1.17. THE INVERSE OPERATOR $\frac{1}{f(D)}$

$\frac{1}{f(D)}$ Q is that function of x, *free from arbitrary constants,* which when operated upon by f(D) gives Q.

$$\text{Thus} \qquad f(D)\left\{\frac{1}{f(D)}Q\right\} = Q$$

$\therefore$ f(D) and $\frac{1}{f(D)}$ are inverse operators.

Note 1. $\frac{1}{D}Q = \int Q\,dx.$

Note 2. $\frac{1}{D-a}Q = e^{ax}\int Qe^{-ax}\,dx.$

1.18. RULES FOR FINDING THE PARTICULAR INTEGRAL (P.I.)

Consider the differential equation, $(D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_{n-1}D + a_n)y = Q$

It can be written as $\qquad f(D)y = Q$

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D)}Q.$$

1.18.1. Case I. When Q = e^{ax} (or e^{ax+b})

Since

$$D\,e^{ax} = a\,e^{ax}$$
$$D^2\,e^{ax} = a^2\,e^{ax}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D^{n-1}\,e^{ax} = a^{n-1}\,e^{ax}$$
$$D^n e^{ax} = a^n\,e^{ax}$$

$$\therefore \quad (D^n + a_1D^{n-1} + \ldots\ldots + a_{n-1}D + a_n)e^{ax} = (a^n + a_1a^{n-1} + \ldots\ldots + a_{n-1}\,a + a_n)e^{ax}$$

or
$$f(D)\,e^{ax} = f(a)\,e^{ax}$$

Operating on both sides by $\frac{1}{f(D)}$;

$$\frac{1}{f(D)}[f(D)\,e^{ax}] = \frac{1}{f(D)}[f(a)\,e^{ax}]$$

or
$$e^{ax} = f(a)\,\frac{1}{f(\mathrm{D})}\,e^{ax}$$

Dividing both sides by $f(a)$, $\dfrac{1}{f(a)}\,e^{ax} = \dfrac{1}{f(\mathrm{D})}\,e^{ax}$, provided $f(a) \neq 0$

Hence
$$\boxed{\frac{1}{f(D)}\,e^{ax} = \frac{1}{f(a)}\,e^{ax}\text{, provided } f(a) \neq 0.}$$

Case of failure: If $f(a) = 0$, the above method fails.

Since $f(a) = 0$, $\mathrm{D} = a$ is a root of $f(\mathrm{D}) = 0$

$\therefore$ $\mathrm{D} - a$ is a factor of $f(\mathrm{D})$.

Let
$$f(\mathrm{D}) = (\mathrm{D} - a)\,\phi(\mathrm{D}),\text{ where } \phi(a) \neq 0 \qquad ...(1)$$

Then
$$\frac{1}{f(\mathrm{D})}\,e^{ax} = \frac{1}{(\mathrm{D}-a)\,\phi(\mathrm{D})}\,e^{ax} = \frac{1}{\mathrm{D}-a}\cdot\frac{1}{\phi(a)}\,e^{ax}$$
$$= \frac{1}{\phi(a)}\cdot\frac{1}{\mathrm{D}-a}\,e^{ax} = \frac{1}{\phi(a)}\,e^{ax}\int e^{ax}\,.\,e^{-ax}\,dx \qquad \text{[by Note 2]}$$
$$= \frac{1}{\phi(a)}\,e^{ax}\int 1\,dx = x\,.\,\frac{1}{\phi(a)}\,e^{ax} \qquad ...(2)$$

Differentiating both sides of (1) w.r.t. D, we get
$$f'(\mathrm{D}) = (\mathrm{D} - a)\,\phi'(\mathrm{D}) + \phi(\mathrm{D})$$
$$\Rightarrow \qquad f'(a) = \phi(a)$$

$\therefore$ From (2), we have $\dfrac{1}{f(\mathrm{D})}\,e^{ax} = x\,.\,\dfrac{1}{f'(a)}\,e^{ax}$ **provided** $f'(a) \neq 0$

Another case of failure:

If $f'(a) = 0$, then $\dfrac{1}{f(\mathrm{D})}\,e^{ax} = x^2\,.\,\dfrac{1}{f''(a)}\,e^{ax}$, provided $f''(a) \neq 0$ and so on.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the P.I. of* $(4D^2 + 4D - 3)y = e^{2x}$.

Sol.
$$\text{P.I.} = \frac{1}{4\mathrm{D}^2 + 4\mathrm{D} - 3}\,e^{2x} = \frac{1}{4(2)^2 + 4(2) - 3}\,e^{2x} \qquad \text{(Replacing D by 2)}$$
$$= \frac{1}{21}\,e^{2x}.$$

Example 2. *Find the P.I. of* $(D^3 - 3D^2 + 4)y = e^{2x}$.

Sol.
$$\text{P.I.} = \frac{1}{\mathrm{D}^3 - 3\mathrm{D}^2 + 4}\,e^{2x}.$$

Here the denominator vanishes when D is replaced by 2. It is a case of failure.

We multiply the numerator by x and differentiate the denominator w.r.t. D.

$\therefore$
$$\text{P.I.} = x\,.\,\frac{1}{3\mathrm{D}^2 - 6\mathrm{D}}\,e^{2x}$$

It is again a case of failure. We multiply the numerator by x and differentiate the denominator w.r.t. D.

$$\therefore \quad \text{P.I.} = x^2 \cdot \frac{1}{6\text{D}-6} e^{2x} = x^2 \cdot \frac{1}{6(2)-6} e^{2x} = \frac{x^2}{6} e^{2x}.$$

Example 3. *Find the P.I. of* $(D+1)^3 y = e^{-x}$.

Sol.

$$\text{P.I.} = \frac{1}{(\text{D}+1)^3} e^{-x} = x \cdot \frac{1}{3(\text{D}+1)^2} e^{-x} \qquad | \text{ Case of failure}$$

$$= x^2 \cdot \frac{1}{3.2(\text{D}+1)} e^{-x} \qquad | \text{ Again case of failure}$$

$$= x^3 \cdot \frac{1}{3.2.1} e^{-x} = \frac{x^3}{6} e^{-x}.$$

Example 4. *Solve:*

(*i*) $(D^3 - 2D^2 + 4D - 8)\, y = 8$ [U.P.T.U. (B.Pharm.) SUM 2009]

(*ii*) $(D-2)^3\, y = 17\, e^{2x}$. (M.T.U. 2011)

Sol. (*i*) Auxiliary equation is

$$m^3 - 2m^2 + 4m - 8 = 0$$

$$\Rightarrow \quad (m^2+4)(m-2) = 0$$

$$\Rightarrow \quad m = 2, \pm 2i$$

$$\therefore \quad \text{C.F.} = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x$$

$$\text{P.I.} = \frac{1}{\text{D}^3 - 2\text{D}^2 + 4\text{D} - 8} (8\, e^{0x}) \qquad | \because \; e^{0x} = 1$$

$$= \frac{1}{(0)^3 - 2(0)^2 + 4(0) - 8} (8e^{0x}) = -1$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x - 1$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

(*ii*) Auxiliary equation is

$$(m-2)^3 = 0$$

$$\Rightarrow \quad m = 2, 2, 2$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x + c_3 x^2)\, e^{2x}$$

$$\text{P.I.} = \frac{1}{(\text{D}-2)^3} 17e^{2x} \qquad | \text{ Case of failure}$$

$$= 17x \cdot \left[\frac{1}{3(\text{D}-2)^2} e^{2x}\right] \qquad | \text{ Again case of failure}$$

$$= \frac{17}{3} x^2 \cdot \left[\frac{1}{2(\text{D}-2)} e^{2x}\right] \qquad | \text{ Again a case of failure}$$

$$= \frac{17}{6} x^3\, e^{2x}$$

Example 8. *Solve:* $(D^2 + D + 1)\, y = (1 + e^x)^2$.

Sol. Auxiliary equation is

$$m^2 + m + 1 = 0$$

$$\Rightarrow \qquad m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i$$

$$\text{C.F.} = e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right)$$

$$\text{P.I.} = \frac{1}{D^2 + D + 1}(1 + e^x)^2 = \frac{1}{D^2 + D + 1}(1 + e^{2x} + 2e^x)$$

$$= \frac{1}{D^2 + D + 1}(e^{0x}) + \frac{1}{D^2 + D + 1}(e^{2x}) + \frac{1}{D^2 + D + 1}(2e^x)$$

$$= \frac{1}{(0)^2 + (0) + 1} e^{0x} + \frac{1}{(2)^2 + (2) + 1} e^{2x} + \frac{2}{(1)^2 + (1) + 1} e^x = 1 + \frac{1}{7} e^{2x} + \frac{2}{3} e^x$$

Hence complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^{-\frac{x}{2}}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right) + 1 + \frac{1}{7} e^{2x} + \frac{2}{3} e^x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 9. *Solve :* $(D + 2)(D - 1)^2\, y = e^{-2x} + 2 \sinh x$.

Sol. Auxiliary equation is

$$(m + 2)(m - 1)^2 = 0 \quad \Rightarrow \quad m = -2,\ 1,\ 1$$

$$\therefore \qquad \text{C.F.} = c_1 e^{-2x} + (c_2 + c_3 x)\, e^x$$

$$\text{P.I.} = \frac{1}{(D + 2)(D - 1)^2}(e^{-2x} + 2 \sinh x)$$

$$= \frac{1}{(D + 2)(D - 1)^2}(e^{-2x} + e^x - e^{-x}) \qquad \left[\because \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

Now, $$\frac{1}{(D+2)(D-1)^2}e^{-2x} = \frac{1}{D+2}\left[\frac{1}{(D-1)^2}e^{-2x}\right] = \frac{1}{D+2}\left[\frac{1}{(-2-1)^2}e^{-2x}\right]$$

$$= \frac{1}{9}\cdot\frac{1}{D+2}e^{-2x} \qquad |\text{ Case of failure}$$

$$= \frac{x}{9}e^{-2x}$$

$$\frac{1}{(D+2)(D-1)^2}e^{x} = \frac{1}{(D-1)^2}\left[\frac{1}{D+2}e^{x}\right] = \frac{1}{(D-1)^2}\left[\frac{1}{1+2}e^{x}\right]$$

$$= \frac{1}{3}\cdot\frac{1}{(D-1)^2}e^{x} \qquad |\text{ Case of failure}$$

$$= \frac{1}{3}\cdot x\,\frac{1}{2(D-1)}e^{x} \qquad |\text{ Case of failure}$$

$$= \frac{1}{3}\cdot x^2\cdot\frac{1}{2}e^{x} = \frac{1}{6}x^2e^{x}$$

$$\frac{1}{(D+2)(D-1)^2}e^{-x} = \frac{1}{(-1+2)(-1-1)^2}e^{-x} = \frac{1}{4}e^{-x}$$

$\therefore$ $$\text{P.I.} = \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^{x} + \frac{1}{4}e^{-x}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1e^{-2x} + (c_2 + c_3x)e^{x} + \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^{x} + \frac{1}{4}e^{-x}.$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 10. *Solve the differential equation*

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = e^x + 2.$$

Sol. The given equation is

$$(D^3 - 3D^2 + 3D - 1)y = e^x + 2$$

or $$(D-1)^3 y = e^x + 2$$

Auxiliary equation is

$$(m-1)^3 = 0 \quad\Rightarrow\quad m = 1, 1, 1$$

$\therefore$ $$\text{C.F.} = (c_1 + c_2x + c_3x^2)e^x$$

$$\text{P.I.} = \frac{1}{(D-1)^3}(e^x+2) = \frac{1}{(D-1)^3}e^x + \frac{1}{(D-1)^3}(2e^{0.x})$$

$$= x\cdot\frac{1}{3(D-1)^2}e^x + \frac{1}{(0-1)^3}(2e^{0.x}) = x^2\cdot\frac{1}{3.2\,(D-1)}(e^x) - 2$$

$$= x^3\cdot\frac{1}{3.2.1}(e^x) - 2 = \frac{x^3}{6}e^x - 2$$

$\therefore$ The complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2x + c_3x^2)e^x + \frac{x^3}{6}e^x - 2$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $\frac{d^3y}{dx^3} + y = 3 + 5e^x$

2. $\frac{d^2y}{dx^2} - 4y = (1 + e^x)^2$

3. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$

4. $(2D + 1)^2 y = 4e^{-x/2}$

5. $(D^2 - 2kD + k^2)\, y = e^{kx}$

6. $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^{-x}$

7. $(D + 2)(D - 1)^3 y = e^x$

8. $\frac{d^2y}{dx^2} + 31\frac{dy}{dx} + 240y = 272\, e^{-x}$

9. $\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + (p^2 + q^2)\, y = e^{2x}$

10. $(D^4 + D^3 + D^2 - D - 2)\, y = e^x$

11. $\frac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$

12. $y'' + 4y' + 13y = 18e^{-2x}$; $y(0) = 0$, $y'(0) = 9$.

Answers

1. $y = c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right) + 3 + \frac{5}{2}e^x$

2. $y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4} - \frac{2}{3}e^x + \frac{1}{4}xe^{2x}$

3. $y = e^{-2x}(c_1\cos x + c_2\sin x) - \frac{1}{10}e^x - \frac{1}{2}e^{-x}$

4. $y = \left(c_1 + c_2x + \frac{x^2}{2}\right)e^{-x/2}$

5. $y = (c_1 + c_2x)\, e^{kx} + \frac{x^2}{2}e^{kx}$

6. $y = (c_1 + c_2x + c_3x^2)\, e^{-x} + e^{-x}\cdot\frac{x^3}{6}$

7. $y = (c_1 + c_2x + c_3x^2)\, e^x + c_4\, e^{-2x} + \frac{x^3 e^x}{18}$

8. $y = c_1e^{-15x} + c_2e^{-16x} + \frac{136}{105}e^{-x}$

9. $y = e^{-px}(c_1\cos qx + c_2\sin qx) + \frac{e^{2x}}{(2 + p)^2 + q^2}$

10. $y = c_1\, e^x + c_2e^{-x} + e^{-x/2}\left[c_3\cos\frac{\sqrt{7}}{2}x + c_4\sin\frac{\sqrt{7}}{2}x\right] + \frac{1}{8}xe^x$

11. $y = c_1\, e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right) + 3 + \frac{5}{9}e^{2x} + \frac{1}{3}xe^{-x}$

12. $y = e^{-2x}(-2\cos 3x + 3\sin 3x + 2)$.

1.18.2. Case II. When Q = sin (ax + b) or cos (ax + b)

$$D\sin(ax + b) = a\cos(ax + b)$$

$$D^2\sin(ax + b) = (-a^2)\sin(ax + b)$$

$$D^3\sin(ax + b) = -a^3\cos(ax + b)$$

$$D^4\sin(ax + b) = a^4\sin(ax + b)$$

or $$(D^2)^2 \sin(ax+b) = (-a^2)^2 \sin(ax+b)$$

....

....

In general, $$(D^2)^n \sin(ax+b) = (-a^2)^n \sin(ax+b)$$

$\therefore$ $$f(D^2) \sin(ax+b) = f(-a^2) \sin(ax+b)$$

Operating on both sides by $\dfrac{1}{f(D^2)}$,

$$\frac{1}{f(D^2)} \{f(D^2) \sin(ax+b)\} = \frac{1}{f(D^2)} \{f(-a^2) \sin(ax+b)\}$$

or $$\sin(ax+b) = f(-a^2) \frac{1}{f(D^2)} \sin(ax+b).$$

Dividing both sides by $f(-a^2)$,

$$\frac{1}{f(D^2)} \sin(ax+b) = \frac{1}{f(-a^2)} \sin(ax+b), \text{ provided } f(-a^2) \neq 0$$

Similarly, $$\frac{1}{f(D^2)} \cos(ax+b) = \frac{1}{f(-a^2)} \cos(ax+b), \text{ provided } f(-a^2) \neq 0$$

Steps: When $Q = \sin(ax+b)$ **or** $\cos(ax+b)$,

1. Replace D^2 by $-a^2$,

D^4 by a^4,

D^6 by $-a^6$,

D^8 by a^8 and so on.

2. By doing so, following possibilities arise:

(*a*) If denominator reduces to a constant, it will be final step in finding P.I.

(*b*) If denominator reduces into D only, we are then only to integrate the given function Q once.

(*c*) If denominator reduces to a factor of the form $\alpha D + \beta$ then we operate by its conjugate $\alpha D - \beta$ on both numerator and denominator from left hand side such as

$$\frac{\alpha D - \beta}{\alpha D - \beta} \cdot \left[\frac{1}{(\alpha D + \beta)} \sin(ax+b)\right]$$

By doing so, denominator will become $\alpha^2 D^2 - \beta^2$ which in turn reduces to a constant by replacing D^2 by $-a^2$.

Now, we operate $\sin(ax+b)$ by $(\alpha D - \beta)$ and consequently, find the required particular integral.

Case of failure: If $f(-a^2) = 0$, the above method fails. Then we proceed as follows:

$$\frac{1}{f(D^2)} \cos(ax+b) = x \cdot \frac{1}{f'(D^2)} \cos(ax+b), \text{ provided } f'(-a^2) \neq 0$$

$$\frac{1}{f(D^2)} \sin(ax+b) = x \cdot \frac{1}{f'(D^2)} \sin(ax+b), \text{ provided } f'(-a^2) \neq 0$$

If $f'(-a^2) = 0$, then

$$\frac{1}{f(D^2)} \sin(ax+b) = x^2 \cdot \frac{1}{f''(D^2)} \sin(ax+b), \text{ provided } f''(-a^2) \neq 0$$

$$\frac{1}{f(D^2)} \cos(ax+b) = x^2 \cdot \frac{1}{f''(D^2)} \cos(ax+b), \text{ provided } f''(-a^2) \neq 0$$

and so on.

Steps:

1. When $f(-a^2) = 0$, we differentiate the denominator w.r.t. D and multiply the expression by x simultaneously in the same step.
2. When $f'(-a^2) = 0$ (*i.e.*, step 1 fails) we again differentiate the reduced denominator in D w.r.t. D and again multiply the remaining expression by x simultaneously.
3. If there is another case of failure, above process is to be repeated again and again until we reach a constant in the denominator or any other possibility(ies) which we have discussed before in the same article.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following differential equation:*

$(D^2 + 4)y = \sin 3x + \cos 2x$. [U.P.T.U. (SUM) 2008]

Sol. Auxiliary equation is

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$$\therefore \quad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4} (\sin 3x) + \frac{1}{D^2+4} (\cos 2x)$$

$$= \frac{1}{-(3)^2+4} \sin 3x + x \cdot \frac{1}{2D} (\cos 2x)$$

$$= -\frac{1}{5} \sin 3x + \frac{x}{2}\left(\frac{\sin 2x}{2}\right) = -\frac{1}{5} \sin 3x + \frac{x}{4} \sin 2x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{5} \sin 3x + \frac{x}{4} \sin 2x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 2. *Find the P.I. of $(D^3 + 1)y = \sin(2x + 1)$.*

Sol. $$\text{P.I.} = \frac{1}{D^3+1}\sin(2x+1) = \frac{1}{D(-2^2)+1}\sin(2x+1) \qquad [\text{Putting } D^2 = -2^2]$$

$$= \frac{1}{1-4D}\sin(2x+1)$$

Operating N^r and D^r by $(1 + 4D)$

$$= \frac{1+4D}{(1+4D)(1-4D)}\sin(2x+1) = \frac{1+4D}{1-16D^2}\sin(2x+1)$$

$$= \frac{1+4D}{1-16(-2^2)}\sin(2x+1) \qquad [\text{Putting } D^2 = -2^2]$$

$$= \frac{1}{65}[\sin(2x+1) + 4D\sin(2x+1)]$$

$$= \frac{1}{65}[\sin(2x+1) + 8\cos(2x+1)] \qquad \left[\because D \equiv \frac{d}{dx}\right]$$

Example 3. *Solve the following differential equations:*

(*i*) $\dfrac{d^2y}{dx^2} + a^2y = \sin ax$

(*ii*) $(D^2 + 4)y = \cos^2 x.$ (U.P.T.U. 2008)

Sol. (*i*) The auxiliary equation is [M.T.U. (B. Pharm.) 2011]

$$m^2 + a^2 = 0 \Rightarrow m = \pm ai$$

$$\therefore \quad \text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2+a^2}(\sin ax) = x \cdot \frac{1}{2D}\sin ax$$

$$= \frac{x}{2}\left[\frac{-\cos ax}{a}\right] = -\frac{x}{2a}\cos ax$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1\cos ax + c_2 \sin ax - \frac{x}{2a}\cos ax$$

where c_1 and c_2 are arbitrary constants of integration.

(*ii*) The auxiliary equation is

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$$\therefore \quad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4}\cos^2 x = \frac{1}{2}\left[\frac{1}{D^2+4}(1+\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+4}(e^{0x}) + \frac{1}{D^2+4}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + x \cdot \frac{1}{2D}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{x}{4}\sin 2x\right] = \frac{1}{8}(1 + x\sin 2x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}(1 + x\sin 2x)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 4. *Solve:* $\dfrac{d^4y}{dx^4} - m^4y = \cos mx.$

Sol. Auxiliary equation is

$$M^4 - m^4 = 0$$

$$(M^2 - m^2)(M^2 + m^2) = 0$$

$$M = \pm m, \pm mi$$

$$\therefore \quad \text{C.F.} = c_1e^{mx} + c_2e^{-mx} + c_3 \cos mx + c_4 \sin mx$$

$$\text{P.I.} = \frac{1}{D^4 - m^4}(\cos mx) = x \cdot \frac{1}{4D^3}\cos mx$$

$$= \frac{x}{4} \cdot \frac{1}{D^2}\left(\frac{\sin mx}{m}\right) = -\frac{x}{4m^2}\left(\frac{\sin mx}{m}\right) = -\frac{x}{4m^3}\sin mx$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1e^{mx} + c_2e^{-mx} + c_3\cos mx + c_4 \sin mx - \frac{x}{4m^3}\sin mx$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 5. *Solve:* $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 2y = e^x + \cos x.$ (U.P.T.U. 2006)

Sol. Auxiliary equation is

$$m^3 - 3m^2 + 4m - 2 = 0$$

$$\Rightarrow \quad (m^2 - 2m + 2)(m - 1) = 0 \quad \Rightarrow \quad m = 1, 1 \pm i$$

$$\therefore \quad \text{C.F.} = c_1e^x + e^x(c_2 \cos x + c_3 \sin x)$$

$$\text{P.I.} = \frac{1}{(D^3 - 3D^2 + 4D - 2)}e^x + \frac{1}{(D^3 - 3D^2 + 4D - 2)}\cos x$$

$$= x \cdot \frac{1}{3D^2 - 6D + 4}(e^x) + \frac{1}{(-D + 3 + 4D - 2)}(\cos x)$$

$$= x \cdot \frac{1}{(7-6)} e^x + \frac{1}{3D+1}(\cos x) = x e^x + \frac{3D-1}{9D^2-1}(\cos x)$$

$$= xe^x - \frac{1}{10}(-3 \sin x - \cos x)$$

$\therefore$ Complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x) + xe^x + \frac{1}{10}(3 \sin x + \cos x)$$

where c_1, c_2, c_3 are arbitrary constants of integration.

Example 6. *Solve:* $(D^2 - 4D + 1)y = \cos x \cos 2x + \sin^2 x$.

Sol. Auxiliary equation is

$$m^2 - 4m + 1 = 0$$

$$\Rightarrow \qquad m = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$$\therefore \quad \text{C.F.} = e^{2x}(c_1 \cosh \sqrt{3}\, x + c_2 \sinh \sqrt{3}\, x)$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 1}(\cos x \cos 2x) + \frac{1}{D^2 - 4D + 1}(\sin^2 x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 4D + 1}(\cos 3x) + \frac{1}{D^2 - 4D + 1}(\cos x)\right] + \frac{1}{D^2 - 4D + 1}\left(\frac{1 - \cos 2x}{2}\right)$$

$$= \frac{1}{2}(P_1 + P_2) + P_3 \qquad \text{...(1)}$$

where, $$P_1 = \frac{1}{D^2 - 4D + 1}(\cos 3x)$$

$$= \frac{1}{-9 - 4D + 1}(\cos 3x) = -\frac{1}{4(D+2)} \cos 3x$$

$$= -\frac{1}{4}\frac{D-2}{(D^2-4)} \cos 3x = -\frac{1}{4}\frac{(D-2)}{(-9-4)} \cos 3x = \frac{1}{52}(-3 \sin 3x - 2 \cos 3x)$$

$$P_2 = \frac{1}{D^2 - 4D + 1}(\cos x) = \frac{1}{-1 - 4D + 1} \cos x = -\frac{1}{4} \sin x$$

$$P_3 = \frac{1}{2}\left[\frac{1}{D^2 - 4D + 1}(1) - \frac{1}{D^2 - 4D + 1}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 4D + 1}(e^{0x}) - \frac{1}{-4 - 4D + 1}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{(0)^2 - 4(0) + 1}(e^{0x}) + \frac{1}{4D+3}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[1 + \frac{4D-3}{16D^2-9}(\cos 2x)\right] = \frac{1}{2}\left[1 + \frac{4D-3}{(-64-9)}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{73}(-8\sin 2x - 3\cos 2x)\right] = \frac{1}{2}\left[1 + \frac{1}{73}(8\sin 2x + 3\cos 2x)\right]$$

$\therefore$ From (1),

$$\text{P.I.} = -\frac{1}{104}(3\sin 3x + 2\cos 3x) - \frac{1}{8}\sin x + \frac{1}{2} + \frac{1}{146}(8\sin 2x + 3\cos 2x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= e^{2x}(c_1 \cosh \sqrt{3}\,x + c_2 \sinh \sqrt{3}\,x) + \frac{1}{2} - \frac{1}{8}\sin x$$

$$- \frac{1}{104}(3\sin 3x + 2\cos 3x) + \frac{1}{146}(8\sin 2x + 3\cos 2x)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 7. *Solve:* $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37\sin 3x = 0$ *and find the value of* y *when* $x = \frac{\pi}{2}$ *being given that* $y = 3$, $\frac{dy}{dx} = 0$ *when* $x = 0$. (G.B.T.U. 2011)

Sol. We have,

$$(D^2 + 2D + 10)\,y = -37\sin 3x$$

Auxiliary equation is

$$m^2 + 2m + 10 = 0$$

$$\Rightarrow \quad m = -1 \pm 3i$$

$$\therefore \quad \text{C.F.} = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 10}(-37\sin 3x)$$

$$= (-37)\frac{1}{2D+1}(\sin 3x) \qquad | \text{ Replacing } D^2 \text{ by } -9$$

$$= (-37)\frac{2D-1}{4D^2-1}(\sin 3x) \qquad | \text{ Operating by } 2D-1$$

$$= (-37)\frac{(2D-1)}{(-37)}(\sin 3x) \qquad | \ D^2 = -9$$

$$= 6\cos 3x - \sin 3x$$

Hence the general solution is

$$y = \text{C.F.} + \text{P.I.} = e^{-x}(c_1 \cos 3x + c_2 \sin 3x) + 6\cos 3x - \sin 3x \qquad \ldots(1)$$

Applying the condition $y(0) = 3$ in (1),

$$3 = c_1 + 6 \quad \Rightarrow \quad c_1 = -3$$