Closed-form Solution for American Options*

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Abstract

We introduce an exact closed-form pricing formula for American options when stock returns follow a normal distribution or Lévy processes. It uses a non-homogeneous partial differential equation for American options and the optimality from early exercise general solution, and hence a Feynman Kac martingale expectation valuation with a known exercise boundary value at expiry. Under instantaneous information, it is Kim's (1990) solution. Modifying the pricing formula solves closed-form pricing problems for American (put or call) options with discrete dividend payouts and Bermudan options without the curse of dimensionality.

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1. Introduction

An option gives its buyers the right to buy or sell the underlying securities at a future date and an agreed price. European options only allow settlement to occur at the expiration date. American options permit termination of the options any time before the expiration date. Both are actively traded plain-vanilla options with high transaction values. Most organized exchange-traded index options are European (e.g., SPX, RUT, NDX). Almost all exchange-traded equity options and options on ETFs (e.g., SPY, IWM, AAPL) are American. Exact closed-form solutions exist for American call options with a single dividend payout or no dividend yield. American options are typically valued using numerical techniques: lattice methods, Monte Carlo simulations, or approximate closed-form solutions.

Valuing American options has been a puzzle since its inception (Brennan and Schwartz, 1977). Nonetheless, many financial economists have thoroughly examined their properties (see Broadie and Detemple, 2004), such as the smooth paste property and put-call symmetry (Schroder, 1999). They developed numerical techniques to locate the American option values using binomial lattices (Cox, Ross, and Rubinstein, 1979), finite difference methods, and Monte Carlo simulations (Longstaff and Schwartz, 2001; Ibáñez and Zapatero, 2004). Closed-form approximation formulas were also available (Barone-Adesi and Whaley, 1987; Ju, 1998; Ju and Zhong, 1999; Laprise, Fu, Marcus, Lim, and Zhang, 2006; Chung and Chang, 2007; Zhang and Li, 2010). Kim (1990) introduced an exact closed-form formula requiring knowledge of the optimal exercise boundary curve (Kim, 1990; Jacka, 1991; Carr, Jarrow, and Myneni, 1992; Frontczak and Schöbel, 2008). It cannot be known as a priori but must be determined numerically.

The rationale of this study is to recast the parabolic variational inequality equation for American options (Jaillet, Lamberton, and Lapeyre, 1990) to a non-homogeneous parabolic partial differential equation (PDE). The analytic solution to the PDE problem may be Feynman Kac's integral solution or a general solution of a complementary and particular solution that is the same as a European option. Hence, Feynman Kac's discounted expected terminal value is a European option. While Feynman Kac's solution provides a structure for the analytic solution, the optimality of the general solution makes it computable. It equates the rate of time change of the complementary solution to the negative of the corresponding rate of time change of the particular solution. As it is the European option, the early exercise premium, at any time, is the total European option time decay. The European option time decay depends only on a unique critical early exercise price.

The analytic solution is consistent with Kim's solution when finite difference approximates the time decay symmetrically around the time-to-maturity. We find it by numerical search based on the internally consistent application of the analytic solution to the early exercise condition. We developed analytical finite difference solutions based on a unique early exercise boundary value and the Black-Scholes European option. They represent the numerical series approximations of the early exercise value function at the expiry date T with variable time t. The exercise value is its parameter determined by the critical exercise boundary condition. In the limit, the finite-difference solutions converge to the analytic solution as the number of finite-difference points increases. It gives the intuitive formula for the analytic solution. Its early-exercise premium is the time-average return of the European option at maturity. The values of the analytic solution are close to those of the refined binomial lattice with 10,000 iterations. Under the assumption of return normality, it is still difficult to calculate the analytical finite-difference solutions. However, the formulation of the fully analytic solution is simple and general. It has the same complexity as the European options. When we can express the European options in closed form, the American options will also be analytic. It obeys the put-call symmetry of the American options. The behavior of the early exercise boundary curves (Evans, Kuske, and Keller, 2002) is consistent with those of other numerical methods near and away from expiry. When the early exercise decision horizon becomes instantaneous, the solution provides the instantaneous critical values on the early exercise boundary curve. It is Kim's integral solution.

We organize the remainder of the paper as follows. Section 2 discusses the theoretical structures for solving the problem of American options. Section 3 describes the closed-form pricing formula for American options when the security prices follow geometric Brownian motion. It also discusses the numerical results and error analysis. Section 4 presents an integral formula for pricing American options when the security prices follow geometric Lévy processes. Section 5 concludes.

2. The Properties of the Problem

Consider the parabolic variational inequality equation for American options

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_A}{\partial S^2} + (r-q)S\frac{\partial V_A}{\partial S} + \frac{\partial V_A}{\partial t} - rV_A \leq 0. \tag{1}$$

Where S is the stock price, σ is the volatility, r and q are the constant interest rate and the dividend yield, t is the time, T is the date of maturity, $\tau = T - t, \mu = r - q - \frac{\sigma^2}{2}$, ϕ is an indicator value equal to 1 for call options and –1 for put options, $V_A(S, \tau; \phi)$ is the solution to the American option problem, and $V(S, \tau; \phi)$ is the solution to the early exercise premium of the American option problem. In the following, we show some of the established findings. The boundary conditions are

(a) $V_A(0, \tau; 1) = 0$,

(b) $V_A(\infty, \tau; -1) = 0$,

(c) $V(S, 0; 1) = 0 = V(S, 0; -1)$.

The free boundary conditions are

(d) $V_A(G(\tau), \tau; \phi) = \phi(G(\tau) - K)$,

(e) $\frac{\partial V_A}{\partial S}|_{S=G(\tau)} = \phi$,

(f) $\frac{\partial V_A}{\partial \tau}|_{S=G(\tau)} = 0$.

The critical exercise value condition of perpetual American options is

(g) $G(\tau) = \frac{\lambda_\phi(r,q)}{\lambda_\phi(r,q)-1} K$ as $\tau \to \infty$,

where $\lambda_\phi(r, q) = \frac{-\tilde{b}+\phi\sqrt{\tilde{b}^2-4\tilde{c}}}{2}$, $\tilde{b} = \frac{2\mu}{\sigma^2}$, and $\tilde{c} = -\frac{2r}{\sigma^2}$. Condition (d) is the critical exercise boundary condition. Condition (e) is the high contact condition. Condition (f) is of Bunch and Johnson (2000). It suggests that the early exercise value is independent of time decay.

2.1 Feynman Kac Formulation

Over the life of the American option contract, the critical points of exercise and non-exercise create an early exercise boundary curve. The knowledge of the whole curve is unknown a priori except the point at the expiry. Throughout the life of the American option contract, we overcome the free-boundary nature of the problem by repeated application of the martingale expectation using only the boundary curve value at expiry. Using a non-homogeneous PDE, we demonstrate how to execute the idea.

The insight of this paper is to introduce a new non-homogeneous PDE in Eq. (2). Unlike the formulation in the previous contributions (Kim, 1990; Jacka, 1991; Carr, Jarrow, and Myneni, 1992; Frontczak and Schöbel, 2008), our introduction of Eq. (2) enables us to directly calculate an exact analytic closed-form solution for American options notably under geometric Brownian motion.

When we recast the American option problem in Inequality (1) as the Feynman-Kac problem,

$$-\frac{\partial V_A}{\partial \tau}+\frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_A}{\partial S_t^2}+(r-q)S_t\frac{\partial V_A}{\partial S_t}-rV_A+f^*(S_t,\tau)=0 \quad (2)$$

subject to

(h) $\quad V_A(S_T,0;\phi)=V_E(S_T,0;\phi)$,

where $V_E(S_T,0;\phi)$ is the European option value at time t. The terminal boundary condition determines the nature of a financial derivatives problem. For the European option, the expiration date is certain, and the only uncertainty is on the terminal payoff because of the uncertain future security prices. Therefore, the terminal boundary condition is the option's intrinsic value at expiration. Assuming overall optimality whereby there is timing indifference to exercise early, the Feynman Kac solution of Eq. (2) is

$$V_A(S_0,T;\phi)=\frac{1}{T}\int_0^T e^{-rt}\,E_{x_t}[f(S_0e^{x_t},T-t)]dt+e^{-rT}E_{x_T}[V_E(S_0e^{x_T},0;\phi)] \qquad (3)$$

where $S_t=S_0e^{x_t}$, and $x_0=0$. The Feynman-Kac formulation summarizes the dynamics into a single equation that includes exercise values at expiration and before. On the other hand, the solution $V_A(S_t,T-t)$ of a non-homogeneous PDE is a sum of the particular solution $V_A^P(S_t,T-t)$, and the complementary solution $V_A^C(S_t,T-t)$. In particular, the particular solution of the non-homogeneous equation problem in Inequality (1) is the solution of the homogeneous equation

$$-\frac{\partial V_A^P}{\partial \tau}+\frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_A^P}{\partial S_t^2}+(r-q)S_t\frac{\partial V_A^P}{\partial S_t}-rV_A^P=0, \qquad (4)$$

subject to the boundary conditions

$$V_A^P(S_T, 0; \phi) = \max\left(\phi(S_T - K), 0\right).$$

The setup in Eq. (4) is similar to that of the European option evaluation. Consequently, the particular solution is the second term in Eq. (3), and the complementary solution is the first term in Eq. (3). Hence, we exercise the particular solution of the option at maturity. In contrast, the complementary solution is an early-exercise value. The optimal stopping time solution requires $\frac{\partial V_A(S_t, T-t)}{\partial t} = 0$ for $t \in (0, T)$. Under this optimality condition, $\frac{\partial V_A^C(S_t, T-t)}{\partial t} = -\frac{\partial V_A^P(S_t, T-t)}{\partial t}$. However, the optimality condition using $\frac{\partial V_A^P(S_t, y)}{\partial y}$ is composed of two parts, i.e., $\frac{\partial V_A^P(S_t, y)}{\partial y} = f_t(y) + f_N(y)$. The value change due only to time decay is denoted by $f_t(y)$, whereas its stochastic value variation is the $f_N(y)$. In the context where the stock returns follow normal processes,

$$E_{x_t}[f_t(y)] = -\phi\left\{qS_0 e^{-qy+\left(\mu+\frac{\sigma^2}{2}\right)t} N\left(\phi \frac{\ln\left(\frac{S_0}{G(0)}\right)+(\mu+\sigma^2)(t+y)}{\sigma\sqrt{t+y}}\right) - rKe^{-ry} N\left(\phi \frac{\ln\left(\frac{S_0}{G(0)}\right)+\mu(t+y)}{\sigma\sqrt{t+y}}\right)\right\}, \quad (5)$$

$$E_{x_t}[f_N(y)] = \phi\left\{S_0 e^{-qy+\left(\mu+\frac{\sigma^2}{2}\right)t} \frac{\partial}{\partial y} N\left(\phi \frac{\ln\left(\frac{S_0}{G(0)}\right)+(\mu+\sigma^2)(t+y)}{\sigma\sqrt{t+y}}\right) - Ke^{-ry} \frac{\partial}{\partial y} N\left(\phi \frac{\ln\left(\frac{S_0}{G(0)}\right)+\mu(t+y)}{\sigma\sqrt{t+y}}\right)\right\}. \quad (6)$$

Because of stochastic calculus, we ignore any random changes in values induced by less than the square root of infinitesimally small changes in time. Thus, their time value $E_{x_t}[f_N(y)]$ should be 0. Kim (1990) has pointed out this fact, provided that $G(0)$ is in the money. Instead of $G(0) = K$, we calculate an equivalent exercise price at maturity. If early exercise occurs at time t, we compare the loss of the total premium value due to time decay $E_{x_t}\left[\int_0^\tau - f_t(y)dy\right]$ and the gain from the value of early exercise payoffs $V_A^C(S_t, \tau)$. In Appendix A1, Eq. (3) with the optimal stopping time condition leads to $f(S_t, \tau) = \int_0^{\tau} -f_t(y)dy$. We rewrite the integral part of Eq. (3) as

$$\frac{1}{T}\int_0^T e^{-rt}E_{x_t}[f(S_t, T-t)]dt = \frac{1}{T}\int_0^T E_{x_t}\left[\int_0^{\tau} -e^{-rt}f_t(y)dy\right]dt = V_A^C(S_0, T).$$

The integral part of Eq. (3) represented by $V_A^C(S_t, T-t)$ is, therefore, the total of a stream of expected forward-start total early-exercise American option values $E_{x_t}\left[\int_0^{\tau} \frac{\partial V_A^C(S_t,y)}{\partial y} dy\right]$ with $\tau = T - t$.

The analytic Feynman Kac solution follows the martingale property. It calculates the discounted expected values $f_t(y)$ based on only the payoffs at the date of maturity $t + y$ for $y\epsilon[0, T-t]$. Hence, it requires only a single value $G^f(0)$ on the optimal exercise boundary curve at expiry. The solution thus offers an exact closed-form American option value under geometric Gaussian security price processes. It is a univariate integral valuation for American options under geometric Lévy security price processes.

3. The American Option Value under Geometric Gaussian Security Price Processes

We calculate the particular solution based on the condition (h) that gives value in Eq. (7)

$$V_A^P(S_0, T; \phi) = V_E(S_0, T; \phi); \tag{7}$$

where

$$V_E\left(S_t, T-t; \phi, G^f(0)\right)$$

$$= \phi[e^{-q(T-t)} S_t N\left(\phi d_1\left(S_t, G^f(0)\right)\right)$$

$$- e^{-r(T-t)} K N\left(\phi d_2\left(S_t, G^f(0)\right)\right)],$$

$$d_1\left(S_t, G^f(0)\right) = \frac{\ln\left(\frac{S_t}{G^f(0)}\right) + (\mu - \sigma)(T-t)}{\sigma\sqrt{T-t}}, d_2\left(S_t, G^f(0)\right) = d_1\left(S_t, G^f(0)\right) - \sigma\sqrt{T-t},$$

and $N(\cdot)$ is a cumulative univariate normal distribution function. As a result,

$$V_A^P(S_t, y; \phi) = \phi\left[S_0 e^{-q(t+y)} N\left(\phi \frac{\ln\left(\frac{S_0}{G^f(0)}\right) + (\mu + \sigma^2)(t+y)}{\sigma\sqrt{t+y}}\right) - \right.$$

$$\left. e^{-r(t+y)} K N\left(\phi \frac{\ln\left(\frac{S_0}{G^f(0)}\right) + \mu(t+y)}{\sigma\sqrt{t+y}}\right)\right]. \quad (8)$$

We re-group the time premium to focus on the instantaneous time premium (see Appendix A2)

$$-\lim_{\tau\to 0} E_{x_t} \int_0^\tau e^{-rt} f_t(y)\, dy = -\lim_{\tau\to 0} E_{x_0} \int_t^{t+\tau} f_0(x)\, dx =$$

$$\lim_{\tau\to 0} \int_{T-\tau}^{T} \phi\left\{q S_0 e^{-qx} N\left(\phi \frac{\ln\left(\frac{S_0}{G^f(0)}\right) + (\mu + \sigma^2)x}{\sigma\sqrt{x}}\right) - \; rKe^{-rx} N\left(\phi \frac{\ln\left(\frac{S_0}{G^f(0)}\right) + \mu x}{\sigma\sqrt{x}}\right)\right\} dx \quad (9)$$

where $t + y = x$. We justify the validity of the time premium formula by whether the values of the American option are similar to those of the binomial lattice and whether their early exercise boundary curves have the expected convexity behaviors, especially near maturity (Evans, Kuske, and Keller, 2002; Lauko and Ševčovič, 2010). The American options are constructed based on adding the early exercise premium to the European option values (Kim, 1990; Frontczak and Schöbel, 2008)

$$V_A(S_0, T; \phi) = V_E(S_0, T; \phi) +$$

$$\frac{\phi}{T}\left\{q S_0 \int_0^T \lim_{\tau\to 0} \int_{T-\tau/2}^{T+\tau/2} e^{-qx} N\left(\phi \frac{\ln\left(\frac{S_0}{G^f(0)}\right) + (\mu + \sigma^2)x}{\sigma\sqrt{x}}\right) dx\, d\tau - \right.$$

$$rK \int_0^T \lim_{\tau\to 0} \int_{T-\tau/2}^{T+\tau/2} e^{-rx} N\left(\phi \frac{\ln\left(\frac{S_0}{G^f(0)}\right)+\mu x}{\sigma\sqrt{x}}\right) dx\, d\tau \Bigg\}. \tag{10}$$

For the analytic solution, $\int_0^T \lim_{\tau\to 0} \int_{T-\tau/2}^{T+\tau/2} e^{mx} N\left(\frac{c+bx}{\sqrt{x}}\right) dx\, d\tau = e^{mT} N\left(\frac{c+bT}{\sqrt{T}}\right)\frac{T^2}{2}$. To ensure the critical exercise price occurs in the money, we first calculate $G^f(0)$, which is the early exercise critical price derived at the time to maturity T,

$$\phi[G^f(0) - K] = V_E(G^f(0), T; K, \phi) +$$

$$\frac{\phi}{T}\Bigg\{qG^f(0) \int_0^T \lim_{\tau\to 0} \int_{T-\frac{\tau}{2}}^{T+\frac{\tau}{2}} e^{-qx} N\left(\phi \frac{(\mu+\sigma^2)\sqrt{x}}{\sigma}\right) dx\, d\tau -$$

$$rK \int_0^T \lim_{\tau\to 0} \int_{T-\frac{\tau}{2}}^{T+\frac{\tau}{2}} e^{-rx} N\left(\phi \frac{\mu\sqrt{x}}{\sigma}\right) dx\, d\tau\Bigg\}. \tag{11}$$

Eq. (11) is a consequence of condition (d). Consequently, the formula in Eq. (10) is $V_A(S_0, T; G^f(0))$. In Appendix A2, we show schematically the pricing process. For $r > 0$, the American option value is

$$\hat{V}_A(S_0, T; \phi) = \max\left[\phi(S_0 - K), V_A(S_0, T; \phi), V_E(S_0, T; \phi)\right]. \tag{12}$$

Eq. (10) is an analytic solution because it depends on a known optimal exercise boundary value at the expiry $G^f(0)$. Note that the double integral in Eq. (10) is of the form $\int_0^T \lim_{\tau\to 0} \int_{T-\tau/2}^{T+\tau/2} e^{mx} N\left(\frac{c+bx}{\sqrt{x}}\right) dx\, d\tau$, which can be evaluated analytically for $m \le 0$. We evaluate the integral in Appendix A3 under two scenarios: $m, c \neq 0$, and $c = 0, m \neq 0$. Specifically, for the first and second integrals of Eq. (10), $b = \phi\frac{\mu+\sigma^2}{\sigma}$, $\phi\frac{\mu}{\sigma}$; $c = \phi\left[\frac{\ln\left(\frac{S_0}{G^f(0)}\right)}{\sigma}\right]$, $\phi\left[\frac{\ln\left(\frac{S_0}{G^f(0)}\right)}{\sigma}\right]$; and $m = -q, -r$; respectively. The coefficients of the double integrals are qS_0 and rK, respectively. Therefore, $m = 0$, we may regard the double integrals as having a value of zero, and thus, we do not evaluate $\int_0^T \int_{T-\tau/2}^{T+\tau/2} N\left(b\sqrt{x} + \frac{c}{\sqrt{x}}\right) dx\, d\tau$ in that situation. The analytical integral evaluation affirms that the finite-difference solutions converge to the analytic solution. Fig. 1 demonstrates the convergence of the finite-difference solutions to the analytic solution through their early exercise boundary curves. The range of the integral limits about T in Appendix A3 contracts as we move into the higher-order finite-difference approximations with more sampling points. For instance, the two-point forward difference approximation has a range $[T, 2T]$, whereas it is $[\frac{5}{6}T, \frac{7}{6}T]$ for the four-point central difference. Fig. 2 plots early exercise boundary curves due to Eq. (11) for the American call and put options with maturity of up to 5 years.

When $q = 0$ and $\phi = 1$, the early exercise premium term in Eq. (10) is negative, and $V_A(S_0, T\,; \phi) < V_E(S_0, T\,; \phi)$. Therefore, we will not exercise the non-dividend-paying American call option early. This situation does not occur when $q = 0$ and $\phi = -1$ because the early exercise premium term in Eq. (10) becomes positive.

3.1 *Kim's Integral Solution*

We write the total time premium of Eq. (9) as

$$V(S_0, T\,; \phi) = \lim_{\Delta T \to 0} - \int_0^{T+\Delta T} \frac{1}{T+\Delta T} \int_{T+\Delta T - \frac{\tau+\Delta\tau}{2}}^{T+\Delta T + \frac{\tau+\Delta\tau}{2}} E_{x_t}\left[e^{-rt} f_t\big(y, G(0)\big)\right] dy d(\tau+\Delta\tau).$$

When $\Delta T \to 0$, where $\Delta T \geq \Delta\tau$, the instantaneous rate of time change of $-\int_0^{T+\Delta T} \frac{1}{\Delta T} \int_{\Delta T - \frac{\Delta\tau}{2}}^{\Delta T + \frac{\Delta\tau}{2}} E_{x_t}\left[e^{-rt} f_t\big(T+y, G(T)\big)\right] dy d(\Delta\tau)$ is $-E_{x_t}[e^{-rt} f_t\big(T, G(T)\big)]$ (see Appendix A4). Therefore, we can express the early exercise premium as $V_A(S_0, T\,; \phi) = -\int_0^T f_0(\tau, G(\tau)\, d\tau$. It is Kim's integral solution (Kim (1990), Jacka (1991), Carr, Jarrow, and Myneni (1992)). Granted that we can evaluate them analytically, the two solutions are equivalent and have the same value for an American option. However, Kim's (1990) valuation of the early-exercise premium hinges on a priori knowledge of the complete optimal exercise boundary curve, $G(T - t, r, q, K; \phi)$ for $t \in [0, T]$. Kim's (1990) solution is exact theoretically. However, the lack of a priori knowledge of the entire optimal exercise boundary curve renders his solution to an inexact and slow iterative numerical approximation. The single early-exercise price analytic solution is surprisingly simple. The early-exercise premium is the terminal rate of the European option return multiplied by the average time. Due to the uncertainty throughout the options life, the time average may be the best solution for a rational decision.

3.2. Properties of the Formula

The closed-form formula in Eq. (10) has the following properties.

i) The tail properties in conditions (a) and (b) are satisfied. The early exercise premium and the European option value both go to zero at the tails.

ii) The maturity Property in condition (c) is satisfied because the early exercise premium goes to zero as $T \to 0$.

iii) Based on Eq. (10), the perpetual American option value is zero (see Fig. 3). When the time to maturity goes to infinity, the early-exercise premium approaches zero. Therefore, the strike price of the perpetual American option approaches K (see Appendix A5).

iv) We only exercise non-dividend-paying American call options at expiration because the early exercise premium would then be negative.

v) The put-call symmetry condition is satisfied. The European options satisfy the put-call symmetry conditions because of martingale. Therefore, the derived time decay premium and the early exercise premium must also follow the put-call symmetry condition.

3.3 Numerical Results

Table 1 compares the critical exercise prices $G(T)$ close to expiry. Tables 1 and 2 of Lauko and Ševčovič (2010) provide the information. Evans, Kuske, and Keller (2002) (EKK), Stamicar, Ševčovič, and Chadam (1999) (SSCh-A), Lauko and Ševčovič (2010) (SSCh), Zhu (2006) (ZHU), projected successive over-relaxation (PSOR) (see Kwok, 1998) developed the critical exercise prices. Our pricing technique (Analytic) calculates them based on Eq. (10). The parameters are $\sigma = 0.3, r = 0.1, q = 0, K = 100,$ and $\phi = -1.$ Lauko and Ševčovič (2010) demonstrated that the critical early exercise prices of Zhu (2006) are not accurate near expiry. In Table 1, the values under the column ZHU are the lowest. In contrast, the critical exercise prices due to the analytic pricing technique are slightly larger than all the listed values. However, they are close to the values of PSOR for a very short time-to-maturity and SSCh-A for a longer time-to-maturity.

For a reasonable time to maturity away from expiry, Table 2 in the Appendix shows numerical examples using Eq. (10) regarding the American option values evaluated from other techniques. We divide the table into an upper panel for call options and a lower panel for put options. We list values computed from Eq. (10) in columns under $V_A^{Analytic}$. The values using the Barone-Adesi and Whaley (1987) formula, a binominal lattice of 10,000-time steps, and the Longstaff and Schwartz

(2001) Monte Carlo simulation approach are listed in columns under V_A^{BW}, V_A^{BL}, and V_A^{LS}, respectively. We present the corresponding European option values in columns under V_E.

As the time to maturity shortens, the $V_A^{Analytic}$ values can be higher or lower than the binomial lattice values. The absolute percentage difference decreases, which probably occurs because each time step represents a more refined approximation of the normal process during the shortening of the time to maturity. They are less than the absolute percentage errors of 0.3%. While the values of V_A^{BW} are higher than the V_A^{BL}, the analytic values are all smaller than those of the V_A^{BW}. During the short time to maturity, they are close. The absolute percentage of errors is mostly less than 0.6%. All the analytic values fall within a 95% confidence interval of V_A^{LS}. Rows 2 and 8 of the two panels show that only the analytic and binomial-lattice approaches satisfy the relationship of put-call parity symmetry for American options. They occur because both methods are due to the martingale approach of discounted expected value calculation of European option values, which obey the put-call symmetry (Schroder, 1999).

3.4 Error Analysis

We calculate the solution by an average of the discounted expected payoffs. We evaluate the derivatives by backward, forward, and central differences conditional on a unique early exercise value. Although they are analytical formulas, in a strict sense, the scheme is numerical. Unlike the $O(\frac{\tau}{N})$ order of the finite difference methods, the solution is of order $O(h^m)$, where h and m depend on the choice of finite differences to approximate the derivative. However, the analytic solution is the limit of the finite difference approximation.

For comparison, Table 3 uses the binomial lattice V_A^{BL} with 10,000 iterations. Its order of accuracy is $O\left(\frac{\tau}{10000}\right)$. Therefore, it should be correct to 3 to 4 significant digits. The 2-point forward and the backward difference may be accurate to 1 significant digit. In Table 3, the 2-point forward and backward differences option prices are accurate to 2 significant digits. The 2-point central difference is of order $O\left((\frac{\tau}{2})^2\right)$. Therefore, it could be correct to 2 digits. In Table 3, the results of the 2-point central difference follow the implication of the error analysis. The 4-point forward difference has an order $O\left((\frac{\tau}{2})^3\right)$. It should be accurate to 2 to 3 significant digits. The table shows a 2 to 3-digit significant accuracy and, for a longer time-to-maturity, they are closer to the binomial lattice figures than those of a 2-point forward difference. The 4-point central difference is of $O\left((\frac{\tau}{2})^4\right)$. It should be accurate to 4 to 5 significant digits. When we compare the result of the 4-point central difference to the 4-point forward difference, it achieves 4 to 5 significant digit accuracy. Compared with the 4-point central difference, the analytic solution is accurate to 7 to 9 significant digits. Based on the binomial lattice results, when $G(0)$ is not equal to K, the analytic American option prices have a 3 to 4-significant digit accuracy in Table 2. When $G(0)$ and K are equal, they have 2 to 3 significant digit accuracy. We may regard the finite difference and the binomial tree values as subsequences of the converging sequence to the limiting value $V_A^{Analytic}$. The analytic solution corroborates with the error prediction and convergence.

The derivation of the closed-form formula does not involve approximation and is theoretically intact. The binomial tree is a numerical technique with the curse of dimensionality to understand its limiting value. Closed-form approximation formulas often involve series truncation and cannot derive the limiting formula and calculate their limiting values. Simple closed-form formulas for American options are usually inaccurate. By comparing it to existing closed-form formulas in Table 4, we further demonstrate the accuracy of our formulation. We excerpt the parameters from Table 2 of Viegas and Azevedo-Pereira (2020) for American put option pricing. We compare the 1000 time steps binomial values reported in Aitsahlia and Carr(1997) under column AC, the results reported in Bjerksund and Stensland (1993) under column BJST, the values obtained in Ingersoll (1998) under column IB, and the MLE (n = 4) results of Viegas and Azevedo-Pereira (2020) under column VA, with the results of the 2-point forward, backward, and the analytic formulas in this paper. The 2-point values usually capture the other estimates. When it is a deep-in-the-money option, has high interest rates, or has a long time to maturity, the values under AC, BJST, IB, and VA are over-priced relative to our analytic value.

3.5 Bermudan and Discrete-Dividend-Payout American Options

A closed-form for American call option for known dividends (Whaley (1981)) exists. We can always translate the known dividends problem into a zero-dividend-yield problem. Suppose we pay dividend D_i at time T_i where $i = 1, \cdots, n-1$. We reduce the initial security price S_0 by the total discounted dividend amount $\hat{S}_{0,j-1} = S_0 - \sum_{i=1}^{j-1} D_i\, e^{-rT_i}$ where $0 < T_1 < T_2 < \cdots < T_n = T$ for $j = 1, \cdots, n$. We can then compute the American option values using Eq. (10) but setting $q = 0$. There is no closed-form solution for American put options with known dividends up to date. Under different scenarios of dividend payouts and dividend-paying periods, we obtain their maximum value as our American option value, i.e., $\max\left[\phi(S_0 - K), V_A(\hat{S}_{0,j-1}, T_j\,; \phi)\ \forall j \in \{1, \cdots, n\}\right]$. The valuation will apply regardless of American put or call options (Cosma, Galluccio, Pederzoli, and Scaillet, 2020).

Exact formulations for valuing Bermudan options may use the compound options approach. However, this approach involves multivariate distribution and numerical search at every monitoring date. We change the integral to a discrete sum at finite time points at which the Bermudan options are monitored and evaluated. In contrast to the conventional multivariate distribution evaluation approaches, it overcomes the curse of dimensionality and repetitive numerical searches. It leads to stable, efficient, and analytic results, even for frequent monitoring dates. For Bermudan options, Eq. (10) changes to[1]:

$$V_A(S_0, T\,;\phi) = \begin{cases} V_E(S_0, T\,;\phi) & \text{for } n = 1, \\ V_E(S_0, T\,;\phi) + \frac{\phi T}{2}\{qS_0 e^{-qt_{n-1}} N\left(\phi \frac{\ln(\frac{S_0}{Gf(0)}) + (\mu+\sigma^2)t_{n-1}}{\sigma\sqrt{t_{n-1}}}\right) & \\ -rKe^{-rt_{n-1}} N\left(\phi \frac{\ln(\frac{S_0}{Gf(0)}) + \mu t_{n-1}}{\sigma\sqrt{t_{n-1}}}\right)\} & \text{for } n > 1, \end{cases} \quad (13)$$

where the early exercise dates $t_0 = 0 < t_1 < t_2 < \cdots < t_n = T$, $\Delta = \frac{T}{n}$, and $t_i = i\Delta$.

4. The American Option Value under the Geometric Lévy Security Price Processes

According to Lewis (2001), the European option payoff under exponential Lévy security price processes is

$$V_E(S_0, T) = \frac{e^{-rT}}{2\pi}\int_{iv-\infty}^{iv+\infty} e^{-izY}\,\psi(-z\,;T)\hat{w}(z)dz, \quad (14)$$

where $\hat{w}(z) = -\frac{K^{1+iz}}{z^2 - iz}$, $v = \operatorname{Im} z$ (the imaginary part of z), $Y = \ln S_0 + (r-q)T$, $S_T = S_0 e^{(r-q)T + x_T}$, and $\psi(z\,;T)$ is the characteristic function of the Lévy processes. For European call options, $\operatorname{Im} z > 1$, and for European put options, $\operatorname{Im} z < 1$. The function $\hat{w}(z)$ has poles at $z = 0$ and $z = i$. By applying an analytic function transformation by

[1] $V(S_0, T\,;\phi) = \frac{\phi}{t_{n-1}}\left\{qS_0 e^{-qt_{n-1}} N\left(\phi \frac{\ln(\frac{S_0}{Gf(0)}) + (\mu+\sigma^2)t_{n-1}}{\sigma\sqrt{t_{n-1}}}\right) - rKe^{-rt_{n-1}} N\left(\phi \frac{\ln(\frac{S_0}{Gf(0)}) + \mu t_{n-1}}{\sigma\sqrt{t_{n-1}}}\right)\right\} \sum_{i=1}^{n-1} t_i \Delta$, for $n > 1$.

choosing

$$z = \begin{cases} -e^{-i\theta} & \text{for } \frac{i}{z}dz = d\theta, \\ i + e^{-i\theta} & \text{for } \frac{i}{z-i}dz = d\theta, \end{cases}$$

the European call option formula is

$$V_E(S_0, T\,; 1) = \frac{e^{-rT}}{2\pi}[S_0\psi(-i\,; T)\int_{-\pi}^{\pi} e^{-ie^{-i\theta}\bar{y}}\,\psi(-e^{-i\theta}\,; T)d\theta -$$

$$K\int_{-\pi}^{\pi} e^{ie^{-i\theta}\bar{y}}\,\psi(e^{-i\theta}\,; T)d\theta] \tag{15}$$

where $\bar{y}(S_0) = \ln(\frac{S_0}{K}) + (r-q)T$. By applying an analytic function transformation

$$z = \begin{cases} -e^{i\theta} & \text{for } -\frac{i}{z}dz = d\theta, \\ i + e^{i\theta} & \text{for } -\frac{i}{z-i}dz = d\theta, \end{cases}$$

the European put option formula is

$$V_E(S_0, T\,; -1) = \frac{e^{-rT}}{2\pi}[K\int_{-\pi}^{\pi} e^{ie^{i\theta}\bar{y}}\,\psi(e^{i\theta}\,; T)d\theta -$$

$$S_0\psi(-i\,; T)\int_{-\pi}^{\pi} e^{-ie^{i\theta}\bar{y}}\,\psi(-e^{i\theta}\,; T)d\theta]. \tag{16}$$

The characteristic function of Lévy processes of the security rate of returns under Lévy-Khinchin representation is $E(e^{iz\,ln(S_t/S_0)}) = \psi(z\,; \tau)$ with

$$\Psi(z) \equiv \frac{\ln\psi\,(z\,; \tau)}{\tau} = -\frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{\infty}(e^{izy} - 1 - izy1_{|y|\le 1})v(dy)$$

and $\gamma = r - q - \frac{\sigma^2}{2}$ under the risk-neutral measure. Based on $\phi[G(0) - K] = \lim_{\tau\to 0} V_E(G(0), \tau; \phi)$, we obtain $G(0)$ under the Lévy processes of the security rate of returns,

$$G(0) = \frac{\lim_{\tau\to 0}[1 - \frac{\phi e^{-r\tau}}{2\pi}\int_{iv-\infty}^{iv+\infty} e^{-iz(\ln(\frac{G(0)}{K}) + (r-q)\tau)}\psi(-z\,; \tau)\frac{i}{z}dz]}{\lim_{\tau\to 0}[1 - \frac{\phi e^{-r\tau}}{2\pi}\int_{iv-\infty}^{iv+\infty} e^{-i(z-i)(\ln(\frac{G(0)}{K}) + (r-q)\tau)}\psi(-z\,; \tau)\frac{i}{z-i}dz]}K. \tag{17}$$

Let $G_P(0)$ and $G_C(0)$ be the values of $G(0)$ for the put and call options, respectively. Then, $G_C(0) - K = \lim_{\tau \to 0} V_E(G_C(0), \tau\,; 1)$ implies

$$G_C(0) = \begin{cases} \frac{r}{r - \Psi(-i)} K & \text{for } \psi(-i\,; \tau) > 1 \\ K & \text{for } \psi(-i\,; \tau) \leq 1 \end{cases} \tag{18}$$

When the rate of returns follows a risk-neutral Gaussian process,

$$G_C(0) = \begin{cases} \dfrac{rK}{q} & \text{for } r > q, \\ K & \text{for } r \leq q. \end{cases}$$

Similarly, $K - G_P(0) = \lim_{\tau \to 0} V_E(G_P(0), \tau\,; -1)$ implies

$$G_P(0) = \begin{cases} \frac{r}{r - \Psi(-i)} K & \text{for } \psi(-i\,; \tau) < 1 \\ K & \text{for } \psi(-i\,; \tau) \geq 1 \end{cases} \tag{19}$$

When the rate of returns follows a risk-neutral Gaussian process,

$$G_P(0) = \begin{cases} \dfrac{rK}{q} & \text{for } r < q, \\ K & \text{for } r \geq q. \end{cases}$$

4.1 Calculation of the American Options

When the security returns follow Lévy processes, the numerical procedures for calculating the American options based on Section 2.1 remain unchanged. The probability density function is $g(x_T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x_T} \psi(\omega) d\omega$, which is over the real-number domain. Our approaches consider the time decay premium of European options at future times t whose dynamic paths reach their respective maturity dates $t + y$ for $y \in [0, T - t]$. Based on Lewis' (2001) European option formula, we derive the early exercise premium formula in Appendix A6. Hence, the American option formula is

$$V_A(S_0, T; \phi) = V_E(S_0, T; \phi) + V(S_0, T; \phi) = V_E(S_0, T; \phi) + \frac{\phi T e^{-rT}}{4\pi} [(r - \Psi(-i)) S_0 \psi(-i; T) \int_{-\pi}^{\pi} e^{-ie^{-\phi i\theta} \tilde{y}} \psi(-e^{-\phi i\theta}; T) d\theta - rK \int_{-\pi}^{\pi} e^{ie^{-\phi i\theta} \tilde{y}} \psi(e^{-\phi i\theta}; T) d\theta] \tag{20}$$

where $\tilde{y} = ln(\frac{S_0}{G^f(0)}) + (r-q)T$. Hence, we derive $G^f(0)$ from

$$\phi[G^f(0) - K] = V_E(G^f(0), T; \phi) + \frac{\phi T e^{-rT}}{4\pi}[(r - \Psi(-i))G^f(0)\psi(-i;T)\int_{-\pi}^{\pi} e^{-ie^{-\phi i\theta}(r-q)T}\,\psi(-e^{-\phi i\theta};T)d\theta \; - rK\int_{-\pi}^{\pi} e^{ie^{-\phi i\theta}(r-q)T}\,\psi(e^{-\phi i\theta};T)d\theta].$$

Using Eq. (20), we can evaluate the American option value in univariate integrals when the rate of security returns follows Lévy processes.

5. Concluding Remarks

American options are some of the most actively traded financial options. Many of their properties have been investigated and used in various methods of American option pricing. Unlike the Black-Scholes-Merton (1973) European option pricing formula, a simple exact pricing formula does not exist for American option pricing. Under geometric Gaussian security price processes, this study developed a closed-form formula for pricing American options. It uses the Feynman Kac formulation under a non-homogeneous parabolic differential equation (Eq. (2)) with a rate of time decay function and only a known early exercise boundary value at expiry. The time decay function emerges from a representation of the concept of optimal stopping time evaluated by an in-the-money early exercise time decay premium symmetric around the time-to-maturity. Under normally distributed stock returns and a unique early exercise boundary value, we developed several analytical finite-difference representations of the solution. We demonstrate the convergence of the finite-difference solutions to the analytic solution theoretically and numerically. The analytic solution is general and provides a simple early-exercise premium. It is the time average based on the rate of return of the terminal European option. It should apply to American options with Gaussian and non-Gaussian returns. It has the same complexity as the corresponding European options. As the pricing of the Black-Scholes European option is analytic, our pricing of the American option is in closed form.

The analytic solution is theoretically equivalent to Kim's (1990) early-exercise boundary curve-dependent formula. Numerical examples have shown that the results satisfy put-call symmetry and are similar to those of established methods. Hence, all other properties of American options are arguably satisfied. The boundary curves are well-behaved near and away from expiry. As for the perpetual American options, the early exercise premium under Eq.(10) goes to zero. Hence, the strike price of the perpetual options is the early exercise price. Eq. (12) applies to the actual perpetual American option value.

The method is flexible and can work out exact formulas for other related American options, such as American put or call options with discrete known dividends and computationally efficient Bermudan options, even with frequent monitoring dates (see Section 3.5). Its extension leads to a univariate-integral American option formula under exponential Lévy security price processes, as in Eq. (20). Lewis (2001) developed a similar European option counterpart. In summary, this study introduced a method to tackle American options in which the problem is a free-boundary problem and demonstrated its accuracy, flexibility, and potential extensions.

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Appendix A1. Derivation of the Optimal Non-homogeneous PDE Function

The Feynman Kac solution in Eq. (3) for the non-homogeneous PDE is

$$V_A(S_0,T;\phi)=\frac{1}{T}\int_0^T e^{-rt}\,E_{x_t}[f(S_0e^{x_t},T-t)]dt+V_E(S_0e^{x_T},T;\phi).$$

However, the solution for the non-homogeneous PDE is a sum of the complementary and particular solutions.

$$V_A(S_t,T-t)=V_A^C(S_t,T-t)+V_A^P(S_t,T-t),$$

where we define the complementary solution as

$$V_A^C(S_t,T-t)=\int_0^{T-t} e^{-rs}\,E_{x_s}[f(S_te^{x_s},T-s)]ds$$

and the particular solution is defined as

$$V_A^P(S_t,T-t)=V_E(S_t,T-t;\phi).$$

It is clear that $V_A^C(S_t,T-t)=\int_0^{T-t}\frac{\partial V_A^C(S_t,y)}{\partial y}dy$. The value, $V_A^C(S_t,0)$, should be zero at expiration. We can calculate the particular solution because it is a European option. However, the complementary solution is unknown because we do not know the non-homogeneous function $f(S_te^{x_s},T-s)$. We find the function by applying the optimal stopping time condition $\frac{\partial V_A(S_t,T-t)}{\partial t}=0$. Under this optimality condition,

$$\frac{\partial V_A^C(S_t,T-t)}{\partial t}=-\frac{\partial V_A^P(S_t,T-t)}{\partial t}.$$

Hence, assuming the equally likely chance of early exercise opportunity or a timing indifference to exercise early, using the martingale property of discounted expected payoffs on the complementary solution and the optimality condition leads to

$$V_A^C(S_0,T)=\frac{1}{T}\int_0^T e^{-rt}E_{x_t}\left[\int_0^{T-t}\frac{\partial V_A^C(S_t,y)}{\partial y}dy\right]dt$$
$$=\frac{1}{T}\int_0^T e^{-rt}E_{x_t}\left[\int_0^{T-t}-\frac{\partial V_A^P(S_t,y)}{\partial y}dy\right]dt.$$

Therefore, we establish the computable optimal function

$$E_{x_t}[f(S_te^{x_s},T-t)]=E_{x_t}\left[\int_0^{T-t}-\frac{\partial V_A^P(S_t,y)}{\partial y}dy\right]=E_{x_t}\left[\int_0^{T-t}-f_t(y)dy\right]$$

when $E_{x_t}[f_N(y)]=0$ where $E_{x_t}\left[\frac{\partial V_A^P(S_t,y)}{\partial y}\right]=E_{x_t}[f_t(y)]+E_{x_t}[f_N(y)]$.

Appendix A2. The Schematic Process of the American Option Pricing

The boundary conditions are

(a) $V_A(0,\tau;1)=0$,

(b) $V_A(\infty,\tau;-1)=0$,

(c) $V(S,0;1)=0=V(S,0;-1)$.

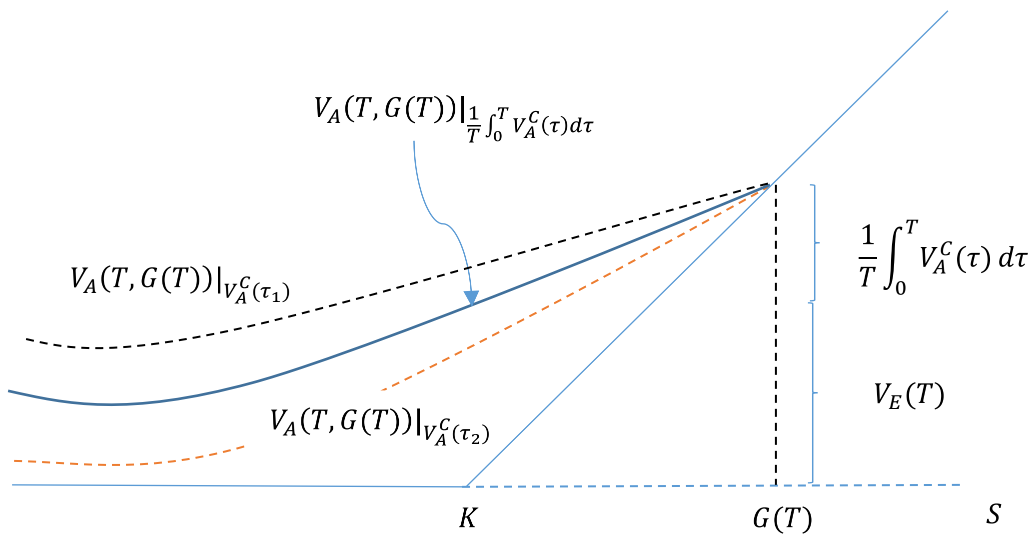
The free boundary conditions are

(d) $V_A(G(\tau),\tau;\phi)=\phi(G(\tau)-K)$,

(e) $\frac{\partial V_A}{\partial S}|_{S=G(\tau)}=\phi$,

(f) $\frac{\partial V_A}{\partial \tau}|_{S=G(\tau)}=0$.

Under condition (f), we can have multiple early-exercise American option functions under $\frac{\partial V_A}{\partial \tau}=0$ due to different early-exercise times with the same critical early-exercise boundary value $G(T)$. Putting conditions (d) and (f) together have the same early-exercise premium at $G(T)$. To capture all the early-exercise opportunities, we use $\frac{1}{T}\int_0^T V_A^C(\tau)d\tau$ for our American option valuation. We illustrate them by a call option diagram.



In finite difference notation,

$$\sum \Delta V_A^C = [V_A^C(\tau) - V_A^C(\tau - \Delta\tau)] + \cdots + [V_A^C(\Delta\tau) - V_A^C(0)] = V_A^C(\tau).$$

Notably, the optimal condition $-\frac{\partial V_A^P}{\partial t}$, determines $V_A^C(i\Delta\tau) - V_A^C\big((i-1)\Delta\tau\big)$, where $V_A = V_A^C + V_A^P$. We can sum two early-exercise premium values as

$w_1 V_A^C(\tau_1) + w_2 V_A^C(\tau_2) = w_1 \sum_{i=1}^{n}\big[V_A^C(i\Delta\tau_1) - V_A^C\big((i-1)\Delta\tau_1\big)\big] +$

$w_2 \sum_{i=1}^{n}[V_A^C(i\Delta\tau_2) - V_A^C\big((i-1)\Delta\tau_2\big)]$ (A2.1)

where w_1, and w_2 are weights. Alternatively,

$w_1 V_A^C(\tau_1) + w_2 V_A^C(\tau_2) = \sum_{i=1}^{n}\{w_1\big[V_A^C(i\Delta\tau_1) - V_A^C\big((i-1)\Delta\tau_1\big)\big] + w_2\big[V_A^C(i\Delta\tau_2) -$

$V_A^C\big((i-1)\Delta\tau_2\big)\big]\} = \sum_{i=1}^{n}\sum_{j=1}^{2} w_j\left[V_A^C\big(i\Delta\tau_j\big) - V_A^C\left((i-1)\Delta\tau_j\right)\right]$ (A2.2)

Eq. (A2.1) is similar to finite difference evaluation in Appendix A3, adding over terms with different $\Delta\tau_i$, whereas Eq. (A2.2) is similar to the continuous-time Eq. (10) imposing $\lim_{\Delta\tau_j \to 0} \sum_{j=1}^{m} w_j\left[V_A^C\big(i\Delta\tau_j\big) - V_A^C\left((i-1)\Delta\tau_j\right)\right]$.

Appendix A3. Analytical Evaluation of the Double Integrals

For a two-point backward difference,

$$\frac{1}{T}\int_0^T\int_{T-\tau}^T e^{mx}\,N(b\sqrt{x}+\frac{c}{\sqrt{x}})dxd\tau=\frac{1}{T}\int_0^T[H(T)-H(T-\tau)]d\tau$$

$$=\frac{1}{T}[H(T)T-\int_{\epsilon\to 0^+}^T H(w)dw].$$

For a two-point forward difference,

$$\frac{1}{T}\int_0^T\int_{T}^{T+\tau} e^{mx}\,N(b\sqrt{x}+\frac{c}{\sqrt{x}})dxd\tau=\frac{1}{T}\int_0^T[H(T+\tau)-H(T)]d\tau$$

$$=\frac{1}{T}[\int_T^{2T}H(w)dw-H(T)T].$$

For two-point central difference,

$$\frac{1}{T}\int_0^T\int_{T-\frac{\tau}{2}}^{T+\frac{\tau}{2}} e^{mx}\,N(b\sqrt{x}+\frac{c}{\sqrt{x}})dxd\tau=\frac{1}{T}\int_0^T[H(T+\frac{\tau}{2})-H(T-\frac{\tau}{2})]d\tau$$

$$=\frac{2}{T}[\int_T^{3T/2}H(w)dw-\int_{T/2}^{T}H(w)dw].$$

For the four-point central difference,

$$\frac{1}{T}\int_0^T[-\int_{T-\frac{2\tau}{12}}^{T+\frac{2\tau}{12}} e^{mx}\, N(b\sqrt{x}+\frac{c}{\sqrt{x}})dx+\frac{24}{3}\int_{T-\frac{\tau}{12}}^{T+\frac{\tau}{12}} e^{mx}\, N(b\sqrt{x}+\frac{c}{\sqrt{x}})dx]d\tau$$

$$=\frac{1}{T}\int_0^T\left[H\left(T-2\left(\frac{\tau}{12}\right)\right)-8H\left(T-\left(\frac{\tau}{12}\right)\right)+8H\left(T+\left(\frac{\tau}{12}\right)\right)\right.$$

$$\left.-H\left(T+2\left(\frac{\tau}{12}\right)\right)\right]d\tau$$

$$=\frac{6}{T}[\int_{\frac{5T}{6}}^{T} H(w)dw-16\int_{\frac{11T}{12}}^{T} H(w)dw+16\int_{T}^{\frac{13T}{12}} H(w)dw$$

$$-\int_{T}^{\frac{7T}{6}} H(w)dw].$$

Consider the limiting case from the forward difference formulation,

$$\frac{1}{T}\int_0^T \lim_{\tau\to 0}\int_T^{T+\tau} e^{mx}\, N(b\sqrt{x}+\frac{c}{\sqrt{x}})dxd\tau=\frac{1}{T}\int_0^T \lim_{\tau\to 0}[H(T+\tau)-H(T)]d\tau$$

$$=\frac{1}{T}\int_0^T \lim_{\tau\to 0}\frac{H(T+\tau)-H(T)}{\tau}\tau d\tau=\frac{H'(T)}{T}\int_0^T \tau d\tau=\frac{H'(T)T}{2}$$

where $H'(T)=e^{mT}N(b\sqrt{T}+\frac{c}{\sqrt{T}})$.

For $m,c\neq 0$,

$$\int e^{mx}\, N(b\sqrt{x}+\frac{c}{\sqrt{x}})dx=H(x)=A[B\mathrm{erf}(\frac{c_{0.5}-xb_m}{\sqrt{2x}})+C_1\mathrm{erf}(\frac{c_{0.5}+xb_m}{\sqrt{2x}})-$$

$$C_2]+Me^{mx}\mathrm{erf}(\frac{bx+c}{\sqrt{2x}})+Me^{mx},$$

where

$$A = -\frac{0.25}{m\sqrt{c^2}\sqrt{b^2-2m}} e^{-\sqrt{c^2}\sqrt{b^2-2m}-bc}, B = c\sqrt{b^2-2m} - b\sqrt{c^2},\ C_1$$
$$= (c\sqrt{b^2-2m} + b\sqrt{c^2}) e^{2\sqrt{c^2}\sqrt{b^2-2m}}, C_2$$
$$= \left(c\sqrt{b^2-2m} + b\sqrt{c^2}\right)\left(e^{2\sqrt{c^2}\sqrt{b^2-2m}} - 1\right), M = \frac{0.5}{m}, c_{0.5}$$
$$= \sqrt{c^2}, b_m = \sqrt{b^2-2m};$$

For $c = 0, m \neq 0$,

$$\int e^{mx} N(b\sqrt{x})dx = H(x) = \frac{1}{2m}[-\frac{\operatorname{erf}(\sqrt{x}\sqrt{0.5b^2-m})b}{\sqrt{b^2-2m}} + e^{mx}\operatorname{erf}(\frac{b\sqrt{x}}{\sqrt{2}}) + e^{mx}].$$

Note that

$$\int \operatorname{erf}\left(\frac{wb+c}{\sqrt{2w}}\right) dw$$
$$= \begin{cases} \{\frac{1}{2b\sqrt{b^2}} e^{-bc-\sqrt{b^2}\sqrt{c^2}}[\left(-\sqrt{b^2}\sqrt{c^2}+bc-1\right)\operatorname{erf}\left(\frac{\sqrt{b^2}w-\sqrt{c^2}}{\sqrt{2w}}\right) \\ +(\sqrt{b^2}\sqrt{c^2}+bc-1)(e^{2\sqrt{b^2}\sqrt{c^2}}\operatorname{erf}\left(\frac{\sqrt{b^2}w+\sqrt{c^2}}{\sqrt{2w}}\right) - e^{2\sqrt{b^2}\sqrt{c^2}}+1)] \\ \quad +w\operatorname{erf}\left(\frac{bw+c}{\sqrt{2w}}\right) + \frac{\sqrt{2w}}{b\sqrt{\pi}} e^{-\frac{(c+bw)^2}{2w}}\} & \text{for } b, c \neq 0, \\ \frac{e^{-\frac{b^2w}{2}}}{b^2}\left[e^{\frac{b^2w}{2}}(b^2w-1)\operatorname{erf}\left(\frac{b\sqrt{w}}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}}b\sqrt{w}\right] & \text{for } b \neq 0, c = 0, \\ e^{-\frac{c^2}{2w}}\left[e^{\frac{c^2}{2w}}(c^2+w)\operatorname{erf}\left(\frac{c}{\sqrt{2w}}\right) + \sqrt{\frac{2}{\pi}}c\sqrt{w}\right] & \text{for } b = 0, c \neq 0, \end{cases}$$

$$\int e^{mw}\operatorname{erf}\left(\frac{wb+c}{\sqrt{2w}}\right)dw$$

$$=\begin{cases} \{\frac{e^{mw}}{m}\operatorname{erf}\left(\frac{wb+c}{\sqrt{2w}}\right)-\frac{1}{2\sqrt{c^2}m\sqrt{b^2-2m}}\{e^{-\sqrt{c^2}\sqrt{b^2-2m}-bc} \\ [\left(c\sqrt{b^2-2m}-b\sqrt{c^2}\right)\operatorname{erf}\left(\frac{\sqrt{c^2}-w\sqrt{b^2-2m}}{\sqrt{2w}}\right)+(\mathrm{c}\sqrt{b^2-2m}+\mathrm{b}\sqrt{c^2}) \\ \quad [e^{2\sqrt{c^2}\sqrt{b^2-2m}}(\operatorname{erf}\left(\frac{w\sqrt{b^2-2m}+\sqrt{c^2}}{\sqrt{2w}}\right)-1)+1]]\}\} & \text{for } m,b,c\neq 0, \\ \frac{1}{m}\left[e^{mw}\operatorname{erf}\left(\frac{b\sqrt{w}}{\sqrt{2}}\right)-\frac{b\operatorname{erf}(\sqrt{w}\sqrt{\frac{b^2}{2}-m})}{\sqrt{b^2-2m}}\right] & \text{for } m,b\neq 0, c=0, \\ \frac{e^{mw}}{m} & \text{for } m\neq 0, b,c=0. \end{cases}$$

Appendix A4. Proof of Equivalence between the Analytic Formula and Kim's (1990) Formula

Consider adding a differential time $\Delta T \rightarrow 0$ to our formula. Therefore,

$$\lim_{\Delta T \rightarrow 0} \frac{1}{T+\Delta T} \int_0^{T+\Delta T} \int_{T+\Delta T-(\tau+\Delta\tau)/2}^{T+\Delta T+(\tau+\Delta\tau)/2} -E_{x_t}[e^{-rt} f_t(y, G(0))] dy d(\tau+\Delta\tau)$$

$$= \frac{1}{T} \int_0^{T} \int_{T-\tau/2}^{T+\tau/2} -E_{x_t}[e^{-rt} f_t(y, G(0))] dy d\tau$$

$$+ \lim_{\Delta T \rightarrow 0} \int_0^{T+\Delta T} \frac{1}{\Delta T} \int_{T+\Delta T-\Delta\tau/2}^{T+\Delta T+\Delta\tau/2} -E_{x_t}[e^{-rt} f_t(y, G(T))] dy d(\Delta\tau)$$

$$= \frac{1}{T} \int_0^{T} \int_{T-\tau/2}^{T+\tau/2} -E_{x_t}[e^{-rt} f_t(y, G(0))] dy d\tau$$

$$+ \lim_{\Delta T \rightarrow 0} \int_0^{T+\Delta T} \frac{1}{\Delta T} \int_{\Delta T-\Delta\tau/2}^{\Delta T+\Delta\tau/2} -E_{x_t}[e^{-rt} f_t(T+y, G(T))] dy d(\Delta\tau).$$

where $\Delta T \geq \Delta\tau$. Therefore, the instantaneous rate at time T is $-E_{x_t}[e^{-rt} f_t(T, G(T))]$. By the argument of recursive sum starting from $T = 0$,

$$\frac{1}{T} \int_0^{T} \int_{T-\tau/2}^{T+\tau/2} -E_{x_t}[e^{-rt} f_t(y, G(0))] dy d\tau = \int_0^{T} -E_{x_t}[e^{-rt} f_t(\tau, G(\tau))] d\tau.$$

Appendix A5. The Perpetual American options

From the Black-Scholes partial differential equation (PDE),

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_A}{\partial S^2} + (r-q)S\frac{\partial V_A}{\partial S} + \frac{\partial V_A}{\partial t} - rV_A = 0.$$

For perpetual options, $\frac{\partial V_A}{\partial t} = 0$. It becomes an ordinary differential equation (ODE)

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_A}{\partial S^2} + (r-q)S\frac{\partial V_A}{\partial S} - rV_A = 0. \qquad \text{(A5.1)}$$

Imposing a solution $V_A(S) = S^n$, we obtain

$$\frac{1}{2}\sigma^2 n(n-1)S^n + (r-q)nS^n - rS^n = 0.$$

After dividing through by S^n, the quadratic equation in n gives a solution $n = \frac{-\tilde{b}\pm\sqrt{\tilde{b}^2-4\tilde{c}}}{2}$, $\tilde{b} = \frac{2\mu}{\sigma^2}$, and $\tilde{c} = -\frac{2r}{\sigma^2}$. It leads to the critical exercise value condition of perpetual American options

(g) $G(\tau) = \frac{\lambda_\phi(r,q)}{\lambda_\phi(r,q)-1}K$ as $\tau \to \infty$,

where $\lambda_\phi(r,q) = \frac{-\tilde{b}+\phi\sqrt{\tilde{b}^2-4\tilde{c}}}{2}$. Barone-Adesi and Whaley (1987) and Bjerksund and Gunnar (1993) adopted it as an assumption for developing their model. However, a trivial solution for Eq. (A5.1) is $V_A = 0$ with $G(\infty) = K$. It results from our analytic pricing. As the theoretical basis between Bjerksund and Gunnar (1993) and our model differs, their lower bound to American options does not apply to our model. In Fig. 3, the critical early boundaries $G(\tau)$ converge to K as $\tau \to \infty$. In Table 4, $V_A^{Analytic}$ can be lower than the value under BJST.

Appendix A6. Calculation of the Early Exercise Premium Formula under Lévy Security Price Processes Formula

Based on the European option formula of Lewis (2001), the particular solution at time 0, similar to Eq. (10), is

$$V_E(S_0, T\,;\phi) = \frac{\phi e^{-rT}}{2\pi}[S_0\psi(-i\,;T)\int_{-\pi}^{\pi} e^{-ie^{-\phi i\theta}\bar{y}}\,\psi(-e^{-\phi i\theta}\,;T)d\theta$$

$$-K\int_{-\pi}^{\pi} e^{ie^{-\phi i\theta}\bar{y}}\,\psi(e^{-\phi i\theta}\,;T)d\theta].$$

Therefore, the time decay premium is

$$f_T(0) = -\frac{\phi e^{-rT}}{2\pi}[(r-\Psi(-i))S_0\psi(-i\,;T)\int_{-\pi}^{\pi} e^{-ie^{-\phi i\theta}\bar{y}}\,\psi(-e^{-\phi i\theta}\,;T)d\theta$$

$$-rK\int_{-\pi}^{\pi} e^{ie^{-\phi i\theta}\bar{y}}\,\psi(e^{-\phi i\theta}\,;T)d\theta].$$

The early exercise premium is

$$V(S_0, T; \phi) = -\frac{1}{T}f_T(0)\int_0^T t\,dt$$

$$= \frac{\phi T e^{-rT}}{4\pi}[(r-\Psi(-i))S_0\psi(-i\,;T)\int_{-\pi}^{\pi} e^{-ie^{-\phi i\theta}\bar{y}}\,\psi(-e^{-\phi i\theta}\,;T)d\theta$$

$$-rK\int_{-\pi}^{\pi} e^{ie^{-\phi i\theta}\bar{y}}\,\psi(e^{-\phi i\theta}\,;T)d\theta].$$

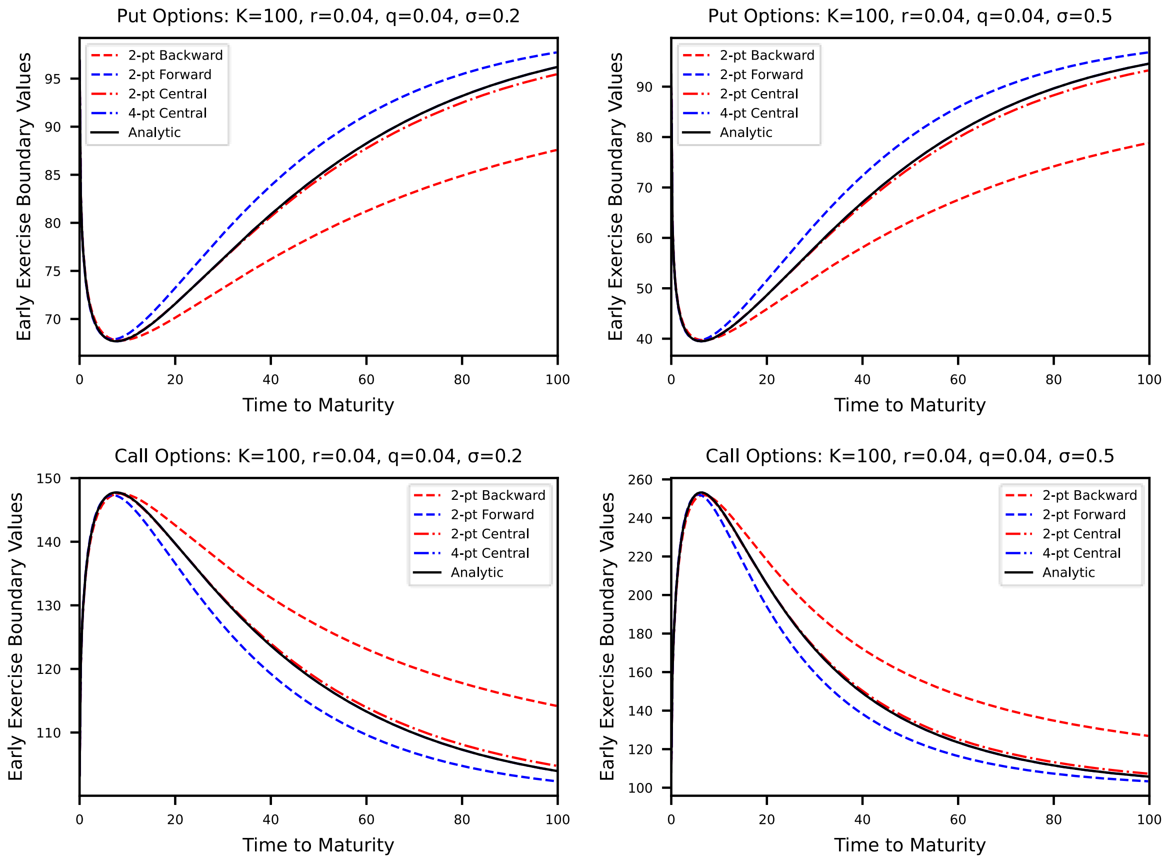


Fig. 1: Plots of boundary curves to show the convergence of finite difference solutions to the analytic solution of American call and put options. The plots use the finite difference and analytic formulas in Appendix A3 and the American option formula in Eq. (10). The critical early exercise boundary value $G(\tau)$ for the time to maturity, τ, is extracted from Eq. (11).

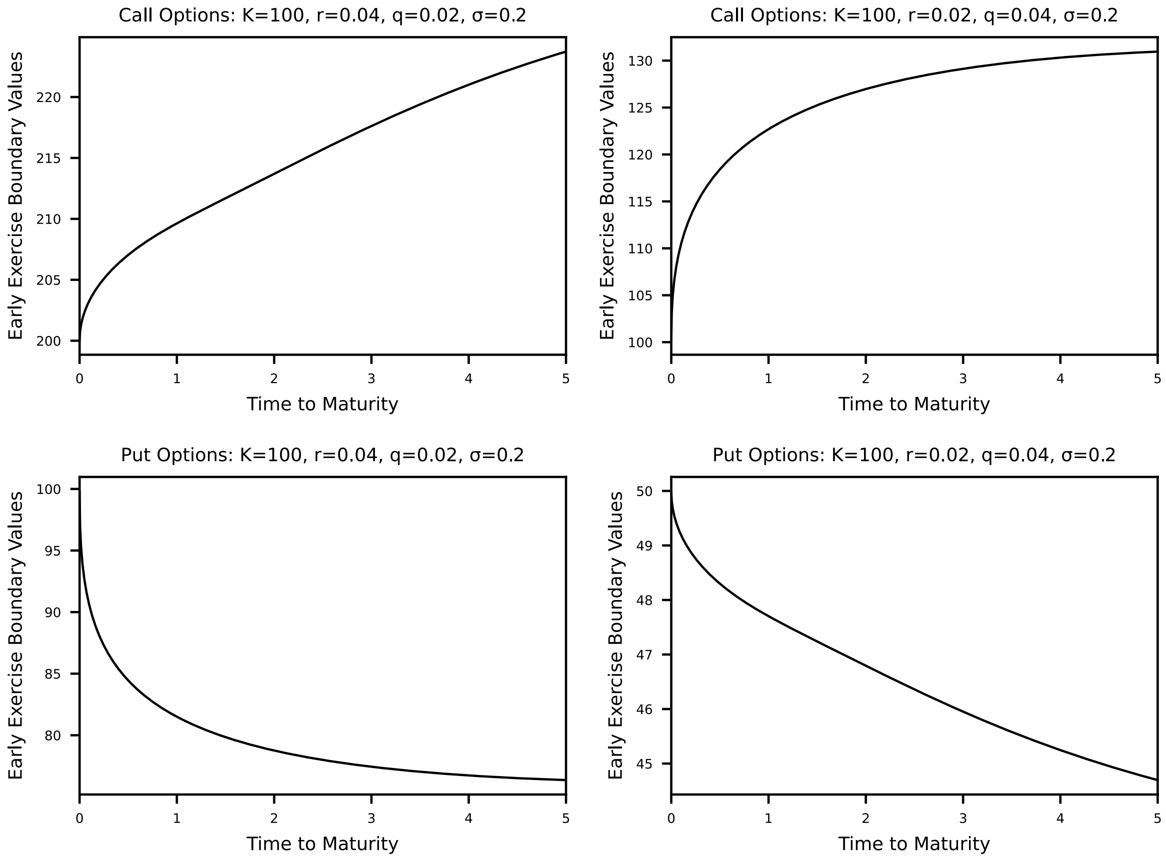


Fig. 2: Plots of boundary curves for American call and put options. The plots use the American option formula in Eq. (10) with the analytic solution. The critical early exercise boundary value $G(\tau)$ for the time to maturity, τ, is extracted from Eq. (11).

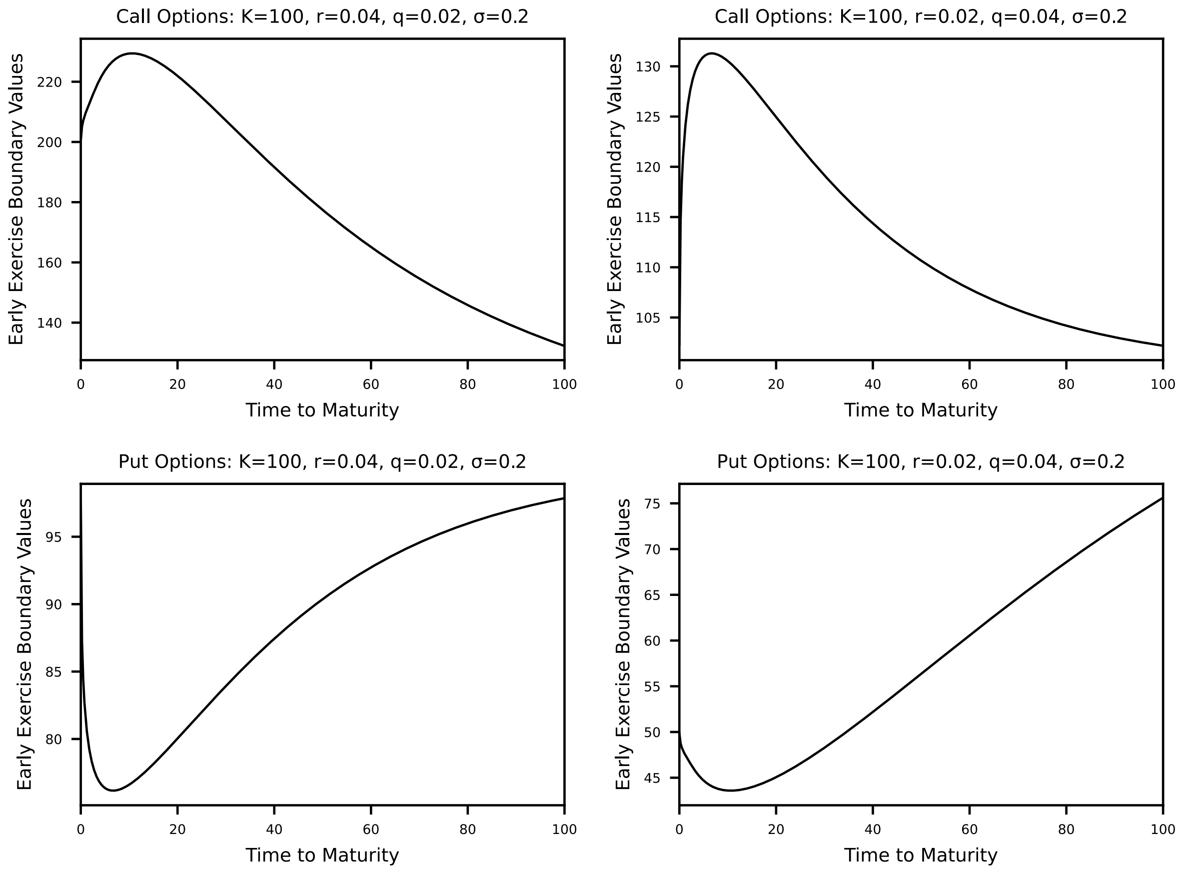


Fig. 3: Plots of long-time-to-maturity boundary curves for American call and put options. The plots use the American option formula in Eq. (10) with the analytic solution. The critical early exercise boundary value $G(\tau)$ for the time to maturity, τ, is extracted from Eq. (11).

Table 1. Comparison of Critical Exercise Prices Close to Expiry

	Early Exercise Boundary					
T	ZHU	EKK	SSCh-A	SSCh	PSOR	Analytic
0.00001	99.51	99.69	99.69	99.69	99.70	99.7357
0.00005	99.03	99.37	99.37	99.36	99.40	99.4623
0.0001	98.72	99.14	99.15	99.11	99.20	99.2737
0.0005	97.57	98.28	98.29	98.27	98.31	98.5635
0.001	96.83	97.70	97.72	97.66	97.73	98.0899
0.005	94.27	95.62	95.69	95.50	95.60	96.3997
0.01	92.73	94.33	94.43	94.07	94.18	95.3425
0.04	88.06	91.12	91.31	90.21	90.30	92.5051
0.1	85.25	89.29	89.42	86.76	86.94	90.1212
0.2	82.38	--	--	83.75	84.30	88.1549
0.4	79.36	--	--	80.48	81.02	86.2256

The parameters are $\sigma = 0.3$, $r = 0.1$, $q = 0$, $K = 100$, and $\phi = -1$. Evans, Kuske, and Keller (2002) (EKK), Stamicar, Ševčovič, and Chadam (1999) (SSCh-A), Lauko and Ševčovič (2010) (SSCh), Zhu (2006) (ZHU), projected successive over-relaxation (PSOR) (see Kwok (1998)) calculated the values. The pricing technique (Analytic) calculates the analytic solution based on Eq. (10). We excerpt the values of ZHU, EKK, SSCH-A, and SSCh from Tables 1 and 2 of Lauko and Ševčovič (2010).

Table 2. Comparing our Analytical American Option Values with Numerically Evaluated Values

Panel A: American Call

r	q	σ	τ	S	K	V_E	$V_A^{Analytic}$	V_A^{BW}	(%)	V_A^{BL}	(%)	V_A^{LS}	C.I.
0.04	0.03	0.20	1.00	120	100	22.2638	22.3238	22.3522	-0.13	22.3185	0.02	22.2555	(21.9751, 22.5360)
0.04	0.03	0.50	1.00	100	100	19.5490	19.6166	19.6711	-0.28	19.6216	-0.03	19.6396	(19.0319, 20.2473)
0.06	0.04	0.50	1.00	80	100	9.5664	9.5931	9.6396	-0.48	9.5896	0.04	9.6364	(9.1893, 10.0835)
0.04	0.03	0.20	0.50	120	100	20.8501	20.8658	20.8746	-0.04	20.8626	0.02	21.0767	(20.8369, 21.3166)
0.06	0.04	0.50	0.50	100	100	14.1784	14.1927	14.2137	-0.15	14.1912	0.01	14.2868	(13.8715, 14.7020)
0.04	0.03	0.20	0.25	120	100	20.2354	20.2388	20.2412	-0.01	20.2375	0.01	20.2067	(20.0869, 20.3265)
0.04	0.03	0.50	0.25	100	100	9.9854	9.9876	9.9926	-0.05	9.9881	-0.01	9.9418	(9.6801, 10.2035)
0.03	0.04	0.20	0.08	100	100	2.2105	2.2124	2.2142	-0.08	2.2157	-0.15	2.1868	(2.1314, 2.2422)
0.04	0.03	0.50	0.08	120	100	20.7417	20.7435	20.7449	-0.01	20.7435	0.00	20.8052	(20.5950, 21.0153)

Panel B: American Put

r	q	σ	τ	S	K	V_E	$V_A^{Analytic}$	V_A^{BW}	(%)	V_A^{BL}	(%)	V_A^{LS}	C.I.
0.03	0.00	0.20	1.00	100	100	6.4580	6.6985	6.7374	-0.58	6.7429	-0.66	6.7464	(6.6032, 6.8895)
0.03	0.04	0.50	1.00	100	100	19.5490	19.6166	19.6703	-0.27	19.6216	-0.03	19.8384	(19.4678, 20.2091)
0.06	0.04	0.50	1.00	120	100	11.7455	11.9365	12.0370	-0.84	11.9479	-0.10	11.8727	(11.5616, 12.1838)
0.03	0.04	0.20	0.50	80	100	20.3748	20.3959	20.4046	-0.04	20.3938	0.01	20.4369	(20.3203, 20.5534)
0.06	0.04	0.50	0.50	100	100	13.2031	13.3114	13.3583	-0.35	13.3455	-0.26	13.2269	(12.9561, 13.4977)
0.03	0.04	0.20	0.25	80	100	20.0856	20.0914	20.0940	-0.01	20.0901	0.01	20.1254	(20.0318, 20.2191)
0.04	0.03	0.50	0.25	100	100	9.7376	9.7524	9.7651	-0.13	9.7663	-0.14	9.7672	(9.5552, 9.9792)
0.04	0.03	0.20	0.08	100	100	2.2105	2.2124	2.2142	-0.08	2.2157	-0.15	2.2262	(2.1737, 2.2786)
0.04	0.03	0.50	0.08	80	100	20.1846	20.2118	20.2231	-0.06	20.2374	-0.13	20.2363	(20.1294, 20.3432)

Values are calculated based on Eq. (10) $V_A^{Analytic}$ with the analytic formula, the Barone-Adesi and Whaley formula V_A^{BW}, the binomial lattice approach of 10,000 iterative steps V_A^{BL}, the Longstaff and Schwartz Monte Carlo simulation in 250-time steps and 10,000 antithetic simulated paths V_A^{LS}, and the European option V_E. Under C.I. are the 95% confidence interval values of V_A^{LS}.

Table 3. Error Analysis for the Prices of American Put Options ($K = 100, r = 0.04, q = 0.04$)

τ	σ	S	V_E	$V_A^{BL}(10{,}000)$	2-point Backward	2-point Forward	4-point Forward	2-point Central	4-point Central	$V_A^{Analytic}$
0.5	0.2	80	19.90697	20.14370	20.0400162	20.0561338	20.0496316	20.0492757	20.0496233	20.0496233
		100	5.52557	5.54618	5.5325863	5.5483452	5.5399519	5.5400343	5.5399604	5.5399605
		120	0.70615	0.70728	0.7062854	0.7079902	0.7066423	0.7067837	0.7066427	0.7066427
0.5	0.5	80	24.53022	24.67754	24.5996450	24.6639719	24.6369463	24.6355721	24.6369184	24.6369184
		100	13.75378	13.80541	13.7716642	13.8114181	13.7903234	13.7905089	13.7903446	13.7903447
		120	7.26827	7.28742	7.2729088	7.2929767	7.2801989	7.2809433	7.2802263	7.2802263
3	0.2	80	22.01419	23.22839	22.8436658	23.0730212	22.9978559	22.9894336	22.9976501	22.9976490
		100	12.19602	12.60497	12.4737925	12.7330416	12.6268856	12.6204274	12.6267733	12.6267733
		120	6.33853	6.48260	6.4273984	6.6017085	6.5133867	6.5129728	6.5134642	6.5134650
3	0.5	80	36.43410	37.97463	37.5222978	38.1470446	37.9227172	37.9015990	37.9222413	37.9222395
		100	29.71134	30.74174	30.4310989	31.0642422	30.8121679	30.7946588	30.8118495	30.8118493
		120	24.48856	25.21384	24.9826418	25.5771002	25.3187496	25.3065937	25.3186293	25.3186304
Order of Accuracy				$O(\frac{\tau}{10000})$	$O(\tau)$	$O(\tau)$	$O((\frac{\tau}{6})^3)$	$O((\frac{\tau}{2})^2)$	$O((\frac{\tau}{12})^4)$	--
CPU Time* (in seconds)				15.072	0.004	0.004	0.006	0.005	0.007	0.004

* The CPU time reported is based on a computer using Python. The CPU time varies with the computer capacity, the software, and the programming style. Nonetheless, the results provide an idea of absolute and relative time.

Table 3 is an error analysis using the binomial lattice as a reference to understand the degree of accuracy of the closed-form solution. It shows that the closed-form solution is the limit of the finite-difference solutions and is real-time efficient.

Table 4: Comparison of American Put Values $(\boldsymbol{K = 100})$

(S,τ,σ,r,q)	2-point Forward	2-point Backward	AC	BJST	IB	VA	$V_A^{Analytic}$
(80;3.0;0.40;0.06;0.02)	28.6593	28.3247	29.26	29.1	29.1	29.24	**28.6092**
(85;3.0;0.40;0.06;0.02)	26.5610	26.1359	26.92	26.77	26.77	26.9	**26.4770**
(90;3.0;0.40;0.06;0.02)	24.6305	24.1345	24.8	24.65	24.65	24.77	**24.5171**
(95;3.0;0.40;0.06;0.02)	22.8544	22.3043	22.88	22.73	22.73	22.85	22.7157
(100;3.0;0.40;0.06;0.02)	21.2202	20.6302	21.13	20.98	20.98	21.1	21.0599
(105;3.0;0.40;0.06;0.02)	19.7163	19.0982	19.54	19.4	19.39	19.5	19.5377
(110;3.0;0.40;0.06;0.02)	18.3316	17.6952	18.08	17.95	17.94	18.05	18.1379
(115;3.0;0.40;0.06;0.02)	17.0561	16.4096	16.76	16.63	16.62	16.72	16.8501
(120;3.0;0.40;0.06;0.02)	15.8807	15.2305	15.54	15.42	15.4	15.51	15.6648
(100;3.0;0.40;0.02;0.02)	26.0009	25.7278	25.89	25.78	25.82	25.86	25.8770
(100;3.0;0.40;0.04;0.02)	23.4649	22.9335	23.3	23.17	23.17	23.26	23.2759
(100;3.0;0.40;0.06;0.02)	21.2202	20.6302	21.13	20.98	20.99	21.1	21.0599
(100;3.0;0.40;0.08;0.02)	19.1511	18.6650	19.27	19.13	19.14	19.25	19.0817
(100;3.0;0.40;0.10;0.02)	17.2171	16.9530	17.66	17.54	17.55	17.65	**17.2767**
(100;3.0;0.30;0.06;0.02)	15.1617	14.7475	15.17	15.04	15.06	15.15	15.0685
(100;3.0;0.35;0.06;0.02)	18.2028	17.6971	18.16	18.02	18.03	18.13	18.0745
(100;3.0;0.40;0.06;0.02)	21.2202	20.6302	21.13	20.98	20.99	21.1	21.0599
(100;3.0;0.45;0.06;0.02)	24.2057	23.5365	24.07	23.91	23.91	24.03	24.0156
(100;3.0;0.50;0.06;0.02)	27.1521	26.4079	26.98	26.8	26.81	26.93	26.9344
(100;0.5;0.40;0.06;0.02)	10.2717	10.1682	10.27	10.21	10.23	10.26	10.2288
(100;1.0;0.40;0.06;0.02)	13.9067	13.6779	13.88	13.78	13.8	13.85	13.8205
(100;1.5;0.40;0.06;0.02)	16.4322	16.0861	16.37	16.25	16.26	16.34	16.3114
(100;2.0;0.40;0.06;0.02)	18.3737	17.9262	18.28	18.15	18.16	18.25	18.2286
(100;2.5;0.40;0.06;0.02)	19.9348	19.4054	19.83	19.69	19.7	19.8	19.7766
(100;3.0;0.40;0.06;0.02)	21.2202	20.6302	21.13	20.98	20.99	21.1	21.0599
(100;3.5;0.40;0.06;0.02)	22.2926	21.6637	22.24	22.08	22.08	22.2	22.1410
(100;4.0;0.40;0.06;0.02)	23.1930	22.5472	23.19	23.03	23.04	23.16	23.0606
(100;4.5;0.40;0.06;0.02)	23.9503	23.3091	24.02	23.87	23.87	23.99	23.8470
(100;5.0;0.40;0.06;0.02)	24.5860	23.9703	24.76	24.61	24.61	24.73	**24.5211**
(100;5.5;0.40;0.06;0.02)	25.1164	24.5465	25.41	25.26	25.26	25.39	**25.0986**
(100;3.0;0.40;0.06;0.00)	19.8028	19.2883	19.85	19.69	19.71	19.83	19.6950
(100;3.0;0.40;0.06;0.02)	21.2202	20.6302	21.13	20.98	20.99	21.1	21.0599
(100;3.0;0.40;0.06;0.04)	22.7106	22.0682	22.49	22.36	22.36	22.45	22.5042

We excerpt the parameters of Table 2 in Viegas and Azevedo-Pereira (2020) for Table 4. We compare the 1000 time steps binomial values reported in Aitsahlia and Carr (1997) under column AC, the results reported in Bjerksund and Stensland (1993) under column BJST, the values obtained in Ingersoll (1998) under column IB, and the MLE (n = 4) results of Viegas and Azevedo-Pereira (2020) under column VA, with the results of the 2-point forward, backward, and the analytic formulas in this paper. The 2-point values usually capture the other estimates, except when the option is deep in the money, has a high interest rate, and has a long time to maturity. Also, they can converge to a limit $V_A^{Analytic}$ outside their range.