CHAPTER

6 Continuity and Differentiability, Methods of Differentiation

Properties of Continuous Functions

Here we present two extremely useful properties of continuous functions;

Let $y = f(x)$ be a continuous function $\forall\, x \in [a, b]$, then following results hold true.

(*i*) f is bounded between a and b. This simply means that we can find real numbers m_1 and m_2 such $m_1 \le f(x) \le m_2\ \forall\ x \in [a, b]$.

(*ii*) Every value between $f(a)$ and $f(b)$ will be assumed by the function atleast once. This property is called intermediate value theorem of continuous function.

In particular if $f(a) \cdot f(b) < 0$, then $f(x)$ will become zero atleast once in (a, b). It also means that if $f(a)$ and $f(b)$ have opposite signs then the equation $f(x) = 0$ will have atleast one real root in (a, b).

Types of Discontinuities

Type-1 : (Removable type of discontinuities)

(***a***) **Missing point discontinuity:** Where $\lim\limits_{x \to a} f(x)$ exists finitely but $f(a)$ is not defined.

(***b***) **Isolated point discontinuity :** Where $\lim\limits_{x \to a} f(x)$ exists & $f(a)$ also exists but; $\lim\limits_{x \to a} f(x) \ne f(a)$.

Type-2 : (Non-Removable type of discontinuities)

(***a***) **Finite type discontinuity :** In such type of discontinuity left hand limit and right hand limit at a point exists but are not equal.

(***b***) **Infinite type discontinuity :** In such type of discontinuity atleast one of the limit viz. LHL and RHL is tending to infinity.

(***c***) **Oscillatory type discontinuity :** Limits oscillate between two finite quantities.

Derivability of Function at a Point

If $f'(a^+) = f'(a^-)$ = finite quantity, then $f(x)$ is said to be **derivable or differentiable at $x = a$.** In such case $f'(a^+) = f'(a^-) = f'(a)$ and it is called derivative or differential coefficient of $f(x)$ at $x = a$.

Note:

(*i*) All polynomial, trigonometric, inverse trigonometric, logarithmic and exponential function are continuous and differentiable in their domains, except at end points.

(*ii*) If $f(x)$ and $g(x)$ are derivable at $x = a$ then the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x).\ g(x)$ will also be derivable at $x = a$ and if $g(a) \ne 0$ then the function $f(x)/g(x)$ will also be derivable at $x = a$.

In short, for a function 'f':

Differentiable $\Rightarrow$ **Continuous;**

Not Differentiable $\Rightarrow$ **Not Continuous**

But Not Continuous $\Rightarrow$ **Not Differentiable**

Continuous $\Rightarrow$ **May or may not be Differentiable**

Derivability Over an Interval

(*a*) $f(x)$ is said to be derivable over an open interval (a, b) if it is derivable at each and every point of the open interval (a, b).

(*b*) $f(x)$ is said to be derivable over the closed interval $[a, b]$ if:

(*i*) $f(x)$ is derivable in (a, b) and

(*ii*) for the points a and b, $f'(a^+)$ & $f'(b^-)$ exist.

Note:

(***i***) If $f(x)$ is differentiable at $x = a$ and $g(x)$ is not differentiable at $x = a$, then the product function $F(x) = f(x).g(x)$ can still be differentiable at $x = a$.

(***ii***) If $f(x)$ & $g(x)$ both are not differentiable at $x = a$ then the product function; $F(x) = f(x).g(x)$ can still be differentiable at $x = a$.

(***iii***) If $f(x)$ & $g(x)$ both are non-derivable at $x = a$ then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function.

(***iv***) If $f(x)$ is derivable at $x = a$ $\not\Rightarrow$ $f'(x)$ is continuous at $x = a$.

Differentiation of Some Elementary Functions

1. $\dfrac{d}{dx}(x^n) = nx^{n-1}$

2. $\dfrac{d}{dx}(a^x) = a^x \ln a$

3. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}$

4. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$

5. $\dfrac{d}{dx}(\sin x) = \cos x$

6. $\dfrac{d}{dx}(\cos x) = -\sin x$

7. $\frac{d}{dx}(\sec x) = \sec x \tan x$

8. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

9. $\frac{d}{dx}(\tan x) = \sec^2 x$

10. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

Basic Theorems

1. $\frac{d}{dx}(f \pm g)(x) = f'(x) \pm g'(x)$

2. $\frac{d}{dx}(k\, f(x)) = k \frac{d}{dx} f(x)$

3. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + g(x)f'(x)$

4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

5. $\frac{d}{dx}\big(f(g(x))\big) = f'(g(x))g'(x)$

Derivative of inverse Trigonometric Functions

$$\frac{d \sin^{-1} x}{dx} = \frac{1}{\sqrt{1-x^2}}, \frac{d \cos^{-1} x}{dx} = -\frac{1}{\sqrt{1-x^2}}, \text{for } -1 < x < 1.$$

$$\frac{d \tan^{-1} x}{dx} = \frac{1}{1+x^2}, \frac{d \cot^{-1} x}{dx} = -\frac{1}{1+x^2} \quad (x \in R)$$

$$\frac{d \sec^{-1} x}{dx} = \frac{1}{|x|\sqrt{x^2-1}}, \frac{d \operatorname{cosec}^{-1} x}{dx} = -\frac{1}{|x|\sqrt{x^2-1}}, \text{ for } x \in (-\infty, -1) \cup (1, \infty)$$

Differentiation Using Substitution

Following substitutions are normally used to simplify these expression.

1. $\sqrt{x^2 + a^2}$ by substituting $x = a \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

2. $\sqrt{a^2 - x^2}$ by substituting $x = a \sin \theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

3. $\sqrt{x^2 - a^2}$ by substituting $x = a \sec \theta$, where $\theta \in [0, \pi]$, $\theta \neq \frac{\pi}{2}$

4. $\sqrt{\frac{x+a}{a-x}}$ by substituting $x = a \cos \theta$, where $\theta \in [0, \pi]$.

Parametric Differentiation

If $y = f(\theta)$ and $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

Derivative of one Function with Respect to Another

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$.

❖ If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where $f, g, h, l, m, n, u, v, w$ are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$