CHAPTER

17 Hyperbola

❖ Standard equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $b^2 = a^2 (e^2 - 1)$

or $a^2 e^2 = a^2 + b^2$ *i.e.* $e^2 = 1 + \frac{b^2}{a^2} = 1+\left(\frac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$

(*a*) Foci:

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(*b*) Equations of Directrices:

$x=\frac{a}{e}$ & $x=-\frac{a}{e}$.

(*c*) Vertices:

$A \equiv (a, 0)$ & $A' \equiv (-a, 0)$.

(*d*) Latus Rectum:

(*i*) Equation: $x = \pm ae$

(*ii*) Length $= \frac{2b^2}{a} = \frac{(\text{Conjugate Axis})}{(\text{Transverse Axis})} = 2a(e^2-1)$

$= 2e$(distance from focus to directrix)

(*iii*) Ends: $\left(ae,\frac{b^2}{a}\right),\left(ae,\frac{-b^2}{a}\right);\left(-ae,\frac{b^2}{a}\right),\left(-ae,\frac{-b^2}{a}\right)$

(*e*) Focal Property:

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis *i.e.* $||PS| - |PS'|| = 2a$. The distance SS' = focal length.

(*f*) Focal Distance:

Distance of any point $P(x, y)$ on hyperbola from foci $PS = ex - a$ & $PS' = ex + a$.

Conjugate Hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ & $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are conjugate hyperbolas of each.

Auxillary Circle: $x^2 + y^2 = a^2$.

Parametric Representation: $x = a \sec\theta$ & $y = b\tan\theta$

Position of *A* point '*P*' w.r.t. *A* Hyperbola:

$S_1 \equiv \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1 >=$ or < 0 according as the point (x_1, y_1) lines inside, on

or outside the curve.

Tangents

(*i*) **Slope Form:** $y = m\times \pm\sqrt{a^2m^2-b^2}$

(*ii*) **Point Form:** at the point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

(*iii*) **Parametric Form:** $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.

❖ **Normal to The Hyperbola** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

(*a*) **Point form:** Equation of the normal to the given hyperbola at the point $P(x_1, y_1)$ on it is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2$.

(*b*) **Slope form:** The equation of normal of slope m to the given hyperbola is $y = mx \mp \frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}$ foot of normal are

$\left(\pm\frac{a^2}{\sqrt{(a^2-m^2b^2)}}, \mp\frac{mb^2}{\sqrt{(a^2-m^2b^2)}}\right)$.

(*c*) **Parametric form:** The equation of the normal at the point P $(a\sec\theta, b\tan\theta)$ to the given hyperbola is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2=a^2e^2$.

Director Circle

Equation to the director circle is: $x^2 + y^2 = a^2 - b^2$.

Chord of Contact

If PA and PB be the tangents from point $P(x_1, y_1)$ to the Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then the equation of the chord of contact AB is

$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ or $T = 0$ at (x_1, y_1).

Equation of Chord with mid Point (x_1, y_1)

The equation of the chord of the ellipse $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, whose mid-point be (x_1, y_1) is $T = S_1$ where T

$=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1,\ S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

i.e. $\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right) = \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right)$.

Asymptotes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

Reflection property of the hyperbola: An incoming light ray aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus.

Rectangular or Equilateral Hyperbola: $xy = c^2$, eccentricity is $\sqrt{2}$.

Vertices: $(\pm c \pm c)$; Focii : $\left(\pm \sqrt{2}c, \pm \sqrt{2}c\right)$. Directrices : $x + y = \pm \sqrt{2}\, c$.

Latus Rectum (l): $= l = 2\sqrt{2}\, c = T.A. = C.A.$

Parametric equation $x = ct$, $y = c/t$, $t \in R - \{0\}$

Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at $P(t)$ is $\frac{x}{t} + ty = 2c$.

Equation of the normal at $P(t)$ is $xt^3 - yt = c(t^4 - 1)$.

Chord with a given middle point as (h, k) is $kx + hy = 2hk$.