# Hyperbola

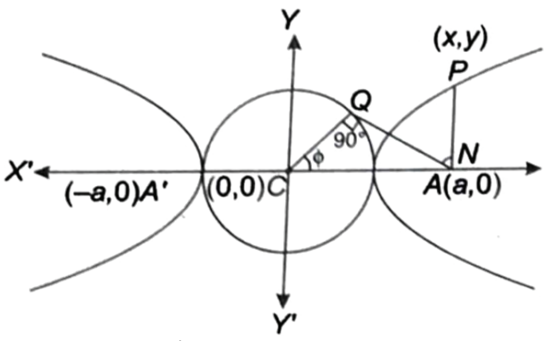
## 01 Definition of Hyperbola

A hyperbola is the particular case of the conic
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
When, $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$
i.e., $\Delta \neq 0$ and $h^2 > ab$

## 03 Auxiliary Circle of Hyperbola

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola with centre C and transverse axis $A'A$. Therefore, circle drawn with centre C and segment $A'A$ as a diameter is called auxiliary circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

**Equation of the auxiliary circle is $x^2 + y^2 = a^2$.**




**Let** $\angle QCN = \phi$

Here, $P$ and $Q$ are the corresponding points on the hyperbola and the auxiliary circle $(0 \le \phi < 2\pi)$.

**Note :** Here $\phi$ is called eccentric angle of point $P$ .

## 04 Parametric equations of hyperbola

The equations $x = a \sec \phi$ and $y = b \tan \phi$ are known as the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

This $(a \sec \phi, \ b \tan \phi)$ lies on the hyperbola for all values of $\phi$.

Position of points Q on auxiliary circle and the corresponding point P which describes the hyperbola and $0 \le \phi < 2\pi$

| $\phi$ varies from | Q$(a \cos \phi, b \sin \phi)$ | P$(a \sec \phi, \ b \tan \phi)$ |
|---|---|---|
| $0 \ to \ \frac{\pi}{2}$ | I | I |
| $\frac{\pi}{2} to \ \pi$ | II | III |
| $\pi \ to \frac{3\pi}{2}$ | III | II |
| $\frac{3\pi}{2} \ to \, 2\pi$ | IV | IV |

## 02 Hyperbola

| Hyperbola Fundamentals | Hyperbola | CONJUGATE HYPERBOLA |
|---|---|---|
| | $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ | $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ |
| Centre | $(0,0)$ | $(0,0)$ |
| Length of transverse axis | $2a$ | $2b$ |
| Length of conjugate axis | $2b$ | $2a$ |
| Foci | $(\pm ae, 0)$ | $(0, \pm be)$ |
| Equation of directrices | $x = \pm a/e$ | $y = \pm b/e$ |
| Eccentricity | $e = \sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$ | $e = \sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$ |
| Length of latus rectum | $\frac{2b^2}{a}$ | $\frac{2a^2}{b}$ |
| Parametric co-ordinates | $(a \sec \phi, \ b \tan \phi)$, $0 \le \phi < 2\pi$ | $(b \sec \phi, \ a \tan \phi)$, $0 \le \phi < 2\pi$ |
| Focal radii | $SP = ex_1 - a$ and $S'P = ex_1 + a$ | $SP = ey_1 - b$ and $S'P = ey_1 + b$ |
| Difference of focal radii $(S'P - SP)$ | $2a$ | $2b$ |
| Tangents at the vertices | $x = -a, x = a$ | $y = -b, \ y = b$ |
| Equation of the transverse axis | $y = 0$ | $x = 0$ |
| Equation of the conjugate axis | $x = 0$ | $y = 0$ |

- If e and e' are the eccentricities of a hyperbola and its conjugate, then $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.
- The foci of a hyperbola and its conjugate are concyclic.

## 05 Position of a Point with Respect to a Hyperbola

Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then , $P(x_1, y_1)$ will lie inside, on or outside the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is positive, zero or negative.

## 06 Equation of the Chord joining Two Points on the Hyperbola

The equation of the chord joining the points $P(a\sec\phi_1, b\tan\phi_1)$ and $(a\sec\phi_2, b\tan\phi_2)$

$$y - b\tan\phi_1 = \frac{b\tan\phi_2 - b\tan\phi_1}{a\sec\phi_2 - a\sec\phi_1}(x - a\sec\phi_1) \quad \frac{x}{a}\cos\left(\frac{\phi_1-\phi_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right) = \cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

### Note

- If the chord joining two points $(a\sec\phi_1, b\tan\phi_1)$ and $(a\sec\phi_2, b\tan\phi_2)$ passes through the focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{1-e}{1+e}$ or $\frac{1+e}{1-e}$.

## 07 Intersection of a Line and a Hyperbola

The straight line $y = mx + c$ will cut the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in two points may be real, coincident or imaginary according as $c^2 >, =, < a^2m^2 - b^2$.

## 08 Condition for tangency

If straight line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $c^2 = a^2m^2 - b^2$.

## 9 Equations of Tangent in Different Forms

**01** **Point form :** The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(x_1, y_1)$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

**02** **Parametric form :** The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(a\sec\phi_1, b\tan\phi_1)$ is $\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1$

**Note**

Point of intersection of tangents drawn at point on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\left(\frac{a\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right).$$

**03** **Slope form :** The equations of tangents of slope $m$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y = mx \pm \sqrt{a^2m^2 - b^2}$ and the co-ordinates of points of contacts are $\left(-\frac{a^2m}{c}, -\frac{b^2}{c}\right)$ where $c^2 = a^2m^2 - b^2$.

Clearly for the existence of tangent with slope $m$ to the hyperbola $|m| > \frac{b}{a}(\text{where } a, b > 0)$.

**Note**

- If the straight line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $a^2l^2 - b^2m^2 = n^2$.
- If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.
- Two tangents can be drawn from an outside point to a hyperbola.

## 10 Equation of Pair of Tangents

If $P(x_1, y_1)$ be any point outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then a pair of tangents $PQ, PR$ can be drawn to it from $P$

The equation of pair of tangents $PQ$ and $PR$ is $SS_1 = T^2$, where,

$$S = \frac{x^2}{a^2}-\frac{y^2}{b^2} - 1,\ S_1 = \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1,\ T = \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1.$$

**Director circle :** The director circle is the locus of points from which perpendicular tangents are drawn to the given hyperbola. The equation of the directo r circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2 + y^2 = a^2 - b^2$.

# 11 Equations of Normal in Different Forms

**01**

**Point form :** The equation of normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(x_1, y_1)$ is $\frac{a^2x}{x_1}+\frac{b^2}{y_1}=a^2+b^2.$

**02**

**Parametric form :** The equation of normal at $(a\ sec\theta, b\ tan\theta)$, to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $ax\ cos\theta+by cot\theta=a^2+b^2.$

**03**

**Slope form :** The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of the slope $m$ of $y=mx\mp\frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$

**04**

**Condition of normality :** If $y=mx+c$ is the normal of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $c=\mp\frac{m(a^2+b^2)}{\sqrt{a^2-m^2b^2}}$ *or* $c=\frac{m^2(a^2+b^2)^2}{(a^2-m^2b^2)}$, which is condition of normality.

# 12 Important Tips

**01**

In general, four normals can be drawn to a hyperbola from any point and if $\alpha,\beta,\gamma,\delta$ be the eccentric angles of these four co-normal points, then $\alpha+\beta+\gamma+\delta$ is an odd multiple of $\pi$.

**02**

if $\alpha,\beta,\gamma$ are the eccentric angles of three points on the hyperbola. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$ at which the normals are concurrent, then

$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0.$

**03**

If the normal at $P$ meets the transverse axis in G, then $SG=e.SP.$

Also the tangent and normal bisect the angle between the focal distances of $P$

**04**

If the normal at $P$ meets the transverse axis in $G$ and conjugate axis at $g$, then

$PG:Pg=b^2:a^2.$

# 13 Equation of Chord of Contact of Tangents drawn from a Point to a Hyperbola

Let $PQ$ and $PR$ be tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ drawn from any external point $P(x_1, y_1)$.

Then equation of chord of contact $QR$ is or $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

or $T=0\ (At\ x_1, y_1)$

# 14 Equation of the Chord of the Hyperbola whose Mid-point $(x_1, y_1)$ is given

Equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$ bisected at the given point $(x_1, y_1)$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1, \quad \text{i.e., } T=S_1.$$

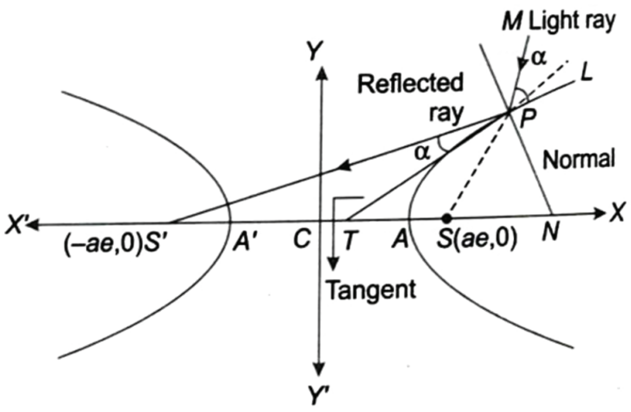
**Note**

The length of chord cut off by hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ from the line

$$y=mx+c\ is\ \frac{2ab\sqrt{\left[c^2-(a^2m^2-b^2)\right](1+m)^2}}{(b^2-a^2m^2)}.$$

# 15 Reflection Property of the Hyperbola

If an incoming light ray passing through one focus $(S)$ strike convex side of the hyperbola, then it will get reflected towards other focus $(S')$. $\angle TPS'=\angle LPM=\alpha$




**Note**

Hyperbola and ellipse are called orthogonal curves to each other iff they are confocal (i.e., they have the same foci).

# 16 Asymptotes of a Hyperbola

The equations of two asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are

$$y=\pm\frac{b}{a}x\ or\ \frac{x}{a}\pm\frac{y}{b}=0.$$

**Important Tips**

The product of length of perpendiculars drawn from any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to the asymptotes is $\frac{a^2b^2}{a^2+b^2}$.

- The tangent at any point $P$ on hyperbola if meet its asymptotes at $Q$ and $R$, then :

(i) the midpoint of QR is always P, (ii)area of triangle $QCR$ is always *"ab"* where $C$ is the of hyperbola and 2a= lengthof transverse axis, 2b= length of conjugate axis of hyperbola.

## 17 Rectangular or Equilateral Hyperbola

**(i) Definition :** A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola. The eccentricity of rectangular hyperbola is always $\sqrt{2}$. The general equation of second degree represents a rectangular hyperbola if $\Delta \neq 0, h^2 > ab$ and coefficient of $x^2$+ coefficient of $y^2 = 0$.

## 18 Parametric coordinates of a point ton he hyperbola $xy = c^2$

If $t$ is non-zero variable, the coordinates of any point on the rectangular hyperbola $xy = c^2$ can be written as $\left(ct, \frac{c}{t}\right)$. The point $\left(ct, \frac{c}{t}\right)$ on the hyperbola $xy = c^2$ is generally referred as the point '$t$'.

For rectangular hyperbola the coordinates of foci are $\left(\pm a\sqrt{2}, 0\right)$ and directrices are $x = \pm a\sqrt{2}$.

For rectangular hyperbola $xy = c^2$, the coordinates of foci are $\left(\pm c\sqrt{2}, \pm c\sqrt{2}\right)$ and directrices are $x + y = \pm c\sqrt{2}$.

## 19 Equation of the chord joining points $t_1$ and $t_2$

The equation of the chord joining two points $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola $xy = c^2$ is $y - \frac{c}{t_1} = \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1}\left(x - ct_1\right)$

$$\Rightarrow x + yt_1t_2 = c\left(t_1 + t_2\right).$$

## 20 Equation of tangent in different forms

**(i) Point form :** The equation of tangent at $(x_1, y_1)$ to the hyperbola $xy = c^2$ is $xy_1 + yx_1 = 2c^2$ or $\frac{x}{x_1} + \frac{y}{y_1} = 2.$

**(ii) Parametric form :** The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $\frac{x}{t} + yt = 2c$ On replacing $x_1$ by ct and $y_1$ by $\frac{c}{t}$ on the equation of the tangent at $(x_1, y_1)$, i.e., $xy_1 + yx_1 = 2c^2$, we get $\frac{x}{t} + yt = 2c.$

**Note**

Point of intersection of tangents at '$t_1$' and '$t_2$' is $\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$.

## 21 Equation of the chord joining points $t_1$ and $t_2$

**(i) Point form :** The equation of the normal at $(x_1, y_1)$ to the hyperbola $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$.

**(ii) Parametric form :** The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$.

**Note**

- The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is a fourth degree in $t$. So, in general,
- If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again in '$t$' then $t' = -\frac{1}{t^3}$.
- Point of intersection of normals at '$t_1$' and '$t_2$' is $\left(\frac{c\left\{t_1t_2\left(t_1^2 + t_1t_2 + t_2^2\right) - 1\right\}}{t_1t_2\left(t_1 + t_2\right)}, \frac{c\left\{t_1^3t_2^3\left(t_1^2 + t_1t_2 + t_2^2\right)\right\}}{t_1t_2\left(t_1 + t_2\right)}\right)$.
- A triangle has its vertices on a rectangular hyperbola; then the orthocentre of the triangle also lies on the same hyperbola
- All conics passing through the intersection of two rectangular hyperbolas are themselves rectangular hyperbolas.

## 22 Intersection of a Circle and a Rectangular Hyperbola

If a circle $x^2 + y^2 + 2gx + 2fy + k = 0$ cuts a rectangular hyperbola $xy = c^2$ in A, B, C and D and the parameters of these four points $t_1$, $t_2$, $t_3$ and $t_4$ respectively, then :

01
(i) $\Sigma t_1 = -\frac{2g}{c}$
(ii) $\Sigma t_1t_2 = \frac{k}{c^2}$
(iii) $\Sigma t_1t_2t_3 = \frac{-2f}{c}$
(iv) $t_1t_2t_3t_4 = 1$
(v) $\Sigma \frac{1}{t_1} = -\frac{2f}{c}$