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Maths Assignment Unit II

Q1 ⇒ Integrate $(1+x^2)(dy/dx) + 2xy - 4x^2 = 0$ to obtain the equation of the curve satisfying this equation and passing through the origin.

Sol:

$$\frac{dy}{dx} + \frac{2x}{(1+x^2)} y = \frac{4x^2}{1+x^2}$$

Above differential equation is linear differential equation.

$$\frac{dy}{dx} + Py = Q.$$

$$I.F = e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2} dx}$$

$$I.F. = (1+x^2)$$

$$y.(I.F) = \int Q (IF)\,dx$$

$$y(1+x^2) = 4\int \frac{x^2}{1+x^2}(1+x^2)\,dx$$

$$y(1+x^2) = \frac{4x^3}{3} + C$$

$$y = \frac{4x^3}{3(1+x^2)} + \frac{C}{(1+x^2)}.$$

Passing through $(0,0)$ $0 = 0 + C$

curve is $y = \frac{4}{3}\frac{x^3}{1+x^2}.$

Q2 ⇒ $x\left(\frac{dy}{dx}\right) + y \log y = xy e^x.$

$$\frac{dy}{dx} + \frac{y}{x}\log y = y e^x$$

$$= \frac{x e^{3x}}{\;} + \frac{1}{2} x^2 \frac{e^x}{\;}$$

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^x.$$

$$\log y = t \qquad \frac{1}{y}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + \frac{1}{x}t = e^x.$$

$$e^{\int \frac{1}{x}dx} = x.$$

$$t x = \int e^x \, x \, dx$$

$$tx = xe^x - \int e^x dx$$

$$tx = xe^x - e^x + c.$$

$$t = \frac{e^x(x-1)}{x} + \frac{c}{x}.$$

$$\log y = \frac{e^x(x-1)}{x} + \frac{c}{x}.$$

Q8 $\Rightarrow$ $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x.$

$$(D^3 - 5D^2 + 7D - 3)y = e^{2x}\left(\frac{e^x + e^{-x}}{2}\right)$$

Auxilliary eq$^n$ is $\quad m^3 - 5m^2 + 7m - 3 = 0$

$$m^2(m-1) - 4m(m-1) + 3(m-1) = 0$$

$$(m^2 - 4m + 3)(m-1) = 0$$

$$m = 1, 1, 3$$

$$C.F. = (c_1 + c_2 x)e^x + c_3 e^{3x}$$

$$P.I. = \frac{1}{2}\frac{(e^{3x} + e^x)}{D^3 - 5D^2 + 7D - 3}$$

$$= \frac{1}{2}\frac{e^{3x}}{D^3 - 5D^2 + 7D - 3} + \frac{1}{2}\frac{e^x}{D^3 - 5D^2 + 7D - 3}$$

$$= \frac{1}{2} x \frac{e^{3x}}{3D^2 - 10D + 7} + \frac{1}{2} x \frac{e^x}{3D^2 - 10D + 7}$$

$$= \frac{x}{2}\frac{e^{3x}}{4} + \frac{1}{2} x^2 \frac{e^x}{6D-10}$$

$$P.I. = \frac{x}{8} e^{3x} - \frac{x^2}{8} e^x.$$

$$y = C.F. + P.I.$$

$$= (c_1 + c_2 x) e^x + c_3 e^{3x} + \frac{x}{8}\left(e^{3x} - xe^x\right)$$

Q4 $\Rightarrow$ $(D^2-1)\, y = xe^x + \cos^2 x.$

Its Auxilliary eq$^n$ is

$$m^2 - 1 = 0.$$

$$m = \pm 1.$$

$$C.F. = c_1 e^x + c_2 e^{-x}$$

$$P.I. = \frac{xe^x}{(D^2-1)} + \frac{\cos^2 x}{(D^2-1)}$$

$$= P_1 + P_2$$

$$P_1 I. = \frac{xe^x}{(D^2-1)} + \frac{1}{2}\frac{(1+\cos 2x)}{(D^2-1)}$$

$$= \frac{xe^x}{(D^2-1)} + \frac{1}{2}\frac{e^{0x}}{(D^2-1)} + \frac{1}{2}\frac{\cos 2x}{(D^2-1)}.$$

$$= \frac{xe^x}{(D^2-1)} + \left(-\frac{1}{2}\right) - \frac{\cos 2x}{10}$$

$$= -\frac{1}{2}\left(1 + \frac{\cos 2x}{5}\right) + \frac{xe^x}{(D^2-1)}$$

$$= \frac{xe^x}{(D+1)^2 - 1} + \frac{1}{2}\left(1 + \frac{\cos 2x}{5}\right)$$

$$= e^x \frac{1}{D^2+2D} x - \frac{1}{2}\left(1 + \frac{\cos 2x}{5}\right).$$

$$\Rightarrow \quad e^x \frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1} x - \frac{\cos 2x}{10} - \frac{1}{2}.$$

$$= \frac{e^x}{2D}\left(1-\frac{D}{2}\right) x - \frac{\cos 2x}{10} - \frac{1}{2}.$$

$$= \frac{e^x}{2D}\left(x-\frac{1}{2}\right) - \frac{\cos 2x}{10} - \frac{1}{2}.$$

$$= \frac{e^x}{2}\left(\frac{x^2}{2}-\frac{x}{2}\right) - \frac{\cos 2x}{10} - \frac{1}{2}$$

$$y = c_1 e^x + c_2 e^{-x} + \frac{e^x}{2}\left(\frac{x^2}{2}-\frac{x}{2}\right) - \frac{1}{2}\left(1+\frac{\cos 2x}{5}\right).$$

Q5 $\Rightarrow \left(\frac{d^2y}{dx^2}\right) - 9y = x + e^{2x} - \sin 2x.$

Its A.E. is

$$m^2 - 9 = 0.$$

$$m = +3, -3.$$

$$C.F = c_1 e^{3x} + c_2 e^{-3x}.$$

$$P.I. = \frac{x + e^{2x} - \sin 2x}{(D^2-9)}$$

$$= \frac{x}{D^2-9} + \frac{e^{2x}}{D^2-9} - \frac{\sin 2x}{D^2-9}$$

$$y = (a_1 x + a_2) + a_3 e^{2x} - (a_4 \sin 2x + a_5 \cos 2x).$$

$$Dy = a_1 + 2a_3 e^{2x} - 2a_4 \cos 2x + 2a_5 \sin 2x$$

$$D^2y = 4a_3 e^{2x} + 4a_4 \sin 2x + 4a_5 \cos 2x.$$

$$-9a_1 x - 9a_2 - 5a_3 e^{2x} + 13a_4 \sin 2x + 13a_5 \cos 2x =$$

$$x + e^{2x} - \sin 2x.$$

$$-5a_3 = 1 \qquad a_3 = -\tfrac{1}{5}.$$

On comparing,

$a_2 = 0$, $a_3 = -\frac{1}{5}$, $a_4 = -\frac{1}{13}$, $a_5 = 0$.

$$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{x}{9} - \frac{e^{2x}}{5} + \frac{\sin 2x}{13}$$

Q6 ⇒ $y''' + y' = 2x^2 + 4\sin x$.

$$(D^3+D)\,y = 2x^2 + 4\sin x.$$

Its A.E, $m^3 + m = 0$ $\quad m(m^2+1) = 0$

$$m = 0, \pm i$$

$$C.F = c_1 e^{0x} + (c_2\cos x + c_3 \sin x)$$

$$P.I. = \frac{2x^2}{(D^3+D)} + \frac{4}{(D^3+D)}\sin x.$$

$$= \frac{1}{D}\frac{2x^2}{(1+D^2)} + \frac{4}{D}\frac{1}{(D^2+1)}\sin x$$

$$= \frac{1}{D}(1+D^2)^{-1}(2x^2) + \frac{4}{D}\frac{x}{2D}\sin x$$

$$= \frac{1}{D}(1-D^2)(2x^2) + \frac{2x}{(-1)}\sin x$$

$$= \frac{2}{D}(x^2-2) - 2x\sin x$$

$$= 2\left[\frac{x^3}{3} - 2x\right] - 2x\sin x$$

$$y = C.F. + P.I. = c_1 e^{0x} + c_2\cos x + c_3\sin x$$
$$+ 2\left(\frac{x^3}{3} - 2x\right) - 2x\sin x.$$

Q7 ⇒ Reduce to homogeneous form by making substitutions $y = z^2$.

$$2x^2 y\frac{d^2y}{dx^2} + 4y^2 = x^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}.$$

$y = z^2 \qquad \frac{dy}{dx} = 2z\frac{dz}{dx} \qquad \frac{d^2y}{dx^2} = 2\left(z\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\right)$

$$2x^2z^2\,2\left(z\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\right) + 4z^4 = 4x^2z^2\left(\frac{dz}{dx}\right)^2 + 2xz^2\,2z\frac{dz}{dx}$$

$$4x^2z^3\frac{d^2z}{dx^2} + 4z^4 = 4xz^3\frac{dz}{dx}$$

$$x^2\frac{d^2z}{dx^2} + z - x\frac{dz}{dx} = 0$$

$$x^2\frac{d^2z}{dx^2} - x\frac{dz}{dx} + z = 0 \quad \text{—(1)}.$$

Let $\quad x = e^t \qquad t = \log x \qquad D = \frac{d}{dt}$.

Eqn (1) becomes,

$$(D(D-1) - D + 1)\,z = 0$$ By homogeneous conversion

$$(D^2 - 2D + 1)\,z = 0.$$

$m^2 - 2m + 1 = 0 \qquad (m-1)^2 = 0 \qquad m = 1, 1.$

$C.F = (c_1 + c_2 t)\,e^t$

Complete soln $\quad z = (c_1 + c_2 t)\,e^t$.

$t = \log x \qquad z = \pm\sqrt{y}$.

$$\pm\sqrt{y} = (c_1 + c_2\log x)\,x.$$

$$y = (c_1 + c_2\log x)^2\,x^2$$

Q 8 ⇒ $\left[(5+2x)^2D^2 - 6(5+2x)D + 8\right]y = x^2 + 1$

Sol: $(5+2x) = e^z \qquad z = \log(5+2x) \qquad D = \frac{d}{dz}$.

then eqn reduces to

$$(D(D-1) - 6D + 8)\,y = \left(\frac{e^z - 5}{2}\right)^2 + 1.$$

$$(D^2 - 7D + 8)\,y = \frac{1}{4}\left(e^{2z} + 25 - 10e^z\right) + 1$$

$$(D^2-7D+8)y = \frac{1}{4}\left(e^{2x} - 10e^{z}\right) + \frac{29}{4}e^{0x}$$

$$m^2 - 7m + 8 = 0$$

$$m = \frac{7 \pm \sqrt{49 - 4\times1\times8}}{2} = \frac{7\pm\sqrt{17}}{2}.$$

$$C.F. = e^{7/2 x}\left(c_1 \cosh\frac{\sqrt{17}}{2}x + c_2 \sinh\frac{\sqrt{17}}{2}x\right)$$

$$P.I. = \frac{1}{4}\left(\frac{e^{2z}}{D^2-7D+8} - \frac{10e^{z}}{(D^2-7D+8)}\right) + \frac{29}{4}\frac{e^{0z}}{D^2-7D+8}$$

$$= \frac{1}{4}\left(\frac{e^{2z}}{-2} - \frac{10e^{z}}{2}\right) + \frac{29}{4}\frac{e^{0z}}{\cancel{D^2-7D+8}\ (+8)}$$

$$= -\frac{1}{8}\left(e^{2z} + 10e^{z} - \frac{29}{4}e^{0z}\right).$$

Complete solution,

$$y = e^{7/2 x}\left(c_1 \cosh\frac{\sqrt{17}}{2}x + c_2 \sinh\frac{\sqrt{17}}{2}x\right) - \frac{1}{8}\left(e^{2z} + 10e^{z} - \frac{29}{4}e^{0z}\right)$$

Q9 ⇒ Solve $tDx + 2(x-y) = t$, $tDy + x + 5y = t^2$

Sol:

$$(tD+2)x - 2y = t \quad \text{—(i)}$$

$$(tD+5)y + x = t^2 \quad \text{—(ii)}$$

Solving (i) and (ii) for $x$, we get

$$(tD+2)(tD+5)x + 2x = 2t^2 + 5t + t$$

$$(t^2D^2 + 7tD + 10)x + 2x = 2t^2 + 6t$$

Let $t = e^k$ $\quad k = \log t \quad \Rightarrow \frac{dk}{dt} = \frac{1}{t}.$

$$\frac{dx}{dt} = \frac{dx}{dk}\cdot\frac{1}{t} \qquad t\frac{dx}{dt} = \frac{dx}{dk}.$$

$$\frac{d^2x}{dt^2} = -\frac{1}{t^2}\frac{dx}{dk} + \frac{1}{t^2}\frac{d^2x}{dk^2}$$

$$\Rightarrow t^2\frac{d^2x}{dt^2} = D(D-1)x \quad \text{where } D = \frac{d}{dk}$$

Q 10 ⇒ Solve. d...

So, we get, $D(D-1)x + 7Dx + 12x = 2e^{2k} + 6e^{k}$.

$$(D^2 + 6D + 12)\, x = 6e^{k} + 2e^{2k}.$$

AE. $m^2 + 6m + 12 = 0$

$$m = \frac{-6 \pm \sqrt{36-48}}{2} \qquad m = -3 \pm \sqrt{3}\, i$$

So, $C.F. = e^{-3k}(c_1 \cos\sqrt{3}k + c_2 \sin\sqrt{3}k)$

P.I. ⇒ $\frac{1}{D^2+6D+12}(6e^{k} + 2e^{2k})$

$$= \frac{6}{19}e^{k} + \frac{e^{2k}}{14}.$$

So, complete solution is

$$y = e^{-3k}\left(c_1 \cos\sqrt{3}k + c_2 \sin\sqrt{3}k\right) + \frac{6}{19}e^{k} + \frac{e^{2k}}{14}$$

$$x = e^{-3\log t}\left\{c_1 \cos(\sqrt{3}\log t) + c_2 \sin(\sqrt{3}\log t)\right\} + \frac{6t}{19} + \frac{t^2}{14}.$$

$$Dx = 3^{-\log t}\left\{-c_1 \sin(\sqrt{3}\log t)\frac{\sqrt{3}}{t} + \frac{\sqrt{3}}{t}c_2 \cos(\sqrt{3}\log t)\right\}$$
$$+ \frac{6}{19} + \frac{t}{7} + \left(c_1 \cos(\sqrt{3}\log t) + c_2 \sin\sqrt{3}\log t\right)\left(\frac{-3}{t^4}\right)$$

So, $$y = \frac{-t + 2x + tDx}{2}.$$

$$x = \frac{1}{t^3}\left\{c_1 \cos(\sqrt{3}\log t) + c_2 \sin(\sqrt{3}\log t)\right\} + \frac{6t}{19} + \frac{t^2}{14}$$

$$2y = -t + \frac{12}{19} + \frac{t^2}{7} + \frac{2}{t^4}\left(c_1 \cos(\sqrt{3}\log t) + c_2 \sin(\sqrt{3}\log t)\right)$$
$$+ \frac{\sqrt{3}}{t^3}\left(c_2 \cos(\sqrt{3}\log t) - c_1 \sin(\sqrt{3}\log t)\right\} + \frac{6t}{19} + \frac{t^2}{7}$$
$$-\frac{3}{t^2}c_1 \cos\sqrt{3}\log t - \frac{3}{t^2}c_2 \sin(\sqrt{3}\log t) - t\Big]$$

Q 10 $\Rightarrow$ Solve $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$ ; $x(0) = 1$
$y(0) = 1$.

Sol:
$$Dx - y = e^t \quad -(i)$$
$$Dy + x = \sin t \quad -(ii)$$

Solving above eq$^n$ for $x$,
$$D^2x + x = \sin t + e^t \Rightarrow (D^2+1)x = e^t + \sin t.$$

A.E, $m^2+1=0 \Rightarrow m = i, -i$.

$$C.F = c_1\cos t + c_2 \sin t.$$

$$P.I. = \frac{1}{D^2+1}(\sin t + e^t) = \frac{e^t}{2} + \frac{t \sin t}{2D}$$
$$= \frac{e^t}{2} - \frac{t\cos t}{2}$$

$$x = C.F. + P.I. = c_1\cos t + c_2 \sin t + \frac{e^t}{2} - \frac{t\cos t}{2}$$

$$Dx = c_2\cos t - c_1 \sin t + \frac{e^t}{2} - \frac{\cos t}{2} + \frac{t}{2}\sin t.$$

So, $y = Dx - e^t$

$$y = c_2 \cos t - c_1 \sin t + \frac{e^t}{2} - \frac{\cos t}{2} + \frac{t}{2}\sin t - e^t.$$

$x(0) = 1$ & $y(0) = 1$.

$1 = \frac{1}{2} + c_1$. $\quad 1 = c_2 + \left(-\frac{1}{2}\right) - \frac{1}{2}$.

$c_1 = 1/2$ $\quad c_2 = 2$.

So, $x = \frac{\cos t}{2} + 2\sin t + \frac{e^t}{2} - \frac{t\cos t}{2}$.

$$y = 2\cos t - \frac{1}{2}\sin t - \frac{e^t}{2} - \frac{\cos t}{2} + \frac{t\sin t}{2}$$

Q 11 $\Rightarrow$ Solve $x^2y'' + xy' - y = 0$ given that $x + 1/x$ is one integral.

Sol: Let $y = uv$ be a sol$^n$. $x + 1/x$ is one sol$^n$.

$$y = \left(x+\frac{1}{x}\right)v \Rightarrow y' = v\left(1-\frac{1}{x^2}\right) + \left(x+\frac{1}{x}\right)\frac{dv}{dx}.$$

$$y'' = 2\left(1-\frac{1}{x^2}\right)\frac{dv}{dx} + \frac{2v}{x^3} + \left(x+\frac{1}{x}\right)\frac{d^2v}{dx^2}$$

So, $x^2\left(2\left(1-\frac{1}{x^2}\right)\frac{dv}{dx} + \frac{2v}{x^3} + \left(x+\frac{1}{x}\right)\frac{d^2v}{dx^2}\right) + x\left(v\left(1-\frac{1}{x^2}\right) + \right.$
$\left.\left(x+\frac{1}{x}\right)\frac{d^2v}{dx^2}\right) - y = 0.$

$$x^2\left(x+\frac{1}{x}\right)\frac{d^2v}{dx^2}+(3x^2-1)\frac{dv}{dx}=0.$$

Put $\frac{dv}{dx}=P. \Rightarrow \frac{d^2v}{dx^2}=\frac{dP}{dx}$

$$x^2\left(x+\frac{1}{x}\right)\frac{dP}{dx}+(3x^2-1)\,P=0.$$

$$\frac{dP}{P}=-\left\{\frac{(3x^2-1)}{(x^2+1)x}\right\}dx$$

$$\int\frac{dP}{P}=-\int\frac{3x^2+1}{x^3+x}-\frac{2}{x(x^2+1)}\,dx$$

$$\log P=-\log(x^3+x)+2\int\frac{1}{x(x^2+1)}dx=-\log(x^3+x)+2\int\frac{1}{x^3(1+1/x^2)}dx$$

$$\log P=-\log(x^3+x)-2\log\left(1+\frac{1}{x^2}\right)-\log c'.$$

$$\log\left(\frac{dv}{dx}\right)=-\log c'\frac{(x^2+1)^2x}{x^2} \Rightarrow \frac{dv}{dx}=\frac{cx}{(x^2+1)^2}$$

$$\int dv=\int\frac{cx}{(x^2+1)^2}dx \Rightarrow v=\frac{1}{2}\left\{\frac{-c}{x^2+1}\right\}+c_1$$

$$y=\left(\frac{x^2+1}{x}\right)\left\{\frac{-c}{2(x^2+1)}+c_1\right\}=\boxed{-\frac{c}{2x}+\frac{c_1}{x}(x^2+1)}.$$

Q12 $\Rightarrow$ $x^2y''+xy'-9y=0.$ given that $y=x^3$ is a solution.

**Sol:** Let $y=x^3v$ be soln,

$$y'=3x^2v+x^3\frac{dv}{dx} \qquad y''=6xv+3x^2\frac{dv}{dx}+x^3\frac{d^2v}{dx^2}$$

So, we get

$$x^2\left(6xv+3x^2\frac{dv}{dx}+x^3\frac{d^2v}{dx^2}\right)+x\left(3x^2v+x^3\frac{dv}{dx}\right)-9x^3v=0$$

$$x^5\frac{d^2v}{dx^2}+\frac{dv}{dx}(6x^4+x^4)=0. \Rightarrow x\frac{d^2v}{dx^2}+7\frac{dv}{dx}=0$$

Put $\frac{dv}{dx}=k \Rightarrow \frac{d^2v}{dx^2}=\frac{dk}{dx}$

$$x\frac{dk}{dx}+7k=0 \Rightarrow \int\frac{dk}{k}=-7\int\frac{dx}{x}$$

$\log k = -7\log x + \log k \qquad kx^7 = c.$

$$\int dv = \int \frac{dx}{x^7} \Rightarrow v = c\left(\frac{x^{-6}}{-6}\right) + c_1$$

$$y = x^3 v \qquad \boxed{y = \frac{-c}{6x^3} + c_1 x^3}$$

Q13 $\Rightarrow x^2 y_2 - 2x(1+x) y_1 + 2(1+x) y = x^3.$

Sol: Comparing with $y_2 + Py_1 + Qy = R.$

$$P = \frac{-2x(1+x)}{x^2} \qquad Q = \frac{2(1+x)}{x^3} \qquad R = x.$$

$[P + Qx = 0]$ $y = x$ is a part of C.F.

So, lets assume solⁿ be $y = xv.$

$$y_1 = v + x\frac{dv}{dx} \quad ; \quad y_2 = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

Putting in eqⁿ,

$$2\frac{dv}{dx} + x\frac{d^2v}{dx^2} - \frac{2x(1+x)}{x^2}\left(v + x\frac{dv}{dx}\right) + \frac{2(1+x)vx}{x^2} = x$$

$$\Rightarrow x^2\frac{d^2v}{dx^2} + \frac{dv}{dx}\{2 - 2(1+x)\} = x$$

Let $\frac{dv}{dx} = k \Rightarrow \frac{d^2v}{dx^2} = \frac{dk}{dx}$

$$\frac{dk}{dx} - 2k = 1 \Rightarrow \int \frac{dk}{2k+1} = \int dx$$

$$\log(2k+1) = x + C \Rightarrow k + \tfrac{1}{2} = e^{2(x+c)} = c_1 e^{2x}$$

$$\frac{dv}{dx} = c_1 e^{2x} - \frac{1}{2} \Rightarrow \int dv = \int \left(c_1 e^{2x} - \frac{1}{2}\right) dx$$

$$v = \frac{c_1}{2} e^{2x} - \frac{x}{2} + c_2 \Rightarrow y = xv.$$

$$\boxed{y = x\left(\frac{c_1}{2} e^{2x} - \frac{x}{2} + c_2\right)}$$

Q14 $\Rightarrow (1-x^2)y'' + xy' - y = x(1-x^2)^{3/2}$

$$y'' + \frac{x}{1-x^2}y' - \frac{1}{1-x^2}y = x(1-x^2)^{1/2}.$$

Comparing with $y'' + Py' + Qy = R$.

$P = \frac{x}{1-x^2}$ $\quad Q = -\frac{1}{1-x^2}$ $\quad R = x(1-x^2)^{1/2}$

$P + Qx = 0$, so $y = x$ is a part of C.F.

Let $y = xv$ be complete sol$^n$.

$y' = v + x\frac{dv}{dx}$ & $y'' = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$

Putting these value in eq$^n$,

$$2\frac{dv}{dx} + x\frac{d^2v}{dx^2} + \frac{x}{1-x^2}\left\{v + x\frac{dv}{dx}\right\} - \frac{1}{(1-x^2)}(vx) = x\sqrt{1-x^2}$$

$$\frac{d^2v}{dx^2} + \frac{dv}{dx}\left[\frac{2}{x} + \frac{x}{1-x^2}\right] = \sqrt{1-x^2}.$$

Let $P = \frac{dv}{dx} \Rightarrow \frac{d^2v}{dx^2} = \frac{dP}{dx}$.

$$\frac{dP}{dx} + P\left(\frac{2}{x} + \frac{x}{1-x^2}\right) = \sqrt{1-x^2}$$

This is linear differential equation.

$$I.F = e^{\int \frac{2-x^2}{x(1-x^2)}dx} = e^{\int \frac{1}{x(1-x^2)} + \frac{1}{x}\,dx} = e^{\log x + \int \frac{1}{x^3(1-1/x^2)}dx}$$

$$= e^{\log x - \frac{1}{2}\log(1-x^2)/x^2} = x^2/\sqrt{1-x^2}$$

$$\frac{Px^2}{\sqrt{1-x^2}} = \int x^2\,dx = \frac{x^3}{3} + C.$$

$$\int dv = \int \frac{x\sqrt{1-x^2}}{3} + C\frac{\sqrt{1-x^2}}{x^2}\,dx$$

$$v = -\frac{1}{3}\left\{\frac{(1-x^2)^{3/2}}{3} + C\int \frac{\sqrt{1-x^2}}{x^2}\,dx\right.$$

For $\int \frac{\sqrt{1-x^2}}{x^2}$ $\quad x = \sin\theta$. $\quad dx = \cos\theta\, d\theta$.

Put $x = \sin\theta$. $dx = \cos\theta\, d\theta$.

$$\int \frac{\sqrt{1-x^2}}{x^2}dx = \int \frac{\cos\theta \cdot \cos\theta}{\sin^2\theta}dx = \int \cot^2\theta\, d\theta = -\cot\theta - \theta.$$

$$= -\frac{\sqrt{1-x^2}}{2} = \sin^{-1}x$$

Q 15 ⇒ Solve $\dfrac{d\left(\cos^2 x \cdot dy/dx\right)}{dx} + y\cos^2 x = 0.$

Sol:

$$\frac{d\left(\cos^2 x \; dy/dx\right)}{dx} = \frac{dy}{dx}\{-\sin 2x\} + \cos^2 x \; \frac{d^2y}{dx^2}.$$

So, we get $\dfrac{d^2y}{dx^2}(\cos^2 x) - \sin 2x \dfrac{dy}{dx} + y\cos^2 x = 0.$

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + y = 0.$$

Comparing with $y'' + Py' + Qy = R.$

$P = -2\tan x \qquad Q = 1 \Rightarrow$ we can try normal form.

$$u = e^{-\int \frac{P}{2} dx} = e^{\int \tan x \, dx} = \sec x.$$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = 1 - \frac{1}{2}(-2\sec^2 x) - \tan^2 x.$$

$[I = 2] \qquad [S = R/u = 0]$

So, $\dfrac{d^2v}{dx^2} + Iv = S. \qquad \left[\dfrac{d^2v}{dx^2} + 2v = 0\right]$

$(D^2+2)v = 0$ where $D = d/dx.$

A.E, $m^2 + 2 = 0 \qquad m = \pm\sqrt{2}\, i$

C.F $= c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x$

P.I. $= 0 \Rightarrow [v_1 = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x]$

Complete solⁿ is $y = uv$

$$\boxed{y = (\sec x)\left(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x\right)}$$

Q 16 ⇒ Solve $xy_2 - y_1 + 4x^3 y = x^5.$

Sol: We have $y_2 - \frac{1}{x}y_1 + 4x^2 y = x^4.$

Comparing with $y_2 + Py_1 + Qy = R.$

$$\left\{P = -\frac{1}{x},\; Q = 4x^2,\; R = x^4\right\}$$

Let $\left(\dfrac{dz}{dx}\right)^2 = 4x^2 \Rightarrow \dfrac{dz}{dx} = 2x \Rightarrow \int dz = \int 2x\,dx$

$z = x^2 \Rightarrow \dfrac{d^2z}{dx^2} = 2.$

So, $P_1 = \dfrac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{(dz/dx)^2}$ , $Q_1 = \dfrac{Q}{(dz/dz)^2}$ , $R_1 = \dfrac{R}{(dz/dx)^2}$

$\Rightarrow \{P_1 = 6,\ Q_1 = 1,\ R_1 = \frac{x^2}{4}\} \Rightarrow P_1 = 0,\ Q_1 = 1,\ R_1 = \frac{z}{4}.$

So, reduced eq$^n$ is

$\frac{d^2y}{dx^2} + y = \frac{z}{4} \Rightarrow (D^2+1)y = \frac{z}{4}$ where $D = \frac{d}{dz}$.

A.E., $m^2 + 1 = 0 \Rightarrow m = \pm i$

C.F. $= c_1 \cos z + c_2 \sin z$

P.I. $= \dfrac{z}{4(1+D^2)} = \frac{1}{4}\{(1-D^2)\}z = \frac{1}{4}z$

So, $y = c_1 \cos z + c_2 \sin z + \frac{z}{4}$.

$\boxed{y = c_1 \cos x^2 + c_2 \sin x^2 + \frac{x^2}{4}}$

Q17 $\Rightarrow$ Solve $(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + 4y = 0.$

Sol: $\frac{d^2y}{dx^2} + \frac{2x}{(1+x^2)} \cdot \frac{dy}{dx} + \frac{4y}{(1+x^2)^2} = 0.$

$P = \frac{2x}{(1+x^2)}$ , $Q = \frac{4}{(1+x^2)^2}$ , $R = 0.$

Let $\left(\frac{dz}{dx}\right)^2 = \frac{4}{(1+x^2)^2} \Rightarrow \frac{dz}{dx} = \frac{2}{1+x^2}$

$z = 2\tan^{-1}x$

$\int dz = 2\int \frac{1}{1+x^2}\,dx$

So, the reduced eq$^n$ will be $\frac{d^2y}{dx^2} + P_1 \frac{dz}{dx} + Q_1 x = R_1$

$P_1 = \dfrac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{(dz/dx)^2}$ , $Q_1 = \dfrac{Q}{(dz/dx)^2}$ , $R_1 = \dfrac{R}{(dz/dx)^2}$

$P_1 = 0$ , $Q_1 = 1$ , $R_1 = 0$.

Reduced eq$^n$ $(D^2+1)y = 0$.

A.E. $m^2 + 1 = 0$ $m = \pm i$.

C.F. $= c_1 \cos z + c_2 \sin z$ $\therefore$, P.I. $= 0$

$y = c_1 \cos z + c_2 \sin z$ $\qquad y = c_1 \cos(2\tan^{-1} x) + c_2 \sin(2\tan^{-1} x)$

Q 18 $\Rightarrow$ $(1-x) y_2 + x y_1 - y = 2(x-1)^2 e^{-x}$ ; $0 < x < t$.

Verify that $e^x$ & $x$ are sol$^n$. Also get general sol$^n$

Sol: $y = c_1 e^x + c_2 x \Rightarrow y_1 = c_1 e^x + c_2$.

$y_2 = c_1 e^x$.

~~LSH~~ LHS $\rightarrow$ $(1-x) c_1 e^x + x(c_1 e^x + c_2) - y = c_1 e^x + c_2 x - c_1 e^x - c_2 x$

$y = c_1 e^x + c_2 x$ is a sol$^n$ ($e^x$ & $x$ are also sol$^n$)

Now, $(1-x) y_2 + x y_1 - y = 2(x-1)^2 e^{-x}$.

$$y_2 + \frac{x}{(1-x)} y_1 - \frac{1}{1-x} y = 2(x-1) e^{-x}$$

$$P = \frac{x}{1-x}, \quad Q = -\frac{1}{1-x} \qquad P + Qx = 0$$

$$\frac{x}{1-x} + \frac{-x}{1-x} = 0 \quad \therefore y = x \text{ is part of C.F.}$$

Let $y = xv \Rightarrow y_1 = x\frac{dv}{dx} + v \Rightarrow y_2 = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$

So, we get $2\frac{dv}{dx} + x\frac{d^2v}{dx^2} + \frac{x}{1-x}\left(v + x\frac{dv}{dx}\right) - \frac{1}{1-x}(vx)$

$= 2(1-x) e^{-x}$.

$$\frac{d^2v}{dx^2} + \left(\frac{2}{x} + \frac{x}{1-x}\right)\frac{dv}{dx} = 2\left(\frac{1}{x} - 1\right) e^{-x}.$$

$P = \frac{dv}{dx}$ $\qquad \frac{dP}{dx} + P\left(\frac{2}{x} + \frac{x}{1-x}\right) = 2\left(\frac{1}{x} - 1\right) e^{-x}$

$$\text{I.F.} = e^{\int \frac{2}{x} + \frac{x}{1-x}\,dx} = e^{2\ln x + \ln(1-x) - x}.$$

$$P e^{\ln(x^2/1-x) - x} = \int \frac{e^{x^2/1-x}}{e^x} \cdot 2\left(\frac{1-x}{x}\right) e^{-x} dx + C_1$$

$$\frac{Px^2}{1-x^2} e^{-x} = -\frac{x}{2} e^{-2x} + \frac{1}{2}\int e^{-2x} dx + c$$

$$P = e^x \left(\frac{1-x}{x^2}\right)\left(-\frac{e^{-2x}}{4} - \frac{x e^{-2x}}{2} + c_1\right).$$

$$\frac{dv}{dx} = P.$$

$\therefore \int dv = \int \frac{e^x}{4}\left(\frac{1}{x}-\frac{1}{x^2}\right) + \frac{e^{-x}}{2}\left(\frac{1}{x}-\frac{1}{x^2}\right) - c_1 e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$

$v = -\frac{e^x}{4x} - \frac{e^{-x}}{2} - \frac{c_1 e^x}{x} + c_2$

$y = xv \quad \Rightarrow \quad \left[y = -\frac{e^{-x}}{4} + \frac{xe^{-x}}{2} - c_1 e^x + c_2 x\right].$

Q 19 $\Rightarrow$ $(x^2+1)y_2 - 2xy_1 + 2y = 6(x^2+1)^2.$

$y_2 - \frac{2xy_1}{(1+x^2)} + \frac{2y}{(1+x^2)} = 6(x^2+1)$

$P + Qx = 0 \Rightarrow$ So part of C.F. $\Rightarrow y = x.$

Complete solⁿ will be $y = vx.$

$y_1 = v + x\frac{dv}{dx} \Rightarrow y_2 = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$

Putting in eqⁿ we get,

$$x\frac{d^2v}{dx^2} + \frac{2}{x(x^2+1)}\frac{dv}{dx} = 6\left(x+\frac{1}{x}\right)$$

$\frac{dv}{dx} = P ; \quad \frac{d^2v}{dx^2} = \frac{dP}{dx}.$

$\frac{dP}{dx} + \frac{2}{x(1+x^2)}P = 6\left(x+\frac{1}{x}\right)$

$I.F. = e^{\int\frac{2}{x(x^2+1)}dx} = e^{-\ln(x^2+1)/x^2} = \frac{x^2}{1+x^2}$

$\therefore, \frac{Px^2}{1+x^2} = 6\int x\,dx + C \Rightarrow \frac{dv}{dx} \cdot \frac{x^2}{1+x^2} = 3x^2 + C.$

$\int dv = \int \frac{3x^2(1+x^2)}{x^2} + \frac{C(1+x^2)}{x^2}dx.$

$v = 3\left(x+\frac{x^3}{3}\right) + C\left(x-\frac{1}{x}\right) + C_1$

$y = 3x\left(x+\frac{x^3}{3}\right) + C(x^2-1) + C_1 x.$

Q 20 $\Rightarrow$ An LR circuit has emf given by 4 sin t, $R = 100\ \Omega$, $L = 4H$, no initial current. Find $i_t$.

**Sol:** $V = iR + L\,di/dt$ As per eqⁿ.

$4\sin t = 100\,i + 4\,di/dt$ $\qquad \frac{di}{dt} + 25i = \sin t$

$I.F. = e^{\int 25\,dt} = e^{25t}$

$i \cdot e^{25t} = \underbrace{\int e^{25t}\sin t\,dt}_{I}$

Now $I = \int e^{25t}\sin t\,dt = \frac{e^{25t}\sin t}{25} - \frac{1}{25}\int \cos t\, e^{25t} + I$

$I = \frac{e^{25t}\sin t}{25} - \frac{e^{25}\cos t}{625} - \frac{I}{625}$

$\frac{626 I}{625} = e^{25t}\left[\frac{\sin t}{25} - \frac{\cos t}{625}\right]$

$i e^{25t} = \frac{625}{626} e^{25t}\left[\frac{\sin t}{25} - \frac{\cos t}{625}\right] + C$

at $t = 0$ $\quad i = 0$ $\quad \therefore C = \frac{1}{626}$

$i e^{25t} = \frac{625}{626} e^{25}\left[\frac{\sin t}{25} - \frac{\cos t}{625}\right] + \frac{1}{626}$

Q21 ⇒ A particle falls under gravity & force of air resistance is proportional to its velocity. Show that velocity can't exceed a particle time.

**Sol:** $F \propto v \Rightarrow F = kv$

$\uparrow F$ $\downarrow mg$

$mg - F = m\frac{dv}{dt}$ $\qquad \frac{dv}{dt} = g - \frac{kv}{m}$

$\int \frac{dv}{g - \frac{kv}{m}} = \int dt$

$\frac{\log\left(g - \frac{kv}{m}\right)}{-k/m} = -t + C$

$\log\left(g - \frac{kv}{m}\right) = -\frac{kt}{m} - \frac{kc}{m}$

$g - \frac{kv}{m} = e^{-k/m(t+c)}$

$$\boxed{v = \frac{m}{k}\left(g - e^{-k/m(t+c)}\right)}$$

v ranges from $(g-1)\,m/R$ to $\frac{gm}{R}$ it can't go beyond $gm/R$.

Q 22 ⇒ Damped LCR circuit is governed by the eqn $\left[L\frac{D^2Q}{dt^2} + R\frac{DQ}{dt} + \frac{1}{C}Q = 0\right]$ ; L, C, R are the constants. Find conditions under which this circuit is overdamped, underdamped & critically damped. Also obtain the critical resistance.

Sol 22: given eqn is $L\frac{d^2Q}{dt^2} + Q\frac{dQ}{dt} + \frac{1}{C}Q = 0$.

$$\left(D^2 + \frac{RD}{L} + \frac{1}{LC}\right)Q = 0 \quad \text{where } D = d/dt.$$

A.E. $\quad m^2 + \frac{Rm}{L} + \frac{1}{LC} = 0$

$$m = \frac{-\frac{R_1}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}$$

Case(1): When $\left(\frac{R}{L}\right)^2 > \frac{4}{LC}$ ⇒ $\left\{R^2 > 4L/C\right\}$

Then we get two real & distinct roots of A.E.

$$x = c_1 e^{\frac{-R/L + \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}} + c_2 e^{\frac{-R/L - \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}}$$

This is case of overdamped motion.

Case (2): When $\left(\frac{R}{L}\right)^2 = \frac{4}{LC}$ ⇒ $R^2 = \frac{4L}{C}$

$$x = (c_1 + x c_2)\, e^{-R/2L}$$

This is case of critically damped motion.

⇒ Critical resistance $= 2\sqrt{\frac{L}{C}}$.

Case (3): When $\left(\frac{R}{L}\right)^2 < \frac{4}{LC}$ or $R^2 < \frac{4L}{C}$.

Then, we get 2 imaginary roots of A.E,

$$m = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}$$

$$m = \frac{-\frac{R}{L} \pm \sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}\, i}{2}$$

$$x = e^{-R/2L} \left\{ C_1 \cos\left(\frac{\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}}{2}\right) + C_2 \sin\left(\frac{\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}}{2}\right)\right.$$

This is the case of underdamped motion.

Q23⇒ Study existence of solutions & uniqueness of solⁿ of following initial value problems:

(a) $y' = \frac{2xy}{4+x^2+y^2}$ $y(-1) = 1$, R $|x+1| \leq 1$, $|y-1| \leq 2$

(a) $y' = \frac{2xy}{4+x^2+y^2}$ $[x_0 = -1, y_0 = 1]$ $\Rightarrow f(x,y) = \frac{2xy}{4+x^2+y^2}$

For the rectangle $-2 \leq x \leq 0$ ; $-1 \leq y \leq 3$.

$f(x,y)$ is continuous everywhere.

$$|f(x,y)| = \left|\frac{2xy}{4+x^2+y^2}\right| < \frac{2(2)(3)}{4+0+1} \leq \frac{12}{5} = M$$

$$\left|\frac{df}{dy}\right| = \frac{(4+x^2+y^2)\,2x - 2xy(2y)}{(4+x^2+y^2)^2}$$

$$= \left|\frac{8x + 2x^3 - 2xy^2}{(4+x^2+y^2)^2}\right|$$

$$\left|\frac{df}{dy}\right| < \frac{8(2) + 2(8) - 2(-2)(3)^2}{(4+0+1)^2}.$$

$$\left|\frac{df}{dy}\right| < \frac{32+36}{75} < \frac{68}{25} = N.$$

Also, $h = \min(a, b/m) = \min\left(1, 2/_{12/5}\right) = \frac{5}{6}.$

So, given problem has a unique solⁿ for the range $\left[\frac{-11}{6}, \frac{-1}{6}\right]$.

(b) $y' = \dfrac{3xy}{2+\cos xy}$ ; $y(0) = 0$ $R = |x| \leq 1$ $|y| \leq 2$.

(b) For the rectangle region $-1 \leq x \leq 1$, $-2 \leq y \leq 2$.

$f(x, y)$ is continuous.

$$|f(x,y)| = \left|\frac{3xy}{2+\cos(xy)}\right| < \frac{3(2)(1)}{2+\cos(2)}$$

$$|f(x,y)| < \frac{6}{2+\cos 2} \Rightarrow M = \frac{6}{2+\cos 2}$$

$$\left|\frac{df}{dy}\right| = \left|\frac{(2+\cos(xy))\,3x + 3xy\,(-\sin xy)\,x}{(2+\cos(xy))^2}\right|$$

$$< \frac{(2+\cos(0))\cdot 3 + 6\cdot 1\cdot 1}{(2+\cos 2)^2}$$

$$\left|\frac{df}{dy}\right| < \frac{15}{(2+\cos 2)^2} = N$$

Also, $h = \min\left(a, \frac{b}{m}\right) = \min\left\{1, \frac{2+\cos 2}{3}\right\}.$

$$h = \frac{2+\cos 2}{3}.$$

So, given problem has unique solⁿ in

Interval $\left[|x| \leq \dfrac{2+\cos 2}{3}\right]$