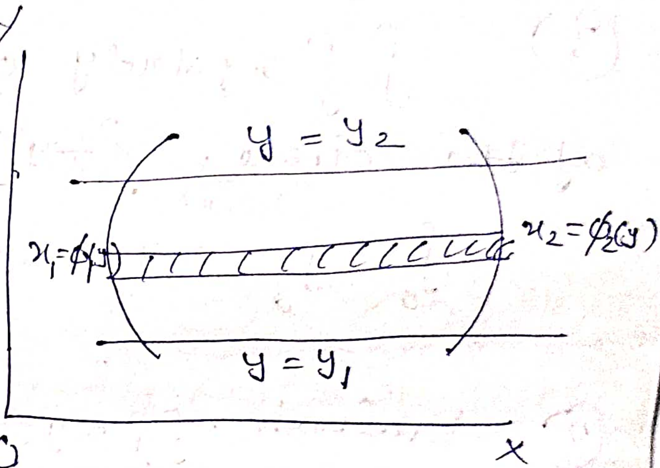
Multiple Integral

Evaluation of Double Integrals :-

The methods of evaluating the double integrals depends upon the nature of the curves bounding the region R. Let the region R be bounded by the curves $x = x_1$, $x = x_2$, and $y = y_1$, $y = y_2$.

i)- when x_1, x_2 are functions of y and y_1, y_2 are constants. AB and CD be the curves $x_1 = \phi_1(y)$, $x_2 = \phi_2(y)$

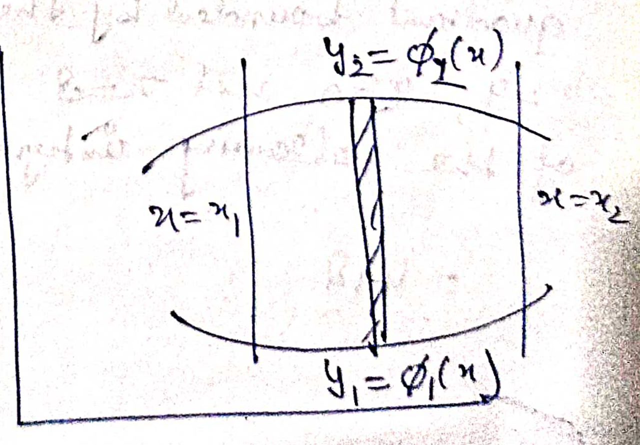


take a horizontal strip of width δy, Here the double integral is evaluated first w.r.t x (treating y as constant) The resulting expression which is a function of y is integrated w.r.t y between the limits $y = y_1$ and $y = y_2$.

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2} \left\{ \int_{x_1=\phi_1(y)}^{x_2=\phi_2(y)} f(x,y)\,dx \right\} dy$$

ii) - when y_1, y_2 are functions of x and x_1, x_2 are constants.

take a ~~horizontal~~ vertical strip of width δx. evaluate first with respect to y then x.

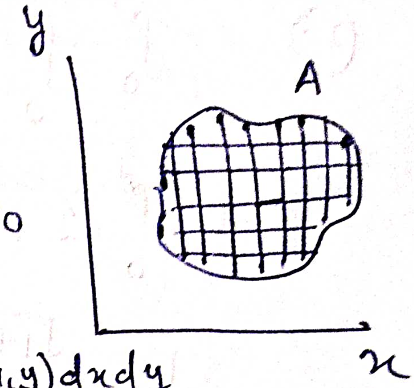


$$\iint_R f(x,y)\,dx\,dy$$

$$= \int_{x_1}^{x_2} \left\{ \int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)} f(x,y)\,dy \right\} dx$$

Double Integral →

double integral of $f(x,y)$ over the region A $= \iint_A f(x,y)\,dA = \iint_A f(x,y)\,dx\,dy$



Properties :-

1). If the region A is partitioned into two parts, says A_1 and A_2 then

$$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy$$

Evaluation of double integrals :-

1. double Integral $\int_a^b \int_c^d f(x,y)\,dx\,dy$

if both limits ~~are~~ constant then you can integrate first either w.r.to x or w.r.to y.
when integrate w.r.t 'x' then y as constant
when integrate w.r.t 'y' then x as constant

2. $\int_a^b \int_{f(x)}^{g(x)} f(x,y)\,dy\,dx$

if variable limit is in 'x' then first integrate w.r.to y and then with respect to 'x'

3. $\int_a^b \int_{f(y)}^{g(y)} f(x,y)\,dx\,dy$

if variable limit is in 'y' then first integrate w.r.t x and then w.r.t 'y'

(1). Evaluate $\int_1^2 \int_0^{3y} y \, dx \, dy$ ③

$= \int_1^2 y \{x\}_0^{3y} dy = \int_1^2 3y^2 dy$.

$= 3 \cdot \left[\frac{y^3}{3}\right]_1^2 = 2^3 - 1^3 = 7$

(2). $\int_0^1 \int_0^2 (x+y) dx dy = \int_0^1 \left[\frac{x^2}{2} + xy\right]_0^2 dy$

$= \int_0^1 (2 + 2y) \, dy = \left[2y + y^2\right]_0^1 = 2+1 = 3$

(3). $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx \, dy}{1+x^2+y^2}$

$\int_0^1 \int_0^{\sqrt{1+x^2}} \left\{\frac{dy}{(\sqrt{1+x^2})^2 + y^2)}\right\} dx$.

$\int_0^1 \left\{\frac{1}{\sqrt{1+x^2}} \cdot \tan^{-1} \left\{\frac{y}{\sqrt{1+x^2}}\right\}\right\}_0^{\sqrt{1+x^2}} dx$

$\int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \{\tan^{-1} 1 - \tan^{-1} 0\} \, dx$

$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} dx$.

$= \frac{\pi}{4} \log \left\{x + \sqrt{1+x^2}\right\}_0^1$

$= \frac{\pi}{4} \log(1+\sqrt{2}) - \frac{\pi}{4} \log 1$

③ Evaluate $\int_0^1 \int_0^x e^{y/x}\,dx\,dy$

$$= \int_0^1 \left[\int_0^x e^{y/x}\,dy \right] dx$$

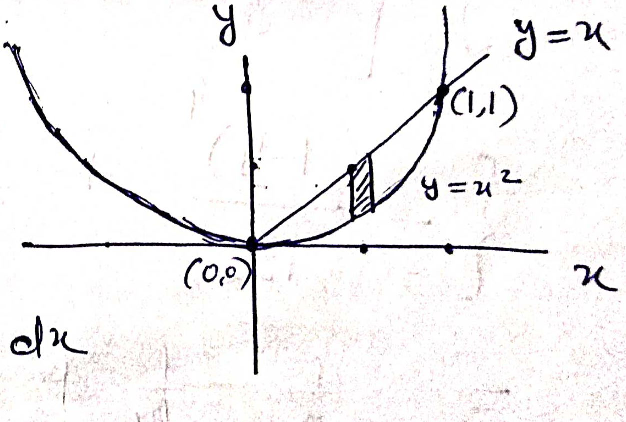
$$\int_0^1 \left\{ x e^{y/x} \right\}_0^x dx = \int_0^1 x(e-1)\,dx$$

$$= (e-1)\int_0^1 x\,dx = (e-1)\left[\frac{x^2}{2}\right]_0^1 = \frac{(e-1)}{2}$$

Area by double integration

$$\boxed{\text{Area} = \iint_A dx\,dy}$$

④ Evaluate $\iint xy(x+y)\,dx\,dy$ over the Area between $y=x^2$ and $y=x$.



$x=0$, $x=1$

$y=x^2$ $y=x$

$$= \int_0^1 \left\{ \int_{x^2}^{x} x^2y + xy^2\,dy \right\} dx$$

$$= \int_0^1 \left\{ \frac{x^2y^2}{2} + \frac{xy^3}{3} \right\}_{x^2}^{x} dx$$

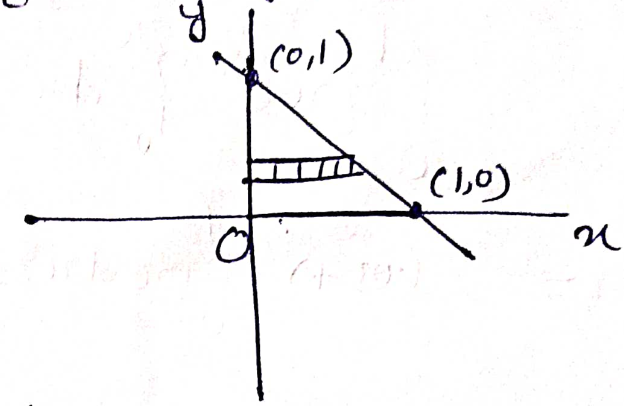
$$\int_0^1 \left\{ \frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right\} dx$$

$$\left\{ \frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24} \right\}_0^1 = \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{3}{56}$$

(1)

(2). Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region in the positive quadrant for which $x+y \le 1$



$y=0$, $y=1$
$x=0$, $x=1-y$

$$\int_0^1 \left[\int_0^{1-y} (x^2+y^2)\,dx\right] dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_0^{1-y} dy$$

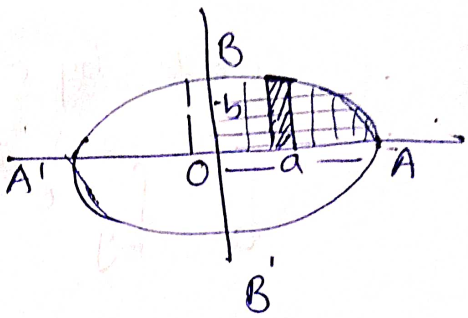
$$\int_0^1 \left[\frac{(1-y)^3}{3} + y^2(1-y)\right] dy$$

$$\left\{-\frac{(1-y)^4}{12} + \frac{y^3}{3} - \frac{y^4}{4}\right\}_0^1$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{1}{6}.$$

(1). Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Area of one quadrant $= \iint dx\,dy$



whole area of ellipse $= 4\iint dx\,dy$

$$= 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx\,dy$$

$$4\int_0^a \{y\}_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx = \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}.\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[0+\frac{a^2}{2}\times\frac{\pi}{2}-0\right] = \frac{\cancel{4}b}{\cancel{a}}\times\frac{a^{\cancel{2}}\pi}{\cancel{2}\times\cancel{2}} = \pi ab.$$

① $\displaystyle\int_0^1 \int_0^y xy \cdot e^{-x^2}\, dx\, dy = \frac{1}{4e}$.

② Evaluate $\iint e^{2x+3y}\, dx\, dy$ over the triangle bounded by $x=0$, $y=0$ and $x+y=1$

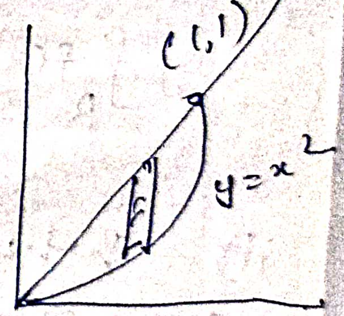
$$= \frac{1}{6}(e-1)^2(2e+1)$$

③ $\iint xy\, dx\, dy$ over the positive quadrant of the circle $x^2+y^2=a^2$.

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx = \frac{a^4}{8}$$

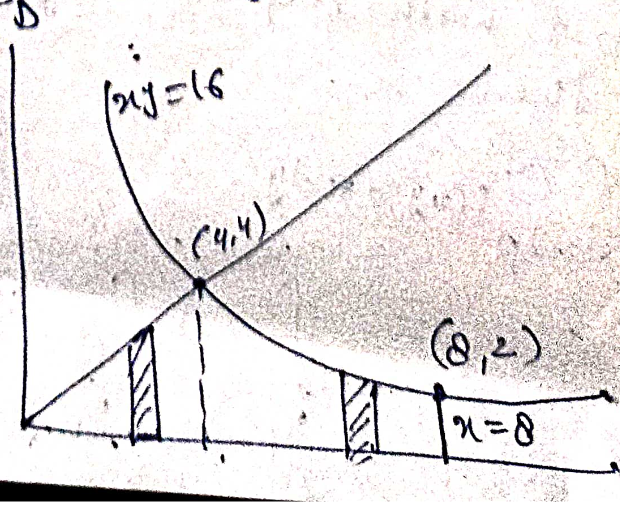
④ Evaluate $\iint xy(x+y)\, dx\, dy$ over the area between $y=x^2$ and $y=x$.

$$\int_0^1 \left\{\int_{x^2}^{x} (x^2y + xy^2)\, dy\right\} dx = \frac{3}{56}$$



⑤ Let D be the region in the first quadrant bounded by the curves $xy=16$, $x=y$, $y=0$ and $x=8$. Sketch the region of integration of the following integral $\iint_D x^2\, dx\, dy$ and evaluate.

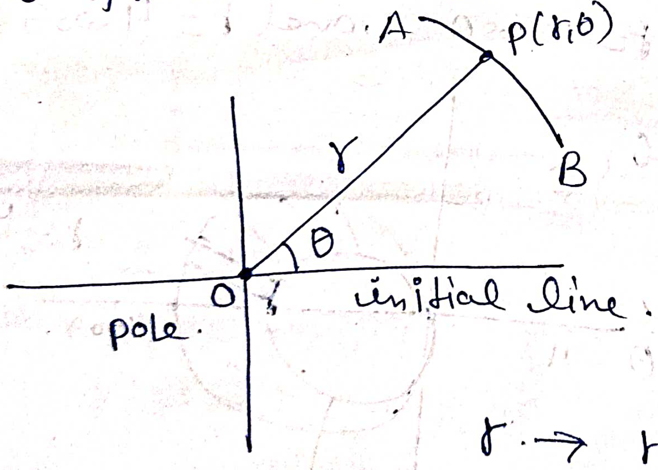
$= 448$

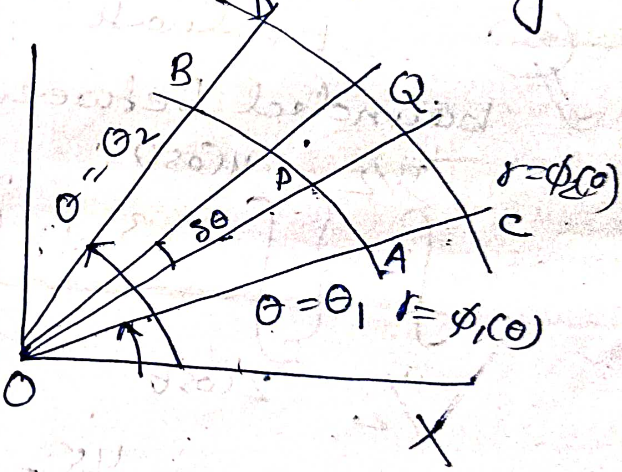


Evaluation of Double Integral in Polar Co-ordi.

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$ over the region bounded by the straight lines $\theta=\theta_1$, $\theta=\theta_2$ and the curves $r=r_1$; $r=r_2$ we first integrate w.r.t r between the limit $r=r_1$ and $r=r_2$ and resulting expression is then integrable w.r.t θ between the limits $\theta=\theta_1$ and $\theta=\theta_2$

$\int_{r_1}^{r_2} f(r,\theta)\,dr$ indicates that the integration is along PQ from P to Q while the integration w.r.t θ corresponds to the turning of PQ from AC to BD





$r \rightarrow$ radius vector.

$\theta \rightarrow$ vectorial angle.

① $\int_0^{\pi}\int_0^{a(1-\cos\theta)} r^2 \sin\theta \,dr\,d\theta = \int_0^{\pi}\left\{\frac{r^3}{3}\right\}_0^{a(1-\cos\theta)} \sin\theta\,d\theta$

$= \frac{4}{3}a^3$

(5)

Area by Double Integration.

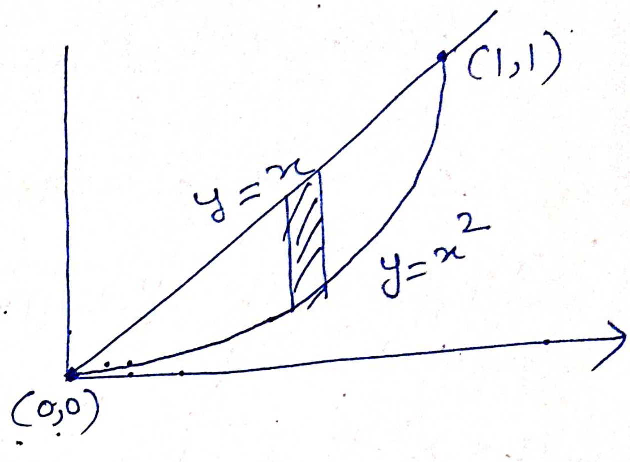
The area A of the region bounded by the curves: $A = \iint_R dx\,dy$.

in polar coordinates. $A = \iint_R r \cdot dr\,d\theta$.

7

16. Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y = x^2$ and $y = x$

$x = 0$, $y = 0$
$x = 1$ $y = 1$
$x = 2$ $y = 4$
$x = 3$ $y = 9$



x varies from 0 to 1
y varies from x^2 to x

$$\int_0^1 \int_{x^2}^{x} xy(x+y)\,dy\,dx$$

$$= \int_0^1 dx \left[\int_{x^2}^{x} (x^2 y + xy^2)\,dy \right]$$

$$= \int_0^1 \left\{ \frac{x^2 y^2}{2} + \frac{xy^3}{3} \right\}_{x^2}^{x} dx$$

$$= \int_0^1 \left(\frac{x^4}{2} + \frac{x^4}{3} \right) - \frac{x^6}{2} - \frac{x^7}{3} \Big] dx$$

$$= \left[\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1$$

$$= \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24} = \frac{3}{56}$$

$x \geq 0 \quad y \geq 0$

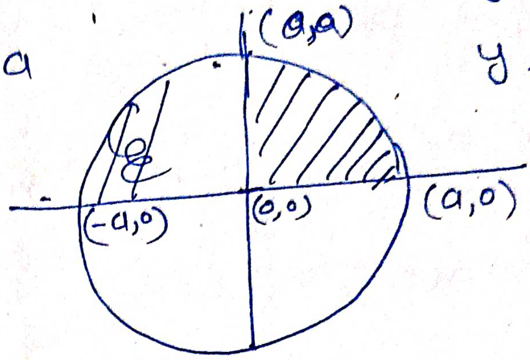
Ex. Evaluate $\iint xy \, dx\, dy$ over the positive quadrand of the circle $x^2+y^2 = a^2$

$y^2 = a^2 - x^2$

$y = \pm\sqrt{a^2-x^2}$

x varies from 0 to a

and y from 0 to $\sqrt{a^2-x^2}$



$$\int_0^a \int_0^{\sqrt{a^2-x^2}} xy \, dy\, dx$$

$$\int_0^a x \cdot \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a x\cdot(a^2-x^2)\, dx$$

$$= \frac{1}{2}\int_0^a \left(a^2 x - x^3\right) dx$$

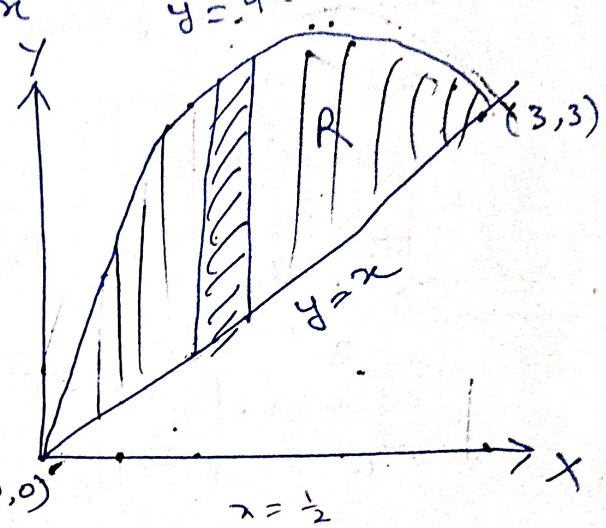
$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{a^4}{8}$$

(6)

Ex): - Evaluate $\iint y \, dx \, dy$ over the part of the plane bounded by the line $y = x$ and the parabola. $y = 4x - x^2$

where $x = 1$, $y = 3$
$x = 2$ $y = 4$
$x = 3$ $y = 3$



The line $y = x$ meets the parabola $y = 4x - x^2$ in two distinct points $(0,0)$ and $(3,3)$

$x = \frac{1}{2}$

$y = x^2$

$0 \le x \le 3$ $\quad x \le y \le 4x - x^2$

$$\iint_R y \, dx \, dx = \int_0^3 \int_x^{4x-x^2} y \, dy \, dx$$

$$= \int_0^3 \left[\frac{y^2}{2}\right]_x^{4x-x^2} dx$$

$$= \frac{1}{2}\int_0^3 \left[(4x-x^2)^2 - x^2\right] dx$$

$$= \frac{1}{2}\int_0^3 \left(16x^2 + x^4 - 8x^3 - x^2\right) dx$$

$$= \frac{1}{2}\left[\int_0^3 (15x^2 - 8x^3 + x^4) \, dx\right.$$

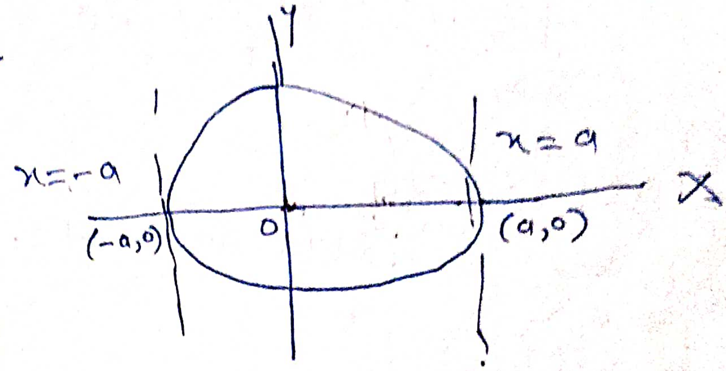
$$\frac{1}{2}\left[\frac{\cancel{15}^{5} x^3}{\cancel{3}} - \frac{\cancel{8}^{2} x^4}{\cancel{4}} + \frac{x^5}{5}\right]_0^3$$

$$\frac{1}{2}\left[5 \times 27 - 2 \times 81 + \frac{243}{5}\right] = \frac{54}{5}$$

Ex) Evaluate $\iint (x+y)^2\, dx\,dy$ over the area bounded by the ellipse. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The region of integration R can be expressed

$-a \le x \le a$



$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$$

$$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$

$$\iint_R (x+y)^2\, dx\,dy = \iint_R (x^2+y^2+2xy)\, dx\,dy$$

$$\int_{-a}^{a}\left[\int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2+y^2)\, dy\right] dx + \int_{-a}^{a}\left[\int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} 2xy\, dy\right] dx$$

(x^2+y^2) is an even func of y

and $2xy$ is an odd fun of y

$$= \int_{-a}^{a} 2\int_{0}^{b/a\sqrt{a^2-x^2}} (x^2+y^2)\, dy\, dx$$

$$= 2\int_{-a}^{a}\left[x^2y+\frac{y^3}{3}\right]_0^{b/a\sqrt{a^2-x^2}} dx$$

$$= 4\int_0^a \left[x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{1}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right] dx$$

$x = a\sin\theta$ and $dx = a\cos\theta \, d\theta$

$$4\int_0^{\pi/2} \left(\frac{b}{a} a^2 \sin^2\theta \cdot a\cos\theta + \frac{b^3}{3a^3} a^3 \cos^3\theta \right) a\cos\theta \, d\theta$$

$$= 4\int_0^{\pi/2} \left[ba^3 \sin^2\theta \cdot \cos^2\theta + \frac{ab^3}{3} \cos^4\theta \right] d\theta$$

$$= 4\left[ba^3 \frac{\Gamma\frac{3}{2}\Gamma\frac{3}{2}}{2\Gamma 3} + \frac{ab^3}{3} \frac{\Gamma\frac{1}{2}\Gamma\frac{5}{2}}{2\Gamma 3} \right]$$

$$\int_0^{\pi/2} \sin^m\theta \cdot \cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2}\Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$$

$$\Gamma n = (n-1)\Gamma n-1$$

$$4ab\left[a^2 \frac{\frac{1}{2}\Gamma\frac{1}{2} \frac{1}{2}\Gamma\frac{1}{2}}{2\cdot 2} + \frac{b^2}{\cancel{3}} \frac{\sqrt{\pi}\,\frac{\cancel{3}}{\cancel{2}}\cdot\frac{1}{\cancel{2}}\sqrt{\pi}}{\cancel{2}\cdot\cancel{2}} \right]$$

$$= \left[\frac{a^2\pi}{16} + \frac{b^2\pi}{16} \right]$$

$$= \boxed{\frac{ab\pi}{4}(a^2+b^2)}$$

CHANGE OF ORDER OF INTEGRATION

Example ①

Evaluate the following integral by changing the order of integration.

$$\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y}$$

$e = 2.7$

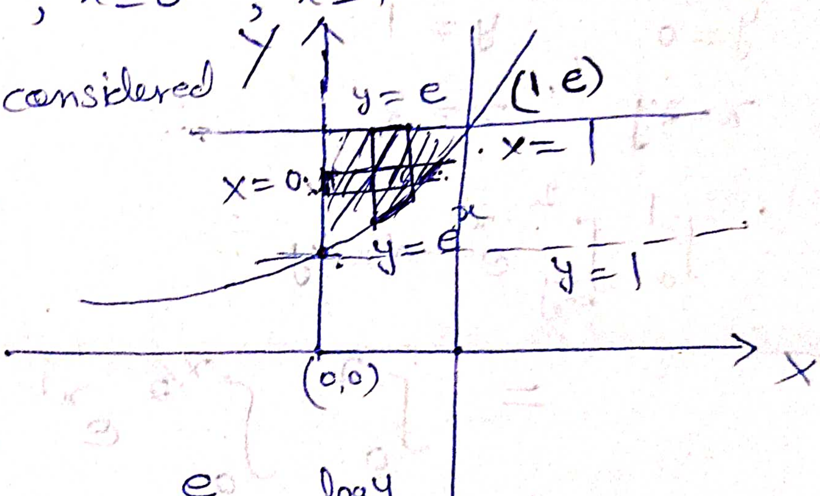
The given limits show that the region of integration is bounded by the curves

$y = e^x$, $y = e$, $x = 0$, $x = 1$

It can be also be considered as bounded by

$x = 0$, $x = \log y$

$y = 1$, $y = e$



$$\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y} = \int_1^e \int_0^{\log y} \frac{1}{\log y}\,dx\,dy$$

$$= \int_1^e \frac{1}{\log y} [x]_0^{\log y}\,dy$$

$$= \int_1^e \frac{1}{\log y} [\log y]\,dy = [y]_1^e$$

$$= e-1$$

② Evaluate by changing the order of integration $\int_0^1 \int_{2y}^{2} e^{x^2}\,dx\,dy$

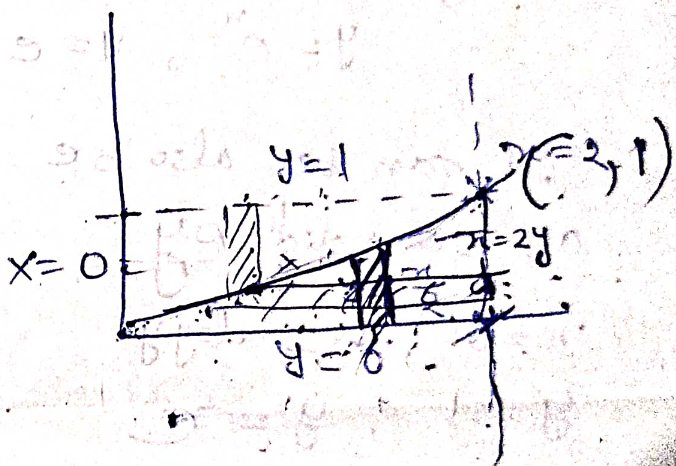
$y=0$, $y=1$, $x=2y$, $x=2$
$\quad\quad\quad\quad\quad\quad\quad y=0 \quad y=1$

we can also consider it as

$y=0$, $y=\frac{x}{2}$, $x=0$, $x=2$

$y=0 \quad y=1$

$x=2y \quad x=2$



$$\int_0^1 \int_{2y}^{2} e^{x^2}\,dx\,dy$$

$$= \int_0^2 \int_0^{x/2} e^{x^2}\,dy\,dx$$

$$= \int_0^2 e^{x^2}\left[\frac{x}{2}\right]dx = \frac{1}{2}\int_0^2 x e^{x^2}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{x^2}}{2}\right]_0^2 = \boxed{\frac{1}{4}(e^4-1)}$$

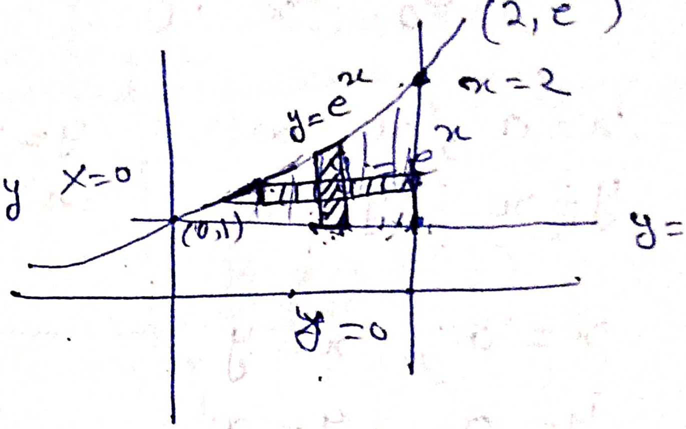
3. $\int_0^2 \int_1^{e^x} dy\,dx$ by changing the order of integration.

$y = 1$, $y = e^x$, $x = 0$, $x = 2$

~~$x = 0$, $x = \log y$~~

$y = 1$ $\quad y = e^x \Rightarrow x = \log y$

$x = 0$ $\quad x = 2$



$x = \log y$ to $x = 2$

$y = 1$ to $y = e^2$

$$\int_0^2 \int_1^{e^x} dy\,dx = \int_1^{e^2} \int_{\log y}^{2} dx\,dy$$

$$= \int_1^{e^2} (2 - \log y)\,dy$$

$$= \left(2y - y\log y + y\right)_1^{e^2}$$

$$= \left(2e^2 - e^2 \log e^2 + e^2\right) - 2 - 1$$

$$= 3e^2 - 2e^2 - 3 = \boxed{e^2 - 3}$$

(4) Evaluate the following integral by changing the order of integration.

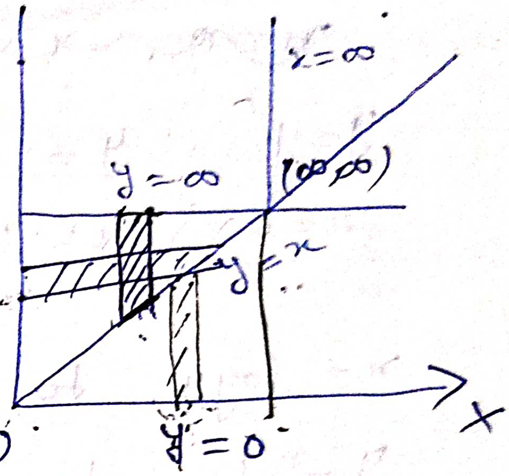
$$\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx$$

$x = 0$, $x = \infty$

$y = x$, $y = \infty$

$y = x$ $y = \infty$

$x = 0$ $x = \infty$



$x = 0$, $x = y$

$y = 0$ $y = \infty$

$$\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx = \int_0^{\infty}\left[\int_0^{y} \frac{e^{-y}}{y}\, dx\right] dy$$

$$= \int_0^{\infty} \frac{e^{-y}}{\cancel{y}} \cdot \cancel{y}\, dy = \left[-e^{-y}\right]_0^{\infty}$$

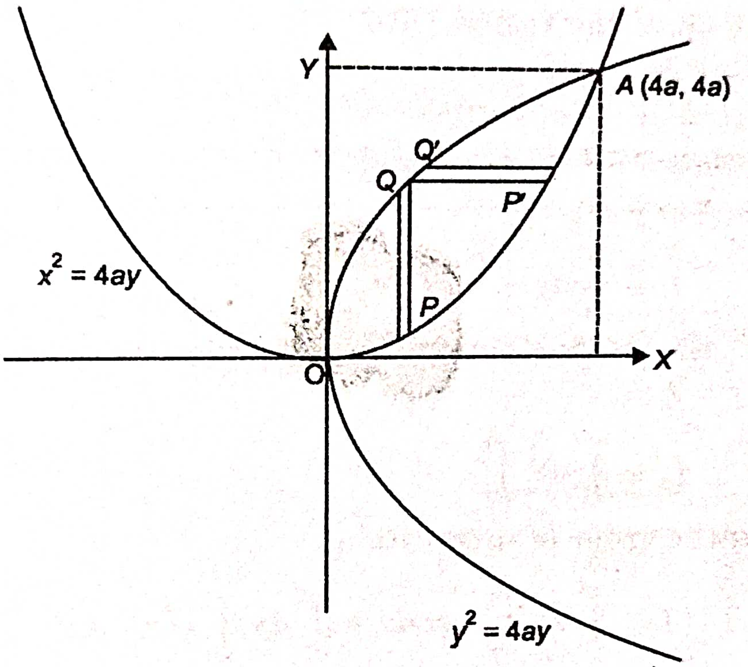
$$= \left[-e^{-\infty} + e^{0}\right] = -\frac{1}{\infty} + 1 = 1$$

Example 5.11

Change the order of integration in the following integral and evaluate:

$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{(ax)}} dy\, dx \qquad \text{(M.D.U., 2000-2001)}$$

Solution: The given limits shows that the region of integration is bounded by the curves $y = x^2/4a$, $y = 2\sqrt{(ax)}$, $x = 0$ and $x = 4a$ which is shown in the figure.



Thus, from the given integral, it is clear that first integrate w.r.t. y and then w.r.t. x.

Now, we have to change the order of integration *i.e.*, we have to integrate first w.r.t. x and then w.r.t. y. Therefore, we have

$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{(ax)}} dy\, dx = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{(ay)}} dx\, dy$$

$$= \int_0^{4a} [x]_{y^2/4a}^{2\sqrt{(ay)}} dy$$

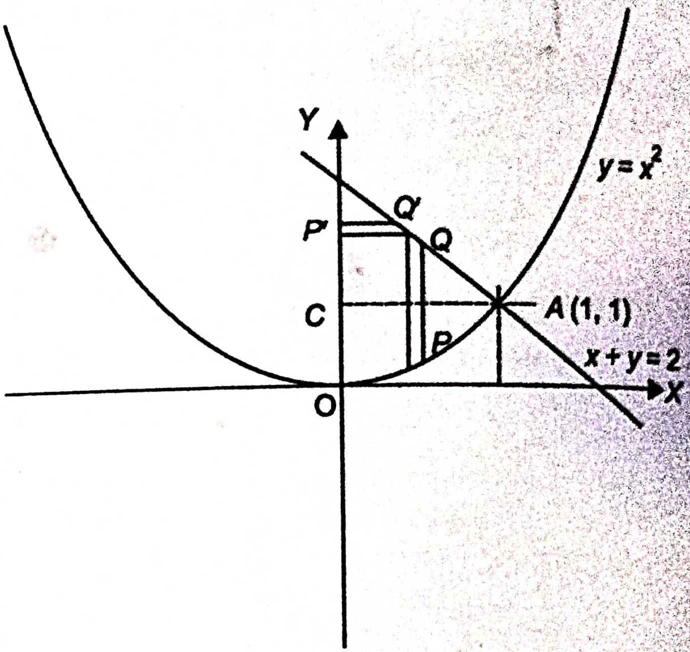
$$= \int_0^{4a} \left[2\sqrt{(ay)} - \frac{y^2}{4a}\right] dy$$

$$= \left[2\sqrt{a}\,.\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{(a)}.(4a)^{3/2} - \frac{64a^3}{12a} = \frac{16a^2}{3}$$

Example 5.12

Change the order of integration and hence evaluate the integral $\int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$.



Solution: Here the integration is first w.r. to y along a vertical strip, which extends from $y = x^2$ to $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as shown.

On changing the order of integration, we first integrate w.r, to x along a horizontal strip and that requires split up of the region OAB into two parts by the line AC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$.

Thus $$I_1 = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx$$

For the region ABC, the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$.

Thus $$I_2 = \int_1^2 dy \int_0^{2-y} xy\, dx$$

Hence on reversing the order of integration,

$$I = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx + \int_1^2 dy \int_0^{2-y} xy\, dx$$

$$= \int_0^1 dy \left| \frac{x^2}{2} \cdot y \right|_0^{\sqrt{y}} + \int_1^2 dy \left| \frac{x^2}{2} \cdot y \right|_0^{2-y}$$

$$= \frac{1}{2} \int_0^1 y^2\, dy + \frac{1}{2} \int_1^2 y\,(2-y)^2\, dy = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

EXERCISES

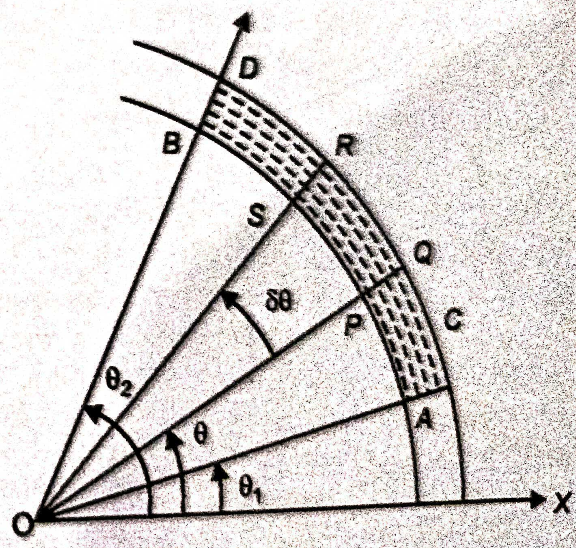
Change the order of integration in the following integrals.

1. $\int_0^4 \int_x^{2\sqrt{x}} f(x, y)\, dx\, dy$
2. $\int_0^a \int_{x^2/a}^{2a-x} f(x, y)\, dx\, dy$
3. $\int_0^a \int_x^{a^2/x} f(x, y)\, dx\, dy$
4. $\int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} f(x, y)\, dy\, dx$
5. $\int_0^{a/2} \int_{x^2/a}^{x-x^2/a} f(x, y)\, dx\, dy$
6. $\int_0^a \int_0^x f(x, y)\, dx\, dy$

5.1.3 Double Integrals in Polar Co-ordinates

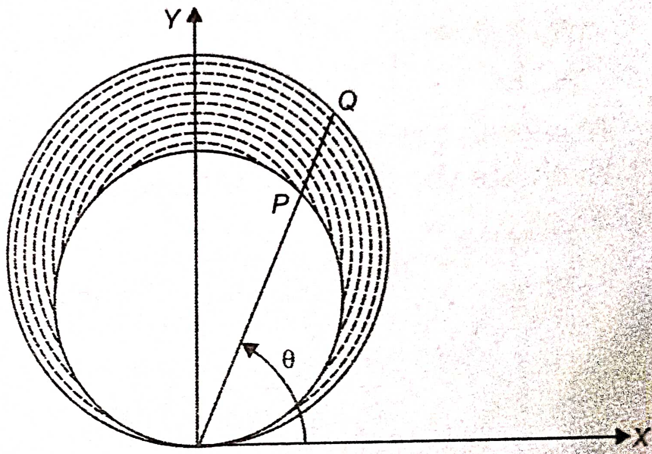
To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta)\, dr\, d\theta$, we first integrate w.r.t. r keeping θ as constants and then the resulting expression w.r.t. θ. In this integral $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ and θ_1, θ_2 are constants.

Thus the whole region of integration is the area ABCD as shown in figure. The order of integration may be changed with appropriate changes in the limits.



Example 5.13

Calculate $\iint r^3 \, dr \, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$. **(J.N.T.U., 1999)**



Solution: Given circles

$$r = 2\sin\theta$$

and $$r = 4\sin$$

are shown as in the figure

The shaded area between these circles is the region of integration.

If we integrate first w.r.t. r, then its limits are from P $(r = 2\sin\theta)$ to θ $(r = 4\sin\theta)$ and to cover the whole region varies from o to π. Thus, the required integral is

$$\int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 dr = \int_0^{\pi} d\theta \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta}$$

$$60\int_0^{\pi} \sin^4\theta \, d\theta = 60\times 2\int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$120\times\frac{3.1}{4.2}\cdot\frac{\pi}{2} = 22.5\,\pi$$

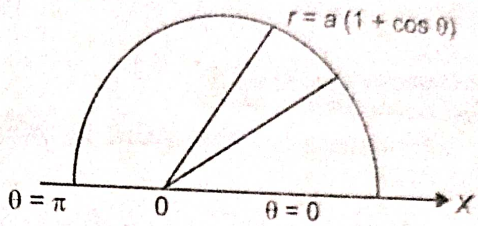
Example 5.14

... integral $\int_0^a \int^{\sqrt{(a^2-x^2)}} y^2 \sqrt{(x^2+y^2)}\, dx\, dy$ by changing to polar co-

Example 5.15

Evaluate $\iint_R r \sin\theta \, dr \, d\theta$ over the area of the cardioid $r = a(1 + \cos\theta)$ above the initial line. **(M.D.U., 2000, 2004)**

Solution: The domain of integration R is covered by radial strips whose ends are $r = 0$ and $r = a(1 + \cos\theta)$, the strips starting from $\theta = 0$ and ending at $\theta = \pi$, as shown.



$$\therefore \quad \iint_R r\sin\theta \, dr\, d\theta = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta \, dr\, d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta \,.\, a^2(1+\cos\theta)^2 \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} 2.\sin\theta/2.\cos\theta/2.\left(2.\cos^2\theta/2\right)^2 d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} 8.\sin\theta/2.\cos^5\theta/2 \, d\theta$$

$$= 4a^2\int_0^{\pi/2} 2\sin\phi\cos^5\phi \, d\phi \qquad \left\{\text{As } \frac{\theta}{2} = \phi \text{ and } d\theta = 2\,d\phi\right\}$$

$$= -8a^2\int_0^{\pi/2} \cos^5\phi\,(-\sin\phi)\, d\phi$$

$$= -8a^2\left[\frac{\cos^6\phi}{6}\right]^{\pi/2} = \frac{4a^2}{3} \text{ s}$$

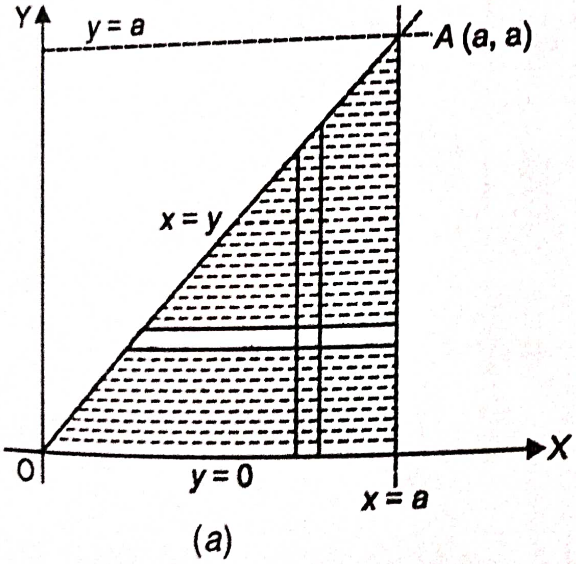
Example 5.18

Evaluate $\int_0^a \int_y^a \frac{x\,dy\,dx}{x^2+y^2}$ by changing to polar coordinates.

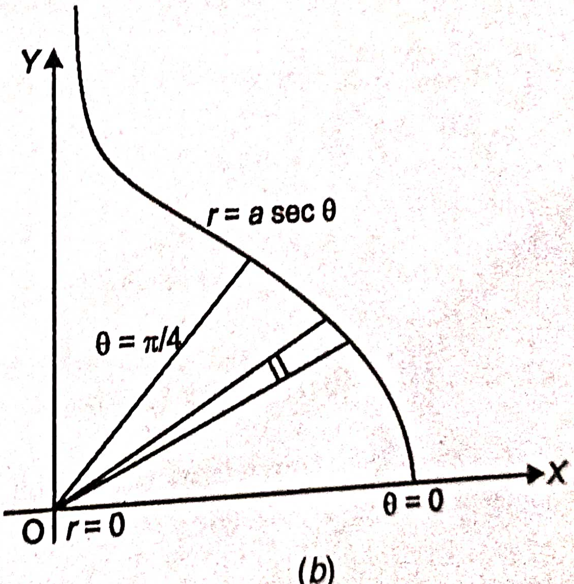
Solution: The domain of the double integral is the region enclosed by the curves $x = y$, $x = a$, $y = 0$ and $y = a$ as shown by shaded portion (a).

Now, to transform in polar coordinates, we put $x = r\cos\theta$; $y = r\sin\theta$, then we have as $x = y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \theta = \pi/4$ and as $y = 0$; $\theta = 0$. *i.e.*, θ varies from 0 to $\pi/4$.

Also r varies from $r = 0$ to $r = a\sec\theta$ as shown by (b).



(a)



(b)

Thus we have

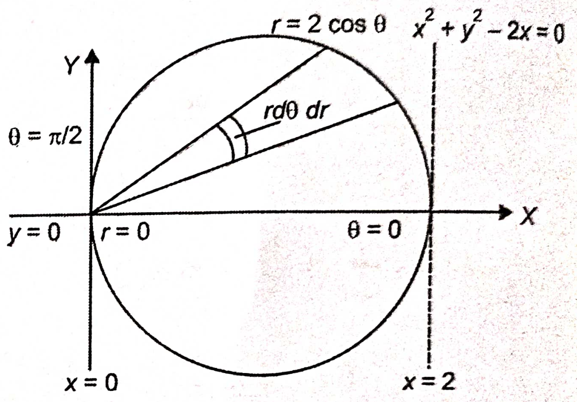
$$\int_0^a \int_y^a \frac{x\,dy\,dx}{x^2+y^2} = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r\cos\theta . r\,dr\,d\theta}{r^2}$$

$$\int_0^{\pi/4} \{\cos\theta\}_0^{a\sec\theta}\, d\theta = \int_0^{\pi/4} a\,d\theta = \frac{a\,\pi}{4}$$

Example 5.17

Evaluate by changing to polar coordinates the following integral

$$\int_0^2 \int_0^{\sqrt{(2x-x^2)}} \left(x^2+y^2\right) dy\, dx$$



Solution: The domain of the given double integral is the region bounded by the curves $x = 0$, $x = 2$, $y = 0$ and $x^2 + y^2 - 2x = 0$ as shown, where, the centre of the circle is (1, 0) and radius = 1.

Let us convert the double integral into polar coordinates by putting.

$$x = r\cos\theta,\ y = r\sin\theta$$

$\therefore \qquad r^2 - 2r\cos\theta = 0$

$\Rightarrow \qquad r = 2\cos\theta$

limits of r are 0 to $2\cos\theta$

limits of θ are 0 to $\pi/2$

Hence we have,

$$\int_0^2 \int_0^{\sqrt{(2x-x^2)}} \left(x^2+y^2\right) dy\, dx = \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\,(r d\theta\, dr)$$

$$\int_0^{\pi/2} \left\{\frac{r^4}{4}\right\}_0^{2\cos\theta} d\theta = 4\int_0^{\pi/2} \cos^4\theta\, d\theta = 4.\frac{3.1.\pi}{4.2.2} = \frac{3\pi}{4}$$

5.1.4 Change of Variables

In the article polar forms of double integral, we have seen that cartesian coordinates being transformed in polar coordinates. Here we shall examine a more general case of the change of variables.

Let the variables in the double integral

$$\iint_A f(x, y)\, dx\, dy \qquad \text{...(1)}$$

be changed from x, y to u, v, where

$$x = \phi(u, v),\; y = \psi(u, v) \qquad \text{...(2)}$$

where $\phi(u, v)$ and $\psi(u, v)$ are continuous and have continuous first order derivatives in some region $A'uv$ in the uv-plant which corresponds to the region Axy in the xy-plane.

Then $\displaystyle\iint_{Axy} f(x, y)\, dx\, dy = \iint_{A'uv} f[\phi(u, v), \varphi(u, v)] \times |J|\, du\, dv$

where $J = \dfrac{\partial(x, y)}{\partial(u, v)} (\neq 0)$ is the Jacobian of Transformation from (x, y) to (u, v) coordinates.

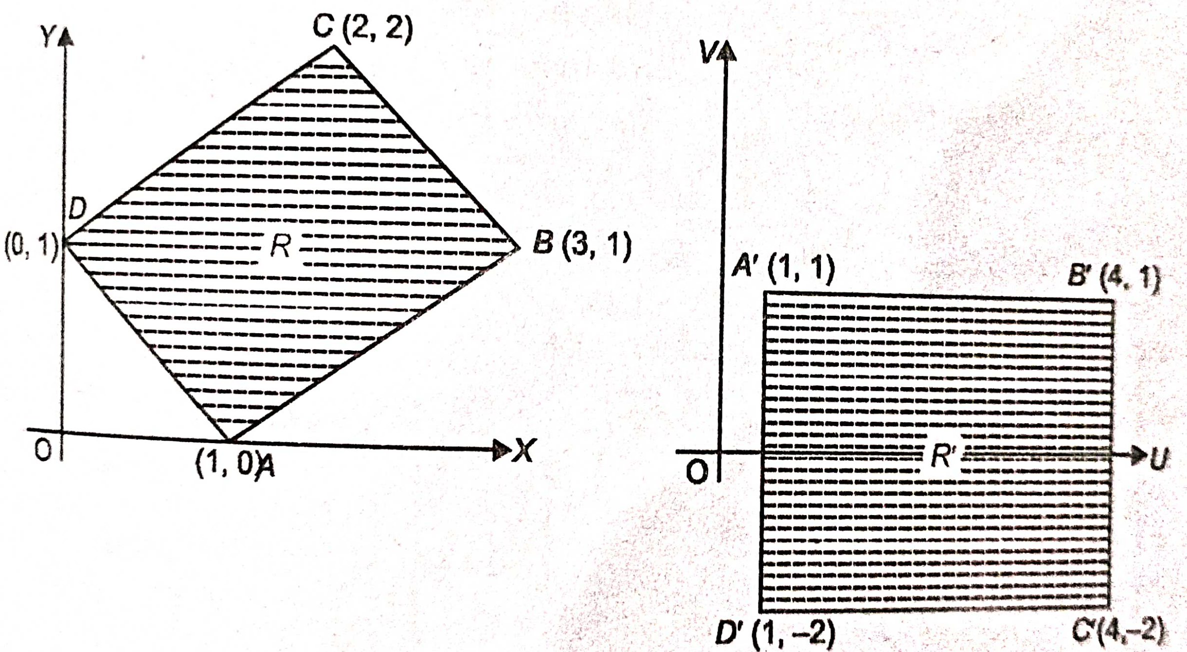
To change (x, y) to (r, θ) i.e., from cartesian system to polar system, we have $x = r\cos\theta$, $y = r\sin\theta$ and $I = r$.

$$\iint_{Axy} f(x, y)\, dx\, dy = \iint_{A'uv} f(r\cos\theta, r\sin\theta)\, r dr\, d\theta$$

Example 5.20

Evaluate $\iint_R (x+y)^2\, dx\, dy$, where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation $u = x + y$ and $v = x - 2y$.

Solution: Here the region R in the xy-plane becomes the region R' in the uv-plane as shown,



As we are given that

$$u = x + y \text{ and } v = x - 2y \qquad \ldots(1)$$

then from (1), we have

$$x = \frac{1}{3}(2u + v), y = \frac{1}{3}(u, v)$$

$\therefore$
$$J(x, y) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = -\frac{1}{3}$$

Hence the given integral

$$= \iint_{R'} u^2 |J| \, du \, dv$$

$$= \int_1^4 \int_{-2}^1 u^2 \cdot \frac{1}{3} du \, dv = \frac{1}{3} \left| \frac{u^3}{3} \right|_1^4 \cdot |v|_{-2}^1 = 21$$

Example 5.21

Find the mass of a plate bounded by the four parabolas $y^2 = 4\,ax$, $y^2 = 4bx$, $x^2 = 4cy$, $x^2 = 4\,dy$, if the density $\rho = kxy$.

Solution: As we know that the mass of the plate $= \iint_A k\,xy\,dx\,dy$, where A is the region bounded by the parabolas. Put

$$u = x^2/y, \; v = y^2/x \qquad ...(1)$$

then in the uv-plane the four parabolas become

$$v = 4a, \; v = 4b, \; u = 4c, \; u = 4d \qquad ...(2)$$

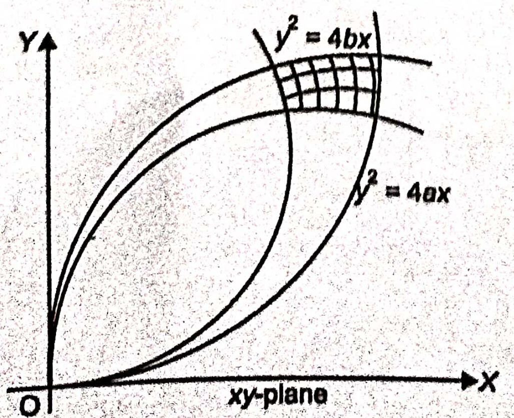
Now, from (1) we have

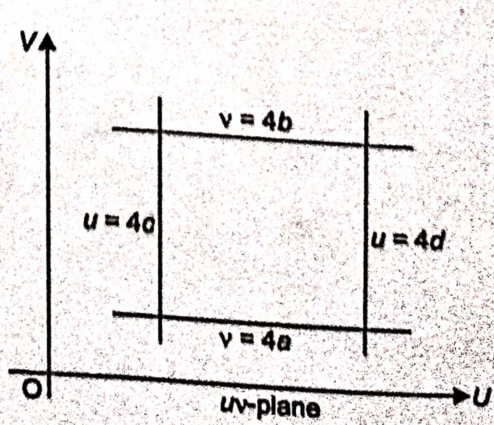
$$x = u^{2/3}.v^{1/3}, \; y = u^{1/3}\, v^{2/3}$$

Therefore

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} u^{-1/3}\, v^{1/3} & 1/3\, u^{-2/3}\, v^{2/3} \\ \frac{1}{3} u^{2/3}\, v^{-2/3} & 1/3\, u^{1/3}\, v^{-1/3} \end{vmatrix}$$

$$= \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$$





Also $xy = u^{2/3}\, v^{1/3}\,.\, u^{1/3}\, v^{2/3} = uv.$

Hence the mass of the plate

$$\iint_A kuv\,.\frac{1}{3}\,du\,dv = \int_{4c}^{4d}\int_{4a}^{4b}\frac{1}{3}\,kuv\,du\,dv \quad \text{using (2),}$$

$$\frac{1}{3}k\int_{4c}^{4d} u\left[\frac{1}{2}v^2\right]_{4a}^{4b} du = \frac{1}{3}k\int_{4c}^{4d} 8\left(b^2-a^2\right)u\,du$$

$$\frac{8}{3}k\left(b^2-a^2\right)\left[\frac{1}{2}u^2\right]_{4c}^{4d} = \frac{64}{3}k\left(b^2-a^2\right)\left(d^2-c^2\right)$$

Example 5.22

Using the transformation $x + y = u$, $y = uv$ show that

$$\int_0^1\int_0^{1-x} e^{y/(x+y)}\,dy\,dx = \frac{1}{2}(e-1)$$

Solution: Let $\quad x + y = u \quad$ ...(1)

$y = uv \quad$ ...(2)

Then from (1) and (2), we have

$$x = u\,(1-v),\ y = uv$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u\end{vmatrix} = u - uv + uv = u$$

Also $\quad dx\,dy = |J|\,du\,dv = u\,du + dv$

$x = 0 \Rightarrow u\,(1-v) = 0$

and $\quad \Rightarrow 0,\ v = 1$

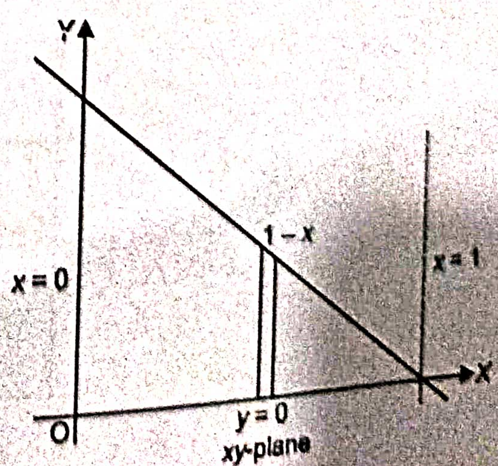
$y = 0 \Rightarrow uv = 0$

Again $\quad \Rightarrow u = 0,\ v = 0$

$x + y = u \Rightarrow x + y = u = 1$ as

$x + 1 - x \Rightarrow 1$

The region of integration of the double integral is shown in the figure.



Now, the above region of integration is being transformed into $u = 0$ to 1 and $v = 0$ to 1, by the given transformation and hence the new region is shown as in the figure.

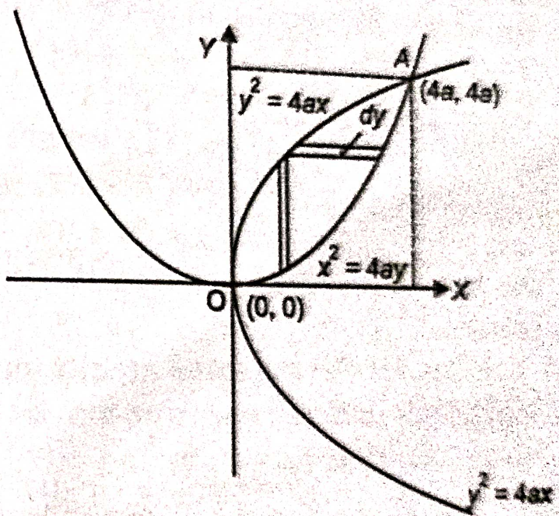
Therefore the area of integration is O′PQR in the uv-plane.

∴ The required value

$$= \int_0^1\int_0^1 e^{uv/u}\,.|J|\,.\,du\,dv$$

Example 5.27

Show by double integration that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\dfrac{16a^2}{3}$.



Solution: Now solving the two parabolas, we see that their point of intersections are at (0, 0) and $(4a, 4a)$ and the area enclosed is shown.

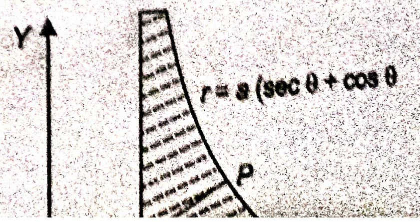
Hence the required area is

$$= \int_0^{4a} \int_{x^2/4a}^{2\sqrt{(ax)}} dy\, dx$$

$$= \int_0^{4a} \left(2\sqrt{(ax)} - \frac{x^2}{4a}\right) dx$$

$$= \left\{2\sqrt{a}\,.\,\frac{2}{3}x^{3/2} - \frac{1}{4a}\cdot\frac{x^3}{3}\right\}_0^{4a}$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16a^2}{3}$$

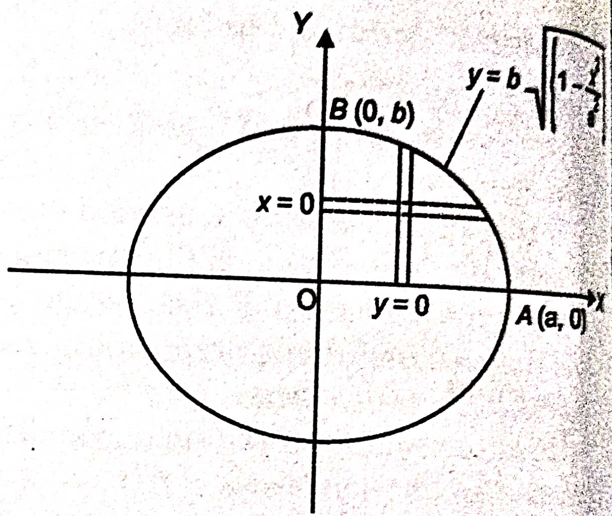


Example 5.28

Example 5.25

Find the are of a plate in the form of a quadrant of the ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



Solution: The area of a plate in the form of a quadrant of the given ellipse is shown.

Now, to find the are of a plate, first we integrate, here, with respect to y *i.e.*, along the horizontal strip of width δy, which slides from

$y = 0$ to $y = b\sqrt{\left(1-\frac{x^2}{a^2}\right)}$ and then along the vertical strip of width δx. We have:

$$= \int_0^a \int_0^{b\sqrt{(1-x^2/a^2)}} dy\, dx = \int_0^a \{y\}_0^{b\sqrt{(1-x^2/a^2)}} dx$$

$$= \frac{b}{a}\int_0^a \sqrt{\left(a^2-x^2\right)}\, dx = \frac{\pi ab}{4}$$

Otherwise, first we integrate along the vertical strip of width δx which slides from $x =$ 0 to $x = a\sqrt{\{1-y^2/b^2\}}$ and then along the horizontal strip of width δy, which slides from $y = 0$ to $y = b$, we have

$$\int_0^b \int_0^{a\sqrt{(1-y^2/a^2)}} dx\, dy = \frac{\pi ab}{4}$$

Note: The change of order of integration never affect the value of the area.

Example 5.26