4.10 Existence and Uniqueness of Solutions

In section 4.3, we made a remark that not all differential equations that we come across have unique solutions. The examples that we considered were

$$\left|\frac{dy}{dx}\right| + |y| = 0$$

which has only a trivial solution $y = 0$ and the differential equation

$$\left|\frac{dy}{dx}\right| + |y| + c = 0, \quad c > 0$$

which has no solution.

Consider now the solution of the initial value problem (IVP)

$$xy' = 3y, \quad y(0) = 1.$$

The differential equation has the general solution $y = cx^3$, but the IVP has no solution. However, if we modify the initial condition as $y(0) = 0$, then the new IVP admits a one parameter family of solutions $y = cx^3$, that is there exists infinite number of solutions. Therefore, in general, an initial value problem may have a unique solution, more than one solution or no solution.

Consider now, the general initial value problem for the first order equation as

$$y' = f(x, y), \quad y(x_0) = y_0. \tag{4.55}$$

Often, the independent variable x is the time denoted by t and x_0(or t_0) is an initial point. An initial value problem then gives the time history of solutions for $t > t_0$.

Often, in mathematical modelling of a physical system, many assumptions are made to simplify the problem. Eventhough, the physical system is well defined and behaves uniquely, the mathematical model may or may not have a solution. Therefore, it is necessary to examine the existence and uniqueness of solutions of the mathematical model. With respect to the IVP given by Eq. (4.55), we would like to find answers for the following questions.

(a) Existence (i) Under what conditions does the IVP given by Eq. (4.55) admits a solution? (ii) If a solution exists, then for what values of x (bounded or unbounded domain) it is defined?

(b) Uniqueness If there is a solution to the IVP given by Eq. (4.55), is it unique?

We present below some results on the existence and uniqueness of the IVP given by Eq. (4.55).

Theorem 4.5 (Existence Theorem) (*Sufficient conditions*) Let $f(x, y)$ be continuous at all points in the closed rectangular region $R: |x - x_0| \le a, |y - y_0| \le b$ (Fig. 4.5) and $|f(x, y)| \le M$. Then, the initial value problem $y' = f(x, y), y(x_0) = y_0$ has at least one solution. This solution is defined at least for all x in the interval

$$I = |x - x_0| < h, \quad \text{where} \quad h = \min\{a, b/M\}$$

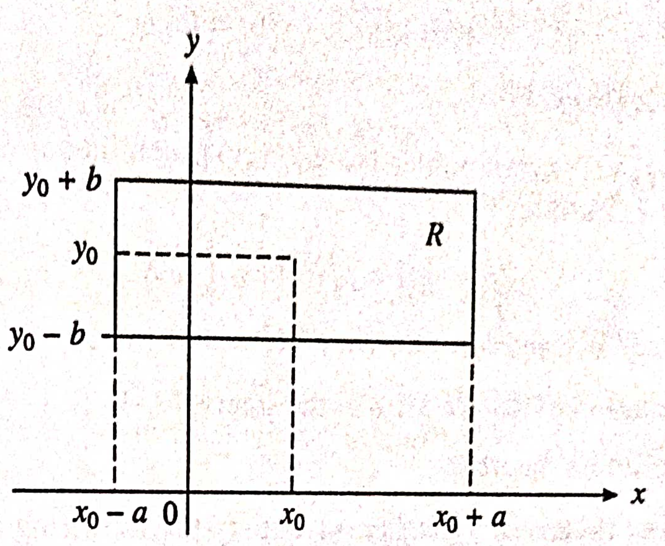

Fig. 4.5. Region where $f(x, y)$ is continuous.

The proof is omitted.

Since $y' = f(x, y)$ and $|f(x, y)| \le M$ implies $|y'| \le M$, the slope of every solution curve lies in the interval $[-M, M]$. If $h = \min\{a, b/M\} = a$, then the solution exists at least in the interval $[x_0 - a, x_0 + a]$. If $h = \min\{a, b/M\} = b/M$, then the solution exists at least in the smaller interval $[x_0 - (b/M), x_0 + (b/M)]$. For values of x outside these ranges, the curve leaves the region R and hence nothing can be said about the existence of solution at these values.

It may be pointed out that the conditions given in the theorem are sufficient and not necessary, that is, even if the conditions are violated, the problem may have a solution.

Example 4.34 Investigate the existence of solutions of the initial value problem $y' = 2x^2 + 3y^2$, $y(0) = 1$ over the rectangle $|x| \le 1, \quad |y - 1| \le 1$.

Solution Here $f(x, y) = 2x^2 + 3y^2$, $x_0 = 0$ and $y_0 = 1$. The given rectangle is $|x - 0| \le 1$ and $|y - 1| \le 1$, or $-1 \le x \le 1$ and $0 \le y \le 2$. Now, $f(x, y)$ is continuous everywhere in the rectangle. Further,

$$|f(x, y)| = |2x^2 + 3y^2| \le 14, \quad -1 \le x \le 1, \quad 0 \le y \le 2.$$

Therefore, at least one solution of the IVP exists. Now, $M = 14$ and

$$h = \min\{a, b/M\} = \min\{1, 1/14\} = 1/14.$$

Solution exists for all x at least in the interval $[-1/14, 1/14]$.

Example 4.35 Study the existence of solutions of the initial value problem

$$xy' = 3y, \quad y(1) = 1$$

over the rectangle $|x - 1| \le 2, \quad |y - 1| \le 4$.

Solution Here, $f(x, y) = 3y/x$, $x_0 = 1$ and $y_0 = 1$. The given rectangle is $-1 \le x \le 3, -3 \le y \le 5$. Now,

$f(x, y)$ is continuous everywhere in the rectangle except at $x = 0$. Hence, Theorem 4.5 cannot be applied, that is, the theorem does not provide any conclusion about the existence of solutions in the region considered. However, solving the differential equation we find that $y = x^3$ is the unique solution of the initial value problem, showing that the conditions in the theorem are sufficient, but not necessary.

Example 4.36 Study the existence of solutions of the initial value problem

$$y' = \sqrt{|y|}, \quad y(0) = 0$$

over the rectangle $|x| < 1, |y| < 1$.

Solution Here $f(x, y) = \sqrt{|y|}$, $x_0 = 0, y_0 = 0$. Now, $f(x, y)$ is continuous everywhere in the rectangle $R: -1 < x < 1, -1 < y < 1$. Further,

$$|f(x, y)| = |\sqrt{|y|}| < 1 \text{ in } R$$

We have $\qquad M = 1$ and $h = \min\{a, b/M\} = \min\{1, 1\} = 1$.

Therefore, at least one solution exists for all x in the interval $(-1, 1)$.

We now present the uniqueness theorem.

Theorem 4.6 (Uniqueness theorem) Let $f(x, y)$ satisfy the following conditions

(i) $f(x, y)$ is continuous at all points in the closed rectangular region R: $|x - x_0| \le a$, $|y - y_0| \le b$ and $|f(x, y) \le M$.

(ii) $f(x, y)$ satisfies the *Lipschitz condition*

$$|f(x, y_2) - f(x, y_1)| \le L|y_2 - y_1| \tag{4.56}$$

where L is the *Lipschitz constant* and $(x, y_1), (x, y_2) \in R$. Then, the initial value problem has a unique solution. This solution is defined at least for all x in the interval

$$I: |x - x_0| < h, \quad \text{where} \quad h = \min\{a, b/M\}.$$

The proof in omitted.

Therefore, the solution of the IVP, $y' = f(x, y)$, $y(x_0) = y_0$ exists and is unique if $f(x, y)$ is continuous and satisfies the Lipschitz condition in R.

Sometimes, it is easy to apply the following stronger condition than the condition given in Eq. (4.56). Let $\partial f/\partial y$ be bounded, that is $|\partial f/\partial y| \le N$ in R. Then the Lipschitz condition is satsified and Eq. (4.56) can be replaced by the condition $|\partial f/\partial y| \le N$. It may be noted that this is a sufficient condition and not a necessary condition. This result is also known as the Picard's theorem, which is stated below.

Theorem 4.7 (Picard's theorem) Let $f(x, y)$ and $\partial f/\partial y$ be continuous at all points in a closed rectangular region R. If (x_0, y_0) is any point in the interior of R, then there exists a positive number h such that the initial value problem (4.55) has a unique solution for all x in the interval $[x_0 - h, x_0 + h]$.

The proof in omitted.

Example 4.37 Test the existence and uniqueness of the solutions of the initial value problem $y' = \sqrt{y}$, $y(1) = 0$, in a suitable rectangle R. If more than one solution exists, then find all solutions.

Solution Here $f(x, y) = \sqrt{y}$, $x_0 = 1$, $y_0 = 0$. Now, $f(x, y)$ is continuous and bounded in R. Hence, by Theorem 4.5, at least one solution exists in some rectangle containing $(1, 0)$. Let us now test the

Lipschitz condition. We have

$$|f(x,y_2)-f(x,y_1)| = |\sqrt{y_2}-\sqrt{y_1}| = \left|\frac{(y_2-y_1)(\sqrt{y_2}-\sqrt{y_1})}{y_2-y_1}\right| = \left|\frac{y_2-y_1}{\sqrt{y_2}+\sqrt{y_1}}\right|$$

or

$$\frac{|f(x,y_2)-f(x,y_1)|}{|y_2-y_1|} = \frac{1}{\sqrt{y_2}+\sqrt{y_1}}.$$

This quantity can be made as large as possible by choosing y_1 and y_2 sufficiently small, that is, a finite value for the Lipschitz constant L cannot be determined.

We have

$$\sqrt{y_1}+\sqrt{y_2} < 2\sqrt{y}, \quad \text{if} \quad y = \max\{y_1, y_2\}$$

and

$$\left|\frac{1}{\sqrt{y_1}+\sqrt{y_2}}\right| > \frac{1}{2\sqrt{y}} > M, \quad \text{if } \sqrt{y} < \frac{1}{2M}.$$

In the neighbourhood of $y = 0$, this criterion is satisfied for every $M > 0$.

Therefore, the initial value problem does not possess a unique solution. Infact, there are two solutions of the IVP, which are

$$y = (x-1)^2/4, \quad \text{and} \quad y \equiv 0.$$

as can be easily verified. For $y \neq 0$, we have from the differential equation

$$\frac{dy}{\sqrt{y}} = dx.$$

Integrating, we have $2\sqrt{y} = x + c$, or $y = \left(\dfrac{x+c}{2}\right)^2$.

The condition $y(1) = 0$ is satisfied both by (i) $y \equiv 0$, and (ii) $[(c+1)^2/4] = 0$, or $c = -1$.

Hence, the solutions are $y = 0$, and $y = (x-1)^2/4$.

Example 4.38 Test for uniqueness of solutions of the initial value problem

$$(x - 2y + 1)\, dy - (3x - 6y + 2)\, dx = 0, \; y(0) = 0$$

over the rectangle R: $|x| \le 1/4$, $|y| \le 1/4$. Determine the interval for x, over which the solution is guaranteed.

Solution We have

$$f(x,y) = \frac{3x-6y+2}{x-2y+1}, \quad x_0 = 0,\; y_0 = 0$$

$$|f(x,y)| = \frac{|3x-6y+2|}{|x-2y+1|} < \frac{3|x|+6|y|+2}{1-|x|-2|y|}$$

$$< \frac{3(1/4)+6(1/4)+2}{1-(1/4)-2(1/4)} = 17 = M$$

$$|f_y| = \left|\frac{(x-2y+1)(-6)-(3x-6y+2)(-2)}{(x-2y+1)^2}\right| = \left|\frac{-2}{(x-2y+1)^2}\right| < \frac{2}{(1/4)^2} = 32 = N.$$

Also $$h = \min\left\{a, \frac{b}{M}\right\} = \min\left\{\frac{1}{4}, \frac{1}{68}\right\} = \frac{1}{68}.$$

Conditions of Theorem 4.7 are satisfied. Therefore, the IVP has a unique solution atleast for all x in the interval $|x| \le 1/68$. From Example 4.10, the solution of the problem is

$$\frac{1}{5}(x - 2y) + \frac{2}{25}\ln\left|x - 2y + \frac{3}{5}\right| + x = \frac{2}{25}\ln\left(\frac{3}{5}\right).$$

It can be seen that the solution is defined in a much larger interval for x than predicted.

4.10.1 Picard's Iteration Method of Solution

Picard's theorem (Theorem 4.7) gives a sufficient condition for the existence and uniqueness of the solution of the initial value problem given by Eq. (4.55). Integrating the differential equation $y' = f(x, y)$, we obtain

$$\int_{x_0}^{x} y'\,dx = \int_{x_0}^{x} f(x, y)\,dx, \quad \text{or} \quad y(x) = y(x_0) + \int_{x_0}^{x} f(x, y)\,dx. \tag{4.57}$$

The solution of the initial value problem defined in Eq. (4.55) is given by Eq. (4.57). Conversely, if $y(x)$ satisfies Eq. (4.57), then it satisfies the initial value problem given by Eq. (4.55). The integral equation (4.57) is solved iteratively assuming the first approximation to $y(x)$ as $y_0(x) = y(x_0) = y_0$. This iterate is called $y_1(x)$ and is given by

$$y_1(x) = y(x_0) + \int_{x_0}^{x} f[x, y_0(x)]\,dx. \tag{4.58}$$

The second approximation $y_2(x)$ is given by

$$y_2(x) = y(x_0) + \int_{x_0}^{x} f[x, y_1(x)]\,dx. \tag{4.59}$$

The sequence of approximations $y_0(x)$, $y_1(x)$, $y_2(x)$, . . . given by

$$y_{n+1}(x) = y(x_0) + \int_{x_0}^{x} f[x, y_n(x)]\,dx \tag{4.60}$$

converges to $y(x)$, for x sufficiently close to x_0. The solution $y(x)$, thus obtained, is the unique solution of the initial value problem defined by Eq. (4.55).

Example 4.39 Find the solution of the initial value problem $y' = 2y - x$, $y(0) = 1$, using the Picard's iteration method. Compare with the exact solution.

Solution We have

$$y(x) = y(x_0) + \int_{x_0}^{x} f(x, y)\,dx = 1 + \int_0^x (2y - x)\,dx.$$

The iteration is defined by

$$y_{n+1}(x) = 1 + \int_0^x [2y_n - x]\,dx, \quad y_0 = 1.$$