Series Solutions and special functions

power series:- An expression of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

is known as power series of the variable x in powers of $(x-x_0)$ or about the point x_0

The power series about $x_0 = 0$ (origin).

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Regular or Ordinary point: Consider the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0 \quad \text{--- (1)}$$

The point $x = a$ is called the regular or ordinary point of equ. (1) if both $P(x)$ and $Q(x)$ exist at the point $x = a$

The point $x = a$ is said to be singular point of equation (1) if either $P(x)$ or $Q(x)$ or both do not exist at a

Regular Singular point → if at $x = a$. $(x-a)\,P(x)$ and $(x-a)^2 Q(x)$ exist at that point.

Irregular Singular point → if $x = a$ is not regular singular point then it is said to be irregular singular point.

Series Solutions:

The solution of ordinary linear D.E of second order with variable coefficients in the form of an infinite convergent series is called solution in series.

Power Series:-

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

called a power series in ascending powers of $x-x_0$

In particular, a power series in ascending powers of x is an infinite series.

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Analytic Function:- A function $f(x)$ definede on an interval containing the point $x=x_0$ is called analytic at x_0 if its Taylor's series $\sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x-x_0)^n$ exists and convergence to $f(x)$ for all x in the interval of convergence

→ A rational function is analytic everywhere except at those values of x at which its denominator is zero.

$$f(x) = \frac{x}{x^2-5x+6}$$ is not analytic at $x=2$, $x=3$

→ all polynomial functions e^x, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$ are analytic everywhere.

Series Solution of the Differential Equation when $x=0$ is an ordinary point

①. Solve in series the D.E

$$(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0 \qquad \text{—} ①$$

$$\frac{d^2y}{dx^2}+\frac{x}{1+x^2}\frac{dy}{dx}-\frac{1}{1+x^2}\cdot y=0$$

$$P(x)=\frac{x}{1+x^2} \qquad Q(x)=-\frac{1}{1+x^2}$$

at $x=0$ $\quad P(x)=0 \quad Q(x)=-1$ both exist

$\Rightarrow$ $x=0$ is an ordinary point of ①

Let solution of ① in power series is

$$y=\sum_{n=0}^{\infty}a_n x^n$$

$$\frac{dy}{dx}=\sum_{n=0}^{\infty}a_n n\cdot x^{n-1}, \quad \frac{d^2y}{dx^2}=\sum_{n=0}^{\infty}a_n n(n-1)x^{n-2}$$

$$(1+x^2)\sum_{n=0}^{\infty}a_n\cdot n(n-1)x^{n-2}+x\sum_{n=0}^{\infty}a_n n x^{n-1}-\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}a_n\cdot n(n-1)x^{n-2}+\sum_{n=0}^{\infty}a_n\cdot n(n-1)x^n+\sum a_n\cdot n x^n$$

$$+\sum_{n=0}^{\infty}a_n\cdot x^n=0$$

$$\sum_{n=0}^{\infty}a_n\cdot n(n-1)x^{n-2}+\sum_{n=0}^{\infty}a_n(n^2-1)x^n=0$$

Equating to zero the coefficients of various powers of x

Coefficient of x^0 $\qquad a_2\cdot 2\cdot 1 + a_0=0$

$$\Rightarrow \boxed{a_2=\frac{a_0}{2}}$$

Coefficient of x

$a_3 \cdot 3 \cdot 2 + 0 = 0 \Rightarrow \boxed{a_3 = 0}$

Coefficient of x^2

$a_4 \cdot 4 \cdot 3 + a_2 \cdot 3 = 0 \Rightarrow a_4 = -\frac{a_2}{4} = \boxed{-\frac{a_0}{8}}$

$\cdot$

---- Coefficient of x^n

$a_{n+2} \cdot (n+2)(n+1) + (n^2-1)a_n = 0$

$\Rightarrow \boxed{a_{n+2} = -\frac{(n-1)}{n+2} \cdot a_n}$

$n = 3 \qquad a_5 = -\frac{2}{5} \cdot a_3 = 0 \qquad \boxed{a_5 = 0}$

$n = 4 \qquad a_6 = -\frac{\cancel{3}}{\cancel{6}2} a_4 = \boxed{\frac{a_0}{16}}$

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$= a_0 + a_1 x + \frac{a_0}{2} x^2 - \frac{a_0}{8} x^4 + \frac{a_0}{16} x^6 + \cdots$

$$\boxed{y = a_0\left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} + \cdots\right) + a_1 x}$$

Ex ②. Solve the following differential equation in series.

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$$

$$y = a_0(1-2x^2) + a_1 x\left(1 - \frac{x^2}{2} - \frac{x^4}{8} + \cdots\right)$$

③. $\frac{d^2y}{dx^2} + xy = 0$

$$y = a_0\left[1 - \frac{x^3}{3!} + \frac{1\cdot 4}{6!}x^6 - \frac{28}{9!}x^9 + \cdots\right] + a_1\left[x - \frac{2}{4!}x^4 + \frac{10}{7!}x^7 - \cdots\right]$$

(3). Solve $xy'' + y' + 2y = 0$ with $y(1) = 2$, $y'(1) = 4$

Since the initial conditions are given at $x = 1$, we obtain a power series solution in powers of $(x-1)$ or about $x = 1$.

$$y = \sum_{n=0}^{\infty} a_n \cdot (x-1)^n$$

put $t = x - 1$

$$y = \sum_{n=0}^{\infty} a_n t^n$$

$$(t+1)\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = 0 \quad \text{with } y(t=0) = 2$$
$$\text{and } y'(t=0) = 4$$

$$(t+1)\sum_{n=0}^{\infty} a_n n(n-1) t^{n-2} + \sum_{n=0}^{\infty} a_n n \cdot t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum a_n \cdot n(n-1) t^{n-1} + \sum a_n \cdot n \cdot t^{n-1} + \sum a_n \cdot n(n-1) t^{n-2} + 2\sum a_n t^n = 0$$

$$\sum a_n n^2 t^{n-1} + \sum a_n \cdot n(n-1) t^{n-2} + 2\sum a_n \cdot t^n = 0$$

Equating to zero the coefficient of various powers of t

Coefficient of t^0

$$a_1 + a_2 \cdot 2 + 2a_0 = 0$$

$$a_2 = -\frac{(2a_0 + a_1)}{2}$$

Coefficient of t^n

$$a_{n+1} \cdot (n+1)^2 + a_{n+2} \cdot (n+2)(n+1) + 2a_n = 0$$

$$a_{n+2} = -\left[\frac{2a_n + (n+1)^2 \cdot a_{n+1}}{(n+1)(n+2)}\right]$$

$$n = 1 \Rightarrow a_3 = -\left[\frac{2a_1 + 4 \cdot a_2}{6}\right] = -\frac{a_1}{3} + \frac{2}{3} \cdot \frac{1}{2}(2a_0 + a_1)$$
$$= \frac{2}{3} a_0$$

Frobenius Method :- Series solution when $x=0$ is a regular singular point of the differential equation :-

$$\frac{d^2y}{dx^2} + P(x)\cdot\frac{dy}{dx} + Q(x)y = 0$$

Assume $\boxed{y = \sum_{n=0}^{\infty} a_n x^{m+n}}$

Substitute $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in the given equation

→ Equate to zero the coefficient of lowest power of x this gives a quadratic equation in m which is known as the Indicial equation.

→ Equate to zero, the coefficient of other powers of x to find $a_1, a_2, ---$ in terms of a_0

→ Substitute the values of $a_1, a_2 ---$ in (1) to get the series solution of the given equation having a_0 as arbitrary constant.

→ Complete solution depends on the nature of roots of the indicial equation.

Case - 1 when Roots are distinct and do not differ by an integer :- $m_1 - m_2 \neq$ an integer (not differ by an integer)

$$y = c_1(y)_{m_1} + c_2(y)_{m_2}$$

Ex ①. $2x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} - (x+1)y = 0$

Solve about $x = 0$ in series

$$\frac{d^2y}{dx^2} + \frac{1}{2x}\frac{dy}{dx} - \frac{(x+1)}{2x^2} y = 0$$

here $x=0$ is not ordinary point

$x \cdot P(x) = \frac{1}{2}$ $\qquad x^2 \cdot Q(x) = \frac{x+1}{x} = \frac{1}{2}$ at $x = 0$

$\Rightarrow x=0$ is a regular singular point.

assume $\quad y = \sum_{n=0}^{\infty} a_n x^{m+n}$

$$2x^2 \sum a_n (m+n)(m+n-1) x^{m+n-2} + x \sum a_n (m+n) x^{m+n-1} - (x+1) \sum a_n x^{m+n} = 0$$

$$2 \sum a_n (m+n)(m+n-1) x^{m+n} + \sum a_n (m+n) x^{m+n} - \sum a_n x^{m+n+1} - \sum a_n x^{m+n} = 0$$

$$\sum a_n \left\{ 2(m+n)(m+n-1) + (m+n) - 1 \right\} x^{m+n} - \sum_{n=0}^{\infty} a_n x^{m+n+1} = 0$$

Equating to zero the coefficient of lowest degree of x

Coefficient of x^m

$$a_0 \left\{ 2m(m-1) + m - 1 \right\} = 0$$

$a_0 \neq 0 \qquad (2m+1)(m-1) = 0$

$m = 1, -\frac{1}{2}$

Coefficient of x^{m+1}

$$a_1 \{2(m+1)(m) + m+1-1\} - a_0 = 0$$

$$\boxed{a_1 = \frac{a_0}{m(2m+3)}}$$

$\because$ Coefficient of x^{m+n}

$$a_n \{(m+n-1)\ (2m+2n+1)\} - a_{n-1} = 0$$

$$a_n = \frac{a_{n-1}}{\{(m+n-1)(2m+2m+1)\}}$$

$$a_2 = \frac{a_1}{(m+1)(2m+5)} = \frac{a_0}{m(m+1)(2m+3)(2m+5)}$$

$$a_3 = \frac{a_2}{(m+2)(2m+7)} = \frac{a_0}{m(m+1)(m+2)(2m+3)(2m+5)(2m+7)}$$

at $m = 1$

$$a_1 = \frac{a_0}{5}\ ; \quad a_2 = \frac{a_0}{2\cdot 5\cdot 7} = \frac{a_0}{70}\ , \quad a_3 = \frac{a_0}{1890}$$

at $m = \frac{-1}{2}$

$$a_1 = -a_0\ , \quad a_2 = \frac{-a_0}{2}\ , \quad a_3 = \frac{-a_0}{18} \ \cdots$$

$$y = c_1\, y_{m_1} + c_2\, y_{m_2}$$

$$= c_1 \{ a_0 + a_1 x^{m+1} + a_2 x^{m+2} + \cdots \} + c_2 \{ \quad \}$$

$$c_1 x^m \{ a_0 + a_1 x + a_2 x^2 + \cdots \} + c_2 x^m \{ a_0 + a_1 x + a_2 x^2 + \cdots \}$$

$$dx^2 \quad + (x+5)\frac{dy}{dx} - 4y = 0$$

Case (III). when $m_1 = m_2$

Find the series solution of the equation

Ex. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + x^2 y = 0$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\cdot\frac{dy}{dx} + xy = 0$$

$$P(x) = \frac{1}{x}, \quad Q(x) = x.$$

$x\,P(x)$ and $x^2Q(x)$ both exist at $x=0$

$\therefore$ $x=0$ is a regular singular point

$$y = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-1}, \quad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n(m+n)(m+n-1)x^{m+n-2}$$

$$\therefore x\sum_{n=0}^{\infty} a_n (m+n)(m+n-1)x^{m+n-2} + \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-1}$$

$$+ x^2 \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-1} + \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-1}$$

$$+ \sum_{n=0}^{\infty} a_n x^{m+n+2} = 0$$

$$\sum_{n=0}^{\infty} a_n \{(m+n)(m+n-1) + (m+n)\} x^{m+n-1} + \sum_{n=0}^{\infty} a_n x^{m+n+2} = 0$$

equating to zero the coefficient of lowest power of x

x^{m-1} $\quad m(m-1) + m$

$$a_0 \{m^2\} = 0 \implies m = 0, 0$$

in this case solution will be

$$y = A(y)_{m=0} + B\left(\frac{\partial y}{\partial m}\right)_{m=0}$$

Now equating the coefficient of x^m, we get

$a_1(m+1)^2 = 0 \quad \Rightarrow \quad a_1 = 0 \quad$ as $m \neq -1$

x^{m+1}

$a_2(m+2)^2 = 0 \quad \Rightarrow \quad a_2 = 0.$

x^{m+2}

$a_3(m+3)^2 + a_0 \quad \Rightarrow \quad a_3 = \frac{-a_0}{(m+3)^2}$

equating the coefficient of x^{m+n}

$$a_{n+1}(m+n+1)^2 + a_{n-2} = 0$$

$$a_{n+1} = -\frac{a_{n-2}}{(m+n+1)^2}$$

$n = 3$

$a_4 = -\frac{a_1}{(m+4)^2} = 0, \quad a_5 = -\frac{a_2}{(m+5)^2} = 0$

$a_6 = -\frac{a_3}{(m+6)^2} = \frac{a_0}{(m+3)^2(m+6)^2}$

$a_7 = -\frac{a_4}{(m+7)^2} = 0, \quad a_8 = -\frac{a_5}{(m+8)^2} = 0$

$a_9 = -\frac{a_6}{(m+9)^2} = -\frac{a_0}{(m+3)^2(m+6)^2(m+9)^2}$

$$y_1 = (y)_{m=0} = \sum_{n=0}^{\infty} a_n x^n$$

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots$

$a_0 - \frac{a_0}{3^2} x^3 + \frac{a_0}{3^2 \cdot 6^2} x^6 \ldots$

$y_1 = a_0\left[1 - \frac{x^3}{3^2} + \frac{1}{3^2 \cdot 6^2} \cdot x^6 \ldots\right]$

$$y = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$= a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots$$

$$= x^m \left[a_0 + a_1 x + a_2 x^2 + \cdots \right]$$

$$= x^m \left[a_0 + \frac{a_0}{(m+3)^2} x^3 + \frac{a_0}{(m+3)^2 (m+6)^2} x^6 + \cdots \right]$$

$$= x^m a_0 \left[1 - \frac{x^3}{(m+3)^2} + \frac{x^6}{(m+3)^2 (m+6)^2} - \cdots \right]$$

~~To get~~

$$\frac{\partial y}{\partial m} = (x^m \log x)\, a_0 \left[1 - \frac{x^3}{(m+3)^2} + \frac{x^6}{(m+3)^2 (m+6)^2} - \cdots \right]$$

$$+ x^m a_0 \left[\frac{2x^3}{(m+3)^3} - \frac{2x^6}{(m+3)^3 (m+6)^2} - \frac{2x^6}{(m+3)^2 (m+6)^3} + \cdots \right]$$

To get first solution, $y_1 = (y)_{m=0}$

$$y_1 = a_0 \left[1 - \frac{x^3}{3^2} + \frac{x^6}{3^2 6^2} - \cdots \right]$$

$$y_2 = \left(\frac{\partial y}{\partial m} \right)_{m=0} = a_0 \log x \left[1 - \frac{x^3}{3^2} + \frac{x^6}{3^2 6^2} \cdots \right]$$

$$+ a_0 \left[\frac{2x^3}{3^3} - \frac{2x^6}{3^3 6^2} - \frac{2x^6}{3^2 6^3} + \cdots \right]$$

general Solution will be

$$y = c_1 a_0 \left[1 - \frac{x^3}{3^2} + \frac{x^6}{3^2 6^2} + \cdots \right] + c_2 a_0 \log x \left[1 - \frac{x^3}{3^2} + \frac{x^6}{3^2 6^2} \right]$$

$$+ c_2 a_0 \left[\frac{2x^3}{3^3} - \frac{2x^6}{3^3 6^2} \cdots \right]$$