1.21. SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

Now we discuss differential equations in which there is one independent variable and two or more than two dependent variables. Such equations are called *simultaneous linear equations.* To solve such equations completely, we must have as many simultaneous equations as the number of dependent variables.

Let x, y be the two dependent variables and t be the independent variable. Consider the simultaneous equations

$$f_1(\mathrm{D})x + f_2(\mathrm{D})y = \mathrm{T}_1 \quad \text{...(1)}$$

and

$$\phi_1(\mathrm{D})x + \phi_2(\mathrm{D})y = \mathrm{T}_2 \quad \text{...(2)}$$

where $\mathrm{D} \equiv \frac{d}{dt}$ and T_1, T_2 are functions of t.

To eliminate y, operating on both sides of (1) by $\phi_2(\mathrm{D})$ and on both sides of (2) by $f_2(\mathrm{D})$ and subtracting,

we get $\quad [f_1(\mathrm{D})\,\phi_2(\mathrm{D}) - \phi_1(\mathrm{D})\,f_2(\mathrm{D})]x = \phi_2(\mathrm{D})\mathrm{T}_1 - f_2(\mathrm{D})\mathrm{T}_2 \quad$ or $\quad f(\mathrm{D})x = \mathrm{T}$

which is a linear equation in x and t and can be solved by the methods already discussed. Substituting the value of x in either (1) or (2), we get the value of y.

Note. We can also eliminate x to get a linear equation in y and t.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following simultaneous differential equations*

$$\frac{dx}{dt} = 3x + 2y,\ \frac{dy}{dt} = 5x + 3y. \quad \text{[M.T.U. (B. Pharm.) 2011; U.P.T.U. (SUM) 2008]}$$

Sol. Let $\frac{d}{dt} \equiv \mathrm{D}$ then the given system of equations become

$$(\mathrm{D} - 3)x - 2y = 0 \quad \text{...(1)}$$

$$-5x + (\mathrm{D} - 3)y = 0 \quad \text{...(2)}$$

Operating eqn. (1) by $(\mathrm{D} - 3)$ and multiplying eqn. (2) by 2 then adding, we get

$$[(\mathrm{D} - 3)^2 - 10]\,x = 0$$

$$(\mathrm{D}^2 - 6\mathrm{D} - 1)\,x = 0$$

Auxiliary equation is

$$m^2 - 6m - 1 = 0$$

$$\Rightarrow \quad m = \frac{6 \pm \sqrt{36+4}}{2} = 3 \pm \sqrt{10}$$

$$\text{C.F.} = e^{3t}(c_1 \cosh \sqrt{10}\,t + c_2 \sinh \sqrt{10}\,t)$$

$$\text{P.I.} = 0$$

$$\therefore \quad x = e^{3t}\,(c_1 \cosh \sqrt{10}\,t + c_2 \sinh \sqrt{10}\,t) \quad \text{...(3)}$$

From (1), $\quad 2y = \frac{dx}{dt} - 3x$

$$= e^{3t}\sqrt{10}\,(c_1 \sinh \sqrt{10}\,t + c_2 \cosh \sqrt{10}\,t)$$
$$+ 3e^{3t}\,(c_1 \cosh \sqrt{10}\,t + c_2 \sinh \sqrt{10}\,t)$$
$$- 3e^{3t}\,(c_1 \cosh \sqrt{10}\,t + c_2 \sinh \sqrt{10}\,t)$$

$$\Rightarrow \quad y = \frac{\sqrt{10}}{2} e^{3t}\,(c_1 \sinh \sqrt{10}\,t + c_2 \cosh \sqrt{10}\,t) \qquad ...(4)$$

Equations (3) and (4), when taken together, give the complete solution. c_1 and c_2 are arbitrary constants of integration.

Example 2. *Solve the following simultaneous differential equations*

$$\frac{dx}{dt} = -wy, \quad \frac{dy}{dt} = wx.$$

Also show that the point (x, y) lies on a circle. [U.P.T.U. (SUM) 2007; U.P.T.U. 2009]

Sol. Let $\frac{d}{dt} \equiv D$ then the given system of equations becomes

$$Dx + wy = 0 \qquad ...(1)$$
$$-wx + Dy = 0 \qquad ...(2)$$

Operating eqn. (1) by D and multiplying eqn. (2) by w then subtracting (2) from (1), we get,

$$(D^2 + w^2)\,x = 0$$

Auxiliary equation is

$$m^2 + w^2 = 0$$
$$\Rightarrow \quad m = \pm wi$$
$$\therefore \quad \text{C.F.} = c_1 \cos wt + c_2 \sin wt$$
$$\text{P.I.} = 0$$
$$\therefore \quad x = \text{C.F.} + \text{P.I.} = c_1 \cos wt + c_2 \sin wt \qquad ...(3)$$

From (3), $$\frac{dx}{dt} = -wc_1 \sin wt + wc_2 \cos wt$$

From (1), $$wy = -Dx = w\,(c_1 \sin wt - c_2 \cos wt)$$
$$\therefore \quad y = c_1 \sin wt - c_2 \cos wt \qquad ...(4)$$

Eliminating w between eqns. (3) and (4), we get

$$x^2 + y^2 = c_1^2 + c_2^2 \qquad ...(5)$$

which is a circle with centre (0, 0) and radius $\sqrt{c_1^2 + c_2^2}$.

Eqn. (5) shows that the point (x, y) lies on a circle.

Example 3. *Solve the following simultaneous differential equations:*

$$\frac{dx}{dt} + 5x - 2y = t, \quad \frac{dy}{dt} + 2x + y = 0$$

given that $x = y = 0$ when $t = 0$. (U.P.T.U. 2008)

Sol. Let $\frac{d}{dt} \equiv D$ then the given system of equations becomes

$$(D + 5)\, x - 2y = t \qquad \text{...(1)}$$

$$2x + (D + 1)\, y = 0 \qquad \text{...(2)}$$

Operating (1) by (D + 1), we get

$$(D^2 + 6D + 5)\, x - 2\,(D + 1)\, y = 1 + t \qquad \text{...(3)}$$

Multiplying (2) by 2, we get

$$4x + 2\,(D + 1)\, y = 0 \qquad \text{...(4)}$$

Adding (3) and (4), we get

$$(D^2 + 6D + 9)\, x = 1 + t$$

Auxiliary equation is

$$m^2 + 6m + 9 = 0 \quad \Rightarrow \quad m = -3, -3$$

$$\therefore \qquad \text{C.F.} = (c_1 + c_2 t)\, e^{-3t}$$

$$\text{P.I.} = \frac{1}{(D+3)^2}(1+t) = \frac{1}{9}\left(1+\frac{D}{3}\right)^{-2}(1+t)$$

$$= \frac{1}{9}\left(1-\frac{2D}{3}\right)(1+t) = \frac{1}{9}\left(1+t-\frac{2}{3}\right) = \frac{1}{9}\left(t+\frac{1}{3}\right)$$

$$\therefore \qquad x = \text{C.F.} + \text{P.I.} = (c_1 + c_2 t)\, e^{-3t} + \frac{1}{9}\left(t+\frac{1}{3}\right) \qquad \text{...(5)}$$

From (5), $$\frac{dx}{dt} = -3\,(c_1 + c_2 t)\, e^{-3t} + c_2 e^{-3t} + \frac{1}{9}$$

From (1), $$2y = \frac{dx}{dt} + 5x - t$$

$$= -3\,(c_1 + c_2 t)\, e^{-3t} + c_2 e^{-3t} + \frac{1}{9} + 5\,(c_1 + c_2 t)\, e^{-3t} + \frac{5}{9}\left(t+\frac{1}{3}\right) - t$$

$$= 2\,(c_1 + c_2 t)\, e^{-3t} + c_2 e^{-3t} - \frac{4}{9}t + \frac{8}{27}$$

$$y = (c_1 + c_2 t)\, e^{-3t} + \frac{c_2}{2} e^{-3t} - \frac{2}{9}t + \frac{4}{27} \qquad \text{...(6)}$$

Eqns. (5) and (6), when taken together, give the general solution.

Applying condition $x(0) = 0$ in (5), we get

$$0 = c_1 + \frac{1}{27} \quad \Rightarrow \quad c_1 = -\frac{1}{27}$$

Applying condition $y(0) = 0$ in (6), we get

$$0 = c_1 + \frac{c_2}{2} + \frac{4}{27} = \frac{c_2}{2} + \frac{1}{9}$$

$$\Rightarrow \qquad c_2 = -\frac{2}{9}$$

Hence the required particular solution is given by

$$x = -\frac{1}{27}(1+6t)e^{-3t} + \frac{1}{9}\left(t+\frac{1}{3}\right)$$

and

$$y = -\frac{2}{27}(2+3t)e^{-3t} - \frac{2}{9}t + \frac{4}{27}$$

Example 4. *Solve:* $\frac{d^2x}{dt^2} + y = \sin t,\ \frac{d^2y}{dt^2} + x = \cos t.$

Sol. Let $\frac{d}{dt} = D$ then the given system of equations become

$$D^2x + y = \sin t \quad ...(1)$$

$$x + D^2y = \cos t \quad ...(2)$$

Operating eqn. (1) by D^2, we get

$$D^4x + D^2y = -\sin t \quad ...(3)$$

Subtracting (2) from (3), we get

$$(D^4 - 1)\,x = -\sin t - \cos t$$

Auxiliary equation is

$$m^4 - 1 = 0$$

$$(m^2 - 1)(m^2 + 1) = 0$$

$\Rightarrow$
$$m = 1, -1, \pm i$$

$\therefore$
$$\text{C.F.} = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t$$

$$\text{P.I.} = \frac{1}{D^4-1}(-\sin t - \cos t)$$

$$= -t.\frac{1}{4D^3}(\sin t + \cos t) = \frac{t}{4}(-\cos t + \sin t)$$

$\therefore$
$$x = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t + \frac{t}{4}(\sin t - \cos t) \quad ...(4)$$

$$Dx = c_1e^t - c_2e^{-t} - c_3\sin t + c_4\cos t + \frac{t}{4}(\cos t + \sin t) + \frac{1}{4}(\sin t - \cos t)$$

$$D^2x = c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t + \frac{t}{4}(-\sin t + \cos t) + \frac{1}{4}(\cos t + \sin t) + \frac{1}{4}(\cos t + \sin t)$$

From (1),
$$y = \sin t - \frac{d^2x}{dt^2}$$

$$y = -c_1e^t - c_2e^{-t} + c_3\cos t + c_4\sin t + \frac{t}{4}(\sin t - \cos t) + \frac{1}{2}(\sin t - \cos t) \quad ...(5)$$

Equations (4) and (5), when taken together, give the complete solution of the given system of equations.

Example 5. *Solve:* $\frac{dx}{dt} + 4x + 3y = t,\ \frac{dy}{dt} + 2x + 5y = e^t.$ (U.P.T.U. 2006)

Sol. Writing D for $\frac{d}{dt}$, the given equations become

$$(D+4)x+3y=t \quad \text{...(1)}$$

and

$$2x+(D+5)y=e^t \quad \text{...(2)}$$

To eliminate y, operating (1) by $(D+5)$ and multiplying (2) by 3 then subtracting, we get

$$[(D+4)(D+5)-6]x=(D+5)t-3e^t$$

or

$$(D^2+9D+14)x=1+5t-3e^t$$

Auxiliary equation is

$$m^2+9m+14=0 \Rightarrow m=-2,-7$$

$\therefore$ C.F. $= c_1e^{-2t}+c_2e^{-7t}$

$$\text{P.I.}=\frac{1}{D^2+9D+14}(1+5t-3e^t)$$

$$=\frac{1}{D^2+9D+14}e^{0t}+5\frac{1}{D^2+9D+14}t-3\frac{1}{D^2+9D+14}e^t$$

$$=\frac{1}{0^2+9(0)+14}e^{0t}+5\cdot\frac{1}{14\left(1+\frac{9D}{14}+\frac{D^2}{14}\right)}t-3\frac{1}{1^2+9(1)+14}e^t$$

$$=\frac{1}{14}+\frac{5}{14}\left[1+\left(\frac{9D}{14}+\frac{D^2}{14}\right)\right]^{-1}t-\frac{1}{8}e^t=\frac{1}{14}+\frac{5}{14}\left[1-\left(\frac{9D}{14}+\frac{D^2}{14}\right)+\ldots\ldots\right]t-\frac{1}{8}e^t$$

$$=\frac{1}{14}+\frac{5}{14}\left(t-\frac{9}{14}\right)-\frac{1}{8}e^t=\frac{5}{14}t-\frac{31}{196}-\frac{1}{8}e^t$$

$$\therefore \quad x=c_1e^{-2t}+c_2e^{-7t}+\frac{5}{14}t-\frac{31}{196}-\frac{1}{8}e^t$$

Now, $\dfrac{dx}{dt}=-2c_1e^{-2t}-7c_2e^{-7t}+\dfrac{5}{14}-\dfrac{1}{8}e^t$

Substituting the values of x and $\dfrac{dx}{dt}$ in (1), we have

$$3y=t-\frac{dx}{dt}-4x$$

$$\Rightarrow \quad 3y=t+2c_1e^{-2t}+7c_2e^{-7t}-\frac{5}{14}+\frac{1}{8}e^t-4c_1e^{-2t}-4c_2e^{-7t}-\frac{10}{7}t+\frac{31}{49}+\frac{1}{2}e^t$$

$$\therefore \quad y=\frac{1}{3}\left[-2c_1e^{-2t}+3c_2e^{-7t}-\frac{3}{7}t+\frac{27}{98}+\frac{5}{8}e^t\right]$$

Hence

$$x=c_1e^{-2t}+c_2e^{-7t}+\frac{5}{14}t-\frac{31}{196}-\frac{1}{8}e^t$$

$$y=-\frac{2}{3}c_1e^{-2t}+c_2e^{-7t}-\frac{1}{7}t+\frac{9}{98}+\frac{5}{24}e^t.$$

Example 6. *Solve the simultaneous differential equations*

$$\frac{dx}{dt}+\frac{dy}{dt}+3x=\sin t \quad \text{and} \quad \frac{dx}{dt}+y-x=\cos t.$$

Example 9. *Solve:* $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} + 3x = e^{-t}$

$$\frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y = \sin 2t.$$ [U.P.T.U. 2007 ; M.T.U. (SUM) 2011]

Sol. Let $D \equiv \dfrac{d}{dt}$ then we have

$$(D^2 + 3)\, x + Dy = e^{-t} \quad \text{...(1)}$$

$$-4Dx + (D^2 + 3)\, y = \sin 2t \quad \text{...(2)}$$

Operating (1) by $(D^2 + 3)$ and (2) by D then subtracting, we get

$$[(D^2 + 3)^2 + 4D^2]\, x = 4e^{-t} - 2\cos 2t$$

$$(D^4 + 10D^2 + 9)\, x = 4e^{-t} - 2\cos 2t$$

Auxiliary equation is

$$m^4 + 10m^2 + 9 = 0 \quad \Rightarrow \quad m = \pm i, \pm 3i$$

$$\text{C.F.} = c_1 \cos t + c_2 \sin t + c_3 \cos 3t + c_4 \sin 3t$$

$$\text{P.I.} = \frac{1}{D^4 + 10D^2 + 9}(4e^{-t}) - \frac{1}{D^4 + 10D^2 + 9}(2\cos 2t)$$

$$= \frac{1}{1+10+9}(4e^{-t}) - \frac{1}{16-40+9}(2\cos 2t)$$

$$= \frac{1}{5}e^{-t} + \frac{2}{15}\cos 2t$$

$$\therefore \qquad x = c_1 \cos t + c_2 \sin t + c_3 \cos 3t + c_4 \sin 3t + \frac{1}{5}e^{-t} + \frac{2}{15}\cos 2t \qquad ...(3)$$

Again operating (1) by 4D and (2) by $(D^2 + 3)$ then adding, we get

$$[(D^2 + 3)^2 + 4D^2]\, y = -4e^{-t} - \sin 2t$$

$$(D^4 + 10D^2 + 9)\, y = -4e^{-t} - \sin 2t$$

Auxiliary equation is

$$m^4 + 10m^2 + 9 = 0 \quad \Rightarrow \quad m = \pm i, \pm 3i$$

$$\text{C.F.} = c_5 \cos t + c_6 \sin t + c_7 \cos 3t + c_8 \sin 3t$$

$$\text{P.I.} = \frac{1}{D^4 + 10D^2 + 9}(-4e^{-t}) - \frac{1}{D^4 + 10D^2 + 9}(\sin 2t)$$

$$= -\frac{1}{5}e^{-t} + \frac{1}{15}\sin 2t$$

$$\therefore \qquad y = c_5 \cos t + c_6 \sin t + c_7 \cos 3t + c_8 \sin 3t - \frac{1}{5}e^{-t} + \frac{1}{15}\sin 2t \qquad ...(4)$$

Equations (3) and (4), when taken together, give the complete solution.

Example 10. *Solve the simultaneous equations:*

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$$

$$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t. \qquad \text{(U.K.T.U. 2011)}$$

Sol. The given equations may be written as

$$Dx + (D - 2)y = 2\cos t - 7\sin t \qquad ...(1)$$

$$(D + 2)x - Dy = 4\cos t - 3\sin t \qquad ...(2)$$

Operating (1) by D and (2) by $(D - 2)$, we get

$$D^2x + D(D - 2)y = -2\sin t - 7\cos t$$

$$(D^2 - 4)x - D(D - 2)y = -4\sin t - 8\cos t - 3\cos t + 6\sin t$$

Adding, we get

$$(2D^2 - 4)x = -18\cos t$$

$$\Rightarrow \qquad (D^2 - 2)x = -9\cos t$$

Auxiliary equation is

$$m^2 - 2 = 0 \quad \Rightarrow \quad m = \pm\sqrt{2}$$

$$\text{C.F.} = c_1 \cosh\sqrt{2}t + c_2 \sinh\sqrt{2}t$$

$$\text{P.I.} = \frac{1}{D^2 - 2}(-9\cos t) = 3\cos t$$

$$\therefore \qquad x = \text{C.F.} + \text{P.I.} = c_1 \cosh\sqrt{2}t + c_2 \sinh\sqrt{2}t + 3\cos t \qquad ...(3)$$

Again, operating (1) by (D + 2) and (2) by D, we get

$$(D+2)\,Dx + (D^2-4)\,y = -2\sin t - 7\cos t + 4\cos t - 14\sin t$$
$$D(D+2)\,x - D^2y = -4\sin t - 3\cos t$$

Subtracting, we get

$$(2D^2-4)\,y = -12\sin t$$
$$(D^2-2)\,y = -6\sin t$$
$$\text{C.F.} = c_3\cosh\sqrt{2}\,t + c_4\sinh\sqrt{2}\,t$$
$$\text{P.I.} = \frac{1}{D^2-2}(-6\sin t) = 2\sin t$$

$\therefore$
$$y = \text{C.F.} + \text{P.I.} = c_3\cosh\sqrt{2}\,t + c_4\sinh\sqrt{2}\,t + 2\sin t \qquad \ldots(4)$$

Eqns. (3) and (4), when taken together, give the complete solution.

TEST YOUR KNOWLEDGE

Solve the following systems of simultaneous differential equations:

1. $\dfrac{dx}{dt} + 7x - y = 0,\ \dfrac{dy}{dt} + 2x + 5y = 0$ [G.B.T.U. (AG) SUM 2010]
2. $\dfrac{dx}{dt} + x - 2y = 0,\ \dfrac{dy}{dt} + x + 4y = 0\ ;\ x(0) = y(0) = 1$ (M.T.U. 2011)
3. $\dfrac{dx}{dt} - 7x + y = 0,\ \dfrac{dy}{dt} - 2x - 5y = 0$ [U.P.T.U. (B. Pharm.) 2009]
4. $\dfrac{dx}{dt} = 3x + 8y,\ \dfrac{dy}{dt} = -x - 3y;\ x(0) = 6,\ y(0) = -2$ (G.B.T.U. 2010)
5. $\dfrac{dx}{dt} - y = t,\ \dfrac{dy}{dt} + x = 1$
6. $\dfrac{dx}{dt} = y + 1,\ \dfrac{dy}{dt} = x + 1.$ [U.P.T.U. (SUM) 2009]
7. $\dfrac{dx}{dt} + y = \sin t,\ \dfrac{dx}{dt} + x = \cos t$; given that $x = 2$ and $y = 0$ when $t = 0$.
8. $\dfrac{dx}{dt} - y = e^t,\ \dfrac{dy}{dt} + x = \sin t;\ x(0) = 1,\ y(0) = 0$ [G.B.T.U. (SUM) 2010 ; G.B.T.U. (C.O.) 2011]
9. $\dfrac{dx}{dt} + 5x + y = e^t,\ \dfrac{dy}{dt} + x + 5y = e^{5t}.$ [U.P.T.U. (C.O.) 2009]
10. $\dfrac{dx}{dt} + 2x - 3y = t,\ \dfrac{dy}{dt} - 3x + 2y = e^{2t}$ [U.K.T.U. 2012]
11. (*i*) $\dfrac{d^2x}{dt^2} + m^2y = 0,\ \dfrac{d^2y}{dt^2} - m^2x = 0$ (*ii*) $\dfrac{d^2x}{dt^2} - 3x - 4y = 0,\ \dfrac{d^2y}{dt^2} + x + y = 0$
12. $\dfrac{dx}{dt} + 2x + 4y = 1 + 4t,\ \dfrac{dy}{dt} + x - y = \dfrac{3}{2}t^2$ [G.B.T.U. 2013]
13. $\dfrac{d^2x}{dt^2} + 16x - 6\dfrac{dy}{dt} = 0,\ 6\dfrac{dx}{dt} + \dfrac{d^2y}{dt^2} + 16y = 0$
14. $(D-1)x + Dy = 2t + 1,\ (2D+1)x + 2Dy = t$
15. $(D^2-1)x + 8Dy = 16e^t$ and $Dx + 3(D^2+1)y = 0$
16. $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = 0,\ \dfrac{dy}{dt} + 5x + 3y = 0$ [G.B.T.U. (C.O.) 2011]

17. $\dfrac{dx}{dt} = -4(x+y),\ \dfrac{dx}{dt} + 4\dfrac{dy}{dt} = -4y$ with conditions $x(0) = 1,\ y(0) = 0.$ *(G.B.T.U. 2011)*

18. $\dfrac{dx}{dt} + \dfrac{2}{t}(x-y) = 1,\quad \dfrac{dy}{dt} + \dfrac{1}{t}(x+5y) = t.$

Answers

1. $x = e^{-6t}(A\cos t + B\sin t),\ y = e^{-6t}[(A+B)\cos t - (A-B)\sin t]$
2. $x = 4e^{-2t} - 3e^{-3t},\ y = -2e^{-2t} + 3e^{-3t}$
3. $x = e^{6t}(c_1\cos t + c_2\sin t),\ y = e^{6t}[(c_1+c_2)\sin t + (c_1-c_2)\cos t]$
4. $x = 4e^t + 2e^{-t},\ y = -e^t - e^{-t}$
5. $x = c_1\cos t + c_2\sin t + 2,\ y = -c_1\sin t + c_2\cos t - t$
6. $x = c_1e^t + c_2e^{-t} - 1,\ y = c_1e^t - c_2e^{-t} - 1$
7. $x = e^t + e^{-t},\ y = e^{-t} - e^t + \sin t$
8. $x = 2\sin t + \dfrac{3}{2}\cos t + \dfrac{t}{2}\cos t - \dfrac{1}{2}e^t,\ y = \dfrac{1}{2}\cos t - \dfrac{3}{2}\sin t + \dfrac{t}{2}\sin t - \dfrac{1}{2}e^t$
9. $x = c_1e^{-6t} + c_2e^{-4t} + \dfrac{6e^t}{35} - \dfrac{e^{5t}}{99},\ y = c_1e^{-6t} - c_2e^{-4t} - \dfrac{1}{35}e^t + \dfrac{10}{99}e^{5t}.$
10. $x = c_1e^{-5t} + c_2e^t + \dfrac{3}{7}e^{2t} - \dfrac{2}{5}t - \dfrac{13}{25},\ y = c_1e^{-5t} + c_2e^t + \dfrac{4}{7}e^{2t} - \dfrac{3}{5}t - \dfrac{12}{25}$
11. (*i*) $x = e^{mt/\sqrt{2}}\left(c_1\cos\dfrac{mt}{\sqrt{2}} + c_2\sin\dfrac{mt}{\sqrt{2}}\right) + e^{-mt/\sqrt{2}}\left(c_3\cos\dfrac{mt}{\sqrt{2}} + c_4\sin\dfrac{mt}{\sqrt{2}}\right);$

    $y = e^{mt/\sqrt{2}}\left(c_1\sin\dfrac{mt}{\sqrt{2}} - c_2\cos\dfrac{mt}{\sqrt{2}}\right) + e^{-mt/\sqrt{2}}\left(c_4\cos\dfrac{mt}{\sqrt{2}} - c_3\sin\dfrac{mt}{\sqrt{2}}\right)$

    (*ii*) $x = (c_1 + c_2t)e^{-t} + (c_3 + c_4t)e^t,\ y = -\dfrac{1}{2}[c_1 + c_2(1+t)]e^{-t} + \dfrac{1}{2}[c_4(1-t) - c_3]e^t$
12. $x = c_1e^{2t} + c_2e^{-3t} + t + t^2,\ y = -c_1e^{2t} + \dfrac{c_2}{4}e^{-3t} - \dfrac{t^2}{2}$
13. $x = c_1\cos 2t + c_2\sin 2t - c_3\cos 8t + c_4\sin 8t,\ y = c_1\sin 2t + c_2\cos 2t - c_3\sin 8t + c_4\cos 8t$
14. $x = -t - \dfrac{2}{3},\ y = \dfrac{1}{2}t^2 + \dfrac{4}{3}t + c$
15. $y = c_1\cos\dfrac{t}{\sqrt{3}} + c_2\sin\dfrac{t}{\sqrt{3}} + c_3\cosh\sqrt{3}\,t + c_4\sinh\sqrt{3}\,t + 2e^t$

    $x = \sqrt{3}\,c_1\sin\dfrac{t}{\sqrt{3}} - \sqrt{3}\,c_2\cos\dfrac{t}{\sqrt{3}} - 3\sqrt{3}\,c_3\sinh\sqrt{3}\,t - 3\sqrt{3}\,c_4\cosh\sqrt{3}\,t - 6e^t - 3t.$
16. $x = \left(\dfrac{c_1 - 3c_2}{5}\right)\sin t - \left(\dfrac{c_2 + 3c_1}{5}\right)\cos t,\ y = c_1\cos t + c_2\sin t$
17. $x = (1-2t)e^{-2t},\ y = te^{-2t}$
18. $x = At^{-4} + Bt^{-3} + \dfrac{t^2}{15} + \dfrac{3t}{10};\ y = -At^{-4} - \dfrac{1}{2}Bt^{-3} + \dfrac{2t^2}{15} - \dfrac{t}{20}.$

1.22. LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

A differential equation of the form $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$ is known as linear differential equation of second order, where P, Q and R are functions of x alone.