CHAPTER

7 Applications of Derivatives

The average rate of change $=\dfrac{\Delta y}{\Delta t}$.

When Limit $\Delta t \to 0$ is applied, the rate of change becomes instantaneous and we get the rate of change of y w.r.t. time at an instant.

i.e., $\lim\limits_{\Delta t \to 0} \dfrac{\Delta y}{\Delta t} = \dfrac{dy}{dt}$.

$\left(\dfrac{dy}{dx}\right)_P = \tan\theta$ = slope of tangent at P.

Equation of Tangent and Normal

Tangent at (x_1, y_1) is given by $(y - y_1) = f'(x_1)(x - x_1)$; when, $f'(x_1)$ is real.

And normal at (x_1, y_1) is $(y - y_1) = -\dfrac{1}{f'(x_1)}(x - x_1)$, when $f'(x_1)$ is nonzero real.

Note:

1. If tangent is parallel to x-axis, $\theta = 0° \Rightarrow \tan\theta = 0$

   $\therefore \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = 0$

2. If tangent is perpendicular to x-axis (or parallel to y-axis) then $\theta = 90° \Rightarrow \tan\theta \to \infty$ or $\cot\theta = 0$

   $\therefore \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = \infty$

Equation of tangent and normal in parametric form

Let the equation of the curve be expressed in the parameteric form $x = g(t)$ and $y = \phi(t)$ where t is the parameter.

The equation of the tangent at a point $P(t)$,

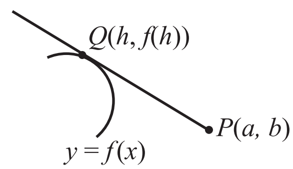
$$y - \phi(t) = \frac{\phi'(t)}{g'(t)}[x - g(t)] \quad \text{and}$$

the equation of normal is $y - \phi(t) = \dfrac{-g'(t)}{\phi'(t)}[x - g(t)]$

Tangent from an External Point

Given a point $P(a, b)$ which does not lie on the curve $y = f(x)$, then the equation of possible tangents to the curve passing through (a, b) can be found by solving for the point of contact Q.

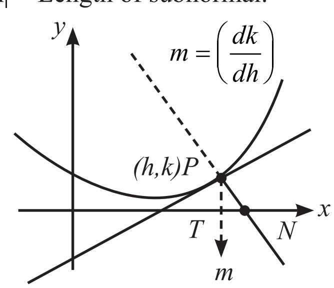
$$f'(h) = \frac{f(h) - b}{h - a}$$



And equation of tangent is $y - b = \dfrac{f(h) - b}{h - a}(x - a)$

Length of Tangent, Normal, Subtangent, Subnormal at P(h,k)

1. $PT = |k|\sqrt{1 + \dfrac{1}{m^2}}$ = Length of Tangent
2. $PN = |k|\sqrt{1 + m^2}$ = Length of Normal
3. $TM = \left|\dfrac{k}{m}\right|$ = Length of subtangent
4. $MN = |km|$ = Length of subnormal.



Angle Between the Curves

Angle between two intersecting curves is defined as the acute angle between their tangents (or normals) at the point of intersection of two curves.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

If $\theta = \pi/2$, then the two curves are said to cut each other orthogonally and the condition for this to happen is:

$$m_1 \times m_2 = -1 \Rightarrow f'(x_0) \times g'(x_0) = -1$$

Shortest Distance between two Curves

Shortest distance between two non-intersecting differentiable curves is always along their common normal. (Wherever defined)

Errors and Approximations

1. **Errors:** Let $y = f(x)$

   From definition of derivative, $\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dx}$

   $\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ approximately or $\Delta y = \left(\frac{dy}{dx}\right) . \Delta x$ approximately

   Definition:

   (i) Δx is known as **absolute error** in x.

   (ii) $\frac{\Delta x}{x}$ is known as **relative error** in x.

   (iii) $\frac{\Delta x}{x} \times 100\%$ is known as **percentage error** in x.

2. **Approximations:** From definition of derivative,

   As Derivative of $f(x)$ at $(x = a) = f'(a)$

   or $f'(a) = \lim_{\delta x \to 0} \frac{f(a + \delta x) - f(a)}{\delta x}$

   or $\frac{f(a + \delta x) - f(a)}{\delta x} \to f'(a)$ (approximately)

   $f(a + \Delta x) - f(a) \to \Delta x f'(a)$ (approximately).

Properties of Monotonic Functions

1. If $f(x)$ is strictly increasing function on an interval $[a, b]$, then f^{-1} exists and it is also a strictly increasing function.
2. If $f(x)$ is strictly increasing function on an interval $[a, b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.
3. If $f(x)$ and $g(x)$ both are monotonically (or strictly) increasing (or decreasing) functions on $[a, b]$, then $gof(x)$ is a monotonically (or strictly) increasing (in either case) function on $[a, b]$.
4. If one of the two functions $f(x)$ and $g(x)$ is strictly (or monotonically) increasing and other a strictly (monotonically) decreasing, then $gof(x)$ is strictly (monotonically) decreasing (in either case) on $[a, b]$.
5. If $f(x)$ is increasing function then $\frac{1}{f(x)}$ is decreasing function for $f(x) \neq 0$.
6. If a function is invertible it has to be either increasing or decreasing.

Rolle's Theorem

If a function f defined on $[a, b]$ is

1. Continuous on $[a, b]$
2. derivable on (a, b) and
3. $f(a) = f(b)$.

Then there exists atleast one c $(a < c < b)$ such that $f'(c) = 0$.

Lagrange's Mean Value Theorem (LMVT)

If a function f defined on $[a, b]$ is

1. continuous on $[a, b]$ is
2. derivable on (a, b)
3. $f(a) = f(b)$,

then there exists at least one real numbers between a and b $(a < c < b)$ such

that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

Special Points

1. **Critical points:** The points of domain for which $f'(x)$ is equal to zero or doesn't exist are called critical points.
2. **Stationary points:** The stationary points are the points of domain where $f'(x) = 0$.

Note: Every stationary point is a critical point but vice-versa is not true.

Significance of the Sign of 2nd order Derivative and Point of Inflection

If $f''(x) > 0 \ \forall \ x \in (a, b)$ then graph of $f(x)$ is concave upward in (a, b). Similarly if $f''(x) < 0 \ \forall \ x \in (a, b)$ then graph of $f(x)$ is concave downward in (a, b).

Useful Formulae of Mensuration to Remember

1. Volume of a cuboid = ℓbh.
2. Surface area of cuboid = $2(\ell b + bh + h\ell)$.
3. Volume of cube = a^3.
4. Surface area of cube = $6a^2$.
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.
6. Curved surface area of cone = $\pi r\ell$ (ℓ = slant height).
7. Curved surface area of a cylinder = $2\pi rh$.
8. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.
9. Volume of a sphere = $\frac{4}{3}\pi r^3$.
10. Surface area of a sphere = $4\pi r^2$.
11. Area of a circular sector = $\frac{1}{2}r^2\theta$, when θ is in radians.
12. Volume of a prism = (area of the base) × (height).
13. Lateral surface area of a prism = (perimeter of the base) × (height).
14. Total surface area of a prism = (lateral surface area) + 2 (area of the base).

    (Note that lateral surfaces of a prism are all rectangle.)
15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).
16. Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height).

    (Note that slant surfaces of a pyramid are triangles).