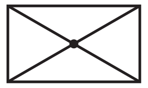
ROTATIONAL MOTION 01

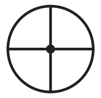
PHYSICS WALLAH

Centre of Mass

- Avg. position of all the parts of the system, weighted according to their mass
- For homogeneous objects, centre of mass lies at their geometric centre

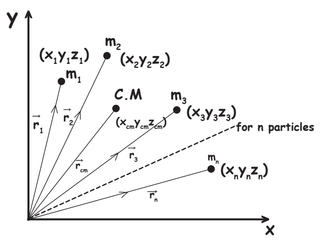


- Centre of mass may or may not lie inside the object



Centre of Mass For System of n Particles

$\vec{r}_{cm} = \frac{\Sigma m\vec{r}}{\Sigma m}$



General Equation

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots m_n\vec{r}_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

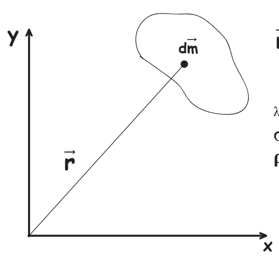
In terms of Cartesian co-ordinates

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots m_nx_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots m_ny_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

$$z_{cm} = \frac{m_1z_1 + m_2z_2 + m_3z_3 + \cdots m_nz_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

Centre Of Mass For Continuous Body

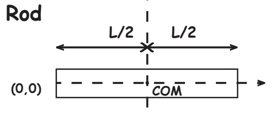
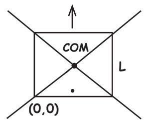
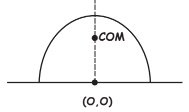
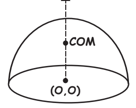
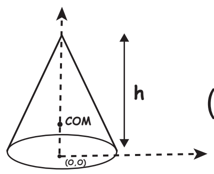
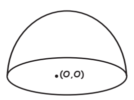


$\vec{r}_{cm} = \frac{\int dm.\vec{r}}{\int dm}$

λ = mass per unit length
σ = mass per unit area
ρ = mass per unit volume

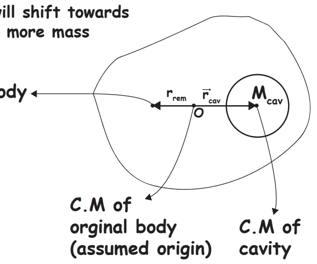
1) Mass distributed over length $\Rightarrow dm = \lambda\, dl$
2) Mass distributed over area $\Rightarrow dm = \sigma\, dA$
3) Mass distributed over volume $\Rightarrow dm = \rho dV$

Centre of mass for various shapes

Uniformly distributed mass	centre of mass
1) Rod 	$\left(\frac{L}{2}, 0\right)$
2) Square Lamina 	$\left(\frac{L}{2}, \frac{L}{2}\right)$
3) Semicircular Ring 	$\left(0, \frac{2R}{\pi}\right)$
4) Hemispherical shell 	$\left(0, \frac{R}{2}\right)$
5) Solid circular cone 	$\left(0, \frac{h}{4}\right)$
6) Solid hemisphere 	$\left(0, \frac{3R}{8}\right)$

Cavity in object

If some mass is removed from a body, COM will shift towards the side with more mass



Assuming COM of original body is at the origin

$$\vec{r}_{rem} = \frac{-M_{cav} \times \vec{r}_{cav}}{M_{rem}}$$

Motion of centre of mass

velocity of centre of mass

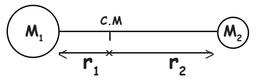
$$\vec{V}_{cm} = \frac{M_1\vec{V}_1 + M_2\vec{V}_2 + M_3\vec{V}_3 + \cdots}{M_1 + M_2 + M_3 + \cdots}$$

Acceleration of centre of mass

$$\vec{a}_{cm} = \frac{M_1\vec{a}_1 + M_2\vec{a}_2 + M_3\vec{a}_3 + \cdots}{M_1 + M_2 + M_3 + M_4 \cdots}$$

Isolated System

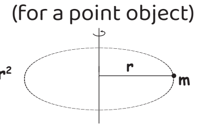
- No net external force acting on the system
- bodies within the system can have mutual force between then



$M_1r_1 = M_2r_2$ $\quad a_{cm} = 0, v_{cm} = \text{constant}$

Moment of Inertia (For a point object)

$I = mr^2$

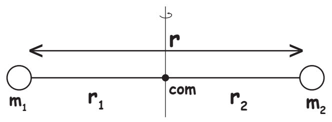


m = Mass of body
r = Perpendicular distance of the body from the axis of rotation

Moment of Inertia
- Tensor Quantity
- $I = mr^2$
- Rotational analogous of mass

Two Point Masses

$$I_{COM} = m_1r_1^2 + m_2r_2^2$$



$$r_1 = \frac{m_2r}{m_1 + m_2}, \quad r_2 = \frac{m_1r}{m_1 + m_2}$$

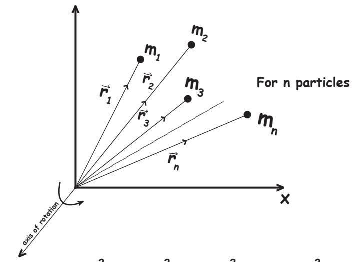
$$I_{com} = m_{red}r^2, \quad m_{red} = \frac{m_1m_2}{m_1 + m_2}$$

Factors Affecting Moment of Inertia

- Mass of the body
- Axis of rotation
- Mass distribution

Moment of Inertia

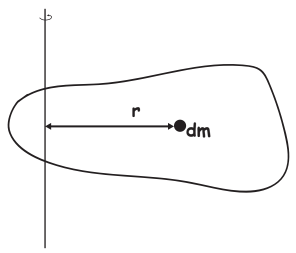
i) for discrete system of particles



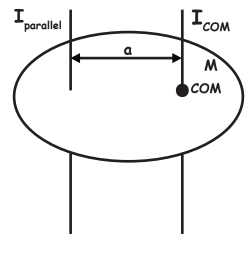
$$I = \frac{m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \cdots m_nr_n^2}{m_1 + m_2 + m_3 + \cdots m_n}$$

ii) for continuous body

$$I = \int dI = \int r^2 dm$$



Parallel Axis Theorm

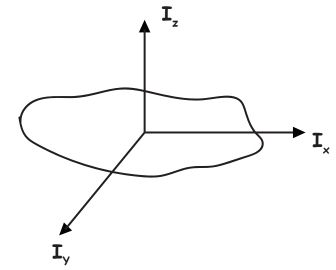


$$I_{parallel} = I_{COM} + Ma^2$$

Conditions:-
1) the two axes must be parallel to each other
2) One of the axis must pass through centre of mass

Perpendicular axis theorem

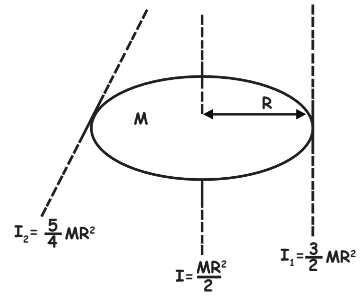
$I_z = I_x + I_y$ (Only valid for laminar bodies)



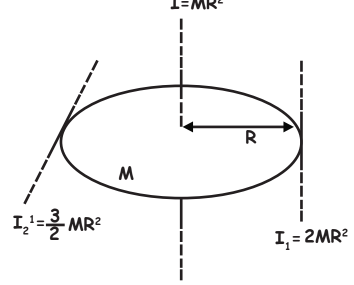
Note:-
X and Y axis must lie in the plane of body
Z- axis must be $\perp$ to the plane of the body
Axes need not pass through center of mass

Moment Of Inertia For Various Objects

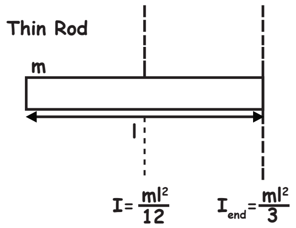
Disc



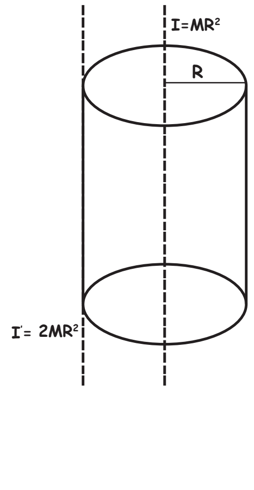
Ring



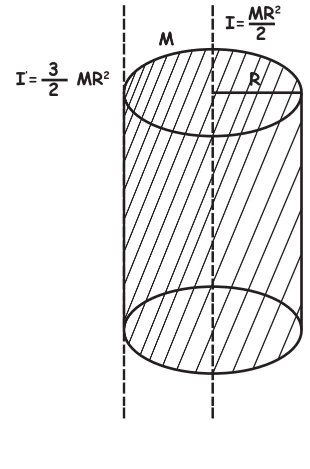
Thin Rod



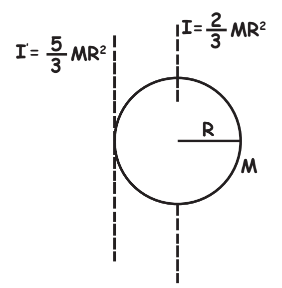
Hollow cylinder



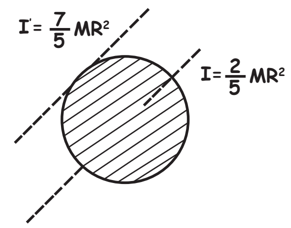
Solid cylinder



Hollow sphere

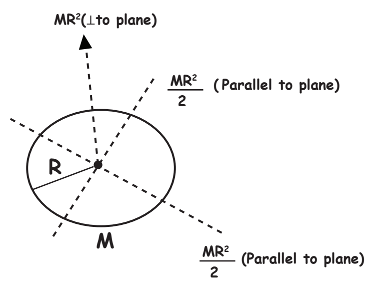


Solid sphere

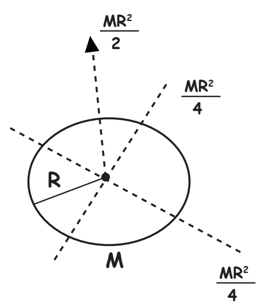


Moment of Inertia along the centre of mass and perpendicular to the plane surface

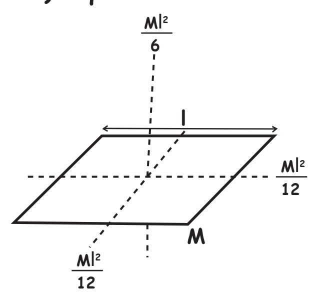
1) Ring



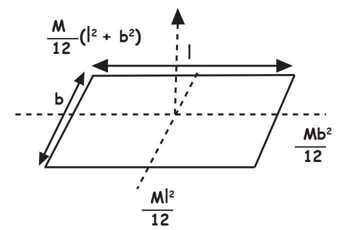
2) Disc



3) Square sheet



4) Rectangular sheet

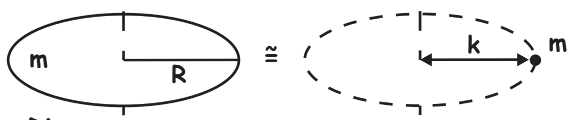


ROTATIONAL MOTION 02

Radius of Gyration

Definition: The distance of a point mass from the axis whose mass is equal to the mass of whole body and whose moment of inertia is equal to moment of inertia of the body about that axis

$I=\frac{mR^2}{2}$ $\qquad I=mk^2$

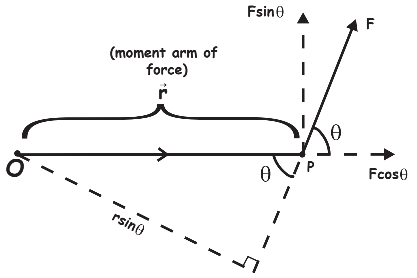


$\frac{mR^2}{2}=mk^2$

$k=\frac{R}{\sqrt{2}}$

k is the radius of gyration

Torque



Torque $\tau_o = r\,F\sin\theta$

$= F\sin\theta \times r = F_\perp r$ (1)

$= F \times r\sin\theta = F\,r_\perp$ (2)

$\vec{\tau}_o = \vec{r}\times\vec{F}$ (Vector form)

If force is radial i.e. $\theta = 0^\circ$ or 180°

Torque $\tau = 0$

If force is tangential and $\perp$ to radius vector i.e. $\theta=90^\circ$

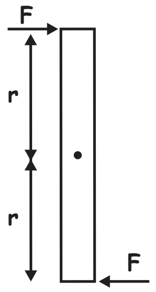
Torque, $\tau = \tau_{max} = rF$

Equilibrium

For translational equilibrium

$F_{net} = 0$

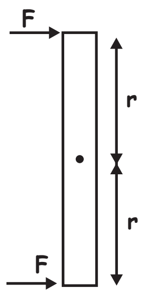
τ_{net} may or maynot be zero



Rotational Equilibrium

$\tau_{net} = 0$

F_{net} may or maynot be zero



Static Equilibrium

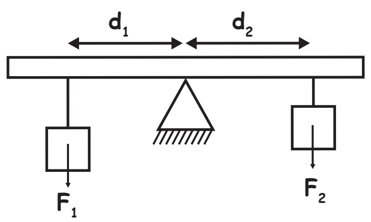
Combination of both translational and rotational equilibrium

$F_{net} = 0 \Rightarrow$ Forces are balanced

$\tau_{net} = 0 \Rightarrow \tau_{clockwise} = \tau_{anticlockwise}$

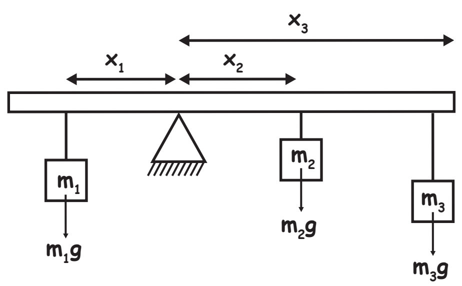
Principle of moments

When a body is in rotational equilibrium sum of clock wise moments about any point is equal to sum of anticlockwise moments about that point

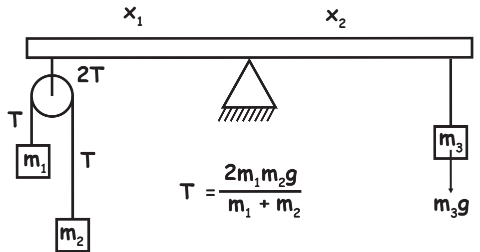


$F_1 \times d_1 = F_2 \times d_2$

Load × load arm = Effort × effort arm



$m_1gx_1 = m_2gx_2 + m_3gx_3$



$T = \frac{2m_1m_2g}{m_1+m_2}$

$2Tx_1 = m_3gx_2$

Angular acceleration

$\tau = I\alpha$

τ - torque

I moment of inertia

α angular acceleration

Initial angular acceleration when a rod is released

Initial angular acceleration when a body is released from an angle θ

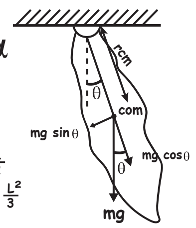
$\tau = I\alpha$

$(mg\sin\theta)\,r_{cm} = I\alpha$

$\alpha = \frac{(mg\sin\theta)r_{cm}}{I}$

For rod $\quad r_{cm} = \frac{L}{2}$

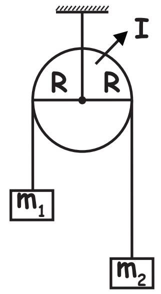
$I = M\frac{L^2}{3}$



Translation - rotation combination

$\alpha = \frac{\tau_{applied} - \tau_{opposition}}{\text{Total } I}$

$\alpha = \frac{m_2Rg - m_1Rg}{m_1R^2 + m_2R^2 + I}$



$m_2 > m_1$

Angular momentum & its conservation

Angular momentum of a point mass:-

Angular momentum about origin

$\vec{L}_o = \vec{r}\times\vec{P}$

$= m(\vec{r}\times\vec{v})$

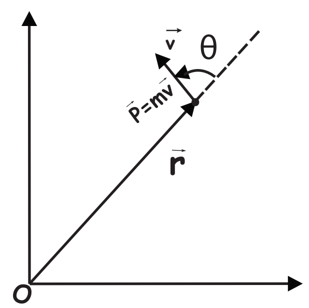
$\vec{P} = m\vec{v}$

1) When $\theta = 0^\circ$ or 180°

$L_o = mvr\sin 180^\circ = 0$

OR $mvr\sin 0^\circ = 0$

Angular momentum is minimum



1) When $\theta = 90^\circ$

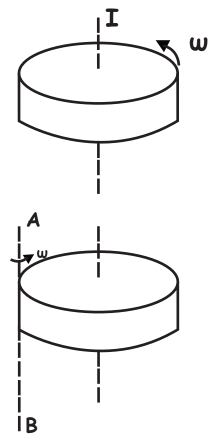
$|\vec{L}| = r\,p\sin\theta$

$= rp\sin 90^\circ$

$= rp$

$= L_{max}$

Spin angular momentum



$L_{axis} = I\omega$

$L_{AB} = I_{AB}\omega$

Conservation of Angular momentum

If there is no external torque, angular momentum is conserved

- $\tau = \frac{dL}{dt}$
- If $\tau = 0 \Longrightarrow \frac{dL}{dt} = 0$

  L=constant

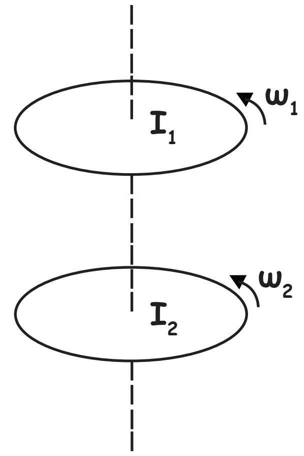
  $I\omega$=constant

- $I_1\omega_1 = I_2\omega_2$
- If moment of inertia increases angular velocity decreases and if moment of inertia decreases angular velocity increases

Moment of inertia when two discs are joined

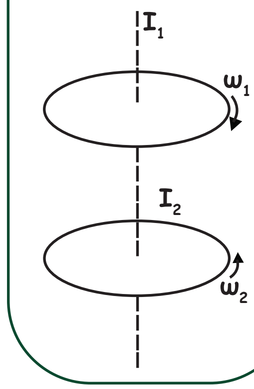
Discs initially rotating in same direction:-

$\omega_f = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$



Discs initially rotating in opposite direction:-

$\omega_f = \frac{I_1\omega_1 - I_2\omega_2}{I_1 + I_2}$



Work, Energy & Power in rotation

1) Work done by a torque,

$W = \tau\theta$ (if torque is uniform)

$= \int \tau\, d\theta$ (if torque is non uniform)

2) K.E for are rotating body $= \frac{1}{2}I\omega^2$

$= \frac{L^2}{2I}$

$= \frac{1}{2}L\omega$

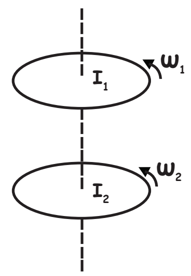
3) Work-Energy theorm

$\Sigma W = \Delta K = \frac{1}{2}I(\omega_2^2 - \omega_1^2)$

Energy loss when 2 discs are joined:-

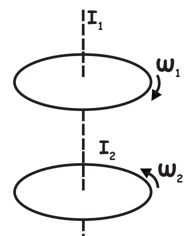
1) Same direction:-

$E_{lost} = \Delta K.E = \frac{I_1I_2}{2[I_1+I_2]}(\omega_1 - \omega_2)^2$

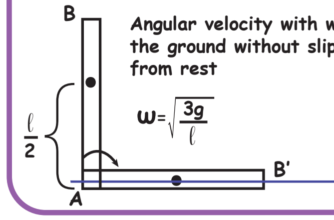


2) Opposite direction:-

$E_{lost} = \Delta K.E = \frac{I_1I_2}{2[I_1+I_2]}(\omega_1 + \omega_2)^2$



Mechanical energy conservation



Angular velocity with which the rod hits the ground without slipping, released from rest

$\omega = \sqrt{\frac{3g}{l}}$

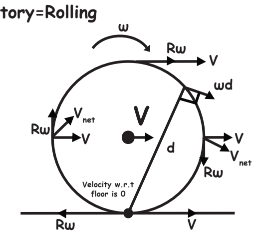
Rolling Motion

Translatory+Rotatory=Rolling

Velocity in rolling

- Condition for rolling without slipping:- $V = R\omega$
- Velocity of any point on rolling object, V_p

  $= \omega d = \frac{v\,d}{R}$



d is the distance from point of contact

Energy in rolling motion

1) Translatory Motion

$T_{K.E} = \frac{1}{2}mv^2$

2) Spinning motion/rotational motion

$R_{K.E} = \frac{1}{2}I\omega^2 = \frac{1}{2}mk^2\frac{V^2}{R^2} = \frac{1}{2}mv^2\times\left(\frac{k^2}{R^2}\right)$

3) Rolling motion

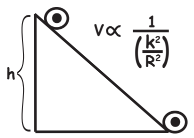
$T_{K.E} + R_{K.E} = \frac{1}{2}mv^2 + \frac{1}{2}mv^2\times\frac{k^2}{R^2} = \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)$

$\frac{k_{total}}{k_{trans}} = \left(1+\frac{k^2}{R^2}\right)$

Motion on an inclined plane

Velocity at bottom $= \sqrt{\frac{2gh}{1+\frac{k^2}{R^2}}}$

$v \propto \frac{1}{\sqrt{1+\frac{k^2}{R^2}}}$



$V \propto \frac{1}{\left(\frac{k^2}{R^2}\right)}$

$\frac{k^2}{R^2}\uparrow \Rightarrow V\downarrow \Rightarrow$ Time $\uparrow$

Velocity: solid sphere > Disc > Hollow > Sphere > Ring

Time to reach bottom: Ring > Hollow sphere > Disc > solid sphere

Value of velocity:-

1) Ring/Hollow cylinder $= \sqrt{gh}$

2) Disc/Solid cylinder $= \sqrt{\frac{4}{3}gh}$

3) Hollow sphere $= \sqrt{\frac{6}{5}gh}$

4) Solid sphere $= \sqrt{\frac{10}{7}gh}$

Acceleration

$a = \frac{g\sin\theta}{1+\frac{k^2}{R^2}} \qquad a \propto \frac{1}{1+\frac{k^2}{R^2}}$

Time of descend:-

$t = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1+\frac{k^2}{R^2}\right)}$

$t \propto \sqrt{1+\frac{k^2}{R^2}}$

Ring > Hollow sphere > Disc > solid sphere

PHYSICS WALLAH