

# System of Particles and Rotational Motion

## CENTRE OF MASS OF A SYSTEM OF 'N' DISCRETE PARTICLES

$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots + m_n \vec{r}_n}{m_1 + m_2 + \dots + m_n}; \quad \vec{r}_{cm} = \frac{\sum_{i=1}^n m_i \vec{r}_i}{\sum_{i=1}^n m_i}, \quad \vec{r}_{cm} = \frac{1}{M} \sum_{i=1}^n m_i \vec{r}_i$$

## CENTRE OF MASS OF A CONTINUOUS MASS DISTRIBUTION

$$x_{cm} = \frac{\int x dm}{\int dm}, \quad y_{cm} = \frac{\int y dm}{\int dm}, \quad z_{cm} = \frac{\int z dm}{\int dm}$$

$$\int dm = M (\text{mass of the body})$$

## MOTION OF CENTRE OF MASS AND CONSERVATION OF MOMENTUM

### Velocity of Centre of Mass of System

$$\vec{v}_{cm} = \frac{m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + m_3 \frac{d\vec{r}_3}{dt} + \dots + m_n \frac{d\vec{r}_n}{dt}}{M}$$

$$= \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \dots + m_n \vec{v}_n}{M}$$

$$\vec{P}_{sys} = M \vec{v}_{cm}$$

### Acceleration of Centre of Mass of System

$$\vec{a}_{cm} = \frac{m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} + m_3 \frac{d\vec{v}_3}{dt} + \dots + m_n \frac{d\vec{v}_n}{dt}}{M}$$

$$= \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2 + m_3 \vec{a}_3 + \dots + m_n \vec{a}_n}{M}$$

$$= \frac{\text{Net force on system}}{M}$$

$$= \frac{\text{Net external force} + \text{Net internal force}}{M}$$

$$= \frac{\text{Net External Force}}{M} \quad (\because \Sigma \text{ Internal force} = 0)$$

$$\vec{F}_{ext} = M \vec{a}_{cm}$$

Impulse of a force F on a body is defined as:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F} dt = \int_{t_i}^{t_f} d\vec{P} = \Delta \vec{P}$$

(Area under the Force vs time curve gives the impulse)

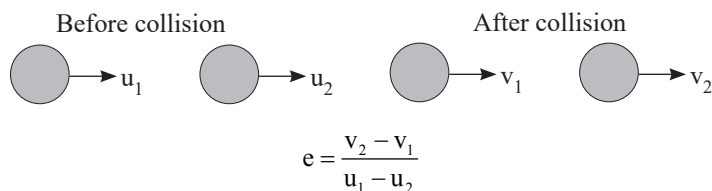
$$\vec{J} = \Delta \vec{P} \quad (\text{impulse} - \text{momentum theorem})$$

### Principle of conservation of linear momentum

$$\diamond \text{ If, } \left( \sum \vec{F}_{ext} \right)_{system} = 0 \Rightarrow (\vec{P}_i)_{system} = (\vec{P}_f)_{system}$$

$$\diamond (KE)_{system} = \frac{1}{2} (m_1 v_1^2 + m_2 v_2^2 + \dots + m_n v_n^2) \neq \frac{1}{2} M V_{com}^2$$

## COEFFICIENT OF RESTITUTION (E)



$$e = \frac{\text{Impulse of reformation}}{\text{Impulse of deformation}} = \frac{\int F_r dt}{\int F_d dt}$$

$$= \frac{\text{Velocity of separation along line of impact}}{\text{Velocity of approach along line of impact}}$$

$$V_1 = \frac{P_i + m_2 e(u_2 - u_1)}{m_1 + m_2}, \quad V_2 = \frac{P_i + m_1 e(u_1 - u_2)}{m_1 + m_2}$$

$$\diamond e = 1$$

+ Impulse of Reformation = Impulse of Deformation

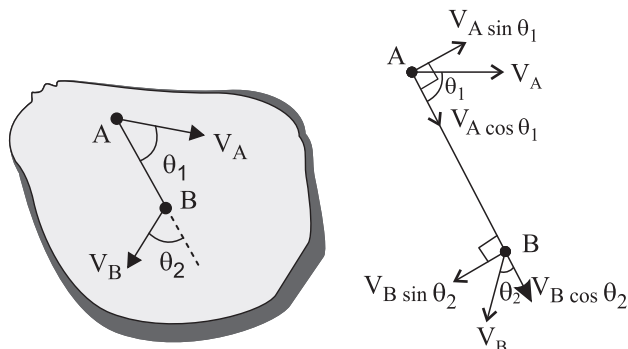
+ Velocity of separation = Velocity of approach

+ Kinetic Energy is conserved in elastic collision.

$$\diamond e = 0$$

- + Impulse of Reformation = 0
- + Velocity of separation = 0
- + Kinetic Energy is not conserved
- + Perfectly Inelastic collision.
- ❖  $0 < e < 1$ 
  - + Impulse of Reformation < Impulse of Deformation
  - + Velocity of separation < Velocity of approach
  - + Kinetic Energy is not conserved
  - + Inelastic collision.

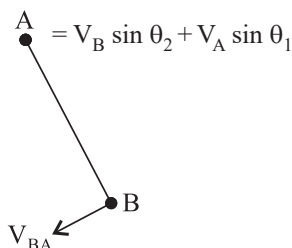
## RIGID BODY



If the above body is rigid

$$V_A \cos \theta_1 = V_B \cos \theta_2$$

$V_{BA}$  = relative velocity of point B with respect to point A.



Types of Motion of rigid body

- Pure Translational Motion
- Pure Rotational Motion
- Combined Translational and Rotational Motion

## MOMENT OF INERTIA (I)

**Definition:** Moment of Inertia is defined as the capability of system to oppose the change produced in the rotational inertia of a body.

Moment of Inertia is a scalar (positive quantity).

$$I = m_1 r_1^2 + m_2 r_2^2 + \dots$$

$$= I_1 + I_2 + I_3 + \dots$$

SI unit of Moment of Inertia is  $\text{Kg m}^2$ .

### Moment of Inertia of

#### A single particle

$$I = mr^2$$

where  $m$  = mass of the particle

$r$  = perpendicular distance of the particle from the axis about which moment of Inertia is to be calculated

For a continuous object

$$I = \int dI = \int r^2 dm$$

where,  $dI$  = moment of inertia of a small element

$dm$  = mass of a small element

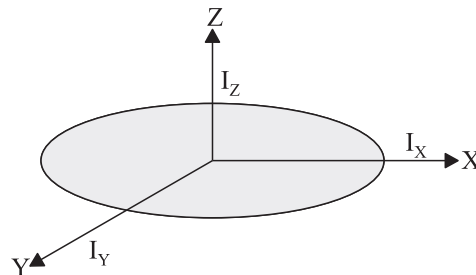
$r$  = perpendicular distance of the particle from the axis

## TWO IMPORTANT THEOREMS ON MOMENT OF INERTIA

### Perpendicular Axis Theorem

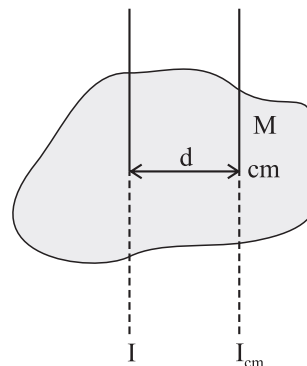
Only applicable to plane lamina (that means for 2-D objects only)

$$I_Z = I_X + I_Y \text{ (when object is in } x-y \text{ plane).}$$



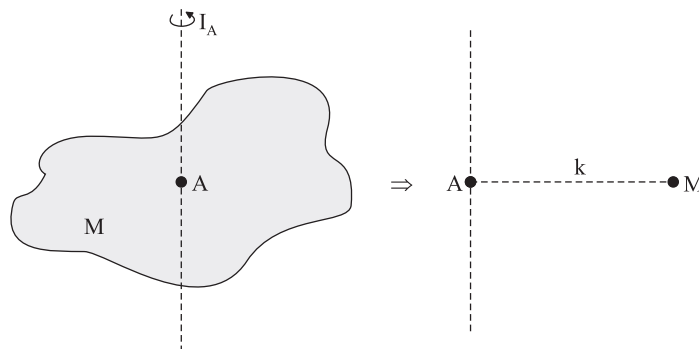
### Parallel Axis Theorem

(Applicable to any type of object) :



$$I = I_{cm} + Md^2$$

## RADIUS OF GYRATION (k)

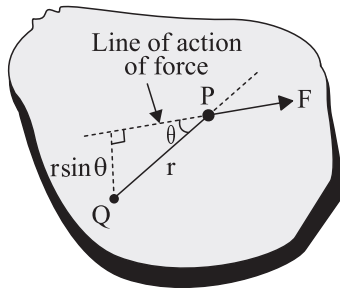


$$I_A = Mk^2$$

$$k = \sqrt{\frac{I_A}{M}}$$

## TORQUE

$$\vec{\tau} = \vec{r} \times \vec{F}$$



$$\vec{P} = M\vec{v}_{CM}$$

$$\vec{F}_{external} = M\vec{a}_{CM}$$

Net external force acting on the body has two components tangential and centripetal.

$$F_C = ma_C = m \frac{v^2}{r_{CM}} = m\omega^2 r_{CM} \quad F_t = ma_t = m\alpha r_{CM}$$

## ROTATIONAL EQUILIBRIUM

For translational equilibrium.

$$\sum F_x = 0$$

$$\text{and } \sum F_y = 0$$

The condition of rotational equilibrium is

$$\sum \vec{\tau} = 0$$

## ANGULAR MOMENTUM ( $\vec{L}$ )

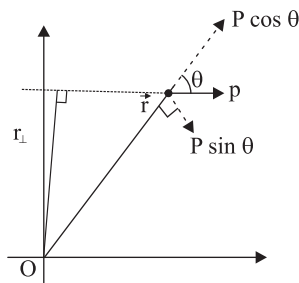
### Angular Momentum of a Particle About a Point

$$\vec{L} = \vec{r} \times \vec{P}$$

$$L = rP \sin \theta$$

$$|\vec{L}| = r_{\perp} \times P$$

$$|\vec{L}| = P_{\perp} \times r$$



### Angular Momentum of a Rigid Body Rotating about Fixed Axis

$$L_H = I_H \omega$$

$L_H$  = angular momentum of object about axis H.

$I_H$  = Moment of Inertia of rigid object about axis H.

$\omega$  = angular velocity of the object.

## Conservation of Angular Momentum

Angular momentum of a particle or a system remains constant if  $\tau_{ext} = 0$  about that point or axis of rotation.

$$L_i = L_f \Rightarrow I_i \omega_i = I_f \omega_f$$

## Relation Between Torque and Angular Momentum

$$\tau = \frac{dL}{dt}$$

Torque is change in angular momentum

## IMPULSE OF TORQUE

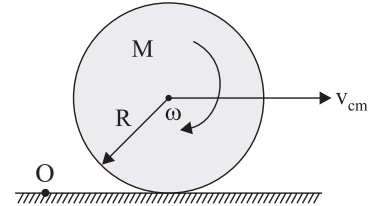
$$\int \vec{\tau} dt = \Delta \vec{J}$$

$\Delta \vec{J}$  = Change in angular momentum.

## Rolling Motion

$$\text{Total kinetic energy} = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I_{CM} \omega^2$$

$$\text{Total angular momentum about O} = M v_{CM} R + I_{CM} \omega, \otimes$$



### Pure Rolling (or Rolling without Slipping) On Stationary Surface

$$\text{Condition: } v_{cm} = R\omega$$

If  $v_{cm} > R\omega$  then rolling with forward slipping.

If  $v_{cm} < R\omega$  then rolling with backward slipping.

Total kinetic energy in pure rolling

$$K_{total} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} (Mk^2) \left( \frac{v_{cm}^2}{R^2} \right) = \frac{1}{2} M v_{cm}^2 \left( 1 + \frac{k^2}{R^2} \right)$$

### Dynamics:

$$\vec{\tau}_{cm} = I_{cm} \vec{\alpha}, \quad \vec{F}_{ext} = M \vec{a}_{cm}, \quad \vec{P}_{system} = M \vec{v}_{cm}$$

$$\text{Total K.E.} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \left( \frac{v_{cm}}{R} \right)^2$$

### Pure rolling motion on an inclined plane

$$\text{Acceleration } a = \frac{g \sin \theta}{1 + k^2 / R^2}$$

$$\text{Minimum frictional coefficient } \mu_{min} = \frac{\tan \theta}{1 + R^2 / k^2}$$

**Angular momentum about axis O** =  $\vec{L}$  about C.M. +  $\vec{L}$  of C.M. about O

$$\vec{L}_O = I_{CM} \vec{\omega} + \vec{r}_{CM} \times M \vec{v}_{CM}$$

