An introduction to variational methods for nonlinear differential equations

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1. Introduction

Central problem in mathematics: Solving Equations.
Already hard for simplest nonlinear equations: polynomials

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

For solving equations, we mean proving existence of solutions.
Most important equations are (nonlinear) differential equations

$$F(x, u, Du, D^2u) = 0, \qquad D^2u = \left\{\partial_{ij} u\right\}_{i,j\in\overline{n}}, \qquad \overline{n} = \{1, 2, \ldots, n\}.$$

They can abstractly be written as operator equations

$$T(u) = 0, \tag{1}$$

where $T : X \to Y$ is operator between suitable function spaces X and Y.
Nonlinear Functional Analysis is powerful tool for solving operator equations.

Fixed Point Theory If $X = Y$ set $f(u) = u + T(u)$, then (1) is just $f(u) = u$.

Fixed point theorems, topological degree (Brouwer, Leray-Schauder)

Critical Point Theory If $Y = X^*$ and $T = f' : X \to X^*$ for some $f : X \to \mathbb{R}$,

then (1) is $f'(u) = 0$, u is critical point of f.

Exm1 (Nonlinear Algebraic Eqns). For $A = (a^i_j)_{n\times n}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, consider

$$Ax = f(x), \qquad \text{i.e.} \qquad \sum_{i=1}^{n} a^i_j x^j = f^i(x^1, \dots, x^n), \qquad i \in \overline{n}. \tag{2}$$

✳ If A is symmetric and $f = \nabla F = (\partial_1 F, \dots, \partial_n F)$ for some $F \in C^1(\mathbb{R}^n)$. Set

$$\Phi : \mathbb{R}^n \to \mathbb{R}, \qquad \Phi(x) = \frac{1}{2} Ax \cdot x - F(x).$$

Then $\nabla\Phi(x) = Ax - \nabla F(x) = Ax - f(x)$, thus critical points of Φ solve (2).

If $A > 0$ and $F(x) = o(|x|^2)$ at ∞, then as $|x| \to \infty$

$$\Phi(x) \geq \frac{1}{3}\lambda_1 |x|^2 + o(|x|^2) \to +\infty,$$

thus Φ attains min at some $\xi \in \mathbb{R}^n$ with $\nabla\Phi(\xi) = 0$. Idea applies to PDE

✳ If $\det A \neq 0$,

$$\lim_{|x|\to\infty} \frac{f(x)}{|x|^2} = 0, \qquad x = A^{-1} f(x) =: g(x) \qquad (f(x) = b, \text{ Cramer})$$

a solution of (2) can be found via Brouwer fixed point theorem.

Nonlinear elliptic PDEs from physics and geometry

Minimal surfaces Let $\Omega \subset \mathbb{R}^2$, the graph of $\varphi : \partial\Omega \to \mathbb{R}$ is a curve in $\mathbb{R}^3$

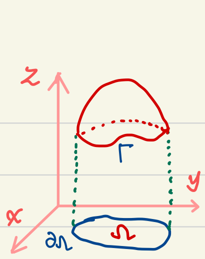
$$\Gamma = \{(x, y, \varphi(x, y)) \mid (x, y) \in \partial\Omega\}.$$

A surface bounded by Γ is graph of $u : \overline{\Omega} \to \mathbb{R}$ with $u|_{\partial\Omega} = \varphi$ and

$$A(u) = \iint_\Omega \sqrt{1 + u_x^2 + u_y^2}\,\mathrm{d}x\mathrm{d}y.$$

To find such surface with minimal area, we need to solve

$$H(u) := \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + u_x^2 + u_y^2}}\right) = 0, \qquad u|_{\partial\Omega} = \varphi.$$

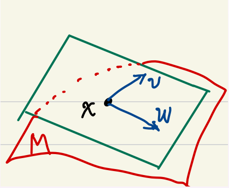


This is obtained from ($h \in C_0^\infty(\Omega)$)

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} A(u + th) = \iint_\Omega \frac{u_x h_x + u_y h_y}{\sqrt{1 + u_x^2 + u_y^2}}\,\mathrm{d}x\mathrm{d}y$$

$$= -\iint_\Omega H(u) h\,\mathrm{d}x\mathrm{d}y.$$

Yamabe problem Let M be compact smooth manifold, $\dim M \geq 3$. A Riemannian metric g is a family of inner products g_x on T_xM for all $x \in M$. Using g we can define $|v|_g = \sqrt{g_x(v,v)}$ for $v \in T_xM$,



$$\angle_g(v,w) = \cos^{-1}\frac{g_x(v,w)}{|v|_g\,|w|_g} \quad \text{for } v,w \in T_xM,$$

$$L(\gamma) = \int_a^b \left|\gamma'(t)\right|_g \mathrm{d}t \quad \text{for } \gamma : [a,b] \to M,$$

$$\mathrm{Vol}(D) = \int_D \mathrm{d}\eta \quad \text{for } D \subset M, \quad \mathrm{d}\eta = \sqrt{\det g_x}\mathrm{d}x \tag{loc}$$

and scalar curvature S_g. $\qquad 2^* = 2n/(n-2)$

A metric $\tilde{g}$ is comformal to g if $\tilde{g} = \varphi^{2^*-2}g$ for some $\varphi : M \to (0,\infty)$. Then, for $\forall x \in M$ and $v,w \in T_xM$,

$$\angle_g(v,w) = \angle_{\tilde{g}}(v,w).$$

Yamabe: Given (M,g), is there a $\tilde{g}$ comformal to g with constant $S_{\tilde{g}}$?

$$-4\frac{n-1}{n-2}\Delta_g\varphi + S_g\varphi = S_{\tilde{g}}\varphi^{2^*-1}.$$

Standing waves of nonlinear Schrödinger equations NLSEs look like

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + U(x)\psi - \tilde{g}(|\psi|)\psi, \qquad t>0,\, x\in\mathbb{R}^N.$$

Standing waves are solutions of the form

$$\psi(t,x) = \mathrm{e}^{-i\omega t}u(x).$$

Given the frequency ω, set $V(x) = U(x) - \omega$ and $g(t) = \tilde{g}(|t|)t$, then

$$\begin{cases} -\Delta u + V(x)u = g(u), & \text{in } \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to \infty. \end{cases}$$

Introductory references to the field

(1) Rabinowitz, P. H. Minimax methods in critical point theory with applications to differential equations, CBMS 65, AMS, Washington, DC, 1986, viii+100

(2) Willem, M. Minimax theorems, Birkhäuser Boston Inc., 1996, x+162

2. Weak solutions

For bounded $\Omega \subset \mathbb{R}^n$, if u solves

$$\Delta u + f(u) = 0, \qquad u|_{\partial\Omega} = 0. \tag{3}$$

For $h \in C_0^\infty(\Omega)$ we have $\int_\Omega h\Delta u + \int_\Omega \nabla h \cdot \nabla u = \int_{\partial\Omega} h\partial_\nu u$

$$0 = \int (\Delta u + f(u))\, h = \int h\Delta u + \int f(u)h = \int f(u)h - \int \nabla u \cdot \nabla h.$$

With $F(t) = \int_0^t f$, set

$$\Phi(u) = \frac{1}{2}\int |\nabla u|^2 - \int F(u). \tag{4}$$

$$\langle \Phi'(u), h \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Phi(u+th) = \int \nabla u \cdot \nabla h - \int f(u)h = 0.$$

Here $\Phi'(u)$ is considered as linear functional acting on h. We have $\Phi'(u) = 0$.
To solve (3) we find critical points of Φ. The classical $C^2(\overline{\Omega})$ is not suitable.
We need Sobolev space $H_0^1(\Omega)$. Critical pts of $\Phi : H_0^1 \to \mathbb{R}$ are weak solutions.
By Regularity Theory, they are classical solutions.

Let $\Omega \subset \mathbb{R}^N$ be open & bdd, on $C_0^\infty(\Omega)$ define

$$\|u\| = \left(\int_\Omega |\nabla u|^2\right)^{1/2}, \qquad |u|_p = \left(\int_\Omega |u|^p\right)^{1/p}.$$

Let $H_0^1(\Omega)$ be the completion of $\left(C_0^\infty, \|\cdot\|\right)$, a Hilbert space.
For $u \in H_0^1$, $\exists\, u_n \in C_0^\infty$ s.t. $u_n \to u$ in H_0^1. Therefore

$$\int_\Omega |\nabla u_m - \nabla u_n|^2 = \int |\nabla(u_m - u_n)|^2 = \|u_m - u_n\|^2 \to 0.$$

So $\{\nabla u_n\}$ is Cauchy in $L^2(\Omega, \mathbb{R}^N)$. $\exists\, v : \nabla u_n \to v$ in $L^2(\Omega, \mathbb{R}^N)$.
It is easy to see that v is indep of $\{u_n\}$. We denote $v = \nabla u$.

Thm1. Set $2^* = 2N/(N-2)$. Then $H_0^1 \subset L^p$, the inclusion

$$i : H_0^1 \to L^p \qquad |u|_p^p \le S_p \|u\|^p$$

is continuous for $p \in [1, 2^*]$ and compact for subcritical $p \in [1, 2^*)$.

Thus, if $u_n \rightharpoonup u$ in H_0^1, then $u_n \to u$ in L^p.
To study

$$\Delta u + f(x, u) = 0,$$

given $u \in H_0^1$ we need to know properites of the new fun $x \mapsto f(x, u(x))$.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be contin, $p_1, p_2 \geq 1$. Given u, $x \mapsto f(x, u(x))$ is a new fun.

Thm2 (Nemytskii operator). If $\exists\, a \in L^{p_2}(\Omega)$ such that

$$|f(x,t)| \leq a(x) + b\,|t|^{p_1/p_2},$$

then $\mathcal{N}_f : L^{p_1} \to L^{p_2}$, $\mathcal{N}_f(u) = f(\cdot, u(\cdot))$ is continuous.

Exm2. If $|F(x,t)| \leq C\,(1 + |t|^p)$ for some $p \in [1, 2^*)$, then $\psi : H_0^1 \to \mathbb{R}$

$$\psi(u) = \int F(x,u) = \int \mathcal{N}_F(u)$$ is weakly continuous.

Pf. If $u_n \rightharpoonup u$ in H_0^1, then $u_n \to u$ in L^p, $\mathcal{N}_F(u_n) \to \mathcal{N}_F(u)$ in L^1 by Thm2. Thus $\psi(u_n) \to \psi(u)$.

Cor1. $\Phi : H_0^1 \to \mathbb{R}$ given in (4) is weakly lower continuous.

Pf. If $u_n \rightharpoonup u$ in H_0^1, $\|u\| \leq \underline{\lim}\, \|u_n\|$. Thus

$$\underline{\lim}\, \Phi(u_n) = \underline{\lim} \left(\frac{1}{2} \int |\nabla u_n|^2 - \int F(u_n) \right) = \frac{1}{2}\, \underline{\lim}\, \|u_n\|^2 - \lim \psi(u_n)$$

$$\geq \tfrac{1}{2} \|u\|^2 - \psi(u) = \Phi(u).$$

3. Minimization

If $D \subset \mathbb{R}^N$ is bdd closed, $f \in C(D)$, then $\exists \xi \in D$ s.t. $f(\xi) = \min_D f$. D compact

Not true in ∞-dim spaces. To ensure minimizer for $f : D \to \mathbb{R}$ on bdd closed $D \subset X$, we impose compactness to f (weakly lower continuous):

$$u_n \rightharpoonup u \text{ in } X \qquad \Longrightarrow \qquad f(u) \le \underline{\lim} f(u_n).$$

Thm3. If X is reflexive Banach space, $f : X \to \mathbb{R}$ weakly lower continuous,

$$\lim_{\|u\| \to \infty} f(u) = +\infty, \qquad \text{(Coercive)}$$

then $\exists v \in X$ s.t. $f(v) = \min_{u \in X} f(u)$.

If $f(v_n) \to \inf f$, $\{v_n\}$ bdd, $v_n \rightharpoonup v$ and $f(v) \le \underline{\lim} f(v_n) = \inf f$.

Rek1. Since $t = 0$ is min of $t \mapsto f(v + th)$, if moreover f is differentiable,

$$\langle f'(v), h \rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(v + th) = 0. \qquad \forall h \in X.$$

Thus v is a critical pt of f.

Exm3. Solutions of $\Delta u + f(u) = 0, \quad u|_{\partial\Omega} = 0$ (5)

are critical pts of w.l.c. functional $\Phi : H_0^1 \to \mathbb{R}$, $|u|_p^p \le S_p \|u\|^p$

$$\Phi(u) = \frac{1}{2}\int |\nabla u|^2 - \int F(u).$$ Cor1

Thus, if $|f(t)| \le C\left(1 + |t|^{p-1}\right)$ for some $p \in (1,2)$, then $|F(t)| \le c(1 + |t|^p)$,

$$\Phi(u) \approx \frac{1}{2}\|u\|^2 - c\int |u|^p \ge \frac{1}{2}\|u\|^2 - cS_p\|u\|^p \to +\infty \qquad \text{as } \|u\| \to \infty.$$

By Rem1, Φ has a critical pt v, which solves (5).

Rek2. If $f(0) = 0$ then $u = 0$ is a trivial solution. It is possible that $v = 0$. To get nontrivial solutions we need conditions on f near 0. For example if

$$\lim_{t\to 0} \frac{F(t)}{t^2} = +\infty,$$

then $v \ne 0$ because $\Phi(0) = 0$ but $\Phi(v) < 0$: for a fixed direction h, as $t \to 0^+$

$$\Phi(th) = \frac{1}{2}\int |\nabla(th)|^2 - \int F(th) = t^2\left(\frac{1}{2}\int |\nabla h|^2 - \int \frac{F(th)}{t^2}\right) < 0.$$

Thm4. Let $f, g : X \to \mathbb{R}$ be C^1. If $v \in M = g^{-1}(1)$, $g'(v) \neq 0$ (g reg at v) and

$$f(v) = \min_{u \in M} f(u),$$

then $\exists \lambda \in \mathbb{R}$ s.t. $f'(v) = \lambda g'(v)$. v is cri pt for $f - \lambda g$.

Exm4. For $p \in (2, 2^*)$, there is $v \neq 0$ solves

$$-\Delta u = |u|^{p-2} u, \qquad u|_{\partial\Omega} = 0. \tag{6}$$

Pf. Solutions are critical pt of $\Phi : H_0^1 \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{p} \int |u|^p, \qquad \text{but } \inf_{H_0^1} \Phi = -\infty, \text{ (note: } \Phi(tu) \to -\infty \text{ at } \infty)$$

Instead, we minimize f over $M = g^{-1}(1)$,

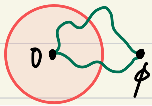
$$f(u) = \frac{1}{2} \int |\nabla u|^2, \qquad g(u) = \frac{1}{p} \int |u|^p.$$

There is $u \in M$, $f(u) = \min f(M)$. Thus $f'(u) = \lambda g'(u)$, u is (weak) solution of

$$-\Delta u = \lambda |u|^{p-2} u, \qquad u|_{\partial\Omega} = 0.$$

Now $v = \lambda^{1/(p-2)} u$ solves (6). Homogeneity of $f(u) = |u|^{p-2} u$.

4. Mountain pass theorem



We need $f \in C^1(X)$ compact: $\{u_n\}$ is precompact if

$$f(u_n) \to c, \qquad f'(u_n) \to 0. \qquad \text{(Palais-Smale)}$$

If f has no cri value in $[a, b]$, then $f^a = \{f \le a\}$ is deformation retract of f^b.

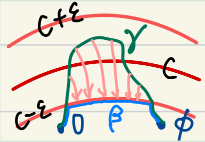
Thm5 (Ambrosetti & Rabinowitz (1973)). If $f \in C^1(X)$ verifies (PS),

$$b = \inf_{\|u\|=\rho} f(u) > f(0) \ge f(\phi), \qquad \text{for some } \rho > 0 \text{ and } \phi \in X \backslash \overline{B}_\rho,$$

then f has a critical value $c \ge b$ given by $c = \inf_{\gamma\in\Gamma} \max_\gamma f$.

Γ consists of paths joining 0 and ϕ. 0 is loc min, $\inf_X f = -\infty$

Pf. If c is not critical, f has no cri val in $[c-\varepsilon, c+\varepsilon]$ $(\varepsilon \ll 1)$



(1) For $\varepsilon \in (0, c)$, $\exists \gamma \in \Gamma$ s.t. $\max_\gamma f \le c + \varepsilon$, i.e. $\gamma \subset f^{c+\varepsilon}$.

(2) By deformation lemma, $\exists \eta : f^{c+\varepsilon} \to f^{c-\varepsilon}$, $\eta|_{f^{c-\varepsilon}} = \mathbf{1}_{f^{c-\varepsilon}}$.

(3) Since $\{0, \phi\} \subset f^{c-\varepsilon}$, $\beta = \eta(\gamma) \in \Gamma$, thus $\max_\beta f \le c - \varepsilon$, a contradiction.

Exm5 (Ambrosetti & Rabinowitz (1973)). For problem (5)

$$\Delta u + f(u) = 0, \qquad u|_{\partial\Omega} = 0, \tag{7}$$

(1) $|f(t)| \le C\left(1 + |t|^{p-1}\right)$ for some $p \in (2, 2^*)$, $\qquad f(t) = |t|^{p-2}t$

(2) $f(0) = f'(0) = 0$,

(3) $\exists \mu > 2$ and $R > 0$, $0 < \mu F(t) \le t f(t)$ for $|t| \ge R$, $\qquad \Longrightarrow F(t) \ge c_1 |t|^\mu - c_2$

then (7) has a nontrivial solution.

Pf. From (1) & (2), $\forall \varepsilon > 0$, $|F(t)| \le \varepsilon t^2 + C_\varepsilon |t|^p$, as $\|u\| \to 0$,

$$\int F(u) = o(\|u\|^2), \qquad \Phi(u) = \frac{1}{2}\int |\nabla u|^2 - \int F(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2).$$

Thus $u = 0$ is strict loc min of Φ.
$\inf \Phi = -\infty$ because from (3), fixed $h \neq 0$, as $t \to +\infty$

$$\Phi(th) = \frac{t^2}{2}\int |\nabla h|^2 - \int F(th) \le \frac{t^2}{2}\int |\nabla h|^2 - c_1 t^\mu \int |h|^\mu + c_2 |\Omega| \to -\infty.$$

We can verify (PS). MPT yields critical value $c > 0$, hence a critical pt $v \neq 0$.

Rek3. Applying MPT to truncated problem

$$\begin{cases} \Delta u + f_+(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{where } f_+(t) = \begin{cases} f(t) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}$$

one obtains a solution v, which is nonnegative because ($v^{\pm} = \max\{0, \pm v\}$)

$$0 = \left\langle \Phi_+(v), v^- \right\rangle = \int \nabla v \cdot \nabla v^- - \int f_+(v) v^- = \int \nabla v \cdot \nabla v^- = \int \left|\nabla v^-\right|^2.$$

Thus $v = v^+ \geq 0$. Since $f(v) = f_+(v)$, v solves (7).
Similarly one gets a nonpositive solution $w \leq 0$ for (7).
A third nontrivial solution was obtained by Wang (1991) via Morse theory.
If f is odd, then

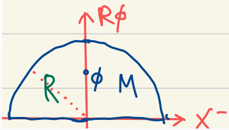
$$\Phi(u) = \frac{1}{2} \int |\nabla u|^2 - \int F(u)$$

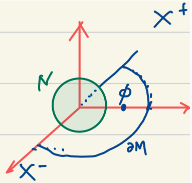
is even and a sequence of solutions $\{u_n\}$ satisfying

$$\Phi(u_n) \to +\infty$$

can be obtain via *Symmetric Mountain Pass Theorem* Ambrosetti & Rabinowitz (1973).

5. Linking theorem





If $u = 0$ is a saddle pt of f, MPT is not applicable.

Thm6 (Rabinowitz (1978)). If $X = X^- \oplus X^+$, $\dim X^- < \infty$; $f \in C^1(X)$ satisfies (PS), for some $R > \rho > 0$ and $\phi \in X^+ \backslash 0$,

$$b = \inf_N f > \sup_{\partial M} f, \qquad \text{where} \quad N = X^+ \cap \partial B_\rho, \quad M = B_R \cap (X^- + \mathbb{R}^+\phi),$$

then f has a critical value $c \geq b$. Since $0 \in \partial M$, $c \geq b > f(0)$

Rek4. It reduces to MPT if $X^- = \{0\}$. In applications, $f(0) = 0$,

(1) $f|_{X^-} \leq 0$, $f(u) \to -\infty$ as $\|u\| \to \infty$ along $X^- \oplus \mathbb{R}\phi$ (anti-coercive), thus

$$\sup_{\partial M} f \leq 0, \qquad \partial M \subset X^- \cup \left(\partial B_R \cap \left(X^- \oplus \mathbb{R}\phi\right)\right)$$

(2) 0 is loc min of $f|_{X^+}$, thus

$$\inf_N f > 0. \qquad (\text{if } \rho \ll 1)$$

Exm6. For $\lambda \in (\lambda_k, \lambda_{k+1})$, $\qquad \Delta u + \lambda u + f(u) = 0, \qquad u|_{\partial\Omega} = 0.$ (8)

(1) $|f(t)| \le C\left(1 + |t|^{p-1}\right)$ for some $p \in (2, 2^*)$,

(2) $f(0) = f'(0) = 0$, $f(t)t \ge 0$,

(3) $\exists \mu > 2$ and $R > 0$, $0 < \mu F(t) \le tf(t)$ for $|t| \ge R$, $\qquad \Longrightarrow F(t) \ge c_1 |t|^\mu - c_2$

then (8) has a nontrivial solution.

Pf. We find critical pt for $\Phi : H_0^1 \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2}\int \left(|\nabla u|^2 - \lambda u^2\right) - \int F(u) = Q(u) - \int F(u).$$

Let $X^- = \operatorname{span}\{\phi_1, \ldots, \phi_k\}$, $X^+ = Y^\perp$. Then $Q(u) \ge \kappa \|u\|^2$ for $u \in X^+$.

(1) Since $F \ge 0$, $\Phi \le Q \le 0$ on X^-; for $u \in X^- \oplus \mathbb{R}\phi_{k+1}$, $\|u\| \to \infty$,

$$\Phi(u) \le \alpha \|u\|^2 - \int F(u) \le \alpha \|u\|^2 - c_1 |u|_\mu^\mu + c_2 |\Omega| \to -\infty. \quad \sup_{\partial M} \Phi \le 0.$$

(2) As before $\int F(u) = o(\|u\|^2)$ as $u \to 0$. For $u \in X^+$, $\|u\| \to 0$,

$$\Phi(u) = Q(u) - \int F(u) \ge \kappa \|u\|^2 + o(\|u\|^2). \qquad 0 \text{ loc min of } \Phi|_X^+$$

6. Preliminaries for working in the field

You can start learning some of these after you become a student in the field.

Multivariable Calculus and Linear Algebra Differential Geometry

Real Analysis ✳ Lebesgue measure

- ✳ measurable functions, convergence of sequence of functions
- ✳ Lebesgue integral and L^p-spaces

Linear Functional Analysis Bressan (2013)

- ✳ Banach space, Hilbert space
- ✳ Linear continuous operators, compact operators and their spectrum

Partial Differential Equations (Evans, 2010, Chapters 5 & 6)

- ✳ Sobolev spaces
- ✳ Elliptic PDEs of second order

Topology Basic topological concepts such as neighborhood, compactness, connectedness, homotopy $h : [0, 1] \times X \to Y$.

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Thank you!

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