Binomial Theorem

$\underbrace{\text{Binomial}}_{\text{2 terms}}$

Binomial Theorem

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2x^1y^1 + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1x^0y^4$$

Observation

① (F.T) ↓

② (S.T) ↑

Pascal's Triangle

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Some Binomial Expansions

$(x+y)^n = \_\_\_\_$

$(x+y)^{\boxed{5}} = 1x^{\boxed{5}}y^{\boxed{0}} + 5x^{\boxed{4}}y^{\boxed{1}} + 10x^{\boxed{3}}y^{\boxed{2}} + 10x^{\boxed{2}}y^{\boxed{3}} + 5x^1y^4 + 1x^0y^5$

<u>Obs.</u>
1. $x \downarrow$
2. $y \uparrow$
3. Sum of powers of x and $y \Rightarrow n$
4. No. of terms $= (n+1)$

Factorial

$n! \rightarrow$ defined

(Whole No)!

$\boxed{\frac{1}{2}! \; / \; (-1)!}$

$3! = 3\times2\times1 = 6$

$4! = 4\times\underline{3\times2\times1} = 24$

$5! = 5\times\boxed{4\times3\times2\times1} = 120$

* $\boxed{0! = 1}$ $\quad$ $\boxed{5! = 5\times4!}$

$1! = 1$

$2! = 2\times1 = 2$

$3! = 3\times2\times1 = 6$

Proof :- $n! = n\boxed{(n-1)(n-2)\,.....\,1}$

$n! = n\,(n-1)!$

$$\boxed{\frac{n!}{n} = (n-1)!}$$

Put $n=1$

$\frac{1!}{1} = 0! \Rightarrow \boxed{0! = 1}$

Binomial Coefficients

$$\boxed{{}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}}$$

Ex: ${}^{4}C_{3} = \frac{4!}{3!\,1!} = \frac{4\times\cancel{3}\times\cancel{2}\times 1}{\cancel{3}\times\cancel{2}\times 1\times 1} = 4$

${}^{8}C_{6} = \frac{8!}{6!\,2!} = \frac{8\times 7\times \cancel{6!}}{\cancel{6!}\times 2} = \boxed{28}$

Imp Note.

${}^{n}C_{0} = {}^{n}C_{n} = 1$

${}^{n}C_{1} = {}^{n}C_{n-1} = n$

$$\frac{n!}{1!\,(n-1)!} = \frac{n\times\cancel{(n-1)!}}{1\times\cancel{(n-1)!}} = n$$

Statement of Binomial Theorem

$$\boxed{(x+y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \dots\dots\dots + {}^nC_n\, x^0 y^n}$$

$$(x+y)^4 = {}^4C_0\, x^4 + {}^4C_1\, x^3 y^1 + {}^4C_2\, x^2 y^2 + {}^4C_3\, x^1 y^3 + {}^4C_4\, x^0 y^4$$

${}^4C_0$	${}^4C_1$	${}^4C_2$	${}^4C_3$	${}^4C_4$
1	4	6	4	1

Number of Terms

$\rightarrow (n+1)$

Expansion of $(1+x)^n$ & $(1-x)^n$

$$(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \ldots\ldots.. + {}^nC_n\, x^0 y^n$$

$$(1+x)^n = {}^nC_0 + {}^nC_1\, x^1 + {}^nC_2\, x^2 + \ldots + {}^nC_n\, x^n$$

\# $$\boxed{(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r}$$

Ex. $$\underbrace{\sum_{r=0}^{20} {}^{20}C_r\, 3^r} = (1+3)^{20} = \underline{4^{20}}$$

$$(1-x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + (-1)^n\, {}^nC_n\, x^n$$

$$\boxed{(1-x)^n = \sum_{r=0}^{n} {}^nC_r\, (-1)^r\, x^r}$$

Let $n \geq 5$ be an integer. If $\boxed{9^n - 8n - 1 = 64\,\alpha}$ and $\boxed{6^n - 5n - 1 = 25\,\beta}$, then $\alpha - \beta$ is equal to:

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A. $1 + {}^nC_2(8-5) + {}^nC_3(8^2 - 5^2) + \ldots + {}^nC_n(8^{n-1} - 5^{n-1})$

B. $1 + {}^nC_3(8-5) + {}^nC_4(8^2 - 5^2) + \ldots + {}^nC_n(8^{n-2} - 5^{n-2})$

C. ${}^nC_3(8-5) + {}^nC_4(8^2 - 5^2) + \ldots + {}^nC_n(8^{n-2} - 5^{n-2})$

D. ${}^nC_4(8-5) + {}^nC_5(8^2 - 5^2) + \ldots + {}^nC_n(8^{n-3} - 5^{n-3})$

$$\alpha = \frac{9^n - 8n - 1}{64} = \frac{(1+8)^n - 8n - 1}{64} = \frac{\left(\cancel{{}^nC_0} + \cancel{{}^nC_1 8} + {}^nC_2 8^2 + \ldots + {}^nC_n 8^n\right) - \cancel{8n} - \cancel{1}}{8^2}$$

$$\therefore \boxed{\alpha = \cancel{{}^nC_2} + {}^nC_3 8 + {}^nC_4 8^2 + \ldots + {}^nC_n 8^{n-2}}$$

$$(1+x)^n = {}^nC_0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \dots + {}^nC_n x^n$$

$$\beta = \frac{(1+5)^n - 5n - 1}{25}$$

$$= \frac{\left(\cancel{{}^nC_0} + \cancel{{}^nC_1 5^1} + {}^nC_2 5^2 + \dots + {}^nC_n 5^n\right) - \cancel{5n} - \cancel{1}}{5^2}$$

$$\boxed{\beta = \cancel{{}^nC_2} + {}^nC_3 5 + {}^nC_4 5^2 + \dots + {}^nC_n 5^{n-2}}$$

$$\alpha - \beta = {}^nC_3(8-5) + {}^nC_4(8^2-5^2) + \dots + {}^nC_n(8^{n-2} - 5^{n-2})$$

Important Formulas

Important Formulas

$^{10}C_{x} = {}^{10}C_{y}$ $\quad$ $\underline{x+y=10}$

(1) $\boxed{{}^{n}C_{r} = {}^{n}C_{n-r}}$

Ex: $\boxed{{}^{8}C_{3} = {}^{8}C_{5}}$

$$\underbrace{\frac{8!}{3!\,5!} = \frac{8!}{5!\,3!}}$$

Ex $\underbrace{{}^{10}C_{4} = {}^{10}C_{6}}$

Ex: $\boxed{{}^{10}C_{x} = {}^{10}C_{3}}$ $\quad$ $\boxed{x = ?}$

$\boxed{x = 3, 7}$

$^{10}C_{3} = {}^{10}C_{7}$ $\quad$ $3+7=10$

$^{10}C_{x} = {}^{10}C_{3}$

$x = 3$

Important Formulas

② $$\boxed{{}^{n}C_r = \frac{n}{r}\,{}^{n-1}C_{r-1}}$$

$$\frac{n!}{r!\,(n-r)!} = \frac{n\,(n-1)!}{r\,(r-1)!\,(n-r)!}$$

$$= \frac{n}{r}\,{}^{n-1}C_{r-1}$$

Ex $$\boxed{{}^{9}C_4 = \frac{9}{4}\,{}^{8}C_3}$$

$${}^{10}C_5 = \frac{10}{5}\,{}^{9}C_4$$

$$= \frac{10}{5} \times \frac{9}{4} \times {}^{8}C_3$$

$$\vdots$$

Important Formulas

PACMAN 

(3) $${}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$$

$$\boxed{{}^{9}C_{3} + {}^{9}C_{4} = {}^{10}C_{4}}$$

$$\boxed{{}^{8}C_{5} + {}^{8}C_{4} = {}^{9}C_{5}}$$

Important Formulas

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$$\boxed{\frac{{}^{n}C_{r}}{{}^{n}C_{r-1}} = \frac{n-r+1}{r}}$$

$$\boxed{\frac{{}^{n}C_{r+1}}{{}^{n}C_{r}} = \frac{n-r}{r+1}}$$

Important Formulas

The number of ordered pairs (r, k) for which $6\ {}^{35}C_r = (k^2 - 3)\ {}^{36}C_{r+1}$, where k is an integer, is

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A. 6

B. 2

C. 3

D. 4 ✓

$$6\ {}^{35}C_r = (k^2-3)\ {}^{36}C_{r+1}$$

$$\Rightarrow \cancel{6} \cdot \cancel{{}^{35}C_r} = (k^2-3)\frac{\overset{6}{\cancel{36}}}{r+1}\cdot \cancel{{}^{35}C_r}$$

$$\Rightarrow \boxed{\frac{r+1}{6}+3=k^2} \qquad \boxed{0 \le r \le 35}$$

$r+1 \Rightarrow 6, \cancel{12}, \cancel{18}, \cancel{24}, \cancel{30}, 36$

$k=\pm 2$ $\qquad k=\pm 3$

(35, 3)

(35, −3)

$$\frac{\boxed{r+1}}{6} + 3 = k^2$$

if $r+1=6$ $\qquad k^2 = 1+3$

$r=5$ $\qquad k^2=4$

$$\boxed{k=\pm 2}$$

$$\left\{\begin{array}{l}(5, +2)\\(5, -2)\end{array}\right.$$

$$k^2 = \frac{12}{6} + 3 = 5$$

The value of $A = \sum_{r=0}^{20} {}^{50-r}C_6$ is equal to $\boxed{\text{PACMAN}}$

Kab? ✓

Kese?

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A. ${}^{51}C_7 - {}^{30}C_7$

B. ${}^{50}C_7 - {}^{30}C_7$

C. ${}^{51}C_7 + {}^{30}C_7$

D. ${}^{50}C_6 - {}^{30}C_6$

$$A = {}^{50}C_6 + {}^{49}C_6 + {}^{48}C_6 + \dots + {}^{30}C_6$$

$$A = {}^{30}C_6 + {}^{31}C_6 + {}^{32}C_6 + \dots + {}^{50}C_6$$

$$A = {}^{30}C_7 + {}^{30}C_6 + {}^{31}C_6 + {}^{32}C_6 + \dots + {}^{50}C_6 - {}^{30}C_7$$

${}^{31}C_7$ ${}^{32}C_7$ ${}^{33}C_7$

$$\boxed{A = {}^{51}C_7 - {}^{30}C_7}$$

If $\left({}^{40}C_0\right)+\left({}^{41}C_1\right)+\left({}^{42}C_2\right)+\ldots+\left({}^{60}C_{20}\right)=\frac{m}{n}\,{}^{60}C_{20}$, m and n are coprime, then m + n is equal to _____ .

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$\boxed{{}^{n}C_r+{}^{n}C_{r+1}={}^{n+1}C_{r+1}}$

A. 105

B. 102

C. 107

D. 109

$${}^{40}C_{40}+{}^{41}C_{40}+{}^{42}C_{40}+\ldots+{}^{60}C_{40}$$

$$\Rightarrow \underbrace{{}^{41}C_{41}+{}^{41}C_{40}}+{}^{42}C_{40}+\ldots+{}^{60}C_{40}$$

$$\Rightarrow \underbrace{{}^{42}C_{41}+{}^{42}C_{40}}_{{}^{43}C_{41}}+\ldots+{}^{60}C_{40} \Rightarrow {}^{61}C_{41}$$

$$\Rightarrow {}^{61}C_{41}$$

$$\Rightarrow \frac{61}{41}\, {}^{60}C_{40}$$

$$\Rightarrow \frac{61}{41}\, {}^{60}C_{20}$$

$$\Rightarrow \frac{m}{n}\, {}^{60}C_{20}$$

$m = 61$

$n = 41$

$\underline{m+n = 102}$

For a positive integer n, $\left(1+\frac{1}{x}\right)^n$ is expanded in increasing power of x. If three consecutive coefficients in this expansion are in the ratio 2 : 5 : 12, then n is equal to

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$$^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 2 : 5 : 12$$

$$\boxed{\frac{^nC_r}{^nC_{r-1}} = \frac{5}{2}} \qquad \boxed{\frac{^nC_{r+1}}{^nC_r} = \frac{12}{5}}$$

$$\boxed{\frac{n-r+1}{r} = \frac{5}{2}} \quad \& \quad \boxed{\frac{n-r}{r+1} = \frac{12}{5}}$$

Exam

* $\frac{n-r+1}{r} = \frac{5}{2}$

* $\frac{n-r}{r+1} = \frac{12}{5}$

$2n - 2r + 2 = 5r$

$\dfrac{2n+2}{7} = r$

$5n - 5r = 12r + 12$

$\dfrac{5n-12}{17} = r$

$$\frac{2n+2}{7} = \frac{5n-12}{17}$$

$$34n + 34 = 35n - 84$$

$$\Rightarrow \boxed{n = 118}$$

JEE Adv. 2013

The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio 5: 10: 14 then n is ___ 6

$(1+x)^{n+5}$

$5 : 10 : 14$

$\frac{(n+5)-r+1}{r} = 2$

$4 \times \boxed{n+6=3r}$

$\boxed{4n+24=12r}$

$\frac{(n+5)-r}{r+1} = \frac{7}{5}$

$5n+25 = 12r+7$

$\boxed{5n+18=12r}$

$4n+24 = 5n+18$

$\therefore \boxed{n=6}$

Important Terms

Important terms in Binomial Expansion

1. General term
2. Middle Term
3. Term independent of x
4. Numerically Greatest Term

General Term

General Term

$$(x + y)^n = \underbrace{{}^nC_0\, x^{n-0}\, y^0}_{T_1} + \underbrace{{}^nC_1\, x^{n-1}\, y^1}_{T_2} + \underbrace{{}^nC_2\, x^{n-2}\, y^2}_{T_3} + \ldots\ldots.. + \underbrace{{}^nC_n\, x^0\, y^n}_{T_{n+1}}$$

$$\boxed{T_{r+1} = {}^nC_r\, x^{n-r}\, y^r}$$

$T_4 = ?$ $r=3$

$T_7 = ?$ $r=6$

Illustration:

$r=3$

Find fourth term in the expansion of $\left(2x-\frac{y}{2}\right)^7$.

$$T_{r+1} = {}^7C_r (2x)^{7-r}\left(-\frac{y}{2}\right)^r$$

$$\boxed{T_4 = {}^7C_3 (2x)^4\left(-\frac{y}{2}\right)^3}$$

If the fourth term in the expansion of $\left(x + x^{\log_2 x}\right)^7$ 4480, the value of x where x ∈ N is equal to

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A. 2

B. 4

C. 3

D. 1

$T_4 = 4480 \qquad x = ?$

$T_{r+1} = {}^7C_r \, (x)^{7-r} \left(x^{\log_2 x}\right)^r$

$(a^m)^n = a^{mn}$

$r = 3$

$T_4 = {}^7C_3 \, (x)^4 \left(x^{\log_2 x}\right)^3 = 4480$

$\frac{7!}{3!\,4!} = \frac{7 \times \cancel{6} \times 5}{\cancel{6}} = 35$

$x^4 \cdot x^{3\log_2 x} = \frac{4480}{35}$

$x^{4+3\log_2 x} = 128$

$$\log_2 x^{\boxed{4+3\log_2 x}} = \log_2 2^7$$

$$(4+3\log_2 x)(\log_2 x) = 7$$

$$(4+3t)t = 7$$

$$3t^2+4t-7=0$$

$$3t^2+7t-3t-7=0$$

$$(t-1)(3t+7)=0$$

$$t = 1, -7/3$$

$$\log_2 x = 1 \quad \text{OR} \quad -7/3$$

$$x = 2^1 \quad \text{OR} \quad \cancel{2^{-7/3}}$$

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Let the coefficients of x^{-1} and x^{-3} in the expansion of $\left(2x^{\frac{1}{5}} - \frac{1}{x^{\frac{1}{5}}}\right)^{15}$, $x > 0$, be m and n respectively. If r is a positive integer such $mn^2 = {}^{15}C_r \, 2^r$, then the value of r is equal to__.

$$T_{r+1} = {}^{15}C_r \left(2x^{\frac{1}{5}}\right)^{15-r} \left(\frac{-1}{x^{1/5}}\right)^r$$

$$= {}^{15}C_r \, 2^{15-r} (-1)^r \, x^{\frac{15-r}{5} - \frac{r}{5}}$$

$$m \times n^2$$

$$= {}^{15}C_{10} \cdot 2^5 \times (-1)^2$$

$$= {}^{15}C_5 \cdot 2^5$$

$$= {}^{15}C_r \cdot 2^r$$

$$r = 5$$

$${}^{15}C_r \; 2^{15-r} \; (-1)^r \; x^{\boxed{\frac{15-2r}{5}}}$$

$$\frac{15-2r}{5} = -3$$

$$r = 15$$

$$\frac{15-2r}{5} = -1$$

$$\Rightarrow \boxed{r = 10}$$

Coeffi of x^{-1} : $\boxed{{}^{15}C_{10} \; 2^5 \; (-1)^{10} = m}$

Coeffi of x^{-3} : $\underbrace{{}^{15}C_{15} \; 2^0 \; (-1)^{15}} = n$

$\therefore \boxed{n = -1}$ $\quad \boxed{m = {}^{15}C_{10} \cdot 2^5}$

Let the ratio of the fifth term from the beginning to the fifth term from the end in the binomial expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^{n}$, in the increasing powers of $\frac{1}{\sqrt[4]{3}}$ be $\boxed{\sqrt[4]{6} : 1}$. If the sixth term from the beginning is $\frac{\alpha}{\sqrt[4]{3}}$, then α is equal to ________.

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$(n-5+2)^{th} = (n-3)^{th}$

T_5 End $= T_{n-3}$ Start

$r=4$

$r = n-3-1$

$= n-4$

$$\frac{T_5}{T_{n-3}} = 6^{\frac{1}{4}} \Rightarrow \frac{\cancel{{}^{n}C_4}\left(2^{\frac{1}{4}}\right)^{n-4}\left(3^{-\frac{1}{4}}\right)^{4}}{\cancel{{}^{n}C_{n-4}}\left(2^{\frac{1}{4}}\right)^{4}\left(3^{-\frac{1}{4}}\right)^{n-4}} = 6^{\frac{1}{4}} \Rightarrow \boxed{n=9}$$

$$T_6 = {}^{9}C_5\left(2^{\frac{1}{4}}\right)^{4}\left(3^{-\frac{1}{4}}\right)^{5}$$

$$^{9}C_5\; 2\left(3^{-1/4}\right)^{4}\left(3^{-1/4}\right)^{1}$$

$$^{9}C_5 \cdot 2 \times \frac{1}{3}\;\frac{1}{\cancel{\sqrt[4]{3}}} = \frac{\alpha}{\cancel{\sqrt[4]{3}}}$$

$$\therefore \boxed{\alpha = \frac{2}{3}\times {}^{9}C_5} = 84$$

Concept

r^{th} term from End $= (n-r+2)^{th}$ term from beginning

$$(x+y)^{7} = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8$$

5^{th} Last $= T_4$ Start

$(7-5+2)$

$\underbrace{3^{rd} \text{ Term from End}} = 6^{th}$ term from Start

3^{rd} Last $= (7-3+2)^{th}$ Start

$= 6^{th}$ term Start

Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1+x)^2+(1+x)^3+\ldots+(1+x)^{49}+(1+mx)^{50}$ is $(3n+1)\,{}^{51}C_3$ for some positive integer n. Then the value of n is

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$x^2: (1+x)^2+(1+x)^3+(1+x)^4+\ldots+(1+x)^{49}+(1+mx)^{50}$

$: \boxed{\underbrace{{}^{3}C_3+{}^{3}C_2+{}^{4}C_2+\ldots+{}^{49}C_2}+{}^{50}C_2\,m^2=(3n+1)\cdot{}^{51}C_3}$

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$$\frac{{}^{50}C_3}{{}^{50}C_2}+\frac{\cancel{{}^{50}C_2}\,m^2}{\cancel{{}^{50}C_2}}=(3n+1)\frac{51}{3}$$

$$\left(\frac{50-3+1}{3}\right)+m^2=(3n+1)17$$

$$16 + m^2 = 51n + 17$$

$$\boxed{m^2 = 51n + 1}$$

$$\boxed{m, n \in \mathbb{Z}^+}$$

$51(1) + 1 = 52$ ✗

$51(2) + 1 = 103$ ✗

$51(3) + 1 =$ ✗

$51(4) + 1 =$ ✗

$m^2 = 51(5) + 1 = 256$ ✓

$\boxed{n = 5}$ $\boxed{m = 16}$

Term Independent of ‘x’

↳ Constant : coeffi of x^0

Example :

Which

Find term independent of x in $\left(x^2+\frac{1}{x^2}-2\right)^{10}$

$$\left(x-\frac{1}{x}\right)^{20}$$

$$\downarrow$$

$$\left(\left(x-\frac{1}{x}\right)^2\right)^{10}$$

$$\downarrow$$

$$\boxed{\left(x-\frac{1}{x}\right)^{20}}$$

$$T_{r+1} = {}^{20}C_r\,(x)^{20-r}\left(\frac{-1}{x}\right)^r$$

$$T_{r+1} = {}^{20}C_r\,(x)^{\boxed{20-2r}}(-1)^r$$

$$20-2r=0$$

$$\therefore \boxed{r=10}$$

$$T_{11} = {}^{20}C_{10}\,(x)^0\,(-1)^{10} \Rightarrow \boxed{T_{11} = {}^{20}C_{10}}$$

If the term independent of x in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is k, then 18k is equal to

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A. 9

B. 11

C. 5

D. 7

$$ {}^9C_r \left(\frac{3x^2}{2}\right)^{9-r} \left(\frac{-1}{3x}\right)^r $$

$$ 18 - 2r - r = 0 $$

$$ \therefore \boxed{r = 6} $$

$$ 18k = {}^9C_6 \left(\frac{3}{2}\right)^3 \left(\frac{-1}{3}\right)^6 \times 18 $$

$$ = \frac{\cancel{9} \times \cancel{8} \times 7}{\cancel{3} \times \cancel{2} \times 1} \times \frac{1}{\cancel{2^3}} \times \frac{1}{\cancel{3} \times \cancel{3} \times \cancel{3}} \times \cancel{3} \times \cancel{3} \times \cancel{2} $$

$$ = 7 $$

If the maximum value of the term independent of t in the expansion of $\left(t^2 x^{\frac{1}{5}} + \frac{(1-x)^{\frac{1}{10}}}{t}\right)^{15}$, $x \geq 0$, is K, then 8K is equal to ______.

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$$30 - 2r - r = 0$$

$$\therefore \boxed{r = 10}$$

$$^{15}C_r \left(t^2 x^{\frac{1}{5}}\right)^{15-r} \left(\frac{(1-x)^{\frac{1}{10}}}{t}\right)^r$$

$$\Rightarrow \; ^{15}C_{10} \left(x^{\frac{1}{5}}\right)^5 \left((1-x)^{\frac{1}{10}}\right)^{10}$$

$$\Rightarrow \; ^{15}C_{10}\, x(1-x)$$

Term Ind. of 't' $\Rightarrow {}^{15}C_{10}\, \underline{x(1-x)}$

$y = x - x^2$

$\frac{dy}{dx} = 0$

$1 - 2x = 0$

$x = 1/2$

max Value $\Rightarrow 8 \cdot {}^{15}C_{10} \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \Rightarrow 8K$

$$\frac{15!}{10!\,5!} = ?$$

$$\frac{8 \times {}^{15}C_{10}}{4} = 8K$$

$$\therefore \boxed{\underbrace{2 \times {}^{15}C_{10}} = 8K}$$

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The term independent of x in the expression of

$$\underbrace{\left(1-x^2+3x^3\right)\left(\frac{5}{2}x^3-\frac{1}{5x^2}\right)^{11}}, x \neq 0 \text{ is}$$

A. $\frac{7}{40}$

B. $\frac{33}{200}$

C. $\frac{39}{200}$

D. $\frac{11}{50}$

$\Rightarrow {}^{11}C_r \left(\frac{5x^3}{2}\right)^{11-r} \left(\frac{-1}{5x^2}\right)^r$

$\Rightarrow {}^{11}C_r \left(\frac{5}{2}\right)^{11-r} \left(\frac{-1}{5}\right)^r (x)^{33-5r}$

$33-5r = -3$

$33-5r = 0$

$33-5r = -2$

$\therefore \boxed{r=7}$

$$\left(1 - x^2 + 3x^3\right)\underbrace{\left(\frac{5x^3}{2} - \frac{1}{5x^2}\right)^{11}}$$

$$1 \times \cancel{x^0}$$

$$-x^2 \times \underline{x^{-2}} \Rightarrow \boxed{(-1) \times {}^{11}C_7 \left(\frac{5}{2}\right)^4 \left(\frac{-1}{5}\right)^7}$$

$$3x^3 \times x^{-3} \Rightarrow \times$$

$$\Rightarrow \frac{33}{200} \quad (B)$$

Rational and Irrational Terms

If n is the number of irrational terms in the expansion of $\left(3^{1/4}+5^{1/8}\right)^{60}$, then $(n-1)$ is divisible by:

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(1) 26

(2) 30

(3) 8

(4) 7

$\boxed{0 \le r \le 60}$

$\underline{1-1}$

$$^{60}C_r \left(3^{\frac{1}{4}}\right)^{60-r} \left(5^{\frac{1}{8}}\right)^{r}$$

$$\Rightarrow {}^{60}C_r \; 3^{\boxed{\frac{60-r}{4}}} \; 5^{\boxed{\frac{r}{8}}}$$

Integral

$LCM(4,8) = 8$

$r \Rightarrow 0, 8, 16, 24, 32, 40, 48, 56$

Total $= 61$

Rational $= 8$

Irrational $= 53 = n$

$n = 53$

$n-1 = 52$

The total number of irrational terms in the binomial expansion of

$$\left(7^{\frac{1}{5}} - 3^{\frac{1}{10}}\right)^{60}$$

JEE M 2019

A. 55

B. 49

C. 48

D. 54

$0 \le r \le 60$

$G.T \Rightarrow {}^{60}C_r \left(7^{\frac{1}{5}}\right)^{60-r} \left(-3^{\frac{1}{10}}\right)^r$

$\Rightarrow {}^{60}C_r \; 7^{\boxed{\frac{60-r}{5}}} (-1)^r (3)^{\boxed{\frac{r}{10}}}$

$LCM(5, 10) = 10$

$r = 0, 10, 20, 30, 40, 50, 60$

Total = 61

Rational = 7

Irrational = 54

Middle Term

Middle Term

$$(x+y)^4 = T_1 + T_2 + T_3 + T_4 + T_5$$

$$(x+y)^6 = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7$$

$$(x+y)^5 = T_1 + T_2 + \boxed{T_3 + T_4} + T_5 + T_6$$

$\frac{5+1}{2}$ $\frac{5+3}{2}$

MT $\Rightarrow T_3$ and T_4

if

n = Even

$$M.T = \left(\frac{n}{2} + 1\right)^{th}$$

if n = odd

$$M.T = \left(\frac{n+1}{2}\right)^{th} \text{ and } \left(\frac{n+3}{2}\right)^{th}$$

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Let the coefficients of the middle terms in the expansion of $\left(\frac{1}{\sqrt{6}}+\beta x\right)^4$, $(1-3\beta x)^2$ and $\left(1-\frac{\beta}{2}x\right)^6$, $\beta > 0$, respectively form the first three terms of an A.P. If d is the common difference of this A.P., then $50-\frac{2d}{\beta^2}$ is equal to ______

$\frac{4}{2}+1 \Rightarrow T_3$

$\frac{2}{2}+1$

$\frac{6}{2}+1$

$T_3 = \cancel{{}^4C_2}\left(\frac{1}{\sqrt{6}}\right)^2(\beta)^2 = \beta^2$

$T_2 = {}^2C_1(1)^1(-3\beta)^1 = -6\beta$

$T_4 = {}^6C_3(1)^3\left(-\frac{\beta}{2}\right)^3 = -\frac{5}{2}\beta^3$

$\beta^2, -6\beta, -\frac{5\beta^3}{2} \rightarrow$ A.P.

$-12\beta = \beta^2 - \frac{5\beta^3}{2}$

$-12 = \beta - \frac{5\beta^2}{2}$

$50-(-7)$

$= \boxed{57}$

$$-24 = 2\beta - 5\beta^2$$

$$5\beta^2 - 2\beta - 24 = 0$$

$$5\beta^2 - 12\beta + 10\beta - 24 = 0$$

$$(\beta + 2)(5\beta - 12) = 0$$

$$\cancel{\beta = -2} \qquad \beta = \frac{12}{5}$$

$$d = -6\beta - \beta^2$$

$$\frac{d}{\beta^2} = \frac{-6}{\beta} - 1$$

$$\frac{d}{\beta^2} = \frac{-6(5)}{12} - 1$$

$$\frac{d}{\beta^2} = \frac{-7}{2}$$

$$\boxed{\frac{2d}{\beta^2} = -7}$$

$$\frac{101+1}{2} = \frac{102}{2} = 51$$

$$(x+y)^{101}$$

MT $\Rightarrow$ 51^{st} and 52^{nd}

Binomial Coefficient vs Coefficient vs Term

Consider the general term for $(ax + by)^n$

$$^nC_r\,(ax)^{n-r}\,(by)^r$$

$$\underbrace{\underbrace{\underbrace{^nC_r}_{\text{B.C.}}\; a^{n-r}\, b^r}_{\text{Coeffi}}\; x^{n-r}\, y^r}_{\text{Term}}$$

Greatest Binomial Coefficient

$(x+y)^4$ ${}^4C_0$ ${}^4C_1$ ${}^4C_2$ ${}^4C_3$ ${}^4C_4$

1 4 6 4 1

M.T

Greatest Binomial Coefficient

$$\boxed{{}^{7}C_{k} = \max}$$

$k = ?$

$k = \frac{7-1}{2}$ OR $\frac{7+1}{2}$

$k = 3$ OR 4

$${}^{7}C_{3} = {}^{7}C_{4}$$

if $n = \text{Even}$

$$\boxed{k = \frac{n}{2}}$$

if $n = \text{odd}$

$k = \frac{n-1}{2}$ OR $\frac{n+1}{2}$

${}^{19}C_{k}$

$\frac{19-1}{2} \mid \frac{19+1}{2}$

${}^{19}C_{9}$ OR ${}^{19}C_{10}$

If a, b and c are the greatest values of ${}^{19}C_p$, ${}^{20}C_q$ and ${}^{21}C_r$ respectively, then:

JEE M 2020

A. a/11 = b/22 = c/42

B. a/22 = b/11 = c/42

C. a/22 = b/42 = c/11

D. a/21 = b/11 = c/22

$a = {}^{19}C_9$

$b = {}^{20}C_{10}$

$c = {}^{21}C_{11}$

$p = \frac{19-1}{2} / \frac{19+1}{2}$

$q = \frac{20}{2}$

$\frac{21-1}{2} / \frac{21+1}{2}$

$a : b : c = \cancel{{}^{19}C_9} : \frac{20}{10}\cancel{{}^{19}C_9} : \frac{21}{11}\times\frac{20}{10}\times\cancel{{}^{19}C_9}$

$= \left(1 : 2 : \frac{42}{11}\right)\times 11$

$= 11 : 22 : 42$

Numerically Greatest Term

$\underline{N.G.T} = \underline{\underline{G.T}}.$

Numerically Greatest Term

Consider the expansion for $(ax + by)^n$. If T_{r+1} is numerically greatest term, then:

$$m-1 \leq r \leq m \quad \text{where } m = \frac{n+1}{1+\left|\frac{ax}{by}\right|}$$

S-1 $m = \dfrac{n+1}{1+\left|\frac{F.T}{S.T}\right|}$

S-2 $\boxed{m-1 \leq r \leq m}$

Numerically Greatest Term

Illustration:

Q. Find the greatest term in the expansion of $(7 - 5x)^{11}$ where $x = 2/3$.

S-1 $$m = \frac{n+1}{1+\left|\frac{FT}{ST}\right|} = \frac{11+1}{1+\left|\frac{7(3)}{-5(2)}\right|} = \frac{12}{1+2.1} = \frac{12}{3.1} = 3.99$$

S-2 $$m-1 \le r \le m$$

$$\boxed{2.99 \le r \le 3.99}$$

$r=3$

$$\boxed{G.T = T_4 = \_\_\_\_}$$

Illustration:

Find numerically greatest ~~term~~(s) in the expansion of $(3 - 5x)^{15}$ when $x = 1/5$

N.G. Coeffi $\Rightarrow$ $x=1$

S-1 $m = \dfrac{15+1}{1+\left|\dfrac{3}{-5\left(\frac{1}{5}\right)}\right|} = \dfrac{16}{4} = 4$

S-2 $m-1 \leq r \leq m$

$\boxed{3 \leq r \leq 4}$ $\rightarrow$ $r = 3$ and 4

T_4 and T_5

$\boxed{T_4 = T_5 = ?}$

$9^{th} \Rightarrow$ N.G.T.

$r=8$

$(3+6x)^n \qquad x = 3/2 \qquad r=8$

$\underline{\underline{1\text{-}1}} \quad m = \dfrac{n+1}{1+\left|\dfrac{\cancel{3}(2)}{\cancel{6}(3)}\right|} = \dfrac{3(n+1)}{4}$

$\underline{\underline{1\text{-}2}} \quad m-1 \le r \le m$

$\dfrac{3(n+1)}{4} - 1 \le 8 \le \dfrac{3(n+1)}{4}$

$\dfrac{3(n+1)}{4} \le 9$

$n+1 \le 12$

$\underline{n \le 11}$

$\dfrac{8\times 4}{3} \le n+1$

$10.\overline{3} - 1 \le n$

$9.\overline{3} \le n$

$\underbrace{9.\overline{3} \le n \le 11}$

$\underbrace{n = 10, 11}$

$n_0 = 10$

Let for the 9^{th} term in the binomial expansion of $(3 + 6x)^n$, in the increasing powers of 6x, to be the greatest for $\boxed{x = \frac{3}{2}}$, the least value of n is n_0. If k is the ratio of the coefficient of x^6 to the coefficient of x^3, then $\boxed{k + n_0}$ is equal to:

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$n_0 = 10$

$(3+6x)^{10} \Rightarrow$

$$k = \frac{\text{Coeff of } x^6}{\text{Coeff of } x^3} = \frac{{}^{10}C_4 (3)^4 (6)^6}{{}^{10}C_3 (3)^7 (6)^3} = \left(\frac{10-4+1}{4}\right) \times \frac{6^3}{3^3} = \frac{7}{4} \times 2^3 = 14$$

$k + n_0$

$= 14 + 10$

$= 24$

Finding Remainders

2022

#NVStyle

#NVStyle Method

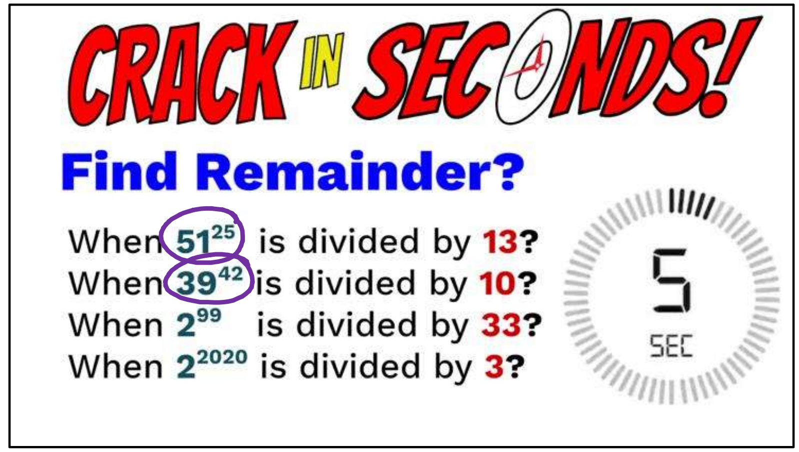
$$\frac{51^{25}}{13}$$

$$= \frac{\overset{(-1)}{51} \times \overset{(-1)}{51} \times \overset{-1}{51} \times \ldots \times \overset{-1}{51}}{13}$$

$$= (-1)^{25}$$

$$= -1 + 13 = 12$$

Video



$$\frac{(52-1)^{25}}{13}$$

$$\frac{{}^{25}C_0(52)^{25} - {}^{25}C_1(52)^{24} + \ldots - + {}^{25}C_{25}(52)^{0}}{13}$$

$$\frac{39^{42}}{10} = \frac{\overset{-1}{39} \times \overset{-1}{39} \times \cdots \times \overset{-1}{39}}{10}$$

$$10 \overline{)\,39\,}\quad 4$$
$$40$$
$$-1$$

$$= (-1)^{42}$$

$$= 1$$

The remainder when 3^{2022} is divided by 5 is

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A. 1

B. 2

C. 3

D. 4

$$\frac{3^{2022}}{5} = \frac{(3^2)^{1011}}{5}$$

$$= \frac{\overset{-1}{9} \times \overset{-1}{9} \times \overset{-1}{9} \times \ldots \times \overset{-1}{9}}{5}$$

$$= (-1)^{1011}$$

$$= -1 + 5 = 4$$

The remainder when $(2021)^{2023}$ is divided by 7 is :

JEE M 2022

A. 1

B. 2

C. 5

D. 6

$$\frac{(2021)^{2023}}{7}$$

$$7\,\overline{)\,2021}$$
$$2023$$
$$-2$$

$$\frac{(-2)^{2023}}{7} = \frac{-(2)^{2022}\cdot 2}{7}$$

$$-2+7=5$$

$$= \frac{-(8)^{674}\cdot 2}{7} = \frac{\overset{-1}{(-1)}\times\overset{1}{8}\times\overset{1}{8}\times\ldots\times\overset{1}{8}\times\overset{2}{2}}{7}$$

The remainder when $(11)^{1011} + (1011)^{11}$ is divided by 9 is

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A. 1

B. 4

C. 6

D. 8

$$\frac{11^{1011}}{9} + \frac{1011^{11}}{9}$$

$$-1 + 0 \Longrightarrow -1+9 \Rightarrow 8$$

$$9\overline{)11} \quad 9 \quad 2$$

$$9\overline{)1011} \quad 9 \quad 11 \quad 9 \quad 21 \quad 18 \quad 3$$

$$= \frac{11 \times 11 \times \ldots \times 11}{9}$$

$$= \frac{2^{1011}}{9} = \frac{(8)^{337}}{9}$$

$$= (-1)^{337} = -1$$

$$\frac{1011 \times 1011 \times 1011 \ldots \times 1011}{9}$$

$$\frac{3^{11}}{9} = \frac{3^{11}}{3^2} = 0$$

The remainder when $(2021)^{2022} + (2022)^{2021}$ is divided by 7 is

JEE M 2022

A. 0

H. W.

B. 1

C. 2

D. 6

The remainder when $7^{2022} + 3^{2022}$ is divided by 5 is :

JEE M 2022

A. 0

B. 2 H.w.

C. 3

D. 4

GP Wale Questions

The coefficient of x^{101} in the expression

$x^0(5+x)^{500}+x^1(5+x)^{499}+x^2(5+x)^{498}+\ldots\ldots x^{500}$,

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$x>0$, is $\quad r=\frac{x}{x+5} \quad a=(5+x)^{500} \quad n=501$

A. ${}^{501}C_{101}(5)^{399}$

B. ${}^{501}C_{101}(5)^{400}$

C. ${}^{501}C_{100}(5)^{400}$

D. ${}^{500}C_{101}(5)^{399}$

$$\Rightarrow \frac{a(1-r^n)}{1-r}$$

$$\Rightarrow \frac{(5+x)^{500}\left(1-\left(\frac{x}{5+x}\right)^{501}\right)}{1-\frac{x}{5+x}}$$

$$\Rightarrow \frac{\cancel{(5+x)^{501}}}{5}\left\{\frac{(5+x)^{501}-x^{501}}{\cancel{(5+x)^{501}}}\right\}$$

$$x^{101} \quad \Rightarrow \frac{1}{5}\left\{(5+x)^{501}-x^{501}\right\}$$

$$\frac{1}{5}\left({}^{501}C_{101}\,(5)^{400}\right)$$

$$= \underbrace{{}^{501}C_{101}\;5^{399}}$$

$${}^{501}C_r\,(5)^{501-r}\,(x)^r$$

$$r=101$$

Multinomial

Multinomial

$$(x_1 + x_2 + x_3)^n =$$

$(x+y)^n$

$$\frac{n!}{r!(n-r)!}(x)^{n-r} y^r$$

$r = r_1$

$n - r = r_2$

$$\frac{n!}{r_1!\, r_2!}(x)^{r_2} y^{r_1}$$

$r_1 + r_2 = n$

$$G.T \Rightarrow \frac{n!}{r_1!\, r_2!\, r_3!}(x_1)^{r_1}(x_2)^{r_2}(x_3)^{r_3}$$

where $\boxed{r_1 + r_2 + r_3 = n}$ ✓

Ex $(3x - 4y + 7)^{10}$

$$\frac{10!}{r_1!\, r_2!\, r_3!}(3x)^{r_1}(-4y)^{r_2}(7)^{r_3}$$

Example:

Find the coefficient of x^4 in $(1 + x + x^2)^{10}$

$$G.T \Rightarrow \frac{10!}{r_1!\, r_2!\, r_3!} (1)^{r_1} (x)^{r_2} (x^2)^{r_3}$$

① $r_2 + 2r_3 = 4$

② $r_1 + r_2 + r_3 = 10$

$0 \le r_1 \le 10$

$0 \le r_2 \le 10$

$0 \le r_3 \le 10$

$$\Rightarrow \frac{10!}{r_1!\, r_2!\, r_3!} (x)^{\boxed{r_2 + 2r_3}}$$

r_1	r_2	r_3	
6	4	0	$\rightarrow \frac{10!}{6!\,4!\,0!}$
7	2	1	$\rightarrow \frac{10!}{7!\,2!\,1!}$
8	0	2	$\rightarrow \frac{10!}{8!\,0!\,2!}$

add

If the constant term in the expansion of $\left(3x^3 - 2x^2 + \frac{5}{x^5}\right)^{10}$ is $2^k \cdot l$, where l is an odd integer, then the value of k is equal to :

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A. 6

B. 7

C. 8

D. 9

$$\frac{10!}{r_1!\,r_2!\,r_3!}(3x^3)^{r_1}(-2x^2)^{r_2}\left(\frac{5}{x^5}\right)^{r_3}$$

r_1	r_2	r_3
1	6	3

① $3r_1 + 2r_2 - 5r_3 = 0$

② $r_1 + r_2 + r_3 = 10$

$3r_1 + 2(10 - r_1 - r_3) - 5r_3 = 0$

$\boxed{20 + r_1 = 7r_3}$

$$\frac{10!}{1!\,6!\,3!}(3)^1(-2)^6(5)^3$$

$$\frac{\cancel{10}^{5}\times 9\times 8\times 7}{\cancel{3}\times\cancel{2}}\times\cancel{3}\times 2^6\times 5^3$$

$$= 2^9\times\left(\underbrace{5\times 9\times 7\times 5^3}_{\text{odd}}\right)$$

$$= 2^k\cdot l$$

Sum of Binomial Coefficients

Sum of Binomial coefficients

$$* \quad \boxed{{}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n}$$

$$(1+x)^n = {}^nC_0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$$

$\underline{x=1}$

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$$

$$\frac{2^n}{2} = 2^{n-1}$$

Sum of odd/ even Binomial coefficients

$$^nC_0 + {}^nC_2 + {}^nC_4 + \dots\dots = 2^{n-1}$$

$$^nC_1 + {}^nC_3 + {}^nC_5 + \dots = 2^{n-1}$$

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$$

$$2^n = {}^nC_0 + \cancel{{}^nC_1} + {}^nC_2 + \cancel{{}^nC_3} + \dots + {}^nC_n$$

$$0 = {}^nC_0 - \cancel{{}^nC_1} + {}^nC_2 - \cancel{{}^nC_3} + \dots + {}^nC_n(-1)^n$$

$$\frac{2^n}{2} = \left({}^nC_0 + {}^nC_2 + {}^nC_4 + \dots\right)$$

If $1 + (2 + {}^{49}C_1 + {}^{49}C_2 + \ldots + {}^{49}C_{49})({}^{50}C_2 + {}^{50}C_4 + \ldots + {}^{50}C_{50})$ is equal to $2^n.m$, where m is odd, then n + m is equal to _____

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$$\Rightarrow 1 + (1 + \underbrace{{}^{49}C_0 + {}^{49}C_1 + {}^{49}C_2 + \ldots + {}^{49}C_{49}}) (\underbrace{{}^{50}C_0 + {}^{50}C_2 + {}^{50}C_4 + {}^{50}C_6 + \ldots + {}^{50}C_{50}} - {}^{50}C_0)$$

$$\Rightarrow 1 + \underbrace{(1 + 2^{49})(2^{49} - 1)}$$

$$\Rightarrow \cancel{1} + (2^{49})^2 - \cancel{1^2}$$

$$\Rightarrow 2^{98} \cdot 1 = 2^n \cdot m$$

$n = 98$

$m = 1$

$n + m = 99$

Sigma Method

① $\sum_{r=1}^{100} 1 = 100$

② $\sum_{r=0}^{n} r \cdot {}^nC_r = \sum_{r=1}^{n} \cancel{r} \cdot \frac{n}{\cancel{r}} \, {}^{n-1}C_{r-1} = n \boxed{\sum_{r=1}^{n} {}^{n-1}C_{r-1}}$

$= n \, 2^{n-1}$

③ $\boxed{\sum_{r=0}^{n} {}^nC_r = 2^n}$

$\cancel{(r-1)} \frac{n-1}{\cancel{r-1}} \, {}^{n-2}C_{r-2}$

④ $\sum_{r=0}^{n} r^2 \, {}^nC_r = \sum_{r=1}^{n} r^2 \frac{n}{\cancel{r}} \, {}^{n-1}C_{r-1}$

$= n\left(\sum_{r=1}^{n} (r-1+1) \, {}^{n-1}C_{r-1}\right) = n\left(\sum (r-1) \, {}^{n-1}C_{r-1} + \sum {}^{n-1}C_{r-1}\right)$

$= n\left((n-1) \cdot 2^{n-2} + 2^{n-1}\right)$

$= n(2)^{n-2}(n-1+2) = \underline{n(n+1)\,2^{n-2}}$

Let m, n ∈ N and gcd(2, n) = 1.

If $30\ {}^{30}C_0 + 29\ {}^{30}C_1 + 28\ {}^{30}C_2 + \ldots\ldots + 2\ {}^{30}C_{28} + 1\ {}^{30}C_{29} = n\ 2^m$ $+ 0 \cdot {}^{30}C_{30}$

then n + m is

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$$\sum_{r=0}^{30} (30-r) \cdot {}^{30}C_r$$

$hcf(2,30) = 2$

$hcf(2,15) = 1$

$$\Rightarrow \sum_{r=0}^{30} 30 \cdot {}^{30}C_r - \sum r\ {}^{30}C_r$$

$$\Rightarrow 30 \boxed{\sum_{r=0}^{30} {}^{30}C_r} - \sum \cancel{r} \frac{30}{\cancel{r}}\ {}^{29}C_{r-1}$$

$\cancel{n = 30}$

$\cancel{m = 29}$

$\cancel{m+n = 59}$

$n = 15$

$m = 30$

$m+n = 45$ ✓

$$\Rightarrow 30 \times 2^{30} - 30 \times 2^{29} \Rightarrow 30 \times 2^{29} = n \times 2^m$$

$$\Rightarrow 15 \times 2^{30} = n \times 2^m$$

If $C_r = {}^{25}C_r$ and ${}^{25}C_0 + 5 \cdot {}^{25}C_1 + 9 \cdot {}^{25}C_2 + \ldots + (101) \cdot {}^{25}C_{25} = 2^{25} \cdot k$, then k is equal to

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$$4\left[25 \cdot 2^{25-1}\right]$$

$$\sum_{r=0}^{25} (4r+1)\, {}^{25}C_r$$

$$\Rightarrow 4 \boxed{\sum r\, {}^{25}C_r} + \sum {}^{25}C_r$$

$$\Rightarrow 4 \sum \cancel{r}\, \frac{25}{\cancel{r}}\, {}^{24}C_{r-1} + \sum {}^{25}C_r$$

$$\Rightarrow 100 \boxed{\sum_{r=1}^{25} {}^{24}C_{r-1}} + \boxed{\sum_{r=0}^{25} {}^{25}C_r}$$

$$100 \times 2^{24} + 2^{25}$$

$$\Rightarrow 2^{25}(51)$$

$$k = 51$$

Multiplying Binomial Coefficients

$\boxed{{}^{m}C_r \cdot {}^{n}C_{n-r}}$

Important Shortcut (#NVStyle)

$$^{m+n}C_r = \boxed{^mC_r\,^nC_0} + {}^mC_{r-1}\,^nC_1 + {}^mC_{r-2}\,^nC_2 + \ldots\ldots\ldots\ldots + {}^mC_0\,^nC_r$$

$r+0 = r-1+1 = r-2+2 = 0+r = r$

Shortcut (kab)

- ☆ B.C. $\rightarrow$ multiplied
- ☆ subscript (sum) = constant
- ☆ 0 to r

$^{m+n}C_r$

Multiplying Binomial expansions

$C_0^2 + C_1^2 + C_2^2 + \ldots\ldots\ldots + C_n^2 = 2n!/n!n!$

$$\left({}^nC_0\right)^2 + \left({}^nC_1\right)^2 + \left({}^nC_2\right)^2 + \ldots + \left({}^nC_n\right)^2$$

$$\Rightarrow {}^nC_0 \cdot {}^nC_n + {}^nC_1 \cdot {}^nC_{n-1} + {}^nC_2 \cdot {}^nC_{n-2} + \ldots + {}^nC_n \cdot {}^nC_0$$

$$\Rightarrow {}^{2n}C_n \Rightarrow \frac{(2n)!}{n!\,n!}$$

Multiplying Binomial expansions

$C_0\,C_1 + C_1C_2 + C_2C_3 + ...+ C_{n-1}\,C_n = {}^{2n}C_{n-1}$

$\Rightarrow {}^nC_0 \cdot {}^nC_1 + {}^nC_1 \cdot {}^nC_2 + {}^nC_2 \cdot {}^nC_3 + \ldots + {}^nC_{n-1}\;{}^nC_n$

$\Rightarrow {}^nC_0\;{}^nC_{n-1} + {}^nC_1\;{}^nC_{n-2} + {}^nC_2\;{}^nC_{n-3} + \ldots + {}^nC_{n-1}\;{}^nC_0$

$\Rightarrow \boxed{{}^{2n}C_{n-1}}$

Multiplying Binomial expansions

$C_0 C_2 + C_1 C_3 + C_2 C_4 + \dots\dots\dots + C_{n-2} C_n = {}^{2n}C_{n-2}$

$\boxed{{}^nC_0 \; {}^nC_{n-2}} + {}^nC_1 \; {}^nC_{n-3} + {}^nC_2 \; {}^nC_{n-4} + \dots + {}^nC_{n-2} \; {}^nC_0$

$\Rightarrow {}^{2n}C_{n-2}$

The value of $\sum_{r=0}^{6} {}^6C_r \, {}^6C_{6-r}$

A. 1124

B. 1324

C. 1024

D. 924

JEE M 2021

$\sum_{r=0}^{6} {}^6C_r \, {}^6C_{6-r}$

$\cancel{r} + 6 - \cancel{r} = 6$

${}^6C_0 \, {}^6C_6 + {}^6C_1 \, {}^6C_5 + {}^6C_2 \, {}^6C_4 + \dots + {}^6C_6 \, {}^6C_0$

$\Rightarrow {}^{12}C_6$

$\Rightarrow {}^{12}C_6$

$\Rightarrow 924$

If $\quad 31-(k-1) = \qquad 30-(k-1)$

$$\underbrace{\sum_{k=1}^{31}\left({}^{31}C_k\right)\left({}^{31}C_{k-1}\right)} - \underbrace{\sum_{k=1}^{30}\left({}^{30}C_k\right)\left({}^{30}C_{k-1}\right)} = \frac{\alpha(60!)}{(30!)(31!)},$$

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Where $\alpha \in R$, then the value of 16α is equal to

A. 1411

B. 1320

C. 1615

D. 1855

$$\sum {}^{31}C_{\boxed{k} \; } {}^{31}C_{\boxed{32-k}} - \sum {}^{30}C_k \, {}^{30}C_{31-k}$$

$$\Rightarrow {}^{62}C_{32} - {}^{60}C_{31}$$

$$\Rightarrow \frac{62!}{32!\,30!} - \frac{60!\,(30)}{31!\,\boxed{29!\,(30)}}$$

$$\Rightarrow \frac{62\times 61\times \cancel{60!}}{32\times 31!\times 30!} - \frac{30\times 60!}{31!\,30!}$$

$$\Rightarrow \frac{\cancel{60!}}{\cancel{31!}\,\cancel{30!}}\left\{\frac{\overset{31}{\cancel{62}}\times 61}{\underset{16}{\cancel{32}}} - 30\right\} = \alpha\,\frac{\cancel{60!}}{\cancel{31!}\,\cancel{30!}}$$

$$\frac{31\times 61 - 30\times 16}{16} = \alpha$$

$$\underline{31\times 61 - 30\times 16} = 16\alpha$$

If $\sum_{k=1}^{10} K^2 \left(10_{C_K}\right)^2 = 22000L$, then L is equal to ____.

JEE M 2022

$$\sum_{k=1}^{10} \cancel{K^2} \; \frac{10}{\cancel{K}} \, {}^{9}C_{K-1} \; \frac{10}{\cancel{K}} \, {}^{9}C_{K-1}$$

$$100 \sum_{k=1}^{10} \boxed{{}^{9}C_{K-1} \; {}^{9}C_{10-K}}$$

$$(\cancel{K}-1)+(10-\cancel{K}) = 9$$

$$1\cancel{0}\cancel{0} \cdot {}^{18}C_{9} = 220\cancel{0}\cancel{0}\,L$$

$$\therefore \boxed{\frac{1}{220} \times {}^{18}C_{9} = L}$$

Let $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \ldots\ldots + 10({}^{10}C_{10})^2$

where ${}^{10}C_r$, $r \in \{1, 2, 3, \ldots, 10\}$ denote binomial coefficients

Then the value of $\frac{1}{1430}X$

JEE Adv 2018

$$\frac{X}{1430} = ?$$

$$X = \sum_{r=0}^{10} r\left({}^{10}C_r\right)^2$$

$$= \sum_{r=0}^{10} \cancel{r}\,\frac{10}{\cancel{r}}\,{}^{9}C_{r-1} \cdot {}^{10}C_r$$

$$= 10 \sum_{r=1}^{10} {}^{9}C_{r-1}\,{}^{10}C_{10-r} = 10 \cdot {}^{19}C_9$$

$$\frac{10 \cdot {}^{19}C_9}{1430}$$

$$= \frac{\cancel{10} \times 19 \times \cancel{18} \times 17 \times \overset{2}{\cancel{16}} \times \overset{3}{\cancel{15}} \times \overset{2}{\cancel{14}} \times \cancel{13} \times \cancel{12} \times \cancel{11}}{(\cancel{11} \times \cancel{13} \times \cancel{10})(\cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2})}$$

$$= 19 \times 17 \times 2$$

$$= \boxed{646}^{*}$$

For nonnegative integers s and r, let

$^{n}C_r = \binom{n}{r}$

$$\binom{s}{r} = \begin{cases} \dfrac{s!}{r!\ (s-r)!} & \text{if } r \le s, \\ 0 & \text{if } r > s. \end{cases}$$

JEE Adv 2020

For positive integers m and n, let

$\boxed{g(m,n) = 2^{m+n}}$

$\boxed{g(n,m) = 2^{n+m}}$

$$g(m,n) = \sum_{p=0}^{m+n} \frac{f(m,n,p)}{\binom{n+p}{p}}$$

$$g(m,n) = \sum_{p=0}^{m+n} \frac{\cancel{^{p+n}C_p}\ {}^{m+n}C_p}{\cancel{^{n+p}C_p}}$$

$$= \sum_{p=0}^{m+n} {}^{m+n}C_p$$

$$= \boxed{2^{m+n}}$$

where for any nonnegative integer p,

ABD

$$f(m,n,p) = \sum_{i=0}^{p} \binom{m}{i} \binom{n+i}{p} \binom{p+n}{p-i}.$$

Then which of the following statements is/are TRUE?

(A) $g(m,n) = g(n,m)$ for all positive integers m, n

(B) $g(m,n+1) = g(m+1,n)$ for all positive integers m, n

(C) $g(2m,2n) = 2\,g(m,n)$ for all positive integers m, n

(D) $g(2m,2n) = (g(m,n))^2$ for all positive integers m, n

$$f(m,n,p) = \sum_{i=0}^{p} {}^{m}C_i \; {}^{n+i}C_p \; {}^{p+n}C_{p-i}$$

$$= \sum_{i=0}^{p} \frac{m!}{i!\,(m-i)!} \times \left(\frac{\cancel{(n+i)!}}{p!\,\underbrace{(n+i-p)!}} \times \frac{(p+n)!}{(p-i)!\,\cancel{(n+i)!}} \right)$$

$$= \sum_{i=0}^{p} {}^{m}C_i \; \boxed{\frac{(p+n)!}{p!\,n!}} \; \frac{n!}{(n+i-p)!\,(p-i)!}$$

$$= \sum_{i=0}^{p} {}^{m}C_i \; {}^{p+n}C_p \; {}^{n}C_{p-i}$$

$$= {}^{p+n}C_p \sum_{i=0}^{p} \boxed{{}^{m}C_i} \; \boxed{{}^{n}C_{p-i}} \Rightarrow {}^{p+n}C_p \; {}^{m+n}C_p$$

$$g(\underline{m}, n) = 2^{m+n}$$

$$g(m, n+1)$$

$$= 2^{m+(n+1)}$$

$$g(m+1, n)$$

$$= 2^{m+1+n}$$

$$g(2\underline{m}, 2n)$$

$$= 2^{2m+2n}$$

$$= \left(2^{m+n}\right)^2$$

$$= \left(g(m,n)\right)^2$$

Use of Differentiation

Use of Differentiation

If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$, then prove that

$$ {}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \cdots + n\,{}^nC_n = n\,2^{n-1} $$

$$ n(1+x)^{n-1} = 0 + C_1(1) + C_2(2x) + C_3(3x^2) + \ldots + C_n(n x^{n-1}) $$

$x = 1$

$$ n\,2^{n-1} = 1 \cdot C_1 + 2C_2 + 3C_3 + \ldots + nC_n $$

Use of Differentiation

If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$, then prove that

$$\underline{C_0 + 2\,C_1 + 3\,C_2 + \cdots + (n+1)\,C_n} = (n+2)\,2^{n-1}$$

$$x(1+x)^n = C_0 x + C_1 x^2 + C_2 x^3 + \ldots + C_n x^{n+1}$$

$$x\,n(1+x)^{n-1} + (1+x)^n = C_0 + C_1(2x) + C_2(3x^2) + \ldots + C_n(n+1)x^n$$

Put $x=1$

$$n(2)^{n-1} + 2^{n-1} \cdot 2 = C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n$$

$$\underline{2^{n-1}(n+2)} = \text{Req}$$

Use of Integration

Use of Integration

If $\int(1+x)^n = \int C_0 + \int C_1 x + \int C_2 x^2 + \cdots + \int C_n x^n$, then prove that

$$\boxed{C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1}} = \underbrace{\frac{2^{n+1}-1}{n+1}}$$

$$\frac{(1+x)^{n+1}}{n+1} = \left[C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \ldots + \frac{C_n x^{n+1}}{n+1}\right] + \frac{1}{n+1}$$

Put $x=0$

$$\Rightarrow \boxed{\frac{1}{n+1} = 0 + c}$$

Put $x=1$

$$\underbrace{\frac{2^{n+1}}{n+1} - \frac{1}{n+1}} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1}$$

Illustration:

Prove that $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \cdots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

$x = (-1)$

Suppose

$$\det\begin{bmatrix} \boxed{\sum_{k=0}^{n} k} & \boxed{\sum_{k=0}^{n} {}^{n}C_k k^2} \\ \boxed{\sum_{k=0}^{n} {}^{n}C_k k} & \boxed{\sum_{k=0}^{n} {}^{n}C_k 3^k} \end{bmatrix} = 0$$

holds for some positive integer n. The $\underbrace{\sum_{k=0}^{n} \frac{{}^{n}C_k}{k+1}}$ equals

______

JEE Adv 2019

(1) $\sum_{k=0}^{n} k = 0+1+2+3+ \ldots + n = \frac{n(n+1)}{2}$

(2) $\sum_{k=0}^{n} \cancel{k} \cdot \frac{n}{\cancel{k}} \, {}^{n-1}C_{k-1} = n(2)^{n-1}$

(3) $\sum_{k=0}^{n} k^{\cancel{2}} \, \frac{n}{\cancel{k}} \, {}^{n-1}C_{k-1} = n\left(\sum (k-1+1)\, {}^{n-1}C_{k-1}\right)$

$$= n\left(\sum \frac{\cancel{(k-1)}}{\cancel{(k-1)}}\, n-1 \; {}^{n-2}C_{k-2} + \sum {}^{n-1}C_{k-1}\right)$$

$$= n\left((n-1) \cdot 2^{n-2} + 2^{n-1}\right)$$

$$= n \cdot 2^{n-2}(n-1+2)$$

$$= \underline{\underline{n(n+1)\, 2^{n-2}}}$$

$$\begin{vmatrix} \frac{n(n+1)}{2} & n(n+1)\, 2^{n-2} \\ n \cdot 2^{n-1} & 2^{2n} \end{vmatrix} = 0$$

$$\frac{\cancel{n}\cancel{(n+1)}}{2} \cdot \cancel{2}^{2n} - n^2 \cancel{(n+1)}\, \cancel{2}^{2n-3} = 0$$

$$\frac{1}{2} - \frac{n}{8} = 0$$

$$\therefore \boxed{n = 4}$$

(4) $$\sum_{k=0}^{n} {}^{n}C_k\, 3^k = (1+3)^n = 2^{2n}$$

To find $\sum_{k=0}^{4} \frac{{}^4C_k}{k+1} = \frac{{}^4C_0}{1} + \frac{{}^4C_1}{2} + \frac{{}^4C_2}{3} + \frac{{}^4C_3}{4} + \frac{{}^4C_4}{5}$

$$= \frac{1}{1} + \frac{4}{2} + \frac{6}{3} + \frac{4}{4} + \frac{1}{5}$$

$$= 1 + 2 + 2 + 1 + 0.2$$

$$= \boxed{6.2}$$ Ans

Double Sigma

S

$$\begin{array}{l}
{}^nC_0\,{}^nC_0 + {}^nC_0\,{}^nC_1 + {}^nC_0\,{}^nC_2 + \dots + {}^nC_0\,{}^nC_n \\
{}^nC_1\,{}^nC_0 + {}^nC_1\,{}^nC_1 + {}^nC_1\,{}^nC_2 + \dots + {}^nC_1\,{}^nC_n \\
{}^nC_2\,{}^nC_0 + {}^nC_2\,{}^nC_1 + {}^nC_2\,{}^nC_2 + \dots + {}^nC_2\,{}^nC_n \\
\vdots \qquad\qquad\qquad\qquad \ddots \\
{}^nC_n\,{}^nC_0 + {}^nC_n\,{}^nC_1 + {}^nC_n\,{}^nC_2 + \dots + {}^nC_n\,{}^nC_n
\end{array} \longrightarrow 2^{2n}$$

S

${}^{2n}C_n$

$$S + S + {}^{2n}C_n = 2^{2n}$$

$$\therefore \boxed{S = \frac{2^{2n} - {}^{2n}C_n}{2}}$$

$$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \dots + ({}^nC_n)^2 = {}^{2n}C_n$$

Illustration:

If $(1 + x)^n = C_0 + C_1x + \ldots\ldots + C_nx^n$, *then the value of* $\sum_{0 \le r}\sum_{< s \le n} C_rC_s$ *is equal to*

A. $\frac{1}{2}[2^{2n} - {}^{2n}C_n]$

B. $\frac{1}{4}[2^{2n} - {}^{2n}C_n]$

C. $\frac{1}{2}[2^{2n} + {}^{2n}C_n]$

D. $\frac{1}{2}[2^{n} - {}^{2n}C_n]$

$$\Rightarrow \boxed{\sum_{0 \le i < j \le n}\sum {}^nC_i\,{}^nC_j = S}$$

$$\boxed{S = \frac{2^{2n} - {}^{2n}C_n}{2}}$$

$$\sum_{\substack{i,j=0 \\ i \neq j}}^{n} {}^{n}C_i \; {}^{n}C_j \text{ is equal to}$$

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A. $2^{2n} - {}^{2n}C_n$

B. $2^{2n-1} - {}^{2n-1}C_{n-1}$

C. $2^{2n} - \frac{1}{2}\,{}^{2n}C_n$

D. $2^{n-1} + {}^{2n-1}C_n$

$$\sum_{\substack{i,j=0 \\ i \neq j}}^{n} {}^{n}C_i \, {}^{n}C_j = \boxed{2^{2n} - {}^{2n}C_n}$$

Binomial for any index

$(x+y)^{5}$

$(x+y)^{10}$

$(x+y)^{-2}$ — ?

$(x+y)^{1/3}$ — ?

Binomial Theorem for any index

When n is a negative integer or a fraction then the expansion of a binomial is possible only when

(i) Its first term is 1, and

(ii) Its second term is numerically less than 1.

$^nC_r = {}^{-2}C_0$

Thus when $n \notin N$ and $|x| < 1$, then it states

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots\dots + \frac{n(n-1)(n-r+1)}{r!}x^r + \dots\dots \infty$$

Kab ?

① First term = 1

② $|x| < 1$

$$\frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{3!}$$

Some Important Expansions

$(1 + x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \dots \infty$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots \infty$

$(1+x)^{n} = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$

Some Important Expansions

$(1 - x)^{-1}$

$x \to (-x) \quad n \to (-1)$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots \infty$$

$$(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \dots \infty$$

$$= 1 + x + x^2 + \dots \infty$$

Some Important Expansions

$$(1 \pm x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots..$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots..$$

$\oplus$ $\ominus$

Illustration:

If $|x| < 2/3$ then the fourth term in the expansion of $\left(1+\boxed{\frac{3}{2}x}\right)^{1/2}$ is–

A. $\frac{27}{128}x^3$

B. $-\frac{27}{128}x^3$

C. $\frac{81}{256}x^3$

D. $-\frac{81}{256}x^3$

$$\frac{n(n-1)(n-2)}{3!}x^3$$

$$\Rightarrow \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{6}\left(\frac{3x}{2}\right)^3$$

$$\Rightarrow \frac{3}{\cancel{48}\ 16} \cdot \frac{\cancel{3}\times3\times3}{2\times2\times2}x^3 = \frac{27}{128}x^3$$

General Term

Coefficient of x^r *in* $(1-x)^{-n}$ *is* $^{n+r-1}C_r$

$(1-x)^{-n}$

Coeffi of x^r : $\boxed{^{n+r-1}C_r}$

$x^3 : (1-x)^{-10}$

$x^3 : {}^{10+3-1}C_3 = {}^{12}C_3$

$x^4 : (1-x)^{-7}$

$x^4 : {}^{7+4-1}C_4$

$^{10}C_4$

Example:

The coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^6$ in powers of x, is

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$$x^4 : (1+x+x^2+x^3)^6$$

$$x^4 : \left(\frac{1(1-x^4)}{1-x}\right)^6$$

$$x^4 : \underbrace{(1-x^4)^6}\ \underbrace{(1-x)^{-6}}_{x^4}$$

M-2

$$(1+x)^6(1+x^2)^6$$

I	II	
x^4	x^0	$\Rightarrow {}^6C_4\,{}^6C_0$
x^2	x^2	$\Rightarrow {}^6C_2 \cdot {}^6C_1$
x^0	x^4	$\Rightarrow {}^6C_0 \cdot {}^6C_2$

$= 120$

$\underline{\underline{x^4}} : \left({}^6C_0 - {}^6C_1\, x^4 + \dots\dots\right)\left(1 + 6x + \dots\dots\right)$

$x^4 : {}^6C_0 \times {}^{6+4-1}C_4 - {}^6C_1 \times 1$

$: 1 \times \frac{\overset{3}{\cancel{9}} \times \cancel{8} \times 7 \times 6}{\cancel{4} \times \cancel{3} \times \cancel{2}} - 6$

$: 42 \times 3 - 6$

$: 120$