1.18.3. Case III. When $Q = x^m$, m being a positive integer.

Here $$\text{P.I.} = \frac{1}{f(D)} x^m$$

Steps:

1. Take out the lowest degree term from $f(D)$ to make the first term unity (so that Binomial Theorem for a negative index is applicable). The remaining factor will be of the form $1 + \phi(D)$ or $1 - \phi(D)$.
2. Take this factor in the numerator. It takes the form $[1 + \phi(D)]^{-1}$ or $[1 - \phi(D)]^{-1}$.
3. Expand it in ascending powers of D as far as the term containing D^m, since $D^{m+1}(x^m) = 0$, $D^{m+2}(x^m) = 0$ and so on.
4. Operate on x^m term by term.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the P.I. of* $(D^2 + 5D + 4)y = x^2 + 7x + 9$.

Sol. $$\text{P.I.} = \frac{1}{D^2 + 5D + 4}(x^2 + 7x + 9) = \frac{1}{4\left(1 + \frac{5D}{4} + \frac{D^2}{4}\right)}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 + \left(\frac{5D}{4} + \frac{D^2}{4}\right)\right]^{-1}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1-\left(\frac{5D}{4}+\frac{D^2}{4}\right)+\left(\frac{5D}{4}+\frac{D^2}{4}\right)^2 - \ldots\ldots\right](x^2+7x+9)$$

$$= \frac{1}{4}\left(1-\frac{5D}{4}-\frac{D^2}{4}+\frac{25D^2}{16}+\ldots\ldots\right)(x^2+7x+9)$$

$$= \frac{1}{4}\left(1-\frac{5D}{4}+\frac{21D^2}{16}\right)(x^2+7x+9) \quad | \text{ Leaving higher powers of D}$$

$$= \frac{1}{4}\left[(x^2+7x+9)-\frac{5}{4}D(x^2+7x+9)+\frac{21}{16}D^2(x^2+7x+9)\right]$$

$$= \frac{1}{4}\left[(x^2+7x+9)-\frac{5}{4}(2x+7)+\frac{21}{16}(2)\right] = \frac{1}{4}\left(x^2+\frac{9}{2}x+\frac{23}{8}\right).$$

Example 2. *Find the P.I. of $y'' - 6y' + 9y = 2x^2 - x + 3$.*

Sol. $$\text{P.I.} = \frac{1}{D^2-6D+9}(2x^2-x+3) = \frac{1}{(D-3)^2}(2x^2-x+3)$$

$$= \frac{1}{9}\left(1-\frac{D}{3}\right)^{-2}(2x^2-x+3) = \frac{1}{9}\left(1+\frac{2D}{3}+\frac{3D^2}{9}+\ldots\ldots\right)(2x^2-x+3)$$

$$= \frac{1}{9}\left(1+\frac{2D}{3}+\frac{D^2}{3}\right)(2x^2-x+3) \quad | \text{ Leaving higher powers of D}$$

$$= \frac{1}{9}\left[2x^2-x+3+\frac{2}{3}(4x-1)+\frac{1}{3}(4)\right] = \frac{1}{9}\left[2x^2+\frac{5}{3}x+\frac{11}{3}\right].$$

Example 3. *Solve: $(D^3 - D^2 - 6D)\, y = 1 + x^2$.*

Sol. Auxiliary equation is

$$m^3 - m^2 - 6m = 0$$

$$\Rightarrow \quad m(m-3)(m+2) = 0 \quad \Rightarrow \quad m = 0, -2, 3$$

$$\therefore \quad \text{C.F.} = c_1 + c_2 e^{-2x} + c_3 e^{3x}$$

$$\text{P.I.} = \frac{1}{D^3-D^2-6D}(1+x^2) = \frac{1}{-6D-D^2+D^3}(1+x^2)$$

$$= -\frac{1}{6D\left\{1+\left(\frac{D-D^2}{6}\right)\right\}}(1+x^2) = -\frac{1}{6D}\left[1+\left(\frac{D-D^2}{6}\right)\right]^{-1}(1+x^2)$$

$$= -\frac{1}{6D}\left[1-\left(\frac{D-D^2}{6}\right)+\left(\frac{D-D^2}{6}\right)^2 - \ldots\ldots\right](1+x^2)$$

$$= -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{D^2}{6}+\frac{D^2}{36}\right](1+x^2) = -\frac{1}{6D}\left[1+x^2-\frac{1}{6}(2x)+\frac{7}{36}(2)\right]$$

$$= -\frac{1}{6D}\left(1+x^2-\frac{x}{3}+\frac{7}{18}\right) = -\frac{1}{6D}\left(x^2-\frac{x}{3}+\frac{25}{18}\right)$$

$$= -\frac{1}{6}\left(\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right) = -\frac{x}{18}\left(x^2-\frac{x}{2}+\frac{25}{6}\right)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 + c_2 e^{-2x} + c_3 e^{3x} - \frac{x}{18}\left(x^2-\frac{x}{2}+\frac{25}{6}\right)$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 4. *Solve* $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$.

Sol. Auxiliary equation is

$$(m-2)^2 = 0 \Rightarrow m = 2, 2$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)\, e^{2x}$$

$$\text{P.I.} = \frac{1}{(D-2)^2}\,[8(e^{2x} + \sin 2x + x^2)]$$

$$= 8\left[\frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}\sin 2x + \frac{1}{(D-2)^2}x^2\right]$$

Now, $\frac{1}{(D-2)^2}e^{2x} = x \,.\, \frac{1}{2(D-2)}e^{2x}$ | Case of failure

$$= x^2 \,.\, \frac{1}{2}e^{2x}$$ | Case of failure

$$\frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2-4D+4}\sin 2x = \frac{1}{-2^2-4D+4}\sin 2x$$ [Putting $D^2 = -2^2$]

$$= -\frac{1}{4D}\sin 2x = -\frac{1}{4}\int \sin 2x\,dx = -\frac{1}{4}\left(-\frac{\cos 2x}{2}\right) = \frac{1}{8}\cos 2x$$

$$\frac{1}{(D-2)^2}x^2 = \frac{1}{(2-D)^2}x^2 = \frac{1}{4\left(1-\frac{D}{2}\right)^2}x^2 = \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}x^2$$

$$= \frac{1}{4}\left[1 + D + \frac{3}{4}D^2 + \ldots\ldots\right]x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

$$\therefore \quad \text{P.I.} = 8\left[\frac{x^2}{2}e^{2x} + \frac{1}{8}\cos 2x + \frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)\right]$$

$$= 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

where c_1 and c_2 are arbitrary constants of integration.

1.18.4. Case IV. When $Q = e^{ax} V$, where V is a function of x

Let u be a function of x, then by successive differentiation, we have

$$D(e^{ax} u) = e^{ax} Du + a e^{ax} u = e^{ax} (D + a)u$$

$$D^2 (e^{ax} u) = D[e^{ax} (D + a) u] = e^{ax} (D^2 + aD) u + ae^{ax} (D + a)u$$

$$= e^{ax} (D^2 + 2aD + a^2) u = e^{ax} (D + a)^2 u$$

Similarly, $\quad D^3 (e^{ax} u) = e^{ax} (D + a)^3 u$

In general, $\quad D^n (e^{ax} u) = e^{ax} (D + a)^n u$

$\therefore \quad f(D) (e^{ax} u) = e^{ax} f(D + a)u$

Operating on both sides by $\frac{1}{f(D)}$,

$$\frac{1}{f(D)} [f(D) (e^{ax} u)] = \frac{1}{f(D)} [e^{ax} f(D + a)u]$$

$$\Rightarrow \quad e^{ax} u = \frac{1}{f(D)} [(e^{ax} f(D + a) u]$$

Now let $\quad f(D + a) u = V$, *i.e.*, $u = \frac{1}{f(D+a)} V$

$\therefore$ From (1) we have $\quad e^{ax} \frac{1}{f(D+a)} V = \frac{1}{f(D)} (e^{ax} V)$

or

$$\boxed{\frac{1}{f(D)} (e^{ax} V) = e^{ax} \frac{1}{f(D+a)} V}$$

Thus e^{ax} which is on the right of $\frac{1}{f(D)}$ may be taken out to the left provided D is replaced by D + a.

ILLUSTRATIVE EXAMPLES

Example 1. *Obtain the general solution of the differential equation*

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}-12y=(x-1)\,e^{2x}.$$

Sol. The given equation is

$$(D^2+4D-12)\,y=(x-1)\,e^{2x} \qquad ...(1)$$

Auxiliary equation is

$$m^2+4m-12=0$$

$$\Rightarrow \quad (m-2)\,(m+6)=0$$

$$\Rightarrow \quad m=2,\,-6$$

$$\therefore \quad \text{C.F.}=c_1e^{2x}+c_2e^{-6x}$$

$$\text{P.I.}=\frac{1}{D^2+4D-12}\,(x-1)\,e^{2x}$$

$$=e^{2x}\,.\,\frac{1}{\{(D+2)^2+4(D+2)-12\}}\,(x-1)$$

$$=e^{2x}\,.\,\frac{1}{D^2+8D}\,(x-1)=e^{2x}\,.\,\frac{1}{8D}\left(1+\frac{D}{8}\right)^{-1}(x-1)$$

$$=e^{2x}\,.\,\frac{1}{8D}\left(1-\frac{D}{8}\right)(x-1) \qquad |\text{ Leaving higher power terms}$$

$$=e^{2x}\,.\,\frac{1}{8D}\left(x-1-\frac{1}{8}\right)=e^{2x}\,.\,\frac{1}{8D}\left(x-\frac{9}{8}\right)=\frac{e^{2x}}{8}\left(\frac{x^2}{2}-\frac{9x}{8}\right)$$

Hence the complete solution is

$$y=\text{C.F.}+\text{P.I.}=c_1e^{2x}+c_2e^{-6x}+e^{2x}\left(\frac{x^2}{16}-\frac{9x}{64}\right)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 2. *Find the complete solution of* $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+2y=xe^{3x}+\sin 2x.$

[G.B.T.U. (SUM) 2010]

Sol. Auxiliary equation is

$$m^2-3m+2=0$$

$$\Rightarrow \quad (m-1)\,(m-2)=0 \quad \Rightarrow \quad m=1,\,2$$

$$\therefore \quad \text{C.F.}=c_1e^{x}+c_2e^{2x}$$

$$\text{P.I.}=\frac{1}{D^2-3D+2}\,(x\,e^{3x}+\sin 2x)$$

$$=\frac{1}{D^2-3D+2}\,(e^{3x}\,.\,x)+\frac{1}{D^2-3D+2}\,(\sin 2x)$$

$$=e^{3x}\,.\,\frac{1}{\{(D+3)^2-3(D+3)+2\}}\,(x)+\frac{1}{-4-3D+2}\,(\sin 2x)$$

$$= e^{3x} \cdot \frac{1}{D^2 + 3D + 2}(x) + \frac{1}{-3D - 2}(\sin 2x)$$

$$= e^{3x} \cdot \frac{1}{2\left[1 + \left(\frac{3D + D^2}{2}\right)\right]}(x) - \frac{(3D - 2)}{9D^2 - 4}(\sin 2x)$$

$$= \frac{e^{3x}}{2} \cdot \left[1 + \left(\frac{3D + D^2}{2}\right)\right]^{-1}(x) - \frac{(3D - 2)}{(-40)}\sin 2x$$

$$= \frac{e^{3x}}{2}\left(1 - \frac{3D}{2}\right)(x) + \frac{1}{40}(6\cos 2x - 2\sin 2x)$$

$$= \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1e^x + c_2e^{2x} + \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 3. *Find the P.I. of* $(D^2 - 3D + 2)y = 2e^x \cos \frac{x}{2}$.

Sol. $$\text{P.I.} = \frac{1}{D^2 - 3D + 2}\left(2e^x \cos\frac{x}{2}\right)$$

$$= 2e^x \cdot \frac{1}{\{(D + 1)^2 - 3(D + 1) + 2\}}\cos\frac{x}{2} = 2e^x \cdot \frac{1}{D^2 - D}\cos\frac{x}{2}$$

$$= 2e^x \cdot \frac{1}{-\frac{1}{4} - D}\cos\frac{x}{2} = -2e^x\left[\frac{\left(\frac{1}{4} - D\right)}{\left(\frac{1}{4} - D\right)\left(\frac{1}{4} + D\right)}\cos\frac{x}{2}\right]$$

$$= -2e^x\frac{\left(\frac{1}{4} - D\right)}{\frac{1}{16} - D^2}\cos\frac{x}{2} = -2e^x \cdot \frac{\frac{1}{4} - D}{\left(\frac{1}{16} + \frac{1}{4}\right)}\cos\frac{x}{2}$$

$$= -\frac{32}{5}e^x\left(\frac{1}{4}\cos\frac{x}{2} + \frac{1}{2}\sin\frac{x}{2}\right) = -\frac{16}{5}e^x\left(\sin\frac{x}{2} + \frac{1}{2}\cos\frac{x}{2}\right).$$

1.18.5. Case V. When Q is any other function of x

Resolve $f(D)$ into linear factors.

Let $$f(D) \equiv (D - m_1)(D - m_2) \ldots\ldots (D - m_n)$$

Then $$\text{P.I.} = \frac{1}{f(D)} Q = \frac{1}{(D - m_1)(D - m_2) \ldots\ldots (D - m_n)} Q$$

$$= \left(\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots\ldots + \frac{A_n}{D - m_n} \right) Q \qquad \text{(Partial Fractions)}$$

$$= A_1 \frac{1}{D - m_1} Q + A_2 \frac{1}{D - m_2} Q + \ldots\ldots + A_n \frac{1}{D - m_n} Q$$

$$= A_1 e^{m_1 x} \int Q e^{-m_1 x} \, dx + A_2 e^{m_2 x} \int Q e^{-m_2 x} \, dx + \ldots\ldots + A_n e^{m_n x} \int Q e^{-m_n x} \, dx$$

Remark. Remember the following formulae:

(*i*) $$\frac{1}{D - \alpha} Q = e^{\alpha x} \int e^{-\alpha x} Q \, dx$$

(*ii*) $$\frac{1}{D + \alpha} Q = e^{-\alpha x} \int e^{\alpha x} Q \, dx$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the complete solution of* $(D^2 + a^2)y = \sec ax$.

[M.T.U. (B. Pharm.) 2011, U.K.T.U. 2011]

Sol. Auxiliary equation is

$$m^2 + a^2 = 0 \Rightarrow m = \pm ai$$

$\therefore$ C.F. $= c_1 \cos ax + c_2 \sin ax$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax$$

$$= \frac{1}{(D - ia)(D + ia)} \sec ax = \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right] \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia}(\sec ax) - \frac{1}{D + ia}(\sec ax)\right] = \frac{1}{2ia}(P_1 - P_2)$$

where $P_1 = \dfrac{1}{D - ia}(\sec ax)$

$$= e^{iax}\int e^{-iax} \sec ax \, dx = e^{iax}\int (\cos ax - i \sin ax) \sec ax \, dx$$

$$= e^{iax}\int (1 - i \tan ax) \, dx = e^{iax}\left\{x + i\left(\frac{\log \cos ax}{a}\right)\right\}$$

$$P_2 = \frac{1}{D + ia}(\sec ax) = e^{-iax}\left\{x - i\left(\frac{\log \cos ax}{a}\right)\right\}$$ | Replacing i by $-i$

$$\therefore \quad \text{P.I.} = \frac{1}{2ia}\left[e^{iax}\left\{x + i\left(\frac{\log \cos ax}{a}\right)\right\} - e^{-iax}\left\{x - i\left(\frac{\log \cos ax}{a}\right)\right\}\right]$$

$$= \frac{1}{2ia}\left[x(e^{iax} - e^{-iax}) + i\left(\frac{\log \cos ax}{a}\right)(e^{iax} + e^{-iax})\right]$$

$$= \frac{1}{2ia}\left[2ix \sin ax + \frac{i}{a}(\log \cos ax)(2 \cos ax)\right] = \frac{1}{a}\left[x \sin ax + \frac{1}{a} \cos ax \log \cos ax)\right]$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos ax + c_2 \sin ax + \frac{1}{a}\left(x \sin ax + \frac{1}{a} \cos ax \log \cos ax\right)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 2. *Find the general solution of the differential ...*