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[UNIT-3]

[Laplace Transformation]

It reduces an ordinary differential eq^n into an algebric eq^n.

Let $f(t)$ be a function of t defined for all $t \geq 0$. Then the laplace transform of $f(t)$ denoted by $L\{f(t)\}$, is defined by –

$$\boxed{L\{f(t)\} = f(p) = \int_0^\infty e^{-pt} f(t)\, dt}$$

* Linearity property –

$$\boxed{L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{f(t)\} + c_2 L\{g(t)\}}$$

→ Laplace transform of some elementary functions :-

① $L(1) = \frac{1}{p}$, $p > 0$

$$L\{1\} = \int_0^\infty e^{-pt} \cdot 1\, dt = \left[\frac{-e^{-pt}}{p}\right]_0^\infty = \frac{1}{p}$$

Similarly,

* $L\{t^n\} = \dfrac{n!}{p^{n+1}}$, $[n = +ve\ int]$

* $L\{e^{at}\} = \dfrac{1}{p-a}$, $(p > a)$.

* $L\{\cos at\} = \dfrac{p}{p^2+a^2}$, $(p > 0)$

* $L\{\sin at\} = \dfrac{a}{p^2+a^2}$, $(p > 0)$

* $L\{\sinh at\} = \dfrac{a}{p^2-a^2}$, $[p > |a|]$

* $L\{\cosh at\} = \dfrac{p}{p^2-a^2}$, $[p > |a|]$

→ Transforms of discontinous function

* First translation property or first shifting property –

* If $L\{F(t)\} = f(p)$ then

• $L\{e^{at} F(t)\} = f(p-a)$.

• $L\{e^{at} F(bt)\} = \dfrac{1}{b} f\left(\dfrac{p-a}{b}\right)$.

* **Second translation property –**

* If $L\{F(t)\} = f(p)$

~~then~~ and
$$G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$$

then

$$\boxed{L\{G(t)\} = e^{-ap} f(p)}$$

→ **Change of scale property –**

If $L\{F(t)\} = f(p)$ then

$$L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right).$$

* Find the laplace transform of $7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t + 2.$

* $L\{7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t + 2\}$

$$= 7L\{e^{2t}\} + 9L\{e^{-2t}\} + 5L\{\cos t\} + 7L\{t^3\}$$
$$+ 5L\{\sin 3t\} + 2L\{1\}$$

$$= 7\cdot\frac{1}{p-2} + 9\cdot\frac{1}{p+2} + 5\cdot\frac{p}{p^2+1} + \frac{7\cdot 3!}{p^4} + \frac{5\cdot 3}{p^2+9} + \frac{2\cdot 1}{p}$$

$$= \frac{7}{p-2} + \frac{9}{p+2} + \frac{5p}{p^2+1} + \frac{42}{p^4} + \frac{15}{p^2+9} + \frac{2}{p}$$

* Find the Laplace transforms of

(i) $\cosh^3 2t$

* $\cosh 6t = 4\cosh^3 2t - 3\cosh 2t$

$$\cosh^3 2t = \frac{1}{4}\{\cosh 6t + 3\cosh 2t\}$$

$$L\{\cosh^3 2t\} = \frac{1}{4}\left[L\{\cosh 6t\} + 3L\{\cosh 2t\}\right]$$

$$= \frac{1}{4}\left[\frac{p}{p^2-36} + 3 \cdot \frac{p}{p^2-4}\right]$$

$$L\{\cosh^3 2t\} = \frac{p(p^2-28)}{(p^2-36)(p^2-4)}$$

(ii) $(1+te^{-t})^3$

* $(1+te^{-t})^3 = 1 + t^3e^{-3t} + 3te^{-t} + 3t^2e^{-2t}$

$$L\{(1+te^{-t})^3\} = L\{1\} + L\{e^{-3t}t^3\} + 3L\{e^{-t}t\} + 3L\{e^{-2t}t^2\}$$

Using first shifting property–

$L\{t^3\} = \frac{3!}{p^4}$ then $L\{e^{-3t}t^3\} = \frac{3!}{(p+3)^4}$

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$$L\{(1+te^{-t})^3\} = \frac{1}{p} + \frac{6}{(p+3)^4} + \frac{3\cdot 1}{(p+1)^2} + \frac{3\cdot 2!}{(p+2)^3}$$

$$L\{(1+te^{-t})^3\} = \frac{1}{p} + \frac{6}{(p+3)^4} + \frac{3}{(p+1)^2} + \frac{6}{(p+2)^3}$$

Note :-

$$\boxed{L\{(t^n)\} = \frac{\Gamma n+1}{p^{n+1}} \quad ; \quad (n \neq +\text{ve int})}$$

* find $L\{F(t)\}$ if

$$F(t) = \begin{cases} \sin(t - \pi/3) & ; \ t > \pi/3 \\ 0 & ; \ t < \pi/3 \end{cases}$$

* Using second shifting property

$$L\{F(t)\} = e^{-p\pi/3} \, L(\sin t) = e^{-p\pi/3} \cdot \frac{1}{p^2+1}$$

* If $L\{(\cos^2 t)\} = \dfrac{p^2+2}{p(p^2+4)}$, find $L\{(\cos^2 at)\}$

* By change of scale property-

$$L\{(\cos^2 at)\} = \frac{1}{a}\left[\frac{(p/a)^2+2}{\frac{p}{a}\{(\frac{p}{a})^2+4\}}\right] = \left(\frac{p^2+2a^2}{p(p^2+4a^2)}\right)$$

* If $L\left\{\left(\frac{\sin t}{t}\right)\right\} = \tan^{-1}\left(\frac{1}{p}\right)$, find $L\left\{\left(\frac{\sin at}{t}\right)\right\}$.

* $L\left\{\left(\frac{\sin at}{at}\right)\right\} = \frac{1}{a}\tan^{-1}\left(\frac{\cancel{1}}{\cancel{p/a}}\right)$.

$$\frac{1}{a} L\left\{\left(\frac{\sin at}{t}\right)\right\} = \frac{1}{a}\tan^{-1}\left(a/p\right)$$

$$L\left\{\left(\frac{\sin at}{t}\right)\right\} = \tan^{-1}\left(a/p\right).$$

* Function of exponential order –

If a constant a and b exists, such that –

$$\lim_{t\to\infty} e^{-bt} f(t) \quad [\text{exists}]$$

or the value of limit is finite then the function f(t) is of exponential order

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* A function of class A—
A function which is sectionally (or piecewise) continous over every finite interval in the range $t \geq 0$ and is of exponential order as $t \to \infty$ is termed as a function of class A.

* Existence theoram—
If $F(t)$ is sectionally continous for $t \geq 0$ and is of exponential order b, then

$$L[F(t)] = f(p) \text{ exists for } \boxed{p > b}.$$

In other words, if $F(t)$ is a function of class A,

$$L[F(t)] \text{ exists.}$$

* Prove that t^n is of exponential order as $t \to \infty$.

* $$\lim_{t \to \infty} (e^{-bt} \cdot t^n) = \lim_{t \to \infty} \frac{t^n}{e^{bt}}$$

$$= \lim_{t \to \infty} \frac{n t^{n-1}}{b e^{bt}} \cdots =$$

$$= \lim_{t \to \infty} \frac{n!}{b^n e^{bt}} = 0 = \text{finite}$$, Hence exp. order

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* Show that e^{t^2} is not of exp order as $t \to \infty$.

* $\lim_{t \to \infty} (e^{-bt} \cdot e^{t^2}) = \lim_{t \to \infty} e^{t(t-b)}$

If $b \geq 0$, or $b < 0$ this limit is always infinite,

Hence e^{t^2} is not of exp. order as $t \to \infty$.

* Laplace Transform of derivatives.

* Theoram - 1 :- If $F(t)$ is cont for all $t \geq 0$ and is of exponential order b as $t \to \infty$, and if $F'(t)$ is of class A then laplace transform of the derivative $F'(t)$ exists when $p > b$ and.

$$L[F'(t)] = pL\{F(t)\} - F(0) = pf(t) - f$$

* Theoram2 :- If $f(t)$ is cont except $t = a$ $(a > 0)$ then

$$L\{F'(t)\} = pL\{F(t)\} - F(0) - e^{-pt}[F(a+0) - F(a-$$

where $F(a+0)$ and $F(a-0)$ are limits of F at $t = a$ as t approaches a from right and from left respectively.

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* Generalization -

$$L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1} F(0) - p^{n-2} F'(0)$$

$$\text{———} \quad p^0 F^{n-1}(0).$$

* Initial value theoram -

$$\lim_{t \to 0} F(t) = \lim_{p \to \infty} p\, L\{F(t)\}$$

* Final value theoram -

$$\lim_{t \to \infty} F(t) = \lim_{p \to 0} p\, L\{F(t)\}$$

* Laplace transform of integrals -

$$L\left\{\int_0^t F(t)\,dt\right\} = \frac{1}{p} L\{F(t)\}.$$

* Multiplication by t^n.

$$L\{t^n \cdot F(t)\} = (-1)^n \frac{d^n}{dp^n}[f(p)]$$

where $f(p) = L\{F(p)\}$.

and $n = 1, 2, 3, \text{———}$

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* Division by t —

If $L\{F(p)\} = f(p)$

then

$$L\left\{\frac{1}{t}F(p)\right\} = \int_p^\infty f(p)\,dp.$$

* If $L\{t \sin \omega t\} = \dfrac{2\omega p}{(p^2+\omega^2)^2}$, evaluate

(i) $L(t\omega \cos \omega t + \sin \omega t)$

(ii) $L(2\cos \omega t - \omega t \sin \omega t)$.

* let,

$F(t) = t \sin \omega t \quad [F(0)=0]$

$F'(t) = \omega t \cos \omega t + \sin \omega t \quad [F'(0)=0]$

$F''(t) = 2\omega \cos \omega t - \omega^2 t \sin \omega t$

* $L\{F'(t)\} = p\{L\{F(t)\}\} - F(0)$

$L[\omega t \cos \omega t + \sin \omega t] = p \times \dfrac{2\omega p}{(p^2+\omega^2)^2} = \dfrac{2\omega p^2}{(p^2+\omega^2)^2}$

$L[F''(t)] = \omega L[2\cos \omega t - \omega t \sin \omega t] = p^2 \cancel{L}\{F(t)\} - F(0) - p^0\{F'(0)$

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$$\omega L[2\cos\omega t - \omega t\sin\omega t] = \frac{2\omega p^3}{(\omega^2+p^2)^2} - p\cdot 0 - 0$$

$$L[2\cos\omega t - \omega t\sin\omega t] = \frac{2p^3}{(p^2+\omega^2)^2}.$$

* Find Laplace transformation of

$$f(t) = \frac{e^{at} - \cos bt}{t}.$$

$$L\{e^{at}\} = \frac{1}{p-a}$$

$$L\{\cos bt\} = \frac{p}{p^2+b^2}$$

$$L\{e^{at} - \cos bt\} = \frac{1}{p-a} - \frac{p}{p^2+b^2}$$

$$L\left\{\frac{e^{at} - \cos bt}{t}\right\} = \int_p^\infty \left(\frac{1}{p-a} - \frac{p}{p^2+b^2}\right) dp$$

$$= \left[\log(p-a) - \frac{1}{2}\log(p^2+b^2)\right]_p^\infty$$

$$= \frac{1}{2}\left[\log(p-a)^2 - \log(p^2+b^2)\right]_p^\infty$$

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$$= \frac{1}{2}\left[\log \frac{(p-a)^2}{p^2+b^2}\right]_p^{\infty}$$

$$= \frac{1}{2}\left[\log \frac{\left(1-\frac{a}{p}\right)^2}{\left(1+\frac{b^2}{p^2}\right)}\right]_p^{\infty}$$

$$L\left\{\frac{e^{at}-\cos bt}{t}\right\} = -\frac{1}{2}\log\frac{(p-a)^2}{(p^2+b^2)}$$

* Find the laplace transform of $\frac{\sin at}{t}$. Does the laplace transform of $\frac{\cos at}{t}$ exist?

* Since $\lim_{t \to 0} \frac{\sin at}{t} = a$, the laplace transform of $\frac{\sin at}{t}$ exists.

$$L(\sin at) = \frac{a}{p^2+a^2}$$

$$L\left\{\frac{\sin at}{t}\right\} = \int_p^{\infty} \frac{a}{p^2+a^2}\,dp = \left[\tan^{-1}(p/a)\right]_p^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}(p/a).$$

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The function $\frac{\cos at}{t}$ is discontinuous at $t=0$, ∴ its laplace transform does not exist.

* Find the laplace transform of

(i) $t^3 e^{-3t}$

* $L(e^{-3t}) = \frac{1}{p+3}$

$$L(t^3 e^{-3t}) = (-1)^3 \frac{d^3}{dp^3}\left(\frac{1}{p+3}\right) = \frac{-(-1)^3 3!}{(p+3)^4}$$

$$= \frac{6}{(p+3)^4}.$$

* (ii) $\frac{1-\cos t}{t^2}$.

* $L[1-\cos t] = \frac{1}{p} - \frac{p}{p^2+1}$

$$L\left[\frac{1-\cos t}{t}\right] = \int_p^\infty \left(\frac{1}{p} - \frac{p}{p^2+1}\right) dp$$

$$= \left[\ln p - \frac{1}{2}\ln(p^2+1)\right]_p^\infty$$

$$= \frac{1}{2}\left[\ln \frac{p^2}{p^2+1}\right]_p^\infty = \left[\frac{1}{2}\ln \frac{1}{\left(1+\frac{1}{p^2}\right)}\right]_p^\infty$$

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$$L\left[\frac{1-\cos t}{t}\right] = \frac{1}{2}\ln\frac{p^2+1}{p^2}$$

$$L\left[\frac{1-\cos t}{t^2}\right] = \int_p^\infty \left[\frac{1}{2}\ln\frac{p^2+1}{p^2}\right]dp$$

$$= \frac{1}{2}\left[p\cdot\ln(p^2+1) - p\cdot 2\log p - \int\left(\frac{2p}{p^2+1} - \frac{2}{p}\right)\cdot p\,dp\right]$$

$$= \frac{1}{2}\left[p\ln\left(\frac{p^2+1}{p^2}\right)\right]_p^\infty + \int_p^\infty \frac{dp}{p^2+1}$$

$$\boxed{L\left[\frac{1-\cos t}{t^2}\right] = -\frac{p}{2}\ln\left(1+\frac{1}{p^2}\right) + \frac{\pi}{2} - \tan^{-1}p}$$

(iii) $t^2 e^{-2t}\cos t$.

* $L[e^{-2t}\cos t] = \frac{p+2}{(p+2)^2+1} = \frac{p+2}{p^2+4p+5}$

$$L[t^2 e^{-2t}\cos t] = (-1)^2 \frac{d^2}{dp^2}\left[\frac{p+2}{p^2+4p+5}\right]$$

$$= \frac{d}{dp}\left[\frac{(p^2+4p+5)\times 1 - (p+2)(2p+4)}{(p^2+4p+5)^2}\right]$$

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$$= \frac{d}{dp}\left[-\frac{p^2+4p+3}{(p^2+4p+5)^2}\right]$$

$$= \frac{2\,(p^3+6p^2+9p+2)}{(p^2+4p+5)^3}.$$

* Evaluate –

(i) $\int_0^\infty e^{-t}\, t^3 \sin t \, dt.$

$$L\{\sin t\} = \frac{1}{p^2+1}$$

$$L\{t^3 \sin t\} = (-1)^3 \frac{d^3}{dp^3}\left(\frac{1}{p^2+1}\right) = -1 \times \frac{d^2}{dp^2}\left[\frac{-2p}{(p^2+1)^2}\right]$$

$$= \frac{d}{dp}\left[\frac{2(1-3p^2)}{(p^2+1)^3}\right]$$

$$L\{t^3 \sin t\} = \frac{24p\,(p^2-1)}{(p^2+1)^4}$$

$$\because L\{t^3 \sin t\} = \int_0^\infty e^{-pt}\, t^3 \sin t \, dt$$

$$\therefore \int_0^\infty e^{-pt}\, t^3 \sin t \, dt = \frac{24p\,(p^2-1)}{(p^2+1)^4}$$

putting $p=1$ $\boxed{\int_0^\infty e^{-t}\, t^3 \sin t \, dt = 0}$

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(ii) $\displaystyle\int_0^\infty e^{-t} \frac{\sin^2 t}{t} \, dt$

* $L\{\sin^2 t\} = L\left\{\dfrac{1-\cos 2t}{2}\right\}$

$$= \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right]$$

$$L\left\{\frac{\sin^2 t}{t}\right\} = \int_p^\infty \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right] dp$$

$$= \frac{1}{2}\left[\ln p - \frac{1}{2}\ln(p^2+4)\right]_p^\infty$$

$$= \frac{1}{2}\times\frac{1}{2}\left[\ln \frac{p^2}{p^2+4}\right]_p^\infty = \frac{1}{4}\left[\ln \frac{1}{(1+4/p^2)}\right]_p^\infty$$

$$L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4}\ln\frac{p^2+4}{p^2} = \int_0^\infty e^{-pt}\frac{\sin^2 t}{t}\,dt$$

putting $p = 1$

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t} = \frac{1}{4}\ln 5 .$$

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* Prove that

$$\int_{t=0}^{\infty}\int_{u=0}^{t} e^{-t}\,\frac{\sin u}{u}\,du\,dt = \pi/4.$$

* $L\{\sin u\} = \frac{1}{p^2+1}$.

$$L\left\{\frac{\sin u}{u}\right\} = \int_p^{\infty}\frac{1}{p^2+1}\,dp = \cancel{\tan^{-1}\left(\frac{1}{p}\right)}$$

$$= \frac{\pi}{2} - \tan^{-1}(p)$$

$$L\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{p}\,L\left\{\frac{\sin u}{u}\right\}$$

$$= \frac{1}{p}\left\{\frac{\pi}{2} - \tan^{-1} p\right\}$$

$$\int_0^{\infty} e^{-pt}\int_0^t \frac{\sin u}{u}\,du\,dt = \frac{1}{p}\left\{\frac{\pi}{2} - \tan^{-1} p\right\}$$

putting $p=1$

$$\int_0^{\infty} e^{-t}\int_0^t \frac{\sin u}{u}\,du\,dt = \frac{\pi}{2} - \pi/4 = \pi/4.$$

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* Unit step function (or Heaviside's Unit step function):

→ Express the following functions in terms of Heaviside's unit step function:-

(i)
$$F(t) = \begin{cases} \sin t , & 0 < t < \pi \\ \sin 2t , & \pi < t < 2\pi \\ \sin 3t , & t > 2\pi \end{cases}$$

* $F(t) = \sin t \{u(t) - u(t-\pi)\} + \sin 2t \{u(t-\pi) - u(t-2\pi)\} + \sin 3t \{u(t-2\pi)\}$

$F(t) = \sin t\, u(t) + (\sin 2t - \sin t)\, u(t-\pi) + (\sin 3t - \sin 2t)\, u(t-2\pi)$

(ii)
$$F(t) = \begin{cases} e^{-t} , & 0 < t < 3 \\ 0 , & t > 3 \end{cases}$$

* $F(t) = e^{-t} \{u(t) - u(t-3)\} + 0 \{u(t-3)\}$

$F(t) = e^{-t} \{u(t) - u(t-3)\}$

(iii)
$$F(t) = \begin{cases} \sin t , & t > \pi \\ \cos t , & 0 < t < \pi \end{cases}$$

* $F(t) = \sin t \{u(t-\pi)\} + \cos t \{u(t) - u(t-\pi)\}$

$= \cos t\, u(t) + (\sin t - \cos t)\, u(t-\pi)$

* Find the laplace transformation of the following.

(i)
$$f(t) = \begin{cases} 2, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \\ \sin t, & t > 2\pi \end{cases}$$

* $f(t) = 2\{u(t) - u(t-\pi)\} + \sin t\{u(t-2\pi)\}$

$= 2\{u(t) - u(t-\pi)\} + \sin(2\pi + t - 2\pi)u(t-2\pi)$

$= 2\{u(t) - u(t-\pi)\} + \sin(t-2\pi)u(t-2\pi)$

Taking laplace transform, we get

$$L\{f(t)\} = 2\left\{\frac{1}{p} - \frac{e^{-\pi p}}{p}\right\} + e^{-2\pi p} L\{\sin t\}$$

$$L\{f(t)\} = 2\left(\frac{1-e^{-\pi p}}{p}\right) + \frac{e^{-2\pi p}}{p^2+1}$$

(ii)
$$f(t) = \begin{cases} 1, & 0 < t < \pi \\ 2, & \pi < t < 2\pi \\ 3, & 2\pi < t < 3\pi \\ - & - \\ - & - \end{cases}$$

(stair case function)

* $f(t) = 1\{u(t) - u(t-\pi)\} + 2\{u(t-\pi) - u(t-2\pi)\}$

$+ 3\{u(t-2\pi) - u(t-3\pi)\} + \ldots$

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$$f(t) = u(t) + u(t-\pi) + u(t-2\pi) + u(t-3\pi) + \cdots$$

$$L\{f(t)\} = \frac{1}{p} + \frac{e^{-\pi p}}{p} + \frac{e^{-2\pi p}}{p} + \cdots$$

(iii)

$$f(t) = \begin{cases} 1, & 0<t<2 \\ 2, & 2<t<4 \\ 3, & 4<t<6 \\ 0, & 6<t \end{cases}$$

$$* \quad f(t) = 1\{u(t)-u(t-2)\} + 2\{u(t-2)-u(t-4)\} + 3\{u(t-4)-u(t-6)\}$$

$$= u(t) + u(t-2) + u(t-4) - 3u(t-6)$$

$$L\{f(t)\} = \frac{1}{p} + \frac{e^{-2p}}{p} + \frac{e^{-4p}}{p} - 3\frac{e^{-6p}}{p}$$

(iv)

$$f(t) = \begin{cases} 0, & 0<t<\pi/2 \\ \sin t, & t>\pi/2 \end{cases}$$

$$* \quad f(t) = \sin t\,\{u(t-\pi/2)\}$$

$$= \sin[\pi/2 + (t-\pi/2)]\,\{u(t-\pi/2)\}$$

$$f(t) = \cos(t-\pi/2)\cdot u(t-\pi/2)$$

$$L\{f(t)\} = \frac{p}{p^2+1}\cdot e^{-\pi/2\, p}$$

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* Find the laplace transform of -

(i) $(t-1)^2 \; u(t-1)$.

* Comparing $(t-1)^2 u(t-1)$ with $F(t-a)u(t-a)$ we have,

$$a = 1 \; , \; F(t) = t^2$$

$$\therefore \; f(p) = L\{F(t)\} = \frac{2}{p^3}$$

$$\therefore \; L\{(t-1)^2 u(t-1)\} = \frac{2e^{-p}}{p^3}.$$

(ii) $\sin t \cdot u(t-\pi)$.

* $$\sin[\pi + (t-\pi)] \cdot u(t-\pi)$$

$$= -\sin(t-\pi) \cdot u(t-\pi)$$

Here, $a = \pi$, $F(t) = -\sin(t)$

$$L\{F(t)\} = -\frac{1}{p^2+1}$$

$$\therefore \; L\{\sin t \cdot u(t-\pi)\} = -\frac{e^{-\pi p}}{p^2+1}$$

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* Periodic Functions –

If $f(t)$ is a periodic function with period T, i.e

$$f(T+t) = f(t)$$

then

$$L\{f(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)\,dt.$$

* Inverse laplace transform :-

1. Find –

(i) $$L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

* we know that –

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\frac{1}{a} L\{\sin at\} = \frac{1}{s^2+a^2}$$

$$L\left\{\frac{\sin at}{a}\right\} = \frac{1}{s^2+a^2}$$

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}.$$

(ii) $L\{\cos at\} = \frac{s}{s^2+a^2}$

$\therefore \quad L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at.$

(iii) $L\{\sinh at\} = \frac{a}{s^2-a^2}$

$\therefore \quad L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}$

(iv) $L\{\cosh at\} = \frac{s}{s^2-a^2}$

$\therefore \quad L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at.$

(v) $L^{-1}\left\{\frac{1}{s}\right\} = 1.$

(vi) $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = ?$

* $L\{t^n\} = \frac{n!}{s^{n+1}} = \frac{\Gamma n+1}{s^{n+1}}$

$\therefore \quad L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma n+1} = \frac{t^n}{n!}$

(vii) $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

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* Find the inverse laplace transform of the following :-

(i) $\frac{5}{s^2} + \left(\frac{\sqrt{s}-1}{s}\right)^2 - \frac{7}{3s+2}$

$$* \Rightarrow L^{-1}\left\{ \frac{5}{s^2} + \frac{1}{s} - \frac{2}{s^{3/2}} + \frac{1}{s^2} - \frac{7}{3}\left(\frac{1}{s+2/3}\right) \right.$$

$$\Rightarrow L^{-1}\left\{\frac{6}{s^2}\right\} + L^{-1}\left\{\frac{1}{s}\right\} - 2L^{-1}\left\{\frac{1}{s^{3/2}}\right\}$$

$$- \frac{7}{3} L^{-1}\left\{\frac{1}{s+2/3}\right\}$$

$$\Rightarrow 6 \cdot \frac{t}{\Gamma 2} + 1 - 2 \cdot \frac{t^{1/2}}{\Gamma 3/2} - \frac{7}{3} e^{-2/3 t}$$

$$= 6t + 1 - 4\sqrt{\frac{t}{\pi}} - \frac{7}{3} e^{-2/3 t}$$

(ii) $f(s) = \frac{3}{s^2-3} + \frac{3s+2}{s^2} - \frac{3s-27}{s^2+9} + \frac{6-30\sqrt{s}}{s^4}$

$$* \quad f(t) = L^{-1}\left\{ \frac{3}{s^2-3} + \frac{3}{s} + \frac{2}{s^2} - 3 \cdot \frac{s}{s^2+9} + \frac{27}{s^2+9} + \frac{6}{s^4} - \frac{30}{s^{7/2}} \right\}$$

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$$F(t) = \sqrt{3}\,\sinh\sqrt{3}t + 3 + 2\cdot\frac{t}{\Gamma 2} - 3\cdot\cos 3t$$

$$+ 9\cdot\sin 3t + 6\cdot\frac{t^3}{3!}$$

$$- 30\cdot\frac{t^{5/2}}{\Gamma 7/2}$$

$$F(t) = \sqrt{3}\,\sinh\sqrt{3}t + 3 + 2t - 3\cos 3t$$

$$+ 9\sin 3t + t^3 - 16t^2\sqrt{\frac{t}{\pi}}.$$

* First Translation or shifting property-

If $L^{-1}\{f(s)\} = F(t)$ then

$\therefore L^{-1}\{f(s-a)\} = e^{at}\cdot L^{-1}\{f(s)\}$

* Second translation or shifting property-

If $L^{-1}\{f(s)\} = F(t)$ and

$$G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

then $L^{-1}\{e^{-as}\cdot f(s)\} = G(t).$

$= F(t-a)\cdot u(t-a).$

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* change of scale property –

If $L^{-1}\{f(s)\} = F(t)$, then

$$L^{-1}\{f(as)\} = \frac{1}{a} F\left(\frac{t}{a}\right).$$

→ Find the inverse laplace transform of the following :–

(i) $$\frac{1}{s^2-6s+10}$$

* $$L^{-1}\left\{\frac{1}{s^2-6s+10}\right\} = L^{-1}\left\{\frac{1}{(s-3)^2+1}\right\} = e^{3t}\cdot \sin t$$

{By 1st shifting property}

(ii) $$\frac{s}{(s+1)^{3/2}}.$$

* $$L^{-1}\left\{\frac{s}{(s+1)^{3/2}}\right\} = L^{-1}\left\{\frac{(s+1)-1}{(s+1)^{3/2}}\right\} = e^{-t}\cdot L^{-1}\left\{\frac{s-1}{(s)^{3/2}}\right\}$$

$$= e^{-t}\left[L^{-1}\left\{\frac{1}{s^{1/2}} - \frac{1}{s^{3/2}}\right\}\right]$$

$$= e^{-t}\left[\frac{t^{1/2-1}}{\Gamma\frac{1}{2}} - \frac{t^{3/2-1}}{\Gamma\frac{3}{2}}\right\}$$

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$$L^{-1}\left\{\frac{s}{(s+1)^{3/2}}\right\} = e^{-t}\sqrt{\frac{1}{t\pi}} - 2e^{-t}\sqrt{\frac{t}{\pi}}$$