1. Let $(X1, X2, \ldots)$ be a random sample of size n taken from a Normal Population with parameters : mean = $\theta 1$ and variance = $\theta 2$. Find the maximum likelihood Estimates of these two parameters.

→ $$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$X_1, X_2, \ldots X_n \rightarrow$ sample of size n

$$L(X_1, X_2, \ldots X_n) = f(x_1) \cdot f(x_2) \ldots f(x_n)$$

$$\Rightarrow \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_1-\mu)^2}{2\sigma^2}}\right) \cdot \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_2-\mu)^2}{2\sigma^2}}\right) \ldots$$

taking ln on both sides

$$\ln(L) = -\frac{n}{2} \ln(2\pi\sigma^2) + \sum_{i=1}^{n} \left(\frac{(x_i-\mu)^2}{2\sigma^2}\right) \quad -(1)$$

take partial derivative w.r.t. μ of above eqn:

$$\frac{\partial \ln(L)}{\partial \mu} = 0 + \sum_{i=1}^{n} -\left(\frac{2(x_i-\mu)}{2\sigma^2}\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} (x_i - \mu) = 0$$

$$\Rightarrow n\bar{X} - n\mu = 0$$

$$\bar{X} = \mu$$

Hence $\theta_1 = \bar{X}$ is therefore sample mean.

Taking derivative w.r.t. σ^2 of eqn (1)

$$\frac{\partial \ln(L)}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \sum_{i=1}^{n} -\frac{(x_i-\mu)^2}{2\sigma^2} = 0$$

$$-n + \sum_{i=1}^{n} -\frac{(x_i - \mu)^2}{\sigma^2} = 0$$

$$n = \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{\sigma^2}$$

$$\sigma^2 = \frac{1}{n}\left(\sum_{i=1}^{n} (x_i - \mu)^2\right)$$

hence $\theta_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$

2. Let $X_1, X_2, \ldots X_n$ be a random sample from $B(m, \theta)$ distribution, where $\theta \in \Theta = (0,1)$ is unknown and 'm' is a known positive integer. Compute value of θ using M.L.E.

→ Binomial distribution → ${}^{n}C_{x_i}\, \theta^{x_i} (1-\theta)^{n-x_i}$

$$L = \prod_{i=1}^{n} {}^{n}C_{x_i}\, \theta^{x_i} (1-\theta)^{n-x_i}$$

log on both sides

$$\log L = \sum_{i=1}^{n} \left(\log({}^{n}C_{x_i}) + \log \theta^{x_i} + \log (1-\theta)^{n-x_i} \right)$$

$$\log L = \sum_{i=1}^{n} \log({}^{n}C_{x_i}) + \log \theta \sum_{i=1}^{n} x_i + \log(1-\theta) \sum_{i=1}^{n} (n-x_i)$$

differentiate w.r.t. θ

$$\frac{d \log(L)}{d\theta} = 0$$

$$\frac{1}{\theta} \sum x_i - \frac{1}{1-\theta} \sum (n - x_i) = 0$$

$$\frac{1}{\theta(1-\theta)} \sum x_i = \frac{n^2}{1-\theta}$$

$$\Rightarrow \theta = \frac{\sum x_i}{n^2}$$