



Special Type of Matrices

- 1. Row Matrix (Row vector):** $A = [a_{11}, a_{12}, \dots, a_{1n}]$ i.e., row matrix has exactly one row.

- 2. Column Matrix (Column vector):** $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ i.e., column matrix has exactly one column.

- 3. Zero or Null Matrix:** ($A = O_{m \times n}$), an $m \times n$ matrix whose all entries are zero.

- 4. Horizontal Matrix:** A matrix of order $m \times n$ is a horizontal matrix if $n > m$.

- 5. Vertical Matrix:** A matrix of order $m \times n$ is a vertical matrix if $m > n$.

- 6. Square Matrix:** (Order n) if number of rows = number of column, then matrix is a square matrix.

Key Note

- The pair of elements a_{ij} and a_{ji} are called Conjugate Elements.
- The elements $a_{11}, a_{22}, a_{33}, \dots, a_{mm}$ are called Diagonal Elements. the line along which the diagonal elements lie is called "Principal or Leading diagonal." The quantity $\sum a_{ii} =$ trace of the matrix written as, $t_r(A)$.

- 7. Unit/Identity Matrix:** A square matrix, in which every non-diagonal element is zero and every diagonal element is 1, is called unit matrix or an identity matrix,

$$\text{i.e. } a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ 1, & \text{when } i = j \end{cases}$$

- 8. Upper Triangular Matrix:** A square matrix $A = [a_{ij}]_{n \times n}$ is called a upper triangular matrix, if $a_{ij} = 0, \forall i > j$.

- 9. Lower Triangular Matrix:** A square matrix $A = [a_{ij}]_{n \times n}$ is called a lower triangular matrix, if $a_{ij} = 0, \forall i < j$.

- 10. Submatrix:** A matrix which is obtained from a given matrix by deleting any number of rows or columns or both is called a submatrix of the given matrix.

- 11. Equal Matrices:** Two matrices A and B are said to be equal, if both having same order and corresponding elements of the matrices are equal.

- 12. Principal Diagonal of a Matrix:** In a square matrix, the diagonal from the first element of the first row to the last element of the last row is called the principal diagonal of a matrix.

e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 6 & 5 \\ 1 & 1 & 2 \end{bmatrix}$, the principal diagonal of A is 1, 6, 2.

- 13. Singular Matrix:** A square matrix A is said to be singular matrix, if determinant of A denoted by $\det(A)$ or $|A|$ is zero, i.e. $|A| = 0$, otherwise it is a non-singular matrix.

Equality of Matrices

Let $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if,

1. Both have the same order.
2. $a_{ij} = b_{ij}$ for each pair of i & j .

Algebra of Matrices

Addition: $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same order.

- 1. Addition of matrices is commutative:** $A + B = B + A$.

- 2. Matrix addition is associative:** $(A + B) + C = A + (B + C)$.

Multiplication of a Matrix By a Scalar

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, then $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

Multiplication of Matrices (Row by Column)

Let A be a matrix of order $m \times n$ and B be a matrix of order $n \times p$ then the matrix multiplication AB is possible if and only if $n = p$.

Let $A_{m \times n} = [a_{ij}]$ and $B_{n \times p} = [b_{ij}]$, then order of AB is $m \times p$ and

$$(AB)_{ij} = \sum_{r=1}^n a_{ir} b_{rj}$$

Characteristic Equation

Let A be a square matrix. Then the polynomial $|A - xI|$ is called as characteristic polynomial of A & the equation $|A - xI| = 0$ is called characteristic equation of A .

Properties of Matrix Multiplication

- If A and B are two matrices such that
 - $AB = BA$ then A and B are said to commute
 - $AB = -BA$ then A and B are said to anticommute
- Matrix Multiplication is Associative:** If A , B & C are conformable for the product AB & BC , then $(AB)C = A(BC)$.
- Distributivity:**
$$\left. \begin{aligned} A(B+C) &= AB+AC \\ (A+B)C &= AC+BC \end{aligned} \right\} \text{, provided } A, B \text{ and } C$$
are conformable for respective products.

Positive Integral Powers of a Square Matrix

- $A^m A^n = A^{m+n}$
- $(A^m)^n = A^{mn} = (A^n)^m$
- $I^m = I$ $m, n \in N$

Orthogonal Matrix

A square matrix is said to be orthogonal matrix if $AA^T = I$.

Key Note

- The determinant value of orthogonal matrix is either 1 or -1.
Hence orthogonal matrix is always invertible.
- $AA^T = I = A^T A$. Hence $A^{-1} = A^T$.

Some Square Matrices

- Idempotent Matrix:** A square matrix is idempotent provided $A^2 = A$. For idempotent matrix note the following:
 - $A^n = A \forall n \geq 2, n \in N$.
 - determinant value of idempotent matrix is either 0 or 1.
 - If idempotent matrix is invertible then its inverse will be identity matrix i.e. I .
- Periodic Matrix:** A square matrix which satisfies the relation $A^{K+1} = A$, for some positive integer K , is a periodic matrix.
The period of the matrix is the least value of K for which this holds true.
Note that period of an idempotent matrix is 1.
- Nilpotent Matrix:** A square matrix is said to be nilpotent matrix of order $m, m \in N$, if $A^m = O, A^{m-1} \neq O$.
Note that a nilpotent matrix will not be invertible.
- Involuntary Matrix:** If $A^2 = I$, the matrix is said to be an involuntary matrix.
Note that $A = A^{-1}$ for an involuntary matrix.

- If A and B are square matrices of same order and $AB = BA$ then

$$(A+B)^n = {}^nC_0 A^n + {}^nC_1 A^{n-1} B + {}^nC_2 A^{n-2} B^2 + \dots + {}^nC_n B^n.$$

Transpose of a Matrix (Changing Rows & Columns)

Let A be any matrix of order $m \times n$. Then A^T or $A' = [a_{ij}]$ for $1 \leq i \leq n$ & $1 \leq j \leq m$ of order $n \times m$.

Properties of Transpose

If A^T & B^T denote the transpose of A and B

- $(A+B)^T = A^T + B^T$; note that A & B have the same order.
- $(AB)^T = B^T A^T$ (Reversal law) A & B are conformable for matrix product AB
- $(A^T)^T = A$
- $(kA)^T = kA^T$, where k is a scalar.

General: $(A_1 \cdot A_2 \cdot \dots \cdot A_n)^T = A_n^T \cdot \dots \cdot A_2^T \cdot A_1^T$ (reversal law for transpose)

Symmetric & Skew Symmetric Matrix

- Symmetric matrix:** For symmetric matrix $A = A^T$.

Note: Maximum number of distinct entries in any symmetric matrix of order n is $\frac{n(n+1)}{2}$.

- Skew symmetric matrix:** Square matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} = -a_{ji} \forall i \& j$. Hence if A is skew symmetric, then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \forall i$.

Thus the diagonal elements of a skew square matrix are all zero, but not the converse.

For a skew symmetric matrix $A = -A^T$.

- Properties of symmetric & skew symmetric matrix:

- Let A be any square matrix then, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew symmetric matrix.
- The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix.
- If A & B are symmetric matrices then,
 - $AB + BA$ is a symmetric matrix.
 - $AB - BA$ is a skew symmetric matrix.

- Every square matrix can be uniquely expressed as a sum or difference of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew symmetric}}$$

and
$$A = \frac{1}{2}(A^T + A) - \frac{1}{2}(A^T - A)$$

Adjoint of a Square Matrix

Let $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix and let the

matrix formed by the cofactors of $[a_{ij}]$ in determinant $|A|$ is

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}. \text{ Then } (\text{adj } A) = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}.$$

Key Note

If A be a square matrix of order n , then

1. $A(\text{adj } A) = |A| I_n = (\text{adj } A) \cdot A$
2. $|\text{adj } A| = |A|^{n-1}$
3. $\text{adj}(\text{adj } A) = |A|^{n-2} A$
4. $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$
5. $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$
6. $\text{adj}(KA) = K^{n-1}(\text{adj } A)$, where K is a scalar

Inverse of a Matrix (Reciprocal Matrix)

A square matrix A (non singular) said to be invertible, if there exists a matrix B such that, $AB = I = BA$.

B is called the inverse (reciprocal) of A and is denoted by A^{-1} . Thus

$$A^{-1} = B \Leftrightarrow AB = I = BA$$

We have, $A \cdot (\text{adj } A) = |A| I_n$

$$A^{-1} \cdot A(\text{adj } A) = A^{-1} I_n |A|$$

$$I_n (\text{adj } A) = A^{-1} |A| I_n$$

$$\therefore A^{-1} = \frac{(\text{adj } A)}{|A|}$$

Note: The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

Theorem: If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Key Note

- If A be an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.
- If A is invertible,
 - (a) $(A^{-1})^{-1} = A$
 - (b) $(A^k)^{-1} = (A^{-1})^k = A^{-k}$; $k \in \mathbb{N}$

System of Equation and Criteria for Consistency

Gauss - Jordan Method

Example:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

$$\Rightarrow \begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow AX = B \Rightarrow A^{-1}AX = A^{-1}B$$

$$\Rightarrow X = A^{-1}B = \frac{\text{Adj } A}{|A|} \cdot B$$

Key Note

- If $|A| \neq 0$, system is consistent having unique solution.
- If $|A| \neq 0$ and $(\text{adj } A) \cdot B \neq O$ (Null matrix), system is consistent having unique non-trivial solution.
- If $|A| \neq 0$ and $(\text{adj } A) \cdot B = O$ (Null matrix), system is consistent having trivial solution.
- If $|A| = 0$, then

Matrix Method Fails

