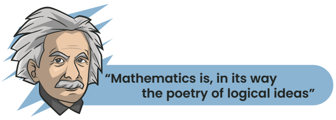


RELATIONS

(1) Types of Relations

1. Empty Relation
A relation in which no element of A is related to any other element of A, i.e., R=φ ⊂ A×A.

2. Universal Relation
A relation in which each element of A is related to every element of A, i.e., R=A×A.

3. Identity Relation
A relation in which each element is related to itself only. $I = \{(a,a), a \in A\}$

4. Reflexive Relation:
$(a, a) \in R$, for every $a \in A$.

5. Symmetric Relation:
$(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

6. Transitive Relation:
$(a_1, a_2) \in R$ & $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.

7. Equivalence Relation :
A relation R in a set A is said to be an equivalence relation if R is reflexive,symmetric & transitive.

8. Inverse Relation
Inverse relation of R from A to B, denoted by R^{-1}, is a relation from B to A is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.

9. Asymmetric Relation
$(x,y) \in R \Rightarrow (y,x) \notin R$

10. Antisymmetric: A relation is antisymmetric if:

- For all $x, y \in X[(x,y) \in R \;\&\; (y,x) \in R] \Rightarrow x = y$
- For all $x, y \in X[(x,y) \in R \;\&\; x \neq y] \Rightarrow (y,x) \notin R$

11. Irreflexive
R is irreflexive iff
$\forall a \in A, ((a,a) \notin R)$

12. Partial order relation
R is a partial order, if R is Reflexive, Antisymmetric and Transitive.

2. EXAMPLE:

A = {1, 2, 3, 4}. Identify the properties of relations.

$R_1 = \{(1,1),(2,2),(3,3),(2,1),(4,3),(4,1),(3,2)\}$

$R_2 = A \times A,\ R_3 = \phi,\ R_4 = \{(1,1),(2,2),(3,3),(4,4)\}$

$R_5 = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(4,3),(3,4)\}$

Relation	Reflexive	Symmetric	Asymmetric	Antisymmetric	Irreflexive	Transitive
R_1	✗	✗	✗	✓	✗	✗
R_2	✓	✓	✗	✗	✗	✓
R_3	✗	✓	✓	✓	✓	✓
R_4	✓	✓	✗	✓	✗	✓
R_5	✓	✓	✗	✗	✗	✓

NOTE
If A={1,2},a relation R={(1,2)} on A is a transitive relation.
using the similar argument a relation R = {(x, y) : x is wife of y} is transitive, where as R = {(x, y) : x is father of y} is not transitive.

3. PROPERTIES

1.
R is not reflexive does not imply R is irreflexive. Counter example:
$A = \{1,2,3\}, R = \{(1,1)\}$

2.
R is asymmetric implies that R is irreflexive. By definition, for all $a, b \in A, (a,b) \in R$ and $(b,a) \notin R$ This implies that for all $(a,b) \in R, a \neq b$ Thus, for all $a \in A, (a,a) \notin R$ Therefore, R is irreflexive.

3.
R is not symmetric does not imply R is antisymmetric. Counter example:
$A = \{1,2,3\}, R = \{(1,2),(2,3),(3,2)\}$

4.
R is not symmetric does not imply R is asymmetric. Counter example:
$A = \{1,2,3\}, R = \{(1,2),(2,2)\}$

5.
R is not antisymmetric does not imply R is symmetric. Counter example:
$A = \{1,2,3\}, R = \{(1,2),(2,3),(3,2)\}$

6.
R is reflexive implies that R is not asymmetric. By definition, for all $a \in A, (a,a) \in R$ This implies that, both (a, b) and (b, a) are in R when a = b. Thus, R is not asymmetric.

(4) COUNTING OF RELATION

- Number of relations from set A to B = 2^{mn}, where $|A| = m, |B| = n$
- Number of Identity relation on a set with 'n' elements = 1
- Number of reflexive relation set on a set with 'n' elements = $2^{n(n-1)}$
- Number of Symmetric relation set on a set with 'n' elements = $2^{n(n+1)/2}$
- The number of antisymmetric binary relations possible on A is $2^n \cdot 3^{(n^2-n)/2}$
- The number of binary relation on A which are both symmetic and antisymmetric is 2^n.
- The number of binary relation on A which are both symmetric and asymmetric is 1.
- The number of binary relation which are both reflexive and antisymmetric on the set A is $3^{(n^2-n)/2}$
- The number of asymmetric binary relation possible on the set A is $3^{(n^2-n)/2}$
- There are at least 2^n transitive relations (lower bound) and at most $2^{n^2} - 2^{\frac{n^2-n}{2}} + 1$ (upper bound)

5. OPERATION ON RELATIONS:

1. $R_1 - R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ and } (a,b) \notin R_2\}$

2. $R_2 - R_1 = \{(a,b) \mid (a,b) \in R_2 \text{ and } (a,b) \notin R_1\}$

3. $R_1 \cup R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ or } (a,b) \in R_2\}$

4. $R_1 \cap R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ and } (a,b) \in R_2\}$

PROPERTIES

1) If R_1 and R_2 are reflexive, and symmetric, then $R_1 \cup R_2$ is reflexive, and symmetric.

2) If R_1 is transitive and R_2 is transitive, then $R_1 \cup R_2$ need not be transitive.

counter example: Let $A = \{1,2\}$ such that $R_1 = \{(1,2)\}$ and $R_2 = \{(2,1)\}$. $R_1 \cup R_2 = \{(1,2),(2,1)\}$ and $(1,1) \notin R_1 \cup R_2$ implies that $R_1 \cup R_2$ is not transitive.

3) If R_1 and R_2 are equivalence relations, then $R_1 \cap R_2$ is an equivalence relation.

4) If R_1 and R_2 are equivalence relations on A,

- $R_1 - R_2$ is not an equivalence relation (reflexivity fails).
- $R_1 - R_2$ is not a partial order (since $R_1 - R_2$ is not reflexive).
- $R_1 \oplus R_2 = R_1 \cup R_2 - (R_1 \cap R_2)$ is neither equivalence relation nor partial order (reflexivity fails)

5) The union of two equivalence relation on a set is not necessarily an equivalence reation on the set.

6) The inverse of a equivalence relation R is an equivalence relation.

6. COMPOSITON OF RELATIONS

Let $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$, Composition of R_2 on R_1, denoted as $R_1 \ R_2$ or simply $R_1 R_2$ is

$$R_1 \ R_2 = \{(a,c) \mid a \in A, c \in C \wedge \exists b \in B \text{ such that } ((a,b) \in R_1, (b,c) \in R_2)\}$$

NOTE

$R_1 (R_2 \cap R_3) \subset R_1 R_2 \cap R_1 R_3$

$R_1 (R_2 \cup R_3) = R_1 R_2 \cup R_1 R_3$

$R_1 \subseteq A \times B, R_2 \subseteq B \times C, R_3 \subseteq C \times D. (R_1 R_2) R_3 = R_1 (R_2 R_3)$

$(R_1 o R_2)^{-1} = R_2^{-1} o R_1^{-1}$

7. EQUIVALENCE CLASS

Equivalence class of $a \in A$ is defined as $[a] = \{x \mid (x,a) \in R\}$, that is all the elements related to a under the relation R.

Example

E=Even integers, O=odd integers.

(i) All elements of E are related to each other and all elements of O are related to each other.

(ii) No element of E is related to any element of O and vice-versa.

(iii) E and O are disjoint and $\mathbf{Z} = E \cup O$

The subset E is called the equivalence class containing zero and is denoted by [0].

Properties: consider an equivalence relation R defiend on a set A.

1. $\bigcup_{\forall a \in A} [a] = A$

2. For every a,b∈A such that $a \in [b], a \neq b$ it follows that $[a] = [b]$

3. $\sum_{\forall x \in A} |[x]| = |R|$

4. For any two equivalence class $[a]$ and $[b]$, either $[a] = [b]$ or $[a] \cap [b] = \phi$

5. For all $a, b \in A$, if $a \in [b]$ then $b \in [a]$

6. For all $a, b, c \in A$, if $a \in [b]$ and $b \in [c]$, then $a \in [c]$

7. For all $a \in A, [a] \neq \phi$

Congruence modulo n given by $a \equiv b \pmod{n}$ if and only if n divides (a – b).

8. BINARY OPERATIONS

Let S be a non-empty set. A function f : S × S → S is called a binary opertion on set S.

Note

Number of binary operations on a set containing n elements is n^{n^2}

**"IN MATHEMATICS
THE ART OF PROPOSING A QUESTION
MUST BE HELD OF HIGHER VALUE THAN SOLVING IT."**

– Georg Cantor

FUNCTION

1 Classification of function

01. Constant function

f(x) = k, k is a constant.

02. Identity function

The function $y = f(x) = x$, $\forall x \in R$

Here domain & Range both R

03. Polynomial function

$y = f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_n$, n is non negative integer, a_i are real constants. Given $a_0 \neq 0$, n is the degree of polynomial function

There are two polynomial functions, $f(x) = 1 + x^n$ & $f(x) = 1 - x^n$ satisfying the relation: $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ where 'n' is a positive integer.

4. Rational functions

It is defined as the ratio of two polynomials.

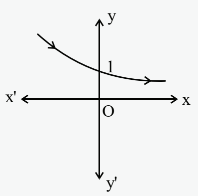
$f(x) = \frac{P(x)}{Q(x)}$ provided $Q(x) \neq 0$

Dom $\{f(x)\}$ is all real numbers except when denominator is zero $[i, e, Q(x) \neq 0]$

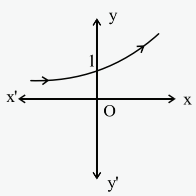
2 Exponential function

f(x) = a^x, a > 0, a ≠ 1.

0<a<1



a > 1

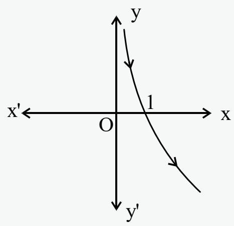


Domain =R, Range = (0, ∞)

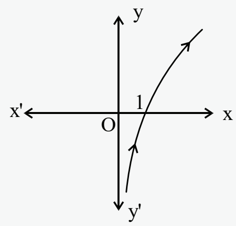
3 Logarithmic function

f(x) = $\log_a x$ [a > 0, a ≠ 1]

0 < a < 1



a > 1



Domain =(0, ∞), Range =R

Proprieties of Log. Functions

1. $\log_a(xy) = \log_a |x| + \log_a |y|$, where $a > 0, a \neq 1$ and $xy > 0$

2. $\log_a x = \frac{1}{\log_x a}$ for $a > 0, a \neq 1$ and $x > 0, x \neq 1$

3. $\log_a\left(\frac{x}{y}\right) = \log_a |x| - \log_a |y|$, where $a > 0, a \neq 1$ and $\frac{x}{y} > 0$

4. $\log_a\left(x^n\right) = n \log_a |x|$, where $a > 0, a \neq 1$ and $x^n > 0$

5. $\log_{a^n} x^m = \frac{m}{n} \log_{|a|} |x|$, where $a > 0, a \neq 1$ and $x > 0$

6. $x^{\log_a y} = y^{\log_a x}$ where $x > 0, y > 0, a > 0, a \neq 1$

7. If a>1, then the values of f(x)= $\log_a x$ increase with the increase in x.
i.e. $x < y \Leftrightarrow \log_a x < \log_a y$

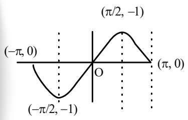
Also, $\log_a x \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1. \end{cases}$

8. If , 0<a<1, then the values of f (x) = $\log_a x$ decrease with the increase in x .
i.e. $x < y \Leftrightarrow \log_a x > \log_a y$

Also, $\log_a x \begin{cases} > 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ < 0 & \text{for } x > 1 \end{cases}$

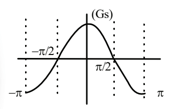
4 Trigonometric Functions

Sine function



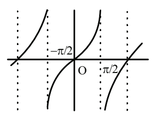
f(x) = sin x.
Dom (f) = R
Ran (f) = [–1, 1]

Cosine function



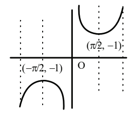
f(x) = cos x
Dom (f) = R
Ran (f) = [-1, 1].

Tangent function



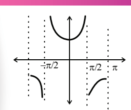
$f(x) = \tan x$
$\text{Dom}(f) = R - \left\{\frac{(2n+1)\pi}{2}, n \in z\right\}$
$\text{Ran}(f) = R$

Cosecant Function



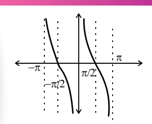
f(x) = cosec x
Dom (f) = R–{mR, n e z }
Ran (f) = R–(–1, 1)

Secant Function



f(x)=sec x
Dom (s) = R - $\{(2n+1)\frac{\pi}{2} | n \in Z\}$
Ran (f) = R - (-1, 1).

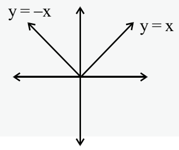
Cotangent function



f(x) = cot (x)
Dom (f) = R –{nπ | n ∈ Z }
Ran (f) = R.

5 Absolute Value Function

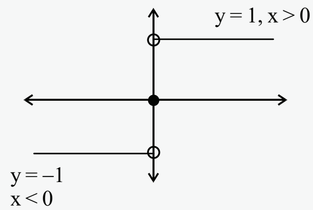
$$y = f(x) = |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$



1. $|x|^2 = x^2$
2. $\sqrt{x^2} = |x|$
3. $|x| = \max\{-x, x\}$
4. $-|x| = \min\{-x, x\}$
5. $\max(a,b) = \frac{a+b}{2} + \left|\frac{a-b}{2}\right|$
6. $\min(a,b) = \frac{a+b}{2} - \left|\frac{a-b}{2}\right|$
7. $|x+y| \le |x| + |y|$
8. $|x+y| = |x| + |y|$ if $xy > 0$
9. $|x-y| = |x| + |y|$ if $xy \leqslant 0$
10. $|x| \geqslant a$ (is – ve) $x \in R$
11. $a \leqslant |x| \leqslant b \Rightarrow -b \le x \le -a$ or $a \leqslant x \leqslant b$. $x \in [-b, -a] \cup [a, b]$.

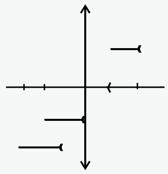
6 Signum Function

$$y = \operatorname{sgn}(x) = \begin{cases} \dfrac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



7 Greatest Integer Function

f(x) = [x] the integral part of x, which is nearest & smaller integer



1. $[x] \leqslant x < [x] + 1$
2. $x - 1 < [x] < x$
3. $I \le x < I + 1 \Rightarrow [x] = I$
4. $[x] - [-x] = \begin{cases} 2x & , x \in I \\ 2x + 1 & , 2 \notin I. \end{cases}$
5. $[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \in I, \end{cases} \Bigg| \begin{matrix} 2x, & x \in I \\ 2x+1, & , x \notin I \end{matrix}$
6. $[x] \le n \Leftrightarrow x < n + 1, n \in I$
7. $[x] < n \Leftrightarrow x < n$
8. $[x] = \left[\frac{x}{2}\right] + \left[\frac{x+1}{2}\right]$
9. $\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \left[\frac{n+4}{8}\right] + \cdots = n$
10. $[x] + [y] \leqslant [x+y] \leqslant [x] + [y] + 1$
11. $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = [nx]$

8 Fractional Part Function

: y={x} fractional past of x.

y = {x} = x - [x]

1. {x} = x, 0 ≤ x < 1.
2. {x} = 0, x ∈ I
3. {-x} = 1-{x}, x ∉ I
4. {x ± integer} = {x}

9 Odd and Even Function

1. if f(-x) =−f(x) ∀ x ∈R then f is an odd function, odd functions are symmetrical in opposite quadrants or about origin.

2. If f(-x) = f(x). then even. It is symmetric about y axis.

Properties

1. Product of two odd or two even function is an even function.
2. Product of odd & even function is an odd function.
3. Every function can be expressed as the sum of an even and odd function, i.e,

$$f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}$$

4. Derivative of an odd function is an even function and of an even is odd.

10 Periodic function

f(x) is periodic if f(x + T) = f(x) $\forall$ x $\in$ R , T = period

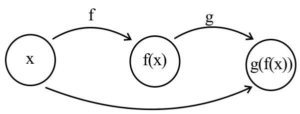
Functions	Period
$\sin^n x, \cos^n x, \sec^n x, \operatorname{cosec}^n x$	π(n is even), 2π(n odd/ fraction)
$\tan^n x, \cot^n x$	π
|trig function|	π
$x-[x]$	1
$f(x)=$ constant	Periodic with no fundamental period.

Properties of Periodic functions

If f(x) is periodic with period T, then

1. c. f(x) is periodic with period T
2. f(x + c) is periodic with period T.
3. f(x) ± c is periodic with period T.
4. $kf(cx+d)$ has period $\frac{T}{|c|}$ period is only affected by coefficient of x.

11 Composition of Function



- h(x) = g f(x) = (gof) (x).
- gof ≠ fog.
- Composition of two bijection is a bijection.

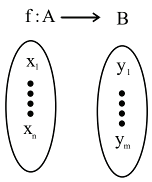
Properties of Composite Function

f	g	fog
even	even	even
odd	odd	odd
even	odd	ever
odd	even	even

12 Kinds of Mapping

1. **One-one/Injective/Homomorphic**: f(x) = f(y) ⇒ x=y, then one-one. Graphically, if no line parallel to x-axis meets the graph of function at more than one point.
2. **Onto/Surjective**: If range = co-domain. Method to show subjectivity: Finding the range of y = f(x) & Showing range of f = c σ-domain of f
3. **Many-one mapping**:If two or more element in domain have same image in co-domain.
4. **Into Function** :There's an element in B not having a pre image in A under f. [f : A → B].

13 Number of functions



Total no of functions $= m^n$

Number of One to one functions $=\begin{cases} {}^mP_n, & m\geq n \\ 0, & m<n \end{cases}$

No. of many one functions $=\begin{cases} m^n-{}^mP_n, & m\geqslant n \\ m^n & ,m<n \end{cases}$

No. of constant function $=m$

No. of onto function $=\begin{cases} \sum_{r=0}^{n}(-1)^r\,{}^mC_r\,(m-r)^n & ,n\geq m \\ m^n & ,n<m \end{cases}$

No. of one-to-one onto functions = n!, if m= n

14 Inverse of a function

g:B→A , f(x) = y ⇔ g(y) = x ∀ x ∈ A and y ∈ B.
Then g is inverse of f

1. Inverse of a bijection is unique.

2. If f : A → B is a bijection g : B →A is the inverse of f, then fog = I_B gof = I_A, where I_A & I_B are identity function

3. The inverse of a bijection is also a bijection $(gof)^{-1} = f^{-1} \text{ o } g^{-1}$.

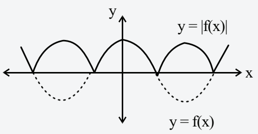
15 Hyperbolic functions

$$\sinh x=\frac{e^x-e^{-x}}{2} \qquad \cosh x=\frac{e^x+e^{-x}}{2} \qquad \tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}} \qquad \coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}}$$

16 Elementary transformation of graphs

01

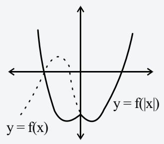
Drawing the graph of y = |f(x)| from the



$$|f(x)|=\begin{cases} f(x) & \text{if} \quad f(x)\geqslant 0 \\ -f(x) & \text{if} \quad f(x)<0. \end{cases}$$

02

Drawing graph of y=f(|x|) from the known graph of y=f(x).

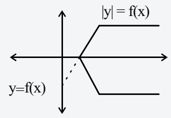


$$f(|x|)=\begin{cases} f(x), & x\geqslant 0 \\ f(-x), & x<0 \end{cases}$$

Neglect the curve for x<0 & take the images of curve for x⩾0 about y axis.

03

Drawing graph of |y|=f(x) from the know m graph of y=f(x).



Remove portion that lies below x axis. Plot the remaining portion of the graph & also its mirror image in x-axis.

17 Things to remember

Range of $a\cos x + b\sin x$ is $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$

Range of $f(x)=\sqrt{a-x}+\sqrt{x-b}$ if $a>b>0$ is $\sqrt{a-b}+\sqrt{2(a-b)}$

Range of $\left(x+\frac{1}{x}\right)+\frac{1}{\left(x+\frac{1}{x}\right)}$ is $(-\infty,-2.5]\cup[2.5,\infty)$

$-\sin 1 < \sin(\cos x) < \sin 1$

$\cos 1 < \cos(\sin x) < 1$

18 Functional Equation

1) $f(x+y)=f(x)f(y)$, then $f(x)=a^x$

2) $f(xy)=f(x)+f(y)$, then $f(x)=\log_a x$

3) $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$, then $f(x)=mx+c$

4) $f(x)f\left(\frac{1}{x}\right)=1$, then $f(x)=\pm x^n$