CHAPTER

4 Determinants

Definition

1. The determinant consisting two rows and two columns is $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, its value is given by:

   $D = a_1 b_2 - a_2 b_2$

2. A determinant which consists of three rows and three columns is called a third-order-determinant.

   Let D = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then its value is

   $$D = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{12}a_{21}a_{33}$$

Minors and Cofactors

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then the minor M_{ij} of the element a_{ij} is the determinant obtained by deleting the i^{th} row and j^{th} column,

i.e. M_{11} = minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

The **cofactor** of the element a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Properties of Minors and Cofactors

1. The sum of the products of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of any other row (or column) is zero,

   i.e., if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

   $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$ and so on.

2. The sum of the product of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of the same row (or column) is Δ,

   i.e., if $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

   $|A| = \Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$.

3. In general, if $|A| = \Delta$, then $|A| = \sum_{i=1}^{n} a_{ij} C_{ij}$ and $|(\text{adj } A)| = \Delta^{n-1}$, where A is a matrix of order $n \times n$.

Properties of Determinants

1. The value of the determinant remains unchanged, if rows are changed into columns and columns are changed into rows.

   e.g., $|A'| = |A|$

2. If $A = [a_{ij}]_{n \times n}$, $n > 1$ and B be the matrix obtained from A by interchanging two of its rows or columns, then

   det $(B) = -$ det (A)

3. If two rows (or columns) of a square matrix A are proportional, then $|A| = 0$.

4. $|B| = k|A|$, where B is the matrix obtained from A, by multiplying one row (or column) of A by k.

5. $|kA| = k^n |A|$, where A is a matrix of order $n \times n$.

6. If each element of a row (or column) of a determinant is the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinant, e.g.,

   $$\begin{vmatrix} a_1 + a_2 & b & c \\ p_1 + p_2 & q & r \\ u_1 + u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$$

7. If the same multiple of the elements of any row (or column) of a determinant are added to the corresponding elements of any other row (or column), then the value of the new determinant remains unchanged, e.g.,

   $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} + ka_{31} & a_{12} + ka_{32} & a_{13} + ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

8. If each element of a row (or column) of a determinant is zero, then its value is zero.

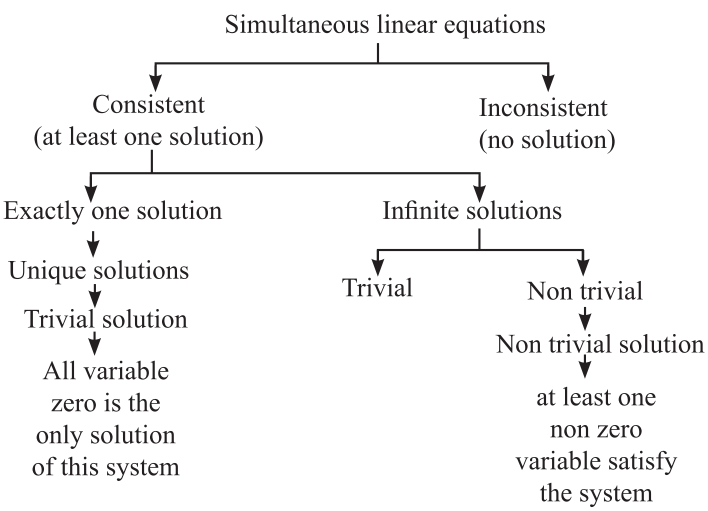
9. If any two rows (or columns) of a determinant are identical, then its value is zero.

10. If r rows (or r columns) become identical, when a is substituted for x, then $(x - a)^{r-1}$ is a factor of given determinant.

Important Results on Determinants

1. $|AB| = |A||B|$, where A and B are square matrices of the same order.
2. $|A^n| = |A|^n$.
3. If A, B and C are square matrices of the same order such that i^{th} columns (or rows) of A is the sum of i^{th} columns (or rows) of B and C and all other columns (or rows) of A, B and C are identical, then $|A| = |B| + |C|$.
4. $|I_n| = 1$, where I_n is identity matrix of order n.
5. $|O_n| = 0$, where O_n is a zero matrix of order n.
6. If $\Delta(x)$ has a third order determinant having polynomials as its elements.
   (*a*) If $\Delta(a)$ has two rows (or columns) proportional, then $(x - a)$ is a factor of $\Delta(x)$.
   (*b*) If $\Delta(a)$ has three rows (or columns) proportional, then $(x - a)^2$ is a factor of $\Delta(x)$.
7. A square matrix A is non-singular, if $|A| \neq 0$ and singular, if $|A| = 0$.
8. Determinant of a skew-symmetric matrix of odd order is zero and of even order is a non-zero perfect square.
9. In general, $|B + C| \neq |B| + |C|$.
10. Determinant of a diagonal matrix = Product of its diagonal elements.
11. If A is a non-singular matrix, then $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$.
12. Determinant of a orthogonal matrix = 1 or – 1.
13. Determinant of a hermitian matrix is purely real.
14. If A and B are non-zero matrices and $AB = O$, then it implies $|A| = 0$ or $|B| = 0$.

System of Equation



Cramer's Rule: [Simultaneous Equations Involving Three Unknowns]

Let $a_1x + b_1y + c_1z = d_1$ ...(i)

$a_2x + b_2y + c_2z = d_2$ ...(ii)

$a_3x + b_3y + c_3z = d_3$ ...(iii)

$$x = \frac{D_1}{D}, y = \frac{D_2}{D}, Z = \frac{D_3}{D}$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}; D_2 \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\& \, D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Note:

1. If $D \neq 0$ and atleast one of D_1, D_2, $D_3 \neq 0$, then the given system of equations are consistent and have unique non trivial solution.
2. If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have trivial solution only
3. If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have infinite solution.

Applications of Determinants in Geometry

Let the three points in a plane be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

1. Area of $\Delta ABC = \left|\dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$

   $= \left|\dfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\right|$
2. If the given points are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$.
3. Let two points are $A(x_1, y_1)$, $B(x_2, y_2)$ and $P(x, y)$ be a point on the line joining points A and B, then the equation of line is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$.