COMPLEX NUMBER

1. Iota

$i^{4n} = 1, i^{4n+1} = i, i^{4n+2} = -1, \ i^{4n+3} = -i \left[i^2 = -1, i^3 = -i, i^4 = 1\right], i = -\frac{1}{i}$

2. Conjugate Complex Number

$\bar{z} = a - ib$ for $z = a + ib$

$z + z' = 2a, \ z\bar{z} = a^2 + b^2.$

- Properties

(i) $(\bar{\bar{z}}) = z$

(ii) $z = \bar{z} \Leftrightarrow z$ is purely real

(iii) $z = -\bar{z} \Leftrightarrow z$ is purely imaginary

(iv) $\text{Re}(z) = \text{Re}(\bar{z}) = \frac{z + \bar{z}}{2}$

(v) $\text{Im}(z) = \frac{z - \bar{z}}{2i}$

(vi) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(vii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(viii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(ix) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(x) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\text{Re}(\bar{z}_1 z_2) = 2\text{Re}(z_1\bar{z}_2)$

(xi) $\overline{(z^n)} = (\bar{z})^n$,

(xii) If $z = f(z_1)$, then $\bar{z} = f(\bar{z}_1)$

3. Modulus and Argument

$z = a + ib$

$a = r\cos\theta, b = r\sin\theta$

$|z| = \sqrt{a^2 + b^2}$, argument $\theta = \tan^{-1}\frac{b}{a}$. $0 \le \theta < 2\pi$

principal values: $-\pi < \theta \le \pi$

Ist quad $\theta = \tan^{-1}\frac{b}{a}$

2nd quad $\theta = \pi - \tan^{-1}\left(\frac{b}{|a|}\right)$

3rd quad $\theta = -\pi + \tan^{-1}\left(\frac{b}{a}\right)$

4th quad $\theta = -\tan^{-1}\left(\frac{|b|}{a}\right)$

NOTE

(i) Arg (0) is not defined.

(ii) If $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ & $\arg z_1 = \arg z_2$.

(iii) If $\arg z = \pi/2$ or $-\pi/2$, z is purely imaginary.

(iv) If $\arg z = 0$ or π, z is purely real.

(v) Any two arguments of a complex number differ by $2n\pi$.,

4. Properties of Modulus

(i) $|z| \ge 0 \Rightarrow |z| = 0$ iff $z = 0$ & $|z| > 0$ of $z \ne 0$

(ii) $-|z| \le \text{Re}(z) \le |z|$ & $-|z| \le \text{Im}(z) \le |z|$

(iii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iv) $z\bar{z} = |z^2|$

(v) $|z_1 \pm z_2| \ge ||z_1| - |z_2||$

(vi) $|z_1z_2| = |z_1|\,|z_2|$,

(vii) $|z^n| = |z|^n$

(viii) $|z_1 \pm z_2| \le |z_1| + |z_2|$

(ix) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)\,(\bar{z}_1 \pm \bar{z}_2) = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2)$.

(x) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

$\theta_1 = \arg(z_1), \ \theta_2 = \arg(z_2)$.

(xi) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left\{|z_1|^2 + |z_2|^2\right\}$

5. Properties of argument

(i) $\arg(z_1z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$

(ii) $\arg z^n = n\arg z + 2k\pi$

(iii) $\arg\frac{z_2}{z_1} = \theta \Rightarrow \arg\frac{z_1}{z_2} = 2k\pi - \theta. \quad k \in I$

(iv) $\arg\bar{z} = -\arg z$

6. De Moivre's Theorem

(a) If n is any rational number, then

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(b) If $z = r(\cos\theta + i\sin\theta)$ then

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi + \theta}{n}\right) + \sin n\left(\frac{2k\pi + \theta}{n}\right)\right]$$

$k = 0, 1, 2, \ldots, (n - 1)$

7. Square Root of a Complex Number

- Square roots of $z = a + ib$ are

$\pm\left[\sqrt{\frac{|z| + a}{2}} + i\sqrt{\frac{|z| - a}{2}}\right]$ for $b > 0$

$$\pm\left[\sqrt{\frac{|z|+a}{2}}-i\sqrt{\frac{|z|-a}{2}}\right] \text{ for } b<0$$

- Square root of $i \rightarrow \pm\left(\frac{1+i}{\sqrt{2}}\right)$
- Square root of $-i \rightarrow \pm\left(\frac{1-i}{\sqrt{2}}\right)$
- Square root of $\omega \rightarrow \pm\omega^2$
- Square root of $\omega^2 \rightarrow \pm\omega$

8. Cube roots of Unity

$$x=\sqrt[3]{1}=1,\frac{-1+i\sqrt{3}}{2},\frac{-1-i\sqrt{3}}{2}.=1,\omega,\omega^2$$

Properties

(i) $1+\omega+\omega^2=0$

(ii) $\omega^3=1$

(iii) $\omega^{3n}=1$, $\omega^{3n+1}=\omega$, $\omega^{3n+2}=\omega^2$,

(iv) $\bar{\omega}=\omega^2$ & $(\bar{\omega})^2=\omega$

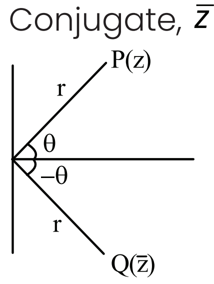
(v) Cube roots of unity lies on vertices of an equilateral triangle inscribed in a unit circle.

(vi) $a+b\omega+c\omega^2=0 \Rightarrow a=b=c$ if a, b, c are real

(vii) $\omega^n+\omega^{n+1}+\omega^{n+2}=0$

9. Geometrical Meanings

1. Conjugate, $\bar{z}$:



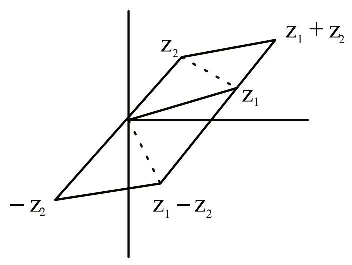
$z=x+iy$

$\bar{z}=x-iy$

$z=(r,\theta)$

$\bar{z}=(r,-\theta)$

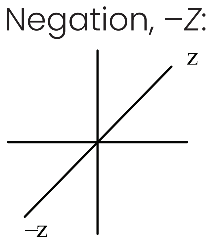
4. Difference:



$z_1=x_1+iy_1$

$z_2=x_2+iy_2$

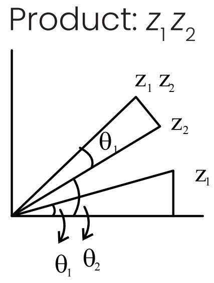
2. Negation, $-Z$:



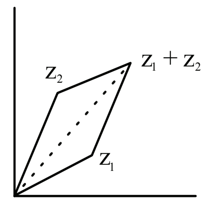
$z=x+iy$

$-z=-x-iy$

5. Product: z_1z_2



3. Sum of complex numbers:

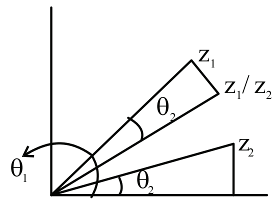


$z_1=x_1+iy_1$

$z_2=x_2+iy_2$

$|z_1|+|z_2|\geq|z_1+z_2|$

6. Quotient:



10. Euler's Formula

$e^{i\theta}=\cos\theta+i\sin\theta$

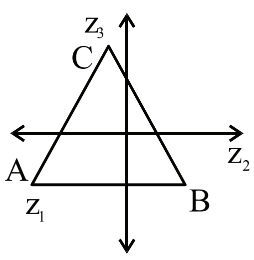
$$=1+\frac{i\theta}{1!}+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\cdots+\cdots\infty$$

$z=re^{i\theta}\Rightarrow \arg z=2n\pi+\theta$

$\log z=\log r+i(\theta+2n\pi)$

$\therefore$ Logarithm of an imaginary number is not unique

11. Concept of Rotation



$AB=|z_2-z_1|$

$BC=|z_3-z_2|$

$CA=|z_3-z_1|$

$$\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)=\arg(z_3-z_1)-\arg(z_2-z_1)=\alpha$$

Also, $\left|\frac{z_3-z_1}{z_2-z_1}\right|=\frac{CA}{AB}$ $\therefore \frac{z_3-z_1}{z_2-z_1}=\frac{CA}{AB}(\cos\alpha+i\sin\alpha)$

If $z_1^2+z_2^2+z_3^2+z_1z_2+z_2z_3+z_3z_1=0$, then the triangle ABC is equilateral.