Permutation & combination

01 Fundamental Principles of - Counting

(1) Multiplication Principle

If an operation can be performed In 'm' different way, following which a second operation can be performed in 'n' different ways, then the two operations in succession can be performed in m × n ways.This can be extended to any finite number of operations

(2) Addition Principle

If an operation can be performed in 'm' different ways & another operation, Which is independent of the first operation, can be performed in 'n' different ways, then either of the two operations can be performed in (m + n) ways. This can be extended to any finite number of mutually exclusive operations.

02 Permutation

Each of the different arrangements which can be made by taking some or all of a number of things is called permutations.

Factorial notation: $n! = n(n-1)(n-2)......3 \cdot 2 \cdot 1$

$n! = n(n-1)!$ $0! = 1! = 1$

$2n! = 2n \times n!\ [1, 3, 5, 7 .... (2n-1)]$

Factorials of negative integers are not defined.

03 Important results

01 Number of permutations of n different things, taking r at a the is denoted by ${}^np_r$ or $P(n,r)$.

$${}^np_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1)$$

Number of permutations of n different things taken all at a time = ${}^np_n = n!$

${}^nP_0 = 1,\ {}^nP_1 = n,\ {}^nP_n = n!$ ${}^nP_r = n({}^{n-1}P_{r-1}) = n(n-1)(n-2)({}^{n-2}P_{r-2})$ ${}^{n-1}P_r = (n-r)^{n-1}P_{n-1}$

$P_n = n!$ ${}^nP_r = {}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$

02 The number of permuation of n things taken all at a time, p are alike of one kind, q are alike of seond kind r are alike of a third kind and n=p+q+r ; $\frac{n!}{p!q!r!}$

03 The number of permutations of n different things taken r at a time when each thing may be repeated any numbr of times is n^r.

04 Number of permutations of n different things taken r at a time when a particular thing is to be always included in each arrangement, is $r\ {}^{n-1}P_{r-1}$ Number of permutations of n different things, taken r at time, when p particular things is to be always included in each arrangement, is $p!\ (r-(p-1))\ {}^{n-p}P_{r-p}$

05 Number of permutation of n different things taken r at a time, when a particular things is never taken in each arrangement is ${}^{n-1}P_r$.

06 Number of permutations of n different things, taken all at a time, when m specified things always come together is $m! \times (n-m+1)!$

07 Number of permutations of n different things, taken all at a time, when m specified things never come together is $n! - m! \times (n-m+1)!$

04 Circular Permutations

Arrangement round a circular table: Number of circular permutations of n different things taken all at a time is $(n-1)!$, if clockwise & anticlockwise orders are taken as different.

Arrangement of beads around a circular necklace: Number of circular permutations of n different things taken all at a time is $\frac{1}{2}(n-1)!$ if clockwise & anticlockwise orders are taken as not different

Number of circular permutations of n different things taken r at a time is-

(i) $\frac{{}^nP_r}{r}$, when anti clockwise & clockwise orders are taken as different. (ii) $\frac{{}^nP_r}{2r}$, when anticlockwise & clockwise orders are not different.

05 Combination

each of the different selections made by taking some or all at a time, irrespective of their arrangements, is called a combination.

The number of all combinations of n objects taken r at a time is denoted by $c(n, r)$ or ${}^nC_r$ or $\binom{n}{r}$

$${}^nC_r = \frac{n!}{r!(n-r)!} \qquad {}^nC_r = \frac{n(n-1)(n-2)......(n-r+1)}{1.2.3.....r} \qquad {}^nC_n = {}^nC_0 = 1 \qquad {}^nC_r = \frac{{}^nP_r}{r!}$$

06 Number of Combinations without Repetition

The number of combination (selections or groups) that can be formed from n different objects taken $r(0 \le r \le n)$ at a time is ${}^nC_r = \frac{n!}{r!(n-r)!}$

- ${}^nC_r$ is a natural number
- ${}^nC_0 = {}^nC_n = 1,\ {}^nC_1 = n$
- ${}^nC_r = {}^nC_{n-r}$
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- ${}^nC_x = {}^nC_y \Leftrightarrow x = y$ or $x + y = n$
- $n.\ {}^{n-1}C_{r-1} = (n-r+1)\ {}^nC_{r-1}$
- If n is even, then the greatest value of ${}^nC_r$ is ${}^nC_{n/2.}$
- If n is odd, then the greatest value of ${}^nC_r$ is $\frac{{}^nC_{n+1}}{2}$ or $\frac{{}^nC_{n-1}}{2}$
- ${}^nC_r = \frac{n}{r} . {}^{n-1}C_{r-1}$

- $\dfrac{{}^{n}C_{r}}{{}^{n}C_{r-1}} = \dfrac{n-r+1}{r}$
- ${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \ldots + {}^{n}C_n = 2^n$
- ${}^{n}C_0 + {}^{n}C_2 + {}^{n}C_4 + \ldots = {}^{n}C_1 + {}^{n}C_3 + {}^{n}C_5 + \ldots = 2^{n-1}$
- ${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n = 2^{2n}$
- ${}^{n}C_n + {}^{n+1}C_n + {}^{n+2}C_n + {}^{n+3}C_n + \ldots + {}^{2n-1}C_n = {}^{2n}C_{n+1}$

07 Total number of divisors of a given natural number

The number of factors of a given natural number greater than 1 we can write as, $N = P_1^{\alpha_1} P_2^{\alpha_2} P_3^{\alpha_3} \ldots P_n^{\alpha_n}$ where $p_1, p_2, \ldots, p_n$ are distinct prime numbers and are non - negative integers. $(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_n + 1)$ ways. Sum of all the divisors of n is given by $\left(\dfrac{p_1^{\alpha_1+1}-1}{p_1-1}\right) \cdot \left(\dfrac{p_2^{\alpha_2+1}-1}{p_2-1}\right) \cdot \left(\dfrac{p_3^{\alpha_3+1}-1}{p_3-1}\right) \ldots \left(\dfrac{p_n^{\alpha_n+1}-1}{p_n-1}\right)$

08 Derangements

Any change in the existing order of things is called a derangement. If 'n' things are arranged in a row, the number of ways in which they can be deranged so that none of them occupies its original place is $n!\left(1-\dfrac{1}{1!}+\dfrac{1}{2!}-\dfrac{1}{3!}+\ldots+(-1)^n\dfrac{1}{n!}\right) = n!\sum_{r=0}^{n}(-1)^r\dfrac{1}{r!}$ And it is denoted by D(n).

09 Multinomial Theorem

Let $x_1, x_2, \ldots x_m$ be integers. Then number of solutions to the equation

$x_1 + x_2 + \ldots + x_m = n$ ...(i)

subject to the conditions ...(ii)

$a_1 \le x_1 \le b_1, a_2 \le x_2 \le b_2, \ldots a_m \le x_m \le b_m$ is equal to the coefficient of x^n in

$(x^{a_1} + x^{a_1+1} + \ldots + x^{b_1})(x^{a_2} + x^{a_2+1} + \ldots + x^{b_2}) \ldots \left(x^{a_m} + x^{a_{m+1}} + \ldots x^{b_m}\right)$ ...(iii)

This is because the number of ways in which sum of m integers in (i) subject to given conditions (ii) equals n is the same as the number of times xn comes in (iii).

10 Distribution

(1) Number of ways of distribution of n distinct balls in r distinct boxes when order is considered

$= n!\ {}^{n-1}C_{r-1}$, if blank (empty) boxes are not allowed. And it is:

$= n!\ {}^{n+r-1}C_{r-1}$ if blank (empty) boxes are allowed.

(2) Number of ways of distribution of n identical balls into r distinct boxes $= {}^{n-1}C_{r-1}$, if blank

(empty) boxes ar not allowed. And it is:

$= {}^{n+r-1}C_{r-1}$, if blank (empty) boxes are allowed.

(3) Number of ways of distribution of n distinct balls into r distinct boxes when order is not considered= r^n, if blank (empty) boxes are allowed, And it is:

$= r^n - {}^{r}C_1 (r-1)^n + {}^{r}C_2 (r-2)^n - {}^{r}C_3 (r-3)^n + \ldots + (-1)^{r-1}\ {}^{r}C_{r-1}$ if blank (empty) boxes are not allowed.

(4) The number of combinations of n objects of which p are identical taken r at a time is $= {}^{n-p}C_r + {}^{n-p}C_{r-1} + \ldots + {}^{n-p}C_0$ if $r \le p$.

(5) The coefficient of x^r in the expansion of $(1-x)^{-n} = {}^{n+r-1}C_r$

11 Multinomial Theorem

1 If there are n_1 objects of one kind, n_2 objects of second kind and so on n_k objects of kth kind, then the number of ways of choosing r objects out of these objects is = coeff of x^r in $(1 + x + x^2 + \ldots + x^{n_1})(1 + x + x^2 + \ldots + x^{n_2}) \ldots (1 + x + x^2 + \ldots)$.

2 If one object of each kind is to be included in selection of (1), then the number of ways of choosing r objects is:

= coeff of x^r in $(x + x^2 + \ldots + x^{n_1})(x + x^2 + \ldots + x^{n_2}) \ldots (x + x^2 + \ldots + x^{n_k})$

3 The number of possible arrangements permutations of p objects out of n_1 objects of kind 1, n_2 of kind 2 and so on is = p! times the coefficient of x^p in the expansion

$$\left(1+x+\frac{x^2}{2!}+\ldots+\frac{x^{n_1}}{n_1!}\right)\ldots\left(1+x+\frac{x^2}{2!}+\ldots+\frac{x^{n_k}}{n_k!}\right)$$