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MATHS - I

Assignment - 2

Unit - 2

Q1. If $u = \log(x^2 + xy + y^2)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$.

$u = \log(x^2 + xy + y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+xy+y^2}\cdot(2x+y) \quad ; \quad \frac{\partial u}{\partial y} = \frac{1}{x^2+xy+y^2}\cdot(2y+x)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \Rightarrow \frac{x(2x+y)}{x^2+xy+y^2} + \frac{y(2y+x)}{x^2+xy+y^2}$$

$$\Rightarrow \frac{2x^2+xy+2y^2+xy}{x^2+xy+y^2} = 2.$$ Hence, Proved

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Q2 verify Euler's theorem for following functions :-

(i) $u = \log\left(\frac{x^2+y^2}{xy}\right)$

$$u(x,y) = \log\left(\frac{x^2+y^2}{xy}\right) \quad ; \quad u(tx,ty) = \log\left(\frac{t^2x^2+t^2y^2}{t^2xy}\right)$$

$$= t^0 \log\left(\frac{x^2+y^2}{xy}\right)$$

or, $u(tx,ty) = t^0 u(x,y)$.

so, $u(x,y)$ is a homogeneous func of degree '0'.

then, by Euler's theorem :-

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{(where, } n \text{ is order / degree)}$$

$$x\left(\frac{\frac{2x\cdot yx - y(x^2+y^2)}{x^2y^2}}{x^2+y^2/xy}\right) + y\left(\frac{\frac{2y\cdot xy - x(x^2+y^2)}{x^2y^2}}{x^2+y^2/xy}\right) = 0.\log\left(\frac{x^2+y^2}{xy}\right)$$

$$\frac{1}{(x^2+y^2)(xy)}\left[\cancel{2x^3y} - \cancel{x^3y} - \cancel{xy^3} + \cancel{2y^3x} - \cancel{yx^3} + - \cancel{xy^3}\right] = 0$$

$\boxed{0 = 0}$ Hence, Euler's theorem verified.

(ii) $u = \frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$

$$u(x,y) = \frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}} \quad ; \quad u(tx,ty) = \frac{t^{1/3}(x^{1/3}+y^{1/3})}{t^{1/2}(x^{1/2}+y^{1/2})}$$

$$= t^{-1/6}\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)$$

or, $u(tx,ty) = t^{-1/6}u(x,y)$.

so, u is homogeneous func of order '$-1/6$' in x and y.

then, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n(u)$ — by Euler's theorem.

$$\frac{\partial u}{\partial x} = \frac{\frac{1}{3}x^{-2/3}(x^{1/2}+y^{1/2}) - \frac{1}{2}x^{-1/2}(x^{1/3}+y^{1/3})}{(x^{1/2}+y^{1/2})^2}$$

$$x\frac{\partial u}{\partial x} = \frac{\frac{1}{3}x^{1/3}(x^{1/2}+y^{1/2}) - \frac{1}{2}x^{1/2}(x^{1/3}+y^{1/3})}{(x^{1/2}+y^{1/2})^2} \quad \text{— (I)}$$

$$\frac{\partial u}{\partial y} = \frac{\frac{1}{3}y^{-2/3}(x^{1/2}+y^{1/2}) - \frac{1}{2}y^{-1/2}(x^{1/3}+y^{1/3})}{(x^{1/2}+y^{1/2})^2}$$

$$y\frac{\partial u}{\partial y} = \frac{\frac{1}{3}y^{1/3}(x^{1/2}+y^{1/2}) - \frac{1}{2}y^{1/2}(x^{1/3}+y^{1/3})}{(x^{1/2}+y^{1/2})^2} \quad \text{— (II)}$$

add (I) and (II) :- $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} =$

$$\frac{\frac{1}{3}x^{5/6} + \frac{1}{3}y^{1/2}x^{1/3} - \frac{1}{2}x^{5/6} - \frac{1}{2}x^{1/2}y^{1/3} + \frac{1}{3}y^{5/6} + \frac{1}{3}y^{1/3}x^{1/2} - \frac{1}{2}y^{1/2}x^{1/3} - \frac{1}{2}y^{5/6}}{(x^{1/2}+y^{1/2})^2}$$

$$\frac{\left(\frac{1}{3}-\frac{1}{2}\right)\cancel{(x^{1/2}+y^{1/2})}(x^{1/3}+y^{1/3})}{(x^{1/2}+y^{1/2})^{\cancel{2}}} = \frac{-1}{6}\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)$$

or, $\boxed{x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} \Rightarrow -\dfrac{1}{6}u(x,y).}$ Hence, verified.

Q3 If $u = f(r)$, where $r^2 = x^2+y^2$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r}f'(r)$.

To prove :- $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r}f'(r)$.

Proof:- $u = f(r)$; $r^2 = x^2+y^2$.

$$\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x} = \cancel{f'(r)\cdot\frac{x}{r}}\ f'(r)\cdot\frac{x}{r}$$

again, differentiating it partially wrt x :-

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(f'(r)\cdot\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\cdot\frac{x}{\sqrt{x^2+y^2}}\cdot\frac{x}{r} + f'(r)\cdot\frac{\sqrt{x^2+y^2} - \frac{2x^2}{2\sqrt{x^2+y^2}}}{(x^2+y^2)}$$

$$= f''(r)\cdot\frac{x^2}{r^2} + f'(r)\cdot\frac{y^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\cdot\frac{x^2}{r^2} + f'(r)\cdot\frac{y^2}{r^3} \quad \text{--- (I)}$$

similarly, $\frac{\partial u}{\partial y} = f'(r)\cdot\frac{y}{r}$ or $f'(r)\cdot\frac{y}{\sqrt{x^2+y^2}}$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(f'(r)\cdot\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\cdot\frac{y}{\sqrt{x^2+y^2}}\cdot\frac{y}{r} + f'(r)\cdot\frac{\sqrt{x^2+y^2} - \frac{2y^2}{2\sqrt{x^2+y^2}}}{(x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\cdot\frac{y^2}{r^2} + f'(r)\frac{x^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\cdot\frac{y^2}{r^2} + f'(r)\cdot\frac{x^2}{r^3} \quad \text{--- (II)}$$

add (I) and (II) :-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\cdot\frac{x^2+y^2}{r^2} + f'(r)\cdot\frac{x^2+y^2}{r^3}$$

$$= f''(r)\cdot 1 + f'(r)\cdot\frac{r^2}{r^3}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r).}$$ Hence, proved.

Q4. If $u = \sec^{-1}\left(\frac{x^3-y^3}{x+y}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$.

Also, Evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

$u = \sec^{-1}\left(\frac{x^3 - y^3}{x+y}\right)$ and To prove :- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$.

Proof :- $u(x, y) = \sec^{-1}\left(\frac{x^3 - y^3}{x+y}\right)$

or $\sec u = \frac{x^3 - y^3}{x+y}$

so, from above eq^n we can say that $\frac{x^3 - y^3}{x+y}$ is homogeneous

func of order '2'.

then, Let $f(u) = \sec u$

By property :- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)}$ (where, n is order)

so, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2 \cdot \frac{\cancel{\sec u}}{\cancel{\sec u} \cdot \tan u}$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u}$$ Hence, proved

then, again by property :-

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \phi(u)[\phi'(u) - 1].$$

where, $\phi(u) = \frac{n f(u)}{f'(u)}$

$\therefore \phi(u) = 2\cot u$.

so, $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = 2\cot u[-2\operatorname{cosec}^2 u - 1]$.

$\cancel{= -4\cot u \cdot \operatorname{cosec}^2 u}$.

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = -2\cot u[2\operatorname{cosec}^2 u + 1].}$$

Ans

Q5 If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$, prove that $6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0$.

$u = f(2x-3y,\ 3y-4z,\ 4z-2x)$.

or, $u = f(a, b, c)$

where, $a = 2x-3y$, $b = 3y-4z$, $c = 4z-2x$.

then,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial x} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\cdot 2 + \frac{\partial u}{\partial b}\cdot(0) + \frac{\partial u}{\partial c}\cdot(-2) \text{ (multiply it by 6)}.$$

$$6\frac{\partial u}{\partial x} = 12\cdot\frac{\partial u}{\partial a} - 12\cdot\frac{\partial u}{\partial c} \quad \text{--- (I)}$$

similarly, $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial y} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\cdot(-3) + \frac{\partial u}{\partial b}\cdot(3) + \frac{\partial u}{\partial c}\cdot(0) \text{ (Multiply it by 4)}.$$

$$4\frac{\partial u}{\partial y} = -12\frac{\partial u}{\partial a} + 12\frac{\partial u}{\partial b} \quad \text{--- (II)}$$

and, $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial z} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial z} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a}\cdot(0) + \frac{\partial u}{\partial b}\cdot(-4) + \frac{\partial u}{\partial c}\cdot(4) \text{ (Multiply it by 3)}$$

$$3\frac{\partial u}{\partial z} = -12\frac{\partial u}{\partial b} + 12\cdot\frac{\partial u}{\partial c} \quad \text{--- (III)}$$

Add (I), (II) and (III) eqn :-

$$6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = \cancel{12\cdot\frac{\partial u}{\partial a}} - \cancel{12\frac{\partial u}{\partial c}} + \cancel{12\frac{\partial u}{\partial b}} - \cancel{12\frac{\partial u}{\partial a}} - \cancel{12\frac{\partial u}{\partial b}} + \cancel{12\cdot\frac{\partial u}{\partial c}}$$

$$\boxed{6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0}$$ Hence, proved

Q6. If $w = f(x, y)$, where $x = e^u \cos v$, $y = e^u \sin v$, show that -

$$y\frac{\partial w}{\partial u} + x\frac{\partial w}{\partial v} = e^{2u}\frac{\partial w}{\partial y}$$

$w = f(x, y)$, where $x = e^u \cos v$, $y = e^u \sin v$.

By differentiating w partially wrt x and y respectively.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot e^u \cos v + \frac{\partial w}{\partial y}\cdot e^u \sin v$$

(multiply it by y).

$$y\frac{\partial w}{\partial u} = y\frac{\partial w}{\partial x}\cdot e^u \cos v + y\frac{\partial w}{\partial y}\cdot e^u \sin v \quad \text{--- (i)}$$

similarly, $$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot(-e^u \sin v) + \frac{\partial w}{\partial y}\cdot e^u \cos v.$$

(multiply it by x).

$$x\frac{\partial w}{\partial v} = x\frac{\partial w}{\partial x}(-e^u \sin v) + x\cdot\frac{\partial w}{\partial y} e^u \cos v \quad \text{--- (ii)}$$

add (i) and (ii) :-

$$y\frac{\partial w}{\partial u} + x\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\left(ye^u \cos v - xe^u \sin v\right) + \frac{\partial w}{\partial y}\left(e^u \sin v\, y + e^u \cos v . x\right)$$

$$= \frac{\partial w}{\partial x}\left(\cancel{e^{2u} \sin v \cos v} - \cancel{e^{2u} \sin v . \cos v}\right)$$

$$+ \frac{\partial w}{\partial y}\left(e^{2u} \sin^2 v + e^{2u} \cos^2 v\right).$$

$$= \frac{\partial w}{\partial y} e^{2u}\left(\sin^2 v + \cos^2 v\right)$$

$$\boxed{y\frac{\partial w}{\partial u} + x\frac{\partial w}{\partial v} = e^{2u}\frac{\partial w}{\partial y}}$$ Hence, proved

Q7. If $w = \sqrt{x^2+y^2+z^2}$ and $x = u\cos v$, $y = \sin v . u$, $z = uv$, then prove that $u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1+v^2}}$.

$w = \sqrt{x^2+y^2+z^2}$; $x = u\cos v$, $y = u\sin v$, $z = uv$.

then,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}.\cos v + \frac{\partial w}{\partial y}.\sin v + \frac{\partial w}{\partial z}.v.$$

(Multiply it by u)

$$u\frac{\partial w}{\partial u} = u\sin v\frac{\partial w}{\partial y} + uv\frac{\partial w}{\partial z} + \frac{\partial w}{\partial x}u\cos v \quad \text{— (I)}$$

similarly, $$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial v}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}u(-\sin v) + \frac{\partial w}{\partial y}(u\cos v) + \frac{\partial w}{\partial z}(u)$$

(multiply it by v)

$$v\frac{\partial w}{\partial v} = -uv\sin v\frac{\partial w}{\partial x} + uv\cos v\frac{\partial w}{\partial y} + uv\frac{\partial w}{\partial z} \quad \text{— (II)}$$

subtract (II) from (I)

$$u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = u\sin v\frac{\partial w}{\partial y} + \cancel{uv\frac{\partial w}{\partial z}} + uv\sin v\frac{\partial w}{\partial x} - uv\cos v\frac{\partial w}{\partial y} + u\cos v\frac{\partial w}{\partial x} - \cancel{uv\frac{\partial w}{\partial z}}$$

$$= \frac{\partial w}{\partial x}(u\cos v + uv\sin v) + \frac{\partial w}{\partial y}(u\sin v - uv\cos v)$$

and, $$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \; ; \; \frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

so, $$\Rightarrow \frac{xu\cos v + xuv\sin v + yu\sin v - yuv\cos v}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \frac{u^2\cos^2 v + u^2 v\sin v\cos v + u^2\sin^2 v - u^2 v\sin v\cos v}{\sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2v^2}}$$

$$\boxed{u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1+v^2}}}$$ Hence, proved

Q8 Show that $u = x+y+z$, $v = x^2+y^2+z^2$; $w = x^3+y^3+z^3-3xyz$, prove that u, v and w are the not independent and hence find the relation between them.

$u = x+y+z$, $v = x^2+y^2+z^2$, $w = x^3+y^3+z^3-3xyz$.

By Jacobian's theorem:-

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= 1(6yz^2 - 6xy^2 - 6y^2z + 6xz^2) - 1(6xz^2 - 6x^2y - 6x^2z + 6yz^2) + 1(6xy^2 - 6x^2z - 6yx^2 + 6y^2z)$$

$$= 0.$$

Hence, u, v and w are dependent.

So, Relation between u, v and w are :-

$$x^3+y^3+z^3-3xyz \Rightarrow (x+y+z)\left[x^2+y^2+z^2 - \left(\frac{(x+y+z)^2-(x^2+y^2+z^2)}{2}\right)\right]$$

$$w \Rightarrow u\left[v - \frac{u^2-v}{2}\right] = \frac{u[3v-u^2]}{2}$$

$$\boxed{2w = u(3v-u^2)}$$

Q9. If $x = v^2+w^2$, $y = w^2+u^2$, $z = u^2+v^2$, then show that

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1.$$

Sol. $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix}$$

$= -2v(0-4uw) + 2w(4uv-0)$

$= 8uvw + 8uvw = 16uvw.$

since, $u^2 = \frac{y+z-x}{2}$, $v^2 = \frac{x+z-y}{2}$, $w^2 = \frac{x+y-z}{2}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -1/4u & 1/4u & 1/4u \\ 1/4v & -1/4v & 1/4v \\ 1/4w & 1/4w & -1/4w \end{vmatrix}$$

$$= \frac{1}{4u} \cdot \frac{1}{4v} \cdot \frac{1}{4w} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \Rightarrow \frac{1}{64uvw} \times 4 \Rightarrow \frac{1}{16uvw}.$$

so, $\frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = 16uvw \cdot \frac{1}{16uvw}$

$$\boxed{\frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1.}$$ Hence, proved

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Q10. If $u = x^2-y^2$, $v = 2xy$ and $x = r\cos\theta$ and $y = r\sin\theta$, then show that - $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$.

By the property of Jacobians :-

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

so, $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2)$$

and, $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

and we know :- $x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$(x^2+y^2 = r^2)$

Hence, $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = 4r^2 \cdot r$

or $\boxed{\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = 4r^3}$

Q11. If $u^3+v^3 = x+y$ and $u^2+v^2 = x^3+y^3$, then show that -

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{y^2-x^2}{2uv(u-v)}.$$

On partial differentiation :- $3u^2\frac{\partial u}{\partial x} + 3v^2\frac{\partial v}{\partial x} = 1.$

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 3x^2.$$

writing in form of matrix :- $\begin{bmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{bmatrix}\begin{bmatrix} \partial u/\partial x \\ \partial v/\partial x \end{bmatrix} = \begin{bmatrix} 1 \\ 3x^2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \partial u/\partial x \\ \partial v/\partial x \end{bmatrix} = \frac{1}{6uv(u-v)}\begin{bmatrix} 2v & -2u \\ -3v^2 & 3u^2 \end{bmatrix}\begin{bmatrix} 1 \\ 3x^2 \end{bmatrix}$$

$$\begin{bmatrix} \partial u/\partial x \\ \partial v/\partial x \end{bmatrix} = \frac{1}{6uv(u-v)}\begin{bmatrix} 2v-6ux^2 \\ 9u^2x^2-3v^2 \end{bmatrix}$$

So, we get :- $\frac{\partial u}{\partial x} = \frac{2v-6ux^2}{6uv(u-v)}$ and $\frac{\partial v}{\partial x} = \frac{9u^2x^2-3v^2}{6uv(u-v)}$

And, due to symmetry :- $\frac{\partial u}{\partial y} = \frac{2v-6uy^2}{6uv(u-v)}$

$$\frac{\partial v}{\partial y} = \frac{9u^2y^2-3v^2}{6uv(u-v)}$$

So, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{2v-6ux^2}{6uv(u-v)} & \frac{2v-6uy^2}{6uv(u-v)} \\ \frac{9u^2x^2-3v^2}{6uv(u-v)} & \frac{9u^2y^2-3v^2}{6uv(u-v)} \end{vmatrix}$

$$= \frac{1}{2}\,\frac{y^2-x^2}{uv(u-v)}$$ Hence, Proved

Q12. Expand $u(x,y) = x^2y + 3y - 2$ in powers of $x-1$ and $y+2$?

Expansion of $f(x,y)$ in powers of $(x-a)$ and $(y-b)$ is given by-

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b)$$

$$+ 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \frac{1}{3!}[(x-a)^3 f_{xxx}(a,b)$$

$$+ 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b) + \ldots \quad \text{——} (1)$$

Here, $f(x,y) = u(x,y) = x^2y + 3y - 2$. and $a = 1$, $b = -2$

$f(1,-2) \Rightarrow -10$.

$f_x = 2xy$ $\qquad f_y = x^2 + 3$.

$f_{xx} = 2y$ $\qquad f_{yy} = 0$.

$f_{xxx} = 0$. $\qquad f_{xyy} = 0$.

$f_{xy} = 2x$ $\qquad f_{yyy} = 0$.

$f_{xxy} = 2$.

and all higher order partial derivatives vanish.

$\therefore$ from (1) we have,

$$x^2y + 3y - 2 = f(x,y) = u(x,y)$$

$$= -10 + [(x-1)(-4) + (y+2)4] + \frac{1}{2!}[(x-1)^2(-4) + 2(x-1).$$

$$.(y+2)2 + (y+2)^2(0)] + \frac{1}{3!}[(x-1)^3(0) + 3(x-1)^2(y+2)2$$

$$+ 3(x-1)(y+2)^2(0) + (y-b)^3(0)].$$

so,

$$\boxed{f(x,y) \Rightarrow -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)}$$

Q13. Expand $u(x,y) = x^4 + x^2y^2 - y^4$ about the point $(1,1)$ upto terms of second degree.

In general, for upto second degree :-

$$u(x,y) = u(a,b) + [(x-a)u_x(a,b) + (y-b)u_y(a,b)] + \frac{1}{2!}[(x-a)^2 u_{xx}(a,b) + (y-b)^2 u_{yy}(a,b)$$

$$+ 2(x-a)(y-b)u_{xy}(a,b)].$$

so, $u(a,b) = u(1,1) = 1$

$u_x = 4x^3 + 2xy^2$; $u_x(1,1) = 6$

$u_y = 2yx^2 - 4y^3$; $u_y(1,1) = -2$

$u_{xx} = 12x^2 + 2y^2$; $u_{xx}(1,1) = 14$

$u_{yy} = 2x^2 - 12y^2$; $u_{yy}(1,1) = -10$

$u_{xy} = 4xy$; $u_{xy}(1,1) = 4$

so,

$$u(x,y) = x^4 + x^2y^2 - y^4$$
$$= 1 + [(x-1)6 + (y-1)(-2)] + \frac{1}{2!}[(x-1)^2(14) + (y-1)^2(-10)$$
$$+ 2.(x-1)(y-1).4].$$

$$\boxed{u(x,y) = 1 + 6(x-1) - 2(y-1) + 7(x-1)^2 - 5(y-1)^2 + 4(x-1)(y-1)}$$

Q14 The diameter and height of a right circular cylinder are found by measurement to be 8.0 cm and 12.5 cm respectively with possible errors of 0.05 in each measurent. Find the maximum possible approximate error in the computed volume?

Let the diameter and height of right circular cylinder denoted by- D and H respectively.

Then, radius $(r) = \frac{D}{2}$.

Volume of right circular cylinder 'V' $= \pi r^2 h$

$$V = \frac{\pi D^2 . H}{4}.$$

$$dV = \frac{\partial V}{\partial D} . dD + \frac{\partial V}{\partial H} . dH = \frac{\pi}{2} DH . dD + \frac{\pi}{4} D^2 . dH$$

$$= \frac{\pi}{2}(8)(12.5)(0.05) + \frac{\pi}{4}(8)^2(0.05)$$

$$= \frac{\pi}{4}[16(0.625) + 64(0.05)] = 3.3\pi . cm^3.$$

or, $\boxed{dV = 10.36 \ cm^3}$ Ans

Q15. The power 'P' required to propel a steamer of length 'l' at a speed 'u' is given by $P = \lambda u^3 l^3$, where λ is constant. If u is increased by 3% and l is decreased by 1%, find the corresponding increase in 'P'?

$$P = \lambda u^3 l^3.$$

Taking log both sides;

$$\log P = \log \lambda + 3\log u + 3\log l.$$

Differentiation yields, $\frac{\delta P}{P} = 0 + 3.\frac{\delta u}{u} + 3.\frac{\delta l}{l}$.

$$\frac{\delta P}{P} \times 100 = 3.\frac{\delta u}{u} \times 100 + 3.\frac{\delta l}{l} \times 100.$$

$$\frac{\delta P}{P} \times 100 = 3(3) + 3(-1). = 9 - 3 = 6\%.$$

Hence, there is 6% increase in power 'P'.

Q16. Evaluate the following integrals:-

(i) $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} dx.$

$$\int_0^\infty \frac{x^8}{(1+x)^{24}} dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}} dx.$$

and we know,

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}.$$

So, $\frac{\Gamma 9 \, \Gamma 15}{\Gamma 24} - \frac{\Gamma 15 \, \Gamma 9}{\Gamma 24}$

$\Rightarrow 0$ Ans

(ii) $\int_0^\infty x^{2n-1} e^{-ax^2} dx$

assume, $ax^2 = t$. $\Rightarrow x = \sqrt{t/a}$

$2ax\,dx = dt$

$$\Rightarrow \int_0^\infty \frac{e^{-t}}{2a} \cdot \left(\frac{t}{a}\right)^{n-1} dt.$$

$$\Rightarrow \frac{1}{2a^n} \int_0^\infty e^{-t} . t^{n-1} dt$$

$$\Rightarrow \frac{1}{2a^n} \Gamma n \quad \left[\text{As } \int_0^\infty e^{-x} x^{n-1} dx = \Gamma n\right].$$

$$\Rightarrow \frac{\Gamma n}{2a^n} \text{ Ans}$$

(iii) $\int_0^1 x^3 (1-x)^{4/3} dx$

$$\int_0^1 x^{4-1} (1-x)^{7/3-1} dx \Rightarrow B(4, 7/3) \left[\text{as, } \int_0^1 x^{m-1}(1-x)^{n-1} dx = B(m,n)\right]$$

then, by Beta and Gamma relation: $B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$

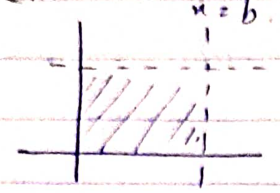
$$\Rightarrow \frac{\Gamma 4 \, \Gamma 7/3}{\Gamma 19/3} \Rightarrow \frac{3 \, \cancel{\Gamma 7/3}}{\frac{16}{3} \times \frac{13}{3} \cdot \frac{10}{3} \cdot \frac{7}{3} \cancel{\Gamma 7/3}} \Rightarrow 3^4$$

Q17. Change the order of integration and evalutate the following integral.

(i) $\int_0^b \int_0^a \frac{x}{x^2+y^2}\, dy\, dx.$

$x = 0 \quad , y = 0$

$x = b \quad , y = a$



change order :-

$$\int_0^a \int_0^b \frac{x}{x^2+y^2}\, dx\, dy.$$

assume $x^2+y^2 = t \Rightarrow 2x\,dx = dt$.

$$\int_0^a \int_{y^2}^{b^2+y^2} \frac{dt}{2t}\, dy \Rightarrow \int_0^a \left.\frac{\log t}{2}\right]_{y^2}^{b^2+y^2} dy$$

$$\int_0^a \frac{1}{2}[\log(b^2+y^2) - \log y^2]\, dy$$

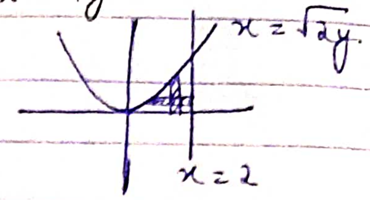
$$\int_0^a \frac{1}{2}\log(b^2+y^2)\, dy - \int_0^a \frac{\log y^2}{2}\, dy$$

$$= \frac{1}{2}\int_0^a (\log(b^2+y^2) - \log y^2)\, dy$$

using by part we can integrate it :-

$$\Rightarrow \frac{1}{2}\left[a\log(a^2+b^2) - 2\left(a - b\tan^{-1}\frac{a}{b}\right) - 2(a\log a - a)\right] \text{ Ans}$$

(ii) $\int_0^2 \int_{\sqrt{2y}}^2 \frac{x^2}{\sqrt{x^4-4y^2}}\, dx\, dy$



By changing order of integration :-

$$\int_0^2 \int_0^{x^2/2} \frac{x^2}{\sqrt{x^4-(2y)^2}}\, dy\, dx$$

$$\int_0^2 \frac{x^2}{2}\left[\sin^{-1}\frac{2y}{x^2}\right]_0^{x^2/2} dx$$

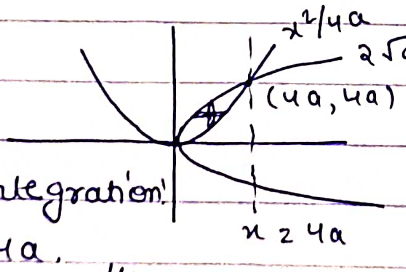
$$\int_0^2 \frac{x^2}{2}\left[\sin^{-1}\frac{\cancel{2}x^2}{\cancel{2}\cdot x^2} - \sin^{-1}0\right] dx$$

$$\int_0^2 \frac{x^2}{2}\,\frac{\pi}{2}\, dx$$

$$= \frac{\pi}{4}\left.\frac{x^3}{3}\right]_0^2 \Rightarrow \frac{\pi}{\cancel{12}}\left[\frac{8}{3} - 0\right]$$

$$\Rightarrow \frac{2\pi}{3} \text{ Ans}$$

(iii) $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx.$



By changing order of integration!

then, $y = 0$ to $y = 4a$.

$$\int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dx\, dy \Rightarrow \int_0^{4a} \left. x\right]_{y^2/4a}^{2\sqrt{ay}} dy \Rightarrow \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$\Rightarrow \int_0^{4a} 2\sqrt{ay}\, dy - \int_0^{4a} \frac{y^2}{4a}\, dy \Rightarrow \left.\frac{4a^{1/2}y^{3/2}}{3}\right]_0^{4a} - \left.\frac{y^3}{12a}\right]_0^{4a}$$

$$\Rightarrow \frac{32a^2}{3} - 0 - \left(\frac{64a^2}{12} - 0\right) \Rightarrow \frac{32a^2}{3} - \frac{16a^2}{3} \Rightarrow \frac{16a^2}{3} \text{ Ans}$$

Q18. Solve following double integrals.

(i) $\int_0^1 \int_{y^2}^{y} (1+xy^2)\,dx\,dy$.

$$\int_0^1 \left[x + \frac{x^2y^2}{2}\right]_{y^2}^{y} dy$$

$$\int_0^1 \left(y - y^2 + \frac{y^4}{2} - \frac{y^6}{2}\right) dy$$

$$\int_0^1 \left(y - y^2 + \frac{y^4}{2} - \frac{y^6}{2}\right) dy$$

$$\left[\frac{y^2}{2} - \frac{y^3}{3} + \frac{y^5}{10} - \frac{y^7}{14}\right]_0^1$$

$$\left(\frac{1}{2} - 0\right) - \left(\frac{1}{3} - 0\right) + \left(\frac{1}{10} - 0\right) - \left(\frac{1}{14} - 0\right)$$

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{10} - \frac{1}{14} \Rightarrow \frac{41}{210} \text{ Ans}$$

(ii) $\int_0^1 \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}\,dx\,dy$.

$$\int_0^1 \frac{1}{\sqrt{1-y^2}} \left[\sin^{-1}\frac{x}{1}\right]_0^1 dy$$

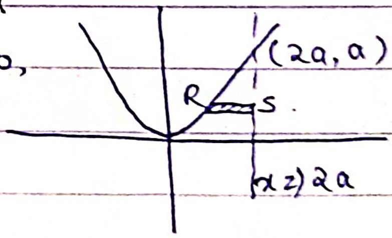
$$\int_0^1 \frac{1}{\sqrt{1-y^2}} \frac{\pi}{2}\, dy$$

$$\frac{\pi}{2}\left[\sin^{-1}\frac{y}{1}\right]_0^1$$

$$\frac{\pi}{2}\left(\frac{\pi}{2} - 0\right) \Rightarrow \frac{\pi^2}{4} \text{ Ans}$$

Q19. Evaluate $\iint_S xy\,dx\,dy$, where S is domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$?

Integrating first over horizontal strip RS so,



$$\iint_S xy\,dx\,dy \Rightarrow \int_0^a \int_{2\sqrt{ay}}^{2a} xy\,dx\,dy$$

$$\Rightarrow \int_0^a y \left.\frac{x^2}{2}\right]_{2\sqrt{ay}}^{2a} dy \Rightarrow \int_0^a \left(\frac{4a^2y}{2} - \frac{4ay^2}{2}\right) dy$$

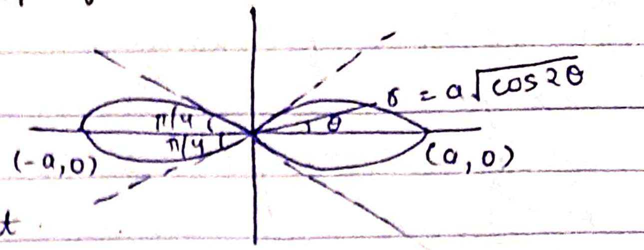
$$\Rightarrow 2a\int_0^a (ay - y^2)\,dy \Rightarrow 2a\left[\frac{ay^2}{2} - \frac{y^3}{3}\right]_0^a$$

$$\Rightarrow 2a\left[\frac{a^3}{2} - \frac{a^3}{3}\right] \Rightarrow \frac{\cancel{2}a^4(3-2)}{\cancel{6}\,3}$$

$$\Rightarrow \frac{a^4}{3} \text{ Ans}$$

Q20. Evaluate $\iint \frac{r}{\sqrt{a^2+r^2}}\, dr\, d\theta$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$.

$$I = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{a^2+r^2}}\, dr\, d\theta$$



assume, $a^2 + r^2 = t \Rightarrow 2r\,dr = dt$

$$I = \int_{-\pi/4}^{\pi/4}\int_{a^2}^{2a^2\cos^2\theta} \frac{dt}{2\sqrt{t}}\, d\theta \Rightarrow \int_{-\pi/4}^{\pi/4} \left[t^{1/2}\right]_{a^2}^{2a^2\cos^2\theta} d\theta$$

$$I = \int_{-\pi/4}^{\pi/4} (\sqrt{2}\,a\cos\theta - a)\, d\theta \Rightarrow \left[\sqrt{2}a\sin\theta - a\theta\right]_{-\pi/4}^{\pi/4}$$

$$I = \sqrt{2}a\left[\sin\frac{\pi}{4} - \sin\left(-\frac{\pi}{4}\right)\right] - a\left[\frac{\pi}{4}+\frac{\pi}{4}\right]$$

$$= \sqrt{2}a\left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right] - \frac{a\pi}{2} \Rightarrow a\left(2-\frac{\pi}{2}\right) \text{ Ans}$$

Q21. Evaluate the following integrals :-

(i) $\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{2} x^2yz\, dx\,dy\,dz$

$$\int_{x=0}^{1}\int_{y=0}^{2} x^2y\left(\frac{z^2}{2}\right)_1^2 dx\,dy$$

$$\int_{x=0}^{1}\int_{y=0}^{2} x^2y\left[2-\frac{1}{2}\right]dx\,dy$$

$$\int_{x=0}^{1}\int_{y=0}^{2} \frac{3}{2}x^2y\,dy$$

$$\int_{x=0}^{1} \frac{3}{2}x^2\left[\frac{y^2}{2}\right]_0^2 dx$$

$$\int_0^1 \frac{3}{2}x^2(2-0)\,dx$$

$$\int_0^1 3x^2\,dx$$

$$\left.\frac{3x^3}{3}\right]_0^1 \Rightarrow 1-0 \Rightarrow 1 \text{ Ans}$$

(ii) $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz.$

$$\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$$

$$\int_0^a\int_0^x \left.e^{x+y+z}\right]_0^{x+y} dy\,dx$$

$$\int_0^a\int_0^x \left(e^{2(x+y)} - e^{x+y}\right)dy\,dx$$

$$\int_0^a \left[\frac{e^{2(x+y)}}{2} - e^{x+y}\right]_0^x dx$$

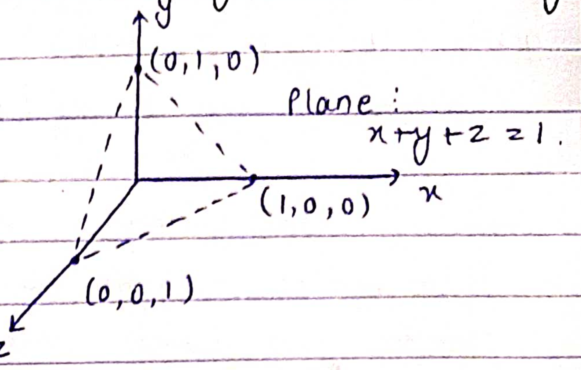
$$\int_0^a \frac{e^{4x}}{2} - \frac{e^{2x}}{2} - \left(e^{2x}-e^x\right)dx$$

$$\left[\frac{e^{4x}}{8} - \frac{e^{2x}}{4} - \frac{e^{2x}}{2} + e^x\right]_0^a$$

$$\frac{e^{4a}}{8} - \frac{1}{8} - \frac{e^{2a}}{4} + \frac{1}{4} - \frac{e^{2a}}{2} + \frac{1}{2} + e^a - 1$$

$$\Rightarrow \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8} \text{ Ans}$$

Q22. Evaluate :- $\iiint (x+y+z+1)^4 \, dx\,dy\,dz$ over the region determined by $x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1$.



$z = 0 \quad , \quad z = 1 - x - y.$

$y = 0 \quad , \quad y = 1 - x$

$x = 0 \quad , \quad x = 1.$

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z+1)^4 \, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left. \frac{(x+y+z+1)^5}{5} \right]_0^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{2^5}{5} - \frac{(x+y+1)^5}{5} \right] dy\, dx$$

$$\Rightarrow \int_0^1 \left[\frac{2^5}{5} y - \frac{(x+y+1)^6}{30} \right]_0^{1-x} dx$$

$$\Rightarrow \int_0^1 \left(\frac{2^5}{5}(1-x) - 0 - \frac{2^6}{30} + \frac{(x+1)^6}{30} \right) dx$$

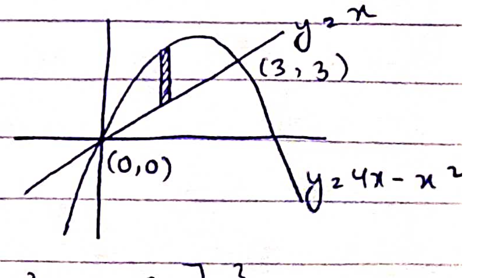
$$\Rightarrow \left[-\frac{2^5}{5} \frac{(1-x)^2}{2} - \frac{2^6}{30} x + \frac{(x+1)^7}{210} \right]_0^1$$

$$\Rightarrow -\frac{2^4}{5}(0) + \frac{2^4}{5} - \frac{2^6}{30} + 0 + \frac{2^7}{210} - \frac{1}{210}$$

$$\Rightarrow \frac{16}{5} - \frac{64}{30} + \frac{128}{210} - \frac{1}{210} \Rightarrow \frac{672 - 448 + 128 - 1}{210}$$

$$\Rightarrow \frac{351}{210} \text{ Ans}$$

Q23. Find the area lying between the parabola $y = 4x - x^2$ and the curve $y = x$.



$y = 4x - x^2$ & $y = x$

so, our interval of integration is :- $0 \leq x \leq 3$.

$$\text{Area} = \int_0^3 \int_x^{4x-x^2} dy\, dx \Rightarrow \int_0^3 y \Big]_x^{4x-x^2} dx$$

$$\Rightarrow \int_0^3 (4x - x^2 - x)\, dx \Rightarrow \int_0^3 (3x - x^2)\, dx \Rightarrow \left. \frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3$$

$\Rightarrow 9/2$ sq. units Ans

Q24. Find the volume bounded by cylinder $x^2+y^2=4$ and the plane $y+z=4$ and $z=0$?

Volume $= \iiint_D dx\,dy\,dz$

so, z varies from 0 to $4-y$

Taking polar co-ordinate $y = r\sin\theta$ where, θ varies from 0 to 2π and r varies from 0 to 2

$$V = \int_0^{2}\int_0^{2\pi}\int_0^{4-r\sin\theta} dz\, r\,dr\,d\theta$$

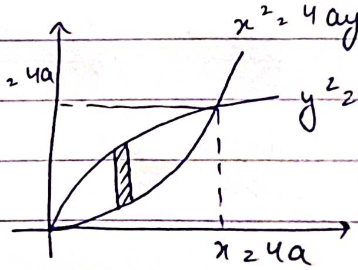
$$= \int_0^{2}\int_0^{2\pi}[z]_0^{4-r\sin\theta}\, d\theta\, r\,dr$$

$$= \int_0^{2}\int_0^{2\pi}(4-r\sin\theta)\,d\theta\, r\,dr = \int_0^{2}(4\theta + r\cos\theta)_0^{2\pi}\, r\,dr$$

$$= \int_0^{2}(8\pi + r - r)\,r\,dr \Rightarrow \int_0^{2} 8\pi r\,dr \Rightarrow \left[\frac{{}^{4}\cancel{8}\pi r^2}{\cancel{2}}\right]_0^2$$

$\Rightarrow 4\pi(4-0) \Rightarrow 16\pi$ cubic unit Ans

Q25 Find the mass of lamina bounded by curve $y^2=4ax$ and $x^2=4ay$; where density of lamina at any pt. varies as the square of distance from the origin.



$\because y^2=4ax$ and $x^2=4ay$.

$\rho = x^2+y^2$

$$\text{Mass} = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}}(x^2+y^2)\,dy\,dx$$

$$\Rightarrow \int_0^{4a}\left(x^2y+\frac{y^3}{3}\right)_{x^2/4a}^{2\sqrt{ax}}dx \Rightarrow \int_0^{4a}\left(2x^2\sqrt{ax}+\frac{8ax\sqrt{ax}}{3}-\frac{x^4}{4a}-\frac{x^6}{64a^3\cdot 3}\right)dx$$

$$\Rightarrow \int_0^{4a}\left(2\sqrt{a}\,x^{5/2}+\frac{8a^{3/2}x^{3/2}}{3}-\frac{x^4}{4a}-\frac{x^6}{192a^3}\right)dx$$

$$\Rightarrow \left[\frac{2\sqrt{a}\,x^{7/2}}{7/2}+\frac{8a^{3/2}}{3}\frac{x^{5/2}}{5/2}-\frac{x^5}{20a}-\frac{x^7}{1074a^3}\right]_0^{4a}$$

$$\Rightarrow \frac{4\sqrt{a}\,4^{7/2}a^{7/2}}{7}+\frac{16}{15}a^4\,4^{5/2}-\frac{4^5a^5}{20a}-\frac{4^7a^7}{1074a^3}$$

$\boxed{\text{Mass} \Rightarrow \frac{4608}{105}a^4}$ Ans