CHAPTER

2 Inverse Trigonometric Functions

Principal Values and Domains of Inverse Trigonometric/circular Functions

	Function	Domain	Range
(*i*)	$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(*ii*)	$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
(*iii*)	$y = \tan^{-1} x$	$x \in R$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(*iv*)	$y = \text{cosec}^{-1} x$	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2};\ y \ne 0$
(*v*)	$y = \sec^{-1} x$	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi;\ y \ne \frac{\pi}{2}$
(*vi*)	$y = \cot^{-1} x$	$x \in R$	$0 < y < \pi$

Properties of Inverse circular Functions

P-1:

(*i*) $y = \sin(\sin^{-1} x) = x$, $x \in [-1, 1]$, $y \in [-1, 1]$, y is aperiodic.

(*ii*) $y = \cos(\cos^{-1} x) = x$, $x \in [-1, 1]$, $y \in [-1, 1]$, y is aperiodic.

(*iii*) $y = \tan(\tan^{-1} x) = x$, $x \in R$, $y \in R$, y is aperiodic.

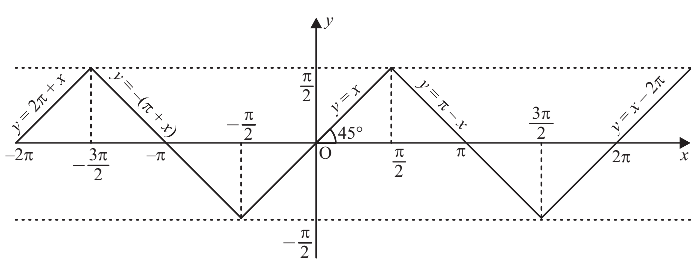
(*iv*) $y = \cot(\cot^{-1} x) = x$, $x \in R$, $y \in R$, y is aperiodic.

(*v*) $y = \text{cosec}(\text{cosec}^{-1} x) = x$, $|x| \ge 1$, $|y| \ge 1$, y is aperiodic.

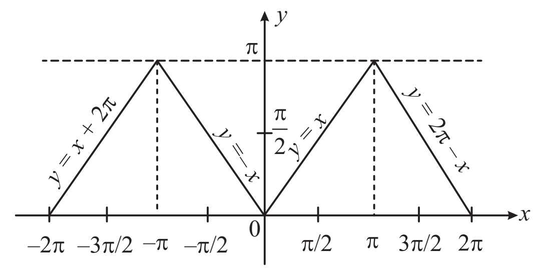
(*vi*) $y = \sec(\sec^{-1} x) = x$, $|x| \ge 1$; $|y| \ge 1$, y is aperiodic.

P-2:

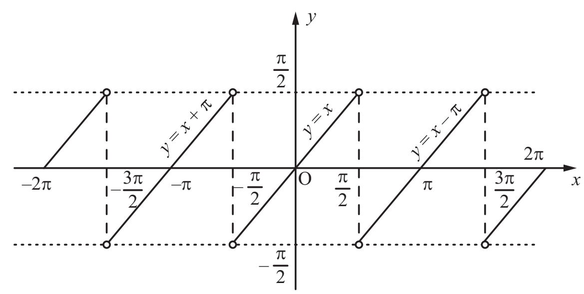
(*i*) $y = \sin^{-1}(\sin x)$, $x \in R$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Periodic with period 2π.



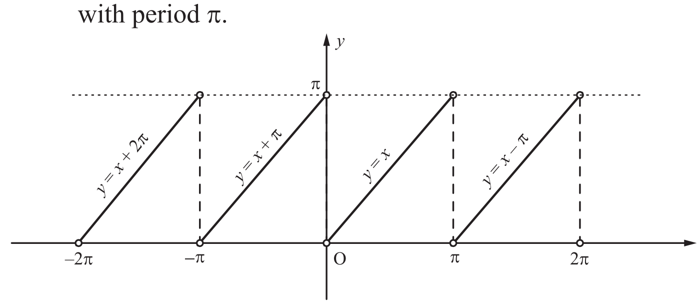
(*ii*) $y = \cos^{-1}(\cos x)$, $x \in R$, $y \in [0, \pi]$, periodic with period 2π.



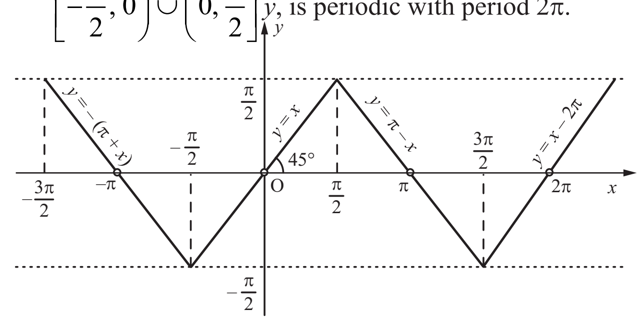
(*iii*) $y = \tan^{-1}(\tan x)$, $x \in R - \left\{\frac{n\pi}{2}\right\}$, $n \in I$



(*iv*) $y = \cot^{-1}(\cot x)$, $x \in R - \{n\pi\}$, $n \in I$, $y \in (0, \pi)$, periodic with period π.

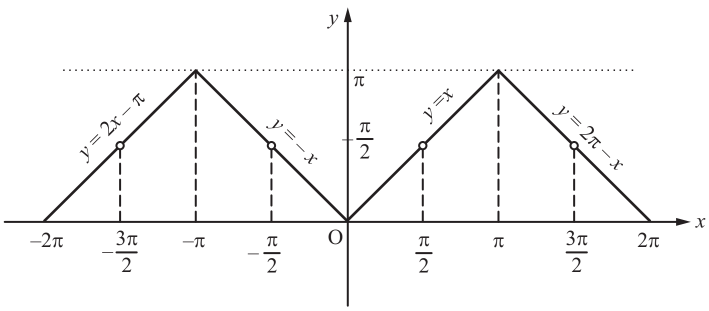


(*v*) $y = \text{cosec}^{-1}(\text{cosec } x)$, $x \in R - \{n\pi\}$, $n \in I$, $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ y, is periodic with period 2π.



(*vi*) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π

$$x \in R - \left\{(2n-1)\frac{\pi}{2}\right\}, n \in I,\ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$



P-3:

(*i*) $\operatorname{cosec}^{-1} x = \sin^{-1}\dfrac{1}{x}$; $\quad x \le -1, x \ge 1$

(*ii*) $\sec^{-1} x = \cos^{-1}\dfrac{1}{x}$; $\quad x \le -1, x \ge 1$

(*iii*) $$\cot^{-1} x = \begin{cases} \tan^{-1}\dfrac{1}{x}; & x > 0 \\ \pi + \tan^{-1}\dfrac{1}{x}; & x < 0 \end{cases}$$

P-4:

(*i*) $\sin^{-1}(-x) = -\sin^{-1} x,\ -1 \le x \le 1$

(*ii*) $\tan^{-1}(-x) = -\tan^{-1} x,\ x \in R$

(*iii*) $\cos^{-1}(-x) = \pi - \cos^{-1} x,\ -1 \le x \le 1$

(*iv*) $\cot^{-1}(-x) = \pi - \cot^{-1} x,\ x \in R$

(*v*) $\sec^{-1}(-x) = \pi - \sec^{-1} x,\ x \le -1$ or $x \ge 1$

(*vi*) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x,\ x \le -1$ or $x \ge 1$

P-5:

(*i*) $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$; $\quad -1 \le x \le 1$

(*ii*) $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$; $\quad x \in R$

(*iii*) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \dfrac{\pi}{2}$; $\quad |x| \ge 1$

P-6:

(*i*) $$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\dfrac{x+y}{1-xy}, & \text{where } x > 0, y > 0 \text{ and } xy < 1 \\ \pi + \tan^{-1}\dfrac{x+y}{1-xy}, & \text{where } x > 0, y > 0 \text{ and } xy > 1 \\ \dfrac{\pi}{2}, & \text{where } x > 0, y > 0 \text{ and } xy = 1 \end{cases}$$

(*ii*) $$\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1}\dfrac{x-y}{1+xy}, & xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{where } x > 0, y > 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{where } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

(*iii*) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$,

where $x \ge 0, y \ge 0$ & $(x^2 + y^2) < 1$

Note that: $x^2 + y^2 < 1 \Rightarrow 0 < \sin^{-1} x + \sin^{-1} y < \dfrac{\pi}{2}$

(*iv*) $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$,

where $x > 0, y > 0$ and $x^2 + y^2 > 1$.

Note that: $x^2 + y^2 > 1 \Rightarrow \dfrac{\pi}{2} < \sin^{-1} x + \sin^{-1} y < \pi$.

(*v*) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

where $x > 0, y > 0$.

(*vi*) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$; $x, y \ge 0$

(*vii*) $\cos^{-1} x - \cos^{-1} y =$

$$\begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right), & x > 0, y > 0 \text{ and } x < y \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right), & x > 0, y > 0 \text{ and } x > y \end{cases}$$

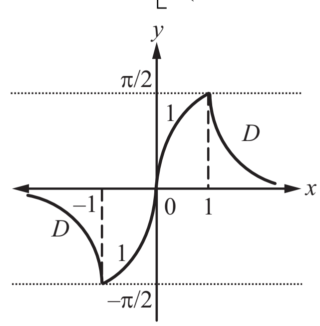
(*viii*) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left[\dfrac{x + y + z - xyz}{1 - xy - yz - zx}\right]$

if $x > 0, y > 0, z > 0$ & $xy + yz + zx < 1$.

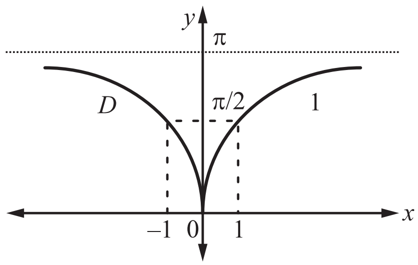
Note that: In the above results x & y are taken positive. In case if these are given as negative, we first apply *P*-4 and then use above results.

Simplified Inverse Trigonometric Functions

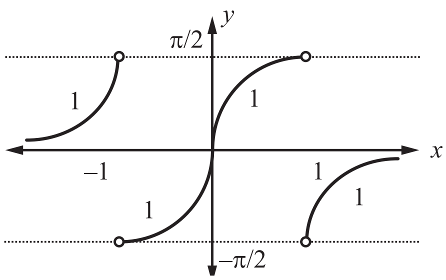
(*a*) $$y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1} x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1} x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1} x) & \text{if } x < -1 \end{cases}$$



(*b*) $y = f(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1} x \text{ if } x \ge 0 \\ -2\tan^{-1} x \text{ if } x < 0 \end{cases}$

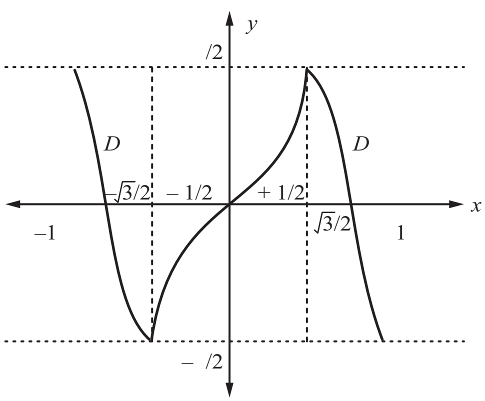


(*c*) $y = f(x) = \tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} 2\tan^{-1} x & \text{if} \quad |x| < 1 \\ \pi + 2\tan^{-1} x & \text{if} \quad x < -1 \\ -(\pi - 2\tan^{-1} x) & \text{if} \quad x > 1 \end{cases}$



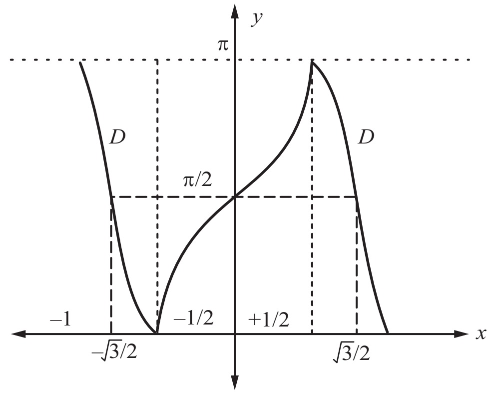
(*d*) $y = f(x) = \sin^{-1}(3x - 4x^3)$

$$= \begin{cases} -(\pi + 3\sin^{-1} x) & \text{if} \quad -1 \le x \le -\dfrac{1}{2} \\ 3\sin^{-1} x & \text{if} \quad -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1} x & \text{if} \quad \dfrac{1}{2} \le x \le 1 \end{cases}$$



(*e*) $y = f(x) = \cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} 3\cos^{-1} x - 2\pi & \text{if} \quad -1 \le x \le -\dfrac{1}{2} \\ 2\pi - 3\cos^{-1} x & \text{if} \quad -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 3\cos^{-1} x & \text{if} \quad \dfrac{1}{2} \le x \le 1 \end{cases}$$



(*f*) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} -(\pi + 2\sin^{-1} x) & -1 \le x \le -\dfrac{1}{\sqrt{2}} \\ 2\sin^{-1} x & -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1} x & \dfrac{1}{\sqrt{2}} \le x \le 1 \end{cases}$