CHAPTER

10 Binomial Theorem

Important terms in the binomial expansion are

(a) **General term:** The general term or the $(r + 1)^{\text{th}}$ term in the expansion of $(x + y)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r} \,.\, y^r$$

(b) **Middle term:** The middle term (s) is the expansion of $(x + y)^n$ is (are):

(i) If n is even, there is only one middle term which is given by

$$T_{(n+2)/2} = {}^nC_{n/2} \,.\, x^{n/2} \,.\, y^{n/2}$$

(ii) If n is odd, there are two middle terms which are

$$T_{(n+1)/2} \text{ and } T_{[(n+1)/2]+1}$$

(c) **Term independent of x:** Term independent of x contains no x ; Hence find the value of r for which the exponent of x is zero.

If $(\sqrt{A} + B)^n = I + f$, where I & n are positive integers and $0 \le f < 1$, then

(a) $(I + f) \cdot f = K^n$ if n is odd & $A - B^2 = K > 0$

(b) $(I + f)(1 - f) = k^n$ if n is even & $\sqrt{A} - B < 1$

Some results on binomial coefficients

(a) ${}^nC_x = {}^nC_y \;\Rightarrow\; x = y$ or $x + y = n$

(b) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

(c) $C_0 + \dfrac{C_1}{2} \aleph \dfrac{C_2}{3} \quad \ldots \dfrac{C_n}{n+1} \quad \dfrac{2^{n+1}-1}{n+1}$

(d) $C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \dfrac{C_3}{4} \ldots + \dfrac{(-1)^n C_n}{n+1} = \dfrac{1}{n+1}$

(e) $C_0 + C_1 + C_2 + \ldots = C_n = 2^n$

(f) $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

(g) $C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \ldots + C_n^{\,2} = {}^{2n}C_n = \dfrac{(2n)!}{n!n!}$

(h) $C_0 \cdot C_r + C_1 . C_{r+1} + C_2 \cdot C_{r+2} + \ldots + C_{n-r} C_n = \dfrac{(2n)!}{(n+r)!\,(n-r)}$

Greatest coefficient and Greatest Term in Expansion of (x + a)^n

(a) If n is even greatest coefficient is ${}^nC_{n/2}$.

If n is odd greatest coefficient is ${}^nC_{\left(\frac{n-1}{2}\right)}$ or ${}^nC_{\left(\frac{n+1}{2}\right)}$

(b) **For greatest term:** Greatest term

$$= \begin{cases} T_p \text{ and } T_{p+1} & \text{if } \dfrac{n+1}{\left|\dfrac{x}{a}\right| + 1} \text{ is an integer} \\[2ex] T_{q+1} & \text{if } \dfrac{n+1}{\left|\dfrac{x}{a}\right| + 1} \text{ is non integer and} \in (q, q+1),\, q \in I \end{cases}$$

Multinomial Theorem

For any $n \in N$,

(i) $(x_1 + x_2 + \ldots + x_k)^n = \displaystyle\sum_{r_1 + r_2 + \ldots + r_k = n} \frac{n!}{r_1!r_2!\ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$

(ii) The general term in the above expansion is

$\dfrac{n!}{r_1!r_2!\ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$

Here total number of terms in the expansion = ${}^{n+k-1}C_{k-1}$.

Binomial Theorem for Negative or Fractional Indices

If $n \in Q$, then $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots \infty$ provided $|x| < 1$.

Notes

(i) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots \infty$

(ii) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots \infty$

(iii) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \infty$

(iv) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \infty$

Exponential series

(a) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots \infty$; where x may be any real or complex number and $e = \displaystyle\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$.

(b) $a^x = 1 + \dfrac{x}{1!}\ell na + \dfrac{x^2}{2!}\ell n^2 a + \dfrac{x^3}{3!}\ell n^3 a + \ldots \infty$, where $a > 0$.

Logarithmic Series

(a) $ln\,(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots \infty$, where $-1 < x \le 1$.

(b) $ln\,(1 - x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \ldots \infty$, where $-1 \le x < 1$.

(c) $ln\,\dfrac{(1 + x)}{(1 - x)} = 2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots \infty\right)$, $|x| < 1$.