Pricing an American Option by Approximating Its Early Exercise Boundary As a Piece-Wise Exponential Function

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Abstract

This paper proposes to price an American option by approximating its early exercise boundary as a piece-wise exponential function. Closed-form formulas are obtained in terms of the bases and exponents of the piece-wise exponential function. It is demonstrated that a three-point extrapolation scheme has the accuracy of an 800-time-step binomial tree, but about 130 times faster. An intuitive argument is given to indicate why this seemingly crude approximation works so well. Our method is very simple and easy to implement. Comparisons with other leading competing methods are also included.

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Pricing and hedging American options have been a challenging problem. McKean (1965) and Merton (1973) demonstrate that the pricing of American options is a free-boundary problem. The difficulty in pricing such options stems from the possibility of early exercise of them and the early exercise boundary must be determined as part of the solution. Even though it is highly desirable, a closed-form formula has not been found and it is not likely that one will be found any time soon. Therefore the efforts have been concentrated on approximate methods. Numerical methods such as the finite difference of Brennan and Schwartz (1977) and the binomial tree model of Cox, Ross and Rubinstein (1979) are among the earliest approximate methods and still widely used ones. Even though these numerical methods are quite flexible and simple to implement, they are also among the most time-consuming ones. Academics and practitioners alike have been trying to find other more efficient methods.

MacMillan (1986) and Barone-Adesi and Whaley (1987) develop an approximate analytical formula for American options. Their approximation is very efficient and many times faster than most of other methods. A serious shortcoming of their method is that it is not very accurate, especially for long maturity options. Therefore the applicability of their method is quite limited. Another drawback of this analytical method is that it is not convergent because there is no parameter to control to improve the accuracy.

Because lower bound and upper bound exist for American options, some researchers have attacked the problem of pricing American option from yet another angle. Generally they first prove a lower bound and an upper bound for the American option price. Then they adopt an interpolation scheme to price the American option. The accuracy of these methods depends crucially on the tightness of the bounds and the interpolation scheme. These methods include Johnson (1983), and Broadie and Detemple (1996). These methods can be quite efficient, but an undesirable feature is that they all need regression coefficients which in turn require computing a large set of options accurately. These methods may also require recalibration of the regression coefficients for a new set of parameters of options. Like the analytical approximation, methods based on interpolation schemes are not convergent.

A fourth group of methods includes those which are approximate schemes based on exact representations of the free-boundary problem of pricing American options or the partial differential equation (PDE) satisfied by the American option prices. This group includes Geske and Johnson (1984), Bunch and Johnson (1992), Carr (1996), and Huang, Subrahmanyam and Yu (1996). These methods can be made as accurate as desired if more and more terms are included in the approximations. A serious problem is that, as more and more terms are included, the methods become less and less efficient. A potential advantage of an exact representation of the American option problem is that it can be subjected to various approximation schemes. Geske and Johnson (1984) use a four-point Richardson extrapolation scheme to approximate their American option formula which involves an infinite series of multivariate cumulative normal functions. Because multivariate cumulative normal functions require the computation of multi-dimensional integrals, the possibility to include more terms in their approximation to gain accuracy is very limited. Bunch and Johnson (1992) implement a modified two-point Geske-Johnson approach to avoid the multi-dimensional integrals. Carr (1996) discretizes only the time dimension of the evaluation PDE. Carr, Jarrow and Myneni (1992), Jacka (1991) and Kim (1990) obtain formulas representing the early exercise premium of an American option as an integral (hereafter the integral representation method) which has the early exercise boundary in it. To avoid having to compute many early exercise points, Huang, Subrahmanyam and Yu (1996) implement a four-point Richardson extrapolation scheme to integral representation method. Only six points on the boundary are needed. Since the integral representation method involves only univariate cumulative normal function, their method is very efficient, but it is not very accurate, especially for moderate and long maturity options, for example, the LEAPS [1].

In this paper, we propose another approximation based on the integral representation method. The key insight of our approximation is that the early exercise boundary B_t appears only as an argument to the logarithmic function in the integral for the early exercise premium[2]. Therefore the integral does not depend on B_t critically. Consequently we propose to approximate the early exercise boundary as a piece-wise exponential function. Fortunately the resulting integral can then be evaluated in closed-form. Because the most important time dependences in the premium integral are integrated analytically, a very efficient and accurate method is obtained.

The layout of the remainder of this paper is as follows. We detail our approximation in section I. Numerical results are presented in section II. We demonstrate there that a three-point extrapolation scheme attains penny accuracy for a large set of options (over 3,000) and is much more accurate than the methods of Geske and Johnson (1984), and Huang, Subrahmanyam and Yu (1996), especially for long maturity options. Our method is also more accurate and efficient than the six-point extrapolation scheme of the randomization method of Carr (1996). It will become clear in section II that our method is as efficient as the lower and upper bound approximation (LUBA) of Broadie and Detemple (1996) and is more accurate than the latter. Our method also has the advantage that it does not involve regression coefficients and is convergent. Our approximation has several other useful features. It shares with many other methods that it is exact in the limit as the time-to-maturity goes to zero. It also shares the property with LUBA that it is exact as the time-to-maturity goes to infinity, the perpetual case. Because our method is exact in both extreme cases, it is not very surprising that it yields good results for intermediate maturity options. On the other hand, as the time-to-maturity becomes longer, most of the other methods necessarily become less accurate. Unlike many other methods, both the "value-match" condition and the "high-contact" condition are satisfied in our method at the points where the early exercise boundary values are estimated. Section III summarizes and concludes the paper.

I. Derivation and Implementation

Under the usual assumptions of constant riskless interest rate r, dividend yield δ, volatility σ and log-normal process for the underlying asset, Carr, Jarrow and Myneni (1992), Jacka (1991) and Kim (1990) obtain the following formula for the price of an American put [3]:

$$\begin{aligned} P_A &= P_E + \int_0^T \left[rKe^{-rt}N(-d_2(S,B_t,t)) - \delta S e^{-\delta t} N(-d_1(S,B_t,t)) \right] dt \\ &= P_E + K(1-e^{-rT}) - S(1-e^{-\delta T}) - K\int_0^T re^{-rt}N(d_2(S,B_t,t))dt \\ &\quad + S\int_0^T \delta e^{-\delta t} N(d_1(S,B_t,t))dt, \end{aligned} \tag{1}$$

where

$$\begin{aligned} d_1(x,y,t) &= \frac{\log(x/y) + (r-\delta+\sigma^2/2)t}{\sigma\sqrt{t}}, \\ d_2(x,y,t) &= d_1(x,y,t) - \sigma\sqrt{t}, \end{aligned}$$

and P_E is the price of the corresponding Black-Scholes (1973) European option formula. The early exercise boundary B_t solves the following integral equation:

$$\begin{aligned} K - B_t &= P_E(B_t,K,T-t) + K(1-e^{-r(T-t)}) - K\int_t^T re^{-r(s-t)}N(d_2(B_t,B_s,s-t))ds \\ &\quad - B_t(1-e^{-\delta(T-t)}) + B_t\int_t^T \delta e^{-\delta(s-t)}N(d_1(B_t,B_s,s-t))ds. \end{aligned} \tag{2}$$

Once B_t is obtained, the price of the American put can be calculated easily. But solving B_t is a very time-consuming process.

Huang, Subrahmanyam and Yu (1996) use the Richardson extrapolation method to tackle this difficulty. They only calculate a few points on the exercise boundary B_t. A sequence of approximate values of the option is obtained and then extrapolated to yield the value of P_A. To generate the approximate sequence, they evaluate the integrals in equations (1) and (2) by approximating the integrands as step functions. This approximation amounts to the assumption that these integrands have no time-dependence during each sub-interval. Approximating the integrands as step functions is obviously not very accurate. To gain enough accuracy, extrapolation is needed.

However, there is a special feature in equation (1) which to our knowledge has not been utilized in the literature. Note that B_t appears only as $\log(S/B_t)$ in the definitions of $d_1(\cdot,\cdot,\cdot)$ and $d_2(\cdot,\cdot,\cdot)$. Therefore the integral in equation (1) does not depend on the exact values of B_t critically. To make use of this property, instead of assuming everything to be a constant during each sub-interval, we only assume that B_t is an exponential function during each sub-interval. Fortunately the integrals in equations (1) and (2) can then be evaluated in closed-form. Since the integral does not depend on B_t critically and the other time dependences are integrated analytically, we show in section II that our approximation results in a very simple, accurate and efficient method.

It should be pointed out that Omberg (1987) has also used exponential early exercise boundary to price American options but in a different way. Because he does not use the optimal boundary, his method always underprices the options. On the other hand, we use the exponential boundary as a calculating device to evaluate the premium integral in an exact formulation. There is no inherit underpricing in our use of the exponential boundary.

As a matter of fact, Broadie and Detemple (1996) prove that a certain approximate early exercise boundary in the integral representation method yields an upper boundary for the option price. It should also be noted that a piece-wise exponential boundary presents no problem in our application, but the first passage time problem in Omberg (1987) is probably impossible analytically.

Assume B_t to be an exponential function $B\exp(bt)$ for the interval $[t_1, t_2]$. Let's consider the integral

$$I_1 = \int_{t_1}^{t_2} re^{-rt} N(d_2(S, Be^{bt}, t))dt. \tag{3}$$

Let's define $x_1 = (r - \delta - b - \sigma^2/2)/\sigma$, $x_2 = \log(S/B)/\sigma$. Then $d_2(S, B_t, t) = x_1 t^{1/2} + x_2 t^{-1/2}$. Integration by parts yields

$$\begin{aligned} I_1 &= -e^{-rt} N(x_1 t^{1/2} + x_2 t^{-1/2})|_{t_1}^{t_2} + \int_{t_1}^{t_2} e^{-rt} n(x_1 t^{1/2} + x_2 t^{-1/2})(\frac{x_1}{2} t^{-1/2} - \frac{x_2}{2} t^{-3/2})dt = \\ & e^{-rt_1} N(x_1 t_1^{1/2} + x_2 t_1^{-1/2}) - e^{-rt_2} N(x_1 t_2^{1/2} + x_2 t_2^{-1/2}) + \frac{e^{-x_1 x_2}}{\sqrt{2\pi}} \int_{t_1}^{t_2} e^{-\frac{1}{2}((x_1^2+2r)t + x_2^2 t^{-1})} \cdot \\ & (\frac{x_1}{2} t^{-1/2} - \frac{x_2}{2} t^{-3/2})dt = e^{-rt_1} N(x_1 t_1^{1/2} + x_2 t_1^{-1/2}) - e^{-rt_2} N(x_1 t_2^{1/2} + x_2 t_2^{-1/2}) + \\ & + \frac{e^{-x_1 x_2}}{\sqrt{2\pi}} \int_{t_1}^{t_2} e^{-\frac{1}{2}(x_3^2 t + x_2^2 t^{-1})} (\frac{x_1}{2} t^{-1/2} - \frac{x_2}{2} t^{-3/2})dt, \end{aligned} \tag{4}$$

where $x_3 = \sqrt{x_1^2 + 2r}$. The above integral can be evaluated analytically by making use of the following identities:

$$dN(x_3 t^{1/2} + x_2 t^{-1/2}) = \frac{e^{-x_3 x_2}}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_3^2 t + x_2^2 t^{-1})} (\frac{x_3}{2} t^{-1/2} - \frac{x_2}{2} t^{-3/2}),$$
$$dN(x_3 t^{1/2} - x_2 t^{-1/2}) = \frac{e^{-x_3 x_2}}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_3^2 t + x_2^2 t^{-1})} (\frac{x_3}{2} t^{-1/2} + \frac{x_2}{2} t^{-3/2}).$$

It is now straightforward to show that

$$I_1 = e^{-rt_1} N(x_1\sqrt{t_1} + \frac{x_2}{\sqrt{t_1}}) - e^{-rt_2} N(x_1\sqrt{t_2} + \frac{x_2}{\sqrt{t_2}}) +$$

$$\frac{1}{2}(\frac{x_1}{x_3}+1)e^{x_2(x_3-x_1)}\left(N(x_3\sqrt{t_2}+\frac{x_2}{\sqrt{t_2}})-N(x_3\sqrt{t_1}+\frac{x_2}{\sqrt{t_1}})\right)+$$
$$\frac{1}{2}(\frac{x_1}{x_3}-1)e^{-x_2(x_3+x_1)}\left(N(x_3\sqrt{t_2}-\frac{x_2}{\sqrt{t_2}})-N(x_3\sqrt{t_1}-\frac{x_2}{\sqrt{t_1}})\right). \quad (5)$$

If we define $y_1=(r-\delta-b+\sigma^2/2)/\sigma$, $y_2=\log(S/B)/\sigma$, and $y_3=\sqrt{y_1^2+2\delta}$, similar derivation would yield

$$\begin{aligned}I_2 &= \int_{t_1}^{t_2}\delta e^{-\delta t}N(d_1(S,Be^{bt},t))dt = e^{-rt_1}N(y_1\sqrt{t_1}+\frac{y_2}{\sqrt{t_1}})-e^{-rt_2}N(y_1\sqrt{t_2}+\frac{y_2}{\sqrt{t_2}}) \\ &+\frac{1}{2}(\frac{y_1}{y_3}+1)e^{y_2(y_3-y_1)}\left(N(y_3\sqrt{t_2}+\frac{y_2}{\sqrt{t_2}})-N(y_3\sqrt{t_1}+\frac{y_2}{\sqrt{t_1}})\right) \\ &+\frac{1}{2}(\frac{y_1}{y_3}-1)e^{-y_2(y_3+y_1)}\left(N(y_3\sqrt{t_2}-\frac{y_2}{\sqrt{t_2}})-N(y_3\sqrt{t_1}-\frac{y_2}{\sqrt{t_1}})\right). \quad (6)\end{aligned}$$

Finally, if we define

$$\begin{aligned}I(t_1,t_2,x,y,z,\phi,\nu) \quad &= e^{-\nu t_1}N(z_1\sqrt{t_1}+\frac{z_2}{\sqrt{t_1}})-e^{-\nu t_2}N(z_1\sqrt{t_2}+\frac{z_2}{\sqrt{t_2}})+ \\ &\frac{1}{2}(\frac{z_1}{z_3}+1)e^{z_2(z_3-z_1)}\left(N(z_3\sqrt{t_2}+\frac{z_2}{\sqrt{t_2}})-N(z_3t_1^{1/2}+\frac{z_2}{t_1^{1/2}})\right)+ \\ &\frac{1}{2}(\frac{z_1}{z_3}-1)e^{-z_2(z_3+z_1)}\left(N(z_3\sqrt{t_2}-\frac{z_2}{\sqrt{t_2}})-N(z_3\sqrt{t_1}-\frac{z_2}{\sqrt{t_1}})\right), \quad (7)\end{aligned}$$

where

$$\begin{aligned}z_1 &= \frac{r-\delta-z+\phi\sigma^2/2}{\sigma}, \\ z_2 &= \frac{\log(x/y)}{\sigma}, \\ z_3 &= \sqrt{z_1^2+2\nu},\end{aligned}$$

then I_1 and I_2 can be expressed neatly as

$$I_1 = I(t_1,t_2,S,B,b,-1,r), \quad (8)$$

$$I_2 = I(t_1,t_2,S,B,b,1,\delta). \quad (9)$$

When $t_1 = 0$, $x > y$, we define

$$J(t,x,y,z,\phi,\nu) = I(0,t,x,y,z,\phi,\nu) = 1 - e^{-\nu t} N(z_1\sqrt{t} + \frac{z_2}{\sqrt{t}}) - \frac{1}{2}(\frac{z_1}{z_3}+1)e^{z_2(z_3-z_1)}$$
$$+\frac{1}{2}(\frac{z_1}{z_3}+1)e^{z_2(z_3-z_1)} N(z_3\sqrt{t} + \frac{z_2}{\sqrt{t}}) + \frac{1}{2}(\frac{z_1}{z_3}-1)e^{-z_2(z_3+z_1)} N(z_3\sqrt{t} - \frac{z_2}{\sqrt{t}}). \quad (10)$$

If we define P_1, P_2, P_3 etc as the approximate option values corresponding to approximating the early exercise boundary as a one-piece-wise exponential function ($B_{11}e^{b_{11}t}$), a two-piece-wise exponential function ($B_{22}e^{b_{22}t}$, $B_{21}e^{b_{21}t}$), and a three-piece-wise exponential function ($B_{33}e^{b_{33}t}$, $B_{32}e^{b_{32}t}$, $B_{31}e^{b_{31}t}$), etc [4], then the P's are given by

$$P_1 = \begin{cases} P_E + K(1-e^{-rT}) - S(1-e^{-\delta T}) - KJ(T,S,B_{11},b_{11},-1,r) + \\ SJ(T,S,B_{11},b_{11},1,\delta) \ \text{ if } S > B_{11}, \\ K - S \quad \text{if } S \le B_{11}. \end{cases} \quad (11)$$

$$P_2 = \begin{cases} P_E + K(1-e^{-rT}) - S(1-e^{-\delta T}) - KJ(T/2,S,B_{22},b_{22},-1,r) + \\ SJ(T/2,S,B_{22},b_{22},1,\delta) - KI(T/2,T,S,B_{21},b_{21},-1,r) + \\ SI(T/2,T,S,B_{21},b_{21},1,\delta) \ \text{ if } S > B_{22}, \\ K - S \quad \text{if } S \le B_{22}. \end{cases} \quad (12)$$

$$P_3 = \begin{cases} P_E + K(1-e^{-rT}) - S(1-e^{-\delta T}) - KJ(T/3,S,B_{33},b_{33},-1,r) + \\ SJ(T/3,S,B_{33},b_{33},1,\delta) - KI(T/3,2T/3,S,B_{32},b_{32},-1,r) + \\ SI(T/3,2T/3,S,B_{32},b_{32},1,\delta) - KI(2T/3,T,S,B_{31},b_{31},-1,r) + \\ SI(2T/3,T,S,B_{31},b_{31},1,\delta) \quad \text{if } S > B_{33}, \\ K - S \quad \text{if } S \le B_{33}. \end{cases} \quad (13)$$

Other P's follow similar patterns. To determine the B's and b's, we apply the "value-match" and "high-contact" conditions. For example, to determine B_{21} and b_{21}, let $B^* = B_{21}e^{b_{21}T/2}$. Applying the "value-match" and "high-contact" conditions at $t = T/2$ yields

$$K - B^* = P_E(B^*,K,T/2) + K(1-e^{-rT/2}) - B^*(1-e^{-\delta T/2}) -$$
$$KJ(T/2,B^*,B^*,b_{21},-1,r) + B^*J(T/2,B^*,B^*,b_{21},1,\delta), \quad (14)$$

$$-1 = -e^{-\delta T/2}N(-d_1(B^*,K,T/2)) - (1-e^{-\delta T/2}) - KJ_S(T/2,B^*,B^*,b_{21},-1,r) +$$
$$J(T/2,B^*,B^*,b_{21},1,\delta) + B^*J_S(T/2,B^*,B^*,b_{21},1,\delta). \quad (15)$$

Similarly, to determine B_{22} and b_{22}, applying the "value-match" and "high-contact" conditions at $t = 0$ yields

$$K - B_{22} = P_E(B_{22}, K, T) + K(1 - e^{-rT}) - B_{22}(1 - e^{-\delta T}) -$$
$$KJ(T/2, B_{22}, B_{22}, b_{22}, -1, r) + B_{22}J(T/2, B^{22}, B_{22}, b_{22}, 1, \delta) -$$
$$KI(T/2, T, B_{22}, B_{21}, b_{21}, -1, r) + B_{22}I(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) \quad (16)$$
$$-1 = -e^{-\delta T}N(-d_1(B_{22}, K, T)) - (1 - e^{-\delta T}) - KJ_S(T/2, B_{22}, B_{22}, b_{22}, -1, r) +$$
$$J(T/2, B_{22}, B_{22}, b_{22}, 1, \delta) + B_{22}J_S(T/2, B_{22}, B_{22}, b_{22}, 1, \delta) -$$
$$-KI_S(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) + I(T/2, T, B_{22}, B_{21}, b_{21}, 1, \delta) +$$
$$B_{22}I_S(T/2, T, B_{22}, B21, b_{21}, 1, \delta). \quad (17)$$

The functions $J_S(\cdots)$ and $I_S(\cdots)$ are defined by

$$J_S(t, S, B, b, \phi, \nu) = \frac{\partial J}{\partial S} = -\frac{1}{2}(\frac{z_1}{z_3} + 1)e^{z_2(z_3 - z_1)}\frac{z_3 - z_1}{S\sigma} - e^{-\nu t}n(z_1\sqrt{t} + \frac{z_2}{\sqrt{t}})\frac{1}{S\sigma\sqrt{t}} +$$
$$\frac{1}{2S\sigma}(\frac{z_1}{z_3} + 1)e^{z_2(z_3 - z_1)}\left(N(z_3\sqrt{t} + \frac{z_2}{\sqrt{t}})(z_3 - z_1) + n(z_3\sqrt{t} + \frac{z_2}{\sqrt{t}})\frac{1}{\sqrt{t}}\right) -$$
$$\frac{1}{2S\sigma}(\frac{z_1}{z_3} - 1)e^{-z_2(z_3 + z_1)}\left(N(z_3\sqrt{t} - \frac{z_2}{\sqrt{t}})(z_3 + z_1) + n(z_3\sqrt{t} - \frac{z_2}{\sqrt{t}})\frac{1}{\sqrt{t}}\right), \quad (18)$$
$$I_S(t_1, t_2, S, B, b, \phi, \nu) = \frac{\partial I}{\partial S} = \left(\frac{e^{-\nu t_1}}{\sqrt{t_1}}n(z_1\sqrt{t_1} + \frac{z_2}{\sqrt{t_1}}) - \frac{e^{-\nu t_2}}{\sqrt{t_2}}n(z_1\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}})\right)\frac{1}{\sigma S} +$$
$$\frac{1}{2\sigma S}e^{z_2(z_3 - z_1)}\left(N(z_3\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}) - N(z_3 t_1^{1/2} + \frac{z_2}{t_1^{1/2}})\right)(z_3 - z_1) +$$
$$\frac{1}{2\sigma S}e^{z_2(z_3 - z_1}\left(n(z_3\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}})\frac{1}{\sqrt{t_2}} - n(z_3 t_1^{1/2} + \frac{z_2}{t_1^{1/2}}))\frac{1}{\sqrt{t_1}})\right) -$$
$$\frac{1}{2\sigma S}e^{-z_2(z_3 + z_1)}\left(N(z_3\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}}) - N(z_3 t_1^{1/2} + \frac{z_2}{t_1^{1/2}})\right)(z_3 + z_1) -$$
$$\frac{1}{2\sigma S}e^{-z_2(z_3 + z_1)}\left(n(z_3\sqrt{t_2} + \frac{z_2}{\sqrt{t_2}})\frac{1}{\sqrt{t_2}} - n(z_3 t_1^{1/2} + \frac{z_2}{t_1^{1/2}}))\frac{1}{\sqrt{t_1}})\right). \quad (19)$$

The B's and b's can be easily and efficiently obtained using the 2-dimensional Newton-

Raphson method [5]. For example, to find B_{11} and b_{11}, the approximation of MacMillan (1986) and Barone-Adesi and Whaley (1987) provides a good estimate for B_{11}, and $b = 0.0$ provides a good estimate for b_{11}. Once B_{11} and b_{11} are found, the values on the one-piece-wise exponential function $B_{11}e^{b_{11}t}$ can be used as good initial values for finding B_{21} and b_{21}. To find B^* ($B_{21} = B^*e^{-b_{21}T/2}$) and b_{21} in Equations (14) and (15), $B = B_{11}e^{b_{11}T/2}$ and $b = b_{11}$ are good initial starting values. To find B_{22} and b_{22}, $B = B_{21}$ and $b = b_{21}$ are good initial estimates. When the difference between the perpetual boundary $B(\infty)$ and the terminal boundary $B(0) = \min(K, Kr/\delta)$ is smaller than 10% of their average, the early exercise boundary will be very flat. This occurs when the interest rate is much smaller than the dividend yield. In these cases, the early exercise premium will be very small, and we recommend replacing the piece-wise exponential functions by the the piece-wise constants (step functions). Therefore in the Newton-Raphson search, the exponents are initialized to zero and not updated.

In this article, we recommend a three-point Richardson scheme to price American options. If P_1, P_2 and P_3 are the values given in equations (11), (12) and (13), then the American put price is approximated by

$$\hat{P}_A = 4.5P_3 - 4P_2 + 0.5P_1$$

We demonstrate in the following section that this results in a very accurate and efficient method for pricing American options. In fact, we also demonstrate that even the unextrapolated values P_1, P_2 and P_3 are very good approximate values for the true American option prices. This is of course confirms our argument that the true values do not depend on the exact values of the early exercise boundary critically.

II. Numerical Results and Discussions

In this section we report numerical results to demonstrate the efficiency and accuracy of our approximation scheme. We compare our three-point piece-wise exponential function method (hereafter EXP3) with an 800-time-step binomial tree model (hereafter BT800), the four-point extrapolation scheme of Geske and Johnson (1984) (hereafter GJ4), the modified two-point Geske and Johnson method of Bunch and Johnson (1992) (hereafter MGJ2), the four-point and six-point recursive methods of Huang, Subrahmanyam and Yu (1996) (hereafter HSY4 and HSY6), the fine-tuned three-point and six-point randomization methods of Carr (1996) (hereafter RAN3 and RAN6), and the lower and upper bound approximation of Broadie and Detemple (1996) (hereafter LUBA).

To assess the accuracies of these methods, we choose a binomial model with N = 10,000 time steps as our benchmark for the true values of the options considered. We use the root of mean squared errors (RMSE) to measure the overall accuracy of a set of options and maximum absolute error (MAE) to measure the maximum possible error. The computational efficiency is measured using the total CPU time (in seconds) required to price the the whole set of the options considered [6]. Because there are different techniques to compute the multivariate cumulative normal functions, no attempt is made to optimize the computations for GJ4. But it is reasonable to assume that it is much less efficient than MGJ2. We follow Bunch and Johnson (1992) in the implementation of MGJ2. A more sophisticated optimization routine is not likely to affect the efficiency and accuracy significantly.

To test the accuracies of these methods for short and moderate maturity options, Table I reports the results for the 20 options in Table 1 in Broadie and Detemple (1996). To test the accuracies of these methods for longer maturity options, Table II reports the results for the 20 options in Table V in Barone-Adesi and Whaley (1987). From these tables it is clear that the performance of EXP3 is truly amazing. The maximum error for the 40 options in Tables I and II is only 0.36 cents. Overall, BT800, EXP3, LUBA and RAN6 appear to have about the same accuracies, but EXP3 does appear to be a little bit more accurate than LUBA and RAN6 [7]. RAN6 is clearly less efficient than EXP3 and LUBA. Nevertheless, RAN6 is a very efficient and accurate method. Obviously, MGJ2 is the least accurate method. Even for the relatively short maturity options in Table II, the pricing errors of MGJ2 are quite substantial. HSY4 and HSY6 appear to be dominated by EXP3, LUBA, and RAN6 in term of efficiency and accuracy.

To understand better why EXP3 works so well, Table III considers the convergent behavior of the unextrapolated values of the approximate option prices. From the table, it is not difficult to see the reason. Even the unextrapolated values EXP-P1, EXP-P2 and EXP-P3 are quite close to the true option prices. Specifically, the one-point unextrapolated value EXP-P1 is much more accurate than the six-point unextrapolated values HSY-P6 and RAN-P6. As a matter of fact, EXP-P1 is only slightly less accurate than the six-point extrapolated value HSY6. Even for these long maturity options, the unextrapolated values EXP-P1, EXP-P2 and EXP-P3 are quite acceptable for many applications. Again, this behavior confirms the observation that the early exercise premium does not depend on the the early exercise boundary critically. Since the other time dependences are integrated analytically, very good approximate values are obtained. This is the key insight of this paper. On the other hand, even the *six*-point unextrapolated values HSY-P6 and RAN-P6 have huge pricing errors.

Table IV considers the accuracies of the these methods for calculating the hedging ratio Δ. To compute Δ using the binomial tree model, we use the extended tree method described in Pelsser and Vorst (1994). We do not consider the efficiencies here because it is obvious that all the methods take roughly the same amount of time to compute the Δ as to compute the price. If the price is also needed, little extra calculations are needed to compute the Δ.

The general behavior concerning the prices also hold here for the hedging ratios. BT800, EXP3, LUBA and RAN6 all provide accurate hedging ratios, but EXP3 is clearly the most accurate method. Even though LUBA has about the same price errors as EXP3, its hedging errors are more than ten times larger than those of EXP3. This is probably due to the fact that LUBA is an interpolation method and is minimized for pricing errors. This points out a potential drawback of methods based on regression coefficients. These methods are fine-tuned for the best performance according to one objective function. They may not provide the best performance in other applications.

Tables I and II should provide reliable information about the efficiencies of these methods because the computing time should be roughly proportional to the number of options computed. In order to have a more complete assessment concerning the accuracies of these methods, Table V reports the results from 3,000 randomly generated American put options. The parameters are generated as follows: Volatility σ is uniform between 0.1 and 0.6; time-to-maturity τ is uniform between 0.0 and 3.0 years; the strike price is fixed at $K = 100.0$; current stock price S is uniform between 70.0 and 130.0; both the riskless interest rate and the dividend yield are uniform between 0.0 and 0.15. These values represent a wide range of parameter values. The maximum error of EXP3 is not only smaller than 1 cent for the 40 options in Tables I and II, it is also smaller than 1 cent for the 3,000 options in Table V. BT800 has only one occurrence with a pricing error just at 1 cent. All other methods have multiple occurrences with pricing errors equal or bigger than 1 cent. Nevertheless, the performances of LUBA and RAN6 are really impressive, especially RAN6. RAN6 has a maximum error of 1.79 cents for the 3,040 options in Tables I, II and V. In light of the tests in Tables I, II and V, EXP3, LUBA and RAN6 are all extremely efficient and accurate methods for pricing American options.

III. Concuding Remarks

We have proposed an efficient and accurate approximation for pricing American options. Our approximation is based on the observation that the early exercise premium does not depend on the exact values of the early exercise boundary crucially. This insight allows us to approximate the early exercise boundary as a piece-wise exponential function. Because the resulting integral of the early exercise premium can be evaluated analytically, an efficient and accurate approximation is obtained. Numerical results show that our approximation based a three-point extrapolation scheme has the accuracy of an 800-time-step binomial tree model, but about 130 times faster. We also have demonstrated that among the competing methods, our method appear to have the best performance in terms of accuracy, efficiency and flexibility.

Even though we have only discussed standard American stock options, our approximation obviously applies to other standard American options such as futures options, quanto options, index options and currency options. It is also worthwhile to point out that the present approximation applies equally well to other non-standard American options when the early exercise premium can be represented as integrals involving the cumulative normal function. One particular example is the American barrier options. In the setting of section I, Gao, Huang and Subrahmanyam (1996) have derived analytical formulas for American barrier options. Their formulas are similar to integral representation for the standard American stock options. Therefore our method easily applies.

Appendix

In this appendix, we present the 2-dimensional version of the Newton-Raphson method for finding roots. Suppose we want to find the roots of $g_1(x,y)=0$, $g_2(x,y)=0$. Define

$$g_1^0 = g_1(x_0,y_0) \qquad h_{11} = \frac{\partial g_1(x_0,y_0)}{\partial x} \qquad h_{12} = \frac{\partial g_1(x_0,y_0)}{\partial y}$$
$$g_2^0 = g_2(x_0,y_0) \qquad h_{21} = \frac{\partial g_2(x_0,y_0)}{\partial x} \qquad h_{22} = \frac{\partial g_2(x_0,y_0)}{\partial y}$$

Expanding $g_1(x,y)$ and $g_2(x,y)$ around x_0 and y_0 yields

$$g_1(x,y) = g_1^0 + h_{11}(x-x_0) + h_{12}(y-y_0) + \cdots \tag{20}$$
$$g_2(x,y) = g_2^0 + h_{21}(x-x_0) + h_{22}(y-y_0) + \cdots \tag{21}$$

Because x and y are the roots, the left hand sides of (20) and (21) are zero. If (x_0,y_0) are good initial estimates of (x,y), the higher terms in (20) and (21) can be neglected. Solve x and y, we have

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}^{-1} \begin{bmatrix} g_1^0 \\ g_2^0 \end{bmatrix} \\
&= \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \frac{1}{\Sigma} \begin{bmatrix} h_{22} & -h_{12} \\ -h_{21} & h_{11} \end{bmatrix} \begin{bmatrix} g_1^0 \\ g_2^0 \end{bmatrix} \\
&= \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \frac{1}{\Sigma} \begin{bmatrix} h_{22}g_1^0 - h_{12}g_2^0 \\ h_{11}g_2^0 - h_{21}g_1^0 \end{bmatrix},
\end{aligned}$$

where $\Sigma = h_{11}h_{22} - h_{12}h_{21}$.

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Footnotes

1. Long-term Equity AnticiPation Securities (LEAPS) are long-term exchange-traded options. They last up to 3 years.

2. See equations (1) and (2) in section I of this paper.

3. We consider American puts only. American calls can be evaluated similarly or can be priced using the parity result of McDonald and Schroder (1990) for American options: $C(S, K, r, \delta, \sigma, T) = P(K, S, \delta, r, \sigma, T)$.

4. The intervals are divided into equal sub-intervals. For example, if the early exercise boundary $\{B_t\}$ is approximated by a two-piece-wise exponential function, then $B_t = B_{22}e^{b_{22}t}$ if $t \in [0, T/2)$, and $B_t = B_{21}e^{b_{21}t}$ if $t \in [T/2, T]$.

5. It is to be noted that little additional computation is required beyond what is needed to compute $I(\cdots)$ and $J(\cdots)$. In the polynomial approximation of the cumulative normal function, the normal density function is computed. In applying the Newton-Raphson method to find the B's and b's, the computation does not go beyond computing the cumulative normal functions in $I(\cdots)$ and $J(\cdots)$.

6. This is confirmed in the larger scale test in Table V.

7. The calculations are performed on a Sparc-20 in FORTRAN. To have a more reliable measure of the CPU, the calculation is repeated 100 times even though the reported CPU is for one run.

Table I
Values of American call options ($K = \$100$, $\tau = 0.5$ Years)

(1) (S, σ, r, δ)	(2) TRUE	(3) BT800	(4) GJ4	(5) MGJ2	(6) HSY4	(7) HSY6	(8) LUBA	(9) RAN3	(10) RAN6	(11) EXP3
(80, 0.2, 0.03, 0.07)	0.2194	0.2194	0.2191	0.2186	0.2199	0.2197	0.2195	0.2237	0.2188	0.2196
(90, 0.2, 0.03, 0.07)	1.3864	1.3874	1.3849	1.3818	1.3898	1.3895	1.3862	1.3732	1.3844	1.3872
(100, 0.2, 0.03, 0.07)	4.7825	4.7818	4.7851	4.7862	4.8044	4.7804	4.7821	4.7792	4.7794	4.7837
(110, 0.2, 0.03, 0.07)	11.0978	11.0986	11.0889	11.2553	11.0686	11.0983	11.0976	11.1050	11.0957	11.0993
(120, 0.2, 0.03, 0.07)	20.0004	20.0000	20.0073	20.0000	20.0531	20.0006	20.0000	20.0720	20.0000	20.0005
(80, 0.4, 0.03, 0.07)	2.6889	2.6887	2.6864	2.6827	2.6897	2.6909	2.6893	2.6681	2.6857	2.6899
(90, 0.4, 0.03, 0.07)	5.7223	5.7238	5.7212	5.7163	5.7361	5.7286	5.7231	5.7146	5.7168	5.7237
(100, 0.4, 0.03, 0.07)	10.2385	10.2365	10.2451	10.2351	10.2752	10.2372	10.2402	10.2406	10.2323	10.2404
(110, 0.4, 0.03, 0.07)	16.1812	16.1828	16.1831	16.2107	16.2012	16.1854	16.1817	16.1874	16.1745	16.1831
(120, 0.4, 0.03, 0.07)	23.3598	23.3597	23.3419	23.4771	23.3288	23.3850	23.3574	23.3697	23.3542	23.3622
(80, 0.3, 0.00, 0.07)	1.0373	1.0371	1.0351	1.0317	1.0374	1.0388	1.0373	1.0261	1.0352	1.0381
(90, 0.3, 0.00, 0.07)	3.1233	3.1222	3.1240	3.1146	3.1438	3.1282	3.1232	3.1068	3.1193	3.1247
(100, 0.3, 0.00, 0.07)	7.0354	7.0343	7.0375	7.0413	7.0667	7.0332	7.0355	7.0295	7.0307	7.0371
(110, 0.3, 0.00, 0.07)	12.9552	12.9568	12.9339	13.0637	12.9091	12.9759	12.9531	12.9571	12.9509	12.9574
(120, 0.3, 0.00, 0.07)	20.7173	20.7147	20.7423	20.4380	20.7268	20.6969	20.7208	20.7502	20.7154	20.7194
(80, 0.3, 0.07, 0.03)	1.6644	1.6636	1.6644	1.6644	1.6644	1.6644	1.6644	1.6630	1.6644	1.6644
(90, 0.3, 0.07, 0.03)	4.4947	4.4926	4.4946	4.4947	4.4947	4.4947	4.4947	4.4997	4.4947	4.4947
(100, 0.3, 0.07, 0.03)	9.2504	9.2481	9.2509	9.2506	9.2506	9.2506	9.2506	9.2845	9.2506	9.2506
(110, 0.3, 0.07, 0.03)	15.7977	15.7993	15.7973	15.7975	15.7975	15.7975	15.7975	15.8450	15.7975	15.7975
(120, 0.3,0.07, 0.03)	23.7061	23.7059	23.7082	23.7062	23.7062	23.7062	23.7062	23.7739	23.7065	23.7062
RMSE		0.0012	0.0090	0.0805	0.0231	0.0089	0.0012	0.0279	0.0035	0.0013
MAE		0.0023	0.0250	0.2793	0.0527	0.0252	0.0035	0.0716	0.0067	0.0025
CPU (sec)		8.6e0	$\gg$ 6.6e-2	6.6e-2	2.9e-2	1.1e-1	6.2e-2	2.4e-2	1.5e-1	6.0e-2

The "TRUE" value is based on the binomial tree model with $N = 10,000$ time steps. Columns 3-11 represent the binomial tree model with $N = 800$ time steps, the four-point method of Geske and John (1984), the modified two-point Geske-Johnson method of Bunch and John (1992), the four-point and six-point recursive schemes of Huang, Subrahmanyam and Yu (1996), the lower and upper bound approximation of Broadie and Detemple (1996), the fine-tuned three-point ($P = 4.5P_3 - 4[1 - 0.0002(5 - T)^+]P_2 + 0.5P_1$) and six-point randomization methods of Carr (1996) and the three-point piece-wise exponential early excise boundary method of this article, respectively. RMSE is the root of mean squared errors. MAE is the maximum absolute error. CPU is the total computing time for the whole set of options on a Sparc-20 in FORTRAN.

Table II
American put values ($K = \$100$, $\tau = 3.0$ years, $\sigma = 0.2$, $r = 0.08$)

(1) (S, δ)	(2) TRUE	(3) BT800	(4) GJ4	(5) MGJ2	(6) HSY4	(7) HSY6	(8) LUBA	(9) RAN3	(10) RAN6	(11) EXP3
(80, 0.12)	25.6577	25.6564	25.6529	25.9487	25.6862	25.7021	25.6568	25.6604	25.6577	25.6570
(90, 0.12)	20.0832	20.0839	20.1089	20.2009	20.1275	20.0905	20.0834	20.0983	20.0830	20.0817
(100, 0.12)	15.4981	15.4945	15.5122	15.5495	15.5356	15.5020	15.4985	15.5090	15.4984	15.4970
(110, 0.12)	11.8032	11.8013	11.8023	11.8236	11.8228	11.8121	11.8032	11.8020	11.8032	11.8022
(120, 0.12)	8.8856	8.8873	8.8803	8.8965	8.8937	8.8921	8.8855	8.8527	8.8856	8.8850
(80, 0.08)	22.2050	22.2043	22.2079	22.7106	22.2445	22.1493	22.1985	22.1912	22.2032	22.2084
(90, 0.08)	16.2071	16.2084	16.1639	16.5305	16.1340	16.2578	16.1986	16.1925	16.2034	16.2106
(100, 0.08)	11.7037	11.7013	11.7053	11.8106	11.7175	11.7237	11.6988	11.6852	11.6994	11.7066
(110, 0.08)	8.3671	8.3664	8.3886	8.4072	8.4355	8.3563	8.3630	8.3451	8.3621	8.3695
(120, 0.08)	5.9299	5.9322	5.9435	5.9310	5.9881	5.9323	5.9261	5.8883	5.9252	5.9323
(80, 0.04)	20.3500	20.3489	20.5134	20.0000	20.5225	20.3932	20.3335	20.3641	20.3492	20.3511
(90, 0.04)	13.4968	13.4969	13.5246	14.0246	13.3784	13.4602	13.4982	13.5048	13.4939	13.5000
(100, 0.04)	8.9438	8.9423	8.8414	9.1086	8.8038	8.9891	8.9424	8.9358	8.9403	8.9474
(110, 0.04)	5.9119	5.9115	5.8904	5.9310	5.9186	5.9269	5.9122	5.8947	5.9077	5.9146
(120, 0.04)	3.8975	3.8988	3.9046	3.8823	3.9778	3.8834	3.8980	3.8622	3.8932	3.8997
(80, 0.0)	20.0000	20.0000	19.7313	20.0000	19.8458	19.9484	20.0000	20.0320	20.0000	20.0000
(90, 0.0)	11.6974	11.6955	11.8842	10.1758	11.7606	11.7047	11.6953	11.7242	11.6960	11.6991
(100, 0.0)	6.9320	6.9301	6.9266	6.9394	6.7859	6.9109	6.9346	6.9439	6.9301	6.9346
(110, 0.0)	4.1550	4.1539	4.1033	4.1453	4.0902	4.1897	4.1550	4.1550	4.1528	4.1571
(120, 0.0)	2.5102	2.5103	2.4906	2.4546	2.5591	2.5150	2.5110	2.4920	2.5082	2.5119
RMSE		0.0016	0.0872	0.4014	0.0853	0.0304	0.0048	0.0206	0.0028	0.0023
MAE		0.0036	0.2687	1.5216	0.1725	0.0557	0.0165	0.0416	0.0050	0.0036
CPU time (sec)		8.6e0	$\gg$6.7e-02	6.7e-2	2.5e-02	8.4e-2	5.4e-2	2.2e-2	1.5e-1	6.3e-2

The "TRUE" value is based on the binomial tree model with $N = 10,000$ time steps. Columns 3-11 represent the binomial tree model with $N = 800$ time steps, the four-point method of Geske and John (1984), the modified two-point Geske-Johnson method of Bunch and John (1992), the four-point and six-point recursive schemes of Huang, Subrahmanyam and Yu (1996), the lower and upper bound approximation of Broadie and Detemple (1996), the fine-tuned three-point ($P = 4.5P_3 - 4[1 - 0.0002(5 - T)^+]P_2 + 0.5P_1$) and six-point randomization methods of Carr (1996) and the three-point piece-wise exponential early excise boundary method of this article, respectively. RMSE is the root of mean squared errors. MAE is the maximum absolute error. CPU is the total computing time for the whole set of options on a Sparc-20 in FORTRAN.

Table III
Unextroplated Put Values ($K = \$100$, $\tau = 3.0$ years, $\sigma = 0.2$, $r = 0.08$)

(1) (S, δ)	(2) TRUE	(3) EXP-P1	(4) EXP-P2	(5) EXP-P3	(6) HSY-P2	(7) HSY-P4	(8) HSY-P6	(9) RAN-P2	(10) RAN-P4	(11) RAN-P6
(80, 0.12)	25.6577	25.6404	25.6543	25.6564	25.3230	25.5499	25.6087	24.9528	25.2663	25.3867
(90, 0.12)	20.0832	20.0679	20.0805	20.0821	19.8723	20.0010	20.0429	18.9587	19.4841	19.6752
(100, 0.12)	15.4981	15.4867	15.4964	15.4976	15.3623	15.4412	15.4688	14.1383	14.7863	15.0161
(110, 0.12)	11.8032	11.7949	11.8017	11.8026	11.7170	11.7655	11.7827	10.5027	11.1097	11.3309
(120, 0.12)	8.8856	8.8799	8.8844	8.8850	8.8324	8.8612	8.8718	7.8534	8.3163	8.4930
(80, 0.08)	22.2050	22.1650	22.1916	22.1983	21.5703	22.2163	22.2967	21.7577	21.9402	22.0170
(90, 0.08)	16.2071	16.1473	16.1882	16.1977	15.9365	16.2288	16.2538	15.3822	15.7510	15.8907
(100, 0.08)	11.7037	11.6417	11.6840	11.6938	11.5394	11.6880	11.7147	10.6768	11.1544	11.3263
(110, 0.08)	8.3671	8.3122	8.3488	8.3574	8.2370	8.3353	8.3623	7.3980	7.8407	8.0038
(120, 0.08)	5.9299	5.8857	5.9142	5.9214	5.8224	5.8971	5.9192	5.1787	5.5070	5.6342
(80, 0.04)	20.3500	20.3379	20.3447	20.3469	18.8206	20.1777	20.4122	20.1814	20.2493	20.2787
(90, 0.04)	13.4968	13.4459	13.4781	13.4866	13.1947	13.7667	13.7612	12.9199	13.1819	13.2803
(100, 0.04)	8.9438	8.8747	8.9197	8.9308	8.9527	9.1426	9.1053	8.1677	8.5315	8.6617
(110, 0.04)	5.9119	5.8435	5.8876	5.8985	5.9368	6.0075	5.9956	5.1826	5.5153	5.6381
(120, 0.04)	3.8975	3.8394	3.8761	3.8854	3.8829	3.9342	3.9389	3.3418	3.5822	3.6762
(80, 0.0)	20.0000	20.0000	20.0000	20.0000	16.6525	18.7463	19.2411	20.0000	20.0000	20.0000
(90, 0.0)	11.6974	11.6729	11.6878	11.6919	11.1602	12.0365	12.0819	11.3607	11.5205	11.5781
(100, 0.0)	6.9320	6.8832	6.9145	6.9225	7.1356	7.3556	7.2640	6.3889	6.6529	6.7441
(110, 0.0)	4.1550	4.1020	4.1362	4.1447	4.4050	4.4077	4.3392	3.6412	3.8815	3.9684
(120, 0.0)	2.5102	2.4646	2.4938	2.5010	2.6586	2.6376	2.6079	2.1269	2.2954	2.3607
RMSE		0.0437	0.0151	0.0081	0.8593	0.3260	0.2239	0.7883	0.4258	0.2923
MAE		0.0691	0.0243	0.0134	3.3475	1.2537	0.7589	1.3598	0.7118	0.4820

The "TRUE" value is based on the binomial tree model with $N = 10,000$ time steps. Columns 3-5 represent the one-point, two-point and three-point unextrapolated values of the piece-wise exponential early excise boundary method of this article, respectively. Columns 6-8 denote the two-point, four-point and six-point unextrapolated values of the recursive method of Huang, Subrahmanyam and Yu (1996), respectively. Columns 9-11 are the two-point, four-point and six-point unextrapolated values of the randomization method of Carr (1996), respectively. RMSE is the root of mean squared errors. MAE is the maximum absolute error.

Table IV
Hedge ratios (Δ's) of American puts ($K = \$100$, $\tau = 3.0$ years, $\sigma = 0.2$, $r = 0.08$)

(1) (S, δ)	(2) TRUE	(3) BT800	(4) GJ4	(5) MGJ2	(6) HSY4	(7) HSY6	(8) LUBA	(9) RAN3	(10) RAN6	(11) EXP3
(80, 0.12)	-0.61030	-0.61015	-0.60457	-0.63670	-0.60943	-0.61249	-0.61010	-0.61002	-0.61036	-0.61044
(90, 0.12)	-0.50633	-0.50611	-0.50611	-0.51466	-0.50566	-0.50875	-0.50633	-0.50519	-0.50632	-0.50637
(100, 0.12)	-0.41224	-0.41215	-0.41407	-0.41723	-0.41393	-0.41149	-0.41226	-0.41327	-0.41173	-0.41221
(110, 0.12)	-0.32866	-0.32859	-0.3296	-0.32976	-0.33023	-0.3285	-0.32868	-0.33134	-0.32868	-0.32862
(120, 0.12)	-0.25691	-0.25689	-0.25694	-0.25767	-0.25765	-0.2573	-0.25692	-0.25977	-0.25685	-0.25688
(80, 0.08)	-0.68782	-0.68769	-0.70126	-0.35398	-0.71619	-0.67884	-0.68873	-0.68966	-0.68806	-0.68769
(90, 0.08)	-0.51890	-0.51873	-0.51611	-0.54394	-0.51521	-0.51503	-0.51856	-0.51830	-0.51902	-0.51897
(100, 0.08)	-0.38712	-0.38710	-0.38311	-0.40428	-0.37791	-0.39293	-0.38691	-0.38745	-0.38718	-0.38718
(110, 0.08)	-0.28468	-0.28469	-0.28451	-0.28968	-0.28311	-0.28468	-0.28463	-0.28621	-0.28470	-0.28468
(120, 0.08)	-0.20642	-0.20649	-0.20767	-0.20920	-0.20908	-0.2047	-0.20638	-0.20817	-0.20633	-0.20641
(80, 0.04)	-0.83744	-0.83747	-0.82683	-1.00000	-0.85071	-0.84998	-0.83383	-0.83998	-0.83767	-0.83716
(90, 0.04)	-0.55413	-0.55418	-0.57167	-0.59390	-0.57537	-0.54718	-0.55471	-0.55458	-0.55423	-0.55403
(100, 0.04)	-0.36908	-0.36922	-0.37290	-0.38527	-0.35682	-0.36645	-0.36892	-0.37040	-0.36913	-0.36911
(110, 0.04)	-0.36908	-0.36922	-0.37290	-0.38527	-0.35682	-0.36645	-0.36892	-0.37040	-0.36913	-0.36911
(110, 0.04)	-0.24559	-0.24573	-0.24140	-0.25715	-0.23280	-0.25081	-0.24548	-0.24725	-0.24562	-0.24564
(120, 0.04)	-0.16284	-0.16299	-0.16148	-0.16448	-0.16065	-0.16303	-0.24548	-0.24725	-0.24562	-0.24564
(80, 0.0)	-1.00000	-1.00000	-0.92362	-1.00000	-0.94719	-0.97546	-1.00000	-1.00160	-1.00000	-1.00000
(90, 0.0)	-0.62088	-0.62161	-0.63518	-0.22217	-0.65065	-0.64186	-0.61732	-0.62039	-0.62092	-0.62066
(100, 0.0)	-0.35826	-0.35873	-0.37158	-0.34051	-0.36201	-0.3489	-0.35883	-0.35986	-0.35832	-0.35821
(110, 0.0)	-0.21090	-0.21120	-0.20921	-0.21596	-0.19687	-0.21124	-0.21084	-0.21266	-0.21088	-0.21092
(120, 0.0)	-0.12568	-0.12591	-0.12240	-0.12897	-0.11870	-0.12887	-0.12562	-0.12711	-0.12563	-0.12570
RMSE		0.00023	0.01861	0.12266	0.01703	0.00873	0.00117	0.00159	0.00015	0.00010
MAE		0.00073	0.07638	0.039871	0.05281	0.02454	0.00361	0.00286	0.00051	0.00028

The "TRUE" value is based on the binomial tree model with $N = 10,000$ time steps. Columns 3-11 represent the binomial tree model with $N = 800$ time steps, the four-point method of Geske and John (1984), the modified two-point Geske-Johnson method of Bunch and John (1992), the four-point and six-point recursive schemes of Huang, Subrahmanyam and Yu (1996), the lower and upper bound approximation of Broadie and Detemple (1996), the fine-tuned three-point ($\Delta = 4.5\Delta_3 - 4[1 - 0.0002(5 - T)^+]\Delta_2 + 0.5\Delta_1$) and six-point randomization methods of Carr (1996) and the three-point piece-wise exponential early excise boundary method of this article, respectively. RMSE is the root of mean squared errors. MAE is the maximum absolute error.

Table V
Summary of a large sample of American puts

(1)	(2) BT800	(3) MGJ2	(4) HSY4	(5) HSY6	(6) LUBA	(7) RAN3	(8) RAN6	(9) EXP3
RMSE	0.0024	0.2650	0.0765	0.0299	0.0045	0.0386	0.0049	0.0028
MAE	0.0100	2.3351	0.4593	0.1601	0.0607	0.1230	0.0179	0.0096
$\#(\text{error} \geq 0.01)$	1	1510	1712	1033	143	2168	191	0

3,000 options are randomly generated using the following distribution: Volatility σ is uniform between 0.1 and 0.6; time to maturity τ is uniform between 0.0 and 3.0 years; the strike price is fixed at $K = 100.0$; current stock price S is uniform between 70.0 and 130.0; both the riskless interest rate and the dividend yield are uniform between 0.0 and 0.15. Values based on the binomial tree model with $N = 10,000$ time steps are taken to be the true option prices. Columns 2-8 represent the modified two-point Geske-Johnson method of Bunch and John (1992), the four-point and six-point recursive schemes of Huang, Subrahmanyam and Yu (1996), the lower and upper bound approximation of Broadie and Detemple (1996), the fine-tuned three-point ($P = 4.5P_3 - 4[1 - 0.0002(5-T)^+]P_2 + 0.5P_1$) and six-point randomization methods of Carr (1996) and the three-point piece-wise exponential early excise boundary method of this article, respectively. We did not include the four-point method of Geske and John (1984) in this large scale test because it is too inefficient. RMSE is the root of mean squared errors. MAE is the maximum absolute error. The last row reports the number of pricing errors which are at least 1 cent.