}^i,\ldots x^m)} \frac{\partial(x^1,\ldots,\hat{x}^i,\ldots x^m)}{\partial(u^1,\ldots,u^{m-1})}$$

Thank you!

http://lausb.github.io