Binomial Theorem

01 BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

$(x + y)^n = {}^nC_0\, x^{n-0}\, y^0 + {}^nC_1\, x^{n-1}\, y^1 + {}^nC_2\, x^{n-2}\, y^2 + \ldots +$
${}^nC_r\, x^{n-r}\, y^r + \ldots + {}^nC_{n-1}\, x y^{n-1} + {}^nC_n\, x^0\, y^n$

i.e., $(x + y)^n = \sum_{r=0}^{n} {}^nC_r . x^{n-r} . y^r$ ...(i)

Here ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called binomial coefficients and

${}^nC_r = \dfrac{n!}{r!(n-r)!}$, For $0 \le r \le n$.

NOTE

The binomial coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots$ equidistant from beginning and end are equal, i.e., ${}^nC_r = {}^nC_{n-r}$.

02 SOME USEFUL RELATIONS IN COMBINATORIAL

Some useful relations in combinatorial:

(1) ${}^nC_x = {}^nC_y \Rightarrow$ either $x = y$ or $x + y = n$

(2) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(3) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(4) ${}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \ldots + {}^nC_r = {}^{n+1}C_{r+1}$

(5) ${}^mC_r + {}^{m+1}C_r + {}^{m+2}C_r + \ldots + {}^nC_r = {}^{n+1}C_{r+1} - {}^mC_{r+1}$

(6) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$

(7) ${}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots + (-1)^n\, {}^nC_n = 0$

(8) ${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n = 2^{2n}$

03 SOME IMPORTANT EXPANSIONS

(1) $(1 + x)^n = {}^nC_0\, x^0 + {}^nC_1\, x^1 + {}^nC_2\, x^2 + \ldots + {}^nC_r\, x^r + \ldots + {}^nC_n\, x^n$, i.e., $(1 + x)^n = \sum_{r=0}^{n} {}^nC_r x^r$

(2) $(1 - x)^n = {}^nC_0\, x^0 + {}^nC_1\, x^1 + {}^nC_2\, x^2 - \ldots + (-1)^r\, {}^nC_r\, x^r + \ldots + (-1)^n\, {}^nC_n\, x^n$

i.e., $(1 - x)^n = \sum_{r=0}^{n} (-1)^r\ {}^nC_r\, x^r$

(3) $(x + y)^n + (x - y)^n = 2[{}^nC_0\, x^n\, y^0 + {}^nC_2\, x^{n-2}\, y^2 + {}^nC_4\, x^{n-4}\, y^4 + \ldots]$ and

$(x + y)^n - (x - y)^n = 2[{}^nC_1\, x^{n-1}\, y^1 + {}^nC_3\, x^{n-3}\, y^3 + {}^nC_5\, x^{n-5}\, y^5 + \ldots]$

(4) The coefficient of $(r + 1)^{th}$ term in the expansion of $(1 + x)^n$ is ${}^nC_r$.

(5) If n is odd, then $(x + y)^n + (x - y)^n$ and $(x + y)^n - (x - y)^n$, both have the same number of terms equal to $\left(\dfrac{n+1}{2}\right)$

(6) If n is even, then $(x + y)^n + (x - y)^n$ has $\left(\frac{n}{2}+1\right)$ terms and $(x + y)^n - (x - y)^n$ has $\dfrac{n}{2}$ terms.

04 GENERAL TERMS

$T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$ for $r = 0, 1, 2, 3, \ldots n$ defines respectively first, second, third, ..., nth term.

(1) In the binomial expansion of $(x + y)^n$, the p^{th} term from the end is $(n - p + 2)^{th}$ term from beginning.

(2) The number of terms in the above binomial expansion are ${}^{n+1}C_1 = n + 1$.

(3) The general term in the expansion of trinomial $(x + y + z)^n = \sum_{r,s,t} \dfrac{n!}{r!s!t!}$ $X^r\, y^s\, z^t$ where $n \in N$ and $r, s, t \in \{0, 1, 2, \ldots, n\}$ and $r + s + t = n$. The number of terms in this expansion $= {}^{n+2}C_2$.

(4) In the expansion of $(x + y)^n$, $n \in N$ $\dfrac{T_{r+1}}{T_r} = \left(\dfrac{n-r+1}{r}\right)\dfrac{y}{x}$.

(5) The coefficient of x^{n-1} in the expansion of $(x - 1)(x - 2) \ldots (x - n) = -\dfrac{n(n+1)}{2}$

(6) The coefficient of x^{n-1} in the expansion of $(x + 1)(x + 2) \ldots (x + n) = \dfrac{n(n+1)}{2}$

05 APPLICATION OF GENERAL TERM

I. To Determine a Particular Term in the Expansion

In the expansion of $\left(x^\alpha \pm \dfrac{1}{x^\beta}\right)^n$, if x^m occurs in T_{r+1}, then r is given by

$n\alpha - r(\alpha + \beta) = m \Rightarrow r = \dfrac{n\alpha - m}{\alpha + \beta}$.

Thus in above expansion if constant term which is independent of x, occurs in T_{r+1} then r is determined by $n\alpha - r(\alpha + \beta) = 0 \Rightarrow r = \dfrac{n\alpha}{\alpha + \beta}$

06 MIDDLE TERM

The middle term depends upon the value of n.

(1) When n is even, then total number of terms in the expansion of $(x + y)^n$ is $n + 1$ (odd). So there is only middle term, i.e., $\left(\frac{n}{2}+1\right)^{th}$

(2) When n is odd, then total number of terms in the expansion of $(x + y)^n$ is $n + 1$ (even). So there are two middle terms $T_{\left(\frac{n+1}{2}\right)}$ And $T_{\left(\frac{n+3}{2}\right)}$ given by:

$$T_{\left(\frac{n+1}{2}\right)} = {}^nC_{\frac{n-1}{2}}\, x^{\frac{n+1}{2}}\, y^{\frac{n-1}{2}} \text{ And } T_{\left(\frac{n+3}{2}\right)} = {}^nC_{\frac{n+1}{2}}\, x^{\frac{n-1}{2}}\, y^{\frac{n+1}{2}}$$

07 Number of irrational terms in $(a^{1/p} + b^{1/q})^N\ \forall$ a, b $\in$ prime numbers

Method for finding terms free from radical or rational terms in the expansion of $(a^{1/p} + b^{1/q})^N\ \forall$ a, b $\in$ Prime numbers:

Find the general term

$T_{r+1} = {}^NC_r\, (a^{1/p})^{N-r}\, (b^{1/q})^r = {}^NC_r\, a^{\frac{N-r}{p}} . b^{\frac{r}{q}}$

Putting the values of $0 \le r \le N$, when indices of a and b are integers.

NOTE

(1) When there are two middle terms in the expansion then their binomial coefficients are equal.

(2) Binomial coefficient of middle term is the greatest binomial coefficient.

(3) If n is even, then greatest binomial coefficient is ${}^nC_{\frac{n}{2}}$.

(4) If n is odd, then greatest binomial coefficient are ${}^nC_{\frac{n+1}{2}}$ and ${}^nC_{\frac{n+3}{2}}$.

08 Numerically Greatest Term

Shortcut method: To find the greatest term (numerically) in the expansion of $(1 + x)^n$.

(i) Calculate $m = \frac{|x|(n+1)}{|x|+1}$

(ii) If m is integer, then T_m and T_{m+1} are equal and both are greatest term.

(iii) If m is not integer, then $T_{[m]+1}$ is the greatest term, where [.] denotes the greatest integral part.

09 PROPERTIES OF BINOMIAL COEFFICIENTS

01 In the expansion $(1 + x)^n$, $2^n = C_0 + C_1 + C_2 + \dots + C_n$

02 $0 = C_0 - C_1 + C_2 - C_3 + \dots$

03 Sum of the coefficients of the odd terms in the expansion of $(1 + x)^n$ is equal to sum of the coefficients of even terms and each is equal to 2^{n-1}.

$C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$

10 MULTINOMIAL THEOREM (FOR POSITIVE INTEGRAL INDEX)

If n is positive integer and $a_1, a_2, a_3, \dots a_n \in C$, then

$$(a_1 + a_2 + a_3 + \dots + a_m)^n = \sum \frac{n!}{n_1!\, n_2!\, n_3! \dots n_m!} \dots a_1^{n_1}\ a_2^{n_2}\ a_m^{n_m}$$

Where $n_1, n_2, n_3, \dots n_m$ are all non-negative integers subject to the condition, $n_1 + n_2 + n_3 + \dots + n_m = n$.

(1) The coefficients of $a_1^{n_1}\ a_2^{n_2} \dots a_m^{n_m}$ in the expansion of $(a_1 + a_2 + a_3 + \dots + a_m)^n$ is

$$\frac{n!}{n_1!\, n_2!\, n_3! \dots n_m!}$$

(2) The greatest coefficients in the expansion of $(a_1 + a_2 + a_3 + \dots a_m)^n$ is

$$\frac{n!}{(q!)^{m-r}\left[(q+1)!\right]^r}$$

Where q is the quotient and r is the remainder when n is divided by m.

(3) If n is + ve integer and $a_1, a_2, \dots a_m \in C$, then coefficients of x^r in the expansion of

$(a_1 + a_2 x + \dots + a_m x^{m-1})^n$ is $\sum \frac{n!\left(a_1^{n_1}.a_2^{n_2}\dots a_m^{n_m}\right)}{n_1!\, n_2!\, n_3! \dots n_m!}$

Where $n_1, n_2, \dots n_m$ are all non-negative integers subject to the condition: $n_1 + n_2 + \dots + n_m = n$ and $n_2 + 2n_3 + 3n_4 + \dots + (m - 1) n_m = r$.

(4) The number of distinct or dissimilar terms in the multinomial expansion $(a_1 + a_2 + a_3 + \dots a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

11 BINOMIAL THEOREM FOR ANY INDEX

Statement

$$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \dots \text{ terms up to } \infty,$$

IMPORTANT TIPS

- Expansion is valid only when $-1 < x < 1$.
- ${}^nC_r$ cannot be used because it is defined only for natural number, so ${}^nC_r$ will be written as $\frac{(n)(n-1)\dots(n-r+1)}{r!}$
- The number of terms in the series is infinite.
- If first term is not 1, then make first term unity in the following way, $(x + y)^n = x^n\left[1+\frac{y}{x}\right]^2$, if $\left|\frac{y}{x}\right|<1$.

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$$T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r$$

13 SOME IMPORTANT EXPANSIONS:

(i) $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots$

(ii) $(1 - x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \dots + \frac{n(n-1)(n-2)(n-r+1)}{r!}(-x)^r + \dots$

(iii) $(1 - x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots + \frac{n(n+1)\dots(n+r-1)}{r!}(x)^r + \dots$

(iv) $(1 + x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}X^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots + \frac{n(n+1)\dots(n+r-1)}{r!}(-x)^r + \dots$