"Mathematics is not just about numbers, equations, computations, or algorithms: it is about understanding and implementing"

— William Paul Thurston

QUADRATIC EQUATION

1. Basic Results

The quantity ($D = b^2-4ac$) is known as the discriminant of the quadratic equation

- The quadratic equation has real and equal roots if and only if $D=0$, i.e, $b^2-4ac=0$.
- The quadratic equation has real and distinct roots if and only if $D>0$, i.e., $b^2-4ac>0$.
- The quadratic equation has complex roots with non-zero imaginary parts if and only if $D<0$, i.e., $b^2-4ac<0$.
- If $p+iq$ (p and q being real) is a root of the quadratic equation where $i=\sqrt{-1}$, then $p-iq$ is also a root of the quadratic equation.
- If $p+\sqrt{q}$ is an irrational root of the quadratic equation, then $p-\sqrt{q}$ is also a root of the quadratic equation provided that all the coefficients are rational
- The quadratic equation has rational roots if D is a perfect square and a,b,c are rational
- If $a=1$ and b,c are integers and the roots of the quadratic equation are rational, then the roots must be integers.
- If the quadratic equation is satisfied by more than two numbers (real or complex), then it becomes an identity, $i.e, a=b=c=0$.

2. Quick Look

1. $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{b^2-2ac}{a^2}$
2. $\alpha^2-\beta^2=(\alpha+\beta)\sqrt{(\alpha+\beta)^2-4\alpha\beta}=-\frac{b\sqrt{b^2-4ac}}{a^2}$
3. $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=-\frac{b(b^2-3ac)}{a^3}$
4. $\alpha^3-\beta^3=(\alpha-\beta)^3+3\alpha\beta(\alpha-\beta)$
5. $\alpha^4+\beta^4=\{(\alpha+\beta)^2-2\alpha\beta\}^2-2\alpha^2\beta^2=\left(\frac{b^2-2ac}{a^2}\right)^2-2\frac{c^2}{a^2}$
6. $\alpha^4-\beta^4=(\alpha^2-\beta^2)(\alpha^2+\beta^2)=\frac{\pm b(b^2-ac)\sqrt{b^2-4ac}}{a^4}$
7. $\alpha^2+\alpha\beta+\beta^2=(\alpha+\beta)^2-\alpha\beta=\frac{b^2-ac}{a^2}$
8. $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{b^2-ac}{a^2}$
9. $\alpha^2\beta+\beta^2\alpha=\alpha\beta(\alpha+\beta)=-\frac{bc}{a^2}$
10. $\left(\frac{\alpha}{\beta}\right)^2+\left(\frac{\beta}{\alpha}\right)^2=\frac{\alpha^4+\beta^4}{\alpha^2\beta^2}=\frac{(\alpha^2+\beta^2)^2-2\alpha^2+\beta^2}{\alpha^2+\beta^2}$

3. Formation of an equation with given roots

A quadratic equation whose roots are

α and β is given by a

$$(x-\alpha)(x-\beta)=0$$

$$\therefore x^2-Sx+P=0$$

$$i.e., x^2-(sum\ of\ roots)x+(product\ of\ roots)=0$$

$$\therefore x^2-(\alpha+\beta)x+\alpha\beta=0$$

4. Equation in terms of the roots of another equations

If α,β are roots of the equation $ax^2+bx+c=0$, then the equation whose roots are

- $-\alpha,-\beta \Rightarrow ax^2-bx+c=0$ (Replace x by $-x$)
- $\frac{1}{\alpha},\frac{1}{\beta} \Rightarrow cx^2+bx+a=0$ (Replace x by $\frac{1}{x}$)
- $\alpha^n\beta^n, n\in N \Rightarrow a\left(x^{\frac{1}{n}}\right)^2+b\left(x^{\frac{1}{n}}\right)+c=0$ (Replace x by $x^{\frac{1}{n}}$)
- $k\alpha,k\beta \Rightarrow ax^2+kbx+k^2c=0$ (Replace x by $\frac{x}{k}$)
- $\frac{\alpha}{k},\frac{\beta}{k} \Rightarrow k^2ax^2+kbx+c=0$ (Replace x by kx)
- $k+\alpha,k+\beta \Rightarrow a(x-k)^2+b(x-k)+c=0$ (Replace x by (x-k))
- $\alpha^{\frac{1}{n}},\beta^{\frac{1}{n}}; n\in N \Rightarrow a(x^n)^2+b(x^n)^2+c=0$ (Replace x by x^n)

Relation between the Roots of a Polynomial Equation of Degree n

Consider the equation

$$a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\ldots+a_1x+a_0=0$$

$(a_0,a_1\ldots,a_n$ are real coefficients and $a_n\neq 0)$.

Let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be the roots of equation (i). Then

$$a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\ldots+a_1x+a_0=a_n(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

Comparing the coefficients of like powers of x, we get

$$\alpha_1+\alpha_2+\alpha_3+\ldots\alpha_n=-\frac{a_{n-1}}{a_n}$$

$$\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\ldots+\alpha_2\alpha_3+\ldots+\alpha_{n-1}\alpha_n=\frac{a_{n-2}}{a_n}$$

$$\alpha_1\alpha_2\ldots\alpha_r+\ldots+\alpha_{n-r+1}\alpha_{n-r+2}\ldots\alpha_n=(-1)^r\frac{\alpha_{n-r}}{a_n}$$

$$\alpha_1\alpha_2\ldots\alpha_n=(-1)^n\frac{a_0}{a_n}.$$

5.

1. Condition for Two Quadratic Equations to have one Common Root

If $ax^2+bx+c=0$ and $dx^2+ex+f=0$ have a common root $\alpha(say)$.

Then $(dc-af)^2=(bf-ce)(ae-bd)$,

2. Both roots are common

Then required condition is $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

6. Quadratic Expression

The expression $ax^2 + bx + c$ is said to be a real quadratic expression in x where a,b,c are real and $a \neq 0$. Let $f(x) = ax^2 + bx + c$ where $a,b,c, \in R (\alpha \neq 0)$.

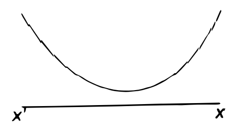
$f(x)$ can be rewritten as

$f(x) = a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\} = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$, where $D = b^2 - 4ac$ is discriminate of the quadratic expression. Then $y = f(x)$ represents a parabola whose axis is parallel to the $y - axis$, with vertex at $A\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$.

7. Sign of f(x)

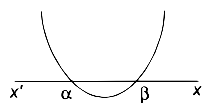
Depending on the sign of a and $b^2 - 4ac, f(x)$ may be positive, negative or zero, This given rise to the following cases:

A $a > 0$ and $b^2 - 4ac < 0 \Leftrightarrow f(x) > 0 \forall x \in R.$



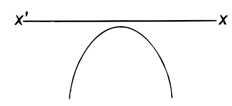
In this case the parabola always remains above the x-axis.

B $a > 0$ and $b^2 - 4ac > 0$. Let $f(x) = 0$ have two real roots α and β $(\alpha < \beta)$.



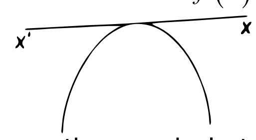
Then $f(x) > 0 \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$, and $f(x) < 0 \forall x \in (\alpha, \beta)$.

C $a < 0$ and $b^2 - 4ac < 0 \Leftrightarrow f(x) < 0 \forall x \in R.$



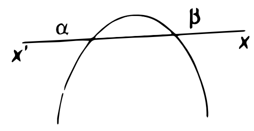
In this case the parabola always remains below the x-axis.

D $a < 0$ and $b^2 - 4ac = 0 \Leftrightarrow f(x) \leq 0 \forall x \in R.$



In this case the parabola touches the x-axis and lies below the x-axis.

E $a < 0$ and $b^2 - 4ac > 0$



Let $f(x) = 0$ have two real roots α and β $(\alpha < \beta)$. Then $f(x) < 0 \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0 \forall x \in (\alpha, \beta)$.

8. Maximum and minimum values of quadratic expression

Maximum and minimum value of quadratic expression can be found out by two methods:

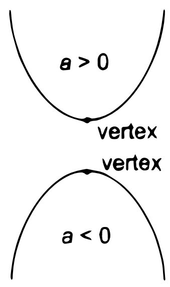
1. **Discriminate method:**

In a quadratic expression $ax^2 + bx + c$.

2. Vertex of the parabola $Y = aX^2$ is $X = 0, Y = 0$.

$i.e.,\ x + \frac{b}{2a} = 0, y + \frac{D}{4a} = 0 \Rightarrow x = -\frac{b}{2a}, y = -D/4a$

Hence, vertex of $y = ax^2 + bx + c$ is $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$.



- For $a > 0, f(x)$ has least value at $x = -\frac{b}{2a}$.

  This least value is given by $f\left(-\frac{b}{2a}\right) = -\frac{D}{4a}$.

- For $a < 0, f(x)$ has greatest value at $x = -\frac{b}{2a}$.

  This greatest value is given by $f\left(-\frac{b}{2a}\right) = -\frac{D}{4a}$.

9. Interval in Which the Roots Lie

In some problems we want the roots of the equation $ax^2 + bx + c = 0$ to lie in a given interval.

For this we impose conditions on a, b and c. Let $f(x) = ax^2 + bx + c$.

1. If both the roots are positive, $i.e.,$ they lie in $(0, \infty)$, then the sum of the roots as well as the product of the roots must be positive.

$$\Rightarrow \alpha + \beta = -\frac{b}{\alpha} > 0 \text{ and } \alpha\beta = \frac{c}{a} > 0 \text{ with } b^2 - 4ac \geq 0.$$

2. Similarly, if both the roots are negative, $i.e.,$ they lie in $(-\infty, 0)$ then the sum of the roots will be negative and the product of the roots must be positive, $i.e.,$ $\alpha + \beta = -\frac{b}{a} < 0$ and $\alpha\beta = \frac{c}{a} > 0$ with $b^2 - 4ac \geq 0$.

a. Both the roots are greater than a given number k if the following three conditions are satisfied: $D \geq 0, -\frac{b}{2a} < k$ and $a.f(k) > 0$.

b. Both the roots will lie in the given interval $(k_p k_2)$ if the following conditions are satisfied. $D \geq 0, k_1 < -\frac{b}{2a} < k_2$ and $a.f(k_1) > 0, a.f(k_2) > 0$.

c. Exactly one of the roots lies in the given interval $(k_p k_2)$ if $f(k_1).f(k_2) < 0$.

d. A given number k will lie between the roots if $a.f(k) < 0$.

In particular, the roots of the equation will be of opposite signs if 0 lies between the roots $\Rightarrow a.f(0) < 0$.

It also implies that the product of the roots is negative.