# TOPICS TO BE COVERED

1. Modulus, Conjugate and Argument of Complex Number
2. Polar Forms of Complex Numbers
3. De-Moivre's Theorem
4. Cube Roots of Unity & $n^{th}$ Roots of Unity
5. Rotation of Complex Number
6. Complex Geometry
7. TYQ's- 2022

## Complex Number ( Introduction )

$Re(z) = x$

$Im(z) = y$

$z = 2 + 3i$

$$\boxed{z = x + iy}$$

- Purely real if $y = 0$ | Purely imaginary if $x = 0$ | Imaginary if $y \neq 0$

$z = 5 + 0i$

$z = 3i$

$\boxed{i = \sqrt{-1}}$ is called the imaginary unit. Also, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ etc. in general $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, where $n \in I$

$i = \sqrt{-1} = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

$i^6 = -1$

- Every real number is a complex with its imaginary part zero.
- $\boxed{0 + 0i}$ is both purely real as well as purely imaginary.

# 3 Important Terms for a C.N

(i) Conjugate of a complex number ($\bar{z}$)

(ii) Modulus of a complex number ($|z|$) ✓

(iii) Argument of a complex number ($\arg(z)$)

If $\boxed{z = x + iy}$

$\bar{z} = x - iy$

Ex→1 If $z = 3 - 4i$

$\bar{z} = 3 + 4i$

Ex→2 If $\boxed{z = 5i}$

$\bar{z} = -5i$

Ex→3 If $\boxed{z = 2}$

$\Rightarrow \bar{z} = 2$

$|z| = \sqrt{x^2 + y^2}$

Ex → $|3 - 4i| = \sqrt{3^2 + 4^2} = 5$

$\boxed{|z| \geq 0}$ Real non-ve.

## Properties of Conjugate

(1) If Z is purely real then $Z = \bar{Z}$

(2) If Z is purely imaginary then $\bar{Z} = -Z$

(3) $\overline{Z_1 + Z_2} = \bar{Z}_1 + \bar{Z}_2$

(4) $\overline{Z_1 - Z_2} = \bar{Z}_1 - \bar{Z}_2$

(5) $\overline{Z_1 Z_2} = (\bar{Z}_1)(\bar{Z}_2)$

(6) $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\bar{Z}_1}{\bar{Z}_2}\ (Z_2 \neq 0)$

(7) $(\overline{Z^n}) = (\bar{Z})^n$

# (8) $Z + \bar{Z} = 2Re(Z)$

# (9) $Z - \bar{Z} = 2iIm(Z)$

$z = x + iy$

$\oplus\ \bar{z} = x - iy$

$z + \bar{z} = 2\,Re(z)$.

$\ominus\ z - \bar{z} = 2iy = 2i\,Im(z)$

$z_1 = z_2 = z \Rightarrow (\overline{z^2}) = (\bar{z})(\bar{z})$.

$(\overline{z^2}) = (\bar{z})^2$.

# Properties of Modulus

$|x|^2 = x^2$ for Real no.

$|z|^2 = z^2$ (X)

$(x^2+y^2)$ $(x+iy)^2$

(i) $|Z| = |-Z| = |\bar{Z}| = |-\bar{Z}|$

## (ii) $Z\bar{Z} = |Z|^2$

If $|z| = 1 \Rightarrow z\bar{z} = 1$

$\boxed{\bar{z} = \frac{1}{z}}$

(iii) $|Z_1Z_2| = |Z_1||Z_2|$

(iv) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$

In general

$|z_1 \pm z_2| \neq |z_1| \pm |z_2|$

$z\bar{z}$

$= (x+iy)(x-iy)$

$x^2 - (iy)^2$

$= x^2 + y^2 = \text{LHS}$

(v) $|Z^n| = |Z|^n$

# (vi) $|Z_1 + Z_2|^2 = |Z_1|^2 + |Z_2|^2 + Z_1\bar{Z}_2 + Z_2\bar{Z}_1$

$= |Z_1|^2 + |Z_2|^2 + 2\text{Re}(Z_1\bar{Z}_2)$

(vii) $|Z_1 - Z_2|^2 = |Z_1|^2 + |Z_2|^2 - Z_1\bar{Z}_2 - Z_2\bar{Z}_1$

$= |Z_1|^2 + |Z_2|^2 - 2\text{Re}(Z_1\bar{Z}_2)$

Proof → (vi).

$$|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$

$$z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$$

$$|z_1|^2 + \underbrace{z_1\bar{z}_2}_{\alpha} + \underbrace{z_2\bar{z}_1}_{\bar{\alpha}} + |z_2|^2$$

$$= \text{RHS}$$

$$\alpha = z_1\bar{z}_2$$

$$\bar{\alpha} = \bar{z}_1 z_2$$

$$\alpha + \bar{\alpha} = 2\,\mathrm{Re}(z_1\bar{z}_2)$$

Q.1 If $z = 2 + 3i$, then $z^5 + (\bar{z})^5$ is equal to

$= 2\,\mathrm{Re}(z^5)$

[2022 Main, 29 July I]

[Ans. A]

(A) 244

(B) 224

(C) 245

(D) 265

$\boxed{(2+3i)^5}$

$$\mathrm{Re}\,z^5 = \underbrace{{}^5C_0\,(2)^5}_{} + \underbrace{{}^5C_1\,2^4(3i)}_{\text{Im.}} + \underbrace{{}^5C_2\,(2)^3(3i)^2}_{\text{Real}}$$

$$= {}^5C_0(2)^5 + {}^5C_2(2)^3(-9) + {}^5C_4(2)^1(3i)^4$$

$$= 32 + 10\times 8\times(-9) + 5\times 2\times 81$$

$$32 - 720 + 810$$

$$32 + 90$$

$$= 122$$

$2\times 122$

$244$

Q.2 The area of the polygon, whose vertices are the non-real roots of the equation $\bar{z} = iz^2$ is:

$x + iy \Rightarrow (x, y)$

[2022 Main, 27 June I]

[Ans. A]

(A) $\frac{3\sqrt{3}}{4}$

(B) $\frac{3\sqrt{3}}{2}$

(C) $\frac{3}{2}$

(D) $\frac{3}{4}$

Let $\boxed{z = x + iy}$ & $\bar{z} = x - iy$

$(x - iy) = i\,[x + iy]^2$

$x - iy = i\,[x^2 - y^2 + 2xy\,i]$

$(x - iy) = i\,(x^2 - y^2) - 2xy$

$x = -2xy$ & $-y = (x^2 - y^2)$ ~ (2)

$x = 0$ or $y = -\frac{1}{2}$

Case → 1 $x = 0$

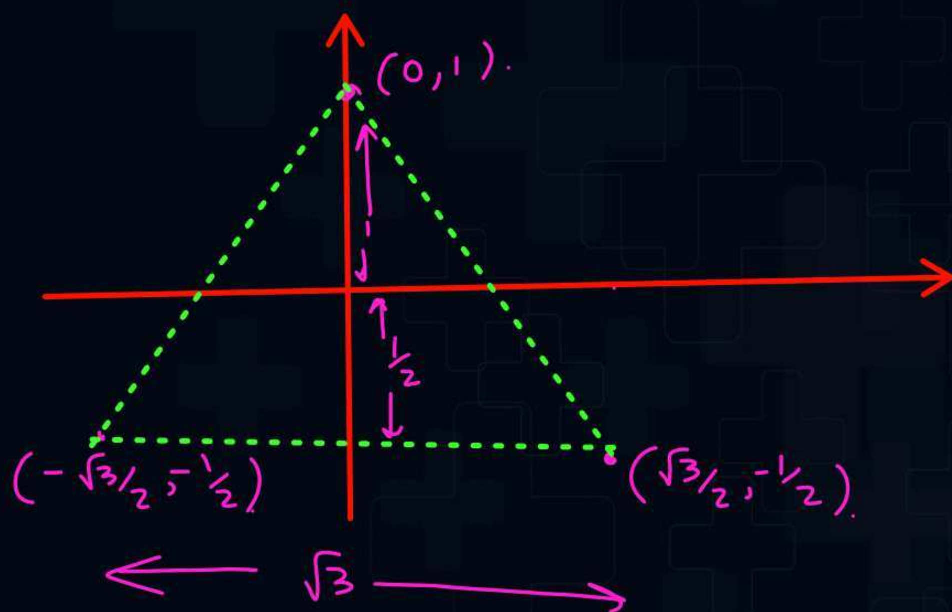
$\Rightarrow -y = -y^2$

$y = 0$ or $y = 1$

Case → 2 $y = -\frac{1}{2}$

$\Rightarrow \frac{1}{2} = x^2 - \frac{1}{4} \Rightarrow x^2 = 3/4 \Rightarrow x = \pm\frac{\sqrt{3}}{2}$

| $x$ | $y$ | |
|---|---|---|
| 0 | 0 | → Reject |
| 0 | 1 | $z = i$ |
| $+\sqrt{3}/2$ | $-1/2$ | $z = \sqrt{3}/2 - i/2$ |
| $-\sqrt{3}/2$ | $-1/2$ | $z = -\sqrt{3}/2 - i/2$ |




$$\text{Area} = \frac{1}{2}\left(\frac{3}{2}\right)\cdot\sqrt{3}$$

$$= \frac{3\sqrt{3}}{4} \quad \text{Ans}$$

Q.3 Sum of squares of modulus of all the complex numbers $z$ satisfying $\bar{z} = iz^2 + z^2 - z$ is equal to

$z = x + iy$.

[2022 Main, 28 June II]

[Ans. 2]

$z + \bar{z} = z^2(1+i)$

$2x = (x+iy)^2(1+i)$

$\frac{(1-i)}{(1-i)} \cdot \frac{2x}{1+i} = (x^2 - y^2 + 2xyi)$

$\frac{(1-i)\cancel{2}x}{\cancel{2}} =$

$x - ix = (x^2 - y^2) + 2xyi$

$x = x^2 - y^2 \rightarrow (1)$

$-\cancel{x} = 2\cancel{x}y$

$x = 0$ or $y = -\frac{1}{2}$

$x = 0 \Rightarrow y = 0$

$y = -\frac{1}{2}$.

$x = x^2 - (-\frac{1}{2})^2$.

$x = x^2 - \frac{1}{4}$.

$4x^2 - 1 = 4x$.

$4x^2 - 4x - 1 = 0 < \begin{matrix} x_1 \\ x_2 \end{matrix}$

$x_1 + x_2 = 1$, $x_1 x_2 = -\frac{1}{4}$.

$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2$

$= 1 + \frac{1}{2} = \frac{3}{2}$.

$|z|^2$

$z_1 = x_1 - \frac{i}{2}$

$z_2 = x_2 - \frac{i}{2}$

$z_3 = 0 + 0i$

$|z|^2 = x^2 + y^2$

Ans $= x_1^2 + \frac{1}{4} + x_2^2 + \frac{1}{4} + 0$

$= x_1^2 + x_2^2 + \frac{1}{2}$.

$= \frac{3}{2} + \frac{1}{2}$

Q.4 Let α and β be the roots of the equation $\underbrace{x^2 + (2i - 1)}$ $= 0$. Then, the value of $|\alpha^8 + \beta^8|$ is equal to:

[2022 Main, 29 June I]

[Ans. A]

(A) 50 ✓

(B) 250

(C) 1250

(D) 1500

$$\boxed{x^2 = 1 - 2i}$$

$$\alpha^2 = 1 - 2i \quad \& \quad \beta^2 = 1 - 2i$$

$$\alpha^8 = (1 - 2i)^4 = \beta^8$$

$$|\alpha^8 + \beta^8| = |2(1 - 2i)^4|$$

$$= 2\,|1 - 2i|^4$$

$$= 2(\sqrt{5})^4$$

$$= 2 \times 25 = 50.$$

Q.5 Let $n$ denote the number of solutions of the equation $\boxed{z^2 + 3\bar{z} = 0}$, where $z$ is a complex number. Then the value of $\sum_{k=0}^{\infty} \frac{1}{n^k}$ is equal to

[2021 Main, 22 July I]

[Ans. B]

(A) 1

(B) $\frac{4}{3}$ ✓

(C) $\frac{3}{4}$

(D) 2

check $z=0$ seperately.

$z=0$ satisfies this given eq$^n$ ✓

$z\bar{z} = |z|^2$

$$\boxed{\bar{z} = \frac{|z|^2}{z}}$$ ✓

$$\Rightarrow z^2 + 3\frac{|z|^2}{z} = 0$$

$$\boxed{z^3 + 3|z|^2 = 0}$$

3 sol$^n$

Total no. of sol$^n$ = $\boxed{4}$

$n = 4$

$$\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k = 1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots\infty$$

$$= \frac{a}{1-r} = \frac{1}{1-\frac{1}{4}} = \frac{1}{3/4} = 4/3.$$

Q.6 Let $\bar{z}$ denote the complex conjugate of a complex number z and let $i = \sqrt{-1}$. In the set of complex numbers, the number of distinct roots of the equation $\bar{z} - z^2 = i(\bar{z} + z^2)$ is ………… 4

[JEE Adv 2022]

$$\bar{z} - i\bar{z} = iz^2 + z^2.$$

$$\bar{z}(1-i) = z^2(1+i).$$

$$\bar{z} = z^2 \frac{(1+i)}{(1-i)}$$

$z = 0$ Satisfies ✓ or $\bar{z} = \frac{|z|^2}{z}$.

$$\frac{|z|^2}{z} = z^2 \frac{(1+i)}{1-i}$$

3 Solⁿ

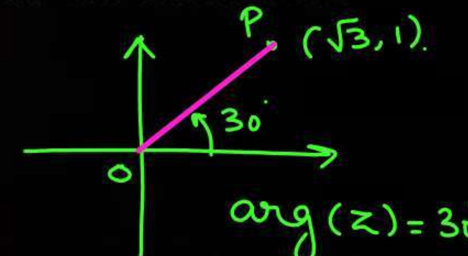
# Argument of a Complex Number

Ex → 1
$Z = \sqrt{3} + i$
$\tan^{-1}(y/x) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$

**Definition :**

The angle made by the line joining origin to the point representing complex number in the argand plane with positive direction of X axis in any direction is called Argument of a Complex Number.

It is represented by arg (Z).



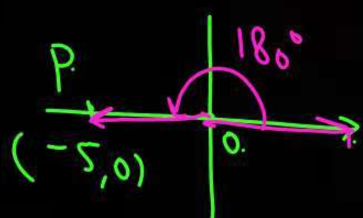

$\arg(z) = 30^\circ$

**Principal Argument:**

The value of θ such that $\boxed{-\pi < \theta \leq \pi}$ is called Principal argument.

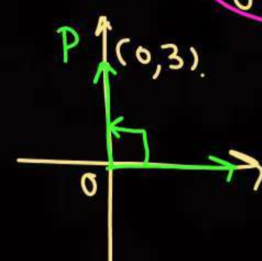
It is represented by amp (Z).

$Z = -5$



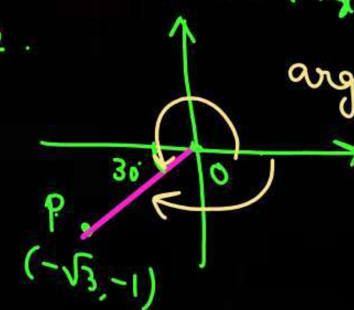

$\arg(z) = \pi$

Ex → 3 $y = 3i$



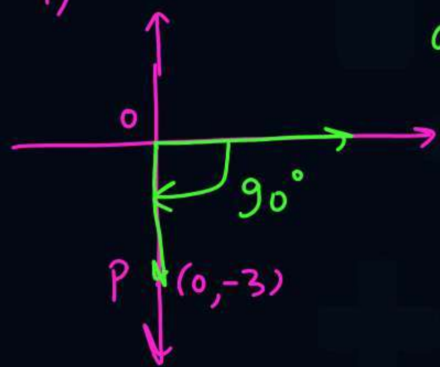

$\arg(z) = \pi/2$

$Z = -\sqrt{3} - i$
$\tan^{-1}(y/x) = \tan^{-1}\frac{1}{\sqrt{3}} = 30^\circ$

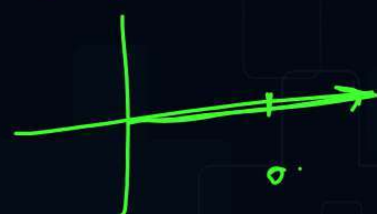



$\arg(z) = 210^\circ$
or
$\boxed{\arg(z) = -150^\circ}$

4) $Z = -3i$.




$\arg(z) = -90°$


$\log(ab) = \log a + \log b$

$\log(a/b) =$ ✓

$\log(a)^b =$

# Properties of Argument of a Complex Number

(i) $\boxed{\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)}$

In general $\arg(z_1 z_2 z_3 \dots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) \dots + \arg(z_n)$

(ii) $\boxed{\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2}$

(iii) $\arg(iy) = \boxed{\frac{\pi}{2}}$ if $y > 0$

$= \boxed{-\frac{\pi}{2}}$ if $y < 0$

✓ (iv) $\boxed{\arg(\bar{z}) = -\arg(z)} = \arg\left(\frac{1}{z}\right)$

(v) $\boxed{\arg(z)^n = n \arg(z)}$

(iv) $\arg(\bar{z})$

Proof. $\arg\left(\frac{|z|^2}{z}\right)$

$= \arg |z|^2 - \arg z$

$\underbrace{\arg |z|^2}_{0} \; (+) \quad - \arg z$

$= -\arg(z)$.

$z = 1 + i$

$\arg z = \pi/4$

$\bar{z} = 1 - i$

$\arg \bar{z} = -\pi/4$

Q.7 Let $z_1$ and $z_2$ be two complex numbers such that $\boxed{\bar{z}_1 = i\bar{z}_2}$ and $\boxed{\arg\left(\frac{z_1}{\bar{z}_2}\right)} = \pi$. Then

[2022 Main, 25 June II]

[Ans. C]

(A) $\arg z_2 = \frac{\pi}{4}$

(B) $\arg z_2 = -\frac{3\pi}{4}$

(C) $\arg z_1 = \frac{\pi}{4}$ ✓

(D) $\arg z_1 = -\frac{3\pi}{4}$

$\arg \bar{z}_1 = \underbrace{\arg i}_{\pi/2} + \arg \bar{z}_2$

$\boxed{\arg z_1 - \arg \bar{z}_2 = \pi}$ ~ 2

$\underbrace{\arg \bar{z}_1} = \pi/2 + \arg \bar{z}_2$

$\boxed{-\arg z_1 = \pi/2 + \arg \bar{z}_2}$ ~ 1

1 + 2

$-\arg \bar{z}_2 = 3\pi/2 + \arg \bar{z}_2$.

$\boxed{\arg \bar{z}_2 = -3\pi/4} \Rightarrow \boxed{\arg z_2 = 3\pi/4}$.

$-\arg z_1 = \pi/2 - 3\pi/4$.

$-\arg z_1 = -\pi/4$

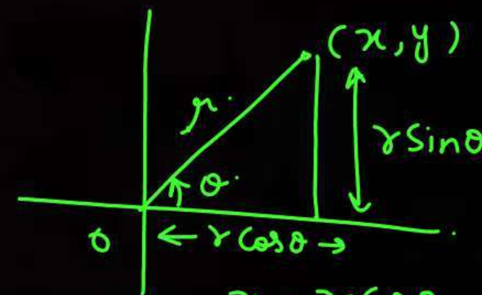
$\arg z_1 = \pi/4$.

# Polar Forms of Complex Numbers

$(r, \theta)$

## Polar Form of Complex Number:

The form of complex number in terms of $(r, \theta)$ where $\boxed{|z| = r;}$ $\boxed{\arg z = \theta}$ is known as Polar Form.




There are two types of Polar Forms.

Trigonometric Form and Exponential Form.

$\boxed{z = x + iy}$

$x = r\cos\theta, \; y = r\sin\theta$

### 1. Trigonometric Form

$\boxed{z = r(\cos\theta + i\sin\theta)}$ is known as Trigonometric Form of complex number.

In short $z = r(\cos\theta + i\sin\theta)$ can also be written as $\boxed{z = r\,\mathrm{CiS}\,\theta}$ where $|z| = r;\ \arg z = \theta$

# Polar Forms of Complex Numbers

**2. Exponential Form:** ( Euler's Form)

Let $z$ be a complex number then the exponential form of complex number is given by $\boxed{z = r \cdot e^{i\theta}}$

Proof

$\boxed{z = 1 + i}$

$r = \sqrt{2}$

$\theta = \pi/4$

$\boxed{z = \sqrt{2} e^{i\pi/4}}$

OR

$z = \sqrt{2}\left(\text{Cis } \pi/4\right)$

## Some Common Polar Forms of C.N

1) $z = 1 \Rightarrow \boxed{r = 1 ,\ \theta = 0} \Rightarrow 1 \cdot e^{i0}$.

# 2) $z = i \Rightarrow r = 1 ,\ \theta = \pi/2 \Rightarrow z = 1 \cdot e^{i\pi/2} = \boxed{e^{i\pi/2} = i}$.

$z = -i \Rightarrow$ $\boxed{e^{-i\pi/2} = -i}$

$z = -1 \Rightarrow r = 1 ,\ \theta = \pi \Rightarrow z = 1 \cdot e^{i\pi}$.

In general: $\cancel{r}(\cos\theta + i\sin\theta) = \cancel{r}e^{i\theta}$.

# $\boxed{\cos\theta + i\sin\theta = e^{i\theta}} \leftarrow$

$\theta \to -\theta$.

$\boxed{\cos\theta - i\sin\theta = e^{-i\theta}}$

Q.8

The value of $\left(\frac{-1+i\sqrt{3}}{1-i}\right)^{30}$ is :

[Main Sep. 05,2020 (II)]

(A) $-2^{15}$

(B) $2^{15}i$

(C) $-2^{15}i$

(D) $6^5$

$-1+\sqrt{3}i = 2e^{i2\pi/3} = 2e^{i2\pi/3}$

$1-i = \sqrt{2}e^{-i\pi/4}$

$$\frac{\left(2e^{i2\pi/3}\right)^{30}}{\left(\sqrt{2}\,e^{-i\pi/4}\right)^{30}}$$

$$\frac{(\sqrt{2})^{30}\, e^{i\,2\pi/3\times 30}}{e^{-i\pi/4\times 30}}$$

$$(2)^{15}\frac{e^{i20\pi}}{e^{-i15\pi/2}} = (2)^{15}\cdot e^{i\cancel{20\pi}+15\pi/2} = (2)^{15}\cdot e^{i15\pi/2} = (2)^{15} e^{i15\pi/2-6\pi} = (2)^{15} e^{i3\pi/2}$$

$e^{i3\pi/2} = -i$

$$e^{i3\pi/2} = \underbrace{\cos 3\pi/2}\, + i\sin 3\pi/2$$

$= 0 + -i$

$= -i$

Q.9 Let $\alpha$ and $\beta$ be two roots of the equation $\boxed{x^2 + 2x + 2 = 0}$, then $\alpha^{15} + \beta^{15}$ is equal to:

$$x = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

[JEE Mains 2019]

(C)

$\alpha = -1 + i$, $\beta = -1 - i = \bar{\alpha}$

(A) 512

(B) –512

(C) –256 ✓

(D) 256

$\alpha = \sqrt{2}\, e^{i3\pi/4}$

$\alpha^{15} = (\sqrt{2})^{15}\, e^{i\frac{3\pi}{4} \times 15}$

$\alpha^{15} = 2^7 \cdot \sqrt{2} \cdot e^{i\left(\frac{45\pi}{4} - 10\pi\right)}$

$2^7 \sqrt{2}\, e^{i5\pi/4}$

$\alpha^{15} = \underbrace{2^7 \sqrt{2} \left[\cos\right.}\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\left.\right]$

$Re(\alpha^{15}) = 2^7 \sqrt{2} \cdot \left(-\frac{1}{\sqrt{2}}\right) = -128$

$\alpha^{15} + (\bar{\alpha})^{15}$

$Ans = 2Re(\alpha^{15})$

$= 2(-128)$

$= -256$

## Demoivre's Theorem

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

LHS: $(e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta = \text{RHS}$

# $\cos(n\theta) + i\sin(n\theta)$ is the only value of $(\cos\theta + i\sin\theta)^n$ if '$n$' is integer.

Ex → $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ only.

$\cos(n\theta) + i\sin(n\theta)$ is one of the values of $(\cos\theta + i\sin\theta)^n$ if $n$ is a non integral rational number

$$(\cos\theta + i\sin\theta)^{2/5} = \underbrace{\cos\frac{2\theta}{5} + i\sin\frac{2\theta}{5}}_{\text{one of the values.}}$$

Q.10 If $(\sqrt{3}+i)^{100} = 2^{99}(p+iq)$ then $p$ and $q$ are roots of the equation

[JEE Mains (Aug) 2021]

(A)

(A) $x^2-(\sqrt{3}-1)x-\sqrt{3}=0$ ✓

(B) $x^2-(\sqrt{3}+1)x+\sqrt{3}=0$

(C) $x^2+(\sqrt{3}-1)x-\sqrt{3}=0$

(D) $x^2-(\sqrt{3}+1)x+\sqrt{3}=0$

$\sqrt{3}+i = 2\left(\cos \pi/6 + i\sin \pi/6\right)$.

$(\sqrt{3}+i)^{100} = 2^{100}\left[\cos \pi/6 + i\sin \pi/6\right]^{100}$

$2^{100}\left[\cos \frac{100\pi}{6} + i\sin \frac{100\pi}{6}\right]$.

$2^{100}\left[\cos 50\pi/3 + i\sin 50\pi/3\right]$.

$2^{100}\left[\cos 2\pi/3 + i\sin 2\pi/3\right]$ $\quad 50\pi/3 - 16\pi$

$2^{100}\left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$. $\quad 2\pi/3$.

$= 2^{99}\left[-1+\sqrt{3}i\right]$

$p=-1,\ q=\sqrt{3}$.

Sum of roots $= (\sqrt{3}-1)$

Product $= -\sqrt{3}$

$x^2-(\text{Sum})x+\text{Pdt}=0$

11. Let $i=\sqrt{-1}$. If $\boxed{\frac{(-1+i\sqrt{3})^{21}}{(1-i)^{24}}+\frac{(1+i\sqrt{3})^{21}}{(1+i)^{24}}=k}$ and $n=[|k|]$ be the greatest $\rightarrow 0.\quad n=0$

integral part of $|k|$. Then $\sum_{j=0}^{n+5}(j+5)^2-\sum_{j=0}^{n+5}(j+5)$ is equal to _____.

$\sum_{j=0}^{5}(j+5)^2-\sum_{j=0}^{5}(j+5)=?$

[JEE Mains (Feb) 2021]

[Ans. 310]

$$z=\frac{(1+\sqrt{3}i)^{21}}{(1+i)^{24}} \Rightarrow \bar{z}=\frac{(1-\sqrt{3}i)^{21}}{(1-i)^{24}} \Rightarrow \boxed{(z-\bar{z})=2i\,\underbrace{\mathrm{Im}(z)}_{\downarrow\, 0}}$$

$$\Rightarrow z=\frac{\left\{2\left[\cos\pi/3+i\sin\pi/3\right]\right\}^{21}}{\left[\sqrt{2}\left(\cos\pi/4+i\sin\pi/4\right)\right]^{24}}=\frac{2^{21}(\cos 7\pi+i\sin 7\pi)}{2^{12}(\cos 6\pi+i\sin 6\pi)}$$

$\Rightarrow k=0.$

$6\pi=0$

$$=\frac{2^9\cdot(-1)}{1}=-2^9=\text{Purely Real.}$$

$$\sum_{j=0}^{5}(j+5)^2 - \sum_{j=0}^{5}(j+5)$$

$$\left[5^2+6^2+\ldots 10^2\right] - \left[5+6+7+\ldots 10\right]$$

$$\sum_{r=1}^{10} r^2 - (1^2+2^2+3^2+4^2) - \frac{6}{2}[5+10]$$

$$\frac{10\cdot 11\times 21}{6} - (30) - 45$$

$$5\times 11\times 7 - 75$$

$$= \quad 385 - 75 = 310 \text{ Ans}$$

# Cube Roots of Unity

$\checkmark\ Z = (1)^{1/3}$

$Z = (\cos 0 + i\sin 0)^{1/3}$

$Z = (\cos 2n\pi + i\sin 2n\pi)^{1/3}$

$Z = \left(\cos \frac{2n\pi}{3} + i\sin \frac{2n\pi}{3}\right)$

$n = 0 \Rightarrow z_1 = \cos 0 + i\sin 0 = 1 \quad = 1$

$n = 1 \Rightarrow z_2 = \cos 2\pi/3 + i\sin 2\pi/3 = -\frac{1}{2} + \frac{\sqrt{3}i}{2} = \omega$

$n = 2 \Rightarrow z_3 = \text{cis}(4\pi/3) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = \omega^2.$

$\omega = (1)^{1/3}$

$\underline{\underline{\omega^3 = 1}}$

$1, \omega, \omega^2$ are 3 cube roots of Unity.

$$\boxed{\boxed{\omega^3 = 1} \quad \boxed{1 + \omega + \omega^2 = 0}}$$

$\omega^4 = \omega$

$\omega^5 = \omega^2.$

$\omega^6 = 1$

In general

$$\boxed{\begin{aligned} \omega^{3n} &= 1 \checkmark \\ \omega^{3n+1} &= \omega. \\ \omega^{3n+2} &= \omega^2. \end{aligned}}$$

12 If $z^2 + z + 1 = 0$, $z \in C$, then $\left|\sum_{n=1}^{15}\left(Z^n + (-1)^n \frac{1}{Z^n}\right)^2\right|$ is equal to ____.

$\frac{-1 \pm \sqrt{3}i}{2} = \omega, \omega^2$

$|-2| = 2$ Ans.

[2022 Main, 26 June II]

[Ans. 2]

$\boxed{\omega^{30} = 1}$

$z = \omega$

$\left(\omega^n + (-1)^n \cdot \frac{1}{\omega^n}\right)^2 = \omega^{2n} + \frac{1}{\omega^{2n}} + 2(-1)^n$

$\sum_{n=1}^{15} \omega^{2n} + \sum \frac{1}{\omega^{2n}} + \sum_{n=1}^{15} 2(-1)^n \rightarrow -\cancel{2} + \cancel{2} - \cancel{2} + \cancel{2} \ldots$

$-2$

$-2 \longrightarrow 8$

$2 \longrightarrow 7$

$\left[\omega^2 + \omega^4 + \omega^6 + \ldots \omega^{30}\right]$

$\omega^2 \frac{\left[(\omega^2)^{15} - 1\right]}{\omega^2 - 1}$

$= \omega^2 \frac{\left[\omega^{30} - 1\right]}{\omega^2 - 1}$

$\frac{1}{\omega^2} + \frac{1}{\omega^4} + \frac{1}{\omega^6} + \ldots \frac{1}{\omega^{30}}$

$\frac{1}{\omega^2} \frac{\left[1 - \left(\frac{1}{\omega^2}\right)^{15}\right]}{1 - \frac{1}{\omega^2}} = (\omega^{30} - 1)$

$\frac{\omega}{\omega} \cdot \frac{1}{\omega^2} = \omega$

0

13. Let $z = \frac{1-i\sqrt{3}}{2}$, $i = \sqrt{-1}$. Then the value of

$$21 + \left(z+\frac{1}{z}\right)^3 + \left(z^2+\frac{1}{z^2}\right)^3 + \left(z^3+\frac{1}{z^3}\right)^3 + \ldots\ldots + \left(z^{21}+\frac{1}{z^{21}}\right)^3$$ is

[2021(26 Aug Shift 1)]

[Ans. 13]

$\underbrace{1+\omega}+\omega^2 = 0 \Rightarrow -1-\omega^2 = \omega$

$z = -\omega$

$\left(-\omega-\frac{1}{\omega}\right)^3$, $\left(\frac{-\omega^2-1}{\omega}\right)^3$, $\left(\frac{\omega}{\omega}\right)^3 = 1$

$\left(\omega^2+\frac{1}{\omega^2}\right)^3$, $\left(\frac{\omega^4+1}{\omega^2}\right)^3$, $\left(\frac{\omega+1}{\omega^2}\right)^3$, $\left(\frac{-\cancel{\omega^2}}{\cancel{\omega^2}}\right)^3 = -1$

$\left(-\omega^3+\frac{1}{-\omega^3}\right)^3$, $(-1-1)^3 = (-2)^3 = -8$

$\left(z^4+\frac{1}{z^4}\right)^3$, $\left(\omega^4+\frac{1}{\omega^4}\right)^3$, $\left(\omega+\frac{1}{\omega}\right)^3$, $\left(\frac{\omega^2+1}{\omega}\right)^3$, $\left(\frac{-\omega}{\omega}\right)^3 = -1$

$\left(z^5+\frac{1}{z^5}\right)^3 \rightarrow 1$

$\left(\omega^6+\frac{1}{\omega^6}\right)^3 = 8$

$-\cancel{8}+\cancel{8}-\cancel{8}+\cancel{8}-\cancel{8}+\cancel{8}-8$

$= -8$

$21-8 = 13$

# $n^{th}$ Roots of Unity

$$\boxed{z = (1)^{1/n}}$$

$$\alpha_3 = e^{i6\pi/n} = (\alpha_1)^3.$$

$$z = (\cos 2m\pi + i\sin 2m\pi)^{1/n}.$$

$$z = \left(\cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}\right)$$

$m = 0 \Rightarrow z_1 = 1$

$m = 1 \Rightarrow z_2 = \text{cis}\, 2\pi/n = e^{i2\pi/n} = \alpha_1$

$m = 2 \Rightarrow z_3 = \text{cis}\, 4\pi/n = e^{i4\pi/n} = \alpha_2 = (\alpha_1)^2$

$\vdots$

$m = n-1 \Rightarrow z_n = \text{cis}\,\frac{2(n-1)\pi}{n} = e^{i2(n-1)\pi/n} = \alpha_{n-1} = (\alpha_1)^{n-1}$

$$\boxed{1, \alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_{n-1}}$$ are the n nth roots of Unity.

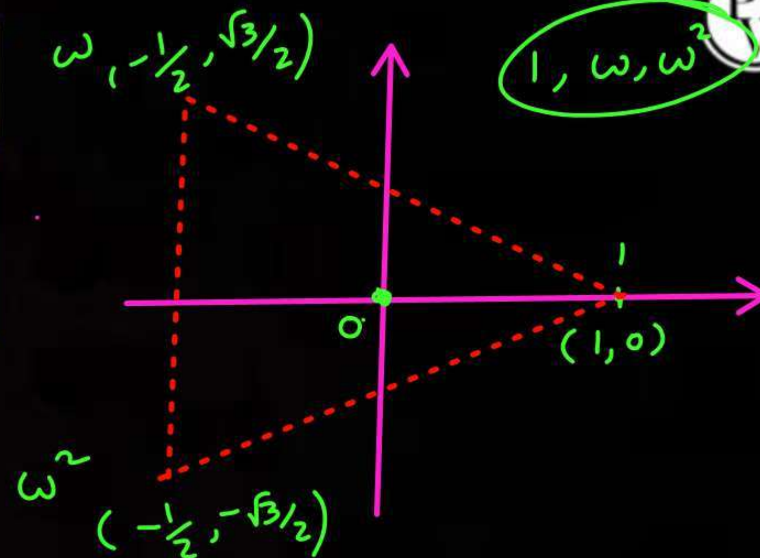
$\alpha_2 \to (\alpha_1)^2$, $\alpha_3 \to (\alpha_1)^3$, $\alpha_{n-1} \to (\alpha_1)^{n-1}$

# Properties of $n^{th}$ Roots of Unity

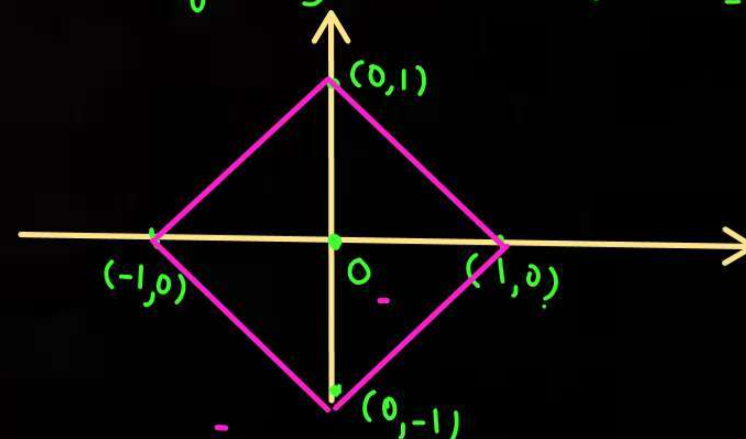
**(P-1)** Then $n^{\text{th}}$ roots of unity are in G.P. with common ratio $e^{i(2\pi/n)} = \alpha_1$

**(P-2)** If $\alpha$ *is any* $\text{n}^{\text{th}}$ root of unity then $\boxed{\alpha^n = 1}$

**(P-3)** The points represented by $n$, $n^{\text{th}}$ roots of unity are located at the vertices of a regular polygon of $n$ sides inscribed in a unit circle having center at origin, one vertex being on positive real axis.




$4^{th}$ roots of Unity $\Rightarrow z = \pm 1$, $z = \pm i$



## Properties of $n^{th}$ Roots of Unity

$z = (1)^{1/n}$

$\Rightarrow z^n = 1$

$\boxed{z^n - 1 = 0}$ → $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$

Sum = 0

**(P-4)** The sum of the "n" $n^{th}$ **roots** of unity is always **zero**.

**(P-5)** The product of the "n" $n^{th}$ **roots** of unity is $-1$ if $n$ is even and 1 if $n$ is odd.

**(P-6)** $1^P + \alpha_1^P + \alpha_2^P + \cdots + \alpha_{n-1}^P = 0$ if $p$ is not an integral multiple of $n$

$= n$ if $p$ is an integral multiple of $n$

$4^{th}$ roots → $1, -1, i, -i$ $\quad 1 \times \omega \times \omega^2 = 1$

$(-1)(-i^2) = -1$

# Properties of $n^{th}$ Roots of Unity

**(P-7)** $(1-\alpha_1)(1-\alpha_2)\ldots\ldots(1-\alpha_{n-1}) = \mathrm{n}$

#.

14. If $\alpha$ is the $n^{\text{th}}$ root of unity, then $1+2\alpha+3\alpha^2+4\alpha^3+\cdots$ upto $n$ terms is equal to $\boxed{\alpha^n = 1}$

(A) $\dfrac{-n}{(1-\alpha)^2}$

(B) $\dfrac{-n}{1-\alpha}$

(C) $\dfrac{-2n}{1-\alpha}$

(D) $\dfrac{-2n}{(1-\alpha)^2}$

$$S = 1 + 2\alpha + 3\alpha^2 + 4\alpha^3 + \cdots + n\alpha^{n-1}$$

$$\alpha S = \alpha + 2\alpha^2 + 3\alpha^3 + \cdots + n\alpha^n$$

$$S - \alpha S = (\underbrace{1 + \alpha + \alpha^2 + \alpha^3 + \cdots}_{\text{GP}} \alpha^{n-1}) - n\alpha^n$$

$$S(1-\alpha) = \frac{1(\alpha^n - 1)}{\alpha - 1} - n\alpha^n \quad \alpha^n \to 1$$

$$S(1-\alpha) = -n$$

$$S = -\frac{n}{1-\alpha}$$

14. If $\alpha$ is the $n^{\text{th}}$ root of unity, then $1+2\alpha+3\alpha^2+4\alpha^3+\cdots$ upto $n$ terms is equal to $\boxed{\alpha^n = 1}$

(A) $\dfrac{-n}{(1-\alpha)^2}$

(B) $\dfrac{-n}{1-\alpha}$

(C) $\dfrac{-2n}{1-\alpha}$

(D) $\dfrac{-2n}{(1-\alpha)^2}$

$S = 1 + 2\alpha + 3\alpha^2 + 4\alpha^3 + \cdots + n\alpha^{n-1}$

Consider: $x + x^2 + x^3 + \cdots + x^n = x\left(\dfrac{x^n - 1}{x - 1}\right)$

diff w.r.t. $x$.

& put $x = \alpha$.

16. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $\underbrace{x^4 + x^3 + x^2 + x + 1}_{\text{sum of roots} = -1} = 0$, then $\alpha^{2021} + \beta^{2021} + \gamma^{2021} + \delta^{2021}$ is equal to:

$\alpha + \beta + \gamma + \delta$

$\boxed{\alpha^5 = 1}$

[2022 Main, 25 July I]

[Ans. B]

(A) –4

(B) –1

(C) 1

(D) 4

$\alpha^{2021} = \alpha^{2020} \cdot \alpha^1 = \alpha$

$1 + x + x^2 + x^3 + x^4$

$n = 5$

$a = 1, \quad r = x$

$\dfrac{1[x^5 - 1]}{x - 1} = 0$

$\Rightarrow x^5 - 1 = 0$

$x^5 = 1$

$x = (1)^{1/5}$

$\boxed{1 + \alpha + \beta + \gamma + \delta = 0}$

$\alpha + \beta + \gamma + \delta = -1$

*Brain Teaser*

Q. If '$w$' is a non real root of the equation $z^{28} = 1$ such that $|w + 1|$ is maximum. If $x = \frac{1}{2}\left|w - \frac{1}{w}\right|$ then find the value of $\boxed{8x^4 + 4x^3 - 8x^2 - 3x + 4.}$

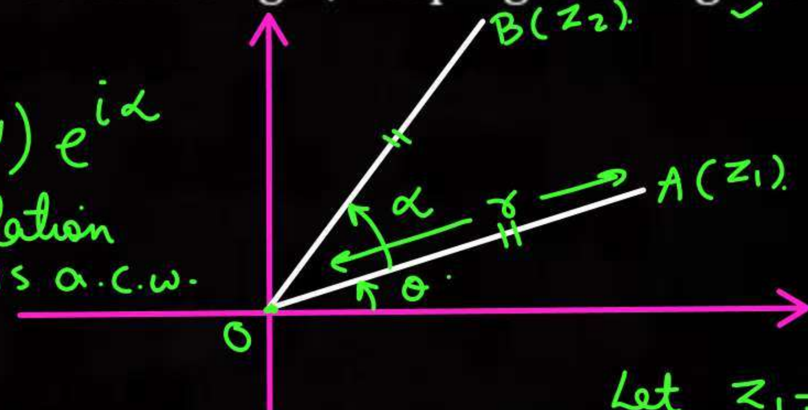
[Ans. 3]

# Rotation of Complex Numbers

**Case 1:** When Rotation is done about origin, keeping the magnitude same.

$$(\text{Final C.N}) = (\text{Initial C.N})\, e^{i\alpha}$$

if rotation is a.c.w.




Let $z_1 = re^{i\theta}$

$\Rightarrow z_2 = re^{i(\theta+\alpha)}$

$z_2 = re^{i\theta} e^{i\alpha}$

$\Rightarrow z_2 = z_1 e^{i\alpha}$

## Note that

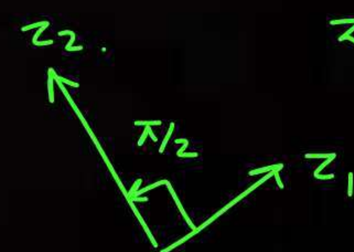
1. **(Final C.N) = (Initial CN)** $e^{i\theta}$ if rotation is anticlockwise.

2. **(Final C.N) = (Initial CN)** $e^{-i\theta}$ if rotation is clockwise.

$$e^{i\pi/2} = \cos\pi/2 + i\sin\frac{\pi}{2} = 0 + i$$

$$e^{i\pi/2} = i$$

\# 3. If a complex number is multiplied with $i$ then it gets rotated by $90^0$ in anti clock wise direction.




$$z_2 = z_1 e^{i\pi/2} \to i$$

$$\boxed{z_2 = i z_1}$$

# Rotation of Complex Numbers

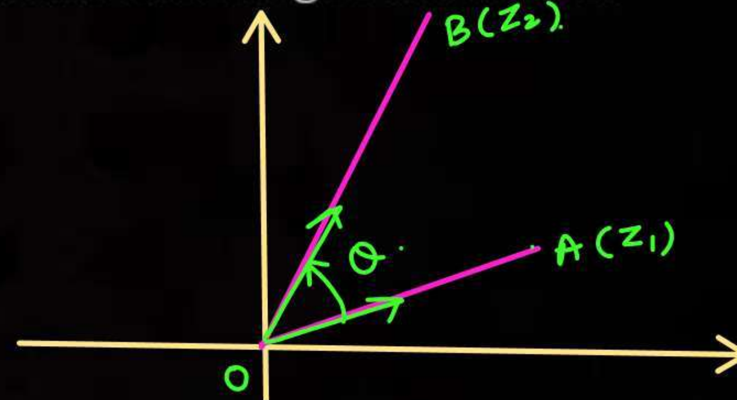
**Case 2:** When Rotation is done about origin and the magnitude is also changed.

$$\hat{OA} \longrightarrow \hat{OB}$$

$$\hat{OB} = \hat{OA}\, e^{i\theta}$$

$$\frac{z_2}{|z_2|} = \frac{z_1}{|z_1|}\, e^{i\theta}$$

$$\boxed{z_2 = \frac{|z_2|}{|z_1|}\, z_1 e^{i\theta}}$$



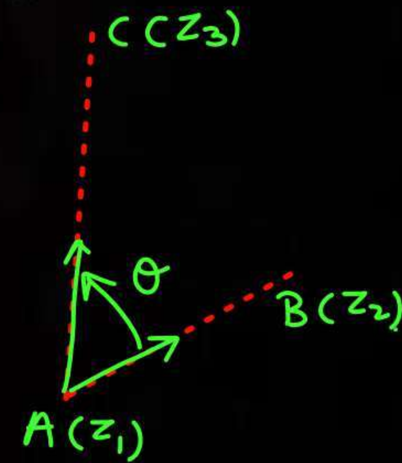

$$\hat{OA} = \frac{z_1}{|z_1|}$$

$$\hat{OB} = \frac{z_2}{|z_2|}$$

# Rotation of Complex Numbers

**Case 3:** When rotation is done about any general point and magnitude is also changed.

In this case also, we first find unit vectors along the two vectors $AB$ and $AC$ and then rotate them using Case 2.




$$\vec{AB} = z_2 - z_1 \Rightarrow \hat{AB} = \frac{z_2 - z_1}{|z_2 - z_1|}.$$

$$\hat{AC} = \frac{z_3 - z_1}{|z_3 - z_1|}.$$

$$\hat{AC} = \hat{AB}\, e^{i\theta}.$$

$$\boxed{\frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_2 - z_1}{|z_2 - z_1|}\, e^{i\theta}}\ \checkmark$$

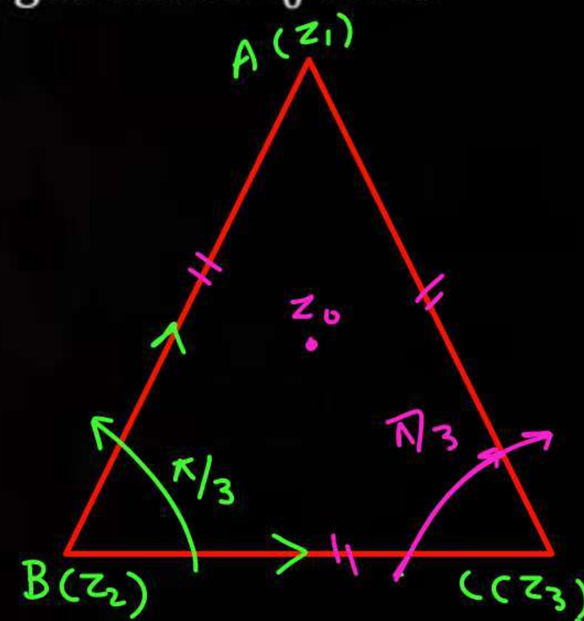
## Note that

(ii) $3z_0 = z_1 + z_2 + z_3$

S.B.S.

If $z_1, z_2, z_3$ represent the vertices of an equilateral triangle where $z_0$ is its circum-centre then

1. $\boxed{z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1}$ ✓
2. $\boxed{z_1^2 + z_2^2 + z_3^2 = 3z_0^2}$ ✓




$\overrightarrow{BA} = \overrightarrow{BC}\, e^{i\pi/3}.$

$(z_1 - z_2) = (z_3 - z_2)e^{i\pi/3}$ —— ①

$\overrightarrow{CA} = \overrightarrow{CB}\, e^{-i\pi/3}.$

$(z_1 - z_3) = (z_2 - z_3)e^{-i\pi/3}$ —— ②

① × ②

$\Rightarrow (z_1 - z_2)(z_1 - z_3) = (z_3 - z_2)(z_2 - z_3)$

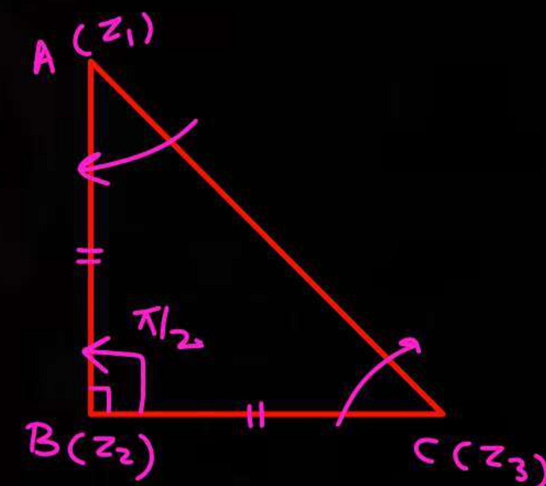
$z_1^2 - z_1z_3 - z_1z_2 + z_2z_3 = -(z_3^2 + z_2^2 - 2z_3z_2)$

## Note that

$$\underbrace{e^{i\pi/2} = i}$$

If Complex numbers $z_1, z_2, z_3$ respectively represent the vertices of an isosceles right angled triangle $ABC$ with right angle at $B$ then

$z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$ ✓




$$(z_1 - z_2) = (z_3 - z_2)\, e^{i\pi/2}$$

$$(z_1 - z_2) = (z_3 - z_2)\, i$$

S.B.S.

$$(z_1 - z_2)^2 = -(z_3 - z_2)^2.$$

$$z_1^2 + z_2^2 - 2z_1z_2 = -z_3^2 - z_2^2 + 2z_2z_3.$$

$$z_1^2 + 2z_2^2 + z_3^2 = 2z_1z_2 + 2z_2z_3.$$

$e^{i\theta}$

17. Let $z_1$, $z_2$ be the roots of the equation $\boxed{z^2 + az + 12 = 0}$ and $z_1$, $z_2$ form an equilateral triangle with origin. Then, the value of $|a|$ is

[2021 Main, 18 March I]

[Ans. 6]

$$z_1^2 + z_2^2 + \cancel{z_3^2} = z_1 z_2 + \cancel{z_1 z_3} + \cancel{z_2 z_3}$$

$z_3 = 0$

$$z_1^2 + z_2^2 = z_1 z_2$$

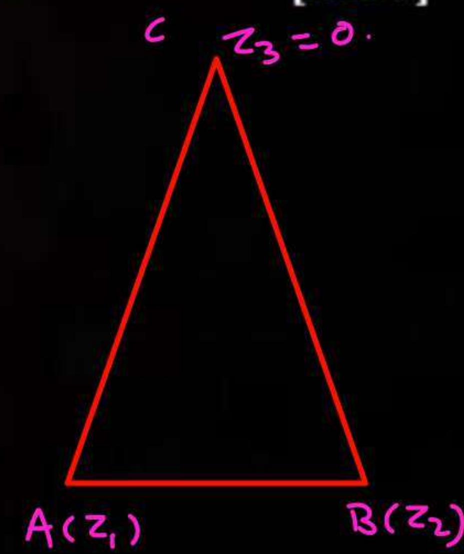
$$(z_1 + z_2)^2 = 3 z_1 z_2.$$

$$(-a)^2 = 3.(12)$$

$$a^2 = 36.$$

$$\boxed{a = \pm 6}$$

$$|a| = 6.$$



$\alpha < 0$

18. Let $O$ be the origin and A be the point $z_1 = 1 + 2i$. If $B$ is the point $z_2$, $\text{Re}(z_2) < 0$, such that $OAB$ is a right angled isosceles triangle with $OB$ as hypotenuse, then which of the following is NOT true?

[2022 Main, 26 July I]

[Ans. D]

(A) $\arg z_2 = \pi - \tan^{-1}3$

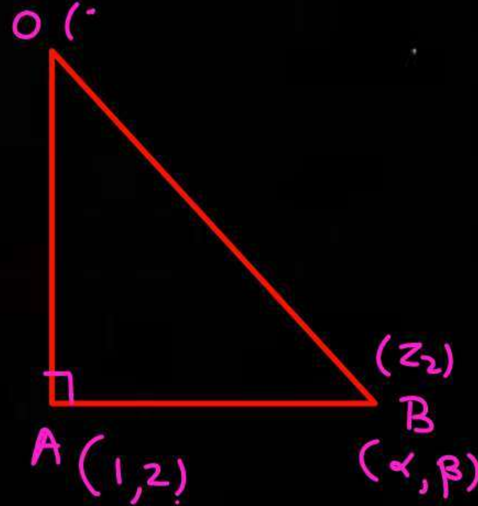
(B) $\arg(z_1 - 2z_2) = -\tan^{-1}(4/3)$

(C) $|z_2| = \sqrt{10}$

(D) $|2z_1 - z_2| = 5$ ✓

$\cancel{z_1^2} + 2z_2^2 + z_3^2 = 2z_2(\cancel{z_1} + z_3)$

$z_1 = 0, \quad z_2 = 1 + 2i, \quad z_3 = z_2$




$2(1+2i)^2 + z_2^2 = 2(1+2i)z_2$

$2[1-4+4i] + z_2^2 = 2(1+2i)z_2$

$2[-3+4i] + z_2^2 = 2(1+2i)z_2$

$z_2^2 - 2(1+2i)z_2 + 2(-3+4i) = 0$

$D = 4(1+2i)^2 - 4\times 2(-3+4i)$

$= 4(-3+4i) - 4\times 2(-3+4i) = 4(-3+4i)(-1) = 2^2(1+2i)^2(i)^2$

$\boxed{\sqrt{D} = 2(1+2i)i}$

$$z_2 = \frac{\cancel{2}(1+2i) \pm \cancel{2}(1+2i)i}{\cancel{2}}$$

$$(1+2i) \pm (i-2).$$

$$1+2i+i-2 \quad \text{or} \quad 1+2i-i+2.$$

$$(-1+3i) \quad \text{or} \quad (3+i).$$

$\checkmark$ $\qquad$ $(\times)$

$$z_2 = -1+3i.$$

D) $z_1 = 1+2i$

$2z_1 = 2+4i$

$$= |2+4i+1-3i| = |3+i| = \sqrt{10}.$$

# Triangle Inequality

$||z_1|-|z_2|| \leq |z_1+z_2| \leq |z_1|+|z_2|$

For any two complex numbers $z_1$ and $z_2$

$$\boxed{\left||\mathbf{z_1}|-|\mathbf{z_2}|\right| \leq |\mathbf{z_1}+\mathbf{z_2}| \leq |\mathbf{z_1}|+|\mathbf{z_2}|}$$

**Note that – Equality Holds**

If $|z_1+z_2| = |z_1|+|z_2| \Rightarrow z_1$ and $z_2$ are parallel.

If $|z_1+z_2| = \left||z_1|-|z_2|\right| \Rightarrow z_1$ and $z_2$ are anti-parallel.

19. If $z \neq 0$ be a complex number such that $\left|z - \frac{1}{z}\right| = 2$, then the maximum value of $|z|$ is:

[2022 Main, 29 July II]

[Ans. D]

(A) $\sqrt{2}$

(B) $1$

(C) $\sqrt{2} - 1$

(D) $\sqrt{2} + 1$ ✓

$z_1 \to z \quad \& \quad z_2 = -\frac{1}{z}$

$$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

$$\left||z| - \left|\tfrac{1}{z}\right|\right| \le \left|z - \tfrac{1}{z}\right| \le |z| + \left|-\tfrac{1}{z}\right|.$$

$|z| = r.$

$$\left|r - \tfrac{1}{r}\right| \le 2 \le r + \tfrac{1}{r}$$

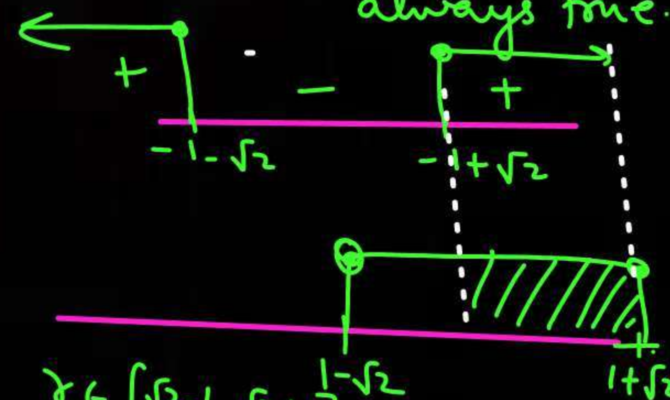
$r + \frac{1}{r} \ge 2$ always true.

$$-2 \le r - \tfrac{1}{r} \le 2$$

$$-2r \le r^2 - 1 \le 2r$$

$$r^2 + 2r - 1 \ge 0 \quad \& \quad r^2 - 2r - 1 \le 0.$$

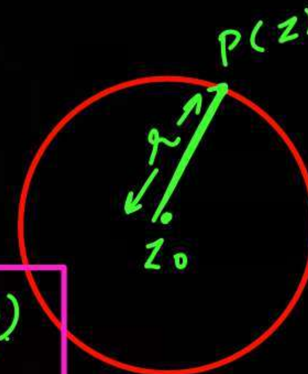
$$\frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2} \qquad \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}.$$




$r \in [\sqrt{2} - 1, \sqrt{2} + 1]$

# Complex Geometry_Introduction

#.




Ex→1 $|z-3i| = 5$. centre $(0,3)$, $r=5$.

Ex→2 $|z+5-3i|=7$

$\Rightarrow |z-(-5+3i)|=7$ $\boxed{\text{centre } (-5,3),\ r=7}$

## NOTE 1

$\boxed{|z-z_0| = r}$ represents a circle with centre $z_0$ and radius $= r$.

Ex→2 Put $z = x+iy$.

$|x+iy+5-3i|=7$

$|(x+5)+i(y-3)|=7$

$\sqrt{(x+5)^2+(y-3)^2}=7$.

$\boxed{(x+5)^2+(y-3)^2=49}$

$(-5,3)$

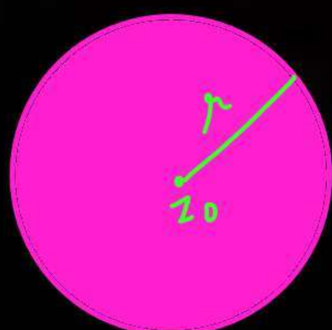
3) $|z+5|=3$

centre $(-5,0)$

$r=3$.

## NOTE 2

$\boxed{|z-z_0| \le r}$ represents a circular disc with centre $= z_0$ and radius $= r$.



20. Let $A = \{z \in C: 1 \leq |z - (1 + i)| \leq 2\}$ and $B = \{z \in A: |z - (1 - i)| = 1\}$. Then, $A \cap B$.

[2022 Main, 24 June I]

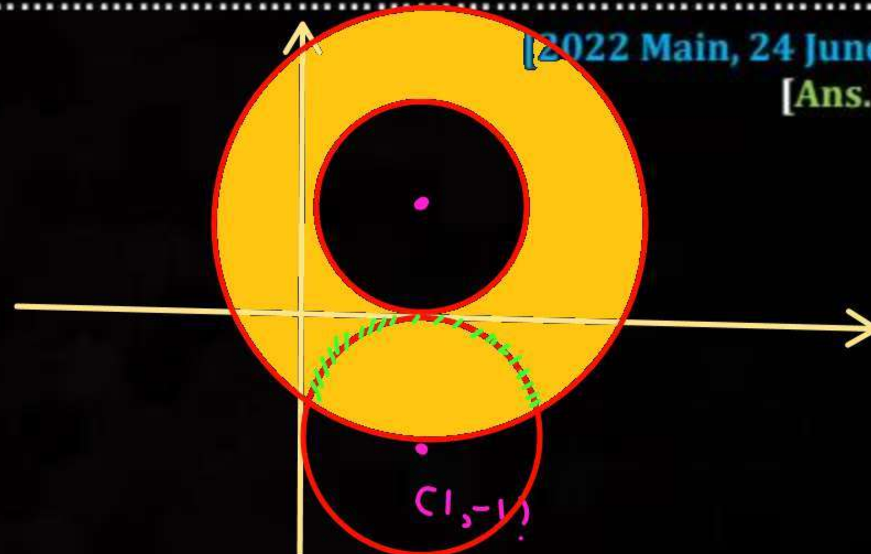
[Ans. D]

(A) is an empty set

(B) contains exactly two elements

(C) contains exactly three elements

(D) is an infinite set



21. If for the complex numbers z satisfying $|z - 2 - 2i| \leq 1$, the maximum value of $|3iz + 6|$ is attained at $a + ib$, then $\underbrace{a + b}$ is equal to . $\boxed{\text{Ans} = 5}$

$a \downarrow 3$, $b \downarrow 2$

[2021(01 Sep Shift 2)]

[Ans. 5]

$$= |3iz - 6i^2|$$

$$= |3i(z - 2i)|$$

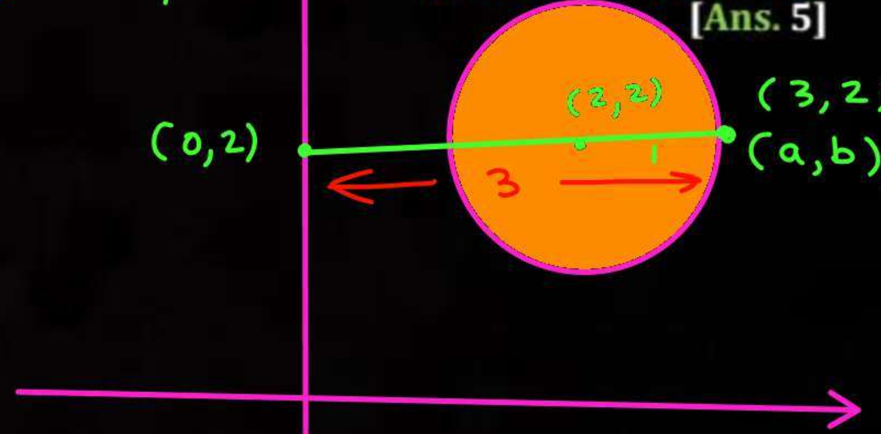
$$= \underbrace{|3i|}\,|z - 2i|$$

$$= 3\,|z - 2i| \rightarrow 3$$

$$3\,|z - z_1|$$

Max value $= 3 \times 3 = 9$

$|z - (2 + 2i)| \leq 1$



22. The number of elements in the set $\{z = a + ib \in \mathbb{C} : \boxed{a, b \in \mathbb{Z}} \text{ and } 1 < |z - 3 + 2i| < 4\}$ is

[2022 Main, 28 June I]

[Ans. 40]

$a - 3 = X$

$b + 2 = Y$

$X, Y \in Z$

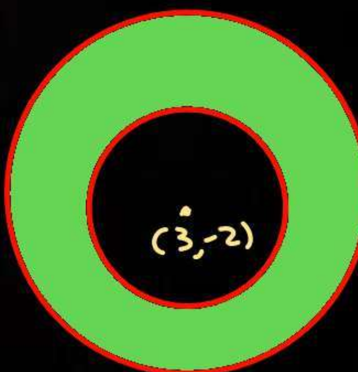
$1 < |z - (3 - 2i)| < 4$

$z = a + ib$

$1 < |(a-3) + i(b+2)| < 4$

$$\boxed{1 < (a-3)^2 + (b+2)^2 < 16}$$

$$\boxed{1 < X^2 + Y^2 < 16}$$




$X = 0, \; Y = \pm 2, \pm 3 \; \} \Rightarrow 4 = 4$

$X = \pm 1, \; Y = \pm 1, \pm 2, \pm 3 \rightarrow (6 \times 2) = 12$

$X = \pm 2, \; Y = 0, \pm 1, \pm 2, \pm 3 \rightarrow (7 \times 2) = 14$

$X = \pm 3, \; Y = 0, \pm 1, \pm 2 \rightarrow (5 \times 2) = 10$

$X = \pm 4, \; Y \Rightarrow (\times)$

Ans = 40

# Various Forms of Straight Lines

Real.

Canonical form of a straight line is given by $\bar{a}z + a\bar{z} + b = 0$, where $a$ is a complex constant, $b$ is a real constant.

Ex→ $(2+3i)z + (2-3i)\bar{z} + 5 = 0$

$2(z+\bar{z}) + 3i(z-\bar{z}) + 5 = 0.$

$2(2x) + 3i(2iy) + 5 = 0$

$$\boxed{4x - 6y + 5 = 0}$$

$z + \bar{z} = 2\,\mathrm{Re}(z)$

$z - \bar{z} = 2i\,\mathrm{Im}(z)$

Ex→2 $(3-4i)z + (3+4i)\bar{z} = 7$

Complex slope $= -\dfrac{(3+4i)}{3-4i}$

# Complex Slope of a Line

For the line $\bar{a}z + a\bar{z} + b = 0$

Complex slope is $\boxed{C = \frac{-a}{\bar{a}}}$

Complex slope in 2 point form $C = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$

23. Let the lines $(2 - i)z = (2 + i)\,\bar{z}$ and $(2 + i)z + (i - 2)\,\bar{z} - 4i = 0$, (here $i^2 - 1$) be normal to a circle $C$. If $iz + \bar{z} + 1 + i = 0$ is tangent to this circle $C$, then its radius is:

$(2-i)z - (2+i)\bar{z} = 0$

[2021 Main, 25 Feb I]

[Ans. 3]

(A) $\dfrac{3}{\sqrt{2}}$

(B) $3\sqrt{2}$

(C) $\dfrac{3}{2\sqrt{2}}$

(D) $\dfrac{1}{2\sqrt{2}}$

$2(z-\bar{z}) - i(z+\bar{z}) = 0$

$2(\cancel{2}\cancel{i}y) - \cancel{i}(\cancel{2}x) = 0$

$\boxed{2y = x}$ ~ (1)

$2(z-\bar{z}) + i(z+\bar{z}) = 4i$

$2(\cancel{2}y\cancel{i}) + \cancel{i}(\cancel{2}x) = 4i$

$\boxed{2y + x = 2}$ ~ (2)

$\Rightarrow x = 1,\ y = \frac{1}{2}.$

$(1, \frac{1}{2}) \rightarrow$ centre.

$z = x + iy.$

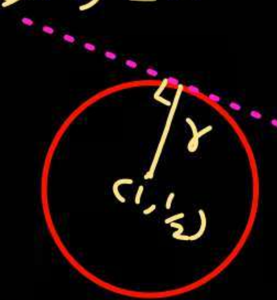
$i(x+iy) + (x-iy) + 1 + i = 0.$

$\underbrace{ix - y +}\ x - iy + 1 + i = 0.$

$(i+\cancel{1})x - y(i\cancel{+1}) + (\cancel{1+i}) = 0$

$\boxed{x - y + 1 = 0}$

$r = \left|\dfrac{1 - \frac{1}{2} + 1}{\sqrt{2}}\right| = \dfrac{3}{2\sqrt{2}}.$



24. Let $S = \{z \in C : \boxed{|z - 3| \leq 1}$ and $z(4 + 3i) + \bar{z}(4 - 3i) \leq 24\}$. If $\alpha + i\beta$ is the point in $S$ which is closest to $4i$, then $25(\alpha + \beta)$ is equal to ____.

[2022 Main, 24 June II]

[Ans. 80]

$4(z + \bar{z}) + 3i(z - \bar{z}) \leq 24.$

$4(2x) + 3i(2iy) \leq 24.$

$8x - 6y \leq 24.$

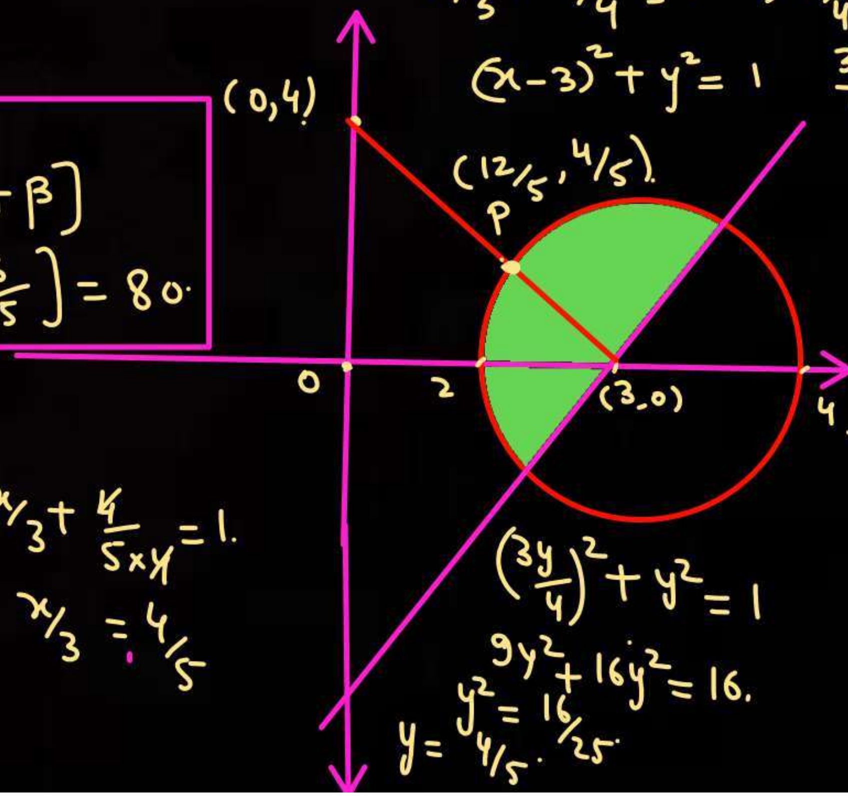
$\boxed{4x - 3y \leq 12}$ ✓

$\boxed{4x - 3y = 12}$

$\boxed{\begin{aligned} \text{Ans} &\rightarrow 25[\alpha + \beta] \\ &25\left[\tfrac{16}{5}\right] = 80. \end{aligned}}$

$x/3 + y/4 = 1 \Rightarrow y/4 = 1 - \frac{x}{3}$

$(x-3)^2 + y^2 = 1 \qquad \frac{3y}{4} = 3 - x$




$\left(\frac{3y}{4}\right)^2 + y^2 = 1$

$9y^2 + 16y^2 = 16.$

$y^2 = 16/25.$

$y = 4/5.$

$x/3 + \frac{4}{5 \times 4} = 1.$

$x/3 = 4/5$

HW. Let $S = \{z \in C : |z - 2| \le 1, z(1 + i) + \bar{z}(1 - i) \le 2\}$. Let $|z - 4i|$ attains minimum and maximum values, respectively, at $z_1 \in S$ and $z_2 \in S$.
If $(|z_1|^2 + |z_2|^2) = \alpha + \beta\sqrt{5}$, where $\alpha$ and $\beta$ are integers, then the value of $\alpha + \beta$ is equal to ____.

HW

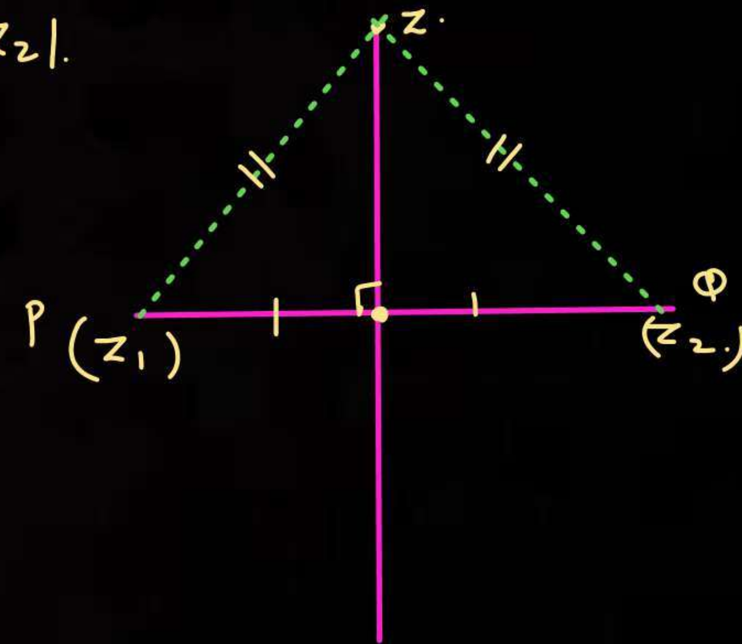
[2022 Main, 29 June I]

[Ans. 26]

# Perpendicular Bisector Form

$|z - z_1| = |z - z_2|$ represents the perpendicular bisector of the line joining the points $Z_1$ & $Z_2$.

$|z - z_1| = |z - z_2|.$



25. If $z = x + iy$ satisfies $|z| - 2 = 0$ and $|z - i| - |z + 5i| = 0$, then

$|z - (-5i)|$

[2022 Main, 26 July II]

[Ans. C]

$|z| = 2$ $\qquad$ $|z - i| = |z + 5i|$

$|z - z_1| = |z - z_2|$

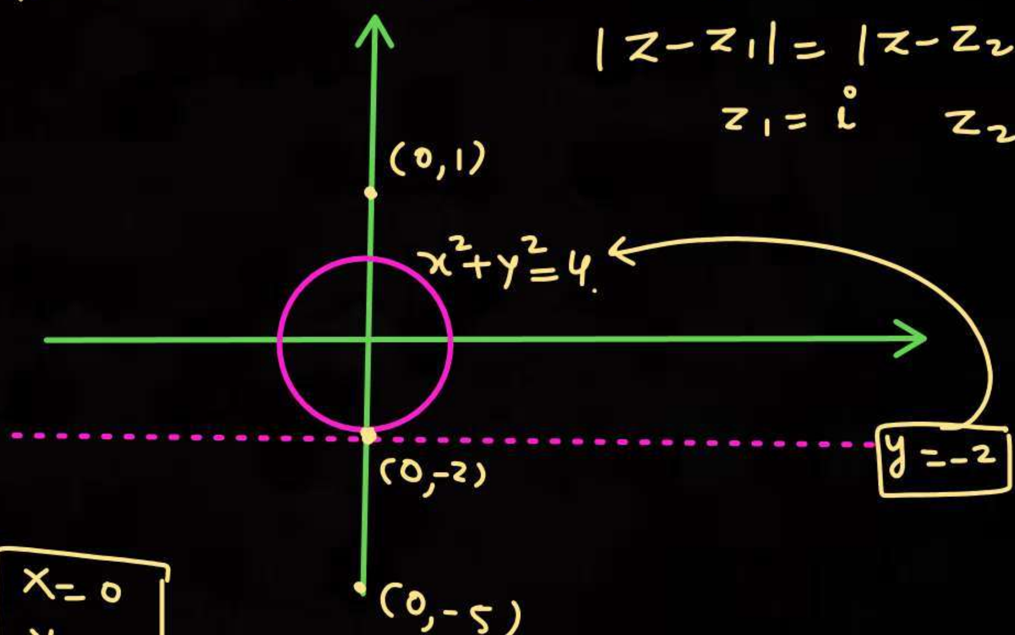
$z_1 = i \qquad z_2 = -5i$

(A) $x + 2y - 4 = 0$

(B) $x^2 + y - 4 = 0$

(C) $x + 2y + 4 = 0$ ✓

(D) $x^2 - y + 3 = 0$

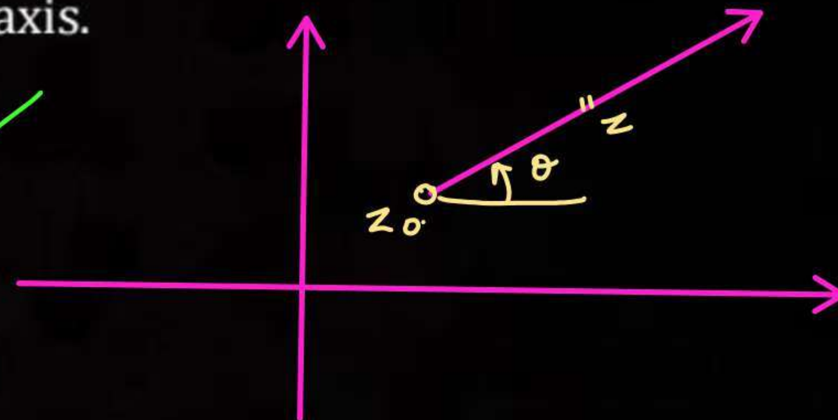



$\boxed{\begin{array}{l} x = 0 \\ y = -2 \end{array}}$

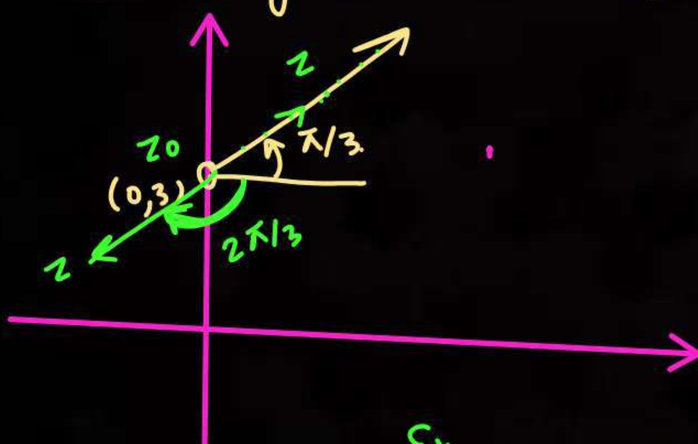
# Argument Form

$\arg(0) = \text{N.D.}$

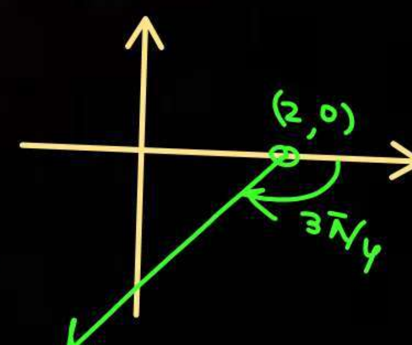
$\arg(z - z_0) = \theta$ represents a '**ray**' originating from point $z_0$ & inclined at an angle of $\theta$ with positive direction of $x$–axis.




Ex → $\arg(z - 3i) = \pi/3$

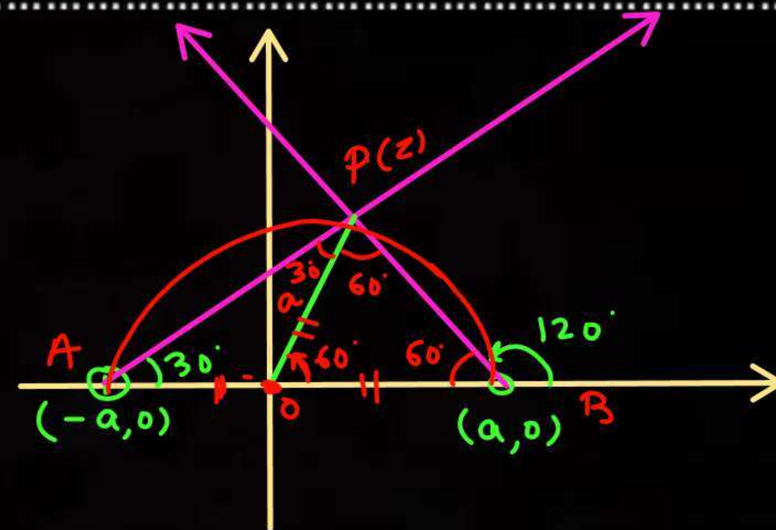



Ex → 2 $\arg(z - 2) = -3\pi/4$



26. If arg $(z + a) = \pi/6$ and arg $(z - a) = 2\pi/3$ $(a \in R^+)$, then

$\arg[z - (-a)] = \pi/6.$




(A) $|z| = a$

(B) $|z| = 2a$

(C) $\arg(z) = \dfrac{\pi}{2}$

(D) $\arg(z) = \dfrac{\pi}{3}$

$OP = a \Rightarrow |z| = a.$

$\arg(z) = 60°$

# Various forms of Circle

$z\bar{z} = |z|^2 = x^2 + y^2$

**NOTE 1**

$\boxed{|z - z_0| = r}$ represents a circle with centre $z_0$ and radius $= r$.

$x^2 + y^2 + 2x + 3y + 1 = 0$

**NOTE 2**

$\boxed{|z - z_0| \le r}$ represents a circular disc with centre $= z_0$ and radius $= r$.

$\boxed{\boxed{z\bar{z}} + \bar{a}z + a\bar{z} + b = 0}$ represents a circle with centre $(-a)$ & radius $= \sqrt{|a|^2 - b}$, where $a$ is a complex constant & $b$ is a real constant

27. If Re $\left(\frac{z-1}{2z+i}\right) = 1$, where $z = x + iy$, then the point $(x, y)$ lies on $a$ :

(HW) $z = x + iy.$

[JEE Main-2020 (January)]

(A) straight line whose slope is $-2/3$

(B) straight line whose slope $3/2$

(C) circle whose diameter is $\sqrt{5}/2$

(D) circle whose centre is at $(-1/2, -3/2)$

28. If z is a complex number such that $\frac{z-i}{z-1}$ is purely imaginary, then the minimum value of $|z-(3+3i)|$ is:

(HW)

[2021(31 Aug Shift 2)]

[Ans. D]

(A) $2\sqrt{2}-1$

(B) $3\sqrt{2}$

(C) $6\sqrt{2}$

(D) $2\sqrt{2}$

HW. If $S = \left\{ z \in C : \frac{z-i}{z+2i} \in R \right\}$, then :

(HW)

[2021(27 Aug Shift 1)]

[Ans. D]

A. $S$ contains exactly two elements

B. $S$ contains only one element

C. $S$ is a circle in the complex plane

D. $S$ is a straight line in the complex plane

# Various forms of Circle

## Argument form

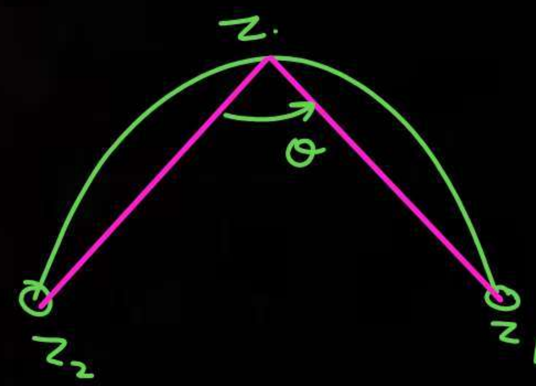
$\arg\left(\frac{z-z_1}{z-z_2}\right) = \theta$ (where $z_1$ & $z_2$ are complex constant & $\theta$ is a real constant) represents a circular arc subtending an angle of $\theta$ at the circumference.

$$\arg \frac{(z-z_1) \to \text{Final}}{(z-z_2) \to \text{Initial.}} = \theta.$$

$$\arg(z-z_1) - \arg(z-z_2) = \theta$$

$\arg(z-z_1)$ ↓ Final C.N.

$(z-z_2)$ Initial



29. The equation $arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ represent a circle with:

[2021(26 Aug Shift 1)]

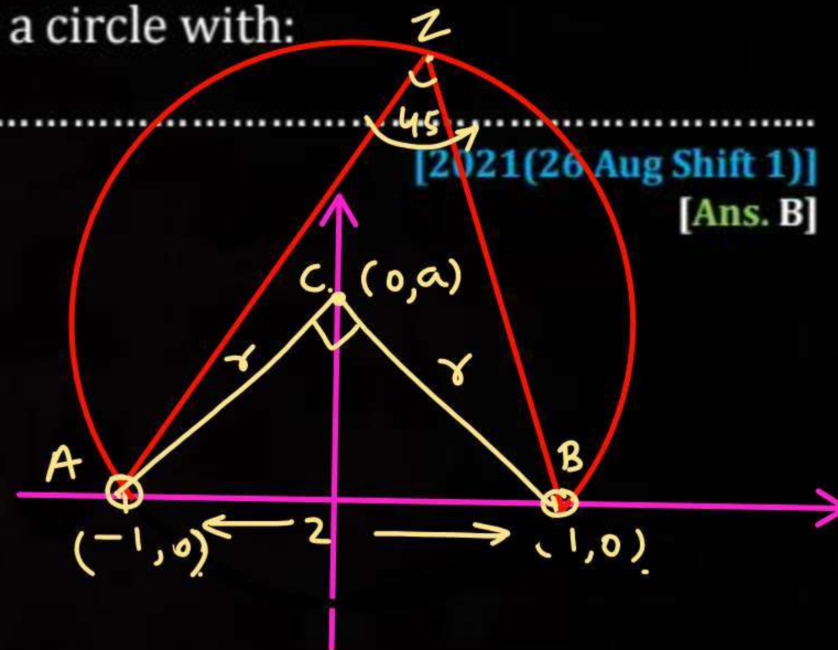
[Ans. B]

(A) Centre at $(0, -1)$ and radius $\sqrt{2}$

(B) Centre at $(0, 1)$ and radius $\sqrt{2}$

(C) Centre at $(0, 0)$ and radius $\sqrt{2}$

(D) Centre at $(1, 1)$ and radius $\sqrt{2}$




$$r^2 + r^2 = 2^2.$$
$$2r^2 = 4$$
$$r = \sqrt{2}.$$

M-2

$$\arg(z-1) - \arg(z+1) = \pi/4.$$

$$z = x+iy$$

$$\arg(x-1+iy) - \arg(x+1+iy) = \pi/4$$

$$\tan^{-1}\frac{y}{x-1} - \tan^{-1} y/_{x+1} = \pi/4$$

$$\tan^{-1}\left[\frac{y/_{x-1} - \frac{y}{x+1}}{1 + \frac{y^2}{x^2-1}}\right] = \pi/4$$

$$\frac{y(x+1) - y(x-1)}{x^2+y^2-1} = 1$$

$$y + y = x^2+y^2-1$$

$$\boxed{x^2+y^2-2y-1=0} \checkmark$$

$(0,1)$

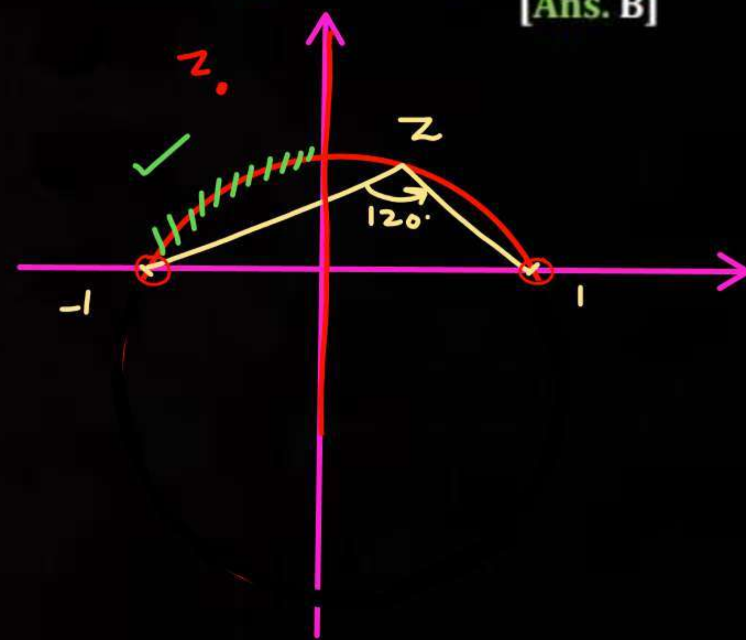
30. Let $A = \left\{z \in C : \left|\frac{z+1}{z-1}\right| < 1\right\}$ and $B = \left\{z \in C : \arg\left(\frac{z-1}{z+1}\right) = \frac{2\pi}{3}\right\}$. Then $A \cap B$ is:

$|z+1| < |z-1|$. ✓

$\boxed{|z+1| = |z-1|} \longrightarrow \perp$ bisector

[2022 Main, 26 June I]

[Ans. B]




(A) a portion of a circle centered at $\left(0, -\frac{1}{\sqrt{3}}\right)$ that lies in the second and third quadrants only

(B) a portion of a circle centered at $\left(0, -\frac{1}{\sqrt{3}}\right)$ that lies in the second quadrants only ✓

(C) an empty set

(D) a portion of a circle of radius $\frac{2}{\sqrt{3}}$ that lies in the third quadrant only

HW. A point z moves in the complex plane such that $arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}$, then the minimum value of $|z - 9\sqrt{2} - 2i|^2$ is equal to.

(HW)

[2021(31 Aug Shift 1)]

[Ans. 98]

# Various Forms of Conics

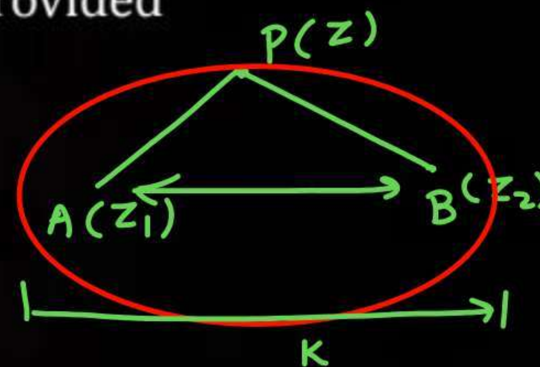
$\boxed{|z - z_1| + |z - z_2| = k}$ represents the equation of ellipse provided

$k > |z_2 - z_1|$ ✓

$|z - z_1| = PA$

$|z - z_2| = PB$ $\Rightarrow \boxed{PA + PB = K}$

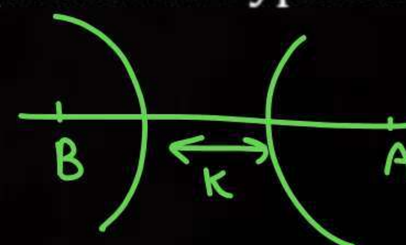
$K > AB$




$||z - z_1| - |z - z_2|| = k$ represents the equation of Hyperbola provided

$k < \underbrace{|z_2 - z_1|}_{K < AB}$

$|PA - PB| = K$



32. The number of points of intersection of $|z-(4+3i)|=2$ and $|z|+|z-4|=6$, $z \in C$ is:

[2022 Main, 27 June II]

[Ans. C]

(A) 0

(B) 1

(C) 2

(D) 3

$PA + PB = 6$ ✓

$A(0,0)$, $B(4,0)$

$2a = 6$

$a = 3$ ✓

$2ae = 4 \Rightarrow ae = 2.$

$b^2 = a^2 - a^2e^2.$

$b^2 = 9 - 4$

$b = \sqrt{5}.$