CAN YOU IMAGINE YOUNG PEOPLE NOWADAYS MAKING A STUDY OF TRIGONOMETRY FOR THE FUN OF IT? WELL, I DID

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TRIGONOMETRIC RATIOS AND EQUATIONS

01 IMPORTANT TRIGONOMETRIC RATIOS

- $\sin 15^\circ$ or $\sin\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}=\cos 75^\circ$ or $\cos\frac{5\pi}{12}$
- $\cos 15^\circ=\cos\frac{\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}=\sin 75^\circ=\sin\frac{5\pi}{12}$
- $\sin\frac{\pi}{10}$ or $\sin 18=\frac{\sqrt{5}-1}{4}$
- $\tan 15^\circ=\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}=\cot 75^\circ$
- $\cos\frac{\pi}{5}$ or $\cos 36^\circ=\frac{\sqrt{5}+1}{4}$
- $\tan 75^\circ=\frac{\sqrt{3}+1}{\sqrt{3}-1}=2+\sqrt{3}=\cot 15^\circ$
- $\sin 18^\circ=\frac{\sqrt{5}-1}{4}$, $\sin 36^\circ=\frac{\sqrt{10-2\sqrt{5}}}{4}$
- $\cos 18^\circ=\frac{\sqrt{10+2\sqrt{5}}}{4}$, $\cos 36^\circ=\frac{\sqrt{5}+1}{4}$
- $\tan 15^\circ=2-\sqrt{3}$, $\tan 22.5^\circ=\sqrt{2}-1$, $\tan 67.5^\circ=\sqrt{2}+1$,
- $\tan 75^\circ=2+\sqrt{3}$.

04 Transformation of Product into Sum or Difference of Sines or Cosines

01 $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

02 $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

03 $2\cos A\cos B=\cos(A+B)+\cos(A-B)$

04 $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

05 Some Special Series

01

$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots+\sin\{\alpha+(n-1)\beta\}=$

$$\frac{\sin\left\{\frac{2\alpha+(n-1)\beta}{2}\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

02

$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos\{\alpha+(n-1)\beta\}=$

$$\frac{\cos\left\{\frac{2\alpha+(n-1)\beta}{2}\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

02 Trigonometrical Equations

- $\cos n\pi=(-1)^n, \sin n\pi=0, n\in I$
- $\cos\frac{n\pi}{2}=0, \sin\frac{n\pi}{2}=(-1)^{\frac{n-1}{2}}, n$ is odd integer.
- $\cos(n\pi+\theta)=(-1)^n\cos\theta, n\in I$

  $\sin(n\pi+\theta)=(-1)^n\sin\theta, n\in I$
- $\cos\left(\frac{n\pi}{2}+\theta\right)=(-1)^{\frac{n+1}{2}}\sin\theta, n$ is odd integer
- $\sin\left(\frac{n\pi}{2}+\theta\right)=(-1)^{\frac{n-1}{2}}\cos\theta, n$ is odd integer

03 Factorisation of the sum or Difference of sine and cosine with two variables

- $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$
- $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$
- $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$
- $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B$
- $\sin 2\theta=2\sin\theta\cos\theta=\frac{2\tan\theta}{1+\tan^2\theta}$
- $\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$.
- $\cos^2\theta=\frac{1+\cos 2\theta}{2}, \sin^2\theta=\frac{1-\cos 2\theta}{2}$ $\tan^2\theta=\frac{1-\cos 2\theta}{1+\cos 2\theta}$
- $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$.
- $\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$
- $\sin 3\theta=3\sin\theta-4\sin^3\theta$, $\cos 3\theta=4\cos^3\theta-3\cos\theta$, $\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$
- $\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

  $\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

  $\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$
- $\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$.

  $\tan A+\tan B=\frac{\sin(A+B)}{\cos A\cos B}$

  $\tan A-\tan B=\frac{\sin(A-B)}{\cos A\cos B}$.

06 QUICK LOOK

For any three angles A, B, C

- $\sin(A+B+C)=\sin A\cos B\cos C+\sin B\cos C\cos A+\sin C\cos A\cos B+\sin A\sin B\sin C$
- cos (A + B +C) = cos A cos B cos, C- cos A sin B sin C – sin A cos B sin C – sin A sin B cos C.
- $\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$
- $\tan(A+B+C+D\ldots)=\frac{S_1-S_3+S_5-\ldots}{1-S_2+S_4-\ldots}$

  $S_1=\sum\tan A$, $S_2=\sum\tan A\tan B,\ldots$

- $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$
- $\tan(nA) = \dfrac{{}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - ....}{1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - {}^nC_6 \tan^6 A - ....}$
- $\sin(B+C) = \sin A, \cos B = -\cos(C+A)$ for Δ ABC
- $\cos(A+B) = -\cos C, \sin C = \sin(A+B)$ for Δ ABC
- $\tan(C+A) = -\tan B, \cot A = -\cot(B+C)$ for Δ ABC
- $\cos\dfrac{A+B}{2} = \sin\dfrac{C}{2}, \cos\dfrac{C}{2} = \sin\dfrac{A+B}{2}$ for Δ ABC
- $\sin\dfrac{C+A}{2} = \cos\dfrac{B}{2}, \sin\dfrac{A}{2} = \cos\dfrac{B+C}{2}$ for Δ ABC
- $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$ for Δ ABC
- $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$ for Δ ABC
- $\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$ for Δ ABC
- $\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$ for Δ ABC
- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ for Δ ABC
- $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$ for Δ ABC
- $\tan\dfrac{A}{2}\tan\dfrac{B}{2} + \tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} = 1$ for Δ ABC
- $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$ for Δ ABC
- $\sin 2mA + \sin 2mB + \sin 2mC = (-1)^{m+1} \cdot 4\sin mA \sin mB \sin mC$
- $\cos mA + \cos mB + \cos mC = 1 \pm 4\sin\dfrac{mA}{2}\sin\dfrac{mB}{2}\sin\dfrac{mC}{2}$.
  according as m is of the form 4n + 1 or 4n + 3.
- cos A + cos B + cos C + cos (A+B+C)= $4\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{B+C}{2}\right)\cos\left(\dfrac{C+A}{2}\right)$.
- sin A + sin B + sin C – sin (A+B+C) = $4\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{B+C}{2}\right)\sin\left(\dfrac{C+A}{2}\right)$

07 Trigonometrical Ratio of allied Angles

Equation of the type $a\cos\theta + b\sin\theta = c$

Case 1:

if $c \le \sqrt{a^2+b^2} \Rightarrow \theta = \alpha + 2n\pi \pm \cos^{-1}\left(\dfrac{c}{\sqrt{a^2+b^2}}\right)$

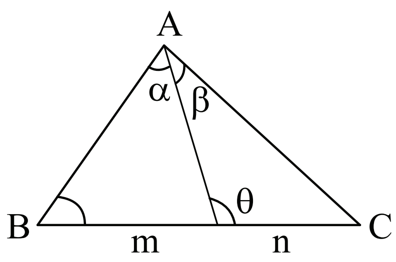
Case 2:

if $c > \sqrt{a^2+b^2}$

No solution

08 m-n Rule: in any triangle,

$(m+n)\cot\theta = m\cot\alpha - n\cot\beta. = n\cot B - m\cot C.$



09 Standard General Solutions of Trigonometrical Equations

- $\sin\theta = 0 \Leftrightarrow \theta = n\pi, \cos\theta = 0 \Leftrightarrow \theta = \left(2n\pi + \dfrac{\pi}{2}\right)$
- $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi + \alpha, \eta \in z$
- $\sin^2\theta = \sin^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha, \cos^2\theta = \cos^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha$
- $\cos\theta = \cos\alpha = \theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$
- $\cos\theta = 1 \Leftrightarrow \theta = 2n\pi, \cos = -1 \Leftrightarrow \theta = (2n+1)\pi$
- $\tan\theta = 0 \Leftrightarrow \theta = n\pi, \sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n\alpha$, where $\alpha \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
- $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha$, where $\alpha \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
- $\tan^2\theta = \tan^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha, \sin\theta = 1 \Leftrightarrow \theta = (4n+1)\dfrac{\pi}{2}$