Untitled

Stéphane Laurent

6 mai 2020

Let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a probability space, E a Polish space, and X an E-valued random variable. Denote by $\mathcal{B}_E$ the Borel σ-field on E. It is well-known that the law of X can be disintegrated over any σ-field $\mathfrak{B} \subset \mathfrak{A}$. That means that there exists a *probability kernel* $K \colon \Omega \times \mathcal{B}_E \to [0,1]$ from Ω to E such that for every suitable function $f \colon E \to \mathbb{R}$,

$$\mathbb{E}\big[f(X)\big] = \int_\Omega K_\omega(f) \mathrm{d}\mathbb{P}(\omega)$$

where we denote by K_ω the probability $A \mapsto K(\omega, A)$ and we set $\mu(f) = \int f \mathrm{d}\mu$ for a probability μ. The map $\omega \mapsto K_\omega$ is called the *conditional law of X given* $\mathfrak{B}$, and is denoted by $\mathcal{L}(X \mid \mathfrak{B})$.

Recall that a *probability kernel* from a probability space Ω to a measurable space $(E, \mathcal{B})$ is an application $K \colon \Omega \times \mathcal{B} \to [0,1]$ such that

(i) for fixed $\omega \in \Omega$, the map $A \mapsto K(\omega, A)$ is a probability on $(E, \mathcal{B})$;

(ii) for fixed $A \in \mathcal{B}$, the map $\omega \mapsto K(\omega, A)$ is measurable.

It is less known that there exists a Polish topology on the set $\mathrm{Pr}(E)$ of probabilities on E such that the conditional law $\mathcal{L}(X \mid \mathfrak{B})$ is a $\mathrm{Pr}(E)$-valued random variable.

I am unable to find a self-contained reference for this result. I aim to prove it in these notes, in a self-contained way.

Polish spaces

A topological space E is said to be *Lindelôf* if there is a countable subcover of every open cover of E. It is said to be *second-countable* when it has a countable basis.

Theorem. *Every second-countable topological space is Lindelöf.*

Proof. Let E be a second-countable topological space. Take a countable basis $\{O_n\}_{n \in \mathbb{N}}$ of the topology on E. Let $\mathcal{C} = \{C_i\}_{i \in I}$ be an open cover of E. For every $x \in E$, take $i(x) \in I$ such that $x \in C_{i(x)}$, and let $n(x) \in \mathbb{N}$ such that $x \in O_{n(x)} \subset C_{i(x)}$. Then $\{O_{n(x)}\}_{x \in X}$ is a countable open cover of E. For every $\ell \in n(E) \subset \mathbb{N}$, the set $\{j \in I \mid O_\ell \subset C_j\}$ is non-empty, therefore one can select $j(\ell) \in I$ such that $O_\ell \subset C_{j(\ell)}$ by using the axiom of countable choice. The set $\{C_{j(\ell)}\}_{\ell \in n(E)}$ is a countable subcover of $\mathcal{C}$. $\square$

Theorem. *For a metrizable space, the notions of separability, second-countable and Lindelöf are equivalent.*

Proof. Let E be a metric space. Denote by $B_r(x)$ the open ball of radius r centered at x.

Assume E is separable. Let $(x_n)_{n \geqslant 1}$ be a dense sequence of E. Then $\big\{B_r(x_n) \mid r \in \mathbb{Q}, n \geqslant 1\big\}$ is a countable basis of E, hence E is second-countable.

By the previous theorem, the second-countable property implies the Lindelöf property. Now assume E is Lindelöf. Consider the open cover $\big\{B_\epsilon(x)\big\}_{x \in E}$ for a given $\epsilon > 0$. By the Lindelöf property, there exists a countable subcover $\big\{B_\epsilon(x_{\epsilon,k})\big\}_{k \in \mathbb{N}}$. Then $\bigcup_{n \geqslant 1} \big\{x_{1/n,k}\big\}_{k \in \mathbb{N}}$ is dense in E, thereby showing that E is separable. $\square$

Corollary. *A subspace of a metrizable separable space is itself separable.*

Proof. It is clear that a subspace of a second-countable space is itself second-countable. Therefore the result follows from the previous theorem. $\square$

Given a metric space (E,d), a non-empty set $A \subset E$ and $\epsilon > 0$, we define the set

$$A^\epsilon = \big\{x \in E \mid d(x,A) < \epsilon\big\}.$$

It is well-known that $x \mapsto d(x,A)$ is continuous. This straigthforwardly results from the inequality $\big|d(x,A) - d(y,A)\big| \leqslant d(x,y)$ (see [2]). Hence A^ϵ is open. It is also well-known that $d(x,A) = 0$ if and only if $x \in \overline{A}$.

Proposition. *An open subset of a Polish space is Polish.*

Proof. Let E be a Polish metric space, and $A \subset E$ be open. Then A is separable by the previous corollary. Assume $A \neq E$ and let d be a compatible metric on E under which E is complete. For $x, y \in E$, define

$$d'(x,y) = d(x,y) + \left|\frac{1}{d(x,A^c)} - \frac{1}{d(y,A^c)}\right|.$$

It is easy to check that d' is a metric on A. Using the continuity of $x \mapsto d(x,A^c)$, it is easy to see that convergence for d is equivalent to convergence for d'.

It is also clear that a d'-Cauchy sequence $\{u_n\}$ in A is a d-Cauchy sequence. Let u_∞ be the limit of $\{u_n\}$. Then $u_\infty \in A$, otherwise $d(u_n, A^c) \to 0$ and $d'(u_n, u_m) \to \infty$ as $m, n \to \infty$ □

More generally, a subset of a Polish space is a Polish space if and only if it is a G_δ set - see [6].

A topological space is said to be *strongly Lindelöf* if every open set is Lindelöf.

Corollary. *A Polish space is strongly Lindelöf.*

Proof. A Polish space is metrizable and separable, therefore it is Lindelöf by the second theorem. Since an open subset of a Polish space is Polish, we see that a Polish space is strongly Lindelöf. □

Lemma. *The continuous image of a strongly Lindelöf space is a strongly Lindelöf space.*

Proof. Let E_1 and E_2 be two topological spaces with E_1 strongly Lindelöf, and let $f\colon E_1 \to E_2$ be a continuous function. Let $O_2 \subset E_2$ be an open subset of $f(E_1)$ and let $\mathcal{C}_2$ be an open cover of O_2. Then $\big\{f^{-1}(C) \mid C \in \mathcal{C}_2\big\}$ is an open cover of the open set $O_1 := f^{-1}(O_2)$. Then it has a countable subcover: $O_1 \subset \big\{f^{-1}(C) \mid C \in \mathcal{D}_2\big\} =: \mathcal{D}_1$ where $\mathcal{D}_2$ is a countable subset of $\mathcal{C}_2$. One has $f\big(f^{-1}(C)\big) = C$ for every $C \subset f(E_1)$, therefore $f(\mathcal{D}_1) = \mathcal{D}_2$ is an open subcover of $f(O_1) = O_2$. Thus $f(E_1)$ is strongly Lindelöf. □

Proposition. *A Souslin space is strongly Lindelöf.*

Proof. Let X be a Souslin space, and $f\colon Y \to X$ be a surjective continuous function from a Polish space Y to X. By the previous corollary, Y is strongly Lindelöf, therefore X is strongly Lindelöf by the previous lemma. □

Space of probability measures

For a topological space E, we denote by $C_b(E)$ the space of real-valued bounded continuous functions on E and by $\Pr(E)$ the topological space of probabilities on E equipped with the narrow topology.

The *canonical basis of neighborhoods* at $\mu \in \Pr(E)$ is the family of open sets

$$\Big\{\nu \in \Pr(E) \mid \big|\nu(f_i) - \mu(f_i)\big| < \epsilon, i \in [\![1,k]\!]\Big\}$$

for $k \geqslant 1$ and $f_i \in C_b(E)$.

Lemma. *Let $\mu \in \Pr(E)$, $f \in C_b(E)$ and $\epsilon > 0$. There exist an integer $k \geqslant 1$, a number $\eta > 0$ and some closed sets $F_1 \subset E, \ldots, F_k \in E$ such that the set*

$$\Big\{\nu \mid \big|\nu(f) - \mu(f)\big| < \epsilon\Big\}$$

contains the set

$$\big\{\nu \in \Pr(E) \mid \nu(F_i) < \mu(F_i) + \eta, i \in [\![1, k]\!]\big\}.$$

Proof. Using a linear transformation, one can write

$$\Big\{\nu \mid \big|\nu(f) - \mu(f)\big| < \epsilon\Big\}$$

as

$$\Big\{\nu \mid \big|\nu(g) - \mu(g)\big| < \epsilon'\Big\}$$

where $g \in C_b(E)$ is such that $0 < g(x) < 1$ for all $x \in E$ and $\epsilon' > 0$. Take an integer $k > \frac{2}{\epsilon'}$ and take the closed sets $F_i = \big\{x \in E \mid g(x) \geqslant \frac{i-1}{k}\big\}$ for $i \in [\![1, k]\!]$ and $F_{k+1} = \varnothing$.

Define the intervals $J_i = \big[\frac{i-1}{k}, \frac{i}{k}\big[$. For every $P \in \Pr(E)$, one has

$$\sum_{i=1}^{k} \frac{i-1}{k} P\big(g^{-1}(J_i)\big) \leqslant P(g) \leqslant \sum_{i=1}^{k} \frac{i}{k} P\big(g^{-1}(J_i)\big).$$

The sum on the right is

$$\sum_{i=1}^{k} \frac{i}{k}\big(P(F_i) - P(F_{i+1})\big) = \frac{1}{k} + \frac{1}{k}\sum_{i=2}^{k} P(F_i)$$

and the sum on the left is $\frac{1}{k}\sum_{i=2}^{k} P(F_i)$. Therefore $\nu(g) < \frac{\epsilon'}{2} + \frac{1}{k}\sum_{i=2}^{k} \nu(F_i)$ and $\frac{1}{k}\sum_{i=2}^{k} \mu(F_i) \leqslant \mu(g)$. Thus, if $\nu(F_i) < \mu(F_i) + \eta$, one has

$$\nu(g) < \frac{\epsilon'}{2} + \mu(g) + \eta$$

and then $\nu(g) < \mu(g) + \epsilon'$ if we take $\eta = \frac{\epsilon'}{2}$. Applying the same mathematics with $1 - g$ instead of g, we finally get $\big|\nu(g) - \mu(g)\big| < \epsilon'$. ☐

Theorem. *The probabilities on E with finite support form a dense subset of* $\Pr(E)$*.*

Proof. Let $\mu \in \Pr(E)$ and

$$V = \Big\{\nu \in \Pr(E) \mid \big|\nu(f_i) - \mu(f_i)\big| < \epsilon, i \in [\![1, k]\!]\Big\}$$

be an open set in the canonical basis of neighborhoods of μ. By the previous lemma, V contains a set

$$U = \big\{\nu \in \Pr(E) \mid \nu(F_i) < \mu(F_i) + \epsilon', i \in [\![1, k']\!]\big\}$$

with $k' \geqslant 1$, $\epsilon' > 0$, and $F_i \subset E$ closed. The F_i generate a finite partition $\mathcal{P} = \{B_1, \ldots, B_J\}$ with $B_j \neq \varnothing$ closed. For each $B_j \in \mathcal{P}$, we pick a point $b_j \in B_j$. The probability $\sum_{j=1}^{J} \mu(B_j)\delta_{b_j}$ belongs to U. That shows the result. ☐

Theorem. *If E is separable, then so is* $\Pr(E)$*.*

Proof. Let $D \subset E$ be a countable dense subset of E. We are going to show that the set of probability measures with finite support contained in D and which have a rational mass at each point of their support, is dense in $\Pr(E)$. Since this set is countable, this will prove the proposition.

By the preceding theorem, we know that the probabilities with finite support form a dense subset of $\Pr(E)$. Let $\mu = \sum_{j=1}^{J} p_j \delta_{x_j}$ be such a probability. Take an open neighborhood of μ

$$\mathcal{V} = \Big\{\nu \in \Pr(E) \mid \big|\nu(f_i) - \mu(f_i)\big| < \epsilon, i \in [\![1, k]\!]\big\},$$

with $k \geqslant 1$, $\epsilon > 0$ and $f_i \in C_b(E)$. Note that $\mu(f_i) = \sum_{j=1}^{J} p_j f_i(x_j)$.

Let $K = \max\{\|f_i\|_\infty\}$. Let $q_j \in \mathbb{Q} \cap [0,1]$ such that $\sum_{j=1}^J q_j = 1$ and $|q_j - p_j| < \frac{\epsilon}{2JK}$. Let $y_j \in D$ such that $\left|f_i(y_j) - f_i(x_j)\right| < \frac{\epsilon}{2}$ for all $j \in [\![1, J]\!]$.

We set $\nu = \sum_{j=1}^J q_j \delta_{y_j}$, so that one has $\nu(f_i) = \sum_{j=1}^J q_j f_i(y_j)$, and then

$$\left|\nu(f_i) - \mu(f_i)\right| \leqslant \sum_{j=1}^J \left|q_j f_i(y_j) - p_j f_i(x_j)\right|$$

Let us bound from above each term of the sum:

$$\begin{aligned}
&\left|q_j f_i(y_j) - p_j f_i(x_j)\right| \\
&\quad \leqslant \left|q_j f_i(y_j) - p_j f_i(y_j)\right| + \left|p_j f_i(y_j) - p_j f_i(x_j)\right| \\
&\quad \leqslant |q_j - p_j| \|f_i\| + p_j \left|f_i(y_j) - f_i(x_j)\right| \\
&\quad < \frac{\epsilon}{2J} + p_j \frac{\epsilon}{2}.
\end{aligned}$$

By summing, one gets $\left|\nu(f_i) - \mu(f_i)\right| < \epsilon$, hence $\nu \in \mathcal{V}$. □

Theorem. *Assume E is metric and denote by d the metric on E. Let $F \subset E$ be closed and $\epsilon > 0$. Then there is a function $f \in C_b(E)$ such that $f(x) = 1$ for $x \in F$, $f(x) = 0$ if $d(x, F) \geqslant \epsilon$ and $0 \leqslant f(x) \leqslant 1$ for all $x \in E$. This function f may be taken to be uniformly continuous.*

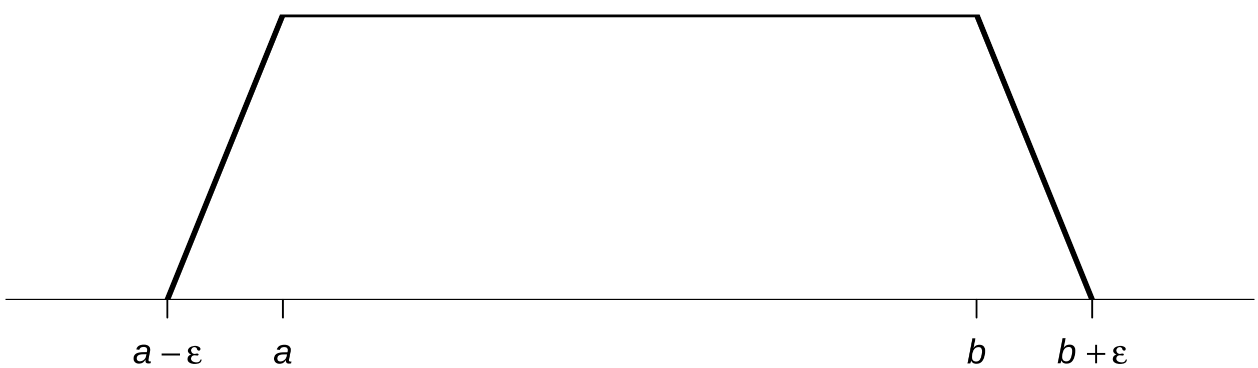
Proof. Define the continuous function $\phi\colon \mathbb{R} \to [0,1]$ by

$$\phi(t) = \begin{cases} 1 & \text{if } t \leqslant 0 \\ 1 - t & \text{if } 0 \leqslant t \leqslant 1 \\ 0 & \text{if } t \geqslant 1 \end{cases}.$$

Then define

$$f(x) = \phi\left(\frac{1}{\epsilon} d(x, F)\right).$$

This function f has the required properties. Here is the graph of f for $F = [a, b] \subset \mathbb{R}$:



□

Proposition (neighborhood bases of the narrow topology). *If E is metric space, then each of the three families of sets below form a neighborhood basis of the narrow topology at $\mu \in \Pr(E)$.*

(i) $\left\{\nu \in \Pr(E) \mid \nu(F_i) < \mu(F_i) + \epsilon, i \in [\![1, k]\!]\right\}$, *where $k \geqslant 1$ is an integer, the F_i are closed in E, and $\epsilon > 0$.*

(ii) $\left\{\nu \in \Pr(E) \mid \nu(O_i) > \mu(O_i) - \epsilon, i \in [\![1, k]\!]\right\}$, *where $k \geqslant 1$ is an integer, the O_i are open in E, and $\epsilon > 0$.*

(iii) $\Big\{\nu \in \Pr(E) \mid \big|\nu(A_i) - \mu(A_i)\big| + \epsilon, i \in [\![1,k]\!]\Big\}$*, where $k \geqslant 1$ is an integer, the A_i are Borelian in E and satisfy $\mu(\partial A_i) = 0$, and $\epsilon > 0$.*

Proof. One obviously gets equality between the families *(i)* and *(ii)* with the help of the complementary operation. By the previous lemma, each neighborhood at μ in the canonical basis contains a set in the family *(ii)*. Let us show the converse. Denote by d the metric on E and for $A \subset E$ and $\eta > 0$, define $A^\eta = \big\{x \in E \mid d(x, A) < \eta\big\}$. Choose $\eta_i > 0$ such that $\mu(F_i^{\eta_i}) < \mu(F_i) + \frac{\epsilon}{2}$. Apply the previous theorem to get $f_i \in C_b(E)$ with value 1 on F_i, value 0 outside $F_i^{\eta_i}$, and value everywhere contained in $[0, 1]$. If $\big|\nu(f_i) - \mu(f_i)\big| < \frac{\epsilon}{2}$, then

$$\nu(F_i) \leqslant \nu(f_i) < \mu(f_i) + \frac{\epsilon}{2} \leqslant \mu(F_i^{\eta_i}) + \frac{\epsilon_i}{2} < \mu(F_i) + \epsilon.$$

Thus, each set in the canonical neighborhood at μ is contained in a set of family *(i)*.

It remains to treat the family *(iii)*. If $A \subset E$ is Borel and $\mu(\partial A) = 0$, then for ν in a set of family *(i)* we have

$$\nu(A) \leqslant \nu(\overline{A}) < \mu(\overline{A}) + \epsilon = \mu(A) + \epsilon$$

and for ν in a set of family *(ii)* we have

$$\nu(A) \geqslant \nu(A^\circ) > \mu(A^\circ) - \epsilon = \mu(A) - \epsilon.$$

Since *(i)* is a neighborhood basis at μ, each set of family *(iii)* contains a set of family *(i)*. On the other hand, if $F \subset E$ is closed, then for any $\epsilon > 0$ there is $\eta > 0$ such that $\mu(\partial F^\eta) = 0$ and $\mu(F^\eta) < \mu(F) + \frac{\epsilon}{2}$. Indeed, firstly one can obviously take λ such that $\mu(F^\lambda) < \mu(F) + \frac{\epsilon}{2}$. Since the ∂F^η for $\eta \in]0, \lambda[$ have empty intersection (since $\partial F^\eta = \big\{x \in E \mid d(x, F) = \eta\big\}$), there is $\eta \in]0, \lambda[$ such that $\mu(\partial F^\eta) = 0$. Obviously $\mu(F^\eta) \leqslant \mu(F^\lambda)$. The claim is proved. Now take ν such that $\big|\nu(F^\eta) - \mu(F^\eta)\big| < \frac{\epsilon}{2}$. Then

$$\nu(F) \leqslant \nu(F^\eta) < \mu(F^\eta) + \frac{\epsilon}{2} < \mu(F) + \epsilon.$$

Thus each set of family *(i)* contains a set of family *(iii)*. □

Lemma (regularity of probabilities on metric spaces) *Assume E is metric. Then for every probability μ on E and every Borel set $A \subset E$,*

$$\begin{aligned}\mu(A) &= \inf\big\{\mu(O) \mid O \text{ open}, O \supset A\big\} \\ &= \sup\big\{\mu(F) \mid F \text{ closed}, F \subset A\big\}.\end{aligned}$$

Proof. The first equality is obviously true for an open set A. Let us show that the second equality is true for any open set A. There exist some closed sets $F_n \uparrow A$: take $F_n = \big\{x \mid d(x, A^c) \geqslant \frac{1}{n}\big\}$. Since we have $\mu(F_n) \uparrow \mu(A)$, it is easy to see that the second equality holds for A. Let $\mathcal{A}$ be the set of all Borel sets A fulfilling the stated property. To prove that the lemma is true, it remains to show that $\mathcal{A}$ is a σ-algebra. Let us show this is the case. It is clear that $\mathcal{A}$ is stable under complementation. It is easy to see that $A \in \mathcal{A}$ if and only if for every $\epsilon > 0$ there exist an open set $O \supset A$ and a closed set $F \subset A$ such that $\mu(O \setminus F) < \epsilon$.

Let (A_n) be a sequence in $\mathcal{A}$. For every n, let $F_n \subset A_n \subset O_n$ with $\mu(O_n \setminus F_n) \leqslant \epsilon/3^n$. Let n_0 such that

$$\mu\Big(\bigcup F_n \setminus \bigcup_{k=1}^{n_0} F_k\Big) < \frac{\epsilon}{2}.$$

Take $O = \bigcup_n O_n$ and $F = \bigcup_{k=1}^{n_0} F_k$. Then $F \subset \bigcup_n A_n \subset O$ and since $O \setminus \bigcup_n F_n \subset \bigcup(O_n \setminus F_n)$,

$$\mu(O \setminus F) \leqslant \mu\Big(O \setminus \bigcup F_n\Big) + \mu\Big(\bigcup F_n \setminus F\Big) < \sum \mu(O_n \setminus F_n) + \frac{\epsilon}{2} < \epsilon.$$

□

Lemma. *If the inequality $\mu(A) \leqslant \nu(A^\epsilon) + \epsilon$ holds for $A = E \setminus B^\epsilon$, then $\nu(B) \leqslant \mu(B^\epsilon) + \epsilon$.*

Proof. Note that the two inclusions $A \subset E \setminus B^\epsilon$ and $B \subset E \setminus A^\epsilon$ are equivalent because each one is equivalent to $d(x,y) \geqslant \epsilon$ for all $x \in A$ and $y \in B$. Hence

$$\mu(B^\epsilon) = 1 - \mu(A) \geqslant 1 - \nu(A^\epsilon) - \epsilon = \nu(E \setminus A^\epsilon) - \epsilon \geqslant \nu(B) - \epsilon.$$

□

We will not use equality (2) of the following proposition.

Proposition and definition (Prohorov distance). *Let E be a metric space. The Prohorov distance between $\mu \in \Pr(E)$ and $\nu \in \Pr(E)$ is defined by*

$$d_P(\mu,\nu) = \inf\big\{\epsilon > 0 \mid \forall A \in \mathcal{B}_E, \mu(A) \leqslant \nu(A^\epsilon) + \epsilon \text{ and } \nu(A) \leqslant \mu(A^\epsilon) + \epsilon\big\}.$$

One has

$$\begin{aligned} d_P(\mu,\nu) &\overset{(1)}{=} \inf\big\{\epsilon > 0 \mid \forall A \in \mathcal{B}_E, \mu(A) \leqslant \nu(A^\epsilon) + \epsilon\big\} \\ &\overset{(2)}{=} \inf\big\{\epsilon > 0 \mid \forall \text{ closed } F \subset E, \mu(F) \leqslant \nu(F^\epsilon) + \epsilon\big\}. \end{aligned}$$

Proof. Let $\epsilon > 0$ such that $\mu(A) \leqslant \nu(A^\epsilon) + \epsilon$ for all $A \in \mathcal{B}_E$. Let $B \in \mathcal{B}_E$. Then $E \setminus B^\epsilon \in \mathcal{B}_E$ since B^ϵ is open. Applying the previous lemma, we get $\nu(B) \leqslant \mu(B^\epsilon) + \epsilon$. Thus the two inequalities in the definition of $d_P(\mu,\nu)$ hold for all $B \in \mathcal{B}_E$. Equality (1) is proved.

Now we prove equality (2). Let $\epsilon > 0$ such that $\mu(F) \leqslant \nu(F^\epsilon) + \epsilon$ for all closed sets $F \subset E$. We will first prove that $\nu(F) \leqslant \mu(F^\epsilon) + \epsilon$ for all closed sets $F \subset E$. Let $H \subset E$ be closed and $G = E \setminus H^\epsilon$. Then G is closed. Hence $\nu(H) \leqslant \mu(H^\epsilon) + \epsilon$ by the previous lemma. Thus it suffices to prove that $\epsilon > 0$ satisfies

$$\mu(A) \leqslant \nu(A^\epsilon) + \epsilon \text{ and } \nu(A) \leqslant \mu(A^\epsilon) + \epsilon$$

for all Borel A if and only if it satisfies these inequalities for all closed A. Assume these inequalities hold for all closed A. Let $B \subset E$ be Borel. Take $\eta > 0$. By the lemma *"regularity of probabilities on metric spaces"*, there exists a closed $C \subset B$ such that $\mu(C) \geqslant \mu(B) - \delta$. Hence

$$\mu(B) - \delta \leqslant \mu(C) \leqslant \nu(C^\epsilon) + \epsilon \leqslant \nu(B^\epsilon) + \epsilon.$$

Therefore $\mu(B) \leqslant \nu(B^\epsilon) + \epsilon$ since δ was arbitrary. The other inequality is similarly proved. □

Lemma. *Let E be a metric space, and $\mu \in \Pr(E)$. For every $\eta > 0$, one can cover E by open balls $\{S_i\}$ satisfying $\mu(\partial S_i) = 0$ and diam$(S_i) < \eta$ for all i. If E is separable, one can cover E by countably many such balls.*

Proof. Denote by $S_\eta(x)$ the open sphere of radius η centered at x, and by $B_\eta(x) = \partial S_\eta(x) = \big\{y \mid d(x,y) = \eta\big\}$ the boundary of this sphere. One has $S_\eta(x) = \bigcup_{0 \leqslant \eta' < \eta} B_\eta(x)$. This is an uncountable disjoint union of Borel sets, and hence all but a countable number of them must have μ-measure zero. Therefore every sphere $S_\eta(x)$ contains a sphere $S_{\eta'}(x)$, $\eta' \leqslant \eta$ such that $\mu\big(B_{\eta'}(x)\big) = 0$. Picking such a sphere for every $x \in E$, one gets the first claim of the lemma. If E is separable, one picks such a sphere for every x in a dense subset of E. □

Theorem. *The Prohorov distance defines a metric on* $\Pr(E)$. *The topology it induces is finer than the narrow topology. These two topologies coincide when E is separable.*

Proof. It is clear that $d_P(\mu,\nu) = d_P(\nu,\mu) \geqslant 0$ and $d_P(\mu,\mu) = 0$. If $A \subset E$ is closed then $A^\epsilon \downarrow A$ as $\epsilon \downarrow 0$ (it is easy to see that $d(x,A) = 0$ for every $x \in \cap_{\epsilon>0} A^\epsilon$, and this is equivalent to $x \in \overline{A}$), hence $\mu(A^\epsilon) \downarrow \mu(A)$. Therefore $d_P(\mu,\nu) = 0$ implies $\mu(A) = \nu(A)$ for all closed A, and therefore $\mu = \nu$. Now let us prove the triangular inequality. Let $\mu_1, \mu_2, \mu_3 \in \Pr(E)$. For any $\epsilon > d_P(\mu_1,\mu_2)$ and $\eta > d_P(\mu_2,\mu_3)$, we have

$$\mu_1(A) < \mu_2(A^\epsilon) + \epsilon, \quad \mu_2(A^\epsilon) < \mu_3(A^{\epsilon+\eta}) + \eta$$

for every Borel set $A \subset E$. This implies $\mu_1(A) < \mu_3(A^{\epsilon+\eta}) + \epsilon + \eta$. Similarly, $\mu_3(A) < \mu_1(A^{\epsilon+\eta}) + \epsilon + \eta$. Therefore $d_P(\mu_1, \mu_3) \leqslant \epsilon + \delta$. Letting $\epsilon \to d_P(\mu_1, \mu_2)$ and $\eta \to d_P(\mu_2, \mu_3)$, we get $d_P(\mu_1, \mu_3) \leqslant d_P(\mu_1, \mu_2) + d_P(\mu_2, \mu_3)$.

Now let us show that the topology induced by d_P is finer than the narrow topology.

Let $\mu \in \Pr(E)$, $F \subset E$ closed and $\epsilon > 0$. Let $\eta \in]0, \epsilon[$ such that $\mu(F^\eta) < \mu(F) + \epsilon$. If $d_P(\mu, \nu) < \eta$, then $\nu(F) < \mu(F^\eta) + \eta < \mu(F) + 2\epsilon$. Thus each set of the family *(i)* in proposition *"neighborhood bases of the narrow topology"* contains a d_P-ball. The separability assumption has not been used here.

Now we show the converse under the separability assumption on E . We will show that every open d_P-ball contains a set of the family *(iii)* in proposition *"neighborhood bases of the narrow topology"*. Take $\eta \in]0, \epsilon/2[$. Apply the previous lemma: cover E by open balls $\{S_i\}_{i \geqslant 1}$ satisfying $\mu(\partial S_i) = 0$ and $\operatorname{diam}(S_i) < \eta$ for all i. Let $A_1 = S_1$ and $A_n = S_n \setminus \big(\bigcup_{m=1}^{n-1} S_m\big)$ for $n \geqslant 2$. Take $k \geqslant 1$ such that $\mu\big(\bigcup_{i=1}^k A_i\big) > 1 - \eta$. One has $\bigcup_{i=1}^k A_i = \bigcup_{i=1}^k S_i$, therefore $\mu\Big(\partial\big(\bigcup_{i=1}^k A_i\big)\Big) = 0$. Denote by $\mathcal{A}$ the set of all Borel A that can be written as a union of sets from $\{A_1, \ldots, A_k\}$. Let N be a neighborhood of μ of type *(iii)* in proposition *"neighborhood bases of the narrow topology"*, described by

$$N = \Big\{\nu \in \Pr(E) \mid \forall A \in \mathcal{A}, \big|\mu(A) - \nu(A)\big| < \eta\Big\}.$$

For any Borel set $B \subset E$, let A' be the union of sets in $\{A_1, \ldots, A_k\}$ which intersect B. Then $\big|\mu(A') - \nu(A')\big| < \eta$ if $\nu \in N$.
Let us check that $A' \subset B^\eta$. If $x \in A'$, then there is j such that $x \in A_j$ and $A_j \cap B \neq \varnothing$. Pick $y \in A_j \cap B$. Then $d(x, B) \leqslant d(x, y)$. Since $\operatorname{diam}(A_j) < \eta$, one has $x \in B^\eta$.

One has $B \subset A' \cup \big(\bigcup_{i=1}^k A_i\big)^c$. Thus

$$\mu(B) \leqslant \mu(A') + \mu\Big(\big(\bigcup_{i=1}^k A_i\big)^c\Big) \leqslant \nu(A') + 2\eta \leqslant \nu(B^\eta) + 2\delta,$$

hence $d_P(\mu, \nu) \leqslant 2\delta < \epsilon$. That is, every open d_P-ball contains a set of type *(iii)* in proposition *"neighborhood bases of the narrow topology"*. $\square$

We refer the reader to [3] for a proof of the following theorem.

Theorem (Prohorov). *Let E be a metric space. If $\Lambda \subset \Pr(E)$ is tight, then every sequence in Λ has a convergent subsequence.*

Theorem. *If E is Polish, the Prohorov metric is complete.*

Proof. Let $\{\mu_n\}$ be a Cauchy sequence in $\Pr(E)$ with respect to d_P. Let $\epsilon > 0$ and $k \geqslant 1$ be an integer. Pick $n(k) \geqslant 1$ such that $d_P(\mu_n, \mu_{n(k)}) < \epsilon 2^{-k}$ for $n \geqslant n(k)$. Consider a complete metric on E compatible with the topology. Since E is separable, it can be covered by countably many open balls of diameter $\epsilon 2^{-k}$. Let $B_1^k, \ldots, B_{m(k)}^k$ be finitely many open balls of diameter $\epsilon 2^{-k}$ satisfying

$$\mu_{n(k)}\left(\bigcup_{i=1}^{m(k)} B_i^k\right) > 1 - \frac{\epsilon}{2^k}.$$

For each $i \in [\![1, m(k)]\!]$, let A_i^k be the open ball concentric with B_i^k and with twice the radius. Since

$$\left(\bigcup_{i=1}^{m(k)} B_i^k\right)^{\frac{\epsilon}{2^k}} \subset \bigcup_{i=1}^{m(k)} A_i^k$$

and $d_P(\mu_n, \mu_{n(k)}) < \frac{\epsilon}{2^k}$ for $n \geqslant n(k)$,

$$\mu_n\left(\bigcup_{i=1}^{m(k)} A_i^k\right) > 1 - \frac{2\epsilon}{2^k}$$

for $n \geqslant n(k)$.

By adding if necessary finitely many open balls $A^k_{m(k)+1}, \dots, A^k_{j(k)}$ of radius $\frac{2\epsilon}{2^k}$, we have

$$\mu_n\left(\bigcup_{i=1}^{j(k)} A_i^k\right) > 1 - \frac{2\epsilon}{2^k}$$

for $n \geqslant 1$. It is easy to see that the set

$$K = \bigcap_{k\geqslant 1} \bigcup_{i=1}^{j(k)} A_i^k$$

is totally bounded. Therefore its closure $\overline{K}$ is totally bounded as well. Since it is complete, $\overline{K}$ is compact. Also, $\mu_n(\overline{K}) \geqslant 1 - 2\epsilon$ for all n. Therefore the sequence $\{\mu_n\}$ is tight. By Prohorov's theorem, it has a convergent subsequence. Since it is Cauchy, it converges. □

Corollary. *If E is Polish,* $\mathrm{Pr}(E)$ *is Polish.*

Proof. This is a consequence of the two previous theorems. □

Kernels as random probabilities

Now we show the result stated in the introduction, namely that a probability kernel from a probability space to a Polish space E is a random variable in the Polish space $\mathrm{Pr}(E)$.

Lemma. *Let E be a Polish space and Ω be a probability space. Then a map $\Gamma\colon \Omega \to \mathrm{Pr}(E)$ is measurable if and only if $\omega \mapsto \Gamma_\omega(f)$ is measurable for every $f \in C_b(E)$, where we denote by Γ_ω the map $\mathcal{B}_E \ni A \mapsto \Gamma(\omega)(A)$.*

Proof. The "only if" part is obvious (and does not require the Polish assumption). Let us show the converse. Since E is Polish, $\mathrm{Pr}(E)$ is Polish, hence it is strongly Lindelöf. Therefore every open set $O \subset \mathrm{Pr}(E)$ is a countable union of sets

$$\Big\{\nu \mid \big|\nu(f_i) - \mu(f_i)\big| < \epsilon, i \in [\![1,k]\!]\Big\}$$

where $\mu \in \mathrm{Pr}(E)$, $k \geqslant 1$ and $f_i \in C_b(E)$. The maps $\omega \mapsto \Gamma_\omega(f_i)$ are measurable, therefore the set

$$\Big\{\omega \mid \big|\Gamma_\omega(f_i) - \mu(f_i)\big| < \epsilon, i \in [\![1,k]\!]\Big\}$$

is measurable. Thus $\Gamma^{-1}(O)$ is a measurable set. □

Theorem. *Let E be a metric space. Let Ω be a probability space, and $\Gamma\colon \Omega \to \mathrm{Pr}(E)$ be a map. We denote by Γ_ω the image of ω by Γ. Define the map $K\colon \Omega \times \mathcal{B}_E \to [0,1]$ by $K(\omega, A) = \Gamma_\omega(A)$. If Γ is measurable, then K is a probability kernel from Ω to E. If E is Polish, the converse is true.*

Proof. Assume Γ is measurable. Then $\omega \mapsto \Gamma_\omega(f)$ is measurable for every $f \in C_b(E)$. Let $F \subset E$ be a closed set and for every integer $n \geqslant 1$ define $f_n \in C_b(E)$ by $f_n(x) = \max\big\{0, 1 - nd(x,F)\big\}$ where d is the metric on E. Then $f_n(x) \downarrow \mathbf{1}_F(x)$ for every $x \in E$. By monotone convergence, $\Gamma_\omega(f_n) \to \Gamma_\omega(F)$ and then $\omega \mapsto \Gamma_\omega(F)$ is measurable. It is easy to see that the set

$$\big\{A \in \mathcal{B}_E \mid \omega \mapsto \Gamma_\omega(A) \text{ is measurable}\big\}$$

is a λ-system. Since it contains the π-system of closed sets, it is equal to $\mathcal{B}_E$.

Now assume that E is Polish and K is a probability kernel. The set

$$V = \big\{f\colon E \to \mathbb{R} \mid \omega \mapsto \Gamma_\omega(f) \text{ is measurable}\big\}$$

is a vector space containing the indicator function $\mathbf{1}_A$ for every $A \in \mathcal{B}_E$, hence it contains all finite linear combinations of such indicators functions. Let $f \in C_b(E)$. For every integer $n \geqslant 1$, define

$$f_n(x) = \sum_{j=1}^{\infty} \frac{j-1}{n} \mathbf{1}_{]\frac{j-1}{n}, \frac{j}{n}]}\big(f(x)\big).$$

This is a finite sum since f is bounded, and $\big|f_n(x) - f(x)\big| \leqslant \frac{1}{n}$ for every $x \in E$. Therefore $f_n(x) \leqslant 1 + \|f\|_\infty$ and $f_n(x) \to f(x)$ for every $x \in E$. By dominated convergence, $\Gamma_\omega(f_n) \to \Gamma_\omega(f)$, hence $f \in V$. By the previous lemma, Γ is measurable. $\square$

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