SEQUENCE & SERIES

ERROR IS EVER THE SEQUENCE OF HASTE
- Duke of Wellington

1 Arithmetic progression (AP)

A sequence $\{a_n\}$ of real numbers is called an arithmetic progression (AP) if $a_{n+1} - a_n$ is constant for all positive integers $n \geq 1$, and this constant number is called the common difference of the AP.

2 General form of AP:

The terms of an AP with first term 'a' and common difference d are a, a + d, a + 2d, a + 3d, ..., and the n^{th} term being $a_n + (n - 1)$ d.

Quick Look

- If $\{a_n\}$ is an AP and k is any real number, then $\{a_n+k\}$ is also an AP with same common difference and $\{k\,a_n\}$ is also an AP.
- If $\{a_n\}$ and $\{b_n\}$ are arithmetic progressions, then $\{a_n+b_n\}$ is also an AP.
- Product of two arithmetic progressions is also an AP if and only if one of them is a constant sequence
- Arithmetic means (AM's): If a, A_1, A_2, ... A_n, b are in AP, then A_1, A_2, ... A_n are called n AM's between a and b. The K th mean A_k is given by $A_k = a + K\frac{(b-a)}{n+1}$ for K = 1, 2, ...,n
- Sum to first n terms of an AP: Let s_n be the sum to first n terms of an AP with first term 'a' and common difference 'd'. Then $s_n = \frac{n}{2}[2a + (n - 1) d]$ or $S_n = \frac{n}{2}$ [first term + n^{th} term]
- If $A_1, A_2, ..., A_n$ are n AM's between a and b then $A_1 + A_2 + ... + A_n = \frac{n(a+b)}{2}$
- Ratio of n^{th} terms of two AP's: Let t_n be the n^{th} term of an AP whose first term is a and common difference d and S_n is its sum to first n terms. Let $t_{n'}$ be the n^{th} term of another AP with first term b and common difference d whose sum of first n terms is $S_{n'}$ Then

$$\frac{t_n}{t'_n} = \frac{S_{2n-1}}{S'_{2n-1}}$$

- Characterization of an AP: A sequence of real numbers is an arithmetic progression if and only if its sum of the first n terms is a quadratic expression in n with constant term zero.
- Helping points:
  (1) Three numbers in AP can be taken as a − d, a, a + d.
  (2) Four numbers in AP can be taken as a − 3d, a − d, a + d, a + 3d.
  (3) Five numbers in AP can be taken as a − 2d, a − d, a, a + d, a + 2d.

1 Geometric progression (GP)

A sequence $\{a_n\}$ of non-zero real numbers is called GP if $a_n / a_{n-1} = a_{n+1} / a_n$ for $n \geq 2$

2. That is the ratio a_{n+1} / a_n is constant for $n \geq 1$ and this constant ratio is called the common ratio of the GP and is generally denoted by r.

2 General form

GP with first term a, ar, ar^2 .... whose n^{th} term is ar^{n-1}

Quick Look

- If three numbers are in GP, then they can be taken as a/r, a, ar.
- If four numbers are in GP, then they can be taken as a/r^3, a/r ar, ar^3.
- Sum to first n-terms of a GP: The sum of the first n-terms of a GP with first term 'a' and common ratio $r \neq 1$ Is $\frac{a(1-r^n)}{1-r}$
- Sum to infinity of a GP: If − 1 < r < 1 is the common ratio of a GP whose first term is a, then S_n=a/1 − r is called sum to infinity of the GP.
- Geometric mean and geometric means: if three numbers a, b and c are in GP, then b is called the Geometric mean (GM) between a and c and b^2 = ac. If x and y are positive real numbers, then x, $\sqrt{xy}$,y are in GP. If a, $g_1, g_2, ... g_n$, b are in GP, then $g_1, g_2, ... g_n$ are called n geometric means between a and b.
- k th GM g_k is given by $g_k = a(b/a)^{k/n+1}$ for k = 1, 2, ..., n
- Product of n GM's between a and b is $\left(\sqrt{ab}\right)^n$

3 Arithmetic geometric progression (AGP):

Sequence of numbers of the form a, (a + d)r, (a + 2d)r², + ... is called AGP

sum to n terms of an AGP is

$$\frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)r^n}{1-r}$$

and $\frac{a}{1-r} + \frac{dr}{(1-r)^2}$ is the sum to infinity.

4 AM − GM inequality:

Let $a_1, a_2, ...,a_n$ be positive reals. Then $\frac{a_1+a_2+....+a_n}{n}$ is called AM of $a_1, a_2, ..., a_n$ and $(a_1, a_2, ..., a_n)^{1/n}$ is called their GM. Further $\frac{a_1+a_2+....+a_n}{n} \geq (a_1, a_2, ..., a_n)^{1/n}$ and equality holds if an only if $a_1 = a_2 = a_3 = ... = a_n$

5. Harmonic progression (HP)

A sequence of non-zero reals is said to be in HP, if their reciprocals are in AP.

General form of an HP

Sequence of real numbers

$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots, \frac{1}{a+(n-1)d}, \ldots$

can be taken as general form of an HP.

Harmonic mean & Harmonic means:

(1) If a, b, c are in HP, then b is called the Harmonic mean (HM) between a and c and in this case $b = \frac{2ac}{a+c}$.

(2) If a, $h_1, h_2, \ldots, h_2, \ldots, h_n$, b are in HP, then $h_1, h_2, \ldots, h_2, \ldots, h_n$ are called HM's between a and b further

$h_k = \frac{ab(n+1)}{b(n+1)+k(a-b)}$ for k = 1, 2, ..., n

Theorem:

Let $a_1, a_2, \ldots, a_n$ be positive real and A, G be AM and GM of the given numbers. Then,

$H = \frac{n}{1/a_1 + 1/a_2 + \ldots + 1/a_n}$

which is called harmonic mean of $a_1, a_2, \ldots, a_n$

Note: $A \geq G \geq H$ and

equality holds if and only if $a_1 = a_2 = a_3 = \ldots = a_n$

6. Some Useful Formulae

1. Telescopic Series: Suppose that we have to find the sum to n terms of a series $u_1 + u_2 + u_3 + \ldots$.

$$u_k = a_k - a_{k+1} \text{ for all k}$$

$$\text{then } u_1 + u_2 + \ldots u_n = a_1 - a_{n+1}$$

For example, consider $\frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots$ here, we have

$$\frac{1}{K(K+1)} = \frac{1}{K} - \frac{1}{K+1} \quad \text{for all } K \geq 2$$

2. Suppose that the term u_n of a given series is the product of r successive terms of an AP beginning with the n^{th} term of the AP; such that

$$u_n = [a+(n-1)d][a+nd]\ldots[a+(n+r-2)d]$$

By choosing $a_n = u_n[a+(n+r-1)d]$, we can write

$$u_n = \frac{1}{(r+1)d}[a_n - a_{n-1}]$$

so that the sum to n terms is equal to $\frac{1}{(r+1)d}(a_n - a_0)$

For example, consider

(i) 1.2.3 + 2.3.4 + 3.4.5 + ...

(ii) 1.3.5.7 + 3.5.7.9 + 5 5.7 .9 .11 +....

3.Suppose that the n^{th} term of a series is the reciprocal of the n^{th} term of the series given in II; that is,

$$u_n = \frac{1}{[a+(n-1)d][a+nd]\ldots[a+(n+r-2)d]}$$

Then, we can choose $a_n = u_n[a+(n-1)d]$

so that $u_n = \frac{1}{(r-1)d}(a_{n-1} - a_n)$ and sum to n terms is given by

$$\frac{1}{(r-1)d}(a_0 - a_n)$$

For example, consider

(i) $\frac{1}{1.4.7} + \frac{1}{4.7.10} + \frac{1}{7.10.13} + \ldots$

(ii) $\frac{1}{1.3.5.7} + \frac{1}{3.5.7.9} + \frac{1}{5.7.9.11} + \ldots$