1.19. HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS (EULER-CAUCHY EQUATIONS)

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_{n-1} x \frac{dy}{dx} + a_n y = Q \qquad \ldots(1)$$

where a_i's are constants and Q is a function of x, is called Cauchy's homogeneous linear equation.

Such equations can be reduced to linear differential equations with constant coefficients by the substitution

$$x = e^z \quad \text{or} \quad z = \log x$$

so that
$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} \quad \text{or} \quad x\frac{dy}{dx} = \frac{dy}{dz} = Dy, \text{ where } D \equiv \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} \qquad \left(\because \frac{dz}{dx} = \frac{1}{x}\right)$$

or
$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D^2y - Dy = D(D-1)y$$

Similarly, $x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on.

Substituting these values is equation (1), we get a linear differential equation with constant coefficients, which can be solved by the methods already discussed.

1.19.1. Steps for Solution

1. Put $x = e^z$ so that $z = \log x$ and Let $D \equiv \frac{d}{dz}$

2. Replace $x\frac{d}{dx}$ by D,

$x^2 \frac{d^2}{dx^2}$ by $D(D-1)$

$x^3 \frac{d^3}{dx^3}$ by $D(D-1)(D-2)$ and so on.

3. By doing so, this type of equation reduces to linear differential equation with constant coefficients which is then solved as before.

1.20. LEGENDRE'S LINEAR DIFFERENTIAL EQUATION

An equation of the form

$$(a + bx)^n \frac{d^n y}{dx^n} + a_1(a + bx)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_{n-1}(a + bx)\frac{dy}{dx} + a_n y = Q \qquad \ldots(1)$$

where a_i's are constants and Q is a function of x, is called Legendre's linear differential equation.

Such equations can be reduced to linear differential equations with constant co-efficients by the substitution

$$a + bx = e^z \text{ i.e. } z = \log(a + bx) \text{ so that } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{b}{a+bx}\frac{dy}{dz}$$

or
$$(a + bx)\frac{dy}{dx} = b\frac{dy}{dz} = b\,\mathrm{D}y, \text{ where } \mathrm{D} \equiv \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{b}{a+bx}\frac{dy}{dz}\right) = -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\frac{d^2y}{dz^2} \cdot \frac{dy}{dx}$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\frac{d^2y}{dz^2} \cdot \frac{b}{a+bx} = \frac{b^2}{(a+bx)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

or
$$(a + bx)^2\frac{d^2y}{dx^2} = b^2(\mathrm{D}^2y - \mathrm{D}y) = b^2\,\mathrm{D}(\mathrm{D} - 1)y$$

Similarly, $(a + bx)^3 \dfrac{d^3y}{dx^3} = b^3\,\mathrm{D}(\mathrm{D} - 1)(\mathrm{D} - 2)y.$

Substituting these values in equation (*i*), we get a linear differential equation with constant coefficients, which can be solved by the methods already discussed.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right).$ [U.P.T.U. (C.O.) 2009]

Sol. Put $x = e^z$ so that $z = \log x$ and let $\mathrm{D} \equiv \dfrac{d}{dz}$ then the given differential equation reduces to

$$[\mathrm{D}(\mathrm{D} - 1)(\mathrm{D} - 2) + 2\mathrm{D}(\mathrm{D} - 1) + 2]y = 10(e^z + e^{-z})$$

or
$$(\mathrm{D}^3 - \mathrm{D}^2 + 2)y = 10\,(e^z + e^{-z})$$

which is a linear equation with constant coefficients.

Its Auxiliary equation is

$$m^3 - m^2 + 2 = 0 \quad \text{or} \quad (m + 1)(m^2 - 2m + 2) = 0$$

$$\therefore \quad m = -1, \frac{2 \pm \sqrt{4-8}}{2} = -1, 1 \pm i$$

$$\therefore \quad \text{C.F.} = c_1e^{-z} + e^z(c_2 \cos z + c_3 \sin z) = \frac{c_1}{x} + x\,[c_2 \cos(\log x) + c_3 \sin(\log x)]$$

$$\text{P.I.} = 10\,\frac{1}{D^3 - D^2 + 2}\,(e^z + e^{-z}) = 10\left(\frac{1}{D^3 - D^2 + 2}\,e^z + \frac{1}{D^3 - D^2 + 2}\,e^{-z}\right)$$

$$= 10\left(\frac{1}{1^3 - 1^2 + 2}\,e^z + z\,.\,\frac{1}{3D^2 - 2D}\,e^{-z}\right) = 10\left(\frac{1}{2}\,e^z + z\,.\,\frac{1}{3(-1)^2 - 2(-1)}\,e^{-z}\right)$$

$$= 5e^z + 2ze^{-z} = 5x + \frac{2}{x}\log x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = \frac{c_1}{x} + x\,[c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2}{x}\log x$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 2. *Solve:* $x^3\,\dfrac{d^3y}{dx^3} + 3x^2\,\dfrac{d^2y}{dx^2} + x\,\dfrac{dy}{dx} + y = x + \log x.$

Sol. Put $x = e^z$ so that $z = \log x$ and let $D \equiv \dfrac{d}{dz}$ then the given differential equation reduces to

$$[D(D-1)(D-2) + 3D(D-1) + D + 1]\,y = e^z + z$$

$$\Rightarrow \qquad (D^3 + 1)y = e^z + z$$

Auxiliary equation is

$$m^3 + 1 = 0$$

$$\Rightarrow \qquad (m+1)(m^2 - m + 1) = 0 \qquad \Rightarrow \qquad m = -1,\ \frac{1 \pm \sqrt{3}i}{2}$$

$$\therefore \qquad \text{C.F.} = c_1 e^{-z} + e^{z/2}\left(c_2 \cos\frac{\sqrt{3}}{2}z + c_3 \sin\frac{\sqrt{3}}{2}z\right)$$

$$\text{P.I.} = \frac{1}{D^3 + 1}\,(e^z + z) = \frac{1}{D^3 + 1}\,(e^z) + \frac{1}{1 + D^3}\,(z)$$

$$= \frac{e^z}{2} + (1 + D^3)^{-1}\,(z) = \frac{e^z}{2} + (1 - D^3)\,(z) \qquad \text{| Leaving higher terms}$$

$$= \frac{e^z}{2} + z$$

$\therefore$ The complete solution is

$$y = c_1 e^{-z} + e^{z/2}\left(c_2 \cos\frac{\sqrt{3}}{2}z + c_3 \sin\frac{\sqrt{3}}{2}z\right) + \frac{e^z}{2} + z$$

$$\therefore \qquad y = \frac{c_1}{x} + \sqrt{x}\left[c_2 \cos\frac{\sqrt{3}}{2}(\log x) + c_3 \sin\frac{\sqrt{3}}{2}(\log x)\right] + \frac{x}{2} + \log x$$

where c_1, c_2 and c_3 are the arbitrary constants of integration.

Example 3. *Solve:* $x^2\,\dfrac{d^2y}{dx^2} + x\,\dfrac{dy}{dx} - \lambda^2 y = 0.$ (U.P.T.U. 2007)

where c_1 and c_2 are arbitrary constants of integration.

Example 5. *Solve:* $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1.$

Sol. Given equation is a Legendre's linear differential equation.

Put $3x + 2 = e^z$ *i.e.,* $z = \log(3x+2)$ so that $(3x+2)\frac{dy}{dx} = 3Dy.$

$$(3x+2)^2 \frac{d^2y}{dx^2} = 3^2 D(D-1)y, \quad \text{where } D \equiv \frac{d}{dz}.$$

Substituting these values in the given equation, it reduces to

$$[3^2 D(D-1) + 3.3D - 36]y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

or $$9(D^2-4)y = \frac{1}{3}e^{2z} - \frac{1}{3}$$

or $$(D^2-4)y = \frac{1}{27}(e^{2z}-1)$$

which is a linear equation with constant co-efficients.

Its Auxiliary equation is $\qquad m^2 - 4 = 0 \quad \therefore \quad m = \pm 2$

$$\text{C.F.} = c_1 e^{2z} + c_2 e^{-2z} = c_1 (3x+2)^2 + c_2 (3x+2)^{-2}$$

$$\text{P.I.} = \frac{1}{27} \cdot \frac{1}{D^2-4}(e^{2z}-1) = \frac{1}{27}\left[\frac{1}{D^2-4}e^{2z} - \frac{1}{D^2-4}e^{0z}\right]$$

$$= \frac{1}{27}\left[z.\frac{1}{2D}e^{2z} - \frac{1}{0-4}e^{0z}\right] = \frac{1}{27}\left[\frac{z}{2}\int e^{2z}\,dz + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[\frac{z}{4}e^{2z} + \frac{1}{4}\right] = \frac{1}{108}(ze^{2z}+1) = \frac{1}{108}[(3x+2)^2 \log(3x+2) + 1]$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2 \log(3x+2) + 1].$$

where c_1 and c_2 are arbitrary constants of integration.

Example 6. *By reducing to homogeneous, solve the differential equation*

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\{\log(1+x)\}.$$

Sol. Put $1 + x = e^z$ so that $z = \log(1+x)$ and let $D \equiv \frac{d}{dz}$ then the given differential equation reduces to

$$\{D(D-1) + D + 1\}y = 4\cos z$$

$$\Rightarrow \qquad (D^2+1)y = 4\cos z$$

Auxiliary equation is

$$m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$$

$$\therefore \qquad \text{C.F.} = c_1 \cos z + c_2 \sin z$$

$$\text{P.I.} = \frac{1}{D^2+1}(4\cos z) = 4z.\frac{1}{2D}\cos z = 2z\sin z$$

Hence the complete solution is

$$y = c_1\cos z + c_2 \sin z + 2z\sin z$$

$$= c_1 \cos\{\log(1+x)\} + c_2 \sin\{\log(1+x)\} + 2\log(1+x)\sin\{\log(1+x)\}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 7. *Solve the differential equation:*

$$(3x+2)^2\frac{d^2y}{dx^2}-(3x+2)\frac{dy}{dx}-12y=6x.$$ (G.B.T.U. 2011)

Sol. Put $3x+2=e^z$ so that $z=\log(3x+2)$ and let $D\equiv\frac{d}{dz}$ then the given differential equation reduces to

$$[3^2D(D-1)-3D-12]y=6\left(\frac{e^z-2}{3}\right)$$

$$(9D^2-12D-12)y=2e^z-4 \qquad ...(1)$$

Auxiliary equation is

$$9m^2-12m-12=0$$

$$\Rightarrow \quad (9m+6)(m-2)=0$$

$$\Rightarrow \quad m=2,\ -\frac{2}{3}$$

$$\therefore \quad \text{C.F.}=c_1e^{2z}+c_2e^{-\frac{2}{3}z}$$

$$\text{P.I.}=\frac{1}{9D^2-12D-12}2e^z-\frac{1}{9D^2-12D-12}4e^{0z}=-\frac{2}{15}e^z+\frac{1}{3}$$

Hence complete solution is

$$y=c_1e^{2z}+c_2e^{-\frac{2}{3}z}-\frac{2}{15}e^z+\frac{1}{3}$$

$$=c_1(3x+2)^2+c_2(3x+2)^{-2/3}-\frac{2}{15}(3x+2)+\frac{1}{3}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 8. *Solve:* $(x+1)^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}=(2x+3)(2x+4).$

[M.T.U. (SUM) 2011; G.B.T.U. (C.O.) 2011]

Sol. Put $x+1=e^z$ so that $z=\log(x+1)$ and let $D\equiv\frac{d}{dz}$ then the given differential equation reduces to

$$[D(D-1)+D]y=(2e^z+1)(2e^z+2)$$

$$D^2y=4e^{2z}+6e^z+2$$

Auxiliary equation is

$$m^2=0 \ \Rightarrow \ m=0,0$$

$$\therefore \quad \text{C.F.}=c_1+c_2z$$

$$\text{P.I.}=\frac{1}{D^2}(4e^{2z}+6e^z+2)=e^{2z}+6e^z+z^2$$

Hence complete solution is

$$y=\text{C.F.}+\text{P.I.}=c_1+c_2z+e^{2z}+6e^z+z^2$$

$$=c_1+c_2\log(x+1)+(x+1)^2+6(x+1)+[\log(x+1)]^2$$

where c_1 and c_2 are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve:

1. $\dfrac{d^3y}{dx^3} - \dfrac{4}{x}\dfrac{d^2y}{dx^2} + \dfrac{5}{x^2}\dfrac{dy}{dx} - \dfrac{2y}{x^3} = 1$

2. $x^2 \dfrac{d^2y}{dx^2} - 2y = x^2 + \dfrac{1}{x}$

3. $x^2 \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 20y = (x+1)^2$

4. $x^2 \dfrac{d^3y}{dx^3} - 4x\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = 4$

5. (*i*) $x^4 \dfrac{d^3y}{dx^3} + 2x^3 \dfrac{d^2y}{dx^2} - x^2 \dfrac{dy}{dx} + xy = 1$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = x^2$

6. (*i*) $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 2\log x$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = x$

7. (*i*) $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2 \log x$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x\log x$

8. (*i*) $x^2y'' + xy' - y = x^3 e^x$. (*ii*) $\left(\dfrac{d}{dx} + \dfrac{1}{x}\right)^2 y = x^{-4}$. (M.T.U. 2012)

9. $x^2 \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 12y = x^3 \log x$

10. $(x^2D^2 - xD + 4)y = \cos(\log x) + x\sin(\log x)$

11. (*i*) $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = \sin(\log x)$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 4y = x\log x$

12. (*i*) $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x^{-1}$ (*ii*) $x^3 y''' + xy' - y = 3x^4$

13. (*i*) $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} - 3y = x + x^2$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^m$

14. (*i*) $x^4 \dfrac{d^4y}{dx^4} + 6x^3 \dfrac{d^3y}{dx^3} + 9x^2 \dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = (1 + \log x)^2$

 (*ii*) $[x^2D^2 - (2m-1)xD + (m^2+n^2)]y = n^2x^m \log x$

15. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x} \cdot \dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$

16. $(x+a)^2 \dfrac{d^2y}{dx^2} - 4(x+a)\dfrac{dy}{dx} + 6y = x$

17. $(1+2x)^2 \dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2$

18. $(2x+3)^2 \dfrac{d^2y}{dx^2} - 2(2x+3)\dfrac{dy}{dx} - 12y = 6x$

19. $x^2 \dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$

20. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{(\log x)\sin(\log x) + 1}{x}$.

Answers

1. $y = c_1x^2 + x^{5/2}\left(c_2x^{\sqrt{21}/2} + c_3x^{-\sqrt{21}/2}\right) - \frac{1}{5}x^3$

2. $y = c_1x^2 + \frac{c_2}{x} + \frac{1}{3}\left(x^2 + \frac{1}{x}\right)\log x$

3. $y = c_1x^5 + c_2x^{-4} - \frac{x^3}{14} - \frac{x}{9} - \frac{1}{20}$

4. $y = c_1 + c_2x^3 + c_3x^4 + \frac{2}{3}x$

5. (i) $y = (c_1 + c_2\log x)x + c_3x^{-1} + \frac{1}{4x}\log x$ (ii) $y = c_1x^2 + c_2x^3 - x^2\log x$

6. (i) $y = x(c_1 + c_2\log x) + 2\log x + 4$ (ii) $y = c_1x^2 + c_2x^3 + \frac{x}{2}$

7. (i) $y = c_1x^3 + \frac{c_2}{x} - \frac{x^2}{3}\left(\log x + \frac{2}{3}\right)$ (ii) $y = x[c_1\cos(\log x) + c_2\sin(\log x)] + x\log x$

8. (i) $y = c_1x + c_2 \cdot \frac{1}{x} + \left(x - 3 + \frac{3}{x}\right)e^x$ (ii) $y = x^{-1/2}\left[c_1\cos\frac{\sqrt{3}}{2}(\log x) + c_2\sin\frac{\sqrt{3}}{2}(\log x)\right] + \frac{1}{3}x^{-2}$

9. $y = c_1x^3 + c_2x^{-4} + \frac{x^3}{98}\log x\,(7\log x - 2)$

10. $y = x[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x)] + \frac{3}{13}\cos(\log x) - \frac{2}{13}\sin(\log x) + \frac{1}{2}x\sin(\log x)$

11. (i) $y = x^2[c_1\cos(\log x) + c_2\sin(\log x)] + \frac{1}{8}[\sin(\log x) + \cos(\log x)]$

(ii) $y = \left(\frac{c_1 + c_2\log x}{x^2}\right) + \frac{x}{9}\left(\log x - \frac{2}{3}\right)$

12. (i) $y = c_1x + c_2x^2 + \frac{1}{6x}$ (ii) $y = x[c_1 + c_2\log x + c_3(\log x)^2] + \frac{1}{9}x^4$

13. (i) $y = c_1x + c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x) + \frac{1}{7}x^2 + \frac{1}{4}x\log x$

(ii) $y = c_1x + \frac{c_2}{x} + \frac{x^m}{m^2 - 1}$

14. (i) $y = (c_1 + c_2\log x)\cos(\log x) + (c_3 + c_4\log x)\sin(\log x) + (\log x)^2 + 2\log x - 3$
(ii) $y = x^m[c_1\cos(n\log x) + c_2\sin(n\log x)] + x^m\log x$

15. $y = c_1\log x + c_2 + 2(\log x)^3$

16. $y = c_1(x + a)^2 + c_2(x + a)^3 + \frac{1}{2}(x + a) - \frac{1}{6}a$

17. $y = (1 + 2x)^2[c_1 + c_2\log(1 + 2x) + \{\log(1 + 2x)\}^2]$

18. $y = c_1(2x + 3)^{-1} + c_2(2x + 3)^3 - \frac{3}{4}(2x + 3) + 3$

19. $y = \frac{1}{x}(c_1 + c_2\log x) + \frac{1}{x}\log\left(\frac{x}{1 - x}\right)$

20. $y = x^2[c_1\cosh(\sqrt{3}\log x) + c_2\sinh(\sqrt{3}\log x)] + \frac{1}{61x}[\log x\{5\sin(\log x) + 6\cos(\log x)\}$

$+ \frac{2}{61}\{27\sin(\log x) + 191\cos(\log x)\}] + \frac{1}{6x}$