2.5.3. Case III. When Roots are Distinct, Differ by Integer and Making a Coefficient of y Infinite

Let m_1 and m_2 be the roots such that $m_1 > m_2$.

In this case, if some of the coefficients of y become infinite when $m = m_2$, we modify the form of y by replacing a_0 by $b_0\,(m - m_2)$.

Complete solution is

$$\boxed{y = c_1\,(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_2}.}$$

Remark. We can also obtain two independent solutions by putting $m = m_2$ (value of m for which some coefficients of y become infinite) in modified form of y and $\frac{\partial y}{\partial m}$. The result of putting $m = m_1$ in y will give a numerical multiple of that obtained by putting $m = m_2$.

ILLUSTRATIVE EXAMPLES

Example 1. *Obtain the series solution of the Bessel's equation of order two*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 4)\,y = 0 \quad \textit{near } x = 0.$$

Sol. Comparing the given equation with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \text{ we get}$$

$$P(x) = \frac{1}{x} \quad \text{and} \quad Q(x) = \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2}$$

At $x = 0$, both $P(x)$ and $Q(x)$ are not analytic.

Therefore $x = 0$ is a *singular point.*

Also, $x\,P(x) = 1$ and $x^2\,Q(x) = x^2 - 4$

Both $x\,P(x)$ and $x^2\,Q(x)$ are analytic at $x = 0$

$\therefore$ $x = 0$ is a *regular singular point.*

Let us assume,

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots \qquad \ldots(1)$$

Then, $$\frac{dy}{dx} = m\,a_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + (m+3)\,a_3 x^{m+2} + \ldots$$

and $$\frac{d^2y}{dx^2} = m(m-1)\,a_0\,x^{m-2} + (m+1)\,ma_1\,x^{m-1} + (m+2)(m+1)\,a_2\,x^m + (m+3)(m+2)\,a_3 x^{m+1} + \ldots$$

Substituting these values in the given equation, we get

$$x^2\,[m(m-1)\,a_0\,x^{m-2} + (m+1)\,ma_1\,x^{m-1} + (m+2)(m+1)\,a_2\,x^m + (m+3)(m+2)\,a_3\,x^{m+1} + \ldots]$$
$$+ x\,[m\,a_0\,x^{m-1} + (m+1)\,a_1\,x^m + (m+2)\,a_2\,x^{m+1} + (m+3)\,a_3\,x^{m+2} + \ldots]$$
$$+ (x^2 - 4)\,[a_0\,x^m + a_0\,x^{m+1} + a_2\,x^{m+2} + a_3\,x^{m+3} + \ldots] = 0$$

Now, coefficient of lowest power of $x = 0$

$\Rightarrow$ Coefficient of $x^m = 0$

$\Rightarrow$ $m(m-1)a_0 + m\,a_0 - 4a_0 = 0$ $\Rightarrow$ $(m^2 - 4)\,a_0 = 0$

$\Rightarrow$ $m^2 - 4 = 0$ *(Indicial equation)* $\quad | \because \; a_0 \neq 0$

$$\boxed{m = -2, 2}$$

Roots are distinct and differ by integer.

Now, coefficient of $x^{m+1} = 0$

$(m+1)m\,a_1 + (m+1)\,a_1 - 4a_1 = 0$

$\Rightarrow$ $(m^2 + 2m - 3)\,a_1 = 0$ $\Rightarrow$ $(m+3)(m-1)\,a_1 = 0$

$\Rightarrow$ $\boxed{a_1 = 0}$ $\quad$ | Since $m \neq 1$, and $m \neq -3$

Coefficient of $x^{m+2} = 0$

$\Rightarrow$ $(m+2)(m+1)\,a_2 + (m+2)\,a_2 + a_0 - 4a_2 = 0$

$\Rightarrow$ $(m^2 + 4m)\,a_2 + a_0 = 0$

$\Rightarrow$ $$\boxed{a_2 = \frac{-a_0}{m(m+4)}}$$

Coefficient of $x^{m+3} = 0$

$\Rightarrow$ $(m+3)(m+2)\,a_3 + (m+3)\,a_3 + a_1 - 4a_3 = 0$

$\Rightarrow$ $(m+1)(m+5)\,a_3 = -a_1$

$$\Rightarrow \qquad \boxed{a_3 = 0} \qquad | \because a_1 = 0$$

Also, coefficient of $x^{m+4} = 0$

$$(m+2)(m+6)\, a_4 + a_2 = 0$$

$$\Rightarrow \qquad a_4 = \frac{-a_2}{(m+2)(m+6)} = \frac{a_0}{m(m+2)(m+4)(m+6)}$$

$$\therefore \qquad \boxed{a_4 = \frac{a_0}{m(m+2)(m+4)(m+6)}}$$

Similarly, $a_5 = a_7 = a_9 = \dots = 0$

$$a_6 = \frac{-a_0}{m(m+2)(m+4)^2(m+6)(m+8)} \text{ etc.}$$

Substituting above obtained values in assumed y given by eqn. (1), we get

$$y = a_0 x^m \left[1 - \frac{x^2}{m(m+4)} + \frac{x^4}{m(m+2)(m+4)(m+6)} - \frac{x^6}{m(m+2)(m+4)^2(m+6)(m+8)} + \dots\right] \quad \dots(2)$$

Putting $m = 2$ (the greater of the two roots) in (2), the first solution is

$$y_1 = a_0 x^2 \left(1 - \frac{x^2}{2.6} + \frac{x^4}{2.4.6.8} - \frac{x^6}{2.4.6^2.8.10} + \dots\right)$$

If we put $m = -2$ in (1), the coefficients become infinite due to the presence of the factor $(m+2)$ in the denominator. To overcome this difficulty, let $a_0 = b_0(m+2)$ so that

$$y = b_0 x^m \left[(m+2) - \frac{(m+2)x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} - \frac{x^6}{m(m+4)^2(m+6)(m+8)} + \dots\right]$$

Differentiating partially w.r.t. m, we get

$$\frac{\partial y}{\partial m} = b_0 x^m \log x \left[(m+2) - \frac{(m+2)x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} - \dots\right]$$

$$+ b_0 x^m \left[1 - \frac{(m+2)}{m(m+4)}\left\{\frac{1}{m+2} - \frac{1}{m} - \frac{1}{m+4}\right\} x^2 + \frac{1}{m(m+4)(m+6)}\left\{-\frac{1}{m} - \frac{1}{m+4} - \frac{1}{m+6}\right\} x^4 \dots\right]$$

The second solution is $y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=-2}$

$$= b_0 x^{-2} \log x \left[\frac{x^4}{(-2)(2)(4)} - \frac{x^6}{(-2)(2)^2(4)(6)} \dots\right]$$

$$+ b_0 x^{-2}\left[1 - \frac{x^2}{(-2)(2)} + \frac{1}{(-2)(2)(4)}\left(\frac{1}{2} - \frac{1}{2} - \frac{1}{4}\right) x^4 \dots\right]$$

$$= b_0 x^2 \log x \left[-\frac{1}{2^2.4} + \frac{x^2}{2^3.4.6} \ldots\right] + b_0 x^{-2}\left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \ldots\right]$$

Hence the complete solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= Ax^2\left[\left(1 - \frac{x^2}{2.6} + \frac{x^4}{2.4.6.8} - \frac{x^6}{2.4.6^2.8.10} + \ldots\right)\right] + B\left[x^2 \log x\left(-\frac{1}{2^2.4} + \frac{x^2}{2^3.4.6} \ldots\right)\right.$$

$$\left. + x^{-2}\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \ldots\right)\right]$$

where $A = c_1 a_0$, $B = c_2 b_0$.

Example 2. *Solve in series the differential equation* $x^2 \dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + x^2 y = 0$.

Sol. Comparing the given equation with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\, y = 0, \text{ we get}$$

$$P(x) = \frac{5}{x},\ Q(x) = 1$$

At $x = 0$, since $P(x)$ is not analytic $\therefore$ $x = 0$ is a *singular point*.

Also, $\quad x\,P(x) = 5$

$$x^2\,Q(x) = 0$$

Since both $x\,P(x)$ and $x^2\,Q(x)$ are analytic at $x = 0$ $\therefore$ $x = 0$ is a *regular singular point*.

Let us assume

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots \quad \ldots(1)$$

$$\therefore \quad \frac{dy}{dx} = m\,a_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + \ldots \quad \ldots(2)$$

and $$\frac{d^2y}{dx^2} = m(m-1)\,a_0 x^{m-2} + (m+1)\,m\,a_1 x^{m-1} + (m+2)(m+1)\,a_2 x^m + \ldots \quad \ldots(3)$$

Substituting the above values in given equation, we get

$$x^2\,[m(m-1)\,a_0 x^{m-2} + (m+1)\,m a_1 x^{m-1} + (m+2)(m+1)\,a_2 x^m + \ldots]$$
$$+ 5x\,[m a_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + \ldots]$$
$$+ x^2\,[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \ldots] = 0 \quad \ldots(4)$$

Equating the coefficient of lowest power of x to zero, we get [Coeff. of $x^m = 0$]

$$m(m-1)\,a_0 + 5m a_0 = 0$$

$$\Rightarrow \quad (m^2 + 4m)\,a_0 = 0$$

$$\Rightarrow \quad m(m+4) = 0 \quad \text{(Indicial equation)} \quad (\because\ a_0 \neq 0)$$

$$\Rightarrow \quad \boxed{m = 0, -4}$$

Hence *the roots are distinct and differing by an integer*. Equating to zero, the coefficients of successive powers of x, we get

$$\text{coefficient of } x^{m+1} = 0$$

$$(m+1)\, m\, a_1 + 5(m+1)\, a_1 = 0$$

$$\Rightarrow \quad (m+5)(m+1)a_1 = 0 \quad \Rightarrow \quad \boxed{a_1 = 0} \quad \ldots(5) \quad | \because m \neq -5, -1$$

$$\text{coefficient of } x^{m+2} = 0$$

$$(m+2)(m+1)\, a_2 + 5(m+2)a_2 + a_0 = 0$$

$$(m+2)(m+6)\, a_2 + a_0 = 0$$

$$\boxed{a_2 = \frac{-a_0}{(m+2)(m+6)}} \quad \ldots(6)$$

Again, $\quad$ coefficient of $x^{m+3} = 0$

$$(m+3)(m+2)\, a_3 + 5(m+3)\, a_3 + a_1 = 0$$

$$(m+3)(m+7)\, a_3 + a_1 = 0$$

$$\Rightarrow \quad a_3 = \frac{-a_1}{(m+3)(m+7)}$$

$$\Rightarrow \quad \boxed{a_3 = 0} \quad \ldots(7)$$

Similarly, $\quad a_5 = a_7 = a_9 = \ldots = 0$

Now, $\quad$ coefficient of $x^{m+4} = 0$

$$(m+4)(m+3)\, a_4 + 5(m+4)\, a_4 + a_2 = 0$$

$$\Rightarrow \quad (m+4)(m+8)\, a_4 = -a_2$$

$$a_4 = \frac{-a_2}{(m+4)(m+8)} = \frac{a_0}{(m+2)(m+4)(m+6)(m+8)} \text{ etc.} \quad \ldots(8)$$

These give $\quad y = a_0 x^m \left[1 - \dfrac{x^2}{(m+2)(m+6)} + \dfrac{x^4}{(m+2)(m+4)(m+6)(m+8)} - \ldots\right] \quad \ldots(9)$

Putting $\quad m = 0$ in (9), we get

$$y_1 = (y)_{m=0} = a_0 \left[1 - \frac{x^2}{2.6} + \frac{x^4}{2.4.6.8} - \ldots\right] \quad \ldots(10)$$

If we put $m = -4$ in the series given by eqn. (9), the coefficients become infinite. To avoid this difficulty, we put $a_0 = b_0\,(m+4)$, so that

$$y = b_0 x^m \left[(m+4) - \frac{(m+4)\,x^2}{(m+2)(m+6)} + \frac{x^4}{(m+2)(m+6)(m+8)} - \ldots\right] \quad \ldots(11)$$

Now, $\quad \dfrac{\partial y}{\partial m} = y \log x + b_0 x^m \left[1 + \dfrac{m^2 + 8m + 20}{(m^2 + 8m + 12)^2}\, x^2 - \dfrac{(3m^2 + 32m + 76)}{(m^3 + 16m^2 + 76m + 96)^2}\, x^4 + \ldots\right]$

Second solution is given by

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=-4} = (y)_{m=-4} \log x + b_0 x^{-4} \left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \ldots\right)$$

$$= b_0 x^{-4} \log x \left[0 - 0 + \frac{x^4}{(-2)(2)(4)} - \frac{x^6}{16} + \ldots\right] + b_0 x^{-4} \left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \ldots\right)$$

$$= b_0 x^{-4} \log x \left(\frac{-x^4}{16} - \frac{x^6}{16} - \dots \right) + b_0 x^{-4} \left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \dots \right)$$

Hence the complete solution is given by

$y = c_1 y_1 + c_2 y_2$

$$= c_1 a_0 \left(1 - \frac{x^2}{12} + \frac{x^4}{384} - \dots \right) + c_2 b_0 x^{-4} \log x \left(-\frac{x^4}{16} - \frac{x^6}{16} - \dots \right) + c_2 b_0 x^{-4} \left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \dots \right)$$

$$\therefore \; y = A \left(1 - \frac{x^2}{12} + \frac{x^4}{384} - \dots \right) + Bx^{-4} \left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \dots \right) - B \log x \left(\frac{1}{16} + \frac{x^2}{16} + \dots \right)$$

where $A = c_1 a_0$ and $B = c_2 b_0$.

TEST YOUR KNOWLEDGE

Solve in series:

1. $x(1-x)\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} - y = 0$

2. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2-1)y = 0$ [*M.T.U. (SUM) 2011*]
(*Bessel's equation of order one*)

3. $(x + x^2 + x^3)\dfrac{d^2y}{dx^2} + 3x^2\dfrac{dy}{dx} - 2y = 0$

4. $x(1-x)\dfrac{d^2y}{dx^2} - (1+3x)\dfrac{dy}{dx} + y = 0$.

Answers

1. $y = (A + B \log x)(x + 2x^2 + 3x^3 + 4x^4 + \dots) + B(1 + x + x^2 + x^3 + \dots)$

2. $y = Ax\left(1 - \dfrac{x^2}{2.4} + \dfrac{x^4}{2.4^2.6} - \dots\right) + Bx^{-1}\log x\left(-\dfrac{x^2}{2} + \dfrac{x^4}{2^2.4} - \dots\right) + Bx^{-1}\left[1 + \dfrac{x^2}{2^2} - \dfrac{3}{2^2 2^3}x^4 + \dots\right]$

3. $y = Ax\left[1 + x - \dfrac{1}{2}x^2 - \dfrac{1}{2}x^3 + \dots\right] + B\log x(2x + 2x^2 - x^3 + \dots) + B(1 - x - 5x^2 - x^3 + \dots)$

4. $y = (A + B\log x)(1.2\,x^2 + 2.3x^3 + 3.4x^4 + \dots) + B(-1 + x + 5x^2 + 11x^3 + \dots)$.

2.5.4. Case IV. When Roots are Distinct, Differ by Integer and Making One or More Coefficients Indeterminate

Let the roots be m_1 and m_2. If one of the coefficients (suppose a_1) become indeterminate when $m = m_2$, the *complete solution is given by putting* $m = m_2$ *in* y which then contains two arbitrary constants.

Note. The result contained by putting $m = m_1$ in y merely gives a numerical multiple of one of the series contained in the first solution. Hence we reject the solution obtained by putting $m = m_1$.

Example. *Solve in series the differential equation:* $xy'' + 2y' + xy = 0$.

Sol. Comparing the given equation with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \text{ we get}$$

$$P(x) = \frac{2}{x} \quad \text{and} \quad Q(x) = 1$$

At $x = 0$, $P(x)$ is not analytic $\therefore$ $x = 0$ is a *singular point*.

Also, $xP(x) = 2$ and $x^2 Q(x) = x^2$

At $x = 0$, since $x\,P(x)$ and $x^2\,Q(x)$ are analytic $\therefore$ $x = 0$ is a *regular singular point*.

Let us assume

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots \qquad \ldots(1)$$

Then $$\frac{dy}{dx} = m a_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + (m+3)\,a_3 x^{m+2} + \ldots$$

and $$\frac{d^2y}{dx^2} = m\,(m-1)\,a_0 x^{m-2} + (m+1)\,m\,a_1 x^{m-1} + (m+2)\,(m+1)\,a_2 x^m + (m+3)\,(m+2)\,a_3 x^{m+1} + \ldots$$

Substituting these values in the given equation, we get

$$x\,[m\,(m-1)\,a_0 x^{m-2} + (m+1)\,m\,a_1 x^{m-1} + (m+2)\,(m+1)\,a_2 x^m + (m+3)\,(m+2)\,a_3 x^{m+1} + \ldots]$$
$$+ 2\,[m\,a_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + \ldots]$$
$$+ x\,[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots] = 0$$

Now, coefficient of $x^{m-1} = 0$

$\Rightarrow$ $m\,(m-1)\,a_0 + 2m\,a_0 = 0$

$(m^2 + m)\,a_0 = 0$

$\Rightarrow$ $m^2 + m = 0$ (Indicial equation) $\quad [\because a_0 \neq 0$

$\Rightarrow$ $\boxed{m = 0, -1}$

Hence roots are distinct and differ by an integer.

coefficient of $x^m = 0$

$\Rightarrow$ $(m+1)\,m\,a_1 + 2(m+1)\,a_1 = 0$

$\Rightarrow$ $(m+1)\,(m+2)\,a_1 = 0$

$\Rightarrow$ $(m+1)\,a_1 = 0$ $\quad [\because m + 2 \neq 0$

Since $m + 1$ may be zero, hence a_1 *is arbitrary* (or takes the form $\frac{0}{0}$). In other words, a_1 becomes indeterminate.

Hence the solution will contain a_0 and a_1 as arbitrary constants. The complete solution will be given by putting $m = -1$ in y.

Now, coefficient of $x^{m+1} = 0$

$\Rightarrow$ $(m+2)\,(m+1)\,a_2 + 2(m+2)\,a_2 + a_0 = 0$

$\Rightarrow$ $(m+2)\,(m+3)\,a_2 + a_0 = 0$

$$\boxed{a_2 = \frac{-a_0}{(m+2)(m+3)}}$$

coefficient of $x^{m+2} = 0$

$\Rightarrow$ $(m+3)\,(m+2)\,a_3 + 2(m+3)\,a_3 + a_1 = 0$

$(m+3)\,(m+4)\,a_3 + a_1 = 0$

$$\boxed{a_3 = \frac{-a_1}{(m+3)(m+4)}}$$

coefficient of $x^{m+3} = 0$

$$\Rightarrow \quad (m+4)(m+3)\,a_4 + 2(m+4)\,a_4 + a_2 = 0$$

$$\Rightarrow \quad (m+4)(m+5)\,a_4 = -a_2$$

$$\Rightarrow \quad a_4 = \frac{-a_2}{(m+4)(m+5)}$$

$$\Rightarrow \quad \boxed{a_4 = \frac{a_0}{(m+2)(m+3)(m+4)(m+5)}}$$

coefficient of $x^{m+4} = 0$

$$(m+5)(m+4)\,a_5 + 2(m+5)\,a_5 + a_3 = 0$$

$$(m+5)(m+6)\,a_5 = -a_3$$

$$\boxed{a_5 = \frac{a_1}{(m+3)(m+4)(m+5)(m+6)}}$$

and so on.

Substituting these values in eqn. (1), we get

$$y = x^m\left[a_0 + a_1 x - \frac{a_0}{(m+2)(m+3)}x^2 - \frac{a_1}{(m+3)(m+4)}x^3 + \frac{a_0}{(m+2)(m+3)(m+4)(m+5)}x^4 + \frac{a_1}{(m+3)(m+4)(m+5)(m+6)}x^5 + \ldots\right]$$

$$y = x^m\left[a_0\left\{1 - \frac{x^2}{(m+2)(m+3)} + \frac{x^4}{(m+2)(m+3)(m+4)(m+5)} - \ldots\right\} + a_1\left\{x - \frac{x^3}{(m+3)(m+4)} + \frac{x^5}{(m+3)(m+4)(m+5)(m+6)} - \ldots\right\}\right]$$

Now, $$(y)_{m=-1} = x^{-1}\left[a_0\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots\right) + a_1\left(x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \ldots\right)\right]$$

$$= x^{-1}\,[a_0 \cos x + a_1 \sin x]$$

Hence complete solution is given by

$$y = (y)_{m=-1}$$

$$\Rightarrow \quad y = \frac{1}{x}\,(a_0 \cos x + a_1 \sin x).$$

Note. All those problems, in which $x = 0$ was an ordinary point of $y'' + P(x)\,y' + Q(x)\,y = 0$, can also be solved by Frobenius method as given in *Art.* 2.5.4 and explained in above illustrative example.

TEST YOUR KNOWLEDGE

Solve in series:

1. $x^2 \dfrac{d^2y}{dx^2} + 4x \dfrac{dy}{dx} + (x^2 + 2)\, y = 0$

2. $(1 - x^2) \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + 4y = 0$ (U.P.T.U. 2006)

3. $(1 - x^2) \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} + n(n+1)\, y = 0$.

Answers

1. $y = x^{-2} (a_0 \cos x + a_1 \sin x)$

2. $y = a_0 (1 - 2x^2) + a_1 \left(x - \dfrac{x^3}{2} - \dfrac{x^5}{8} + \dfrac{x^7}{16} - \ldots \right)$

3. $y = a_0 \left[1 - \dfrac{n(n+1)}{2!} x^2 + \dfrac{(n-2)\, n(n+1)(n+3)}{4!} x^4 - \ldots \right]$
$+ a_1 \left[x - \dfrac{(n-1)(n+2)}{3!} x^3 + \dfrac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 + \ldots \right]$.

2.6. LEGENDRE'S DIFFERENTIAL EQUATION

The differential equation $$(1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0 \qquad \ldots(1)$$

where n is real number, is called Legendre's differential equation. This equation is of considerable importance in applied mathematics, particularly in boundary value problems involving spherical configurations.

Though n is a real number, in most physical applications, only integral values of n are required. Also, equation (1) can be solved in series of ascending or descending powers of x. The solution in descending powers of x is more important than the one in ascending powers.

Let $$y = \sum_{k=0}^{\infty} a_k x^{m-k}$$

then $$\frac{dy}{dx} = \sum_{k=0}^{\infty} (m-k)\, a_k x^{m-k-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} (m-k)(m-k-1)\, a_k x^{m-k-2}$$

Substituting for y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$(1 - x^2) \sum_{k=0}^{\infty} (m-k)(m-k-1)\, a_k x^{m-k-2} - 2x \sum_{k=0}^{\infty} (m-k) a_k x^{m-k-1} + n(n+1) \sum_{k=0}^{\infty} a_k x^{m-k} = 0$$

or $$\sum_{k=0}^{\infty} (m-k)(m-k-1)\, a_k x^{m-k-2} - \sum_{k=0}^{\infty} [(m-k)(m-k-1) + 2(m-k) - n(n+1)]\, a_k x^{m-k} = 0$$

or $$\sum_{k=0}^{\infty} (m-k)(m-k-1)\, a_k x^{m-k-2} - \sum_{k=0}^{\infty} (m-k)^2 - n^2 + (m-k) - n]\, a_k x^{m-k} = 0$$

or
$$\sum_{k=0}^{\infty}(m-k)(m-k-1)\,a_k\,x^{m-k-2}-\sum_{k=0}^{\infty}[(m-k-n)(m-k+n+1)\,a_k\,x^{m-k}=0.$$

Equating to zero the coefficient of highest power of x, *i.e.*, x^m, we get the indicial equation
$$(m-n)(m+n+1)\,a_0=0$$

whence $m=n$ or $m=-(n+1)$ since $a_0\neq 0$

Equating to zero the coefficient of the next lower power of x, *i.e.*, x^{m-1}, we get
$$(m+n)(m-n-1)\,a_1=0 \text{ or } a_1=0,$$

since $(m+n)$ and $(m-n-1)$ are not zero for $m=n$ or $-(n+1)$.

Equating to zero the coefficient of x^{m-k}, we get the recurrence relation
$$[m-(k-2)][m-(k-2)-1]\,a_{k-2}-(m-k-n)(m-k+n+1)\,a_k=0$$

or
$$\boxed{a_k=-\frac{(m-k+2)(m-k+1)}{(n-m+k)(n+m-k+1)}a_{k-2}} \qquad \text{...(2)}$$

Since $a_1=0$, therefore, from (2), we get $a_3=a_5=a_7=\ldots\ldots=0$.

Case I. When $m=n$, the recurrence relation (2) reduces to
$$a_k=-\frac{(n-k+2)(n-k+1)}{k(2n-k+1)}a_{k-2}$$

Putting $k=2,4,6,\ldots\ldots$, we get $a_2=-\dfrac{n(n-1)}{2(2n-1)}a_0$,
$$a_4=-\frac{(n-2)(n-3)}{4(2n-3)}a_2=\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}a_0, \text{ etc.}$$

Therefore, one solution of Legendre's equation is given by
$$y_1=a_0\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}-\ldots\right] \qquad \text{...(3)}$$

Case II. When $m=-(n+1)$, the recurrence relation (2) reduces to
$$a_k=\frac{(n+k-1)(n+k)}{k(2n+k+1)}a_{k-2}$$

Putting $k=2,4,6,\ldots\ldots$, we get
$$a_2=\frac{(n+1)(n+2)}{2(2n+3)}a_0,$$
$$a_4=\frac{(n+3)(n+4)}{4(2n+5)}a_2=\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}a_0, \text{ etc.}$$

Therefore, the second solution of Legendre's equation is given by
$$y_2=a_0\left[x^{-n-1}+\frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}x^{-n-5}+\ldots\right] \qquad \text{...(4)}$$

2.7. LEGENDRE'S FUNCTION OF FIRST KIND $P_n(x)$

When n is a positive integer and $a_0 = \dfrac{1.3.5 \ldots\ldots (2n-1)}{n!}$,

the first solution given by (3) is denoted by $P_n(x)$ and is called **Legendre's function of first kind.**

Thus, $$P_n(x) = \frac{1.3.5\ldots\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1).(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \ldots\right]$$

$P_n(x)$ is a terminating series. R.H.S. is known as Zonal Harmonic. $P_n(x)$ gives Legendre's polynomials for different values of n such that $P_n(1) = 1$.

Now, two cases arise:

Case I. When n is even:

No. of terms in the series within bracket $= \dfrac{n}{2} + 1$

$$\text{Last term} = (-1)^{n/2} \cdot \frac{\{n(n-1)(n-2)(n-3)\ldots\ldots 2.1\}}{(2.4.6\ldots n)\{(2n-1)(2n-3)\ldots(n+1)\}}.$$

Case II. When n is odd:

No. of terms in the series within bracket $= \dfrac{n+1}{2}$

$$\text{Last term} = (-1)^{\frac{n-1}{2}} \cdot \frac{n(n-1)(n-2)(n-3)\ldots 3.2}{\{2.4.6\ldots(n-1)\}\{(2n-1)(2n-3)\ldots(n+2)\}}.$$

2.8. LEGENDRE'S FUNCTION OF SECOND KIND $Q_n(x)$

When n is a positive integer and $a_0 = \dfrac{n!}{1.3.5\ldots(2n+1)}$,

the second solution is denoted by $Q_n(x)$ and is called Legendre's function of second kind.

Thus, $$Q_n(x) = \frac{n!}{1.3.5\ldots(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2.(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}x^{-n-5} + \ldots\right]$$

It is a non-terminating series so there is no last term.

2.9. SOLUTION OF LEGENDRE'S EQUATION

Since $y = P_n(x)$ and $y = Q_n(x)$ both are the solutions of the given equation hence the most general solution is given by

$$\boxed{y = AP_n(x) + BQ_n(x)}$$

here A and B are arbitrary constants.

.10. GENERATING FUNCTION FOR $P_n(x)$

Ve shall show that $P_n(x)$ is the coefficient of h^n in the expansion of $(1 - 2xh + h^2)^{-1/2}$ in ascending owers of h.

.e.,
$$\boxed{(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) . h^n}$$

Using Binomial theorem,

$$(1 - t)^{-1/2} = 1 + \frac{1}{2}t + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}t^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}t^3 + \ldots$$

$$= 1 + \frac{1}{2}t + \frac{1.3}{2.4}t^2 + \frac{1.3.5}{2.4.6}t^3 + \ldots + \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots 2n}t^n + \ldots$$

$$\therefore \quad (1 - 2xh + h^2)^{-1/2} = [1 - h(2x - h)]^{-1/2}$$

$$= 1 + \frac{1}{2}h(2x - h) + \frac{1.3}{2.4}h^2(2x - h)^2 + \ldots$$

$$+ \frac{1.3.5\ldots\ldots(2n-5)}{2.4.6\ldots\ldots(2n-4)}h^{n-2}(2x - h)^{n-2} + \frac{1.3.5\ldots\ldots(2n-3)}{2.4.6\ldots\ldots(2n-2)}h^{n-1}(2x - h)^{n-1}$$

$$+ \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)}h^n(2x - h)^n + \ldots$$

Now, the coefficient of h^n in $\frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)}h^n(2x-h)^n$ is

$$= \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)}(2x)^n = \frac{1.3.5\ldots\ldots(2n-1)}{2^n(n)!}(2x)^n = \frac{1.3.5\ldots\ldots(2n-1)}{n!}x^n$$

The coefficient of h^n in $\frac{1.3.5\ldots\ldots(2n-3)}{2.4.6\ldots\ldots(2n-2)}h^{n-1}(2x-h)^{n-1}$ is

$$= \frac{1.3.5\ldots\ldots(2n-3)}{2.4.6\ldots\ldots(2n-2)}[-\,{}^{n-1}C_1(2x)^{n-2}] = -\frac{1.3.5\ldots\ldots(2n-3)}{2.4.6\ldots\ldots(2n-2)}(n-1)\,2^{n-2}.x^{n-2}$$

$$= -\frac{1.3.5\ldots\ldots(2n-3)}{2^{n-1}(n-1)!}(n-1)\,2^{n-2}.x^{n-2}$$

$$= -\frac{1.3.5\ldots\ldots(2n-3)(2n-1)}{2(n)!}\cdot\frac{n(n-1)}{2n-1}x^{n-2}$$

$$= -\frac{1.3.5\ldots\ldots(2n-1)}{n!}\cdot\frac{n(n-1)}{2(2n-1)}x^{n-2}$$

Similarly, the coefficient of h^n in $\frac{1.3.5\ldots\ldots(2n-5)}{2.4.6\ldots\ldots(2n-4)}h^{n-2}(2x-h)^{n-2}$ is

$$= \frac{1.3.5\ldots\ldots(2n-1)}{n!}\cdot\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}$$ and so on.

$\therefore$ The coefficient of h^n in $(1 - 2xh + h^2)^{-1/2}$ is given by

$$\frac{1.3.5\ldots\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \ldots\right] = P_n(x)$$

Thus, in the expansion of $(1 - 2xh + h^2)^{-1/2}$, $P_1(x)$, $P_2(x)$, $P_3(x)$, ..., $P_n(x)$, ... are the coefficients of h, h^2, h^3, ..., h^n, ... respectively.

$$\therefore \quad (1 - 2xh + h^2)^{-1/2} = 1 + P_1(x) . h + P_2(x) . h^2 + \ldots + P_n(x) . h^n + \ldots = \sum_{n=0}^{\infty} P_n(x) . h^n$$

The function $(1 - 2xh + h^2)^{-1/2}$ is called the **generating function for $P_n(x)$.**

ILLUSTRATIVE EXAMPLES

Example 1. *Show that*

(i) $P_n(1) = 1$ *(ii)* $P_n(-x) = (-1)^n P_n(x)$ *(iii)* $P'_n(-x) = (-1)^{n+1} P'_n(x)$.

(G.B.T.U. 2012)

Sol. We know that $$\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-1/2} \quad \ldots(1)$$

(i) Putting $x = 1$ in eqn. (1), we get

$$\sum_{n=0}^{\infty} h^n P_n(1) = (1 - 2h + h^2)^{-1/2} = (1 - h)^{-1}$$

$$= 1 + h + h^2 + \ldots\ldots + h^n + \ldots\ldots = \sum_{n=0}^{\infty} h^n$$

Equating the coefficients of h^n, we have $P_n(1) = 1$.

(ii) Replacing x by $(-x)$ in eqn. (1), we get

$$\sum_{n=0}^{\infty} h^n P_n(-x) = (1 + 2xh + h^2)^{-1/2} \quad \ldots(2)$$

Again, replacing h by $(-h)$ in eqn. (1), we have

$$\sum_{n=0}^{\infty} (-h)^n P_n(x) = (1 + 2xh + h^2)^{-1/2}$$

or $$\sum_{n=0}^{\infty} (-1)^n h^n P_n(x) = (1 + 2xh + h^2)^{-1/2} \quad \ldots(3)$$

From (2) and (3), $$\sum_{n=0}^{\infty} h^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n h^n P_n(x)$$

Equating the coefficients of h^n, we have

$$P_n(-x) = (-1)^n P_n(x).$$

(iii) We have, $$P_n(-x) = (-1)^n P_n(x) \quad | \text{ Proved in } (ii)$$

Differentiating w.r.t. x, we get

$$-P'_n(-x) = (-1)^n P'_n(x)$$

$$\therefore \quad P'_n(-x) = (-1)^{n+1} P'_n(x).$$

$$\Rightarrow \qquad P_n'(-1) = (-1)^{n-1} \cdot \frac{n(n+1)}{2}$$

2.11. RODRIGUE'S FORMULA

[M.T.U. 2013, U.P.T.U. 2007 ; U.P.T.U. (C.O.) 2008 ; G.B.T.U. (C.O.) 2010]

The relation $\quad P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n} (x^2 - 1)^n$ is known as Rodrigue's formula.

To prove it, let $\quad v = (x^2 - 1)^n$ then $v_1 = \dfrac{dv}{dx} = n(x^2 - 1)^{n-1} \cdot 2x$

Multiplying both sides by $(x^2 - 1)$, we get

$$(x^2 - 1)v_1 = 2nx\,(x^2 - 1)^n = 2nxv$$

or

$$(1 - x^2)v_1 + 2nxv = 0$$

Differentiating $(n + 1)$ times by Leibnitz's theorem, we have

$$\left[(1-x^2)v_{n+2} + (n+1)(-2x)v_{n+1} + \frac{(n+1)n}{2!}(-2)v_n\right] + 2n\,[xv_{n+1} + (n+1)\,v_n] = 0$$

or

$$(1 - x^2)v_{n+2} - 2xv_{n+1} + n(n + 1)v_n = 0$$

or

$$(1 - x^2)\frac{d^2(v_n)}{dx^2} - 2x\frac{d(v_n)}{dx} + n(n + 1)v_n = 0$$

which is Legendre's equation and v_n is its solution. But the solutions of Legendre's equation are $P_n(x)$ and $Q_n(x)$.

Since $v_n = \dfrac{d^n}{dx^n}(x^2 - 1)^n$ contains only positive powers of x, it must be a constant multiple of $P_n(x)$.

i.e.,

$$v_n = cP_n(x)$$

or

$$cP_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n$$

$$= \frac{d^n}{dx^n} [(x-1)^n (x+1)^n] \quad \ldots(1)$$

$$= (x-1)^n \frac{d^n}{dx^n} (x+1)^n + {}^nC_1 \,.\, n(x-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} (x+1)^n + \ldots\ldots + (x+1)^n \frac{d^n}{dx^n} (x-1)^n$$

$$= (x-1)^n \,.\, n\,! + {}^nC_1 \,.\, n(x-1)^{n-1} \,.\, \frac{d^{n-1}}{dx^{n-1}} (x+1)^n + \ldots\ldots + (x+1)^n\, n\,!$$

$$= n\,!\,(x+1)^n + \text{terms containing powers of } (x-1)$$

Putting $x = 1$ on both sides, we get

$$cP_n(1) = n\,!\,.\,2^n \quad \text{or} \quad c = 2^n\, n\,!, \text{ since } P_n(1) = 1$$

Substituting in (1), we get

$$\boxed{P_n(x) = \frac{1}{2^n\, n\,!} \frac{d^n}{dx^n} (x^2-1)^n}$$

Putting $n = 0, 1, 2, 3, \ldots\ldots$ in Rodrigue's formula, we get Legendre's polynomials. Thus

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{2} \frac{d}{dx} (x^2-1) = x$$

$$P_2(x) = \frac{1}{2^2\,(2)\,!} \frac{d^2}{dx^2} (x^2-1)^2 = \frac{1}{8} \frac{d^2}{dx^2} (x^4 - 2x^2 + 1) = \tfrac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2^3\,(3)\,!} \frac{d^3}{dx^3} (x^2-1)^3 = \frac{1}{48} \frac{d^3}{dx^3} (x^6 - 3x^4 + 3x^2 - 1) = \tfrac{1}{2}(5x^3-3x)$$

Similarly, $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{6}(231x^6 - 351x^4 + 105x^2 - 5) \text{ etc.}$$

Example 7. *Show that* $x^4 = \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)]$.

Sol. We know that $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$$P_2(x) = \tfrac{1}{2}(3x^2-1), \; P_0(x) = 1$$

$$\therefore \quad \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)] = \frac{1}{35}[35x^4 - 30x^2 + 3) + 10(3x^2-1) + 7] = x^4.$$

Example 8. *Express* $f(x) = x^3 - 5x^2 + x + 2$ *in terms of Legendre's polynomials.*

Sol. We know that $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$\therefore \quad x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} x$$

$$\therefore \quad f(x) = \left[\frac{2}{3}P_3(x) + \frac{3}{5}(x)\right] - 5x^2 + x + 2 = \frac{2}{5}P_3(x) - 5x^2 + \frac{8}{5}x + 2$$

$$= \frac{2}{5}P_3(x) - 5\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + \frac{8}{5}x + 2 \qquad \left[\because \quad P_2(x) = \tfrac{1}{2}(3x^2 - 1) \quad \therefore \quad x^2 = \tfrac{2}{3}P_2(x) + \tfrac{1}{3}\right]$$

$$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}x + \frac{1}{3}$$

$$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}P_1(x) + \frac{1}{3}P_0(x)$$

$$[\because \quad x = P_1(x) \text{ and } 1 = P_0(x)]$$

Example 9. *Prove that:* $\int_{-1}^{1} P_n(x)\,dx = \left.\begin{matrix} 0, & n \neq 0 \\ 2, & n = 0 \end{matrix}\right\}$. [U.P.T.U. (SUM) 2009]

Sol. We know by Rodrigue's formula, $P_n(x) = \dfrac{1}{2^n\, n!}\dfrac{d^n}{dx^n}\{(x^2-1)^n\}$

Integrating, we get

$$\int_{-1}^{1} P_n(x)\,dx = \frac{1}{2^n\, n!}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{1}{2^n\, n!}\left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{1} = 0$$

when $n = 0$, $\quad \int_{-1}^{1} P_0(x)\,dx = \int_{-1}^{1} 1\,dx = 2. \qquad | \because \quad P_0(x) = 1$

Example 10. *Express $4x^3 + 6x^2 + 7x + 2$ in terms of Legendre's polynomials.*

Sol. Let $\quad 4x^3 + 6x^2 + 7x + 2 \equiv \alpha P_3(x) + \beta P_2(x) + \gamma P_1(x) + \xi P_0(x) \qquad ...(1)$

$$\equiv \alpha\left(\frac{5x^3 - 3x}{2}\right) + \beta\left(\frac{3x^2-1}{2}\right) + \gamma(x) + \xi(1)$$

$$\equiv \frac{5\alpha}{2}x^3 + \frac{3\beta}{2}x^2 + \left(\gamma - \frac{3\alpha}{2}\right)x + \left(\xi - \frac{\beta}{2}\right)$$

Equating the coefficients of like powers of x, we get

$$\frac{5\alpha}{2} = 4 \quad \Rightarrow \quad \alpha = \frac{8}{5}$$

$$6 = \frac{3\beta}{2} \quad \Rightarrow \quad \beta = 4$$

$$7 = \gamma - \frac{3\alpha}{2} \quad \Rightarrow \quad 7 = \gamma - \frac{12}{5} \quad \Rightarrow \quad \gamma = \frac{47}{5}$$

$$2 = \xi - \frac{\beta}{2} \quad \Rightarrow \quad 2 = \xi - 2 \quad \Rightarrow \quad \xi = 4$$

Hence from (1),

$$4x^3 + 6x^2 + 7x + 2 = \frac{8}{5}P_3(x) + 4P_2(x) + \frac{47}{5}P_1(x) + 4P_0(x).$$

Example 11. *Prove that:*

$$P_n\left(-\frac{1}{2}\right) = P_0\left(-\frac{1}{2}\right)P_{2n}\left(\frac{1}{2}\right) + P_1\left(-\frac{1}{2}\right)P_{2n-1}\left(\frac{1}{2}\right) + \ldots + P_{2n}\left(-\frac{1}{2}\right)P_0\left(\frac{1}{2}\right).$$

2.12. RECURRENCE RELATIONS

1. $$\boxed{n\,P_n(x) = (2n-1)\,x\,P_{n-1}(x) - (n-1)\,P_{n-2}(x)}$$

Or

$$(n+1)\,P_{n+1}(x) = (2n+1)\,x\,P_n(x) - n\,P_{n-1}(x)$$ (*U.P.T.U. 2007, M.T.U. 2012*)

We know that,

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \qquad \text{...(1)}$$

Differentiating both sides w.r.t. h, we get

$$-\frac{1}{2}(1-2xh+h^2)^{-3/2}(2h-2x) = \sum_{0}^{\infty} n\,h^{n-1} P_n(x)$$

$$\Rightarrow \quad (x-h)(1-2xh+h^2)^{-1/2} = (1-2xh+h^2)\sum_{0}^{\infty} n\,h^{n-1} P_n(x)$$

$$\Rightarrow \quad (x-h)\sum_{0}^{\infty} h^n P_n(x) = (1-2xh+h^2)\sum_{0}^{\infty} n\,h^{n-1} P_n(x)$$

Equating coefficient of h^{n-1} on both sides,

$$x\,P_{n-1}(x) - P_{n-2}(x) = n\,P_n(x) - 2x\,(n-1)\,P_{n-1}(x) + (n-2)\,P_{n-2}(x)$$

$$\Rightarrow \quad n\,P_n(x) = (2n-1)\,x\,P_{n-1}(x) - (n-1)\,P_{n-2}(x)$$

Replacing n by $(n+1)$, we get the other form.

2. $$\boxed{n\,P_n(x) = x\,P_n'(x) - P'_{n-1}(x)}$$

[*U.P.T.U. 2006, 2009; G.B.T.U. (C.O.) 2011*]

We know that, $$(1-2hx+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \qquad \text{...(1)}$$

Differentiating both sides of (1) w.r.t. h, we get

$$-\frac{1}{2}(1-2xh+h^2)^{-3/2}\,.\,(-2x+2h) = \sum_{0}^{\infty} n\,h^{n-1} P_n(x)$$

$$\Rightarrow \quad (x-h)(1-2hx+h^2)^{-3/2} = \sum_0^\infty n h^{n-1} P_n(x) \quad ...(2)$$

Differentiating both sides of (1) w.r.t. x, we get

$$-\frac{1}{2}(1-2hx+h^2)^{-3/2} . (-2h) = \sum_0^\infty h^n P_n'(x)$$

$$\Rightarrow \quad (x-h)(1-2hx+h^2)^{-3/2} = (x-h)\sum_0^\infty h^{n-1} P_n'(x) \quad ...(3)$$

Equating eqns. (2) and (3), we get

$$\sum_0^\infty n h^{n-1} P_n(x) = (x-h)\sum_0^\infty h^{n-1} P_n'(x)$$

Comparing the coefficient of h^{n-1} on both sides, we get

$$n P_n(x) = x P_n'(x) - P'_{n-1}(x)$$

3. $\boxed{(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)}$

[U.P.T.U. (C.O.) 2009; U.P.T.U. (SUM) 2008]

From Recurrence relation (1),

$$(2n+1)\, x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$$

Differentiating w.r.t. x, we get

$$(2n+1)[x P_n'(x) + P_n(x)] = (n+1) P'_{n+1}(x) + n P'_{n-1}(x) \quad ...(1)$$

From Recurrence relation (2), $\quad x P_n'(x) = n P_n(x) + P'_{n-1}(x)$

From (1),

$$(2n+1)[n P_n(x) + P'_{n-1}(x) + P_n(x)] = (n+1) P'_{n+1}(x) + n P'_{n-1}(x)$$

$$\Rightarrow \quad (2n+1)(n+1) P_n(x) = (n+1) P'_{n+1}(x) - (n+1) P'_{n-1}(x)$$

$$\Rightarrow \quad (2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$$

4. $\boxed{(n+1) P_n(x) = P'_{n+1}(x) - x P_n'(x)}$ *(G.B.T.U. 2010)*

From Recurrence relation (3), we have

$$(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \quad ...(1)$$

From Recurrence relation (2), we have

$$n P_n(x) = x P_n'(x) - P'_{n-1}(x) \quad ...(2)$$

Subtraction yields, $(n+1) P_n(x) = P'_{n+1}(x) - x P'_n(x)$

5. $\boxed{(1-x^2) P'_n(x) = n[P_{n-1}(x) - xP_n(x)]}$ *(U.P.T.U. 2007)*

From Recurrence relation (4), we have

$$P_n'(x) - xP'_{n-1}(x) = nP_{n-1}(x) \quad ...(1)$$

From Recurrence relation (2), we have

$$x P_n'(x) - P'_{n-1}(x) = n P_n(x) \quad ...(2)$$

Multiplying (2) by x and subtracting from (1), we get

$$(1-x^2) P_n'(x) = n[P_{n-1}(x) - xP_n(x)]$$

6. $$\boxed{(1-x^2)\, P'_n(x) = (n+1)\,[xP_n(x) - P_{n+1}(x)]}$$

Recurrence relation (1) may be written as

$$(\overline{n+1+n})\, xP_n(x) = (n+1)\, P_{n+1}(x) + nP_{n-1}(x)$$

$$\Rightarrow \quad (n+1)\, xP_n(x) + nxP_n(x) = (n+1)\, P_{n+1}(x) + nP_{n-1}(x)$$

or $$(n+1)\,[xP_n(x) - P_{n+1}(x)] = n[P_{n-1}(x) - xP_n(x)]$$

$$= (1-x^2)\, P_n'(x) \qquad | \text{ Using recurrence relation (5)}$$

$$\therefore \quad (1-x^2)\, P_n'(x) = (n+1)\,[xP_n(x) - P_{n+1}(x)].$$

2.13. BELTRAMI'S RESULT

$$\boxed{(2n+1)\,(x^2-1)\, P_n' = n(n+1)\,(P_{n+1} - P_{n-1})}$$

From Recurrence relation (5), we have

$$n(P_{n-1} - xP_n) = (1-x^2)\, P_n' \qquad \ldots(1)$$

From Recurrence relation (6), we have

$$(n+1)(xP_n - P_{n+1}) = (1-x^2)\, P_n' \qquad \ldots(2)$$

From eqn. (1), $\quad nP_{n-1} - nx\, P_n = (1-x^2)\, P_n'$

$$\Rightarrow \quad xP_n = \frac{nP_{n-1} - (1-x^2)\, P_n'}{n} \qquad \ldots(3)$$

From eqn. (2) $\quad xP_n - P_{n+1} = \dfrac{(1-x^2)\, P_n'}{n+1}$

$$\Rightarrow \quad xP_n = P_{n+1} + \frac{(1-x^2)\, P_n'}{n+1} \qquad \ldots(4)$$

From (3) and (4),

$$\frac{nP_{n-1} - (1-x^2)P_n'}{n} = P_{n+1} + \frac{(1-x^2)\, P_n'}{n+1}$$

$$= \frac{(n+1)\, P_{n+1} + (1-x^2)P_n'}{n+1}$$

$$\Rightarrow \quad (n+1)\,\{nP_{n-1} - (1-x^2)\, P_n'\} = n\{(n+1)\, P_{n+1} + (1-x^2)\, P_n'\}$$

$$\Rightarrow \quad (2n+1)\,(1-x^2)\, P_n' = n\,(n+1)\,\{P_{n-1} - P_{n+1}\}$$

$$\Rightarrow \quad (2n+1)\,(x^2-1)\, P_n' = n(n+1)\,\{P_{n+1} - P_{n-1}\}.$$

2.14. ORTHOGONALITY OF LEGENDRE POLYNOMIALS

[*G.B.T.U. 2013; M.T.U. 2013*]

We shall show that

$$\boxed{\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0, & \text{if } m \neq n \\ \dfrac{2}{2n+1}, & \text{if } m = n \end{cases}}$$

[U.P.T.U. 2008 ; U.P.T.U. (SUM) 2007]

Case I. When $m \neq n$

We know that $P_m(x)$ and $P_n(x)$ are the solutions of the equations

$$(1-x^2)u'' - 2xu' + m(m+1)u = 0 \qquad ...(1)$$

and

$$(1-x^2)v'' - 2xv' + n(n+1)v = 0 \qquad ...(2)$$

Multiplying (1) by v and (2) by u and subtracting, we get

$$(1-x^2)(u''v - v''u) - 2x(u'v - v'u) + [m(m+1) - n(n+1)]\, uv = 0$$

or

$$\frac{d}{dx}[(1-x^2)(u'v - v'u)] + (m-n)(m+n+1)uv = 0$$

or

$$(n-m)(n+m+1)\, uv = \frac{d}{dx}[(1-x^2)(u'v - v'u)]$$

Integrating w.r.t. x from -1 to 1, we get

$$(n-m)(n+m+1)\int_{-1}^{1} uv\, dx = \Big[(1-x^2)(u'v - v'u)\Big]_{-1}^{1} = 0$$

Hence $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0$, since $m \neq n$.

Case II. When $m = n$ [U.P.T.U. 2008 ; G.B.T.U. (C.O.) 2011]

We know that $\qquad (1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

Squaring both sides, we get

$$(1-2xh+h^2)^{-1} = \sum_{n=0}^{\infty} [h^n P_n(x)]^2 = \sum_{n=0}^{\infty} h^{2n}[P_n(x)]^2 + 2\sum_{\substack{m=0 \\ (m\neq n)}}^{\infty}\sum_{n=0}^{\infty} h^{m+n} P_m(x)\, P_n(x)$$

Integrating w.r.t. x between the limits -1 to 1, we have

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}[P_n(x)]^2\, dx + 2\sum_{\substack{m=0 \\ (m\neq n)}}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} h^{m+n} P_m(x)\, P_n(x)\, dx = \int_{-1}^{1}\frac{dx}{1-2xh+h^2}$$

or

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}[P_n(x)]^2\, dx = \int_{-1}^{1}\frac{dx}{1-2xh+h^2}$$

| Since other integrals on the LHS vanish by Case I as $m \neq n$

$$= -\frac{1}{2h}\Big[\log(1-2xh+h^2)\Big]_{-1}^{1} = -\frac{1}{2h}[\log(1-h)^2 - \log(1+h)^2]$$

$$= \frac{1}{h}[\log(1+h) - \log(1-h)] = \frac{1}{h}\left[\left(h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \ldots\right) + \left(h + \frac{h^2}{2} + \frac{h^3}{3} + \frac{h^4}{4} + \ldots\right)\right]$$

$$= \frac{2}{h}\left[h + \frac{h^3}{3} + \frac{h^5}{5} + \ldots\right]$$

or $$\sum_{n=0}^{\infty} h^{2n} \int_{-1}^{1} [P_n(x)]^2 \, dx = 2\left(1+\frac{h^2}{3}+\frac{h^4}{5}+\ldots\ldots+\frac{h^{2n}}{2n+1}+\ldots\ldots\right)$$

Equating the coefficients of h^{2n} on the two sides, we get

$$\boxed{\int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{2}{2n+1}.}$$ (*M.T.U. 2013*)

$$\sum_{m=0} (2m+1)\, P_m(x)\, P_m(y) = \frac{\ldots P_{n+1}(y) P_n(x)]}{x-y}.$$

2.19. EXPANSION OF A FUNCTION IN A SERIES OF LEGENDRE POLYNOMIALS (FOURIER-LEGENDRE SERIES)

The orthogonal property of Legendre polynomials enables us to expand a function $f(x)$, defined from $x = -1$ to $x = 1$ in a series of Legendre polynomials.

Let
$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \ldots \qquad \ldots(1)$$

To determine a_n, multiplying both sides of (1) by $P_n(x)$ and integrating w.r.t. x from -1 to 1, we have

$$\int_{-1}^{1} f(x) P_n(x)\, dx = a_n \int_{-1}^{1} P_n^2(x)\, dx = a_n \left(\frac{2}{2n+1}\right)$$

$$\Rightarrow \quad \boxed{a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x)\, dx}$$

Expansion of $f(x)$ given by (1) is known as *Fourier-Legendre series.*

Example 13. *Prove that:* $\int_{-1}^{1}(1-x^2)\,P_m'\,P_n'\,dx = 0$

where m and n are distinct positive integers and $m \neq n$. (U.P.T.U. 2006)

Sol. $\int_{-1}^{1}(1-x^2)\,P_m'\,P_n'\,dx$

$$= \Big[(1-x^2)\,P_m'\,P_n\Big]_{-1}^{1} - \int_{-1}^{1} P_n\left[\frac{d}{dx}\{(1-x^2)\,P_m'\}\right]dx \qquad \text{| Integrating by parts}$$

$$= -\int_{-1}^{1} P_n\,\frac{d}{dx}\{(1-x^2)\,P_m'\}\,dx$$

$$= -\int_{-1}^{1} P_n\{-m\,(m+1)\,P_m\}\,dx \qquad \text{| From Legendre's differential equation}$$

$$= m(m+1)\int_{-1}^{1} P_n\,P_m\,dx = m(m+1)\,.\,0 = 0$$

Note. We can also prove that $\int_{-1}^{1}(1-x^2)(P_n')^2\,dx = n(n+1)\,.\,\frac{2}{2n+1}$.

Above is same as Ex. 1 with $m = n$.

Example 14. *Prove that:* $\int_{-1}^{1} x^2\,P_{n-1}(x)\,P_{n+1}(x)\,dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$.

Sol. Recurrence relation (1) is

$$(n+1)\,P_{n+1} = (2n+1)\,xP_n - n\,P_{n-1}$$

$$\Rightarrow \qquad (2n+1)\,xP_n = (n+1)\,P_{n+1} + nP_{n-1} \qquad ...(1)$$

Replacing n by $n+1$ and $n-1$ respectively in (1), we get

$$(2n+3)\,xP_{n+1} = (n+2)\,P_{n+2} + (n+1)\,P_n \qquad ...(2)$$

and

$$(2n-1)\,xP_{n-1} = nP_n + (n-1)\,P_{n-2} \qquad ...(3)$$

Multiplying (2) and (3) and integrating within limits -1 and 1, we get

$$(2n+3)\,(2n-1)\int_{-1}^{1} x^2 P_{n+1}P_{n-1}\,dx = n\,(n+1)\int_{-1}^{1} P_n^{\,2}dx$$

$$+\,n(n+2)\int_{-1}^{1} P_n\,P_{n+2}dx + (n^2-1)\int_{-1}^{1} P_{n-2}\,P_n\,dx + (n+2)\,(n-1)\int_{-1}^{1} P_{n-2}\,P_{n+2}\,dx$$

$$= n(n+1)\,.\,\frac{2}{2n+1} \qquad \text{[Using orthogonal properties]}$$

$$\therefore \quad \int_{-1}^{1} x^2\,P_{n+1}\,P_{n-1}\,dx = \frac{2n(n+1)}{(2n+1)(2n-1)(2n+3)}.$$