Basics of Hyperbola

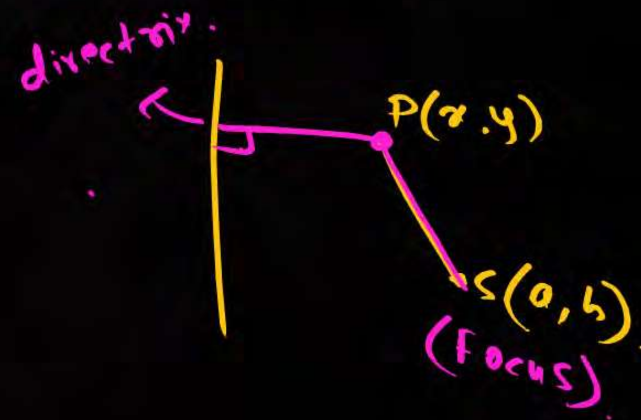
Definition: For Hyperbola $\boxed{e > 1}$

For second degree Equation: $\boxed{ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0}$

Condition: $\boxed{\Delta \neq 0}$ & $\boxed{h^2 > ab}$

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$\frac{SP}{PM} = e$$



Recap

Ellipse :

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

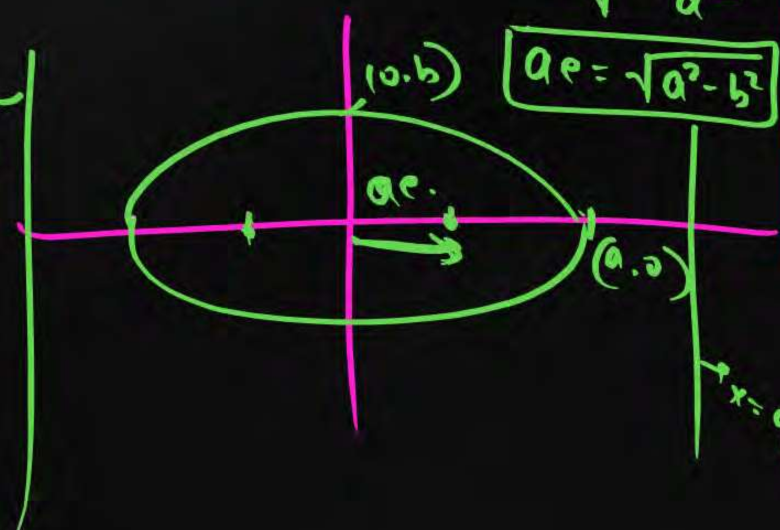
$\boxed{a>b}$

$$e=\sqrt{1-\frac{b^2}{a^2}}$$

$\boxed{ae=\sqrt{a^2-b^2}}$

$(\pm ae, 0)$

$\boxed{x=-\frac{a}{e}}$

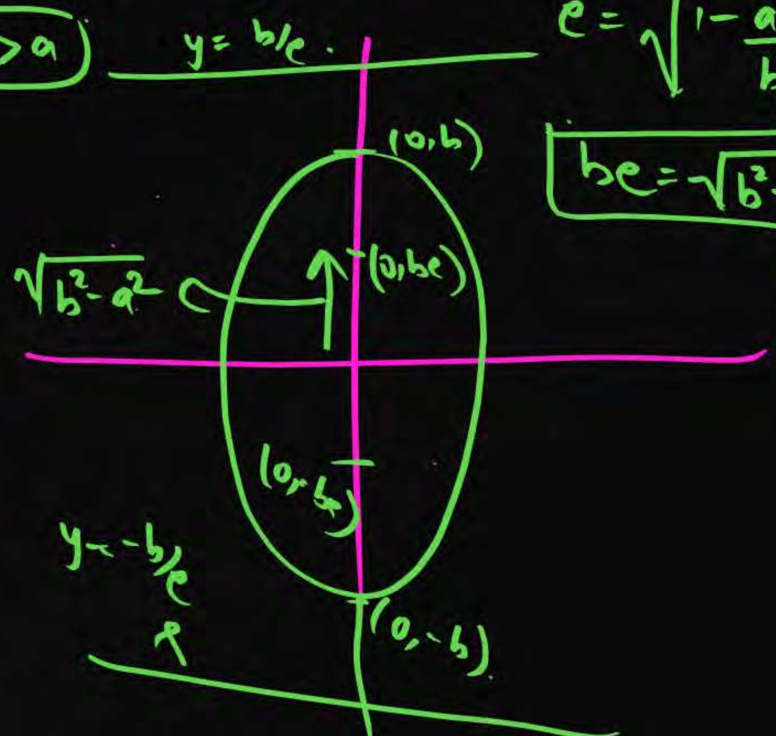


$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

$\boxed{b>a}$

$$e=\sqrt{1-\frac{a^2}{b^2}}$$

$\boxed{be=\sqrt{b^2-a^2}}$



Standard Hyperbolas and Their Equations

Here we will be studying hyperbolas having horizontal or vertical axis.

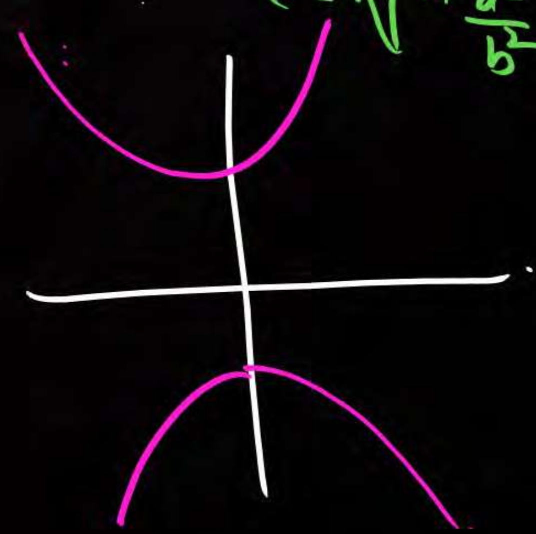
$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

$$e = \sqrt{1 + \frac{b^2}{a^2}}.$$

$$\Rightarrow \boxed{ae = \sqrt{a^2 + b^2}}$$

$$\boxed{-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

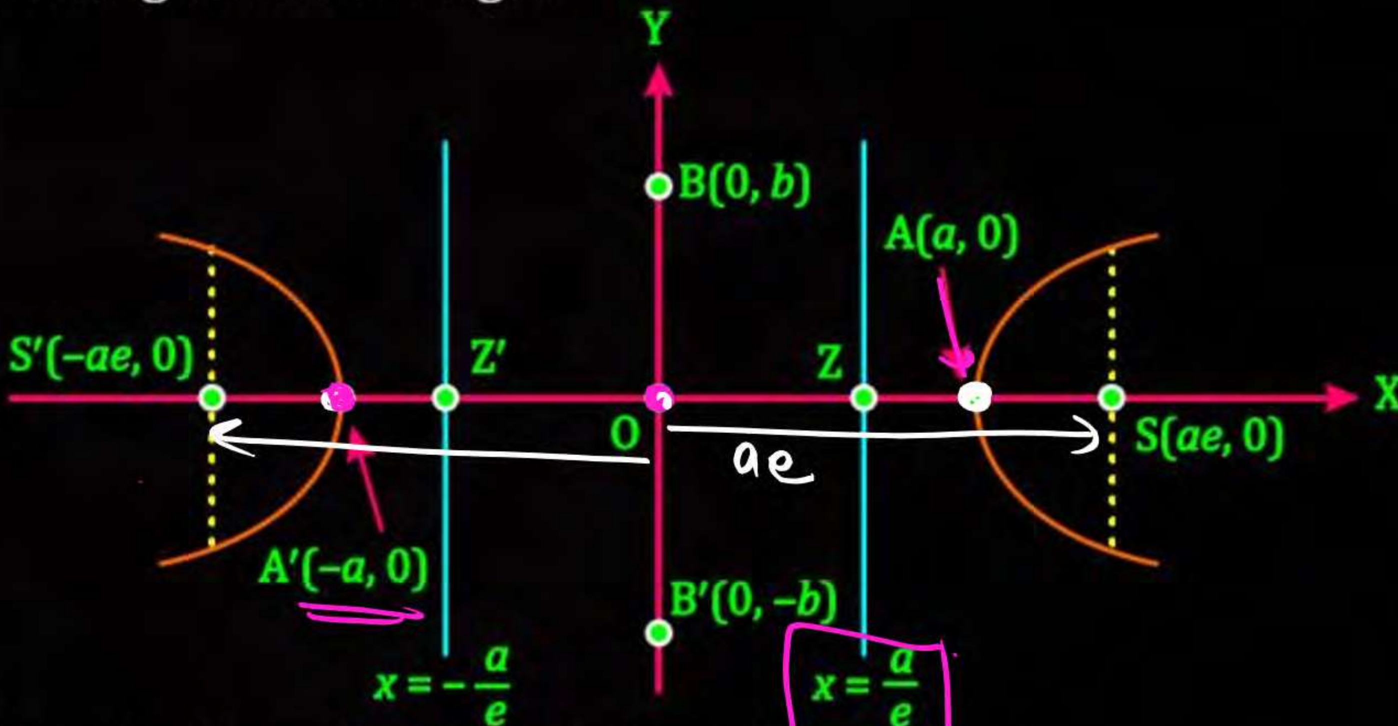
$$e = \sqrt{1 + \frac{a^2}{b^2}} \Rightarrow be = \sqrt{a^2 + b^2}.$$



Standard Hyperbolas and Their Equations

Standard Hyperbolas having centre at origin

(1) $\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$

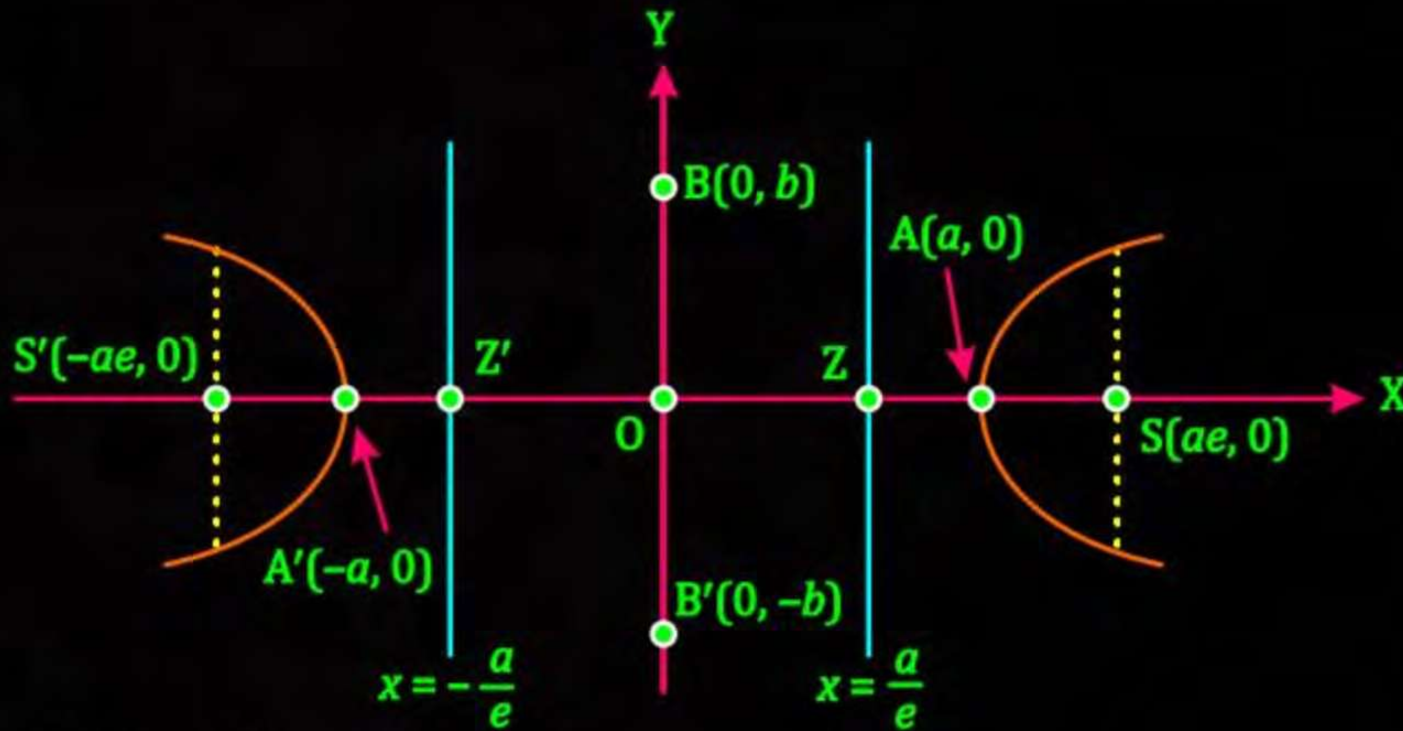


(a) Transverse axis : Along X-axis, and of length $2a$

(b) Conjugate axis : Along Y-axis, and of length $2b$

Standard Hyperbolas and Their Equations

(1) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

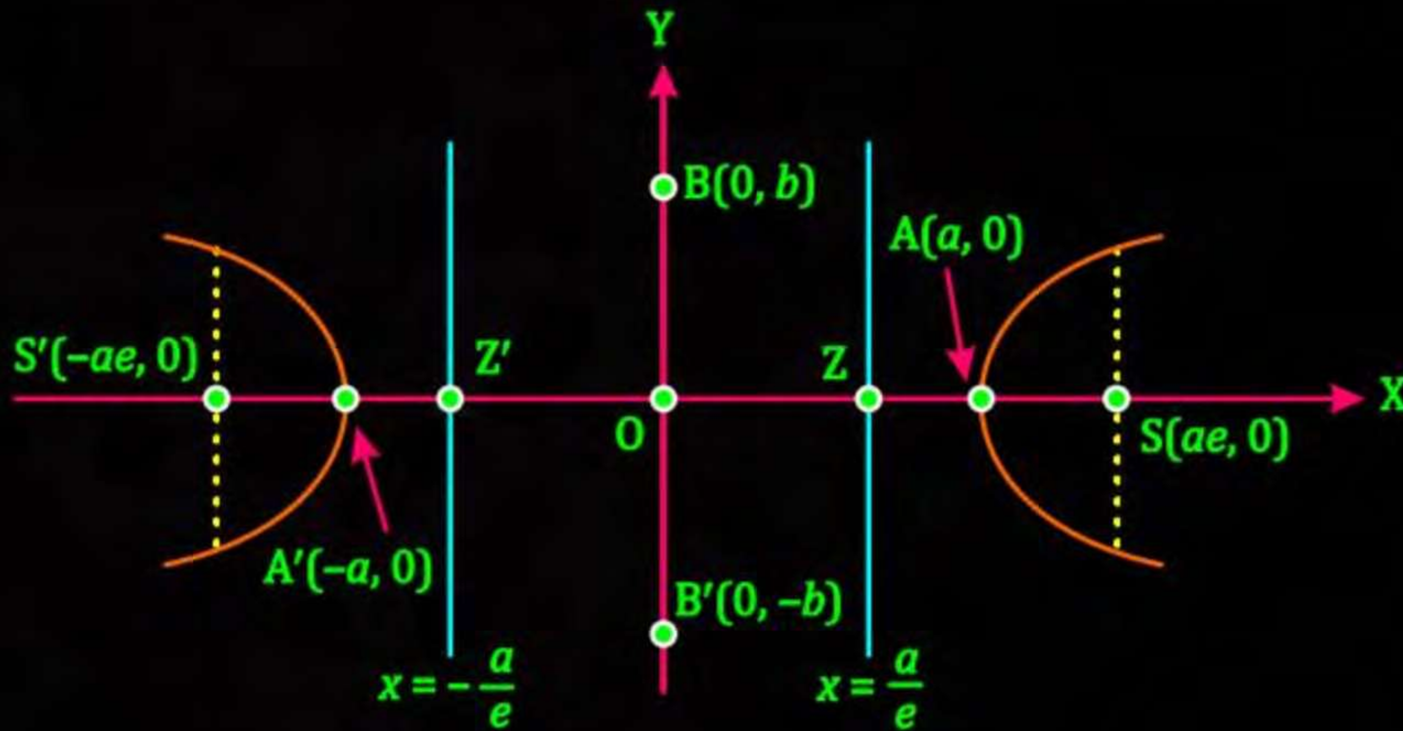


(c) Foci : $S(ae, 0)$ and $S'(-ae, 0)$

(d) Directrices : $x = \frac{a}{e}$ and $x = -\frac{a}{e}$

Standard Hyperbolas and Their Equations

(1) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



(e) Eccentricity $e = \sqrt{1 + \frac{b^2}{a^2}}$

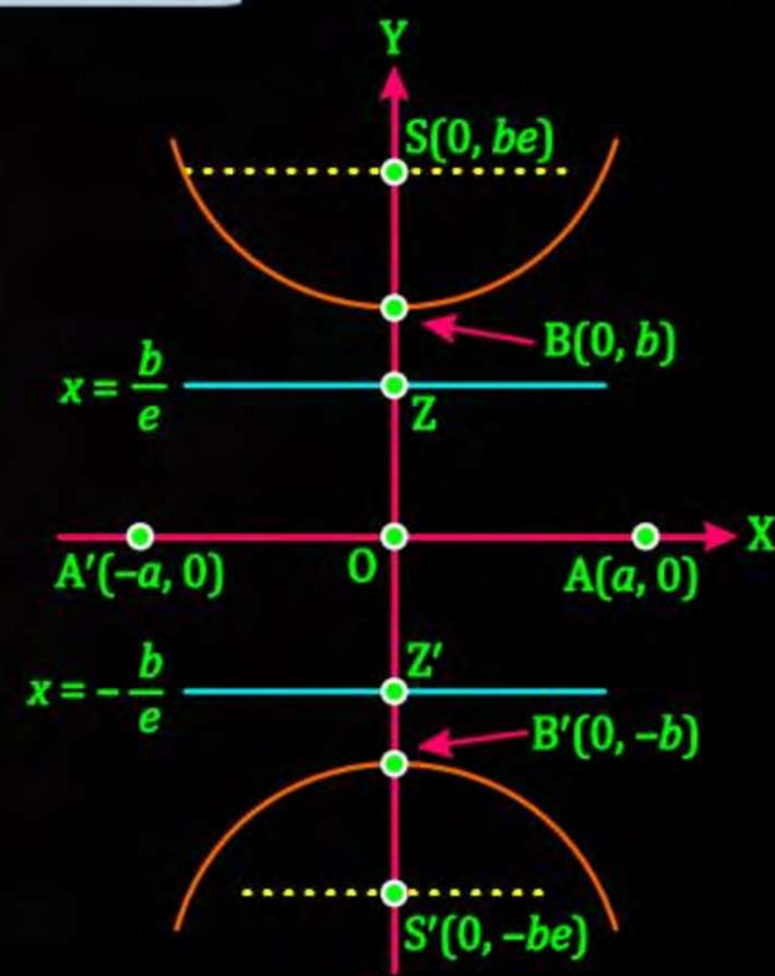
Standard Hyperbolas and Their Equations

Standard Hyperbolas having centre at origin

(1) $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(a) Transverse axis : Along Y-axis, and of length $2b$

(b) Conjugate axis : Along X-axis, and of length $2a$

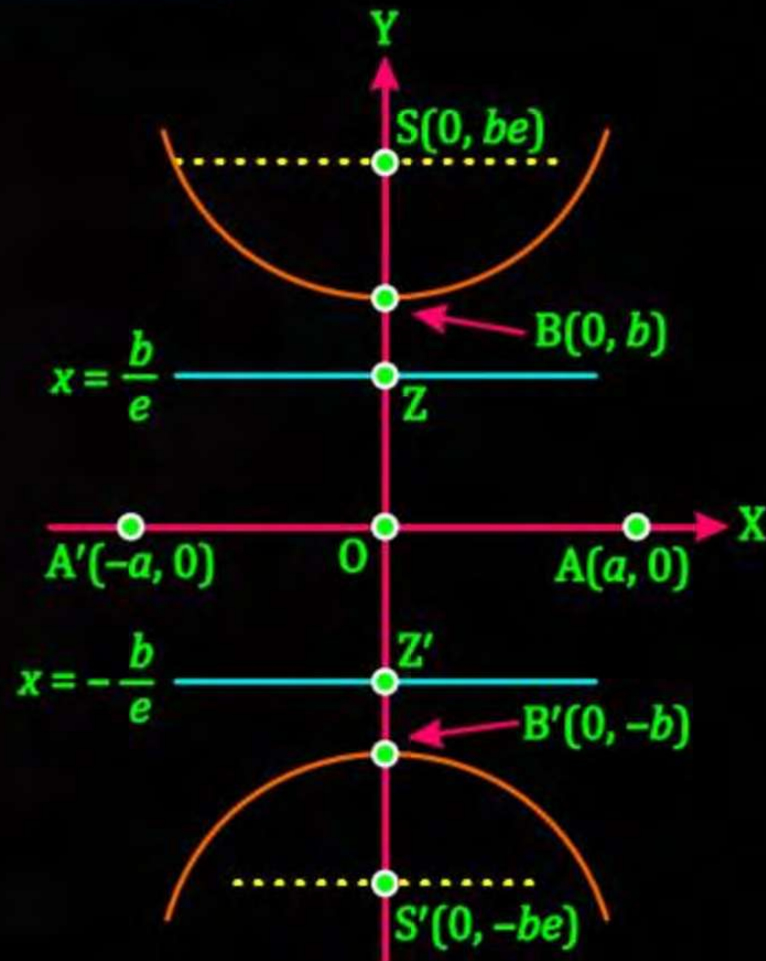


Standard Hyperbolas and Their Equations

(1) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(c) Foci : $S(0, be)$ and $S'(0, -be)$

(d) Directrices : $y = \frac{b}{e}$ and $y = -\frac{b}{e}$



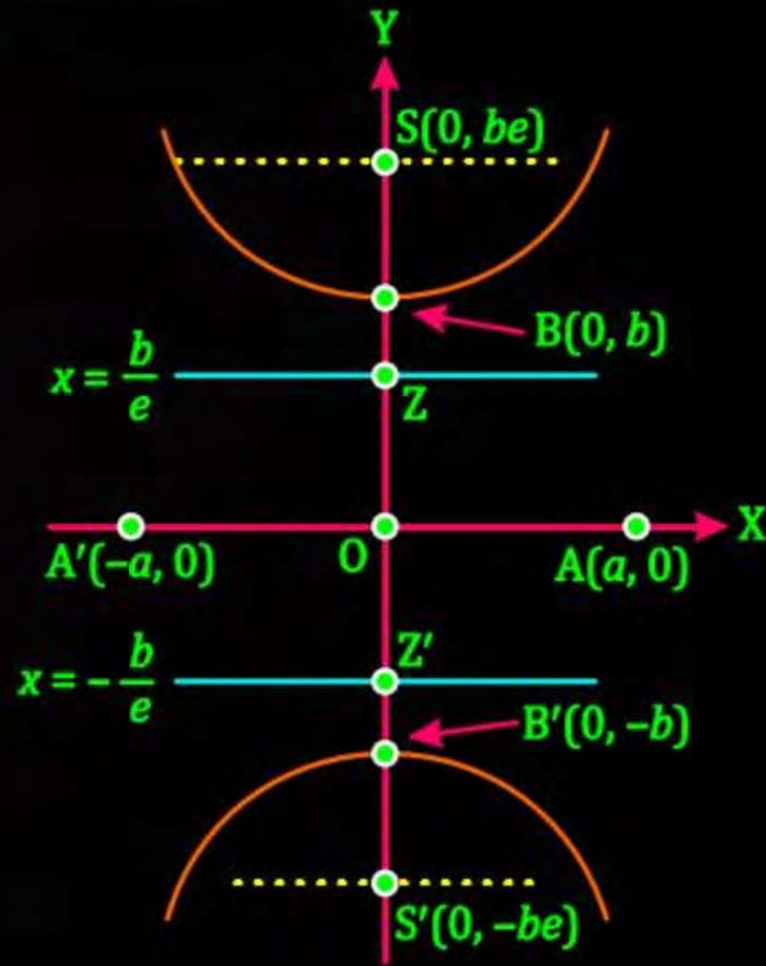
Standard Hyperbolas and Their Equations

(1) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(e) Eccentricity $e = \sqrt{1 + \frac{a^2}{b^2}}$

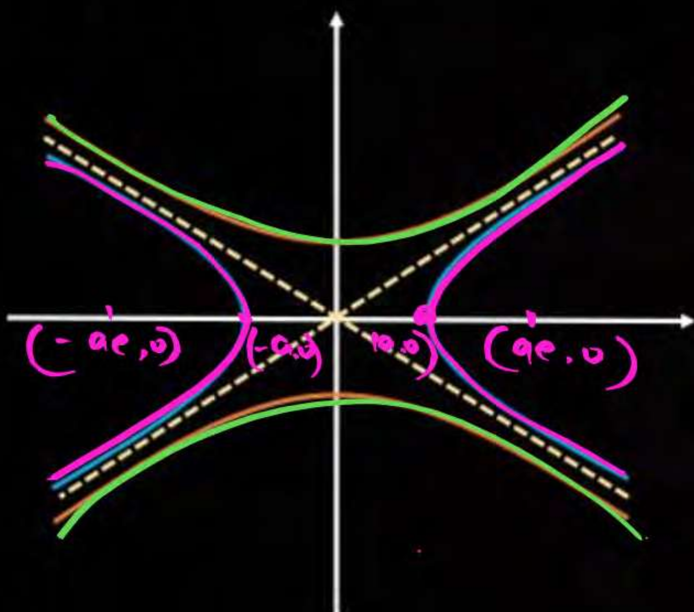
(f) LR : y = be and y = – be

and length of LR = $\frac{2a^2}{b}$



Hyperbola & Conjugate Hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $\qquad$ $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

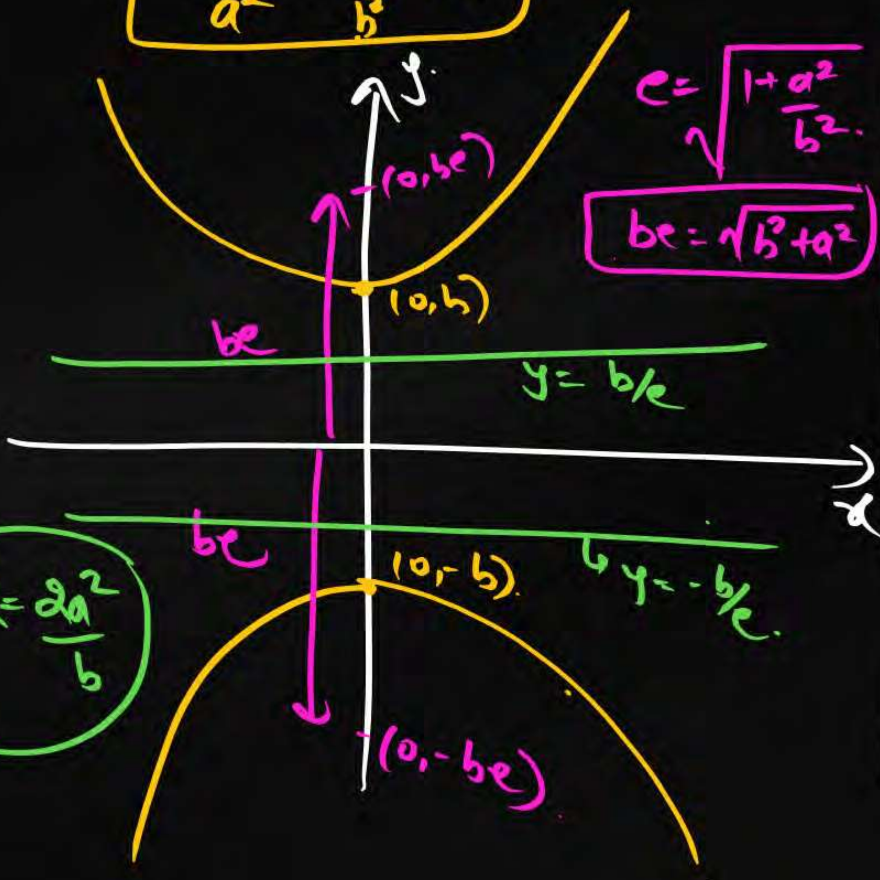


	Hyperbola	Conjugate Hyperbola
Directrices:	$x=\pm\frac{a}{e}$	$y=\pm\frac{b}{e}$
Foci:	$(\pm ae, 0)$	$(0, \pm be)$
Vertices:	$(\pm a, 0)$	$(0, \pm b)$
Principle Axes:	$\boxed{T.A.=2a}$ $C.A=2b$	$T.A.=2b, C.A=2a$
Centre:	$(0,0)$	$(0,0)$
Focal Length:	$2ae$	$2be$
Eccentricity:	$e^2=1+\frac{b^2}{a^2}$	$e^2=1+\frac{a^2}{b^2}$
Latus Rectum:	$LR=\frac{2b^2}{a}$	$LR=\frac{2a^2}{b}$

$$\boxed{-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

$$e=\sqrt{1+\frac{a^2}{b^2}}$$

$$\boxed{be=\sqrt{b^2+a^2}}$$

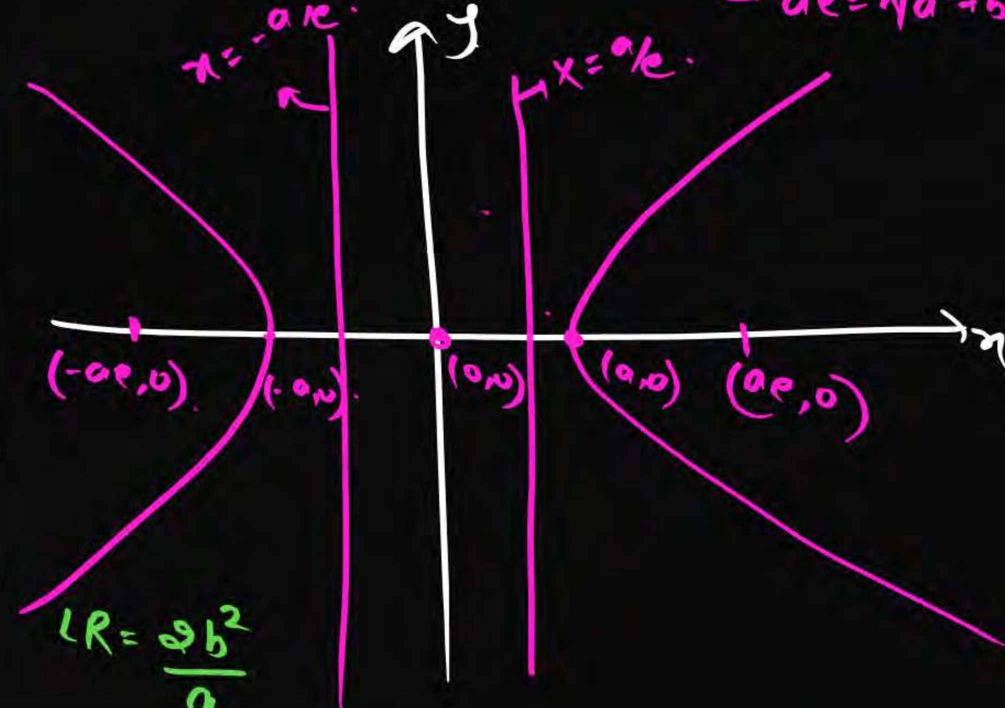


$$LR=\frac{2a^2}{b}$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\boxed{e=\sqrt{1+\frac{b^2}{a^2}}}$$

$$ae=\sqrt{a^2+b^2}$$



$$LR=\frac{2b^2}{a}$$

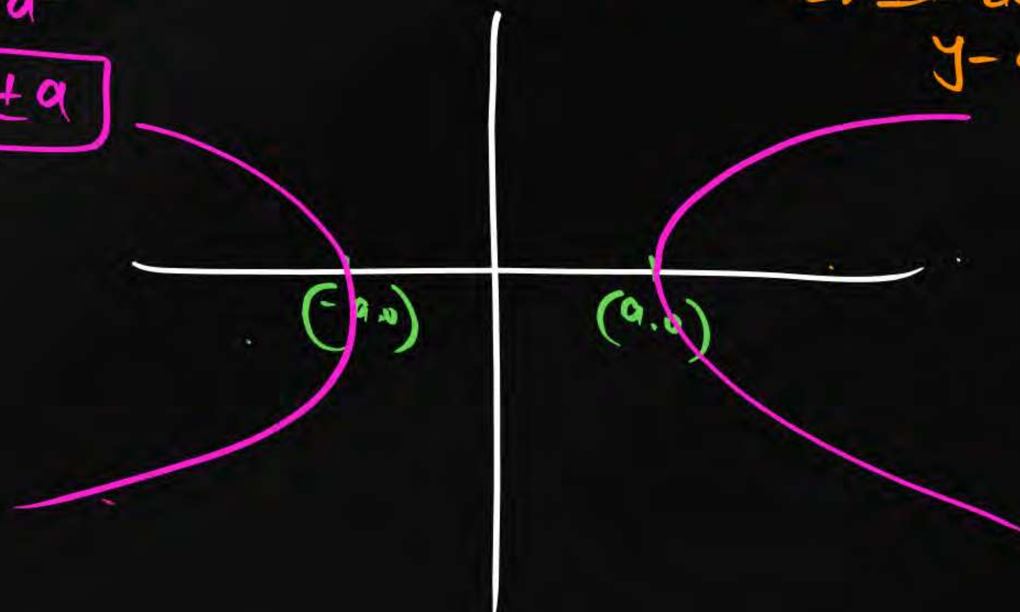
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad x>a$$

put $y=0$, $\frac{x^2}{a^2}=1$

$$\boxed{x=\pm a}$$

put $x=0$, $\frac{-y^2}{b^2}=1 \Rightarrow y^2=-b^2$.

$\Rightarrow$ It does not cut the y-axis.

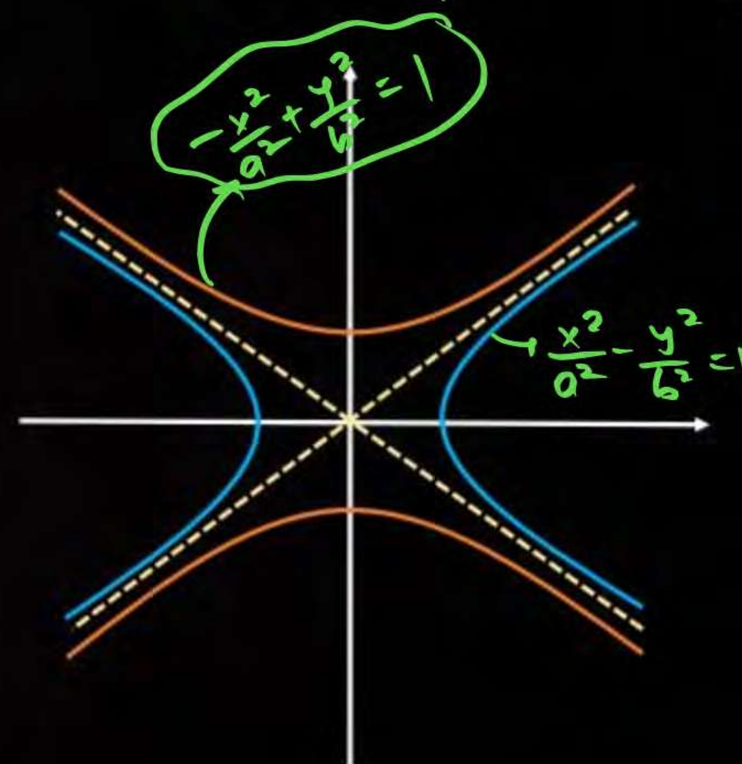


Important Results:

Result-01 : The foci of a hyperbola and its conjugate are concyclic and form the vertices of square.

Result-02 : If e_1 & e_2 are eccentricities of hyperbola and its conjugate then :

$$\boxed{\frac{1}{e_1^2}+\frac{1}{e_2^2}=1}$$

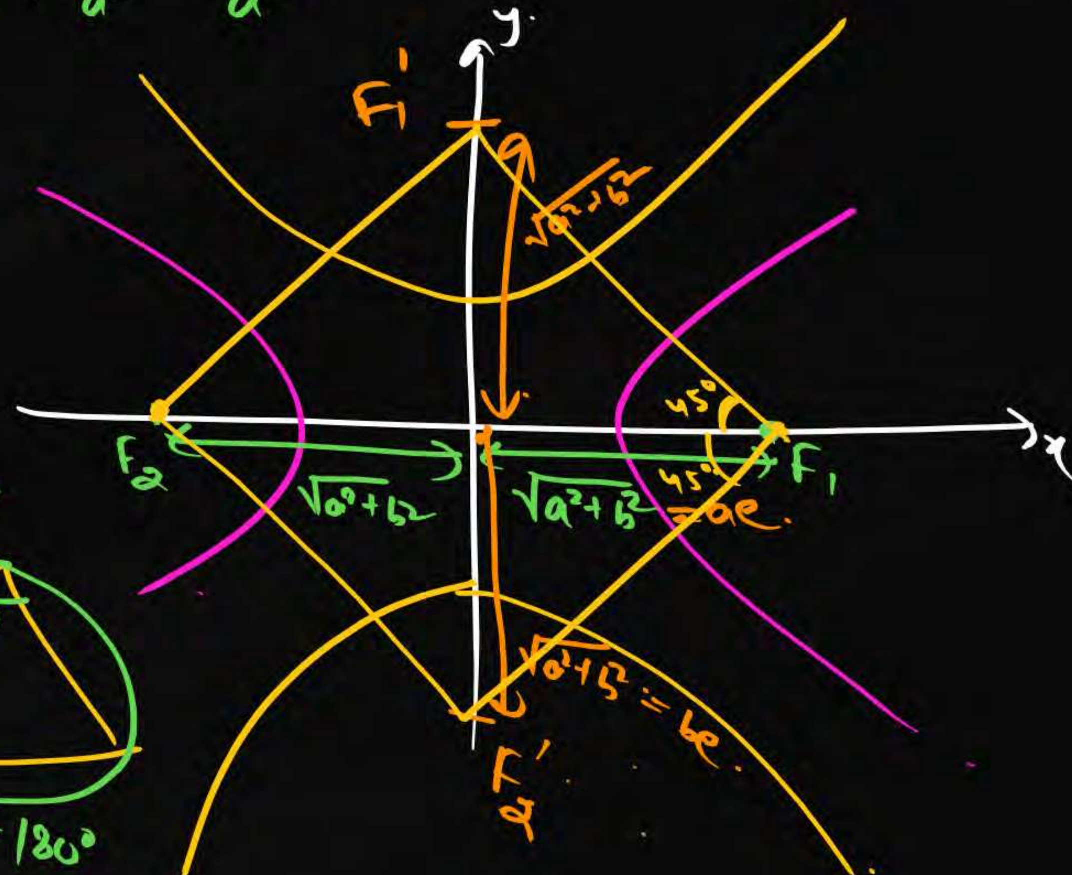


$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \rightarrow e_1^2 = 1+\frac{b^2}{a^2}=\frac{a^2+b^2}{a^2}$$

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \rightarrow e_2^2 = 1+\frac{a^2}{b^2}=\frac{b^2+a^2}{b^2}$$

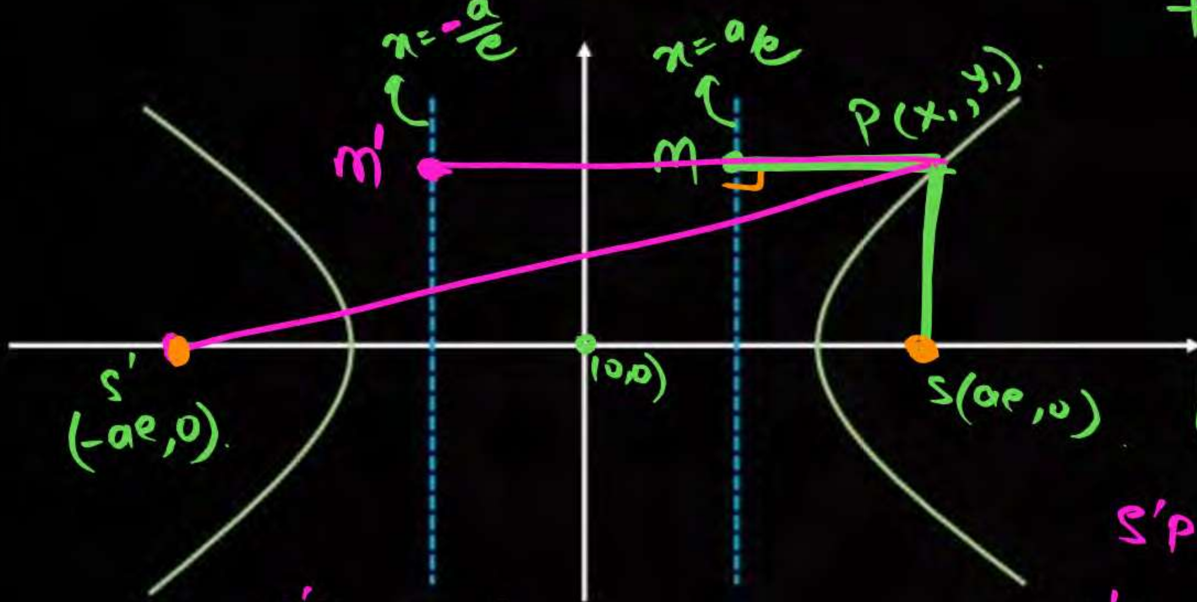
$$\boxed{\frac{1}{e_1^2}+\frac{1}{e_2^2}=1}$$

$\alpha+\beta=180°$



Focal Directrix Property

Focal distance :- dist of a point on HB from focus.



$SP = e\,PM$

$SP = e\left(x_1 - \frac{a}{e}\right)$

$\boxed{SP = ex_1 - a}$

$S'P = e\,PM'$

$S'P = e\left(x_1 + a/e\right)$

$\boxed{S'P = ex_1 + a}$

$S'P - SP = (ex_1 + a) - (ex_1 - a)$

$\boxed{S'P - SP = 2a}$

Second Definition of HB

\# Locus of point which moves such that difference of its distances from two fixed points is constant.

$|S'P - SP| = \underbrace{2a}, < 2ae$

less than dist. b/w. S & S'.

Q $\left| \overbrace{|z-1|} - \overbrace{|z+1|} \right| = k$. What is the locus of z

if (i) $\boxed{k=1} \Rightarrow$ Hyperbola

(ii) $\boxed{k=2} \Rightarrow$ Line segment in left of A or right of B.

(iii) $\boxed{k=3} \Rightarrow$ No locus.

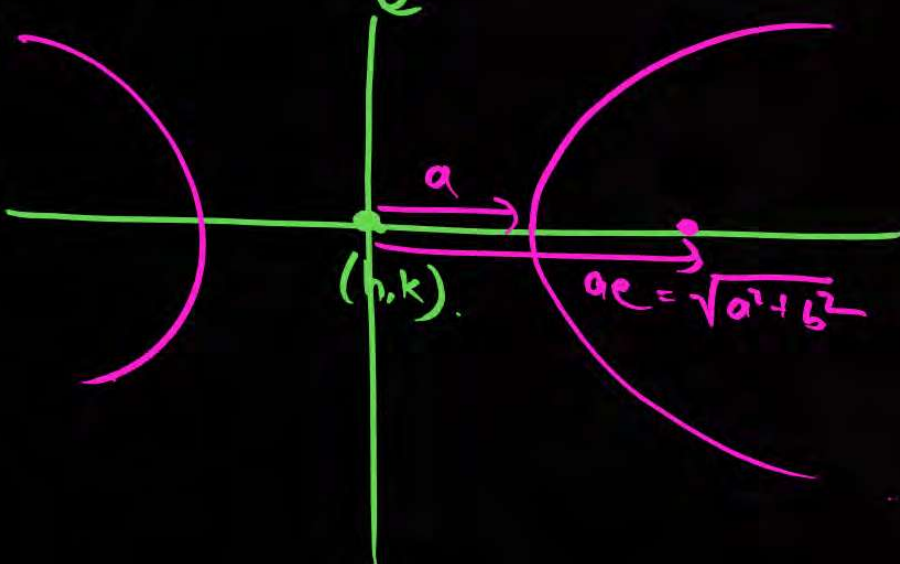
(iv) $k = 3/2 \rightarrow$ Hyperbola

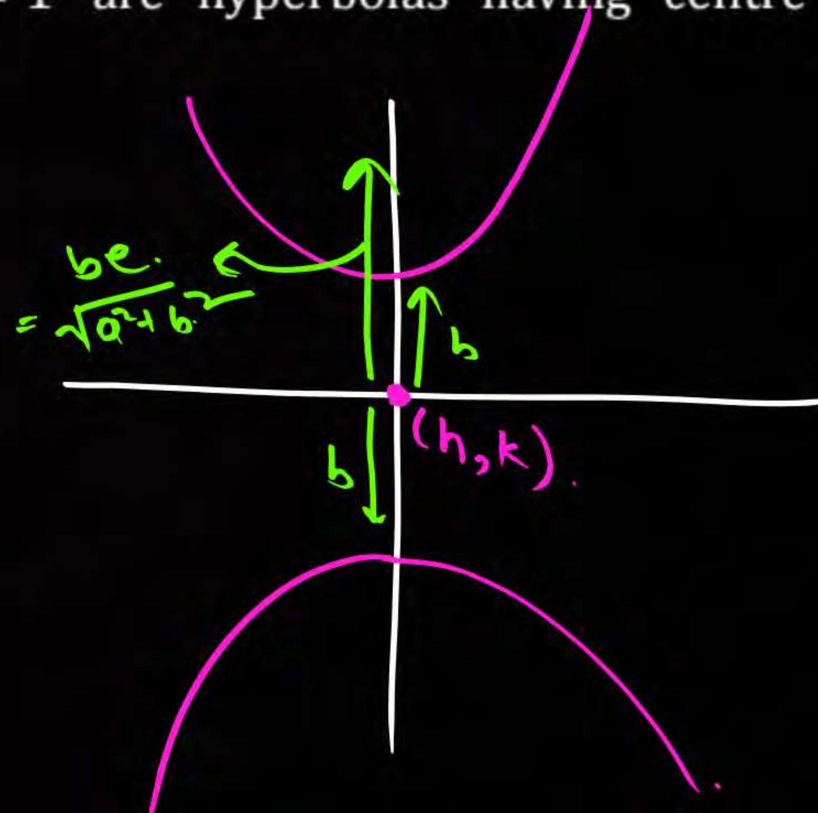
$AC - BC = 2$

Remark

TA = 2a
CA = 2b

$\boxed{\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1}$ and $-\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ are hyperbolas having centre at (h, k).





Q Draw the following hyperbolas and mention their foci, transverse and conjugate axes, LR, directrices and centre.

(a) $\boxed{9x^2 - y^2 = 1}$

$$\boxed{\frac{x^2}{1/9} - \frac{y^2}{1} = 1}$$

$\underbrace{a^2 = \frac{1}{9}}$ $\underbrace{b^2 = 1}$

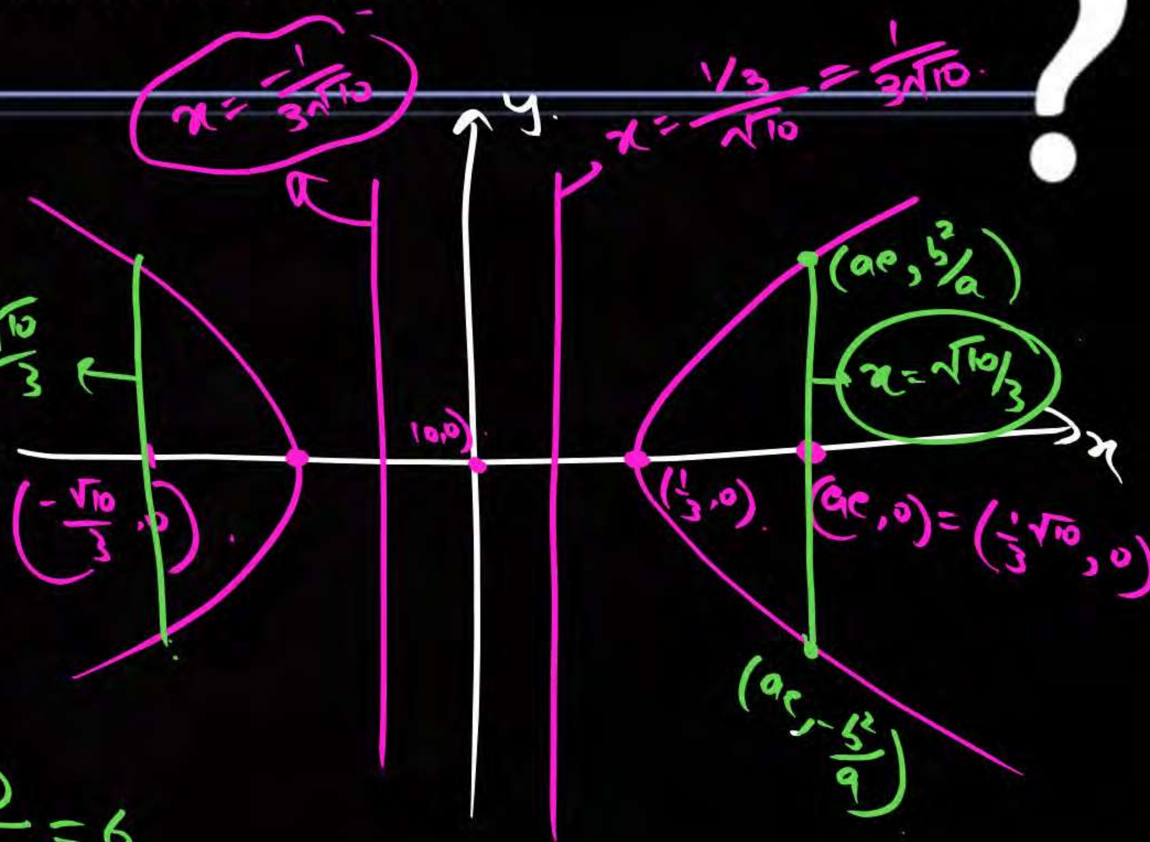
$e = \sqrt{1 + \frac{b^2}{a^2}}$

$= \sqrt{1 + \frac{1}{1/9}}$

$\therefore \boxed{e = \sqrt{10}}$

$LR = \frac{2b^2}{a}$

$= \frac{2(1)}{1/3} = 6.$



Q Draw the following hyperbolas and mention their foci, transverse and conjugate axes, LR, directrices and centre.

(b) $\boxed{16x^2 - 9y^2 = -144}$

$$-\frac{x^2}{9}+\frac{y^2}{16}=1$$

$a^2=9,\quad b^2=16$

$5=\sqrt{a^2+b^2}=be$

$4\cancel{b}e=5$

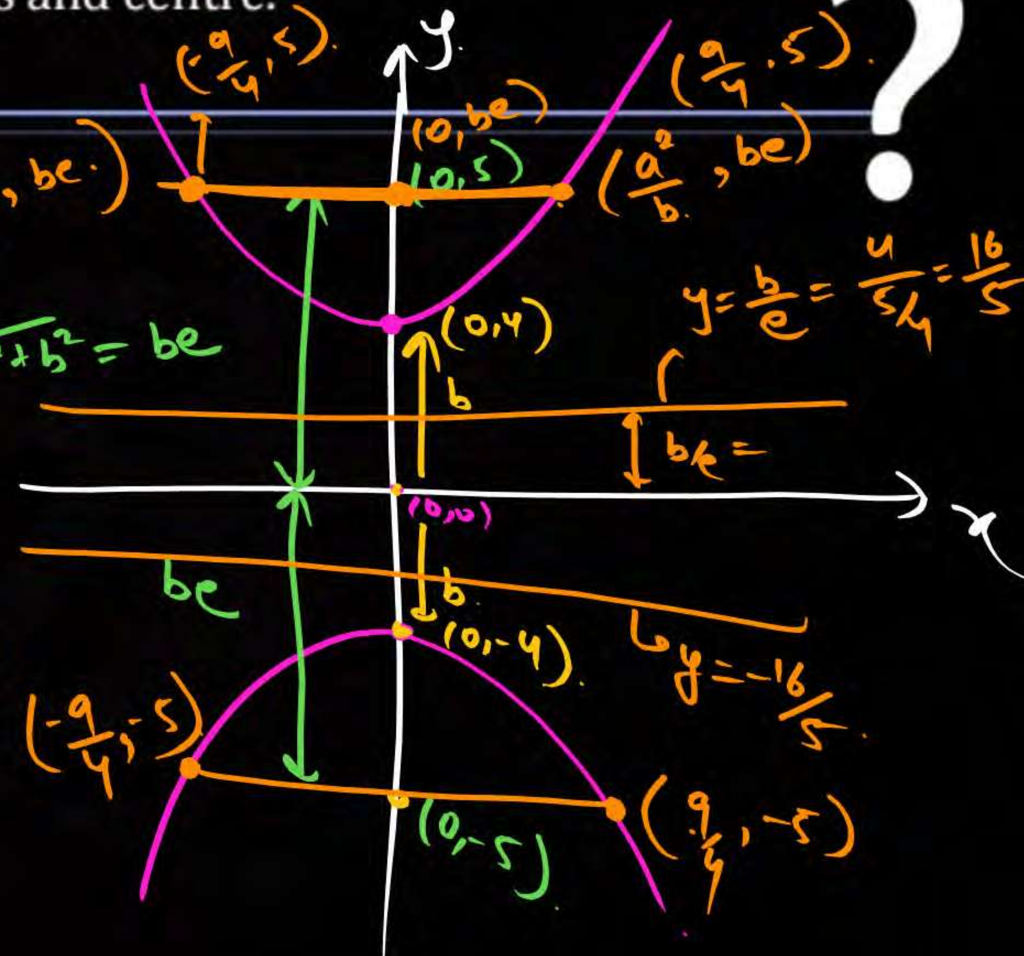
$\boxed{e=5/4}$

$LR=\frac{2a^2}{b}=\frac{2(9)}{4}$

$\boxed{LR=9/2}$

$y=\frac{b}{e}=\frac{4}{5/4}=\frac{16}{5}$

$y=-16/5$



Q Draw the following hyperbolas and mention their foci, transverse and conjugate axes, LR, directrices and centre.

(c) $7y^2 - 9x^2 + 54x - 28y - 116 = 0$

$$7y^2 - 28y - 9x^2 + 54x = 116$$

$$7(y^2 - 4y + 4) - 9(x^2 - 6x + 9) = 116 + 28 - 81$$

$$7(y-2)^2 - 9(x-3)^2 = 63$$

$$-\frac{\cancel{9}(x-3)^2}{\cancel{63}\,7} + \frac{\cancel{7}(y-2)^2}{\cancel{63}\,9} = 1$$

$$\boxed{-\frac{(x-3)^2}{7} + \frac{(y-2)^2}{9} = 1}$$

Q Draw the following hyperbolas and mention their foci, transverse and conjugate axes, LR, directrices and centre.

(c) $7y^2 - 9x^2 + 54x - 28y - 116 = 0$

$LR = \frac{2a^2}{b} = \frac{2(7)}{3} = \frac{14}{3}$

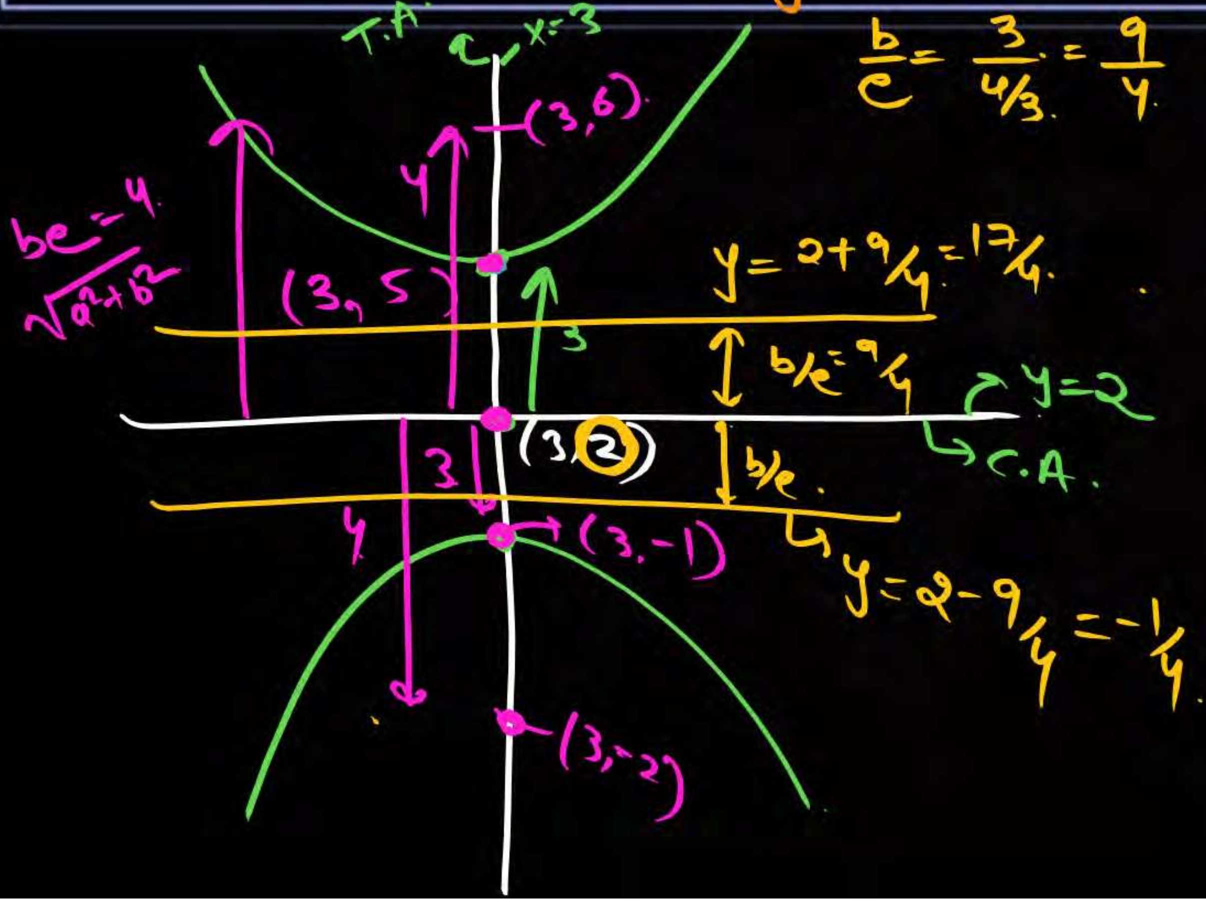
$$\boxed{\frac{-(x-3)^2}{7} + \frac{(y-2)^2}{9} = 1}$$

$a^2 = 7$ $\quad b^2 = 9$

$be = \sqrt{a^2+b^2}$

$3\cancel{b}e = \sqrt{7+9} = 4$

$e = 4/3$



Q Answer the following questions :

(a) If a vertex of a hyperbola is (1, 0), centre is origin and the corresponding focus is (2, 0), then find its equation.

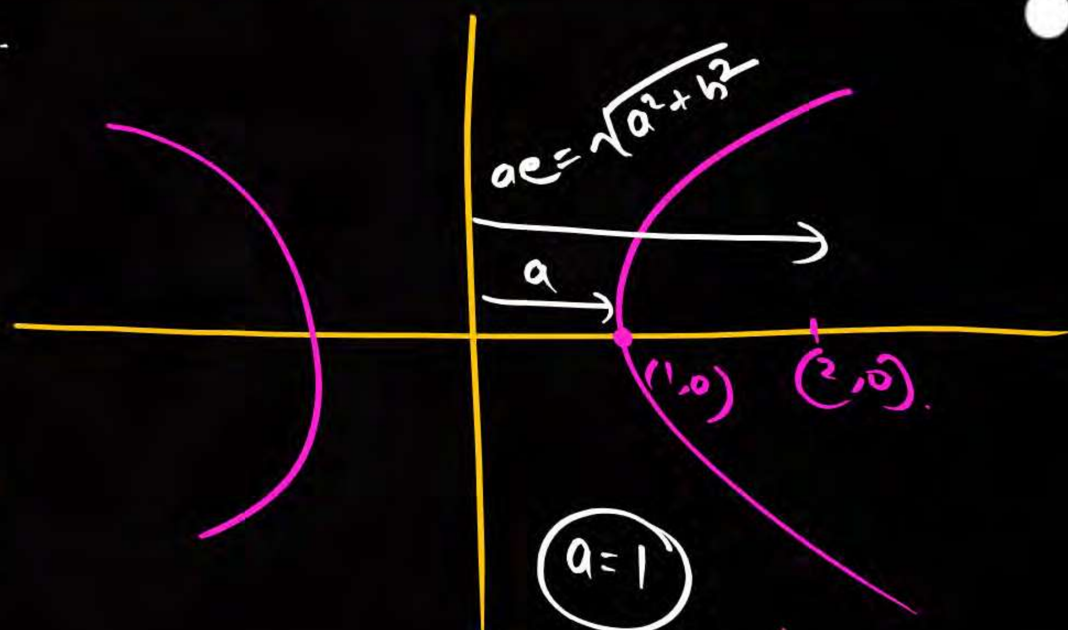
$$\sqrt{a^2+b^2} = 2.$$

$$a^2+b^2 = 4.$$

$$1+b^2 = 4$$

$$b^2 = 3$$

$$\boxed{\frac{x^2}{1} - \frac{y^2}{3} = 1}$$



Q Answer the following questions :

(b) A hyperbola of eccentricity 3, centred at the origin, has transverse axis along the Y-axis. If the distance between its foci is 6 units, then find its equation.

$e = 3$

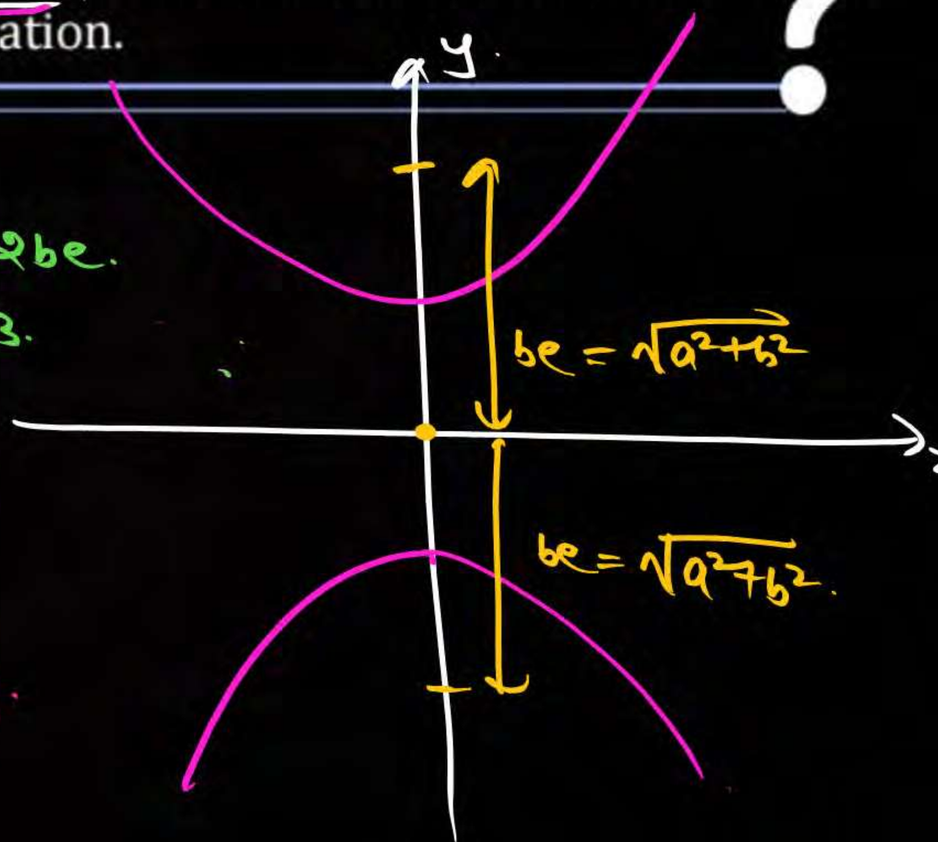
dist. b/w foci $= 2\sqrt{a^2+b^2} = 6 = 2be.$

$\boxed{\sqrt{a^2+b^2} = 3} = be$ $3.$

$\sqrt{1+\dfrac{a^2}{b^2}} = 3.$ $\Rightarrow a^2+b^2 = 9$ $b = 1$

$a^2 = 8$

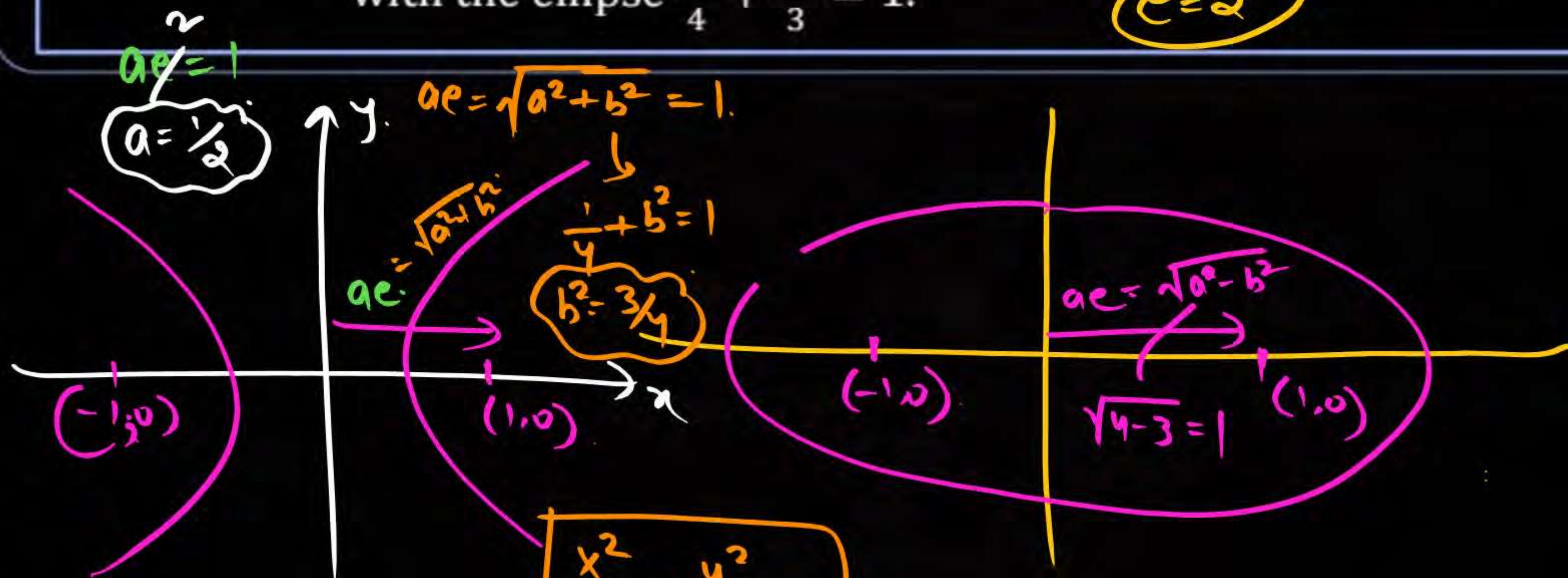
$\boxed{\dfrac{-x^2}{8} + \dfrac{y^2}{1} = 1}$ $\leftarrow$ $\dfrac{-x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$



Q Answer the following questions :

(a) Find equation of hyperbola of eccentricity 2 that is confocal with the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$.

$e=2$



$$\boxed{\frac{x^2}{\frac{1}{4}}-\frac{y^2}{\frac{3}{4}}=1}$$

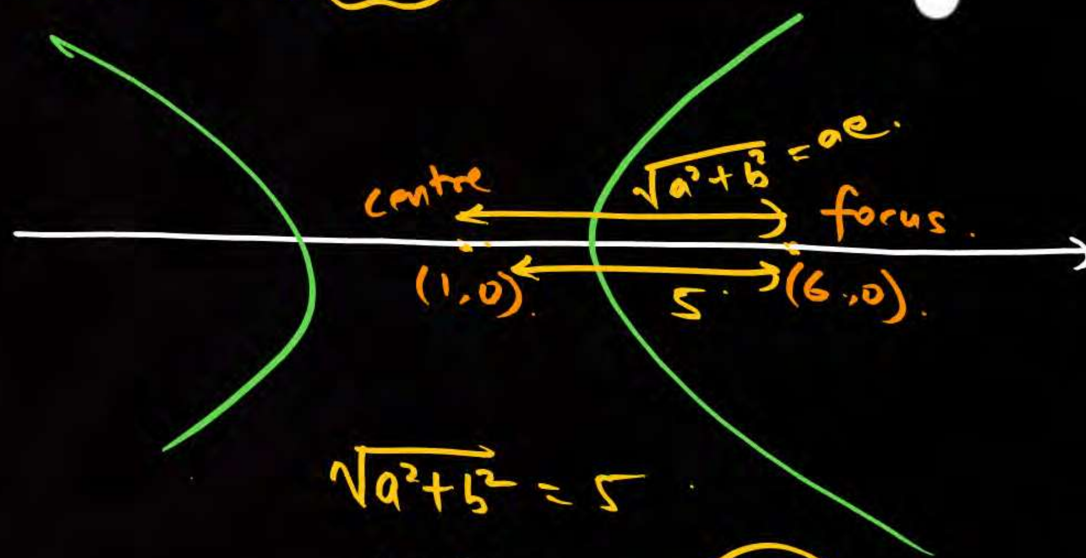
Q Answer the following questions :

(b) Find equation of hyperbola whose centre is (1, 0), one focus is (6, 0) and transverse axis is of length 6 units.

$2a = 6 \Rightarrow a = 3$

$$\boxed{\frac{(x-1)^2}{a^2} - \frac{(y-0)^2}{b^2} = 1}$$

$$\boxed{\frac{(x-1)^2}{9} - \frac{y^2}{16} = 1}$$



$\sqrt{a^2+b^2} = 5$

$9 + b^2 = 25 \Rightarrow b^2 = 16$

Q. Let H : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a > b > 0$, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$.

$a = 2\sqrt{3}$ $b = 2$

[JEE (Adv.)-2018 (Paper-1)]

List-I		List-II	
(A)	The length of the conjugate axis of H is ($2b = 4$)	(P)	8
(B)	The eccentricity of H is $= \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{4}{12}} = \frac{2}{\sqrt{3}}$	(Q)	$\frac{4}{\sqrt{3}}$
(C)	The distance between the foci of H is (or $2\sqrt{a^2+b^2}$) $\to 2ae = 2.(2\sqrt{3}).2/\sqrt{3} = 8$	(R)	$\frac{2}{\sqrt{3}}$
(D)	The length of the latus rectum of H is ($\frac{2b^2}{a}$)	(S)	4

$\tan 30^\circ = \frac{b}{a}$

$\frac{1}{\sqrt{3}} = \frac{b}{a} \Rightarrow \boxed{a = \sqrt{3}\,b}$ — (1)

$a = 2\sqrt{3}$

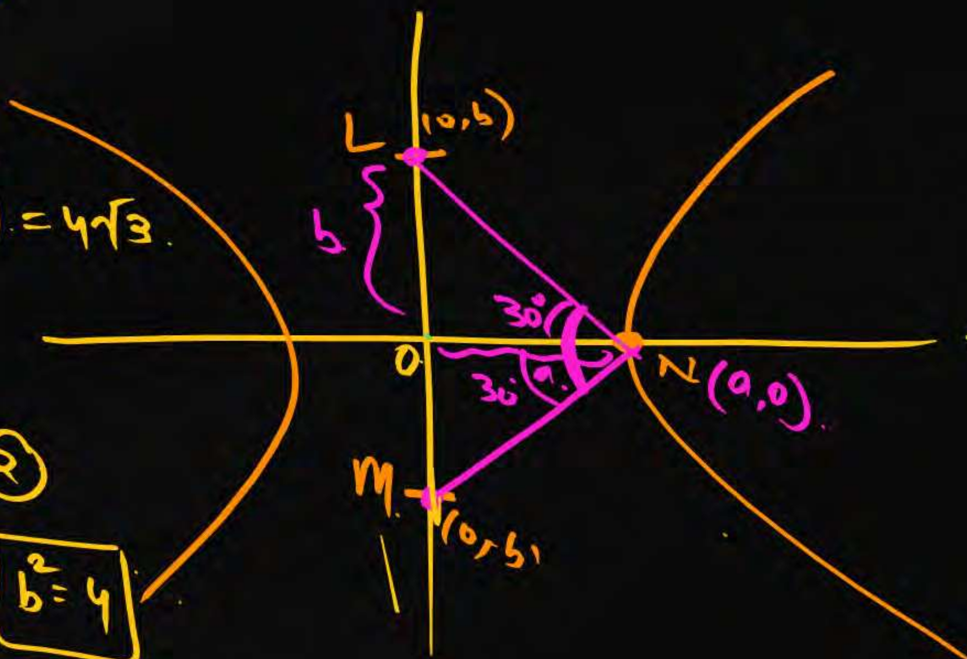
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Area of △LMN: $\frac{1}{2}\overbrace{(LM)}^{2b}\overbrace{(ON)}^{a} = 4\sqrt{3}$

$\frac{1}{\cancel{2}}(\cancel{2}b)(a) = 4\sqrt{3}$

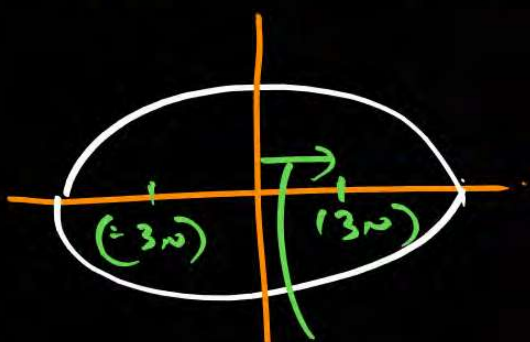
$ab = 4\sqrt{3}$ — (2)

$(\sqrt{3}b)(b) = 4\sqrt{3} \Rightarrow \boxed{b^2 = 4}$



Q. If E : $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ & $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ are confocal then find b^2 =?

$$\frac{x^2}{\underbrace{\frac{144}{25}}_{a^2}}-\frac{y^2}{\underbrace{\frac{81}{25}}_{b^2}}=1$$



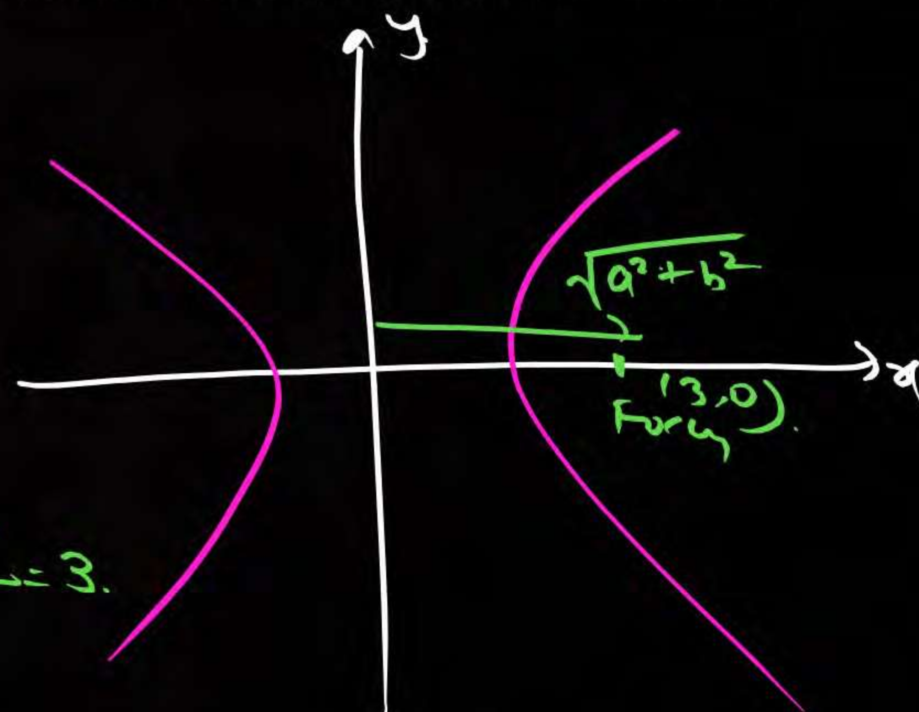
$$\sqrt{16-b^2}=3$$

$$16-b^2=9$$

$$\boxed{b^2=7}$$

$$\sqrt{a^2+b^2}=\sqrt{\frac{144}{25}+\frac{81}{25}}$$

$$=\sqrt{\frac{225}{25}}=\frac{15}{5}=3.$$



Rectangular Hyperbolas

If $a = b$, that is lengths of transverse and conjugate axes are equal, then the hyperbola is called rectangular or equilateral.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$b = a$.

E.g. The hyperbola $x^2 - y^2 = a^2$ is a rectangular hyperbola.

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1+1} = \sqrt{2}.$$

Rrectangular Hyperbola

The Hyperbola whose :

Length of T.A. = Length of C.A. = Length of L.R. $= \frac{2b^2}{a} = \frac{2a^2}{a} = 2a$.

Or

Whose eccentricity (e) = $\sqrt{2}$

Or

Whose asymptotes are perpendicular

Or

Whose director circle is a point circle $\Rightarrow x^2 + y^2 = a^2 - b^2 = 0$.

Or

Whose 'e' is equal to eccentricity of CHB

Or

Whose equation is : $\boxed{x^2 - y^2 = a^2}$

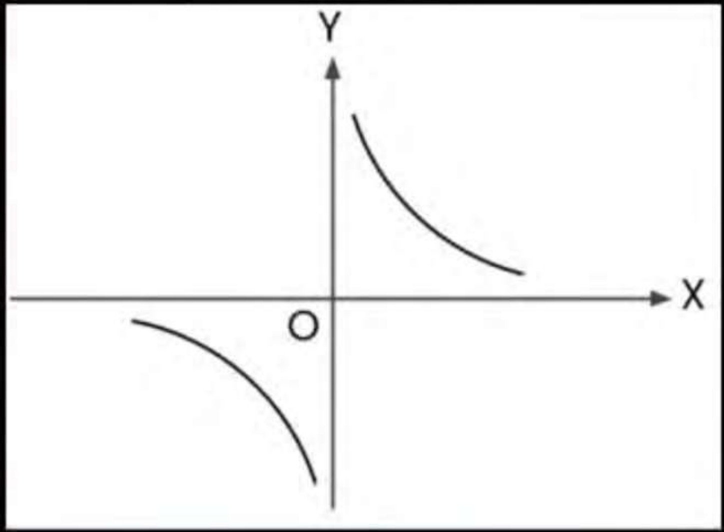
Rectangular Hyperbolas

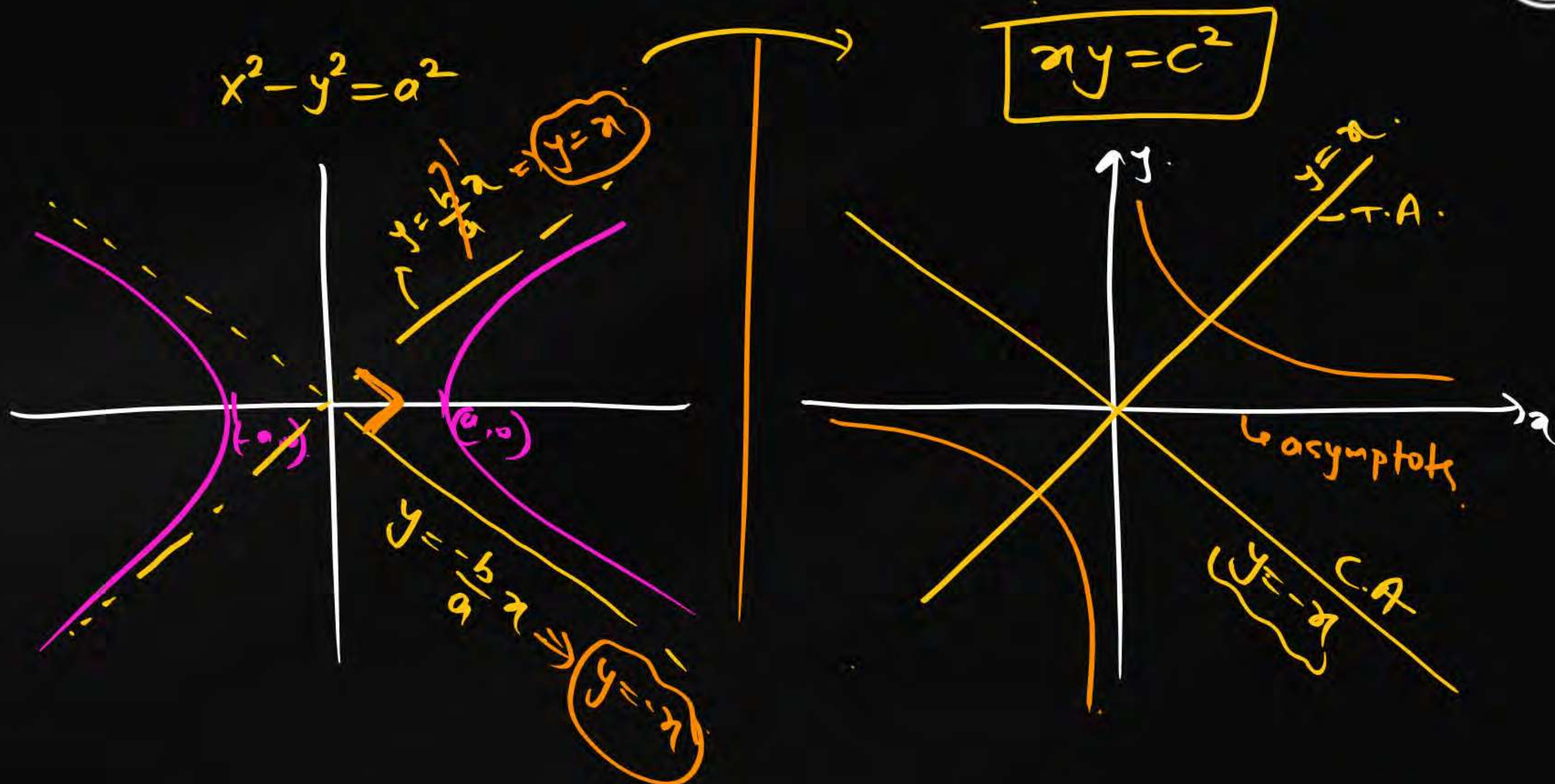
Remark

1. Eccentricity of an equilateral hyperbola is always $\sqrt{2}$.
2. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a rectangular hyperbola if $\Delta \neq 0$ and $a + b = 0$.

Note

$\boxed{xy = c^2}$ is one of the most commonly used rectangular hyperbolas. It's parametric form $\left(ct, \frac{c}{t}\right)$ is very important.





x²−y²=a²
y=(b/a)x = y=x
(−a,0)
(a,0)
y=−(b/a)x ⇒ y=−x
xy=c²
y
y=x
T.A.
x
asymptote
C.A
y=−x

Tangent & Normal

$xy = c^2 \rightarrow (x, y) \equiv \left(ct, \frac{c}{t}\right)$

Tangent

(i) $\boxed{\text{At } P(x_1, y_1)}$ $T: \boxed{\frac{xy_1 + x_1y}{2} = c^2} \Rightarrow xy_1 + x_1y = 2c^2$

$y = -\frac{y_1}{x_1}x + \ldots$

$m_T = -y_1/x_1$

$m_N = x_1/y_1$

(ii) Parametric Form :

$x \rightarrow ct$

$y \rightarrow c/t$.

Normal

(i) At $P(x_1, y_1)$ $\Longrightarrow$ $\boxed{y - y_1 = \frac{x_1}{y_1}(x - x_1)}$.

(ii) Parametric Form :

Q Eccentricity of $xy = 4$ is ________.

A. 1.5

B. 2

C. $1/2$

D. $\sqrt{2}$

Q What are the conditions on λ and μ such that the equation $x^2 + 2\lambda xy - 2x + 2y = 0$ represents a rectangular hyperbola ?

A $\lambda = 0, \mu \neq 1$

B $\lambda = 0, \mu \neq -1$

C $\lambda \neq 0, \mu = -1$

D $\lambda \neq 0, \mu = 1$

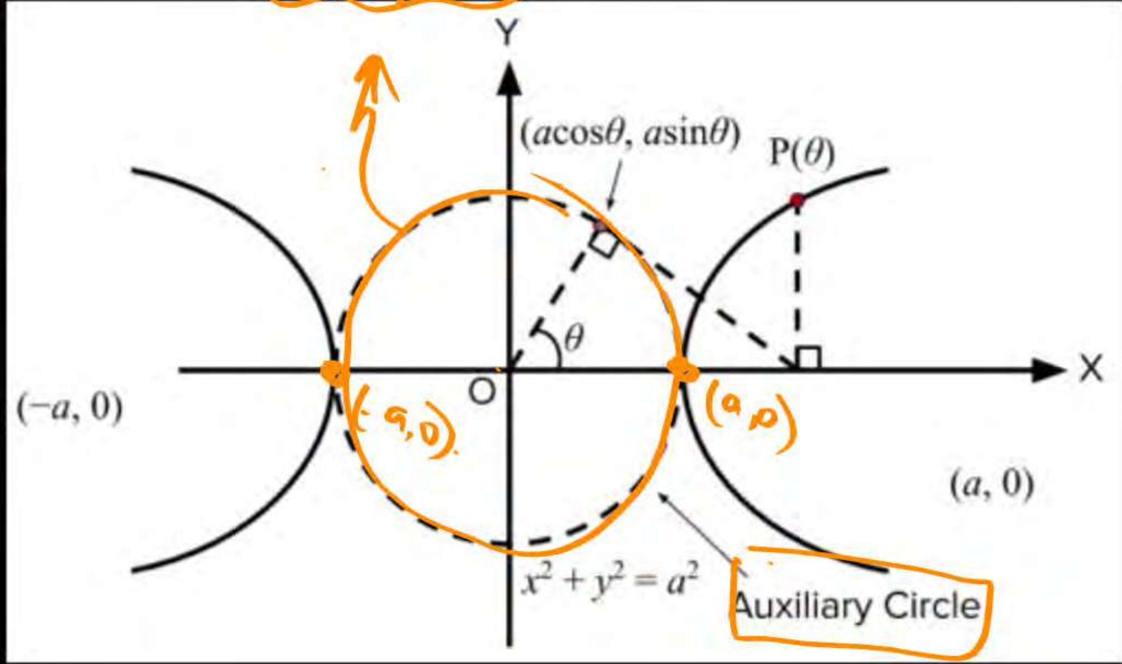
Parametric form of Equation of a Hyperbola

The parametric form of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x = a \sec \theta$, $y = b \tan \theta$

Here θ is a parameter and is called the eccentric angle. Lets see in the figure, which angle q represents.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$x^2+y^2=a^2$



Note :

Hyperbola | **Parametric Point**

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1} \Rightarrow (x,y)\equiv(a\sec\theta, b\tan\theta).$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1 \Rightarrow (x,y)=(a\tan\theta, b\sec\theta).$$

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Position of a Point with respect to a Hyperbola

$$\boxed{S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0}$$

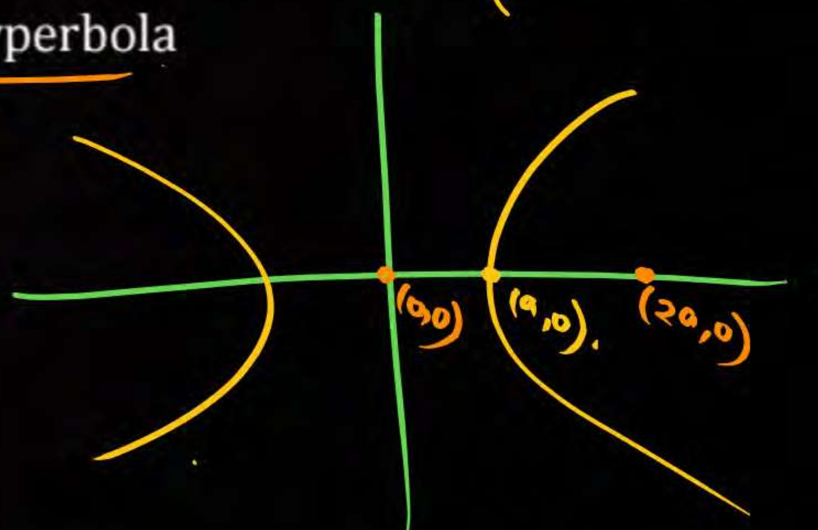
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Result

(1) $\boxed{S_1 >= 0}$ $\Rightarrow$ point lies inside hyperbola

(2) $S_1 == 0$ $\Rightarrow$ point lies on hyperbola

(3) $S_1 <= 0$ $\Rightarrow$ point lies outside hyperbola



Position of a Point with respect to a Hyperbola

Remark

From an external point, two tangents can be drawn.

Equations of Tangents of a Hyperbola

Ellipse

Eqⁿ of tangent

① $\boxed{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1}$ $\qquad y=mx\pm\sqrt{a^2m^2+b^2}$

$b^2 \longrightarrow -b^2$

② $\boxed{\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1}$ $\qquad y=mx\pm\sqrt{a^2m^2-b^2}$

③ $-\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$ $\qquad y=mx\pm\sqrt{-a^2m^2+b^2}.$

Equations of Tangents of a Hyperbola

As we have done earlier, here also we have three standard equations.

(a) Slope form is $y = mx \pm \sqrt{a^2m^2 - b^2}$

(b) Tangent at a point on hyperbola is given by T = 0.

(c) Parametric form is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$T: \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

$x_1 = a\sec\theta,\ y_1 = b\tan\theta$.

Equations of Tangents of a Hyperbola

These are tangents for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

For $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, just expresses it as $\frac{x^2}{(-a^2)} - \frac{y^2}{(-b^2)} = 1$.

Hence, tangent of slope m is given by $y = mx \pm \sqrt{-a^2m^2 + b^2}$.

Q If the line $2x + \sqrt{6}y = 2$ touches the hyperbola $x^2 - 2y^2 = 4$, then the point of contact is ______________.

A $(-2, \sqrt{6})$

B $(-5, 2\sqrt{6})$

C $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$

D $(4, -\sqrt{6})$

$T = 0$

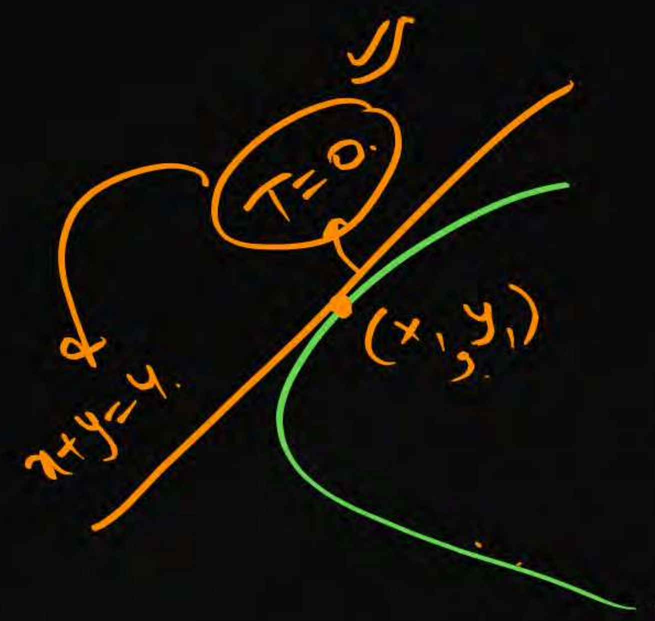
$x(x_1) - 2(yy_1) = 4$

$2x + \sqrt{6}y = 2$

(x_1, y_1)

$\frac{x_1}{2} = \frac{-2y_1}{\sqrt{6}} = \frac{\cancel{4}\,2}{\cancel{2}}$

$x_1 = 4, \quad y_1 = -\sqrt{6}$



T=0.
x+y=4.
(x1, y1)

Q Find equations of common tangents of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $-\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$.

$a^2 \to -b^2$.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \rightarrow y=mx\pm\sqrt{a^2m^2+b^2}$

A $y=\pm x\pm\sqrt{a^2+b^2}$

B $y=\pm x\pm\sqrt{a^2-b^2}$

C $y=\pm x\pm\sqrt{b^2-a^2}$

D None of these

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \rightarrow y=mx\pm\sqrt{a^2m^2-b^2}$

$-\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \rightarrow y=mx\pm\sqrt{-b^2m^2+a^2}$

$\boxed{y=\pm x\pm\sqrt{a^2-b^2}}$

$a^2m^2-b^2=-b^2m^2+a^2$

$m^2(a^2+b^2)=a^2+b^2$

$m^2=1 \Rightarrow m=\pm 1$

Equations of Tangents of a Hyperbola

Result

Equation of director circle of $\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ is $\underbrace{x^2 + y^2 = a^2 - b^2}$.

Proof: $\cancel{y}^{k} = m\cancel{x}^{h} \pm \sqrt{a^2m^2 - b^2}$

$(k - mh)^2 = (\pm\sqrt{a^2m^2 - b^2})^2$

$k^2 + m^2h^2 - 2mkh = a^2m^2 - b^2$

$m^2(\underbrace{h^2 - a^2}) - m(2kh) + \underbrace{k^2 + b^2} = 0$

$\boxed{m_1m_2 = -1}$

$m_1 m_2 = -1$

$\frac{k^2 + b^2}{h^2 - a^2} = -1$

$k^2 + b^2 = -h^2 + a^2$

$k^2 + h^2 = a^2 - b^2$

$y^2 + x^2 = a^2 - b^2$

Director circle.

(1) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$x^2+y^2=a^2+b^2$.

(2) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\boxed{x^2+y^2=a^2-b^2}$ centre $=(0,0)$.

$r=\sqrt{a^2-b^2}$.

(3) $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$\boxed{x^2+y^2=-a^2+b^2}$

Equations of Normals of a Hyperbola

(a) Equation of normal at $P(x_1, y_1)$ on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\boxed{\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2}$$

$a\sec\theta$ $b\tan\theta$

For ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

eqⁿ of Normal: $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$.

Equations of Normals of a Hyperbola

(b) Equation of normal at P(θ) on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$$

Q Equation of the normal to the hyperbola $\frac{x^2}{2} - y^2 = 1$, at the point $(2, 1)$, is ____________.

A $2x + y = 5$

B $2x - y = 3$

C $x + y = 3$

D $x - y = 1$

Q. Tangent are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line $\boxed{2x - y = 1}$. The point of contact of the tangents on the hyperbola are

$m = 2$

$y = mx \pm \sqrt{9m^2 - 4}$

[IIT-JEE-2012 (Paper-1)]

(A) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(B) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(C) $(3\sqrt{3}, -2\sqrt{2})$

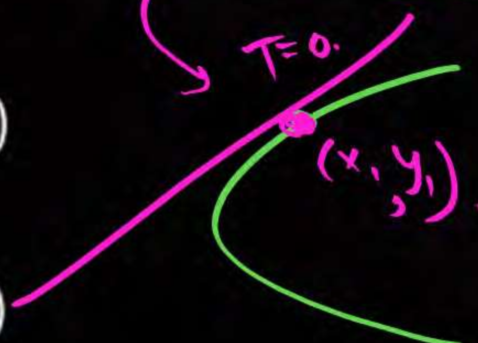
(D) $(-3\sqrt{3}, 2\sqrt{2})$

$y = 2x \pm \sqrt{9(4) - 4}$

$\boxed{y = 2x \pm 4\sqrt{2}}$

$\boxed{\frac{xx_1}{9} - \frac{yy_1}{4} = 1}$

$\boxed{-2x + y = \pm 4\sqrt{2}}$



$\frac{x_1/9}{-2} = \frac{-y_1/4}{1} = \frac{1}{\pm 4\sqrt{2}}$

Q. Let P(6, 3) $=(x_1, y_1)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9, 0), then the eccentricity of the hyperbola is

....

[IIT-JEE-2011 (Paper-2)]

(A) $\sqrt{\frac{5}{2}}$

(B) $\sqrt{\frac{3}{2}}$

(C) $\sqrt{2}$

(D) $\sqrt{3}$

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2.$$

$$\frac{a^2 \cancel{x}^{9}}{6} + \frac{b^2 \cancel{y}^{0}}{3} = a^2 + b^2.$$

$$a^2 \frac{\cancel{9}^{3}}{\cancel{6}_{2}} + 0 = a^2 + b^2$$

$$\frac{3a^2}{2} = a^2 + b^2.$$

$$\boxed{\frac{3}{2} = 1 + \frac{b^2}{a^2} = e^2}$$

Q. Consider a branch of the hyperbola $x^2-2y^2-2\sqrt{2}x-4\sqrt{2}y-6=0$, with vertex at the point A. Let B be one of the end points of its rectum. If C is the focus of the hyperbola nearest to the point A. Then the area of the triangle ABC is

[IIT-JEE-2006 (Paper-2)]

$2(\sqrt{2})(x)$

$$\left(x^2-2\sqrt{2}x+2\right)-2\left(y^2+2\sqrt{2}y+(\sqrt{2})^2\right)=6+2-4$$

$$(x-\sqrt{2})^2-2(y+\sqrt{2})^2=4$$

$$\boxed{\frac{(x-\sqrt{2})^2}{4}-\frac{(y+\sqrt{2})^2}{2}=1}$$

a^2

(A) $1-\sqrt{\frac{2}{3}}$

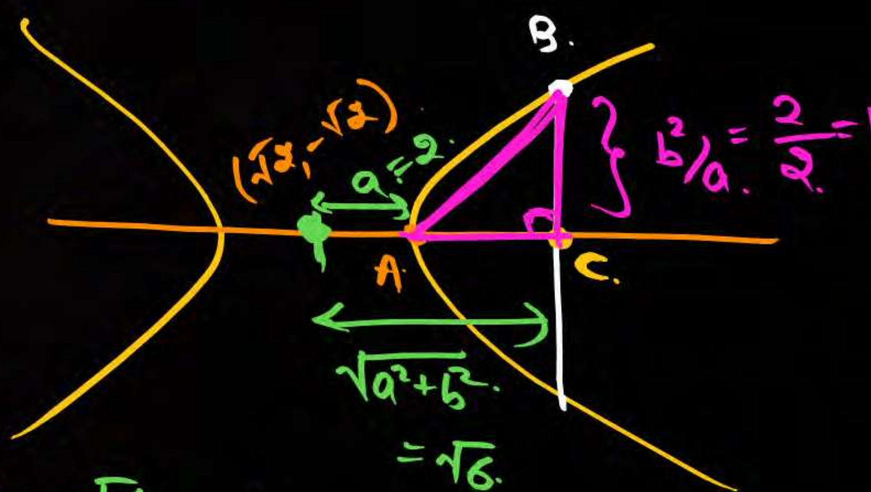
(B) $\sqrt{\frac{3}{2}}-1$

(C) $1+\sqrt{\frac{2}{3}}$

(D) $\sqrt{\frac{3}{2}}+1$

$$\text{Area}=\frac{1}{2}\times(AC)(BC)=\frac{1}{2}(\sqrt{6}-2)=\frac{\sqrt{6}}{2}-1$$

$AC \to (\sqrt{6}-2)$, $BC \to 1$



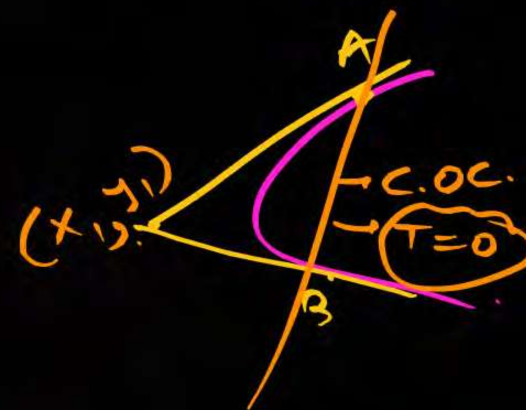
Four Important Terms

1. Chord of Contact: $\boxed{T=0}$

2. Chord with given midpoint: $\boxed{T=S_1}$

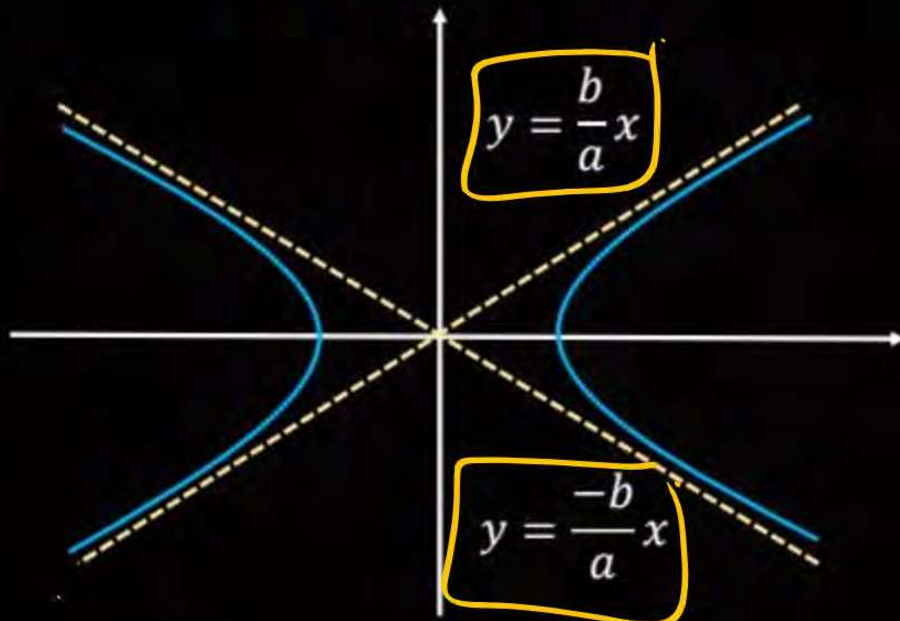
3. Pair of Tangents: $\boxed{SS_1=T^2}$

4. ~~Pole & Polar:~~



Asymptotes

→ Tangents at infinity



$y = mx \pm \sqrt{\underbrace{a^2m^2 - b^2}}$

$a^2m^2 - b^2 \geq 0$

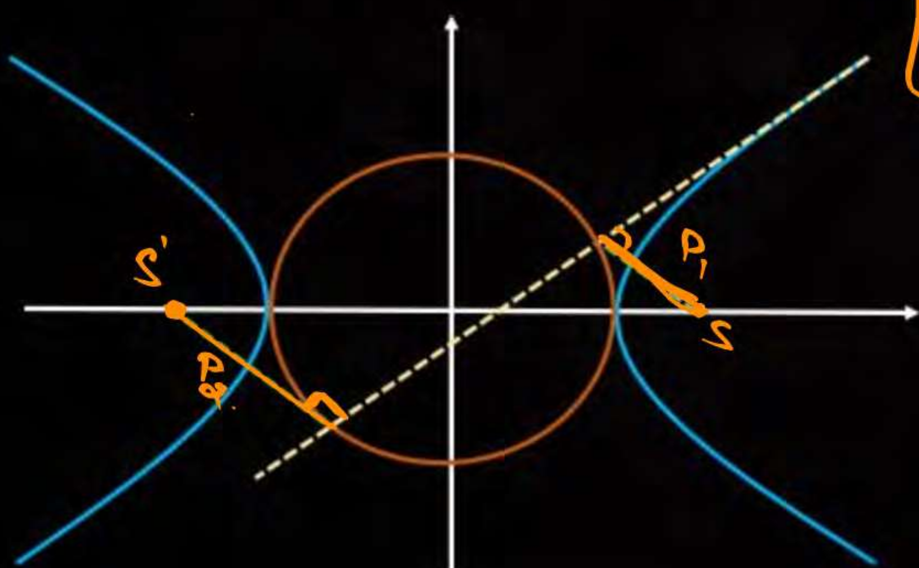
$m^2 \geq b^2/a^2$

$m \geq b/a$ or $m \leq -b/a$

Properties of HB

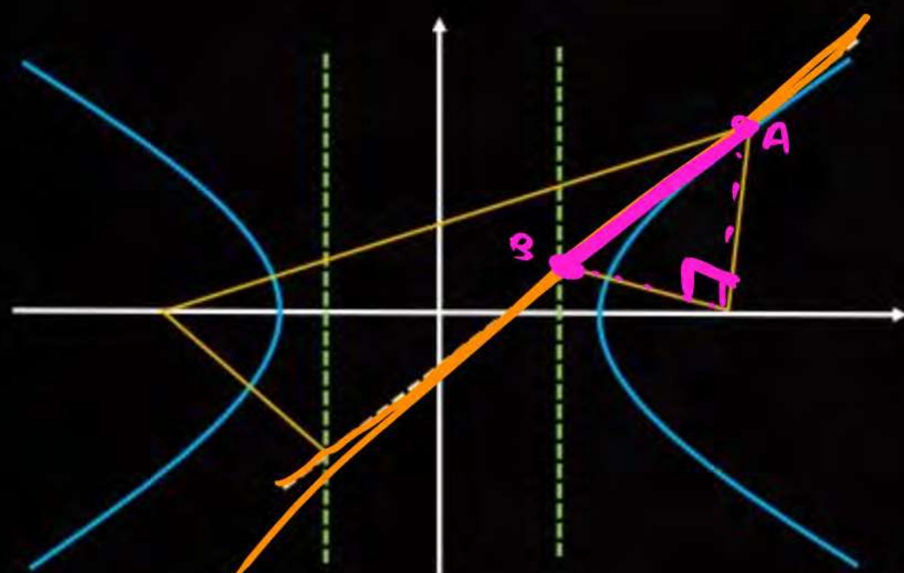
P-1: Locus of foot of perpendicular drawn from foci on any tangent is Auxiliary Circle.

P-2: Product of lengths of perpendiculars from foci on Tangent is always constant & equals to (semi-conjugate axis)^2

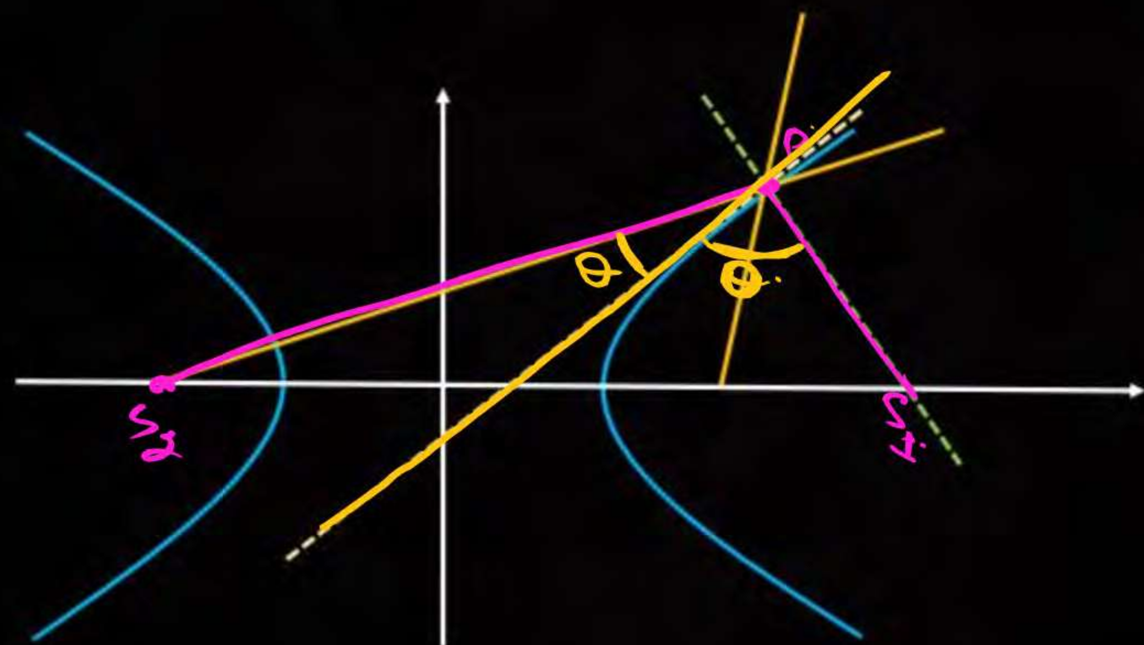


$$\boxed{P_1P_2 = b^2}$$

P-3: Portion of tangent intercepted between point of contact and directrix subtend 90° at corresponding focus.

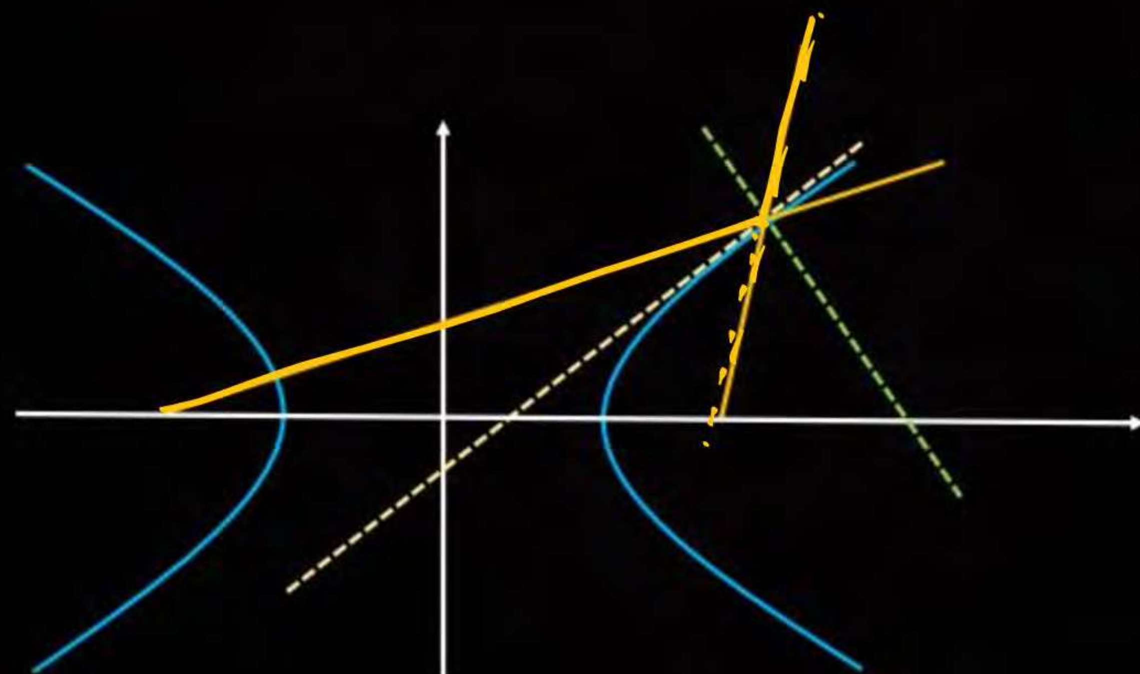


P-4: Tangent and Normal at any point P bisects the angle between focal distances (PS_1 & PS_2).



Reflection Property

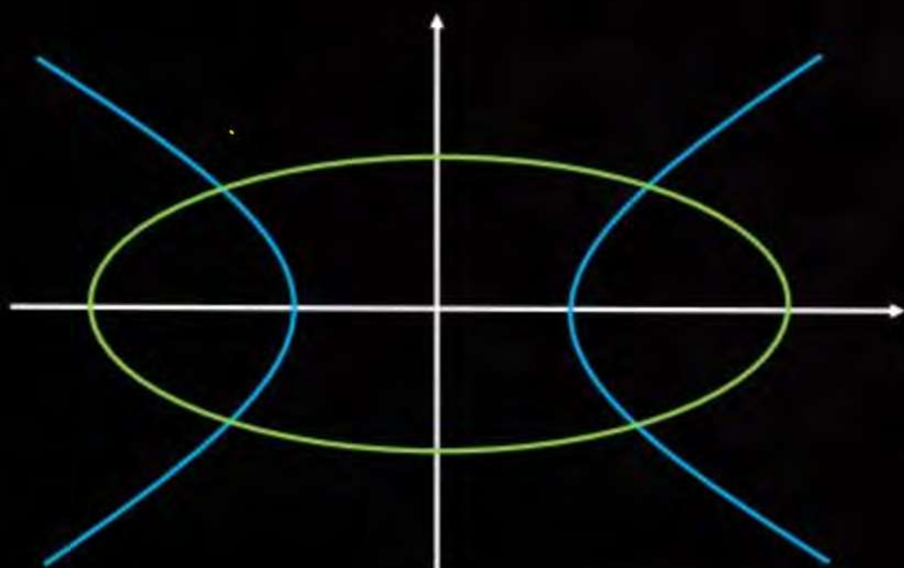
Any ray passing through one focus, after reflection from Hyperbola it passes from another focus.



P-5: Using Reflection Property we can say that:

If Ellipse & Hyperbola are confocal (having same foci) then they are Orthogonal (angle between tangents at point of intersection is 90^0)

Conversely if Ellipse & Hyperbola are Orthogonal they are Confocal.



Q. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then

[IIT-JEE-2009 (Paper-2)]

$e_H = \sqrt{2}$

$e_E = \frac{1}{\sqrt{2}}$

same focus.

$\frac{x^2}{1/2} - \frac{y^2}{1/2} = 1.$

(A) Equation of ellipse is $x^2 + 2y^2 = 2$

(B) The foci of ellipse are $(\pm 1, 0)$

$1/\sqrt{2}$

$a\cancel{e} = 1$

$\Rightarrow \boxed{a = \sqrt{2}}$

(C) Equation of ellipse is $x^2 + 2y^2 = 4$

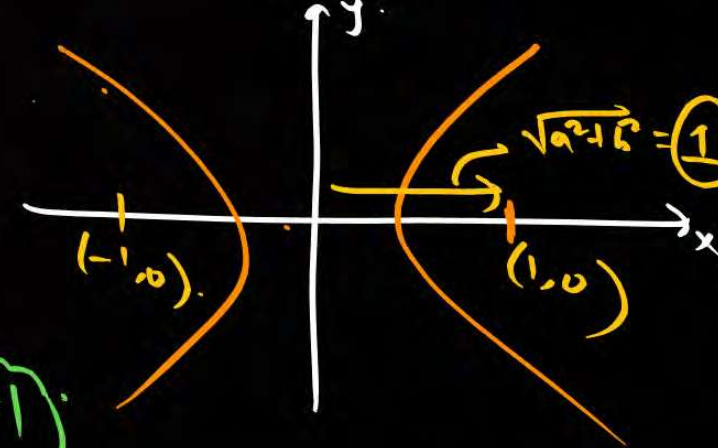
$ae = \sqrt{a^2 - b^2} = 1$

$2 - b^2 = 1$

$\Rightarrow \boxed{b^2 = 1}$

$\frac{x^2}{2} + \frac{y^2}{1} = 1$

(D) The foci of ellipse are $(\pm\sqrt{2}, 0)$



Q. If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, $S(x_4, y_4)$, then **[JEE-1998, 2M]**

$y = c^2/x$

(A) $x_1 + x_2 + x_3 + x_4 = 0$

(B) $y_1 + y_2 + y_3 + y_4 = 0$

(C) $x_1 x_2 x_3 x_4 = c^4$

(D) $y_1 y_2 y_3 y_4 = c^4$

$$x^2 + \left(\frac{c^2}{x}\right)^2 = a^2$$

$$x^4 + c^4 = a^2 x^2$$

$$\boxed{x^4 - a^2 x^2 + c^4 = 0}$$

x_1, x_2, x_3, x_4

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

Sum of roots $= -b/a$

$$\boxed{x_1 x_2 x_3 x_4 = e/a}$$

$$x_1 + x_2 + x_3 + x_4 = \frac{-\cancel{b}^{\,0}}{a} = 0$$

$$\boxed{x_1 x_2 x_3 x_4 = c^4}$$

Q. Let $\lambda x - 2y = \mu$ be a tangent to the hyperbola $a^2x^2 - y^2 = b^2$. Then $\left(\frac{\lambda}{a}\right)^2 - \left(\frac{\mu}{b}\right)^2$ is equal to:

[JEE Main 2022 (24 June – Shift 1)]

(A) –2

(B) –4

(C) 2

(D) 4 ✓

$$\boxed{y = mx \pm \sqrt{\frac{b^2}{a^2}m^2 - b^2}}$$

Tangent

$$\boxed{y = \frac{\lambda}{2}x - \frac{\mu}{2}}$$

$$\frac{a^2x^2}{b^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{\left(\frac{b^2}{a^2}\right)} - \frac{y^2}{(b^2)} = 1$$

$m = \lambda/2$

$$\pm\sqrt{\frac{b^2}{a^2}m^2 - b^2} = -\frac{\mu}{2}$$

$$\frac{b^2}{a^2}m^2 - b^2 = \frac{\mu^2}{4} \Rightarrow \frac{b^2}{a^2}\frac{\lambda^2}{4} - b^2 = \frac{\mu^2}{4} \Rightarrow \boxed{\frac{1}{4}\frac{\lambda^2}{a^2} - 1 = \frac{\mu^2}{4b^2}}$$

Q. Let the hyperbola $H : \boxed{\frac{x^2}{a^2} - y^2 = 1}$ and the ellipse $E : 3x^2 + 4y^2 = 12$ be such that the length of latus rectum of H is equal to the length of latus rectum of E. If e_H and E_E are the eccentricities of H and E respectively, then the value of $12(e_H^2 + e_E^2)$ is equal to :

$$\frac{x^2}{4} + \frac{y^2}{3} = 1 \rightarrow \text{Horizontal ellipse}$$

[JEE Main 2022 (24 June – Shift 2)]

LR : $\frac{2(1)}{a} = \frac{\cancel{2}(3)}{\cancel{2}}$

$\boxed{a = 2/3}$

$e_H = \sqrt{1 + \frac{1}{a^2}} = \sqrt{1 + \frac{9}{4}} = \sqrt{\frac{13}{4}}$

$e_E = \sqrt{1 - 3/4} = \frac{1}{2}$

$12\left(\frac{13}{4} + \frac{1}{4}\right) = 42$ Ans

Q. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be 5/4. If the equation of the normal at the point on $\left(\frac{8}{\sqrt{5}}, \frac{12}{5}\right)$ the hyperbola is $8\sqrt{5}x + \cancel{\beta}^{15} y = \cancel{\lambda}^{100}$, then $\lambda - \beta$ is equal to: $= \boxed{85}$ Ans.

$e = \sqrt{1+\frac{b^2}{a^2}} \Rightarrow a^2e^2 = a^2+b^2$

[JEE Main 2022 (25 June - Shift 2)]

$\boxed{8\sqrt{5}x + 15y = 100}$

$1+\frac{b^2}{a^2} = \frac{25}{16} \Rightarrow \boxed{\frac{b^2}{a^2} = \frac{9}{16}}$

$\sqrt{1+\frac{b^2}{a^2}} = \boxed{e = 5/4}$

$\frac{a^2x}{8/\sqrt{5}} + \frac{b^2y}{12/5} = \underbrace{a^2+b^2}_{a^2e^2}$

divide by a^2

$\frac{\sqrt{5}x}{8} + \frac{5\frac{b^2}{a^2}y}{12} = \frac{25}{16}$

$\frac{\sqrt{5}x}{\cancel{8}_1} + \frac{5\times\cancel{9}^3}{\cancel{16}_2}\frac{y}{\cancel{12}_4} = \frac{25}{\cancel{16}_2}$

$\frac{\sqrt{5}a^2x}{8} + \frac{5b^2y}{12} = a^2\left(\frac{25}{16}\right)$

$\sqrt{5}x + \frac{15}{8}y = \frac{25}{2}$

Q. Let the eccentricity of the hyperbola $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be $\sqrt{\frac{5}{2}}$ and length of its latus rectum be $6\sqrt{2}$, if $y = \underset{m}{2}x + c$ is a tangent to the hyperbola H, then the value of c^2 is equal to:

[JEE Main 2022 (28 June - Shift 1)]

(A) 18

(B) 20 ✓

(C) 24

(D) 32

$e = \sqrt{\frac{5}{2}} = \sqrt{1 + \frac{b^2}{a^2}}$

$\boxed{\frac{b^2}{a^2} = \frac{3}{2}}$

$\frac{2b^2}{a} = 6\sqrt{2}$

$\cancel{2}\left(\frac{3}{\cancel{2}}a^{\cancel{2}}\right) = 6\sqrt{2}\,\cancel{a}$

$a = 2\sqrt{2}$

$\frac{b^2}{8} = \frac{3}{2}$

$b^2 = 12$

$y = \cancel{m}^{2}x \pm \sqrt{a^2m^2 - b^2}$

$y = 2x \pm \sqrt{8(2)^2 - 12}$

$y = 2x \pm \sqrt{20}$

$c = \pm\sqrt{20} \Rightarrow c^2 = 20$

Q. Let $a > 0$, $b > 0$. Let e and l respectively be the eccentricity and length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Let e' and l' respectively the eccentricity and length of the latus rectum of its conjugate hyperbola. If $e^2 = \frac{11}{14} l$ and $(e')^2 = \frac{11}{8} l'$ then the value of $77a + 44b$ is equal to :

[JEE Main 2022 (28 June – Shift 2)]

(A) 100

(B) 110

(C) 120

(D) 130

$1 + \frac{b^2}{a^2} = \frac{11}{14}\left(\frac{2b^2}{a}\right)$

$\boxed{a^2 + b^2 = \frac{11}{7} b^2 a}$

$\frac{\cancel{11}}{7} b^2 a = \frac{\cancel{11}}{4} a^2 b$

$\Rightarrow \boxed{4b = 7a}$

$1 + \frac{a^2}{b^2} = \frac{11}{8}\left(\frac{2a^2}{b}\right)$

$\boxed{b^2 + a^2 = \frac{11}{4} a^2 b}$

$\frac{49\cancel{a^2}}{16} + \cancel{a^2} = \frac{11}{4}\cancel{a^2}\left(\frac{7a}{4}\right)$

$\boxed{65 = 77a}$

$\underbrace{77a}_{=} + 44b$

$65 + \cancel{44}\left(\frac{7a}{\cancel{4}}\right)$

$65 + 77a$ 65

$= 130$

Q. Let $H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $a > 0$, $b > 0$ be a hyperbola such that the sum of lengths of the transverse and the conjugate axes is $4(2\sqrt{2} + \sqrt{14})$. If the eccentricity H is $\frac{\sqrt{11}}{2}$, then value of $\underbrace{a^2}_{32} + \underbrace{b^2}_{4\times 14}$ is equal to :

[JEE Main 2022 (29 June – Shift 1)]

$2a + 2b = 4(2\sqrt{2} + \sqrt{14})$

$\boxed{a + b = 2(2\sqrt{2} + \sqrt{14})} \Rightarrow a + \frac{\sqrt{7}}{2}a = 2\sqrt{2}(2 + \sqrt{7})$

$\Rightarrow a\left(\frac{\cancel{2+\sqrt{7}}}{2}\right) = 2\sqrt{2}\,\cancel{(2+\sqrt{7})}$

$\Rightarrow a = 4\sqrt{2}$

$e_H = \sqrt{1 + \frac{b^2}{a^2}} = \frac{\sqrt{11}}{2}$

$1 + \frac{b^2}{a^2} = \frac{11}{4} \Rightarrow \boxed{\frac{b^2}{a^2} = \frac{7}{4}} \Rightarrow b = \sqrt{\frac{7}{4}}\,a \Rightarrow b = \frac{\sqrt{7}}{\cancel{2}}(\cancel{4}^{2}\sqrt{2}) = 2\sqrt{14}$

Q. Let the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{7}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{\alpha}=\frac{1}{25}$ coincide. Then the length of the latus rectum of the hyperbola is :

$$\frac{x^2}{\frac{144}{25}}-\frac{y^2}{\frac{\alpha}{25}}=1$$

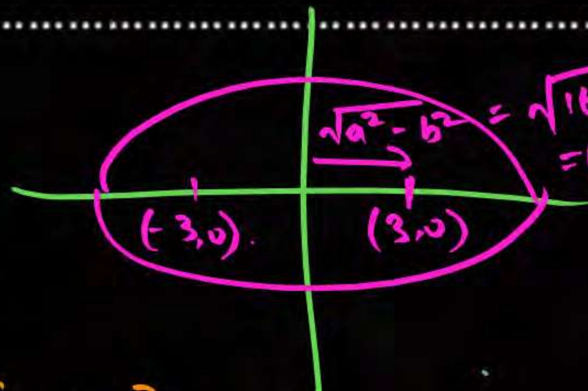
[JEE Main 2022 (25 July – Shift 2)]

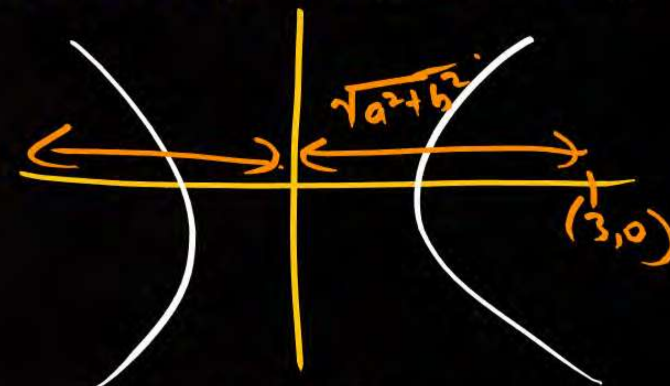
(A) 32/9

(B) 18/5

(C) 27/4

(D) 27/10 ✓





$$\text{LR} = \frac{2b^2}{a} = \frac{2\left(\frac{\alpha}{25}\right)}{\frac{12}{5}} = \frac{\alpha}{30} = \frac{81}{30}$$

$$\sqrt{a^2+b^2}=3$$

$$\frac{144}{25}+\frac{\alpha}{25}=9 \Rightarrow \alpha+144=225$$

$$\alpha = 81$$

Q. Let the tangent drawn to the parabola $y^2 = 24x$ at the point (α, β) is perpendicular to the line $\underbrace{2x + 2y = 5}_{m=-1}$. Then the normal to the hyperbola $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$ at the point $(\alpha + 4, \beta + 4)$ does not pass through the point:

[JEE Main 2022 (26 July – Shift 1)]

(A) (25, 10)

(B) (20, 12)

(C) (30, 8)

(D) (15, 13)

$y^2 = 24x$

$\boxed{m_T = 1}$

$y = mx + \frac{a}{m}$

$\boxed{y = x + 6}$

$yy_1 = \frac{24(x + x_1)}{2}$

$\boxed{\beta y = 12x + 12\alpha}$

$\frac{\beta}{1} = \frac{12}{1} = \frac{12\alpha}{6}$

$\alpha = 6, \beta = 12$

$\boxed{\frac{x^2}{36} - \frac{y^2}{144} = 1}$

Normal at $(10, 16)$

Q. If the line $x - 1 = 0$ is a directrix of the hyperbola $kx^2 - y^2 = 6$, then the hyperbola passes through the point :

HW

[JEE Main 2022 (26 July - Shift 2)]

A $(-2\sqrt{5}, 6)$

B $(-\sqrt{5}, 3)$

C $(\sqrt{5}, -2)$

D $(-2\sqrt{5}, 3\sqrt{6})$

Q. An ellipse $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the vertices of the hyperbola H: $\frac{x^2}{49} - \frac{y^2}{64} = -1$. Let the major and minor axes of the ellipse E coincide with the transverse and conjugate axes of the hyperbola H. Let the product of the eccentricities of E and H be 1/2. If l is the length of the latus rectum of the ellipse E, then the value of 113 l is equal to :

....

[JEE Main 2022 (27 July – Shift 1)]

HW

Q. For the hyperbola H : $x^2 - y^2 = 1$ and the ellipse E: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$, let the

(1) Eccentricity of E be reciprocal of the eccentricity of H, and

(2) The line $y = \sqrt{\frac{5}{2}}x + K$ be a common tangent of E and H.

Then $4(a^2 + b^2)$ is equal to ________.

HW

[JEE Main 2022 (28 July - Shift 1)]

Q. Let the focal chord of the parabola P : $y^2 = 4x$ along the line L : $y = mx + c$, $m > 0$ meet the parabola at the points M and N. Let the line L be a tangent to the hyperbola H : $x^2 - y^2 = 4$. If O is the vertex of P and F is the focus of H on the positive x-axis, then the area of the quadrilateral OMFN is :

$a^2 = 4,\ b^2 = 4.$

$\boxed{y = \frac{2}{\sqrt{3}}x - \frac{2}{\sqrt{3}}}$

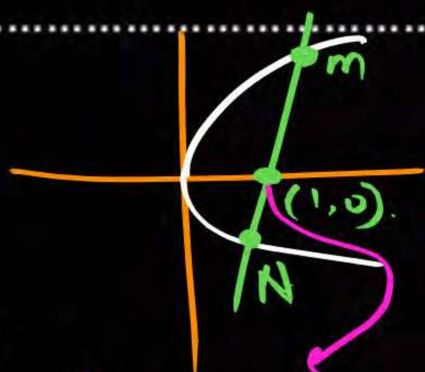
[JEE Main 2022 (29 July – Shift 1)]

(A) $2\sqrt{6}$

(B) $2\sqrt{14}$

(C) $4\sqrt{6}$

(D) $4\sqrt{14}$



$y = \frac{2}{\sqrt{3}}x \pm 2\sqrt{\frac{4}{3} - 1}$

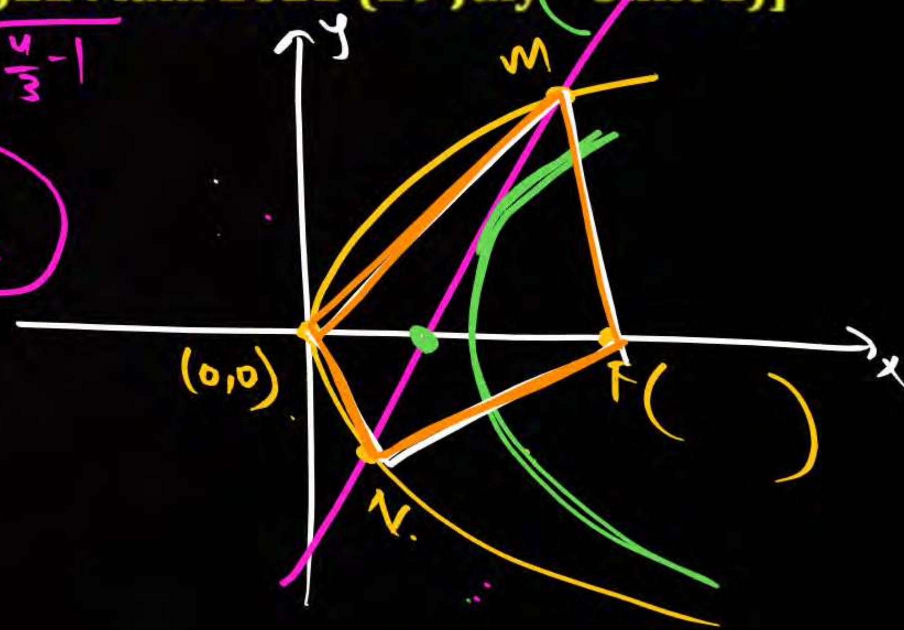
$\boxed{y = \frac{2}{\sqrt{3}}x \pm \frac{2}{\sqrt{3}}}$

$\overset{0}{\cancel{y}} = m\overset{1}{\cancel{x}} \pm \sqrt{4m^2 - 4}$

$0 = m \pm \sqrt{4m^2 - 4}$

$-m = \pm\sqrt{4m^2 - 4} \Rightarrow m^2 = 4/3$

$\boxed{m = 2/\sqrt{3}}$



Q. Let the focal chord of the parabola P : $y^2 = 4x$ along the line L : $y = mx + c$, $m > 0$ meet the parabola at the points M and N. Let the line L be a tangent to the hyperbola H : $x^2 - y^2 = 4$. If O is the vertex of P and F is the focus of H on the positive x-axis, then the area of the quadrilateral OMFN is : ....

$a^2 = 4,\ b^2 = 4$

$\boxed{y = \frac{2}{\sqrt{3}}x - \frac{2}{\sqrt{3}}}$

[JEE Main 2022 (29 July – Shift 1)]

(A) $2\sqrt{6}$

(B) $2\sqrt{14}$

(C) $4\sqrt{6}$

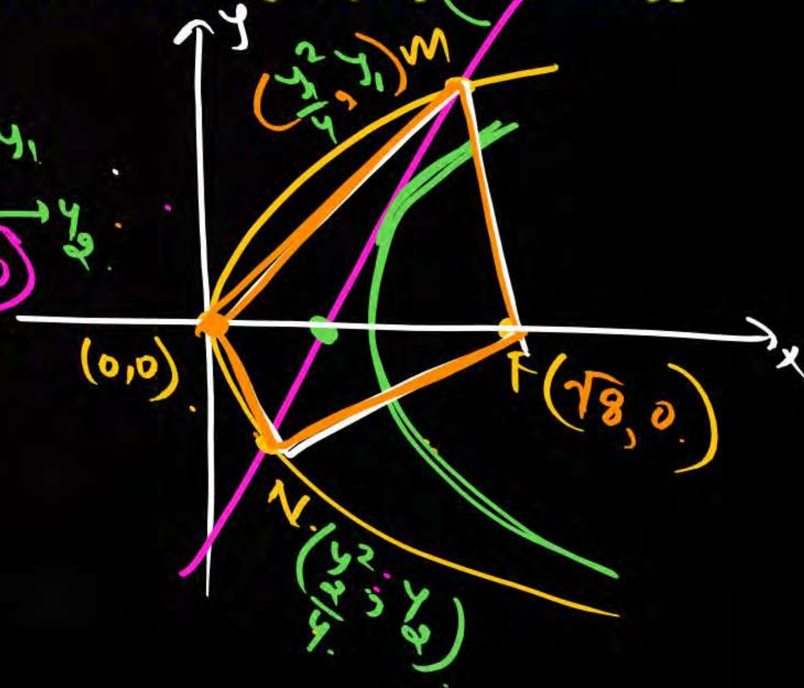
(D) $4\sqrt{14}$

To find M & N solve

$\boxed{y^2 = 4x}$ $\boxed{\sqrt{3}y = 2x - 2}$

$y^2 = 2(\sqrt{3}y + 2) \Rightarrow \boxed{y^2 - 2\sqrt{3}y + 4 = 0}$

y_1, y_2



Q. Let the focal chord of the parabola $P : y^2 = 4x$ along the line $L : y = mx + c$, $m > 0$ meet the parabola at the points M and N. Let the line L be a tangent to the hyperbola $H : x^2 - y^2 = 4$. If O is the vertex of P and F is the focus of H on the positive x-axis, then the area of the quadrilateral OMFN is :

$\boxed{y = \frac{2}{\sqrt{3}}x - \frac{2}{\sqrt{3}}}$

[JEE Main 2022 (29 July – Shift 1)]

(A) $2\sqrt{6}$

(B) $2\sqrt{14}$

(C) $4\sqrt{6}$

(D) $4\sqrt{14}$

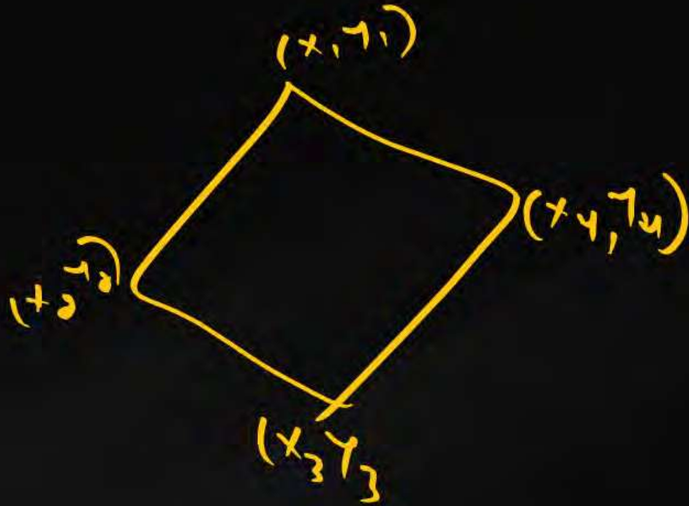
$$\text{Area} = \frac{1}{2}\begin{vmatrix} 0 & 0 \\ \frac{y_1^2}{4} & y_1 \\ \sqrt{8} & 0 \\ \frac{y_2^2}{4} & y_2 \\ 0 & 0 \end{vmatrix}$$

$$= \frac{1}{2}\left(\sqrt{8}y_2 - \sqrt{8}y_1\right)$$

$$= \frac{\sqrt{8}}{2}\left(y_2 - y_1\right)$$

$$= \sqrt{2}\sqrt{28}$$

$$\boxed{y^2 - 2\sqrt{3}y - 4 = 0}$$



$$\text{Area} = \frac{1}{2}\left|\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_1 & y_1 \end{matrix}\right|$$

$$= \frac{1}{2}\,\text{mag}\left(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - x_1y_4 - x_4y_3 - x_3y_2 - x_2y_1\right).$$