CHAPTER

11 Differential Equations

Order of a Differential Equation

The order of highest order derivative appearing in a differential equation is called the order of the differential equation.

Degree of a Differential Equation

The degree of an algebraic differential equation is the degree of the derivative (or differential) of the highest order in the equation, after the equation is freed from radicals and fractions in its derivatives.

Variable Separable Differentiable Equations

A differential equation of the form $f(x) + g(y)\dfrac{dy}{dx} = 0$

Equations Reducible to Variable Separable form

$\dfrac{dy}{dx} = f\,(ax + by + c)$ can be reduced to varible separable form by substitution

$ax + by + c = t$. The reduced variable separable form is:

$$\frac{dt}{bf(t) + a} = dx.$$

Homogeneous Differential Equation

$\dfrac{dy}{dx} = \dfrac{f(x, y)}{g(x, y)}$ where $f(x, y)$ and $g(x, y)$ are both homogeneous function of same degree in x and y.

Substitute $y = vx$ and so $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Equations Reducible to the Homogeneous form

Consider a differential equation of the form:

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}, \text{ where } \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

Put $\quad x = X + h$

$\quad y = Y + k$

Such that, $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$

Linear Equation

An equation of the form $\dfrac{dy}{dx} + Py = Q$

Multiply both sides of the equation by $e^{\int Pdx}$.

$$\therefore \quad \frac{dy}{dx}e^{\int Pdx} + Pye^{\int Pdx} = Qe^{\int Pdx}$$

$$\Rightarrow \quad y.e^{\int Pdx} = \int Q.e^{\int Pdx}dx + C$$

Bernoulli's Equation

An equation of the form $\dfrac{dy}{dx} + Py = Qy^n$,

Putting $\quad y^{-n+1} = v$

$$\Rightarrow \quad \frac{dv}{dx} + (1 - n)\,P\cdot\, y = (1 - n)Q.$$

Following exact differentials must be remembered:

(i) $xdy + ydx = d(xy)$

(ii) $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$

(iv) $\dfrac{xdy + ydx}{xy} = d\left(\ell nxy\right)$

(v) $\dfrac{dx + dy}{x + y} = d(\ell n(x + y))$

(vi) $\dfrac{xdy - ydx}{xy} = d\left(\ell n\dfrac{y}{x}\right)$

(vii) $\dfrac{ydx - xdy}{xy} = d\left(\ell n\dfrac{x}{y}\right)$

(viii) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(ix) $\dfrac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$

(x) $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\ell n\sqrt{x^2 + y^2}\right]$

(xi) $d\left(-\dfrac{1}{xy}\right) = \dfrac{xdy + ydx}{x^2y^2}$

(xii) $d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$

(xiii) $d\left(\frac{e^y}{x}\right) = \frac{xe^y dy - e^y dx}{x^2}$

Orthogonal Trajectory

Any curve which cuts every member of a given family of curves at right angle is called an orthogonal trajectory of the family. For example, each straight line $y = mx$ passing through the origin, is an orthogonal trajectory of the family of the circles $x^2 + y^2 = a^2$.

Procedure for Finding the Orthogonal Trajectory

(*i*) Let $f(x, y, c) = 0$ be the equation, where c is an arbitrary parameter.

(*ii*) Differentiate the given equation w.r.t. x and then eliminate c.

(*iii*) Replace $\frac{dy}{dx} by - \frac{dx}{dy}$ in the equation obtained in (*ii*).

(*iv*) Solve the differential equation in (*iii*).