:- DIFFERENTIAL EQUATION :-

→ ① Ordinary Diff. Eqⁿ
② Partial Diff. Eqⁿ

$y = f(x)$

$$\left.\begin{array}{l} dy/dx \\ d^2y/dx^2 \\ d^3y/dx^3 \end{array}\right\} \quad F(x, y, dy/dx, d^2y/dx^2 \ldots) = 0$$

① **Ordinary Diff. Eqⁿ** :- ① One Independent variable.
② One or more dependent variable.

Ex:- $\frac{dy}{dx} + x^2y = \sin x$, $\frac{dy}{dt} + \frac{dx}{dt} = \sin t$

② **Partial Diff. Eqⁿ** :- ① More than 1 Independent variable.

ORDER AND DEGREE OF DIFFERENTIAL EQⁿ :-

→ Order ⇒ Highest Order Derivative
Degree ⇒ Degree of highest order Derivative.

Ex:- $\left(\frac{d^2y}{dx^2}\right)^1 + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$

Sol:
Order ⇒ 2
Degree ⇒ 1

Ex:- $\sqrt{\left(\frac{dy}{dx}\right)^2 + 1} = \left(\frac{d^2y}{dx^2}\right)$

Sol:
B.S-Square

$$\left(\frac{dy}{dx}\right)^2 + 1 = \left(\frac{d^2y}{dx^2}\right)^2$$

Order ⇒ 2
Degree ⇒ 2

Formation of Ordinary Differential Equation :-

Q. find the D.E

$$y = c_1 \cos 2x + c_2 \sin 2x \longrightarrow \boxed{\text{Arbitrary Constant} \Rightarrow \text{order of D.E}}$$

Sol. Differentiating

$$\frac{dy}{dx} = -2c_1 \sin 2x + 2c_2 \cos 2x$$

$$\frac{d^2y}{dx^2} = -4c_1 \cos 2x - 4c_2 \sin 2x$$

$$\frac{d^2y}{dx^2} \Rightarrow -4\left[c_1 \cos 2x + c_2 \sin 2x\right]$$

$$\frac{d^2y}{dx^2} = -4y$$

Order $\Rightarrow 2 \Rightarrow$ Arbitrary constant

Q. find the DE

$$y = A\cos x^2 + B\sin x^2$$

Sol.

$$\frac{dy}{dx} = -A\sin x^2(2x) + B\cos x^2(2x)$$

$$\frac{dy}{dx} \Rightarrow \underbrace{2x}_{\text{I}} \underbrace{\left[-A\sin x^2 + B\cos x^2\right]}_{\text{II}}$$

$$\frac{d^2y}{dx^2} \Rightarrow 2x\left[-A\cos x^2(2x) - B\sin x^2(2x)\right] + \left[-A\sin x^2 + B\cos x^2\right]2$$

$$\frac{d^2y}{dx^2} \Rightarrow -4x^2\left[A\cos x^2 + B\sin x^2\right] + 2\left[-A\sin x^2 + B\cos x^2\right]$$

$$\frac{d^2y}{dx^2} = -4x^2y + \frac{1}{x}\frac{dy}{dx}$$

$$\boxed{\text{order} \Rightarrow 2}$$

→ Solution of D.E :- General solⁿ / complete solⁿ

→ Particular solⁿ

First Order Ordinary Diff - Eq^n :-

i) Separation of variables.
ii) Exact D.E
iii) Reducible to Exact D.E
iv) Homogenous D.E
v) Reducible of Homogenous D.E
vi) Linear D.E of first order
vii) Bernoulli's DE

i) Separation of variables :-

Consider the D.E

$$\frac{dy}{dx} = f(x,y) \longrightarrow ①$$

Eq ① can be separable of variables if

$$\frac{dy}{dx} = f_1(x) \cdot f_2(y)$$

$$\frac{dy}{f_2(y)} = f_1(x)\,dx$$

Integrating $\int \frac{dy}{f_2(y)} = \int f_1(x)\,dx + C$

Q. Solve :. $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Sol→ $\frac{dy}{dx} = e^{-y}\left[e^x + x^2\right]$

$$\int \frac{dy}{e^{-y}} = \int (e^x + x^2)\,dx$$

$$\boxed{e^y = e^x + x^3/3 + c}$$ Ans.

Q Solve $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

Sol→ $(e^y+1)\cos x\,dx = -e^y \sin x\,dy$

$$\frac{\cos x}{\sin x}\,dx \Rightarrow \frac{-e^y}{e^y+1}\,dy$$

$$\int \cot x\,dx = -\int \frac{e^y}{e^y+1}\,dy \qquad e^y+1 = t$$

$$\log|\sin x| = -\int \frac{dt}{t} \qquad e^y\,dy = dt$$

$$\log \sin x = -\log t + c$$

$$\boxed{\log \sin x = -\log(e^y+1) + c}$$ Ans

Q $(x^2 - yx^2)\,dy + (y^2 + xy^2)\,dx = 0$ Solve :.

Sol→ $x^2(1-y)\,dy + y^2(1+x)\,dx = 0$

$$x^2(1-y)\,dy = -y^2(1+x)\,dx$$

$$\left(\frac{1-y}{y^2}\right)dy = -\left(\frac{1+x}{x^2}\right)dx \longrightarrow \text{Separation of variables}$$

$$\left(\frac{1}{y^2} - \frac{1}{y}\right)dy = -\left(\frac{1}{x^2} + \frac{1}{x}\right)dx$$

Ordinary Diff. Eq^n of First Order → Exact Form :-

Definition :- A first order O.D.E is called exact if it can be derived from its primitive (solution).

Necessary and sufficient cond^n :- A D.E of the form

$$\boxed{M\,dx + N\,dy = 0}, \text{ where } M = M(x,y),\ N = N(x,y)$$

is called exact D.E if and only if $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

Complete Sol^n :- The complete sol^n of Exact D.E $M\,dx + N\,dy = 0$ is given by

$$\boxed{\underbrace{\int M\,dx}_{\text{(y as constant)}} + \underbrace{\int N\,dy}_{\text{(free from x term)}} = C}$$

Remark :-

If D.E. $M\,dx + N\,dy = 0$ is not exact then by multiplying a function $f(x,y)$ which called Integrating factor, The given D.E can be reduced into exact D.E.

Rule
i) By Inspection Method :-

i) $x\,dy + y\,dx = d(xy)$

ii) $x\,dy - y\,dx \Rightarrow I.F = \dfrac{1}{x^2}$

iii) $y\,dx - x\,dy \Rightarrow I.F = \dfrac{1}{y^2}$

Rule
ii) In diff. eq^n $M\,dx + N\,dy = 0$; $Mx + Ny \neq 0$ and M & N are both homogenous, then $\boxed{I.F \Rightarrow \dfrac{1}{Mx+Ny}}$

$$\downarrow$$

$$(x^3 + 3x^2y + y^3)$$

Remark: $Mx + Ny = 0$ & M, N are both homogenous.

$$Mx = -Ny$$

$$\frac{M}{N} = \frac{-y}{x}$$

$$M\,dx + N\,dy = 0$$

$$\Rightarrow \frac{M}{N}dx + dy = 0$$

$$-\frac{y}{x}dx + dy = 0$$

$$-\frac{dx}{x} + \frac{dy}{y} = 0$$

$$-\log x + \log y = \log c$$

$$\log\left(\frac{y}{x}\right) = \log c$$

$$\frac{y}{x} = c \qquad \boxed{y = cx}$$

Rule ③ If the eqⁿ $M\,dx + N\,dy = 0$ has the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$ & $Mx - Ny \neq 0$

then the $\boxed{I.F \Rightarrow \frac{1}{Mx - Ny}}$

Ex: $(1+xy)y\,dx$

$(x^2y^2+2)y\,dx$

Remark If $Mx - Ny = 0$

$$\frac{M}{N} = \frac{y}{x}$$

$$M\,dx + N\,dy = 0$$

$$\frac{M}{N}dx + dy = 0$$

$$\frac{y}{x}dx + dy = 0$$

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

$$\log x + \log y = \log c$$

$$\boxed{xy = c}$$

Rule ④ If in Diff. eq^n $M\,dx + N\,dy = 0$

$$f(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \text{ [function of } x \text{ only]}$$

$$I.F = e^{\int f(x)\,dx}$$

Rule ⑤ If in D.E $M\,dx + N\,dy = 0$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y) \text{ [function of } y \text{ only]}$$

$$I.F = e^{\int f(y)\,dy}$$

Solve :-

① $(1+e^{x/y})dx + e^{x/y}(1 - x/y)\,dy = 0$

Sol^n $M\,dx + N\,dy = 0$

Here $M = 1+e^{x/y}$, $N = e^{x/y}(1-x/y)$

$$\frac{\partial M}{\partial y} = e^{x/y}\left(\frac{-x}{y^2}\right) \qquad \frac{\partial N}{\partial x} = e^{x/y}\left(0 - \frac{1}{y}\right) + \left(1-\frac{x}{y}\right)\cdot e^{x/y}\frac{1}{y}$$

$$\frac{\partial N}{\partial x} = \cancel{\frac{-e^{x/y}}{y}} + \cancel{\frac{e^{x/y}}{y}} - \frac{x}{y^2}e^{x/y}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$ So Given D.E is exact

$\therefore$ Its complete sol^n :-

$$\int_{y = \text{constant}} M\,dx + \int N\,dy = c \quad \text{(free from } x)$$

$$\int 1+e^{x/y}\,dx + \int 0\,dy = c$$

$$x + \frac{e^{x/y}}{1/y} = c$$

$$\boxed{x + ye^{x/y} = c} \text{ Ans}$$

* Linear Diff. Eq^n of first order :-

Standard Form :- $\boxed{\frac{dy}{dx} + Py = Q(x)}$ → when degree of $y = 1$

① $I.F = e^{\int P dx}$

complete solution of D.E :- $y(I.F) = \int Q \cdot (I.F)\, dx + c$

→ $\boxed{\frac{dx}{dy} + Px = Q(y)}$ → when degree of $x = 1$

① $I.F = e^{\int P dy}$

Complete sol^n of D.F $=$ $x \cdot (I.F) = \int Q(I.F)\, dy + c$

Other Form of Linear D.E [BERNOULLI'S FORM] :-

$$\frac{dy}{dx} + Py = Qy^n$$

→ $y^{-n}\frac{dy}{dx} + y^{1-n} P = Q$

Put $y^{1-n} = v$

$(1-n)\, y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$ $\quad y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx}$

$$\frac{1}{1-n}\frac{dv}{dx} + vP \Rightarrow Q$$

$$\frac{dv}{dx} + (1-n)vP = Q(1-n)$$

Q. Solve :- ① $(1+xy^2)\frac{dy}{dx} \Rightarrow 1$

Soln $\cancel{\frac{dy}{dx}} + \cancel{xy^2\frac{dy}{dx}} \Rightarrow \cancel{1}$

Given DE can be written as :-

$$\frac{dx}{dy} \Rightarrow 1 + xy^2$$

$$\frac{dx}{dy} - xy^2 \Rightarrow 1$$

$$\frac{dx}{dy} + Px \Rightarrow Q$$

$$I.F \Rightarrow e^{\int -y^2 dy} \Rightarrow e^{-y^3/3}$$

$$x \cdot e^{-y^3/3} \Rightarrow \int e^{-y^3/3} \cdot 1\, dy + c \text{ Ans}$$

Q. $\frac{dy}{dx} + \frac{y}{x} \Rightarrow x^2$

Soln $P \Rightarrow \frac{1}{x}$ $\quad I.F \Rightarrow e^{\int P dx} \Rightarrow e^{\int \frac{1}{x}dx} \Rightarrow e^{\log x} \Rightarrow x$

$$y \cdot x = \int x^2 \cdot x$$

$$y \cdot x \Rightarrow \frac{x^4}{4} + c \text{ Ans}$$

Q. $\frac{dy}{dx} + \frac{y}{x} \Rightarrow y^2 \sin x$

Soln $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{yx} \Rightarrow \sin x$

Let $\frac{1}{y} \Rightarrow v$ $\qquad (-1)y^{-2}\frac{dy}{dx} \Rightarrow \frac{dv}{dx}$

First Order Differential Equation [Homogenous form] :-

$$\boxed{\frac{dy}{dx} \Rightarrow \frac{f_1(x,y)}{f_2(x,y)}}$$ Homogenous function of same order

$$\frac{y}{x} \Rightarrow v \qquad y \Rightarrow vx$$

$$\frac{dy}{dx} \Rightarrow v + x\frac{dv}{dx}$$

$$\frac{dy}{dx} \Rightarrow f(y/x)$$

$$v + x\frac{dv}{dx} \Rightarrow f(v)$$

$$x\frac{dv}{dx} \Rightarrow f(v) - v$$

$$\int \frac{dv}{f(v)-v} \Rightarrow \int \frac{dx}{x} + c$$

Q. Solve $x(x-y)\,dy + y^2 dx = 0$

Sol $(x^2 - xy)\,dy + y^2 dx = 0$

$$\frac{dy}{dx} \Rightarrow \frac{y^2}{xy - \cancel{y}^2}$$

$$\frac{y}{x} \Rightarrow v \qquad y = vx$$

$$\frac{dy}{dx} \Rightarrow v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{v^2x^2}{x^2v - x^2}$$

$$v + x\frac{dv}{dx} \Rightarrow \frac{v^2\cancel{x^2}}{\cancel{x^2}(v-1)}$$

$$x\frac{dv}{dx} \Rightarrow \frac{v^2}{v-1} - v$$

$$x\frac{dv}{dx} \Rightarrow \frac{\cancel{v^2} - \cancel{v^2} + v}{v-1}$$

$$\int \frac{(v-1)\,dv}{v} \Rightarrow \int \frac{dx}{x}$$

Reducible to Homogenous D.E :-

Eqⁿ of form :- $\dfrac{dy}{dx} = \dfrac{ax+by+c}{lx+my+n}$ which is not Homogenous D.E

Case ① : when $\dfrac{a}{l} = \dfrac{b}{m} = s$

$$\frac{dy}{dx} = \frac{s(lx+my)+c}{lx+my+n}$$

Take $lx+my = v$

$$l + m\frac{dy}{dx} = \frac{dv}{dx}$$

$$l + m\left(\frac{sv+c}{v+n}\right) = \frac{dv}{dx}$$

$$\frac{dv}{dx} = F(v)$$

$$\boxed{\frac{dv}{F(v)} = dx}$$

Case ⑪ : $\dfrac{a}{l} \neq \dfrac{b}{m}$ $\Rightarrow x = X+h,\ y = Y+k$

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{lX+mY+(lh+mk+n)}$$

Choose h, k such that $\boxed{\begin{array}{l} ah+bk+c=0 \\ lh+mk+n=0 \end{array}}$ → Solve for h & k

$\dfrac{dY}{dX} = \dfrac{aX+bY}{lX+mY}$, This can be resolved by taking $Y = vx$

& put $Y = y-k$

$X = x-h$ at the end.

Linear Differential Eq^n with Constant Coefficient :-

The n^{th} order Linear D·E is written as

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = Q \longrightarrow (1)$$

or

$$\left(a_0\frac{d^n}{dx^n} + a_1\frac{d^{n-1}}{dx^{n-1}} + a_2\frac{d^{n-2}}{dx^{n-2}} + \cdots + a_{n-1}\frac{d}{dx} + a_n\right)y = Q \longrightarrow (II)$$

where $a_0, a_1, a_2, \ldots a_n$ are constants & $Q = Q(x)$

use $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, $\ldots$ $D^n = \frac{d^n}{dx^n}$ then (II) become

$$\left(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots a_{n-1} D + a_n\right)y = Q \longrightarrow (III)$$

$$\Rightarrow \boxed{f(D)y = Q} \longrightarrow (4)$$

where $D = \frac{d}{dx}$, $\frac{1}{D} = \int dx$

Remark :- (1) If $Q = 0$, then (4) called Homogenous L·D·E

(2) If $Q \neq 0$ then (4) called non-homogenous L·D·E

Ex :- $(D^2 + 2D + 3)y = \sin x$, $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^{2x}$

General Solution or Complete solution :

The general sol^n of n^{th} order L·D·E $f(D)y = Q \longrightarrow (1)$

consist two parts i) C·F (complementary function)

ii) P·I (Particular Integral) so

Solution of (1) $\boxed{y = C.F + P.I}$

Working Rule for C.F :-

Consider the D.E $(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = Q \rightarrow (1)$

for CF first we find auxiliary eq^n for (1), for this put $\boxed{Q = 0, \; y = 1 \; \& \; D = m}$ into eq (1), so we get a n^{th} degree eq^n in terms of 'm'.

$$\underbrace{a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_{n-1} m + a_n = 0}_{\text{Auxiliary eq}^n} \longrightarrow (2)$$

Let $m = m_1, m_2, m_3 \dots m_n$ are n roots of A.E (2) :.

The value of CF depends on the nature of these roots we have following cases :-

Case-(I) when roots are real and distinct then

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

Case (II) when roots are real & equal then :

2 roots are equal ($m_1 = m_2$) $\quad C.F = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \dots$

3 roots equal ($m_1 = m_2 = m_3$) $\quad C.F = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \dots$

Case (III) when roots are imaginary (in pairs) i.e $\boxed{m = \alpha \pm i\beta}$

$$C.F = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$$

OR

$$CF = c_1 e^{\alpha x} \sin(\beta x + c_2) \quad \text{OR} \quad C.F = c_1 e^{\alpha x} \cos(\beta x + c_2)$$

Case (IV) When roots are irrational $(m = \alpha \pm \sqrt{\beta})$

$$CF = c_1 e^{(\alpha+\sqrt{\beta})x} + c_2 e^{(\alpha-\sqrt{\beta})x}$$

OR

$$CF = c_1 e^{\alpha x} \cosh(\sqrt{\beta}\, x + c_2)$$

★ Rules to find P.I :- Consider the D.E :- $f(D)y = Q \longrightarrow ①$

The P.I of ① as defined as :- $\boxed{P.I = \frac{1}{f(D)} Q} \longrightarrow ②$

① when $f(D) = D \Rightarrow P.I = \frac{1}{D} Q = \int Q\,dx$ $\quad \boxed{\because \frac{1}{D} = \int dx}$

② when $f(D) = D - a \Rightarrow P.I = \frac{1}{D-a} Q = e^{ax} \int e^{-ax} Q\,dx$

[General Rule]

Remark :- $f(D) = (D-a)(D-b)(D-c)$

③ when $Q = e^{ax}$ where a is constant then

$$P.I = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad (D \Rightarrow a)$$

but $\boxed{f(a) \neq 0}$

Remarks :- If $f(a) = 0 \Rightarrow f(D) = (D-a)^n \phi(D)$

where $(\phi(a) \neq 0)$

$$P.I = \frac{1}{(D-a)^n \phi(D)} e^{ax} = \frac{1}{\phi(a)} \left[\frac{1}{(D-a)^n} e^{ax} \right] = \boxed{\frac{1}{\phi(a)}} \times \left[\frac{x^n}{n!} e^{ax} \right]$$

④ when $Q = \sin ax$ or $\cos ax$

$$P.I = \frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$$

Substitute $\left[\begin{array}{l} D^2 \to -a^2 \\ D^3 \to D(-a^2) \end{array} \right]$

if $f(-a^2) \neq 0$

$\boxed{\to \text{Replace } D^2 \text{ by } -a^2}$

⑤ when $Q = x^m$ (where m is +ve integer)

then $P.I = \frac{1}{f(D)} Q = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$

$[f(D)]^{-1} = [1 \pm f(D)]^{-1}$

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⑥ when $Q = e^{ax}$, $v(x) = e^{ax} v$

$$P.I = \frac{1}{f(D)} e^{ax} v = e^{ax} \left[\frac{1}{f(D+a)} v \right]$$

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⑦ (Remark): If $f(D) = D^2 + a^2$

then $P.I = \frac{1}{D^2+a^2} \sin ax = \frac{-x}{2a} \cos ax$

$P.I = \frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$

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⑧ If $Q = x^m v(x)$

then $P.I = \frac{1}{f(D)} x^m v(x) = x^m \frac{1}{f(D)} v + m x^{m-1} \left[\frac{d}{dD} \frac{1}{f(D)} \right] v$

$+ \frac{m(m-1)}{2!} x^{m-2} \left[\frac{d^2}{dD^2} \frac{1}{f(D)} \right] v$

$+ ----$

α —— α

Q.1 Solve :- $\left[(D-1)^2 (D-3)^3\right] y = e^{3x}$

Sol: Auxiliary eq^n :- $D = m$, $y = 1$, $e^{3x} \Rightarrow 0$

$$(m-1)^2 (m-3)^3 \Rightarrow 0$$

$$m = 1, 1, 3, 3, 3$$

$$\therefore C.F = (c_1 + c_2 x) e^{x} + (c_3 + c_4 x + c_5 x^2) e^{3x}$$

$$P.I = \frac{1}{(D-1)^2 (D-3)^3} e^{3x}$$

$$\Rightarrow \frac{1}{(3-1)^2 (D-3)^3} e^{3x} \Rightarrow \frac{1}{4}\left[\frac{1}{(D-3)^3} e^{3x}\right] \Rightarrow \frac{1}{4}\left[\frac{x^3 e^{3x}}{3!}\right]$$

$$\therefore P.I \Rightarrow \frac{1}{24} x^3 e^{3x}$$

Hence complete solution :-

$$y = C.F + P.I$$

$$y = (c_1 + c_2 x) e^{x} + (c_3 + c_4 x + c_5 x^2) e^{3x} + \frac{1}{24} x^3 e^{3x} \quad \text{Ans}$$

Q(2) $\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y \Rightarrow 4\cos^2 x$

Sol: Auxiliary eq^n :- $m^2 + 3m + 2 \Rightarrow 0$

$$(m+2)(m+1) \Rightarrow 0$$

$$m = -2 \quad m = -1$$

$$C.F \Rightarrow c_1 e^{-2x} + c_2 e^{-x}$$

$$P.I = \frac{1}{D^2 + 3D + 2} 4\cos^2 x \Rightarrow \frac{1}{D^2+3D+2} 2 \cdot 2\cos^2 x \Rightarrow \frac{1}{D^2+3D+2} 2 \cdot (1 + \cos 2x)$$

$$P.I = \frac{2}{D^2+3D+2} + \frac{1}{D^2+3D+2} 2\cos 2x$$

$$P.I \Rightarrow \frac{2}{D^2+3D+2} e^{0x} + 2 \cdot \frac{1}{(-2)^2+3D+2} \cos 2x$$

$$P.I \Rightarrow \frac{\cancel{2}}{0+0+\cancel{2}} + 2 \cdot \frac{1}{-4+2+3D} \cos 2x$$

$$P.I \Rightarrow 1 + 2 \cdot \frac{(3D+2)}{(3D-2)(3D+2)} \cos 2x$$

$$\Rightarrow 1 + \frac{2(3D+2)}{9D^2-4} \cos 2x$$

$$\Rightarrow 1 + \frac{2(3D+2)\cos 2x}{9(-2)^2-4}$$

$$\Rightarrow 1 - \frac{1}{20}(3D \cos 2x + 2\cos 2x)$$

$$P.I \Rightarrow 1 - \frac{1}{20}(-6\sin 2x + 2\cos 2x)$$

Complete solⁿ :- $y = c_1 e^{-2x} + c_2 e^{-x} + 1 - \frac{1}{10}(3\sin 2x + \cos 2x)$

Ans

Q. $(D^3+2D^2+D)y = e^{2x} + x^2 + x$

Sol Auxiliary eqⁿ :- $(m^3+2m^2+m) = 0$

$$m(m^2+2m+1) = 0$$

$$m = 0 \quad (m+1)^2 = 0$$

$$m = 0, \quad m = -1, -1$$

$$\therefore C.F \Rightarrow c_1 e^{0x} + (c_2 + c_3 x) e^{-x}$$

$$P.I = \frac{1}{D^3+2D^2+D}(e^{2x}+x^2+x)$$

$$\Rightarrow \frac{e^{2x}}{D^3+2D^2+D} + \frac{x^2+x}{D^3+2D^2+D}$$

$$\Rightarrow \frac{e^{2x}}{(2)^3+2(2)^2+2} + \frac{1}{D(1+D^2+2D)} x^2+x$$

$$\Rightarrow \frac{e^{2x}}{18} + \frac{1}{D}(1+2D+D^2)^{-1}(x^2+x)$$

$$(1+x)^{-1} \Rightarrow 1-x+x^2-x^3+x^4-x^5+\cdots$$

$$P.I \Rightarrow \frac{1}{18}e^{2x} + \frac{1}{D}\left(1-(2D+D^2)+(2D+D^2)^2+\cdots\right)(x^2+x)$$

$$\Rightarrow \frac{1}{18}e^{2x} + \frac{1}{D}(-2D-D^2+1+4D^2)(x^2+x)$$

$$P.I \Rightarrow \frac{1}{18}e^{2x} + \frac{1}{D}(1-2D+3D^2)(x^2+x)$$

$$\Rightarrow \frac{1}{18}e^{2x} + \frac{1}{D}\left[x^2+x-2(2x+1)+3(2+0)\right]$$

$$\Rightarrow \frac{1}{18}e^{2x} + \frac{1}{D}\left[x^2-3x+4\right]$$

$$P.I \Rightarrow \frac{1}{18}e^{2x} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x$$

Complete solⁿ :- $y = C.F + P.I$

Q. Solve :- $(D^2-4D+4)y \Rightarrow 8x^2 e^{2x} \sin 2x$

Sol> A.E :- $m^2-4m+4 \Rightarrow 0$

$(m-2)^2 \Rightarrow 0$

$m = 2, 2$

C.F $\Rightarrow (c_1 + c_2 x) e^{2x}$

P.I $\Rightarrow \dfrac{1}{D^2-4D+4} 8x^2 e^{2x} \sin 2x$

$\Rightarrow \dfrac{1}{(D-2)^2} 8x^2 e^{2x} \sin 2x$

$\Rightarrow 8e^{2x} \dfrac{1}{(D+2-2)^2} x^2 \sin 2x$ [Rule 6 used]

[$D \rightarrow D+a$]

$\Rightarrow 8e^{2x} \dfrac{1}{D^2} x^2 \sin 2x$

$\Rightarrow 8e^{2x} \dfrac{1}{D} \left[\int \underset{I}{x^2} \underset{II}{\sin 2x} \right]$

$\Rightarrow 8e^{2x} \dfrac{1}{D} \left[x^2 \dfrac{-\cos 2x}{2} - 2x \left(\dfrac{-\sin 2x}{4}\right) + 2\left(\dfrac{\cos 2x}{8}\right) \right.$

$\boxed{y = C.F + P.I}$ → complete solⁿ

Q. Solve Ordinary Differential Eqⁿ :-

1) $(D^3-1)y = (e^x+1)^2$

Sol. A.E of given DE :-

$(m^3-1) = 0$

$(m-1)(m^2+m+1) = 0$

$m = 1 \quad m = \frac{-1 \pm \sqrt{1-4}}{2 \cdot 1} = \frac{-1 \pm i\sqrt{3}}{2}$

$m = 1, \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$

$CF = c_1 e^x + e^{-1/2x}\left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right)$

$P.I = \frac{1}{(D^3-1)}(e^x+1)^2 = \frac{1}{D^3-1}(e^{2x}+2e^x+1)$

$P.I = \frac{e^{2x}}{D^3-1} + \frac{2e^x}{D^3-1} + \frac{1}{D^3-1}$

$P.I = \frac{1}{2^3-1}e^{2x} + 2x \cdot \frac{1}{3D^2}e^x + \frac{1}{D^3-1}e^{0x}$

$P.I = \frac{1}{7}e^{2x} + 2x \cdot \frac{1}{3(1)^2}e^x + (-1)$

$P.I = \frac{e^{2x}}{7} + \frac{2}{3}xe^x - 1$

Complete Solution :- $y = C.F + P.I$

$y = c_1e^x + e^{-1/2x}\left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right) + \frac{e^{2x}}{7} + \frac{2}{3}xe^x - 1$

Q $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y \Rightarrow xe^x \sin x$

Sol→ $m^2 - 2m + 1 \Rightarrow 0$

$(m-1)^2 \Rightarrow 0$

$m \Rightarrow 1, 1$

$\therefore$ C.F $\Rightarrow (c_1 + c_2 x)e^x$

P.I $\Rightarrow \frac{1}{(D-1)^2} xe^x \sin x$

P.I $\Rightarrow e^x \cdot \frac{1}{(D+1-1)^2} x \sin x \Rightarrow e^x \cdot \frac{1}{D^2} x \sin x$

P.I $\Rightarrow e^x \cdot \frac{1}{D}\left[\int x \sin x \, dx\right]$

P.I $\Rightarrow e^x \cdot \frac{1}{D}\left[x(-\cos x) - 1(-\sin x)\right]$

P.I $\Rightarrow e^x \int (-x\cos x + \sin x)\, dx$

P.I $\Rightarrow -e^x (x\sin x + 2\cos x)$

$\therefore$ Complete solution of D.E :- $y = \text{C.F} + \text{P.I}$

$y \Rightarrow (c_1 + c_2 x)e^x - e^x (x\sin x + 2\cos x)$ Ans

Q $(D^2 + a^2)y = \sin ax$

Sol→ A.E of given D.E :-

$m^2 + a^2 \Rightarrow 0$ $\quad m^2 \Rightarrow -a^2$

$m \Rightarrow \pm ia$ $\quad m \Rightarrow \pm\sqrt{-a^2}$

C.F $\Rightarrow c_1 \cos ax + c_2 \sin ax$

P.I $\Rightarrow \frac{1}{D^2 + a^2} \sin ax$

P.I $\Rightarrow \frac{x \sin ax}{2D}$

$$P.I \Rightarrow x \cdot \frac{D}{2D^2} \sin ax$$

$$P.I \Rightarrow \frac{x \cdot D \sin ax}{2(-a^2)}$$

$$P.I \Rightarrow \frac{x}{-2a^{\cancel{2}}} \cos ax \,(\cancel{a}) \Rightarrow \frac{-x}{2a} \cos ax$$

$\therefore$ Complete solⁿ of given D.E :- $y = C.F + P.I$

$$y \Rightarrow C_1 \cos ax + C_2 \sin ax - \frac{x}{2a} \cos ax \text{ Ans}$$

④ $(D^2 + a^2)y \Rightarrow \cos ax$

Soln A.E :- $m^2 + a^2 \Rightarrow 0$

$$m^2 \Rightarrow -a^2$$

$$m \Rightarrow \pm\sqrt{-a^2} \Rightarrow \pm ai$$

$$C.F \Rightarrow C_1 \cos ax + C_2 \sin ax$$

$$P.I \Rightarrow \frac{1}{D^2+a^2} \cos ax \Rightarrow \frac{x \cdot \cos ax}{2D} \Rightarrow \frac{x \cdot D \cos ax}{2D^2}$$

$$P.I \Rightarrow \frac{x \cdot D \cos ax}{2(-a^2)} \Rightarrow \frac{\cancel{-}x \cdot \sin ax \,(\cancel{a})}{\cancel{-}2a^{\cancel{2}}} \Rightarrow \frac{x \sin ax}{2a}$$

Complete solⁿ :- $y = C.F + P.I$

$$y = C_1 \cos ax + C_2 \sin ax + \frac{x \sin ax}{2a} \text{ Ans}$$

CAUCHY'S LINEAR HOMOGENOUS DIFFERENTIAL EQUATION :-

Standard form :

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = Q \longrightarrow (1)$$

$$\left(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + a_2 x^{n-2} D^{n-2} + \cdots + a_n\right) y = Q \longrightarrow (2)$$

Put $x = e^z$ i.e $z = \log x$

$$x \frac{d}{dx} = xD = D' \quad \text{where } D' = \frac{d}{dz}$$

$$x^2 \frac{d^2}{dx^2} = x^2 D^2 = D'(D'-1)$$

$$x^3 \frac{d^3}{dx^3} = x^3 D^3 = D'(D'-1)(D'-2) \text{ \& so on.}$$

Q. Solve :- $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \log x$

Sol. Given :- $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \log x \longrightarrow (1)$

Let $x = e^z \Rightarrow z = \log x$

& $x \frac{dy}{dx} \Rightarrow Dy$ where $D = \frac{d}{dz}$

$$x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

So eq (1) become :-

$$D(D-1)y - Dy + y \Rightarrow z$$

$$(D^2 - D - D + 1)y \Rightarrow z$$

$(D^2 - 2D + 1)y \Rightarrow z \longrightarrow$ (2)

Auxiliary eq of (2) :-

$m^2 - 2m + 1 \Rightarrow 0$

$(m-1)^2 \Rightarrow 0$

$m \Rightarrow 1, 1$

$\therefore$ C.F $\Rightarrow (c_1 + c_2 z)e^{z}$

C.F $\Rightarrow (c_1 + c_2 \log x)x$

P.I :- $\dfrac{1}{D^2 - 2D + 1} z$

$\Rightarrow \dfrac{1}{1 + D^2 - 2D} z \Rightarrow (1 + D^2 - 2D)^{-1} z$

$(1+x)^{-1} \Rightarrow 1 - x + x^2 - x^3 + x^4 + \dots$

$(1 + D^2 - 2D)^{-1} \Rightarrow [1 - (D^2 - 2D) + (D^2 - 2D)^2 + \dots] z$

$\Rightarrow z + 2Dz \Rightarrow z + 2$

P.I $\Rightarrow \log x + 2$

$\boxed{y \Rightarrow C.F + P.I}$

Hence complete solⁿ :- $y \Rightarrow (c_1 + c_2 \log x)x + \log x + 2$ Ans

Q. Solve :- $x^2 \dfrac{d^2y}{dx^2} + 2x \dfrac{dy}{dx} - 12y \Rightarrow x^3 \log x$

Sol → Let $x \Rightarrow e^z$ i.e $z \Rightarrow \log x$

$x^2 \dfrac{d^2y}{dx^2} \Rightarrow D(D-1)y$ where $\dfrac{d}{dz} \Rightarrow D$

$x \dfrac{dy}{dx} \Rightarrow Dy$

Solution of Simultaneous Linear Differential Eqⁿ :-

No. of Dependent variable → More than 1.
No. of Independent variable ⇒ 1

x, y → Dependent variables
t → Independent variable

① Elimination Method

② Substitution Method

Solve :- $\frac{dx}{dt} \Rightarrow 2x + 6y \longrightarrow ①$

and $\frac{dy}{dt} \Rightarrow x + y$

ELIMINATION METHOD

Sol. Using Symbol :- $D = \frac{d}{dt}$

$$Dx - 2x - 6y \Rightarrow 0$$
$$-x + Dy - y \Rightarrow 0$$

or

$$(D-2)x - 6y \Rightarrow 0 \longrightarrow ①$$
$$-x + (D-1)y = 0 \longrightarrow ②$$

Multiply ① by $(D-1)$ & ② by 6

$$(D-2)(D-1)x - \cancel{6(D-1)y} \Rightarrow 0$$
$$-6x + \cancel{6(D-1)y} \Rightarrow 0$$

$$(D-2)(D-1)x - 6x = 0$$
$$(D^2 - 3D + 2)x - 6x = 0$$
$$(D^2 - 3D - 4)x = 0 \longrightarrow ③$$

NOW A.E of ③ :- $m^2 - 3m - 4 = 0$

$$(m-4)(m+1) = 0$$
$$m = 4, -1$$

$$\boxed{x = CF \Rightarrow c_1 e^{4t} + c_2 e^{-t}}$$

$(P.I = 0)$

$$\frac{dx}{dt} \Rightarrow 4c_1 e^{4t} - c_2 e^{-t}$$

Put x & $\frac{dx}{dt}$ in eq ①

$$\frac{dx}{dt} = 2x + 6y$$

$$4c_1 e^{4t} - c_2 e^{-t} - 2(c_1 e^{4t} + c_2 e^{-t}) \Rightarrow 6y$$

$$2c_1 e^{4t} - 3c_2 e^{-t} = 6y$$

$$\boxed{y \Rightarrow \frac{1}{3} c_1 e^{4t} - \frac{1}{2} c_2 e^{-t}}$$

Q. Solve : $\frac{dx}{dt} + y \Rightarrow \sin t \longrightarrow$ (I)

$\frac{dy}{dt} + x \Rightarrow \cos t \longrightarrow$ (II)

SUBSTITUTION METHOD

Sol: Differentiating (II) wrt t.

$$\frac{d^2y}{dt^2} + \frac{dx}{dt} = -\sin t$$

$$\frac{dx}{dt} = -\sin t - \frac{d^2y}{dt^2}$$

Put the value of $\frac{dx}{dt}$ in eq (I) :-

$$-\sin t - \frac{d^2y}{dt^2} + y \Rightarrow \sin t$$

$$\frac{d^2y}{dt^2} - y \Rightarrow -2\sin t$$

$$(D^2-1)y \Rightarrow -2\sin t \longrightarrow \text{(III)}$$

Now A.E of (III) :-

$m^2 - 1 = 0$

$m = \pm 1$

$C.F \Rightarrow c_1 e^{t} + c_2 e^{-t}$

& $P.I \Rightarrow \frac{1}{D^2-1}(-2\sin t)$

$P.I \Rightarrow (-2)\frac{1}{(-1)^2-1}\sin t$

$P.I \Rightarrow \cancel{(-2)}\frac{\sin t}{\cancel{(-2)}} \Rightarrow \sin t$

$\therefore y = C.F + P.I$

$$\boxed{y \Rightarrow c_1 e^{t} + c_2 e^{-t} + \sin t}$$

$$\frac{dy}{dt} = c_1 e^{t} - c_2 e^{-t} + \cos t$$

Put $\frac{dy}{dt}$ in eq (2)

$x \Rightarrow \cancel{\cos t} - (c_1 e^{t} - c_2 e^{-t} + \cancel{\cos t})$

$$\boxed{x \Rightarrow c_2 e^{-t} - c_1 e^{t}}$$

SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENT :-

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \longrightarrow (1)$$

where P, Q and R are function of x

$$1 + \frac{-2x(1+x)}{x^2} \cdot + \frac{2(1+x)}{x^2}$$

$$-\frac{2x(1+x)}{x^2} + \frac{2(1+x)x}{x^2}$$

1) One Integral known Method :-

Working Rule :- i) find P, Q, R

ii) Investigate if :- $1+P+Q=0 \Rightarrow u = e^{x}$

if :- $1-P+Q=0 \Rightarrow u = e^{-x}$

if :- $P+Qx = 0 \Rightarrow u = x$

iii) Consider the complete solution as :-

$$\boxed{y = uv}$$

iv) find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ & putting their values into D.E (1)

v) Solve the D.E for 'v' and find the complete solution.

Q. Solve :- $x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0$

Sol: Convert given D.E into $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

Divide by x both sides :

$$\frac{d^2y}{dx^2} - \frac{(2x-1)}{x}\frac{dy}{dx} + \frac{(x-1)}{x}y = 0 \longrightarrow (1)$$

Here :- $P = -2 + \frac{1}{x}$, $Q = 1 - \frac{1}{x}$, $R = 0$

Since $1+P+Q = \cancel{1} - 2 + \cancel{\frac{1}{x}} + 1 - \cancel{\frac{1}{x}} \Rightarrow 0$

$\therefore$ $u = e^{x}$ is a known Integral

SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENT :-

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \longrightarrow (1)$$

where P, Q and R are function of x

$$1 - \frac{2\cancel{x}(1+x)}{x^{\cancel{2}}} \cdot + \frac{2(1+x)}{x^2}$$

$$-\frac{2\cancel{x}(1+x)}{x^{\cancel{2}}} + \frac{2(1+x)x}{x^2}$$

1) One Integral Known Method :-

Working Rule :- i) find P, Q, R.

ii) Investigate if :- $1+P+Q=0 \Rightarrow u = e^{x}$

if :- $1-P+Q=0 \Rightarrow u = e^{-x}$

if :- $P+Qx = 0 \Rightarrow u = x$

iii) Consider the complete solution as :-

$$\boxed{y = uv}$$

iv) find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ & putting their values into D.E (1)

v) Solve the D.E for 'v' and find the complete solution.

2) Removal of First Derivative :- [NORMAL FORM]

Working Rule :- i) Write given D.E into standard form :

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \longrightarrow (1)$$

Find the value of P, Q and R .

ii) Suppose the complete solution of D.E (1) is

$$\boxed{y = vy_1} \longrightarrow (2)$$

where v is the function of x only.

iii) for Removing the first derivative from (1), we have to choose

$$\boxed{y_1 = e^{-1/2\int P\,dx}}$$

iv) The value of v is find by solving following D.E

$$\frac{d^2v}{dx^2} + Q_1 v = \frac{R}{y_1} \longrightarrow (3)$$

Normal eqn of (1)

where $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$

v) Put the value of y_1 and v into complete solution.

3) By Change of Independent variable :-

Working Rule :- i) Write the given DE in Standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \longrightarrow ①$$

ii) for changing the independent variable x to z we assume :-

$$\boxed{\left(\frac{dz}{dx}\right)^2 = |Q|} \Rightarrow \left(\frac{dz}{dx}\right)^2 = cf(x) \longrightarrow ②$$

where z is independent variable

Solving ② $\frac{dz}{dx} = \sqrt{cf(x)}$ (neglect -ve sign)

$$\int dz = \int \sqrt{cf(x)}\, dx$$

$$z = \int \sqrt{cf(x)}\, dx \longrightarrow ③$$

iii) with relation ③, the given D.E ① transform into

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \longrightarrow ④$$

where :- $P_1 = \dfrac{\left(\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}\right)}{\left(\dfrac{dz}{dx}\right)^2}$ $\quad Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}$, $R_1 = R\Big/\left(\dfrac{dz}{dx}\right)^2$

Calculate P_1, Q_1 & R_1 and then solving ④

Put $z = \int\sqrt{cf(x)}\, dx$ into this solution

3) By Change of Independent variable :-

Working Rule :- i) Write the given DE in standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \longrightarrow ①$$

ii) for changing the independent variable x to z we assume :-

$$\boxed{\left(\frac{dz}{dx}\right)^2 = |Q|} \ni \left(\frac{dz}{dx}\right)^2 = cf(x) \longrightarrow ②$$

where z is independent variable

Solving ② $\frac{dz}{dx} = \sqrt{cf(x)}$ (neglect -ve sign)

$$\int dz = \int \sqrt{cf(x)}\, dx$$

$$z = \int \sqrt{cf(x)}\, dx \longrightarrow ③$$

iii) with relation ③, the given D.E ① transform into

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \longrightarrow ④$$

where :- $P_1 = \dfrac{\left(\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}\right)}{\left(\dfrac{dz}{dx}\right)^2}$ $\quad Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}$, $R_1 = R \Big/ \left(\dfrac{dz}{dx}\right)^2$

Calculate P_1, Q_1 & R_1 and then solving ④

Put $z = \int \sqrt{cf(x)}\, dx$ into this solution

VARIATION OF PARAMETER METHOD [WRONSKIAN]

Working Rule :- Let $y'' + Py' + Qy = R$ be a given second order L.D.E $\longrightarrow$ ①

Step ① :- Taking $R = 0$

Consider the eqⁿ :- $y'' + Py' + Qy = 0$

& find the C.F of ①

Let $y = c_1 u + c_2 v$ be C.F of ①

Step ② :- find Wronskian of u and v

i.e $W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

Step ③ :- General Solution of ① is $y = C.F + P.I$

where $P.I = uf(x) + vg(x)$ and

$$f(x) = -\int \frac{v \cdot R}{W(u,v)} dx \quad , \quad g(x) = \int \frac{u \cdot R}{W(u,v)} dx$$

Q. Solve the D.E , $y'' + 4y = 4\tan 2x$

Sol i) A.E :- $m^2 + 4 = 0$

$$\boxed{m = \pm 2i}$$

$$\underset{(C.F)}{y(x)} = c_1 \cos 2x + c_2 \sin 2x$$

$$u = \cos 2x \quad , \quad v = \sin 2x$$

$$W(u,v) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$W(u,v) = 2\cos^2 2x + 2\sin^2 2x \Rightarrow 2$$

$$y_{P.I} = uf(x) + vg(x) = f(x)\cos 2x + \sin 2x\, g(x)$$

$$f(x) = -\int \frac{\sin 2x \cdot 4\tan 2x}{2}\,dx$$

$$f(x) = -2\int \frac{\sin^2 2x}{\cos 2x}\,dx \Rightarrow -2\int \frac{1-\cos^2 2x}{\cos 2x}$$

$$f(x) = -2\int \sec 2x\,dx + 2\int \cos 2x\,dx$$

$$f(x) = -2\log(\sec 2x + \tan 2x) + \cancel{2}\,\frac{\sin 2x}{\cancel{2}}$$

$$f(x) = -2\log(\sec 2x + \tan 2x) + \sin 2x$$

$$g(x) = \int \frac{u \cdot R}{W(u,v)}\,dx = \int \frac{\cos 2x \cdot 4\tan 2x}{2}$$

$$g(x) \Rightarrow 2\int \sin 2x\,dx \Rightarrow -\cos 2x$$

$$P.I = \begin{bmatrix} -2\log(\sec 2x + \tan 2x) \\ + \sin 2x \end{bmatrix} \cdot \cos 2x + (\sin 2x)(-\cos 2x)$$

$$y = C.F + P.I$$

$$\boxed{y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x\,[\log(\sec 2x + \tan 2x)]}$$ Ans

WRONSKIAN :-

Let $y_1(x)$ & $y_2(x)$ are the two solution of second order L.D.E then Wronskian of y_1 & y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Ex :- Let $y_1 = e^x$ & $y_2 = e^{2x}$ are the two solution then

Sol^n
$$W_{(y_1, y_2)} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$$

$$W(y_1, y_2) = 2e^{3x} - e^{3x} \Rightarrow e^{3x}$$

Result :- ① Two solution $y_1(x)$ and $y_2(x)$ of the equation $a_0 y'' + a_1 y' + a_2 y = 0$, $a_0(x) \neq 0$ are linearly dependent if their Wronskian is identically zero.

② Two solution $y_1(x)$ and $y_2(x)$ of the eq^n $a_0 y'' + a_1 y' + a_2 y = 0$ are linearly independent if their Wronskian is not zero on (a, b).