# Ellipse

## 1. Ellipse

The condition for second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent an ellipse is that $h^2 - ab < 0$ and $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

### Standard Equation of the Ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1 - e^2)$

Since $e < 1$, therefore $a^2(1 - e^2) < a^2 \Rightarrow b^2 < a^2$

## 2. Ellipse— Basic fundamentals

| | | |
|---|---|---|
| Equation | $\left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$ | |
| | For a > b | For b > a |
| Centre | (0, 0) | (0, 0) |
| Vertices | $(\pm a, 0)$ | $(0, \pm b)$ |
| Length of major axis | $2a$ | $2b$ |
| Length of minor axis | $2b$ | $2a$ |
| Foci | $(\pm a, 0)$ | $(0, \pm b)$ |
| Equation of directrices | $x = \pm a/e$ | $y = \pm b/e$ |
| Relation in $a$, $b$ and $e$ | $b^2 = a^2(1 - e^2)$ | $a^2 = b^2(1 - e^2)$ |
| Length of latus rectum | $\frac{2b^2}{a}$ | $\frac{2a^2}{b}$ |
| Ends of latus rectum | $\left(\pm ae, \pm\frac{b^2}{a}\right)$ | $\left(\pm\frac{a^2}{b}, \pm be\right)$ |
| Parametric equations | $(a\cos\phi, b\sin\phi)$ | $(a\cos\phi, b\sin\phi)$ $(0 \leq \phi < 2\pi)$ |
| Focal radii | $SP = a - ex_1$ and $S'P = a + ex_1$ | $SP = b - ey_1$ and $S'P = b + ey_1$ |
| Sum of focal radii SP + S′P = | $2a$ | $2b$ |
| Distance between foci | $2ae$ | $2be$ |
| Distance between directrices | $2a/e$ | $2b/e$ |
| Tangents at the vertices | $x = -a, x = a$ | $y = b, y = -b$ |

## 3. Parametric form of the Ellipse

For the equation of ellipse in standard form the parametric form will be given by taking $x = a\cos\phi$, $y = b\sin\phi$, where $\phi$ is the eccentric angle, in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

## 4. Special forms of an Ellipse

1. If the centre of the ellipse is at point $(h, k)$ and the directions of the axes are parallel to the coordinate axes, then its equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$

   If we shift the origin at $(h, k)$ without rotating the coordinate axes, then $x = X + h$ and $y = Y + k$.

2. If the equation of the curve is $\frac{(lx+my+n)^2}{a^2}+\frac{(mx-ly+p)^2}{b^2}=1$ where $lx + my + n = 0$ and $mx - ly + p = 0$ are perpendicular lines, then we substitute $\frac{lx+my+n}{\sqrt{l^2+m^2}}=X, \frac{mx-ly+p}{\sqrt{l^2+m^2}}=Y$, to put the equation in the standard form.

## 5. Position of a Point with Respect to an Ellipse

Let $P(x_1, y_1)$ be any point and let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the equation of an ellipse. The point lies outside, on or inside the ellipse as if $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0,=0,<0.$

## 6. Auxiliary Circle

The circle described on the major of axis of an ellipse as diameter is called an auxiliary circle of the ellipse.

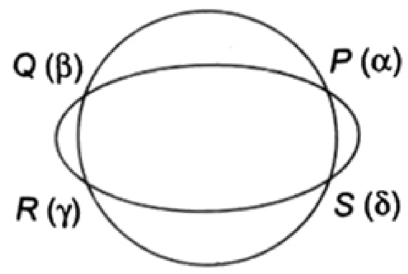
If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an ellipse, then its auxiliary circle is $a^2 + y^2 = a^2$.

## 7. Equation of the Chord Joining Two Points on an Ellipse

The equation of the chord joining two points having eccentric angles $\theta$ and $\phi$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$.

## Concyclic Points

If circle intersects an ellipse in four points. They are called concyclic points and the sum of their eccentric angles is an even multiple $\pi$.




If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the four points $P, Q, R, S$ of circle intersect an ellipse then, $\alpha + \beta + \gamma + \delta = 2n\pi$, where $n$ is any integer.

## 8. Equations of Tangent in Different Forms

**1. Point form:** The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(x_1, y_1)$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

**Note:** Point of intersection of tangent drawn at point $\theta$ and $\phi$ on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$$

**2. Slope from:** If the line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then $c^2 = a^2m^2 + b^2$. Hence, the straight line $y=mx\pm\sqrt{a^2m^2+b^2}$ always represents the tangents to ellipse.

**Points of contact:** Line $y=mx\pm\sqrt{a^2m^2+b^2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $\left(\frac{-a^2m}{c},\frac{b^2}{c}\right)$ where $c^2 = a^2m^2 + b^2$

**3. Parametric form:** The equation of tangent at any point $\phi(a\cos\phi, b\sin\phi)$ is $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$.

## 9. Pair of Tangents

Equation of Pair of Tangents $SS_1 = T^2$

**Pair of tangents:** Let $P(x_1, y_1)$ be any point lying outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let a pair of tangents $PA$, $PB$ can be drawn to it from $P$.

Then the equation of pair of tangents $PA$ and $PB$ is $SS_1 = T^2$

Where

$$S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1,\ S_1\equiv\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1,\ T\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$

## 10. Director Circle

The director circle is the locus of points from which perpendicular tangents are drawn to the ellipse.
equation of director circle is
$x^2 + y^2 = a^2 + b^2$,

It is circle concentric with ellipse and has radius $\sqrt{a^2+b^2}$.

## 11. Equations of Normal in Different Forms

1. **Point form:** The equation of the normal at $(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$.
2. **Parametric form:** The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos\phi, b\sin\phi)$ is $ax\sec\phi - by\operatorname{cosec}\phi = a^2 - b^2$.
3. **Slope form:** If $m$ is the slope of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of normal is

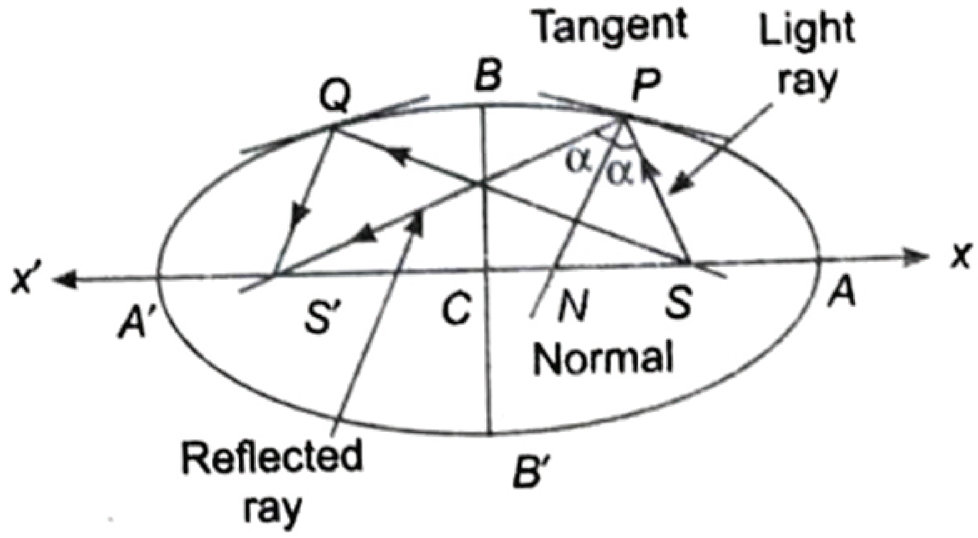
$$y = mx \pm \frac{m\left(a^2-b^2\right)}{\sqrt{a^2+b^2m^2}}$$

## 12. Properties of Eccentric Angles of the Co-normal Points

1. The sum of the eccentric angles of the co-normal points on the ellipse of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is equal to odd multiple of $\pi$.
2. If $\alpha$, $\beta$, $\gamma$ are the eccentric angles of three points on the ellipse, the normals at which are concurrent, then $\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$.

## 13. Reflection Property of an Ellipse

Let $S$ and $S'$ be the foci and $PN$ the normal at the point $P$ of the ellipse, then $\angle SPS' = \angle SQS'$. Hence, if an incoming light ray aimed towards one focus strike the concave side of the mirror in the shape of an ellipse, then it will be reflected towards the other focus.




## 14. Chord of Contact

If $PQ$ and $PR$ be the tangents through point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of the chord of contact $QR$ is $\frac{xx_1}{a^2}+\frac{yy_1}{a^2}=1$ or T = 0 at $(x_1, y_1)$.

### Equation of Chord with Mid-Point $(x_1, y_1)$

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose mid-point be $(x_1, y_1)$ is $T = S_1$, where

$$T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0,\quad S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=0.$$