2.20. BESSEL'S DIFFERENTIAL EQUATION

The differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

is called Bessel's differential equation of order n, where n is a positive constant.

This equation can also be put in the form

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x^2 - n^2)\, y = 0$$

The particular solutions of this equation are called Bessel's functions of order n.

2.21. SOLUTION OF BESSEL'S EQUATION

[U.P.T.U. (SUM) 2007 ; U.P.T.U. (C.O.) 2009]

Bessel's equation is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad \text{...(1)}$$

Comparing equation (1) with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\, y = 0, \text{ we get}$$

$$P(x) = \frac{1}{x} \text{ and } Q(x) = 1 - \frac{n^2}{x^2}$$

At $x = 0$, both $P(x)$ and $Q(x)$ are not analytic $\therefore$ $x = 0$ is a singular point.

Also, $xP(x) = 1$ and $x^2\, Q(x) = x^2 - n^2$

Since both $xP(x)$ and $x^2\, Q(x)$ are analytic at $x = 0$

$\therefore$ $x = 0$ is a regular singular point.

Assume $\qquad y = \sum_{k=0}^{\infty} a_k\, x^{m+k}$

Then $\qquad \frac{dy}{dx} = \sum_{k=0}^{\infty} (m+k)\, a_k\, x^{m+k-1}$

and $\qquad \frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} (m+k)(m+k-1)\, a_k\, x^{m+k-2}$

Substituting for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$x^2 \sum_{k=0}^{\infty} (m+k)(m+k-1) a_k\, x^{m+k-2} + x \sum_{k=0}^{\infty} (m+k)\, a_k\, x^{m+k-1} + (x^2 - n^2) \sum_{k=0}^{\infty} a_k\, x^{m+k} = 0$$

or

$$\sum_{k=0}^{\infty} [(m+k)^2 - (m+k) + (m+k) - n^2]\, a_k\, x^{m+k} + \sum_{k=0}^{\infty} a_k\, x^{m+k+2} = 0$$

or $$\sum_{k=0}^{\infty} [(m+k)^2 - n^2]\, a_k x^{m+k} + \sum_{k=0}^{\infty} a_k x^{m+k+2} = 0$$

The lowest power of x is x^m corresponding to $k = 0$. Equating to zero the coefficient of x^m, we get the indicial equation

$$m^2 - n^2 = 0, \text{ since } a_0 \neq 0 \quad \text{whence } m = \pm n$$

Equating to zero the coefficient of next term *i.e.*, x^{m+1}, we get

$$[(m+1)^2 - n^2]a_1 = 0$$

$\Rightarrow$ $$a_1 = 0, \text{ since } (m+1)^2 - n^2 \neq 0 \text{ for } m = \pm n$$

Equating to zero the coefficient of x^{m+k+2}, we get the recurrence relation

$$[(m+k+2)^2 - n^2]\, a_{k+2} + a_k = 0$$

or $$a_{k+2} = -\frac{a_k}{(m-n+k+2)(m+n+k+2)}$$

Putting $k = 1, 3, 5, \ldots$, we get $a_3 = a_5 = a_7 = \ldots = 0$

Putting $k = 0, 2, 4, \ldots$, we get

$$a_2 = -\frac{a_0}{(m-n+2)(m+n+2)}$$

$$a_4 = -\frac{a_2}{(m-n+4)(m+n+4)} = \frac{a_0}{(m-n+4)(m+n+4)(m-n+2)(m+n+2)}$$

and so on.

$\therefore$ $$\boxed{y = a_0 x^m \left[1 - \frac{x^2}{(m+2)^2 - n^2} + \frac{x^4}{[(m+2)^2 - n^2][(m+4)^2 - n^2]} - \ldots\right]} \quad \ldots(2)$$

Depending upon the values of n, we get different types of solutions.

Case I. When n ≠ 0 or n ≠ an integer.

In this case, we get two independent solutions for $m = n$ and $m = -n$.

For $m = n$, we get

$$y_1 = a_0 x^n \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \ldots\right]$$

$$= a_0 x^n \left[1 + (-1)^1 \frac{x^2}{2^2 (1)!(n+1)} + (-1)^2 \frac{x^4}{2^4 (2)!(n+1)(n+2)} + \ldots\right]$$

$$= a_0 x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k)!(n+1)(n+2)\ldots(n+k)} x^{2k}$$

$\Rightarrow$ $$y_1 = a_0 x^n \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+1)}{2^{2k} . k!\, \Gamma(n+k+1)} x^{2k} \quad \ldots(3)$$

Replacing n by $-n$, the second independent solution corresponding to $m = -n$ is

$$y_2 = a_0 x^n \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(-n+1)}{2^{2k}(k)!\, \Gamma(-n+k+1)} x^{2k} \quad \ldots(4)$$

$\therefore$ The complete solution of equation (1) is $y = c_1 y_1 + c_2 y_2$

Since a_0 is arbitrary, we can choose it in any manner.

Choose $a_0 = \dfrac{1}{2^n\,\Gamma(n+1)}$, then (3) takes the form

$$y_1 = \frac{x^n}{2^n\,\Gamma(n+1)} \sum_{k=0}^{\infty} \frac{(-1)^k\,\Gamma(n+1)}{2^{2k}\,.\,k\,!\,\Gamma(n+k+1)} x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k)\,!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

This is called **Bessel function of the first kind of order n** and is denoted by $\mathbf{J_n(x)}$.

Thus,

$$\boxed{J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k)\,!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}}$$

The solution corresponding to $m = -n$ is

$$\boxed{J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k)\,!\,\Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{-n+2k}}$$

which is called **Bessel function of the first kind of order – n.**

When n is not an integer, $J_{-n}(x)$ is distinct from $J_n(x)$. Hence the complete solution of the Bessel's equation may be expressed as

$$\boxed{y = AJ_n(x) + BJ_{-n}(x)}$$ where A and B are arbitrary constants.

Case II. When n = 0, the Bessel's equation (1) takes the form

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0.$$

This is called **Bessel's equation of order zero.**

The two roots of the indicial equation are equal, each = 0.

From equation (2), putting $n = 0$, we have (assuming $a_0 = 1$)

$$y = x^m \left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} - \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2} + \ldots\ldots\right]$$

which is a solution if $m = 0$.

The first solution is given by

$$\boxed{J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k\,!)^2} \left(\frac{x}{2}\right)^{2k}}, \quad \text{since } \Gamma(k+1) = k\,!$$

which is **Bessel function of the first kind of order zero.**

Now $$\frac{\partial y}{\partial m} = x^m \log x \left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} - \ldots\ldots\right]$$
$$+ x^m \left[\frac{x^2}{(m+2)^2} \cdot \frac{2}{m+2} - \frac{x^4}{(m+2)^2(m+4)^2}\left\{\frac{2}{m+2} + \frac{2}{m+4}\right\} + \ldots\ldots\right]$$

The second independent solution is given by $\left(\frac{\partial y}{\partial m}\right)_{m=0}$

$$= J_0(x)\log x + \left[\frac{1}{2^2}x^2 - \frac{1}{2^2.4^2}\left(1+\frac{1}{2}\right)x^4 + \frac{1}{2^2.4^2.6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^6 - \ldots\ldots\right]$$

$$= J_0(x)\log x + \left[\left(\frac{x}{2}\right)^2 - \frac{1}{(2\,!)^2}\left(1+\frac{1}{2}\right)\left(\frac{x}{2}\right)^4 + \frac{1}{(3\,!)^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)\left(\frac{x}{2}\right)^6 - \ldots\ldots\right]$$

$$= J_0(x)\log x + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k\,!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{k}\right)\left(\frac{x}{2}\right)^{2k}$$

It is denoted by $Y_0(x)$ and is called **Bessel function of the second kind of order zero or Neumann function.**

Thus the complete solution of the Bessel's equation of order zero is

$$\boxed{y = AJ_0(x) + BY_0(x)}$$

Case III. When n is an integer, the two functions $J_n(x)$ and $J_{-n}(x)$ **are not independent** but are connected by the relation

$$\mathbf{J_{-n}(x) = (-1)^n J_n(x).}$$

Now, when n is an integer, y_2 fails to give a solution for positive values of n and y_1 fails to give a solution for negative values of n. Let us find an independent solution of Bessel's equation (1), when n is an integer.

Let $y = u(x)\,J_n(x)$ be a solution of (1) when n is integral.

Then $\qquad \frac{dy}{dx} = u'J_n + uJ_n'$ and $\frac{d^2y}{dx^2} = u''J_n + 2u'J_n' + uJ_n''$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$x^2(u''J_n + 2u'J_n' + uJ_n'') + x(u'J_n + uJ_n') + (x^2 - n^2)uJ_n = 0$$

or $\qquad u[x^2J_n'' + xJ_n' + (x^2-n^2)J_n] + x^2u''J_n + 2x^2u'J_n' + xu'J_n = 0$

or $\qquad x^2u''J_n + 2x^2u'J_n' + xu'J_n = 0$ since J_n is a solution of (1).

Dividing throughout by $x^2u'J_n$, we get

$$\frac{u''}{u'} + 2\frac{J_n'}{J_n} + \frac{1}{x} = 0$$

Integrating w.r.t. x, we get

$$\log(u'J_n^2x) = \log B$$

or $\qquad u'J_n^2x = B$

or $\qquad u' = \frac{B}{xJ_n^2}$ or $u = B\int\frac{dx}{xJ_n^2} + A$

Substituting the value of u in the assumed solution $y = u(x)\,J_n(x)$, we have

$$y = \left[B\int\frac{dx}{xJ_n^2(x)} + A\right]J_n(x)$$

or $$y = AJ_n(x) + BY_n(x), \text{ where } Y_n(x) = J_n(x)\int \frac{dx}{xJ_n^2(x)}$$

The function $Y_n(x)$ is called the **Bessel function of the second kind of order n or Neumann function.**

2.22. SERIES REPRESENTATION OF BESSEL FUNCTIONS

Since $$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

$$\therefore \quad J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}, \quad \because \Gamma(k+1) = k!$$

$$= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots\ldots$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots\ldots$$

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+2)}\left(\frac{x}{2}\right)^{1+2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+1)!}\left(\frac{x}{2}\right)^{1+2k}$$

$$= \frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 - \ldots\ldots = \frac{x}{2} - \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} - \ldots\ldots$$

In particular, $J_0(0) = 1$ and $J_1(0) = 0$.

The values of $J_0(x)$ and $J_1(x)$ are given in 'Jahnke Emde's tables' to four decimal places at intervals of 0.1.

2.23. RECURRENCE RELATIONS FOR J_n (X)

1. $$\boxed{x J_n' = nJ_n - x J_{n+1}}$$

We know that,

$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma n+r+1}\left(\frac{x}{2}\right)^{n+2r}$$

Differentiating w.r.t. x, we get

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r\,(n+2r)}{r!\,\Gamma n+r+1}\cdot\frac{1}{2}\left(\frac{x}{2}\right)^{n+2r-1}$$

Multiplying both sides by x and breaking it into two terms

$$xJ_n' = n\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma n+r+1}\left(\frac{x}{2}\right)^{n+2r} + x\sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\,\Gamma n+r+1}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n + x\sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\Gamma n+s+2}\left(\frac{x}{2}\right)^{n+2s+1} \qquad | \text{ Here } r = s+1$$

$$= nJ_n - x \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma n+s+2} \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$\Rightarrow \qquad x J_n' = nJ_n - xJ_{n+1} \qquad \text{...(1)}$$

2. $\boxed{xJ_n' = -nJ_n + xJ_{n-1}}$ [U.P.T.U. (SUM) 2004 ; U.P.T.U. 2006]

We know that

$$xJ_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,\Gamma n+r+1} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r-n)}{r!\,\Gamma n+r+1} \left(\frac{x}{2}\right)^{n+2r}$$

$$= -n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma n+r+1} \left(\frac{x}{2}\right)^{n+2r} + x \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)}{r!\,\Gamma n+r+1} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= -nJ_n + x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma n+r} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= -nJ_n + x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n-1)+r+1} \left(\frac{x}{2}\right)^{n-1+2r}$$

$$xJ_n' = -nJ_n + xJ_{n-1} \qquad \text{...(2)}$$

3. $\boxed{2J_n' = J_{n-1} - J_{n+1}}$

Adding equations (1) and (2), we get

$$2xJ_n' = x(J_{n-1} - J_{n+1})$$

$$\Rightarrow \qquad 2J_n' = J_{n-1} - J_{n+1} \qquad \text{...(3)}$$

4. $\boxed{2nJ_n = x(J_{n-1} + J_{n+1})}$ [U.P.T.U. 2007 ; U.P.T.U. (SUM) 2007]

Subtracting (2) from (1), we get

$$0 = 2n\,J_n - x\,J_{n-1} - xJ_{n+1}$$

$$\Rightarrow \qquad 2nJ_n = x(J_{n-1} + J_{n+1}) \qquad \text{...(4)}$$

5. $\boxed{\dfrac{d}{dx}(x^{-n} J_n) = -x^{-n} J_{n+1}}$ [U.P.T.U. (C.O.) 2003 ; U.P.T.U. (SUM) 2007]

$\therefore$ Multiplying eqn. (1) by x^{-n-1}, we get

$$x^{-n} J_n' = nx^{-n-1} J_n - x^{-n} J_{n+1}$$

$$\Rightarrow \qquad x^{-n} J_n' - x^{-n-1} \,.\, nJ_n = -x^{-n} J_{n+1}$$

$$\frac{d}{dx}(x^{-n} J_n) = -x^{-n} J_{n+1} \qquad \text{...(5)}$$

6. $\boxed{\dfrac{d}{dx}(x^n J_n) = x^n J_{n-1}}$ [U.P.T.U. 2004, 2005 ; U.P.T.U. (C.O.) 2008]

Multiplying eqn. (2) by x^{n-1}, we get

$$x^n J_n' = -n x^{n-1} J_n + x^n J_{n-1}$$

$$\Rightarrow \quad x^n J_n' + n x^{n-1} J_n = x^n J_{n-1}$$

$$\Rightarrow \quad \frac{d}{dx}(x^n J_n) = x^n J_{n-1} \qquad \text{...(6)}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that :*

$$J_{-n}(x) = (-1)^n J_n(x). \qquad \text{[U.P.T.U. (C.O.) 2005]}$$

Sol. Since $\Gamma - p$ is infinity ($p > 0$), we get terms in $J_{-n}(x)$ equal to zero till $r + 1 - n \geq 1$ so that the series begins when $r \geq n$

Hence we can write,

$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r}{r!\,\Gamma - n + r + 1}\left(\frac{x}{2}\right)^{-n+2r}$$

Putting $r = n + s$, we get

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{n+s}}{(n+s)!\,\Gamma s + 1}\left(\frac{x}{2}\right)^{n+2s}$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{(n+r)!\,\Gamma r + 1}\left(\frac{x}{2}\right)^{n+2r} = (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma n + r + 1\; r!}\left(\frac{x}{2}\right)^{n+2r}$$

$$\Rightarrow \quad J_{-n}(x) = (-1)^n J_n(x).$$

Example 2. *Prove that :* $J_0'(x) = -J_1(x)$.

Sol. We know that

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

$\Gamma 2 = 1!$

$\Gamma n+1 = n!$

Putting $n = 0$, we get

$$\frac{d}{dx}[J_0(x)] = -J_1(x)$$

$$\Rightarrow \quad J_0'(x) = -J_1(x).$$

Example 3. *Prove that :*

(*i*) $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\sin x$ [U.P.T.U. 2009, U.P.T.U. (SUM) 2008] (*ii*) $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cos x$.

Sol. We know that,

$$J_n(x) = \frac{x^n}{2^n\,\Gamma n + 1}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4.(2n+2)(2n+4)} - \ldots\ldots\right] \qquad \text{...(1)}$$

$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

(*i*) Putting $n = \frac{1}{2}$ in (1),

$$J_{1/2}(x) = \frac{\sqrt{x}}{\sqrt{2}\,\Gamma 3/2}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\,.\,\frac{1}{2}\sqrt{\pi}} \cdot \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^3}{5!} - \ldots\right] = \sqrt{\frac{2}{\pi x}}\sin x$$

(*ii*) Putting $n = -\frac{1}{2}$ in (1),

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\,\Gamma 1/2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) = \sqrt{\frac{2}{\pi x}}\cos x.$$

Example 4. *Prove that :*

$$\frac{d}{dx}[J_n^2(x) + J^2_{n+1}(x)] = 2\left[\frac{n}{x}J_n^2(x) - \frac{n+1}{x}J^2_{n+1}(x)\right].$$ (U.P.T.U. 2005)

Sol. $\quad \text{L.H.S.} = 2J_n J_n' + 2J_{n+1} J'_{n+1}$ ...(1)

But $\quad xJ_n' = nJ_n - xJ_{n+1}$ | Recurrence relation (1)

$\therefore \quad J_n' = \frac{n}{x}J_n - J_{n+1}$ ...(2)

and also, $\quad xJ_n' = -nJ_n + xJ_{n-1}$ | Recurrence relation (2)

$\therefore \quad J_n' = -\frac{n}{x}J_n + J_{n-1}$

or $\quad J'_{n+1} = -\left(\frac{n+1}{x}\right)J_{n+1} + J_n$ ...(3)

Substituting these values of J_n' and J'_{n+1} from (2) and (3) in eqn. (1), we get

$$\text{L.H.S.} = 2J_n\left(\frac{n}{x}J_n - J_{n+1}\right) + 2J_{n+1}\left(-\frac{n+1}{x}J_{n+1} + J_n\right)$$

$$= 2\frac{n}{x}J_n^2 - 2\left(\frac{n+1}{x}\right)J^2_{n+1} = \text{R.H.S.}$$

Hence the result.

Example 5. *Prove that :*

(*i*) $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$ [U.P.T.U. (SUM) 2009]

(*ii*) $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{-\cos x}{x} - \sin x\right).$

Sol. By Recurrence relation (4), we have,

$$2n\,J_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)] \quad ...(1)$$

$$\Rightarrow \quad J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x) \quad ...(2)$$

(*i*) Putting $n = 1/2$ in (2), we get

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$= \sqrt{\frac{2}{\pi x}} \left[\frac{\sin x}{x} - \cos x \right] \qquad \text{| Using results of Ex. 3}$$

(*ii*) From equation (1),

$$J_{n-1}(x) = \frac{2n}{x} J_n(x) - J_{n+1}(x) \qquad ...(3)$$

Putting $n = -1/2$ in (3), we get

$$J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x)$$

$$= \sqrt{\frac{2}{\pi x}} \left[\frac{-\cos x}{x} - \sin x \right] \qquad \text{| Using results of Ex. 3}$$

Example 6. *Prove that :*

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\left(\frac{3 - x^2}{x^2} \right) \sin x - \frac{3 \cos x}{x} \right]. \qquad \text{(Q. Bank U.P.T.U. 2002)}$$

Sol. From Recurrence relation (4),

$$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$$

$$\Rightarrow \quad J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \qquad ...(1)$$

Putting $n = 1/2, 3/2$ in (1), we get

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) \qquad ...(2)$$

$$J_{5/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}(x) \qquad ...(3)$$

From (2) and (3),

$$J_{5/2}(x) = \frac{3}{x} \left[\frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) \right] - J_{1/2}(x)$$

$$= \left(\frac{3}{x^2} - 1 \right) J_{1/2}(x) - \frac{3}{x} J_{-1/2}(x)$$

$$= \left(\frac{3 - x^2}{x^2} \right) \sqrt{\frac{2}{\pi x}} \sin x - \frac{3}{x} \cdot \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}} \left[\left(\frac{3 - x^2}{x^2} \right) \sin x - \frac{3}{x} \cos x \right].$$

Example 7. *Prove that :*

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x} \right) J_1(x) + \left(1 - \frac{24}{x^2} \right) J_0(x).$$

Hence or otherwise find $J_6(x)$ in terms of $J_0(x)$ and $J_1(x)$. (U.P.T.U. 2009)

Sol. From Recurrence relation (4),

$$2nJ_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]$$

$$\Rightarrow \quad J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \quad \text{...(1)}$$

Putting $n = 1, 2, 3$ in eqn. (1), we get

$$J_2(x) = \frac{1}{x}[2J_1(x) - xJ_0(x)] \quad \text{...(2)}$$

$$J_3(x) = \frac{1}{x}[4J_2(x) - xJ_1(x)] \quad \text{...(3)}$$

$$J_4(x) = \frac{1}{x}[6J_3(x) - xJ_2(x)] \quad \text{...(4)}$$

From (2) and (3),

$$J_3(x) = \frac{8}{x^2}J_1(x) - \frac{4}{x}J_0(x) - J_1(x) = \left(\frac{8-x^2}{x^2}\right)J_1(x) - \frac{4}{x}J_0(x) \quad \text{...(5)}$$

Again from (4) and (5),

$$J_4(x) = \left(\frac{48-6x^2}{x^3}\right)J_1(x) - \frac{24}{x^2}J_0(x) - \frac{2}{x}J_1(x) + J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x). \quad \text{...(6)}$$

Again, putting $n = 4, 5$ in eqn. (1), we get

$$J_5(x) = \frac{8}{x}J_4(x) - J_3(x) \quad \text{...(7)}$$

$$J_6(x) = \frac{10}{x}J_5(x) - J_4(x) \quad \text{...(8)}$$

From (7) and (8),

$$J_6(x) = \frac{10}{x}\left[\frac{8}{x}J_4(x) - J_3(x)\right] - J_4(x)$$

$$= \left(\frac{80}{x^2} - 1\right)J_4(x) - \frac{10}{x}J_3(x)$$

$$= \left(\frac{80}{x^2} - 1\right)\left[\left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)\right]$$

$$- \frac{10}{x}\left[\left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)\right] \quad | \text{ Using eqn. (5) and (6)}$$

$$\Rightarrow \quad J_6(x) = \left(\frac{3840}{x^5} - \frac{768}{x^3} + \frac{18}{x}\right)J_1(x) + \left(\frac{144}{x^2} - \frac{1920}{x^4} - 1\right)J_0(x).$$

Example 8. *Prove that :*

$$\int J_3(x)\,dx + J_2(x) + \frac{2}{x}J_1(x) = 0. \quad \text{(Q. Bank U.P.T.U. 2002)}$$

Sol. We know that,

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

$$\therefore \quad \int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x) \qquad ...(1)$$

Now,
$$\int J_3(x)\,dx = \int x^2 [x^{-2} J_3(x)]\,dx$$
$$= x^2 . [-x^{-2} J_2(x)] - \int 2x . [-x^{-2} J_2(x)]\,dx$$
$$= -J_2(x) + 2\int x^{-1} J_2(x) dx$$
$$= -J_2(x) + 2[-x^{-1} J_1(x)] = -J_2(x) - \frac{2}{x} J_1(x)$$

or
$$\int J_3(x)\,dx + J_2(x) + \frac{2}{x} J_1(x) = 0.$$

Example 9. *Prove that :*

$$\int xJ_0^2(x)\,dx = \frac{1}{2}x^2 [J_0^2(x) + J_1^2(x)] + c. \qquad \text{(U.P.T.U. 2003, 2004)}$$

Sol. $\int xJ_0^2(x)\,dx = J_0^2(x) . \frac{x^2}{2} - \int 2J_0(x) J_0'(x) . \frac{x^2}{2}\,dx + c$

$$= \frac{x^2}{2} J_0^2(x) - \int x^2 J_0(x) \{-J_1(x)\}\,dx + c \qquad [\because \quad J_0'(x) = -J_1(x)]$$
$$= \frac{x^2}{2} J_0^2(x) + \int xJ_1(x) . xJ_0(x)\,dx + c$$
$$= \frac{x^2}{2} J_0^2(x) + \int xJ_1(x) . \frac{d}{dx}[xJ_1(x)]\,dx + c$$

(Using Recurrence relation)

$$= \frac{x^2}{2} J_0^2(x) + \frac{[xJ_1(x)]^2}{2} + c = \frac{x^2}{2}[J_0^2(x) + J_1^2(x)] + c.$$

Example 10. *Prove that :*

$$4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x). \qquad \text{[U.P.T.U. (C.O.) 2009]}$$

Sol. From Recurrence relation (3), we have

$$2J_n' = J_{n-1} - J_{n+1} \qquad ...(1)$$

Differentiating, $\quad 2J_n'' = J'_{n-1} - J'_{n+1}$

or $\quad 4J_n'' = 2J'_{n-1} - 2J'_{n+1} = (J_{n-2} - J_n) - (J_n - J_{n+2}) \qquad |$ Using (1)

$\Rightarrow \quad 4J_n'' = J_{n-2} - 2J_n + J_{n+2}$

Example 11. *Prove that :*

$$4J_0'''(x) + 3J_0'(x) + J_3(x) = 0. \quad \text{[U.P.T.U. 2002 ; U.P.T.U. (SUM) 2008]}$$

Sol. We know that

$$J_0' = -J_1$$

Diff. gives, $\quad J_0'' = -J_1' = -\frac{1}{2}(J_0 - J_2) \quad$ [By Recurrence relation $2J_n' = J_{n-1} - J_{n+1}$]

Diff. again, $J_0''' = -\frac{1}{2}(J_0' - J_2') = -\frac{1}{2}J_0' + \frac{1}{2}\cdot\frac{1}{2}[J_1 - J_3]$

$$= -\frac{1}{2}J_0' + \frac{1}{4}J_1 - \frac{1}{4}J_3 = -\frac{1}{2}J_0' - \frac{1}{4}J_0' - \frac{1}{4}J_3 \qquad \Big|\because \; J_1 = -J_0'$$

$$= -\frac{3}{4}J_0' - \frac{1}{4}J_3$$

$\Rightarrow \quad 4J_0''' + 3J_0' + J_3 = 0.$

Example 12. *Prove that :*

$$\frac{d}{dx}[xJ_n(x)\,J_{n+1}(x)] = x\,[J_n^2(x) - J_{n+1}^2(x)].$$

Sol. L.H.S. $= \frac{d}{dx}[x^{-n}J_n(x)\,.\,x^{n+1}J_{n+1}(x)]$

$$= x^{-n}J_n(x)\frac{d}{dx}[x^{n+1}J_{n+1}(x)] + x^{n+1}J_{n+1}(x)\,.\,\frac{d}{dx}[x^{-n}J_n(x)]$$

$$= x^{-n}J_n(x)\,.\,x^{n+1}J_n(x) + x^{n+1}J_{n+1}(x)\,[-x^{-n}J_{n+1}(x)]$$

$$\Big|\because \; \frac{d}{dx}(x^n J_n) = x^n J_{n-1}$$

$$= xJ_n^2(x) - xJ_{n+1}^2 = x\,[J_n^2(x) - J_{n+1}^2] = \text{R.H.S.}$$

Example 13. *Prove that :*

$$\lim_{x\to 0}\frac{J_n(x)}{x^n} = \frac{1}{2^n\,\Gamma n+1} \quad ;\ (n > -1).$$

Sol. We know that

$$J_n(x) = \frac{x^n}{2^n\,\Gamma n+1}\left[1 - \frac{x^2}{2.(2n+2)} + \frac{x^4}{2.4.(2n+2)(2n+4)} - \ldots\ldots\right]$$

$$\therefore \quad \lim_{x\to 0}\frac{J_n(x)}{x^n} = \lim_{x\to 0}\frac{1}{2^n\,\Gamma n+1}\left[1 - \frac{x^2}{2.(2n+2)} + \frac{x^4}{2.4.(2n+2)(2n+4)} - \ldots\ldots\right]$$

$$= \frac{1}{2^n\,\Gamma n+1}.$$

Example 14. *Prove that :*

$$J_2'(x) = \left(1 - \frac{4}{x^2}\right)J_1(x) + \frac{2}{x}J_0(x). \qquad \text{(U.P.T.U. 2008)}$$

Sol. By Recurrence relation (2), we have

$$xJ_n' = -nJ_n + xJ_{n-1} \qquad \ldots(1)$$

Putting $n = 2$, $\quad xJ_2' = -2J_2 + xJ_1$

$$\Rightarrow \quad J_2' = -\frac{2}{x}J_2 + J_1 \qquad \ldots(2)$$

By Recurrence relation (1), we have

$$xJ_n' = nJ_n - xJ_{n+1} \qquad \ldots(3)$$

From (1) and (3), we have

$$-nJ_n + xJ_{n-1} = nJ_n - xJ_{n+1}$$

$$\Rightarrow \quad \frac{x}{2} J_n = (n+1) J_{n+1} - \frac{x}{2} J_{n+2} \qquad \text{...(1)}$$

$$\Rightarrow \quad \frac{x}{2} J_{n+2} = (n+3) J_{n+3} - \frac{x}{2} J_{n+4} \qquad \text{...(2)}$$

$\therefore$ From (1) and (2),

$$\frac{x}{2} J_n = (n+1) J_{n+1} - (n+3) J_{n+3} + \frac{x}{2} J_{n+4}$$

Continuing this way,

$$\frac{x}{2} J_n = (n+1) J_{n+1} - (n+3) J_{n+3} + (n+5) J_{n+5} - \ldots..$$

Example 19. *If $n > -1$, show that :*

$$\int_0^x x^{-n} J_{n+1}(x)\, dx = \frac{1}{2^n \Gamma n+1} - x^{-n} J_n(x).$$

Sol. We know that,

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x) \qquad \text{| Recurrence relation}$$

Integrating it between 0 and x, we get

$$\int_0^x x^{-n} J_{n+1}(x)\, dx = -\left[x^{-n} J_n(x)\right]_0^x = -x^{-n} J_n(x) + \underset{x \to 0}{\text{Lt}} \left[\frac{J_n(x)}{x^n}\right]$$

$$= -x^{-n} J_n(x) + \frac{1}{2^n \Gamma n+1}.$$

Example 20. *Prove that :*

$$J_n' = \frac{2}{x}\left[\frac{n}{2} J_n - (n+2) J_{n+2} + (n+4) J_{n+1} - \ldots\ldots\right].$$

Sol. From Recurrence formula (2), we have

$$J_n' = -\frac{n}{x} J_n + J_{n-1}$$

$$= -\frac{n}{x} J_n + \frac{2}{x}[nJ_n - (n+2) J_{n+2} + \ldots] \qquad \text{| using Example 18}$$

$$= \frac{2}{x}\left[\frac{n}{2} J_n - (n+2) J_{n+2} + \ldots\right].$$

2.24. GENERATING FUNCTION FOR J_n (x)

The function $e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}$ is called generating function.

Prove that

(*i*) $$\boxed{e^{\frac{x}{2}\left(z - \frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n J_n(x)}$$

i.e., $J_n(x)$ is the coefficient of z^n in the expansion of $e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}$.

(ii) $$\boxed{e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} (-1)^n J_n(x)\, z^{-n}}$$

i.e., $(-1)^n J_n(x)$ is the coefficient of z^{-n} in the expansion of $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$.

Note. Above results are true if n is an integer.

Proof. We have,

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = e^{\frac{xz}{2}}\, e^{-\frac{x}{2z}}$$

$$= \left[1+\frac{xz}{2}+\left(\frac{x}{2}\right)^2\frac{z^2}{2!}+\ldots\ldots\right]\left[1-\frac{x}{2z}+\left(\frac{x}{2z}\right)^2\frac{1}{2!}-\ldots\ldots\right]$$

(i) Coeff. of z^n in this product

$$= \left(\frac{x}{2}\right)^n\frac{1}{n!}-\left(\frac{x}{2}\right)^{n+2}\cdot\frac{1}{(n+1)!}+\left(\frac{x}{2}\right)^{n+4}\cdot\frac{1}{(n+2)!}\cdot\frac{1}{2!}+\ldots\ldots$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r} = J_n$$

(ii) Coeff. of z^{-n} in this product

$$= \left(-\frac{x}{2}\right)^n\frac{1}{n!}-\left(-\frac{x}{2}\right)^{n+2}\cdot\frac{1}{(n+1)!}+\left(-\frac{x}{2}\right)^{n+4}\cdot\frac{1}{(n+2)!}\cdot\frac{1}{2!}+\ldots\ldots$$

$$= (-1)^n\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r} = (-1)^n J_n.$$

2.25. INTEGRAL FORM OF BESSEL FUNCTION

We know that $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

$$= J_0(x)+tJ_1(x)+t^2J_2(x)+t^3J_3(x)+\ldots\ldots+t^{-1}J_{-1}(x)+t^{-2}J_{-2}(x)+t^{-3}J_{-3}(x)+\ldots\ldots$$

$$= J_0(x)+tJ_1(x)+t^2J_2(x)+t^3J_3(x)+\ldots\ldots-t^{-1}J_1(x)+t^{-2}J_2(x)-t^{-3}J_3(x)+\ldots\ldots$$

$[\because\ J_{-n}(x) = (-1)^n J_n(x)]$

$$= J_0(x)+\left(t-\frac{1}{t}\right)J_1(x)+\left(t^2+\frac{1}{t^2}\right)J_2(x)+\left(t^3-\frac{1}{t^3}\right)J_3(x)+\ldots\ldots \quad \ldots(1)$$

Put $t = \cos\theta + i\sin\theta$

$\therefore\quad t^n = \cos n\theta + i\sin n\theta$ and $\frac{1}{t^n} = \cos n\theta - i\sin n\theta$ | By De Moivre's theorem

so that $\quad t^n + \frac{1}{t^n} = 2\cos n\theta$ and $t^n - \frac{1}{t^n} = 2i\sin n\theta$

Substituting these values in (1), we have

$$e^{ix\sin\theta} = J_0(x) + 2i\sin\theta J_1(x) + 2\cos 2\theta\, J_2(x) + 3i\sin 3\theta\, J_3(x) + \ldots\ldots \quad \ldots(2)$$

Since $e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta)$

$\therefore$ Equating the real and imaginary parts in (2), we get

$$\cos(x\sin\theta) = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \ldots\ldots] \quad \ldots(3)$$

$$\sin(x\sin\theta) = 2[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \ldots\ldots] \quad \ldots(4)$$

These are known as **Jacobi series.**

Multiplying both sides of (3) by $\cos n\theta$ and integrating w.r.t. θ between the limits 0 and π (when n is odd, all terms on the R.H.S. vanish ; when n is even, all terms on the R.H.S. except the one containing $\cos n\theta$ vanish), we get

$$\int_0^\pi \cos(x\sin\theta)\cos n\theta\, d\theta = \begin{cases} 0, & \text{when } n \text{ is odd} \\ \pi J_n(x), & \text{when } n \text{ is even} \end{cases}$$

Similarly, multiplying (4) by $\sin n\theta$ and integrating w.r.t. θ between the limits 0 and π, we get

$$\int_0^\pi \sin(x\sin\theta)\sin n\theta\, d\theta = \begin{cases} \pi J_n(x), & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

Adding, we get $\displaystyle\int_0^\pi [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\, d\theta = \pi J_n(x)$

$\Rightarrow$ $\displaystyle J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta$ for all integral values of n.

ILLUSTRATIVE EXAMPLES

Example 1. *Use Jacobi series to prove that*

$$[J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + 2[J_3(x)]^2 + \ldots\ldots = 1.$$ (U.P.T.U. 2003, 2008)

Sol. The Jacobi series are

$$J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \ldots\ldots = \cos(x\sin\theta) \quad \ldots(1)$$

and $$2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \ldots\ldots = \sin(x\sin\theta) \quad \ldots(2)$$

Squaring (1) and (2) and integrating w.r.t. θ between the limits 0 and π, and remembering that if m, n are integers then

$$\int_0^\pi \cos^2 n\theta\, d\theta = \int_0^\pi \sin^2 n\theta\, d\theta = \frac{\pi}{2}$$

and $$\int_0^\pi \cos m\theta\cos n\theta\, d\theta = \int_0^\pi \sin m\theta\sin n\theta\, d\theta = 0,\ m \neq n, \text{ we get}$$

$$[J_0(x)]^2\pi + 2[J_2(x)]^2\pi + 2[J_4(x)]^2\pi + \ldots\ldots = \int_0^\pi \cos^2(x\sin\theta)\, d\theta$$

$$2[J_1(x)]^2\pi + 2[J_3(x)]^2\pi + \ldots\ldots = \int_0^\pi \sin^2(x\sin\theta)\, d\theta$$

Adding, we have $\pi\{[J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \ldots\ldots\}$

$$= \int_0^\pi [\cos^2(x\sin\theta) + \sin^2(x\sin\theta)]\, d\theta = \int_0^\pi d\theta = \pi\ .$$

$\therefore$ $$[J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \ldots\ldots = 1.$$

$$\frac{d}{dx}\,[x \text{ bei}'(x)] = x - \frac{x^5}{2^2 \,.\, 4^2} + \frac{x^9}{2^2 \,.\, 4^2 \,.\, 6^2 \,.\, 8^2} - \ldots\ldots..$$

$$= x\left(1 - \frac{x^4}{2^2 \,.\, 4^2} + \frac{x^8}{2^2 \,.\, 4^2 \,.\, 6^2 \,.\, 8^2} - \ldots\ldots..\right) = x \text{ ber } (x).$$

2.29. ORTHOGONALITY OF BESSEL FUNCTIONS [U.P.T.U. (SUM) 20

If α and β are the roots of $J_n(x) = 0$, then

$$\int_0^1 x\, J_n(\alpha x) \,.\, J_n(\beta x)\, dx = \begin{cases} 0, & \text{when } \alpha \neq \beta \\ \frac{1}{2} J^2_{n+1}(\alpha), & \text{when } \alpha = \beta \end{cases}$$

Consider the Bessel's equations

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2) u = 0$$

and

$$x^2 v'' + x v' + (\beta^2 x^2 - n^2) v = 0$$

Their solutions are $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ respectively.

Multiplying (1) by $\frac{v}{x}$ and (2) by $\frac{u}{x}$ and subtracting, we get

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)\, xuv = 0$$

or
$$\frac{d}{dx}[x(u'v - uv')] = (\beta^2 - \alpha^2)\, xuv$$

Integrating both sides w.r.t. x between the limits 0 and 1, we get

$$(\beta^2 - \alpha^2)\int_0^1 xuv\, dx = \Big[x(u'v - uv')\Big]_0^1 = \Big[u'v - uv'\Big]_{x=1} \quad \text{...(3)}$$

Since $u = J_n(\alpha x)$

$\therefore$
$$u' = \frac{d}{dx}[J_n(\alpha x)] = \frac{d}{d(\alpha x)}(J_n(\alpha x)]\,.\frac{d(\alpha x)}{dx} = \alpha J_n'(\alpha x)$$

Similarly, $v = J_n(\beta x) \Rightarrow v' = \beta J_n'(\beta x)$

Substituting for u, v, u' and v' in (3), we get

$$\int_0^1 xJ_n(\alpha x)\, J_n(\beta x)\, dx = \frac{\alpha J_n'(\alpha)\, J_n(\beta) - \beta J_n(\alpha)\, J_n'(\beta)}{\beta^2 - \alpha^2} \quad \text{...(4)}$$

If α and β are *distinct* roots of $J_n(x) = 0$, then $J_n(\alpha) = 0$ and $J_n(\beta) = 0$.

Hence, from (4), we have

$$\int_0^1 xJ_n(\alpha x)\, J_n(\beta x)\, dx = 0$$

However, if $\alpha = \beta$, the value of the integral is $\frac{0}{0}$, which is indeterminate.

To evaluate the integral, we assume that α is a root of $J_n(x) = 0$ so that $J_n(\alpha) = 0$ and β is a variable approaching α. Thus, from (4), we have

$$\underset{\beta \to \alpha}{\text{Lt}} \int_0^1 xJ_n(\alpha x)\, J_n(\beta x)\, dx = \underset{\beta \to \alpha}{\text{Lt}} \frac{\alpha J_n'(\alpha)\, J_n(\beta)}{\beta^2 - \alpha^2}$$

or
$$\int_0^1 xJ_n^2(\alpha x)\, dx = \underset{\beta \to \alpha}{\text{Lt}} \frac{\alpha J_n'(\alpha)\, J_n'(\beta)}{2\beta} \quad \text{[by L-Hospital's rule]}$$

$$= \tfrac{1}{2}[J_n'(\alpha)]^2 \quad \text{...(5)}$$

But
$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

$\therefore$
$$xJ_n'(\alpha) = nJ_n(\alpha) - \alpha J_{n+1}(\alpha) = -\alpha J_{n+1}(\alpha), \text{ since } J_n(\alpha) = 0$$

$\Rightarrow$
$$J_n'(\alpha) = -J_{n+1}(\alpha)$$

Hence, from (5), we get $\int_0^1 xJ_n^2(\alpha x)\, dx = \frac{1}{2}J^2{}_{n+1}(\alpha)$.

Note. If the interval is from 0 to a, it can be shown that

$$\int_0^a xJ_n^2(\alpha x)\, dx = \frac{a^2}{2}J^2{}_{n+1}(\alpha a), \quad \text{where } \alpha \text{ is a root of } J_n(\alpha a) = 0.$$

2.30. FOURIER-BESSEL EXPANSION OF f(x)

From the orthogonal property of Bessel functions, we can expand a function $f(x)$ in Fourier-Bessel series in the range 0 to a.

Let $$f(x) = c_1 J_n(\lambda_1 x) + c_2 J_n(\lambda_2 x) + \ldots\ldots + c_n J_n(\lambda_n x) + \ldots\ldots = \sum_{i=1}^{\infty} c_i J_n(\lambda_i x) \qquad \ldots(1)$$

where $\lambda_1, \lambda_2, \ldots\ldots$ are the roots of the equation $J_n(\lambda a) = 0$.

To determine c_i, we multiply both sides of (1) by $xJ_n(\lambda_i x)$ and integrate w.r.t. x between the limits 0 to a. From the orthogonal property of Bessel functions, all integrals on the right hand side will vanish except the one containing c_i and we have

$$\int_0^a xf(x)\, J_n(\lambda_i x)\, dx = c_i \int_0^a xJ_n^2(\lambda_i x)\, dx = c_i \cdot \frac{a^2}{2} J^2_{n+1}(\lambda_i a)$$

$$\therefore \quad c_i = \frac{2}{a^2 J^2_{n+1}(\lambda_i a)} \int_0^a xf(x)\, J_n(\lambda_i x)\, dx$$

Putting $i = 1, 2, 3, \ldots\ldots$ we can find $c_1, c_2, c_3, \ldots\ldots$ and hence the function $f(x)$.