MAT 101

ELEMENTARY MATHEMATICS I

COMPLIED BY DEPARTMENT OF MATHEMATICS, UDUS

January 7, 2025

1 Trigonometry

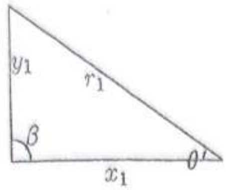


Figure 1: Right triangle with hypotenuse of r_1, opposite of y_1, and adjacent of x_1

- $\sin\theta = \frac{opp}{hyp} = \frac{y_1}{r_1}$
- $\cos\theta = \frac{adj}{hyp} = \frac{x_1}{r_1}$
- $\tan\theta = \frac{opp}{adj} = \frac{y_1}{x_1}$

1. RECIPROCAL RELATION

- $\csc\theta = \frac{hyp}{opp} = \frac{r_1}{y_1} = \frac{1}{\sin\theta}$
- $\sec\theta = \frac{hyp}{adj} = \frac{r_1}{x_1} = \frac{1}{\cos\theta}$
- $\cot\theta = \frac{adj}{opp} = \frac{x_1}{y_1} = \frac{1}{\tan\theta}$

2. Quotient Relations

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$

3. Squared Relations

- $\sin^2\theta + \cos^2\theta = 1$
- $\sec^2\theta = 1 + \tan^2\theta$

- $csc^2\theta = 1 + \cot^2\theta$. From $\triangle$ above

$$x_1^2 + y_1^2 = r_1^2 \tag{1}$$

divide both sides of (1) by r_1^2

$$\therefore \quad \left(\frac{x_1}{r_1}\right)^2 + \left(\frac{y_1}{r_1}\right)^2 = 1 \tag{2}$$

$$\Longrightarrow$$

$$\sin^2\theta + \cos^2\theta = 1 \tag{3}$$

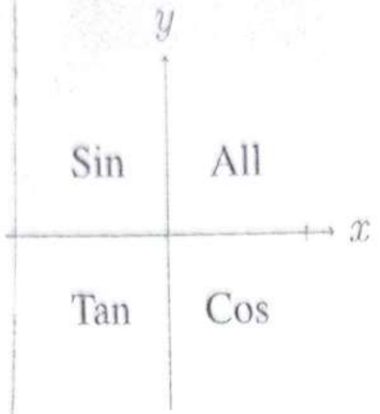
Similarly divide both sides by x_1^2 and y_1^2 respectively, we get

$$1 + \tan^2\theta = \sec^2\theta \tag{4}$$

$$\text{and } \cot^2\theta + 1 = \csc^2\theta \tag{5}$$

1.1 Signs of Trigonometrical Ratios

The sign of a particular t-ratio in any quadrant can be remembered by the word " All-Sin-Tan-Cos" or "Add Sugar To Coffe" or "CAST"



Quadrant	1^{st}	2^{nd}	3^{rd}	4^{th}
$\sin\theta$	+	+	-	-
$\csc\theta$	+	+	-	-
$\cos\theta$	+	-	-	+
$\sec\theta$	+	-	-	+
$\tan\theta$	+	-	+	-
$\cot\theta$	+	-	+	-

Example 1.1 *i. If* $\cos\theta = -\frac{4}{7}$, *in which quadrant does* θ *lie?*

ii. If $\tan\theta = \frac{7}{8}$, *in which quadrant does* θ *lie?*

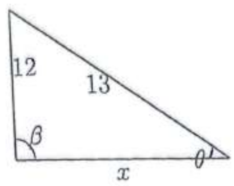
Solution

i. As $\cos\theta$ *is negative,* θ *can be either in the* 2^{nd} *or* 3^{rd} *quadrant*

ii. As $\tan\theta$ *is positive,* θ *can be either in* 1^{st} *or* 3^{rd} *quadrant.*

Example 1.2 *The sine of a certain angle is* $\frac{12}{13}$. *Evaluate other ratio of this angle.*

Solution



Using pathagoras rule $x = \pm 5$

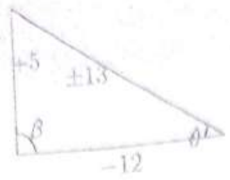
Since $\sin\theta$ is positive then θ will be either in the first quadrant or second.

$$\begin{aligned} \therefore \quad \cos\theta &= \pm\frac{5}{13}, \\ \tan\theta &= \pm\frac{12}{5}, \\ \csc\theta &= \pm\frac{13}{12}, \\ \sec\theta &= \pm\frac{13}{5}, \\ \cot\theta &= \pm\frac{5}{12}, \end{aligned}$$

Example 1.3 *If θ lies in the second quadrant, and $\tan\theta = \frac{-5}{12}$, find the value of $\frac{2\cos\theta}{1-\sin\theta}$*

Solution

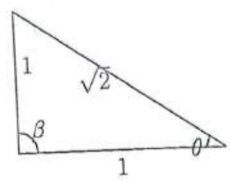
Since θ lies in the second quadrant, $\cos\theta$ is negative and $\sin\theta$ is positive



$$\text{Then } \frac{2\cos\theta}{1-\sin\theta} = \frac{2(-\frac{12}{13})}{1-(\frac{5}{13})}$$
$$= -3$$

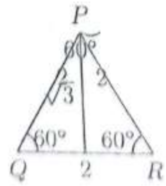
1.2 Trigonometrical Ratios of Special Angles

*** Angle of 45^0 or $\frac{\pi}{4}$**



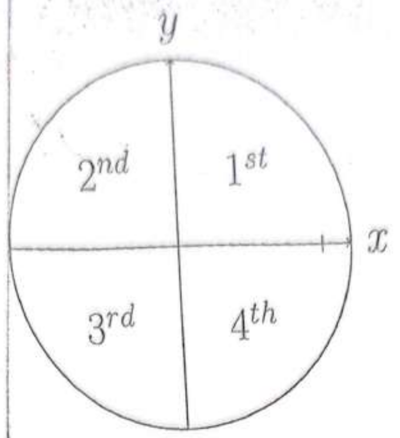
$$\sin 45^0 = \frac{1}{\sqrt{2}}$$
$$\cos 45^0 = \frac{1}{\sqrt{2}}$$
$$\tan 45^0 = 1$$

*** Angles of 30^0 or $\frac{\pi}{6}$ and 60^0 or $\frac{\pi}{3}$**



$$\sin 30^0 = \frac{1}{2}$$
$$\cos 30^0 = \frac{\sqrt{3}}{2}$$
$$\tan 30^0 = \frac{1}{\sqrt{3}}$$
$$\sin 60^0 = \frac{\sqrt{3}}{2}$$
$$\cos 60^0 = \frac{1}{2}$$
$$\tan 60^0 = \sqrt{3}$$

Angle	0^0	30^0	45^0	60^0	90^0
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞



Trigonometrical Ratios for a General Angle

First Quadrant

$$\sin(\theta) = \sin\theta$$

$$\cos(\theta) = \cos\theta$$

$$\tan(\theta) = \tan\theta$$

Second Quadrant

$$\sin(180 - \theta) = \sin\theta$$

$$\cos(180 - \theta) = -\cos\theta$$

$$\tan(180 - \theta) = -\tan\theta$$

Third Quadrant

$$\sin(180+\theta) = -\sin\theta$$

$$\cos(180+\theta) = -\cos\theta$$

$$\tan(180+\theta) = \tan\theta$$

Fourth Quadrant

$$\sin(360-\theta) = -\sin\theta$$

$$\cos(360-\theta) = \cos\theta$$

$$\tan(360-\theta) = -\tan\theta$$

Example 1: Evaluate the following

$(i)\sin 120^0$ $(ii)\cos 150^0$ $(iii)\sin 315^0$ $(iv)\cos 700^0$ and $(v)\tan 675^0$

Solution

$$\begin{aligned}
(i)\sin 120^0 &= \sin(180^0-60^0) = \sin 60^0 = \frac{\sqrt{3}}{2}\\
(ii)\cos 150^0 &= \cos(180^0-30^0) = -\cos 30^0 = -\frac{\sqrt{3}}{2}\\
(iii)\sin 315^0 &= \sin(360^0-45^0) = -\sin 45^0 = -\frac{1}{\sqrt{2}}\\
(iv)\cos 700^0 &= \cos(2\times 360^0-20^0) = \cos 20^0 = 0.9397\\
(v)\tan 675^0 &= \tan(2\times 360^0-45^0) = -\tan 45^0 = -1
\end{aligned}$$

Example 1: Proove that

$(a) \frac{\sec^2\theta - 1}{\tan^2\theta} = 1$ $(b) \sqrt{1 - \sec^2\theta} = \cos\theta$ $(c) \frac{\sin^2\theta + \cos^2\theta}{\sec^2\theta - \tan^2\theta} = 1$

Solution

$$(a) \quad \frac{\sec^2\theta - 1}{\tan^2\theta} = \frac{\frac{1}{\cos^2\theta} - 1}{\frac{\sin^2\theta}{\cos^2\theta}} = \frac{1 - \cos^2\theta}{\cos^2\theta} \times \frac{cos^2\theta}{\sin^2\theta} = 1$$

$$(b) \quad \cos^2\theta + \sin^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \sin^2\theta \Rightarrow \cos\theta = \sqrt{1 - \sec^2\theta}$$

$$(c) \quad \frac{\sin^2\theta + \cos^2\theta}{\sec^2\theta - \tan^2\theta} = \frac{1}{\frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}} = \frac{1}{\frac{1-\sin^2\theta}{\cos^2\theta}} = 1$$

Play Questions

(1) $\tan\theta\sin\theta + \cos\theta = \sec\theta$

(2) $\frac{1}{\tan\theta} + \tan\theta = \frac{1}{\sin\theta\cos\theta}$

(3) If $\sin A = \frac{3}{5}$. Prove that $\tan A + \frac{1}{\cos A} = 2$ or -2

(4) If θ is in the fourth quadrant and $\cos\theta = \frac{5}{13}$ Find value of $\frac{13\sin\theta + 5\sec\theta}{5\tan\theta + 6\csc\theta}$

Compound and Multiple Angles

Addition and Subtraction Formulae

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A + B) &= \cos A \cos B - \sin A \sin B \\
\cos(A - B) &= \cos A \cos B + \sin A \sin B \\
\tan(A + B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
\tan(A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B}
\end{aligned}$$

Example 1: compute $\sin 75^0$ $\cos 75^0$ and $\tan 15^0$ from the ratios of 30^0 and 45^0

Solution

- $\sin 75^0 = \sin(45^0 + 30^0) = \sin 45^0 \cos 30^0 + \cos 45^0 \sin 30^0 = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = 0.966$
- $\cos 75^0 = \cos(45^0 + 30^0) = \cos 45^0 \cos 30^0 - \sin 45^0 \sin 30^0 = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = 0.259$
- $\tan 15^0 = \tan(45^0 - 30^0) = \frac{\tan 45^0 - \tan 30^0}{1 + \tan 45^0 \tan 30^0} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \times \frac{1}{\sqrt{3}}} = 2 - \sqrt{3} = 0.268$

Example 2: If $\cos A = \frac{4}{5}$ and $\sin B = \frac{12}{13}$. Find the vales of

$\sin(A+B)$ and $\cos(A+B)$. (i) If A and B are acute angles. (ii) If $B$is obtuse and A is acute

Solution

If $\cos A = \frac{4}{5}$ then $\sin A = \sqrt{1-\cos^2 A} = \pm\frac{3}{5}$
If $\sin B = \frac{12}{13}$ then $\cos B = \sqrt{1-\sin^2 A} = \pm\frac{5}{13}$

i $\sin(A+B) = \sin A\cos B + \cos A\sin B = \frac{3}{5}\times\frac{5}{13} + \frac{4}{5}\times\frac{12}{13} = \frac{63}{65}$
$\cos(A+B) = \cos A\cos B - \sin A\sin B = \frac{4}{5}\times\frac{5}{13} - \frac{3}{5}\times\frac{12}{13} = -\frac{16}{65}$

ii $\sin(A+B) = \sin A\cos B + \cos A\sin B = \frac{3}{5}\times\left(-\frac{5}{13}\right) + \frac{4}{5}\times\frac{12}{13} = \frac{33}{65}$
$\cos(A+B) = \cos A\cos B - \sin A\sin B = \frac{4}{5}\times\left(-\frac{5}{13}\right) - \frac{3}{5}\times\frac{12}{13} = -\frac{56}{65}$.

Example 3: If $\tan A = \frac{1}{3}$ and $\tan B = \frac{1}{2}$, both A and B being acute, find the value of $\tan(A-B)$ without using table

Solution

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B} = \frac{\frac{1}{3} - \frac{1}{2}}{1 + \left(\frac{1}{3}\right)\left(-\frac{1}{2}\right)} = -\frac{1}{7}$$

Example 4: Express the following as single trigonometric ratios

$(i) \sin 80^0 \cos 20^0 - \cos 80^0 \sin 20^0$ $(ii) \frac{\tan 69^0 + \tan 66^0}{1 - \tan 69^0 \tan 66^0}$

Solution

i $\sin 80^0 \cos 20^0 - \cos 80^0 \sin 20^0 = \sin(80^0 - 20^0) = \sin 60^0 = \frac{\sqrt{3}}{2}$

ii $\frac{\tan 69^0 + \tan 66^0}{1 - \tan 69^0 \tan 66^0} = \tan(69^0 + 66^0) = \tan 135^0 = -1$

Multiple Angles

- $\sin 2A = 2 \sin A \cos A$
- $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
- $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

Half Angles

- $\sin A = \sin\left(\frac{A}{2} + \frac{A}{2}\right) = 2 \sin \frac{A}{2} \cos \frac{A}{2}$
- $\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} = 2\cos^2 \frac{A}{2} - 1 = 1 - 2\sin^2 \frac{A}{2}$
- $\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$

Example 1: Given that $\sin A = \frac{3}{5}$ and A is an acute angle, find without using table, the values of $\sin 2A$, $\cos 2A$ and $\tan 2A$. hence find the value of $\sin 4A$

Solution

$\sin A = \frac{3}{5} \Rightarrow \cos A = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$, $\tan A = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$

- $\sin 2A = 2\sin A\cos A = 2\left(\frac{3}{5}\times\frac{4}{5}\right) = \frac{24}{25}$
- $\cos 2A = 1 - 2\sin^2 A = 1 - 2\times\frac{9}{25} = \frac{7}{25}$
- $\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\times\frac{3}{4}}{1-\frac{9}{16}} = \frac{24}{7}$
- $\sin 4A = 2\sin 2A\cos 2A = 2\times\frac{24}{25}\times\frac{7}{25} = \frac{336}{625}$

Example 2: Prove that $(i)\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \tan x$

$(ii)\ \sin 3A = 3\sin A - 4\sin^3 A$

Solution

i

$$\frac{1-\cos 2x}{1+\cos 2x} = \frac{1-(1-\sin^2 x)}{1+(2\cos^2 x - 1)} = \frac{2\sin^2 x}{2\cos^2 x} = \tan^2 x$$

Take square root of both sides

$$\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \sqrt{\tan^2 x} = \tan x$$

Solution

a

$$\begin{aligned}(1-\cos A)(1+\sec A) &= 1+\sec A-\cos A-\cos A\sec A\\ &= 1+\sec A-\cos A-\cos A\frac{1}{\cos A}\\ &= \sec A-\cos A\\ &= \frac{1-\cos^2 A}{\cos A}=\frac{\sin^2 A}{\cos A}\\ &= \sin A\left(\frac{\sin A}{\cos A}\right)=\sin A\tan A\end{aligned}$$

b

$$\begin{aligned}(\sec\theta+\tan\theta)^2 &= \left(\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right)^2=\frac{(1-\sin\theta)^2}{\cos^2\theta}\\ &= \frac{(1+\sin\theta)^2}{1-\sin^2\theta}=\frac{(1+\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}\\ &= \frac{1-+sin\theta}{1-\sin\theta}\end{aligned}$$

c

$$\sec\theta\cot\theta=\frac{1}{\cos\theta}\times\frac{\sin\theta}{\cos\theta}=\frac{1}{\sin\theta}=\csc\theta$$

Factor Formula

- **Converting Product into Sum of Differences**

Recall that

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \quad (1)$$

$$\sin(A+B) = \sin A\cos B - \cos A\sin B \quad (2)$$

Adding (1) and (2) we have

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

Subtracting (1) from (2)

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

Also

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \quad (3)$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B \quad (4)$$

Adding and subtracting (3) and (4) we get

$$\begin{aligned} 2\cos A\cos B &= \cos(A+B) + \cos(A-B) 2\sin A\sin B \\ &= \cos(A-B) - \cos(A+B) \end{aligned}$$

- **Converting Sums of Differences into Products**

From $\sin(A+B)+\sin(A-B)=2\sin A\cos B$

Let $A+B=C$ and $A-B=D$. Then by addition and subtraction $A=\frac{C+D}{2}$ and $B=\frac{C-D}{2}$

$\therefore \sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

Making the same substitution in the rest of formula above, we have

$$\sin C+\sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \qquad (6)$$
$$\sin C-\sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \qquad (7)$$
$$\cos C+\cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \qquad (8)$$
$$\cos C-\cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right) \qquad (9)$$

Example 1: Find the value of $\sin 75^0 \sin 15^0$

Solution

$$\begin{aligned}
\sin 75^0 \sin 15^0 &= \frac{1}{2}\left[2\sin 75^0 \sin 15^0\right] \\
&= \frac{1}{2}\left[\cos(75^0 - 15^0) - \cos(75^0 + 15^0)\right] \\
&= \frac{1}{2}\left[\cos 60^0 - \cos 90^0\right] = \frac{1}{4}
\end{aligned}$$

Example 2: Prove that $\frac{\sin 7\theta - \sin 5\theta}{\cos 5\theta + \cos 7\theta} = \tan\theta$

Solution

$$\begin{aligned}
\frac{\sin 7\theta - \sin 5\theta}{\cos 5\theta + \cos 7\theta} &= \frac{2\cos\left(\frac{7\theta+5\theta}{2}\right)\sin\left(\frac{7\theta-5\theta}{2}\right)}{2\cos\left(\frac{5\theta+7\theta}{2}\right)\cos\left(\frac{5\theta-7\theta}{2}\right)} \\
&= \frac{\cos 6\theta \sin\theta}{\cos 6\theta \cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta
\end{aligned}$$

Play Question

1 Prove that $\frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$

2 Prove that $\frac{\sin 7x + \sin 3x}{\cos 7x + \cos 3x} = \tan 5x$

3 Prove that $\sin^2\theta(1 + \cot^2 theta) = 1$

2 Mathematical Induction

2.1 Introduction

The principle of mathematical induction is the process of proving a general theorem or formula for particular cases. The method of mathematical induction is useful in proving statements involving all positive integers when it's known. The method of proof in mathematical induction consist of the following steps:-

1. Prove the statement $n = 1$ (or some other $+\mathbb{Z}$)
2. Assume the statement is true for $n = k$, where k is any positive integer.
3. From the assumption in (2), prove that the statement must be true for $n = k + 1$.
4. Since the statement is true for $n = 1$ (from step 1). them it must (from step 3) be true for $n = 1 + 1 = 2$ and from this must be true for all positive integers.

Example 2.1 *Prove by mathematical induction that if n is a positive integer, then*

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

w.w.t.p.t the statement

$$1+2+3+\ldots+n=\frac{n(n+1)}{2} \tag{10}$$

i We prove that the statement is true for $n=1$.

thus when $n=1, \implies$ *l.s.h = 1, and r.h.s* $=\frac{1(1+1)}{2}=1$

$\therefore$ *l.h.s = r.h.s = 1*

ii Next we assume that the statement is true for $n=k$.

when $n=k$, *equation* (10) *becomes*

$$1+2+3+\ldots+k=\frac{k(k+1)}{2} \tag{11}$$

iii. Next, then we shall prove that the statement is true for $n=k+1$, *i.e* $k=k+1$

substitute $k=k+1$ *in eqn* (11)

$$1+2+3+\ldots+k+(k+1)=\frac{k+1(k+2)}{2} \tag{12}$$

Next, to make equation 11 and 12 equal, we add $k+1$ *to both-*

sides of equation 11 $\Longrightarrow$ *eqn 11 becomes*

$$1+2+3+\ldots+k+(k+1) = \frac{k(k+1)}{2}+(k+1)$$

$$\Longrightarrow$$

$$1+2+3+\ldots+k+(k+1) = (k+1)\left[\frac{k}{2}+1\right]$$

$$1+2+3+\ldots+k+(k+1) = (k+1)\left(\frac{k+2}{2}\right) \quad (13)$$

$\Longrightarrow$ *eqn 12 = eqn 13*

Hence its proved.

checking:- if $n=k+1, \Longrightarrow k=n-1$

$$\Longrightarrow \qquad (n-1+1)\frac{n-1+2}{2} = n\frac{n+1}{2} \quad (14)$$

$$\textit{as required} \quad (15)$$

Remarks Hence if the formula is true $n=k$, we have proved it to be true for $n=k+1$. The formula is true for $n=1$, hence it holds for $n=1+1=2$, then it holds $n=2, n=2+1=3$.Thus it holds for all positive integeral values of n.

Example 2.2 *The sum of first odd natural numbers is* n^2, *prove that the induction*

$$1+3+5+\ldots+(2n-1)=n^2$$

Solution

w.w.t.p.t the statement

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2 \tag{16}$$

i We prove that the statement is true for $n = 1$ *i.e*

$$(2(1) - 1) = 1^2 \tag{17}$$

$$1 = 1 \tag{18}$$

hence l.h.s = 1 = r.h.s which is true.

ii We assume that the statement is true for $n = k$ *i.e*

$$1 + 3 + 5 + \ldots + (2k - 1) = k^2 \tag{19}$$

iii Then we shall prove that the statement is also true for $n = k + 1$. *i.e substitute* $k = k + 1$ *in eqn* (19) *wem have*

$$1 + 3 + 5 + \ldots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2$$

$$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = (k + 1)^2 \tag{20}$$

to make eqn (19) *and* (20) *equal, we add* $(2k + 1)$ *to both sides of equation* (19)

$\Longrightarrow$ *eqn* (19) *becomes*

$$\begin{aligned} 1+3+5+\ldots+(2k-1)+(2k+2) &= k^2+2k+1 \\ 1+3+5+\ldots+(2k-1)+(2k+2) &= k^2+k+k+1 \\ &= k(k+1)+1(k+1) \\ &= (k+1)(k+1) \\ &= (k+1)^2 \qquad (21) \end{aligned}$$

Hence eqn (20) *and* (21) *are equal.*

Hence it is proved. ***Checking:-*** $n = k+1 \quad \Longrightarrow \quad k = n-1$

$(n-1+1)^2 = n^2$

Example 2.3 *Prove by mathematical induction that if n is a positive integer then*

$$\frac{1}{1(2)}+\frac{1}{2(3)}+\frac{1}{3(4)}+\ldots+\frac{1}{n(n+1)} = \frac{n}{n+1}$$

$$\frac{1}{1(2)}+\frac{1}{2(3)}+\frac{1}{3(4)}+\ldots+\frac{1}{n(n+1)} = \frac{n}{n+1} \qquad (22)$$

i Prove the statement for $n = 1$

$$\begin{aligned} \frac{1}{1(1+1)} &= \frac{1}{1+1} \\ \frac{1}{2} &= \frac{1}{2} \end{aligned}$$

$\therefore$ l.h.s $= \frac{1}{2} =$ r.h.s which is true

ii assume that it is true for $n = k$ i.e

$$\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1} \tag{23}$$

iii then prove that it is also true for $n = k + 1$

$\implies$ eqn (23) becomes

$$\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2} \tag{24}$$

to make eqn (23) and (24) equal we add $\frac{1}{(k+1)(k+2)}$ to bothsides of eqn (23) $\implies$ eqn (23) becomes

$$\begin{aligned}
\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \ldots + \frac{1}{k(k+1)} &= \frac{k}{k+1} + \frac{1}{k(k+1)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)(k+1)}{(k+1)(k+2)} \\
&= \frac{(k+1)}{(k+2)} \qquad (25)
\end{aligned}$$

$\implies$ eqn (24) and (25) are equal. Hence it is proved as required.

Checking:- $n = k + 1 \quad \implies \quad k = n - 1$

$\frac{n-1+1}{n-1+2} = \frac{n}{n+1}$

Example 2.4 *Prove by mathematical induction*

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2}{4}(n+1)^2$$

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2}{4}(n+1)^2 \tag{26}$$

i verify that the statement is true for $n = 1$

$$\begin{aligned} 1^3 &= \frac{1^2}{4}(1+1)^2 \\ 1 &= \frac{1}{4}(2)^2 \\ 1 &= \frac{4}{4} \\ 1 &= 1 \end{aligned}$$

$\implies$ *l.h.s = r.h.s = 1.*

ii Assume the statement is true for $n = k$ *i.e*

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2}{4}(k+1)^2 \tag{27}$$

iii Then prove that its true for $n = k+1$ *i.e* $k = k+1$

$\implies$ *eqn* (27) *becomes*

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{(k+1)^2}{4}(k+2)^2 \tag{28}$$

to make (27) and (28) equal, add $(k+1)^3$ to bothsides of eqn (27)

$\Longrightarrow$ *eqn (27) becomes*

$$\begin{aligned}
1^3+2^3+3^3+\ldots+k^3+(k+1)^3 &= \frac{(k^2}{4}(k+1)^2+(k+1)^3 \\
&= (k+1)^2\left[\frac{k^2}{4}+(k+1)\right] \\
&= (k+1)^2\left[\frac{4(k+1)+k^2}{4}\right] \\
&= (k+1)^2\left[\frac{k^2+4k+4}{4}\right] \\
&= \frac{(k+1)^2}{4}\left[k^2+2k+2k+4\right] \\
&= \frac{(k+1)^2}{4}\left[(k+2)(k+2)\right] \\
&= \frac{(k+1)^2}{4}\left[(k+1)^2\right] \qquad (29)
\end{aligned}$$

$\Longrightarrow$ *eqn (28) and (29) are equal.*

Hence it is proved.

Example 2.5 *Prove by mathematical induction that, for all positive integer n,*

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Let p_n denotes the statement

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

i We first show that p_1 is true when $n = 1$.

LHS $=1^2 = 1$ *and RHS* $= \frac{1(1+1)(2+1)}{6} = 1$

this p_1 is true.

ii. We now assume that p_k is true. i.e

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Adding $(k+1)^th$ term to both sides of the equation then,

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

Thus:

RHS:$k(k+1)(2k+1) + 6(k+1)^2$

$$= \frac{(k+1)(2K^2+K)+6K+6}{6}$$
$$= \frac{(k+1)(K+2)+(2K+3}{6}$$

which is true for the value of $\frac{n(n+1)(2n+1)}{6}$ *when $(k+1)$ is used to replace n. i.e (LHS) of the equations. Hence the statement is true for all positive integer n.*

Example 2.6 *show by induction that*

$$\sum_{r=1}^{n}(2r+1)(3r+1) = \frac{n}{2}(4n^2+11n+9)$$

for all positive integer n. Let p_n denote the statement:

$$\sum_{r=1}^{n}(2r+1)(3r+1) = \frac{n}{2}(4n^2+11n+9)$$

1 We first show that p_1 is true for $n=1$

LHS $=3(4) = 12$

RHS$= \frac{1}{2}(24) = 12$. *Thus p_1 is true.*

2 We now assume that p_k is true i.e

$$\sum_{r=1}^{k}(2r+1)(3r+1) = \frac{k}{2}(4k^2+11k+9) \qquad \textit{By induction hypothe}$$

to show that p_{k+1} is also true, then

$$\sum_{r=1}^{k+1}(2r+1)(3r+1) = \frac{k+1}{2}(4(k+1)^2+11(k+1)+9)$$

$$\textit{LHS} \quad \sum_{r=1}^{k+1}(2r+1)(3r+1)$$

$$= \sum_{r=1}^{k+1}(2r+1)(3r+1) + 2\left[(k+1)+1\right]\left[3(k+1)+1\right]$$

By adding $(2(k+1)+1)(3(k+1)+1)$ to LHS.

$$\textit{Thus} \quad \frac{k}{2}\left[4k^2+11k+9+16k^2+17k+12\right]$$

By induction hypothesis.

$$= 2k^3 + \frac{23}{2}k^2 + \frac{43}{2}k + 12.$$

Thus the statement is true for all positive integer n, as required.

SEQUENCE AND SERIES

Sequence: A sequence (or progression in British English) is an ordered list of numbers; the numbers in this ordered list are called elements or terms.

A sequence may be named or referred to as A or A_n . The terms or elements of a sequence are usually named as a_i or a_n, where the subscript letter $i \, or \, n$ being the index or counter such that $i \in \mathbb{R} \; or \; n \in \mathbb{R}$

For example if $A = a_i : 1 \leq i \leq 12$

Then $A = a_1, a_2, a_3, ..., a_{12}$

Where a_1 is the first term, a_2 the second term up to a_{12} the twelfth term.

<u>**NB**</u>

Sometimes sequences start with an index of $n = 0$, so the first term is actually a_0. Then the second term would be a_1. The first listed term in such a case would be called the "Zero-eth" term. This method of numbering the terms is used, for example, in Java script arrays.

A sequence $A \, or \, A_n$ with terms $a_i \, or \, a_n$ may also be referred to as $\{a_i\} \; or \; \{a_n\}$, this set is actually an ordered list, not an unordered collection of elements.

Series: A series is the value we get when we add up all the terms of sequences. Or the sum of all the terms of a sequence is called a **series**. To indicate a series, we use either the **Latin capital letter** "S" or else the Greek letter corresponding to the capital letter **"S"**, which is called **"Sigma"** (SIGG-Muh) $\sum$.

For example to show the summation of the first twelfth terms of a sequence $\{a_n\}$

The first twelfth terms of the sequence $\{a_n\}$ can be written as:

$$\sum_{n=1}^{12} a_n$$

Where $n=1$ is the lower index and $n=12$ is the upper index, telling us that a_{12} or the twelfth term will be the last term added in this series .i.e. The summation symbol above means $a_1+a_2+a_3+\cdots+a_{12}$. This is called the "expanded" form of the series, in contrast with the more compact "Sigma" notation.

NB

Any letter can be used for the index, but $i, j, k \, and \, n$ are probably used more than any other letters. Sequence and series: are most useful when there is a formula for their terms. For instance if the formula for $a_n = 2n+3$, then we can find the value of any term by plugging the value of n into the formula which can be read in word as "the nth term is given by two-enn plus three".

Example 1:

Let $A_n = \{1,3,5,7,9\}$

a. What is the value of a_3

b. Find the value of $\sum_{n=1}^{5} a_n$

Solution

a. The index of a_3 is when $n=3$ which is the third term of the sequence

   Therefore the value of a_3 is 5

b. The value of $\sum_{n=1}^{5} a_n$, is the total sum of all a_n from $a_1 \, to \, a_5$

   Therefore the value of the sum: 25

Example 2: list the first four terms of the sequence $\{a_n\} = \{n^2\}$, starting with $n = 1$

Solution

$\{a_1, a_2, a_3, a_4\} = \{1^2, 2^2, 3^2, 4^2\} = \{1, 4, 9, 16\}$

Example 3: list the first four terms of the sequence beginning with $n = 0$

$A_n = \dfrac{(-1)^n}{(n+1)!}, \ where\ 0 \leq n \leq 3$

Solution

By plugging the value of n into the sequence formula we have

$a_0 = 1, a_1 = -\frac{1}{2}, a_2 = \frac{1}{6}, and\ a_3 = -\frac{1}{24}$

Therefore the terms are

$\{a_0, a_1, a_2, a_3\} = \{1, -\frac{1}{2}, \frac{1}{6}, -\frac{1}{24}\}$

Example 4: find the sum of the first six terms of A_n where

$a_n = 2a_{n-1} + a_{n-2}, and\ a_1 = 1, a_2 = 1$

Solution

Given that $a_1 = 1, and\ a_2 = 1$, i.e. the first two terms. Then we use $a_n = 2a_{n-1} + a_{n-2}$ to generate the remaining four terms.

$a_1 = 1$
$a_2 = 1$
$a_3 = 2a_2 + a_1 = 3$

$a_4 = 2a_3 + a_2 = 7$

$a_5 = 2a_4 + a_3 = 17$

$a_6 = 2a_5 + a_4 = 41$

Therefore the sequence is

$A_n = 1, 1, 3, 7, 17, 41$

And the sum of the first six terms is

$$\sum_{n=1}^{6} A_n = 70$$

Example 5: write the series $2-4+6-8+10$ using summation notation, beginning with $n=1$.

Solution

We observe that the relationship between n and the terms in the summation for each term a_n in the series is twice n, so we have $2n$. Also we have figure out that there is alternating sign. If we used $(-1)^n$ we get $-2, 4, -8, \ldots$ which is back words of the given series. But we can switch the signs by throwing in one more factor of $(-1)^1$. Then we get

$$2n(-1)^n(-1)^1 = 2n(-1)^{n+1}$$

Therefore the formula for n^{th} is $2n(-1)^{n+1}$

Since n start from 1, and there are five terms, then the summation is

$$\sum_{n=1}^{5} 2n(-1)^{n+1}$$

Example 6: write the series below using summation notation

$$\frac{5}{6+3}+\frac{5}{7+3}+\frac{5}{8+3}+\cdots+\frac{5}{31+3}$$

Solution

You figure out that the only thing that changes from one term to the next term is one of the numbers in the denominator. Which are 6, 7, 8…this looks like counting, but starting from 6 instead of 1 without any information to the contrary, therefore you need to relate these counting numbers to the counter, the index n. For $n=1$ the number is 6 or $n+5$, for $n=2$ the number is 7 or $n+5$ like that.

Therefore the n^{th} term of the sequence is

$$a_n=\frac{5}{(n+5)+3}$$

Now how many terms do we have in the given series?

The ellipsis means that terms were omitted however, now that we have the general formula for the series terms. We can solve for the counter in the last term of the series

$31=n+5,\Rightarrow n=26$

This shows that there are 26 terms in the given series, therefore the summation notation will be

$$\sum_{n=1}^{26}\frac{5}{(n+5)+3}$$

ARITHMETIC AND GEOMETRIC SEQUENCE

Since we have learned the basic notation and terminologies of sequence and series, we should quickly steps into the two common and straight forward sequence types. They are Arithmetic and Geometric sequence. The Arithmetic and Geometric sequences have a definite pattern that is used to arrive at the sequences terms. While some sequences are simply random values.

ARITHMETIC SEQUENCE (OR PROGRESSION)

An arithmetic sequence goes from one term to the next by always adding (or subtracting) the same value. Or an Arithmetic sequence referred to adding or subtracting a fixed amount from one term to the next term. The fixed amount is called common difference denoted as d, referring to the fact that the difference between two successive terms yields the constant value that was added. To find the common difference subtract the first term from the second term.

Example 1: find the common difference of the sequence $2,5,8,11,14,\ldots$

The common difference d is 3. Since $5-2=3, 8-5=3$ *and* $14-11=3$

This shows that if we add 3 to the first term we get the second term also if we add 3 to the second term we get the third term and so on.

Example 2: Find the common difference of the sequence $7,3,-1,-5,\ldots$

The common difference d is -4. Since if we subtract -4 from the first we get the second term, also if we subtract -4 from the second term we get the third term.

The linear nature of scatter plot of the terms of arithmetic sequence

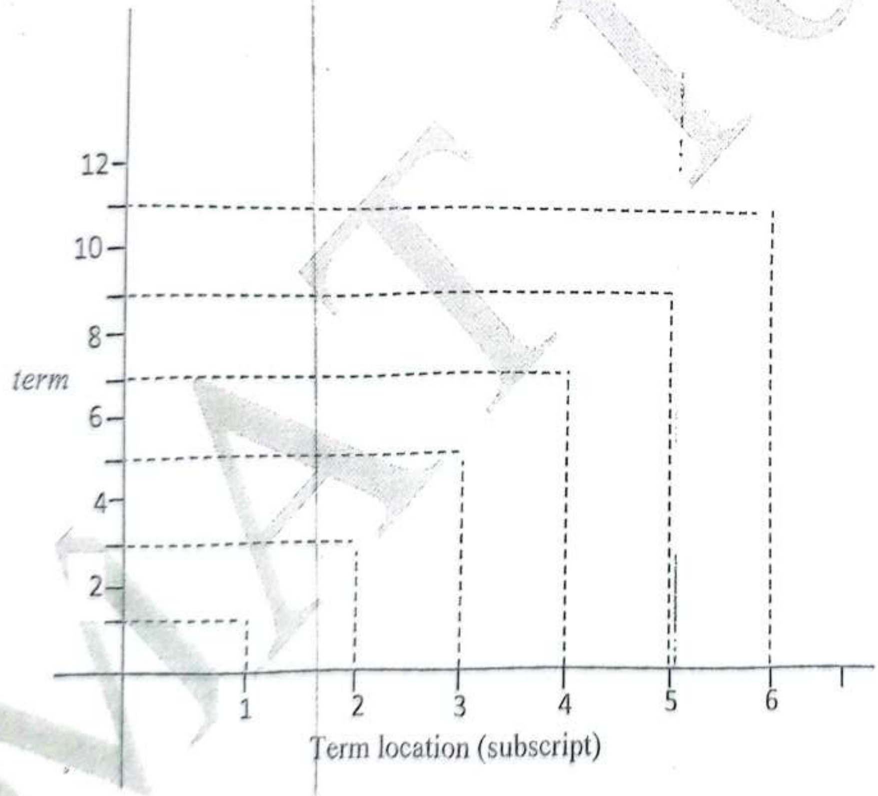
Example

Plot the graph of the sequence $1,4,7,10,13,16,\ldots$

NB

The domain consists of the counting numbers $1, 2, 3, \ldots$ and the range consists of the terms of the sequence. While the x value increases by a constant value 1. The y value increases by a constant value 3. i.e. common difference of the sequence.

X	1	2	3	4	5	6
Y	1	4	7	10	13	16



Since arithmetic sequence is so nice and regular, we have the formula.

The first term in arithmetic is denoted as a_1, since the common difference is d, then the first term a_1, is simply referred to "a". Now to get the second term a_2, we have $a+d=a_2$ and the third term we have $a+d+d=a+2d=a_3$, also for the fourth term $a+d+d+d=a+3d=a_4$. Following this process, the n^{th} term a_n of an A.P will have the form.

$$a_n = a + (n-1)d$$

Where a is the first term of the sequence and d is the common difference also n is the number of the term.

Example 1: Find the seventh term of an arithmetic sequence where $a_1 = 3x \; and \; d = -x$

Solution

Since $a_1 = 3x, d = -x \; and \; n = 7$

$$a_n = a + (n-1)d$$

$$\Rightarrow a_7 = 3x + (7-1)(-x)$$

$$\Rightarrow a_7 = -3x$$

Example 2: find the n^{th} term and the three terms of the arithmetic sequence having $a_4 = 93 \; and \; a_8 = 65$

Solution

Given that $a_4 \; and \; a_8$ are four places a part, also we know from the definition of an arithmetic sequence that $a_8 = a_4 + 4d$, using this we can find the common difference

$$a_8 = a_4 + 4d$$

$$65 = 93 + 4d$$

$$\Rightarrow d = -7$$

Now to find the first term, since

$$a_n = a + (n-1)d$$

$$\Rightarrow a_4 = a + (4-1)(-7)$$

$93 = a + (n-1)(-7)$

$\Rightarrow a = 114$

Therefore now we have the value of first, common difference then the value of the first three terms will be

$a_1 = 114, \Rightarrow a_2 = 114 + (-7) = 107,\ a_3 = 107 + (-7) = 100$

Now to find the general term

$a_n = a + (n-1)d$

$a_n = 114 + (n-1)(-7)$

$\Rightarrow a_n = 121 - 7n$

Example 3: Find the number of terms in the sequence $7, 10, 13, \ldots, 55$

Solution

Given that $a = 7, d = 3, a_n = 55, and\, n = ?$

$a_n = a + (n-1)d$

$55 = 7 + (n-1)(3)$

$\Rightarrow n = 17$

ARITHMETIC SERIES (OR SUM OF AN A.P)

An arithmetic series is the sum of an arithmetic sequence. There are many types of series, but we are unlikely to work with them until we are in calculus. The **partial sum** is the sum of a limited (finite) number of terms, like the first ten terms, or the fifth through the hundredth terms.

The formula for the first n times of an arithmetic sequence starting with $n = 1$ is

$$\sum_{i=1}^{n} a_i = \tfrac{n}{2}(a + a_n),$$

Where a_n, is the last is term and a, is the first term OR $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

The sum is in effect, n times the **average** of the first and last terms. This sum of the first n terms is called the n^{th} **partial sum.**

Example 1: Find the 35^{th} partial sum of $a_n = \tfrac{1}{2}n + 1$

Solution

The 35^{th} partial sum of this sequence is the sum of the first thirty-five terms. Therefore the first few terms of the sequence are:

$a_1 = \tfrac{3}{2},\ a_2 = 2,\ a_3 = \tfrac{5}{2}, \ldots$

The sequence has the common difference $d = \dfrac{1}{2}$, and the last term in the partial sum will be

$\Rightarrow a_{35} = a + (35-1)\left(\tfrac{1}{2}\right)$

$\Rightarrow a_{35} = \tfrac{3}{2} + (35-1)\left(\tfrac{1}{2}\right)$

$\Rightarrow a_{35} = \dfrac{37}{2}, \Rightarrow a_n = \dfrac{37}{2}$

Therefore

$S_n = \tfrac{n}{2}(a + a_n)$

$S_{35} = \tfrac{35}{2}\left(\tfrac{3}{2} + \tfrac{37}{2}\right)$

$\Rightarrow S_{35} = 350$

Example 2: Find the sum of $1+5+9+\cdots+49+53$

Solution

Indeed the given series is arithmetic series, since

$5-1=4,\ 9-5=4, and\, 53-49=4$

$\Rightarrow a_n=53, d=4, a=1, and\, n=?$

Now to find the value of n

$a_n=a+(n-1)d$

$53=1+(n-1)(4)$

$\Rightarrow n=14$

So there are 14^{th} terms in the series, then the sum of the 14^{th} terms is

$S_n=\frac{n}{2}(a+a_n)$

$S_{14}=\frac{14}{2}(1+53)$

$S_{14}=378$

Example 3: If MH has 60 seats in the first row, 68 seats in the second row, 76 seats in the third row, and so on in the same increasing pattern.

Find the sum of the 20 rows in MH.

Solution

The sequences of the rows are $60, 68, 76, \ldots$

$\Rightarrow a=60, n=20, d=8, a_n=?$

To find the number of seats in the last row

$a_n = a + (n-1)d$

$a_{20} = 60 + (20-1)(8)$

$\Rightarrow a_{20} = 212, \Rightarrow a_n = 212$

Now the sum of seats in 20^{th} row is

$S_n = \frac{n}{2}(a + a_n)$

$S_{20} = \frac{20}{2}(60 + 212)$

$\Rightarrow S_{20} = 2720$

Example 4: Find the sum of all natural numbers between *250* and *1000* which are exactly divisible by *5*.

Solution

Clearly, the numbers between *250* and *1000* which are exactly divisible by *5* are *255, 260, 265,...995.* This is an A.P with $a = 255$, $d = 5$ and the last term $a_n = 995$.

$$\begin{aligned} \therefore a_n &= a + (n-1)d \\ 995 &= 255 + (n-1)d \\ \Rightarrow n &= 141 \end{aligned}$$

$$\therefore S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Therefore Required sum i.e $= \frac{141}{2}\left[2 \times 255 + (141-1) \times 5\right]$

$= 85305$

ARITHMETIC MEAN

The arithmetic mean of two members $A\,and\,B$ is the number r, between $A\,and\,B$, such that $A, r, and\ B$ are in A.P then

$r = \frac{A+B}{2}$ where r is the arithmetic mean. In order words the arithmetic mean of two numbers r is simply their average.

Examples:

1. Find the arithmetic mean of 25 and 60.

   Solution

   $A.M = \frac{25+60}{2} = 42.5$

Remark:

If we are required to insert three arithmetic means between two seven numbers $P\,and\,Q$, it means that we have to supply three numbers $A, B, and\ C$ between $P\,and\,Q$ such that $P, A, B, C, and\ Q$ forms an arithmetic progression.

2. Insert three arithmetic means between 8 and 18.

   Solution

   Let the arithmetic means be $A, B, and\ C$

   Therefore $8, A, B, C, 18$ forms an arithmetic progression

   $\Rightarrow a = 8, a+d = A, a+2d = B, a+3d = C, and\ a+4d = 18$ are first, second, third, fourth and fifth term of the arithmetic progression .

   $\Rightarrow a = 8 \qquad (1)$

   $a + d = A \qquad (2)$

   $a + 2d = B \qquad (3)$

   $a + 3d = C \qquad (4)$

   $a + 4d = 18 \qquad (5)$

Solving equations (1) and (5) simultaneously we have $a = 8 \, and \, d = \frac{5}{2}$. By substituting the value of $a \, and \, d$ into equations (2),(3), and (4) we get $A = \frac{21}{2}, B = 13, and \, C = \frac{31}{2}$

$\therefore$ the required numbers are $8, \frac{21}{2}, 13, \frac{31}{2}, 18$

PLAY QUESTIONS 1

1. Insert two arithmetic means between 4 and 18
2. Find the indicated sum for the given A.P
   a. $S_{14} : \frac{3}{10}, \frac{2}{5}, \frac{1}{2}$ b. $S_{12} : -5, -4\frac{5}{8}, -4\frac{1}{4}$
3. Given that 4,P,Q,13 are consecutive terms of an A.P, find the values of P and Q
4. The 6^{th} term of an A.P is -5 and the 10^{th} term is -21. Find the sum of the first thirty terms.
5. The 3^{rd} and 7^{th} terms of an A.P are -1 and 11 respectively, find the n^{th} term and the number of terms which must be added to get a sum of 430.
6. Which term of the sequence $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}...$ is the first negative term?
7. Which term of the sequence $8-6i, 7-4i, 6-2i, ...$ is (i) purely real (ii) purely imaginary?
8. Find the tenth term and the n-th term of the following sequence: $\frac{1}{2}, 1, 2, 4, 8, ...$

GEOMETRIC SEQUENCE (OR G.P)

A geometric sequence goes from one term to the next by multiplying (or dividing) by the same value. So 1, 2, 4, 8, 16,… and 81, 27, 9, 3, 1,… are geometric, since you multiply by 2 and divide by 3, respectively, at each step.

The number multiplied (or divided) at each stage of a geometric sequence is called the "common ratio" which is denoted by "r". Because if you divide successive terms, you will always get this common value. To find the common ratio, divide the second term by the first term.

Example: find the common ratio of the sequence $\frac{2}{9}, \frac{2}{3}, 2, 6, 18, \cdots$

Solution

To find the common ratio we have to divide a pair of terms, it doesn't matter which pair we pick, as long as they are right next to each other.

$\frac{\frac{2}{3}}{\frac{2}{9}} = \frac{2}{3}\times\frac{9}{2} = 3$. Also $\frac{2}{\frac{2}{3}} = 2\times\frac{3}{2} = 3$

Therefore the common ration is 3. So $r = 3$.

For geometric sequences, the common ratio is "r" and first term "a_1" is often referred to simply referred to as "a". Since you get the next term by multiplying by the common ratio, the value of a_2 is just $a\times r = ar = a_2$, the third term is $a\times r\times r = ar^2 = a_3$, also the fourth term is $a\times r^2\times r = ar^3 = a_3$, following this pattern, the n^{th} term a_n or general term of a G.P will have the form

$$a_n = ar^{n-1}$$

Example: find the tenth term of the geometric sequence, $\frac{1}{2}, 1, 2, 4, 8, \cdots$

Solution

$$a_n = ar^{n-1}$$

$$\Rightarrow a = \tfrac{1}{2} \text{ and } r = 2$$

Now to find the tenth term

Since $a = \frac{1}{2}, r = 2$ *and* $n = 10$ plug the above data into the formula

$$a_n = ar^{n-1} = a_{10} = \left(\tfrac{1}{2}\right)(2)^{10-1} = 256$$

The non-linear nature of the scatter plot of the terms of a geometric sequence

Example 1: plot a graph of the sequence 2,4,8,16,32

Solution

The domain consists of the counting numbers 1, 2, 3, 4… and the range consists of the terms of the sequence.

X	1	2	3	4	5
Y	2	4	8	18	32

Plot the graph as a play question. The graph is not a straight line. Try it and see

Example 2: find the n-th term and the 26^{th} terms of the geometric sequence with $a_5 = \frac{5}{4}$ *and* $a_{12}160$

Solution

Since that $a_5 = \frac{5}{4}$ *and* $a_{12} = 160$, the two terms are $12-5=7$ places a part.

Since from the formula for geometric sequence, we know that $a_5 = ar^4 \Rightarrow a_{12} = \left(ar^4\right)r^7$

$\Rightarrow a_{12} = \left(ar^4\right)r^7$

$160 = \left(\frac{5}{4}\right)r^7 \Rightarrow r = 2$

Since $a_5 = ar^4$ we can find the first term

$\Rightarrow a_5 = ar^4$

$\frac{5}{4} = a(2)^4$

$\therefore a = \frac{5}{64}$

Then the 26^{th} is

$a_n = ar^{n-1}$

$a_{26} = \left(\frac{5}{64}\right)(2)^{25}$

$a_{26} = 2621440$

Example 3: find the 11^{th} term of the sequence $1, \frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, ...$

Solution

$\Rightarrow n = 11, a = 1 \; and \; r = \frac{-1}{2}$

$a_n = ar^{n-1}$

$a_{11} = (1)\left(\frac{-1}{2}\right)^{10}$

$$a_{11} = \frac{1}{1024}$$

Example 4: Find the first term of a geometric series, if the third term is 3 and the six term is $\frac{1}{9}$.

Solution

Since $a_3 = 3, \; and \; a_6 = \frac{1}{9} \Rightarrow a_3 = ar^2 = 3$

$\therefore a_6 = \left(ar^2\right)r^3$

$\frac{1}{9} = (3)r^3$

$\Rightarrow r = \frac{1}{3}$

Now to find the first term

Since $a_3 = ar^2 \Rightarrow a = 27$

The sum of geometric progression (series)

geometric series is the sum of a geometric sequence. We can take the sum of a finite number of rms of a geometric sequence. For a geometric sequence with first term $a_1 = a$ and common ation "r", the sum of the first n[th] terms is given by

$$\sum_{i=1}^{n} a_i = S_n = \frac{a\left(r^n - 1\right)}{r-1} \quad ,r > 1$$

n the special case that $r < 1$, the sum is given by

$$\sum_{i=1}^{n} a_i = S_n = \frac{a\left(1 - r^n\right)}{1-r}$$

he sum of an infinite G.P. is given by

$$\sum_{i=1}^{\infty} a_i = S_\infty = \frac{a}{1-r}, \quad |r| < 1$$

Example 1: solve $\sum_{i=1}^{20} 3(-2)^i$

Solution

The first few terms are

For $i = 1 \Rightarrow T_1 = -6$

For $i = 2 \Rightarrow T_2 = 12$

For $i = 3 \Rightarrow T_3 = -24$

$\therefore -6, 12, -24, \ldots$

These are the first few terms and this is a geometric series with common ration $r = -2$ and the first term of the sequence is $a = -6, and\ n = 20$

$$\therefore \sum_{i=1}^{n} a_i = a\left(\frac{1-r^n}{1-r}\right)$$

ii

$$
\begin{aligned}
\sin 3A &= \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A \\
&= 2\sin A \cos A \cos A + (1 - 2\sin^2 A)\sin A \\
&= 2\sin A \cos^2 A + \sin A - 2\sin^3 A \\
&= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A \\
&= 3\sin A - 4\sin^3 A
\end{aligned}
$$

Example 2: Prove that

a $(1 - \cos A)(1 + \sec A) = \sin A \tan A$

b $\frac{1 \sin\theta}{1-\sin\theta} = (\sec\theta + \tan\theta)^2$

c $\sec\theta \cot\theta = \csc\theta$

$$\sum_{i=1}^{20} 3(-2)^i = (-6)\left(\frac{1-(-2)^{20}}{1-(-2)}\right)$$

$$\sum_{i=1}^{20} 3(-2)^i = 2097150$$

So the value of the summation is 2097150

Example 3: Find the sum of 7 terms of the G.P 3, 6, 12,…

Solution

Here $a = 3,\ r = 2\ \therefore S_7 = a\left(\frac{r^7-1}{r-1}\right) = 3\left(\frac{2^7-1}{2-1}\right) = 381$

Example 4: Find the sum to infinity of the G.P. $-\frac{5}{4}, \frac{5}{16}, -\frac{5}{64}, \cdots$

Solution

$$a = -\frac{5}{4}, r = -\frac{1}{4}$$

Here

$$\therefore S_\infty = \frac{a}{1-r} = \frac{-\frac{5}{4}}{1-(-\frac{1}{4})} = -1$$

Example 5: Evaluate S_{10} for *250, 100, 40, 16,…*

Solution: Here

$$a = 250, r = \frac{2}{5}$$

$$S_{10} = 250\left(\frac{1-\left(\frac{2}{5}\right)^{10}}{1-\frac{2}{5}}\right)$$

$$= 250\left(\frac{1-\frac{1024}{9765625}}{\frac{3}{5}}\right)$$

$$= \frac{6509734}{15625}$$

Example 6: Find a_n *if* $S_4 = 26/27$ *and* $r = 1/3$

Solution: Here

$$S_4 = 26/27, r = 1/3, n = 4, a = ?$$

$$S_4 = a\left(\frac{1-r^4}{1-r}\right)$$

$$\frac{26}{27} = a\left(\frac{1-\left(\frac{1}{3}\right)^4}{1-\frac{1}{3}}\right)$$

$$\frac{26}{27} = a\left(\frac{\left(\frac{80}{81}\right)}{\frac{2}{3}}\right)$$

$$\frac{26}{27} = a\left(\frac{40}{27}\right)$$

$$\Rightarrow a = \frac{13}{20}$$

$$but\ \ a_n = ar^{n-1}$$

$$\Rightarrow a_n = \frac{13}{20}\left(\frac{1}{3}\right)^{n-1}$$

Example 7: the sum of the first n[th] terms of a geometric series is 127 and the sum of their reciprocals is $\frac{127}{64}$. The first term is 1. (i) Find the n term and the common ratio (ii) obtain a formula for $\sum_{i=1}^{n}(2-3r)$

Solution

$1, r, r^2, r^3, \cdots, r^n : S_n = a\left(\frac{r^n-1}{r-1}\right) = 127 \; for\, r > 1$

$1, \frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}, \cdots, \frac{1}{r^n} : S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{127}{64}, for\, r < 1$

$S_n : a\left(\frac{r^n-1}{r-1}\right) = 127, \; for\; a = 1$

$S_n : a\left(\frac{1-\frac{1}{r^n}}{1-\frac{1}{r}}\right) = \frac{127}{64}, \; for\; a = 1, r = \frac{1}{r}$

$$\frac{r^n-1}{r-1} = 127 \qquad (1)$$

$$\frac{1-\frac{1}{r^n}}{1-\frac{1}{r}} = \frac{127}{64} \qquad (2)$$

$$\Rightarrow \frac{\frac{r^n-1}{r^n}}{\frac{r-1}{r}} = \frac{r^n-1}{r^n} \times \frac{r}{r-1} = \frac{r(r^n-1)}{r^n(r-1)} = \frac{r}{r^n} \cdot \frac{r^n-1}{r-1} = \frac{127}{64} \qquad (3)$$

Substitute eqn (1) into eqn (3)

$\Rightarrow \frac{r}{r^n} \cdot 127 = \frac{127}{64}$

$$\Rightarrow \frac{r}{r^n} = \frac{1}{64} \Rightarrow r^n = 64r \qquad (4)$$

Substitute eqn (4) into (1)

$\frac{64r-1}{r-1} = 127 \Rightarrow r = 2$

Substitute the value of r into (4)

$r^n = 64r$

$(2)^n = 64(2)$

$(2)^n = 2^7 \Rightarrow n = 7$

Therefore $n = 7\, and\, r = 2$

Solve (ii) as an exercise.

GEOMETRIC MEAN

The geometric mean of two numbers say P and Q is simply the square root of their product. Thus if P, A, Q are in G.P, then $A = \sqrt{PQ}$ where A is the geometric mean.

Example 1: Find the G.M of 25 and 4. Solution: $A = \sqrt{25 \times 4} = 10$

Example 2: Insert two geometric means between 12 and 324

Solution

Let the two G.M's be X and Y, Then 12, X, Y, 324 forms a G.P

$T_1 = a = 12 \qquad (1)$
$T_2 = ar = X \qquad (2)$
$T_3 = ar^2 = Y \qquad (3)$
$T_4 = ar^3 = 324 \qquad (4)$

Dividing (4) by (1) we get $a = 12$ and $r = 3$

$\Rightarrow X = 36, Y = 108$. Hence the required geometric means are 12, 13, 108, 324.

PLAY QUESTIONS 2

1. Find the ninth term and the general term of the progression $\frac{1}{4}, -\frac{1}{2}, 1, -2$
2. Find the sum of the first 8 terms of the G.P with a = 5 and r = 2
3. The 1st and 6th terms of a G.P are 153 and 17/27. Find the sum of the first four terms.
4. If $T_2 = 35$ *and* $T_4 = 875$ respectively. Find the first and fifth terms.
5. Find the sum to infinity of the following series 4, 1, ¼,…..

BINOMIAL EXPANSION

We observe that the coefficients in the above expressions follow a particular pattern as given below

Index of the binomial	Coefficient of various terms

0					1				
1				1		1			
2			1		2		1		
3		1		3		3		1	
4	1		4		6		4		1

It is noticed that each row is bounded by 1 on both sides. Any entry, except the first and the last, in the row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right. This pattern is called or known as ***Pascal's Triangle***.

Therefore, the table of coefficients, which is written in form of the triangle is known as Pascal Triangle, after the French mathematician Blaise Pascal (1623-1662) this table of coefficients could be used to obtain expansions of binomial then n is not large and also positive. Reading from either end of each row, the coefficients are the same.

Example 1: use Pascal's triangle to expand the following expressions

a. $(x+2y)^3$

b. $(x-2)^4$

Solution

a. The coefficient in row three are

Therefore $(x+2y)^3 = \overset{1}{x^3} + \overset{3}{3x^2}(2y) + \overset{3}{3x}(2y)^2 + \overset{1}{(2y)^3}$

$$= x^3 + 6x^2y + 12xy^2 + 8y^3$$

b. The coefficient in row four are

Therefore $(x-2)^4 = \overset{1}{x^4} + \overset{4}{4x^3}(-2) + \overset{6}{6x^2}(-2)^2 + \overset{4}{4x}(-2)^3 + \overset{1}{(-2)^4}$

$$= x^4 - 8x^3 + 24x^2 - 32x + 16$$

Definitions:

1. Factorial of a number say n is defined as $n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$, where n is any counting number.

   For example $5! \rightarrow 5$ factorial

   Therefore $5! = 5\times 4\times 3\times 2\times 1 = 120$

   $4! = 4\times 3\times 2\times 1 = 24$

   $0! = 1$

   $1! = 1$

2. Permutations is the number of arrangements of r objects taken from n different ones and is defined as

   ${}^nP_r = n(n-1)(n-2)\cdots(n-r+1)$

   In general, permutation deals with objects

3. Combination deals with the selection of r objects chosen from n different objects. It's defined as

   $${}^nC_r = \frac{n!}{r!(n-r)!} \qquad (*)$$

   Where $n =$ The number of different object

   $r =$ The number chosen from n different objects

   Equation (*) is the general term of the binomial expansion

Example 2: In how many ways can four boys be chosen from six?

Solution

i. The number of permutation of four boys from six is

   ${}^nP_r = n(n-1)(n-2)\cdots(n-r+1)$

   ${}^6P_4 = 6(6-1)(6-2)(6-3)\cdots(6-4+1)$

   $= 6\times 5\times 4\times 3$

ii. The number of selections or combinations will be

$$^nC_r = \frac{n!}{r!(n-r)!} = \frac{6!}{2!4!} = 15$$

The Binomial theorem

This theorem gives the general expansion of a binomial $(x+a)^n$, where n can take any positive negative integer value and the formula is

$$(x+a)^n = {}^nC_0x^na^0 + {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}a^2 + \cdots + {}^nC_rx^{n-r}a^r + \cdots + {}^nC_nx^na^n$$

$$= x^n + nx^{n-1}a + \frac{n(n-1)x^{n-2}a^2}{2!} + \frac{n(n-1)(n-2)x^{n-3}a^3}{3!} + \cdots + a^r$$

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}a^r \qquad (1)$$

__NB__

1. Since r can have values from $0\ to\ n$, the total number of terms in the expression of $(x+a)^n$ is $(n+1)$ terms
2. Replacing a by –a in eqn (1) we get

   $$(x-a)^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}(-a)^r$$

   $$(x-a)^n = \sum_{r=0}^{n} (-1)^r\,{}^nC_rx^{n-r}(a)^r \qquad (2)$$

3. Add eqn (1) and (2)

   $$(x+a)^n + (x-a)^n = 2\left[{}^nC_0x^na^0 + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + \cdots\right] \qquad (3)$$

   And

   $$(x+a)^n + (x-a)^n = 2\left[{}^nC_1x^{n-1}a^1 + {}^nC_3x^{n-3}a^3 + {}^nC_5x^{n-5}a^5 + \cdots\right] \qquad (4)$$

__NB__

i. If n is odd then equations (3) and (4) both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$

ii. If n is even then equation (3) has $\left(\frac{n}{2}+1\right)$ term and equation (4) has $\left(\frac{n}{2}\right)$ term

Example 3: find the number of terms in the expansion of the following

i. $(3x+5y)^6$

ii. $(1+3x)^7+(1-3x)^7$

iii. $\left(\sqrt{2x}+\sqrt{3y}\right)^{10}+\left(\sqrt{2x}-\sqrt{3y}\right)^{10}$

iv. $\left[(x+6y)^8-(x-6y)^8\right]$

v. $\left(1+2x+x^2\right)^{20}$

Solution

i. The expansion of $(x+a)^n$ has $(n+1)$ terms

Therefore $(3x+5y)^8$ has $(6+1)=7\,terms$

ii. If n is odd, then the expansion of

$\left[(x+a)^n+(x-a)^n\right]$ $contains\left(\frac{n+1}{2}\right)$ $terms$

Therefore $(1+3x)^7+(1-3x)^7\ has\left(\frac{7+1}{2}\right)=4\,terms$

iii. If n is even number, then the expression of

$\left[(x+a)^n+(x-a)^n\right]$ $contains\left(\frac{n}{2}+1\right)$ $terms$

Therefore $\left(\sqrt{2x}+\sqrt{3y}\right)^{10}+\left(\sqrt{2x}-\sqrt{3y}\right)^{10}\ has\left(\frac{10}{2}+1\right)=6\,terms$

iv. $\left[(x+6y)^8-(x-6y)^8\right] has \left(\frac{n}{2}\right)=4 terms$

v. $\left(1+2x+x^2\right)^{20}$ has $(n+1) terms$

Therefore $\left[(1+x)^2\right]^{20}=(1+x)^{40}$

Then the term is $(n+1)=40+1=41 terms$

Example 4: Use the Binomial theorem to expand the following

i. $(2x+3)^5$

ii. $(2x-3y)^4$

iii. $\left(x+\frac{2}{x}\right)^3$

iv. $\left(2+x+x^2\right)^5$

v. $\left(1-\frac{2x}{3}\right)^5$

vi. $(1+x)^{-1}$

vii. $\sqrt{1+2x}$ and $\sqrt{1.03}$

Solution

i. $(2x+3)^5$, $note\ x=2x,\ a=3\ and n=5$

$$(x+a)^n=x^n+\frac{nx^{n-1}a}{1!}+\frac{n(n-1)x^{n-2}a^2}{2!}+\frac{n(n-1)(n-2)x^{n-3}a^3}{3!}+\frac{n(n-1)(n-2)(n-3)x^{n-4}a^4}{4!}$$

$$\Rightarrow(2x+3)^5=(2x)^5+\frac{(5)(2x)^{5-1}(3)}{1!}+\frac{(5)(5-1)(2x)^{5-2}(3)^2}{2!}+\frac{(5)(5-1)(5-2)(2x)^{5-3}(3)^3}{3!}+\frac{(5)(5-1)(5-2)(5-3)(2x)^{5-4}(3)^4}{4!}$$

$$(2x+3)^5=32x^5+240x^4+720x^3+1080x^2+810x$$

ii. $(2x-3y)^4=\sum_{r=0}^{n}{}^nC_rx^{n-r}(-a)^r, \Rightarrow n=4, x=2x, a=-3y$

$$\Rightarrow (2x-3y)^4 = {}^4C_0(2x)^{4-0}(-3y)^0 + {}^4C_1(2x)^{4-1}(-3y)^1 + {}^4C_2(2x)^{4-2}(-3y)^2 + {}^4C_3(2x)^{4-3}(-3y)^3 + {}^4C_4(2x)^{4-4}(-3y)^4$$

$$\Rightarrow (2x-3y)^4 = 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$$

iii. $\left(x+\frac{2}{x}\right)^3 = \sum_{r=0}^{n} {}^nC_r x^{n-r}(a)^r, \Rightarrow n=3, x=x, a=\frac{2}{x}$

$$\Rightarrow \left(x+\frac{2}{x}\right)^3 = {}^3C_0(x)^{3-0}\left(\frac{2}{x}\right)^0 + {}^3C_1(x)^{3-1}\left(\frac{2}{x}\right)^1 + {}^3C_2(x)^{3-2}\left(\frac{2}{x}\right)^2 + {}^3C_3(x)^{3-3}\left(\frac{2}{x}\right)^3$$

$$\Rightarrow \left(x+\frac{2}{x}\right)^3 = x^3 + 4x + 12\frac{1}{x} + 8\frac{1}{x^3}$$

iv. $\left(2+x+x^2\right)^5$

Let $y = x+x^2, \Rightarrow \left(2+x+x^2\right)^5 = (2+y)^5, \Rightarrow x=2, a=y, n=5$

$$\Rightarrow (2+y)^5 = {}^5C_0(2)^{5-0}(y)^0 + {}^5C_1(2)^{5-1}(y)^1 + {}^5C_2(2)^{5-2}(y)^2 + {}^5C_3(2x)^{5-3}(y)^3 + {}^5C_4(2)^{5-4}(y)^4 + {}^5C_5(2)^{5-5}(y)^5$$

$$(2+y)^5 = 32 + 80x + 160x^2 + 200x^3$$

v. $\left(1-\frac{2x}{3}\right)^5 \Rightarrow x=1,\ a=-\frac{2x}{3},\ n=5$

$$\left(1-\frac{2x}{3}\right)^5 = \sum_{r=0}^{5} {}^5C_r(1)^{5-r}\left(-\frac{2x}{3}\right)^r$$

$$\left(1-\frac{2x}{3}\right)^5 = \sum_{r=0}^{5} {}^5C_r\left(-\frac{2x}{3}\right)^r$$

$$\Rightarrow \left(1-\frac{2x}{3}\right)^5 = {}^5C_0\left(-\frac{2x}{3}\right)^0 + {}^5C_1\left(-\frac{2x}{3}\right)^1 + {}^5C_2\left(-\frac{2x}{3}\right)^2 + {}^5C_3\left(-\frac{2x}{3}\right)^3 + {}^5C_4\left(-\frac{2x}{3}\right)^3 + {}^5C_5\left(-\frac{2x}{3}\right)^5$$

$$\Rightarrow \left(1-\frac{2x}{3}\right)^5 = 1 - \frac{10}{3}x + \frac{40}{9}x^2 - \frac{80}{27}x^3 + \frac{80}{81}x^4 - \frac{32}{243}x^5$$

NB

The binomial expansion of any negative or fractional index is given by

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots$$

vi. $(1+x)^{-1}$, using the above formula $n = -1$

$$(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-1-1)x^2}{2!} + \frac{(-1)(-1-1)(-1-2)x^3}{3!} + \cdots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \pm \cdots$$

vii. Obtain the first five terms of the expansion of $\sqrt{1+2x}$ and evaluate $\sqrt{1.03}$ to 5 significant figures

Solution

$\sqrt{1+2x} = (1+2x)^{\frac{1}{2}}, \Rightarrow n = \frac{1}{2},\ x = 2x$

$$(1+2x)^{\frac{1}{2}} = 1 + \left(\tfrac{1}{2}\right)(2x) + \frac{\left(\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)-1\right)(2x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)-1\right)\left(\left(\frac{1}{2}\right)-2\right)(2x)^3}{3!} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(2x)^4}{4!}$$

$$(1+2x)^{\frac{1}{2}} = 1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 - \tfrac{5}{8}x^4 \qquad (*)$$

Next to evaluate $\sqrt{1.03}$, we have

$$\sqrt{1.03} = (1.03)^{\frac{1}{2}} \qquad (**)$$

Evaluate $1 + x = 1.03, \Rightarrow x = 0.03$

We could re write $(1.03)^{\frac{1}{2}} = (1+0.03)^{\frac{1}{2}}$

Now comparing $(1+2x)^{\frac{1}{2}} = (1+0.03)^{\frac{1}{2}}, \Rightarrow 2x = 0.03, \Rightarrow x = 0.015$

But from equation (*)

$$(1+2x)^{\frac{1}{2}} = 1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 - \tfrac{5}{8}x^4$$

$$(1+2x)^{\frac{1}{2}} = 1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 - \tfrac{5}{8}x^4$$

Substitute $x = 0.015$

$(1+2x)^{\frac{1}{2}} = 1+(0.015)-\frac{1}{2}(0.015)^2+\frac{1}{2}(0.015)^3-\frac{5}{8}(0.015)^4$

$(1+2x)^{\frac{1}{2}} = 1.0149$, 5 significant figures

General term in a binomial expansion

$T_{r+1} = {}^{n}C_r x^{n-r} a^r$

Finding m^{th} term

In finding m^{th} term from the given expansion r must be found from $r = m-1$

Example 5:

i. Find the 6^{th} term in the expansion of $(x+y)^{15}$

Solution

$(x+y)^{15}, \Rightarrow n=15, x=x, a=y, let\, m=6, \Rightarrow r=m-1=6-1=5$

$T_{r+1} = {}^{n}C_r x^{n-r} a^r$

$T_6 = {}^{15}C_5 x^{15-5} y^5$

$T_6 = 3003x^{10}y^5$

ii. Find the 8^{th} term in the expansion of $\left(\frac{2}{x}+3x^2\right)^{15}$

Solution

$\left(\frac{2}{x}+3x^2\right)^{15}, \Rightarrow n=15, x=\frac{2}{x}, a=3x^2, let\, m=8, \Rightarrow r=m-1=8-1=7$

$T_{r+1} = {}^{n}C_r x^{n-r} a^r$

$T_8 = {}^{15}C_7\left(\frac{2}{x}\right)^{15-7}\left(3x^2\right)^7$

$T_8 = 3602776220x^6$

iii. Find the 11^{th} term in the expansion of $\left(\frac{2}{3}x^2 - \frac{3}{2x}\right)^{20}$

Solution

$\left(\frac{2}{3}x^2 - \frac{3}{2x}\right)^{20}$, $\Rightarrow n = 20,\ x = \frac{2}{3}x^2,\ a = \frac{3}{2x}, let\ m = 11, \Rightarrow r = m - 1 = 11 - 1 = 10$

$T_{r+1} = {}^nC_r x^{n-r} a^r$

$T_{11} = {}^{20}C_{11}\left(\frac{2}{3}x^2\right)^{20-10}\left(\frac{3}{2x}\right)^{10}$

$T_{10} = 184756x^{10}$

Finding the coefficients

Example 6: Find the coefficient of the following

i. $x^{12}\ in\left(2x+3\right)^{15}$

ii. $x^{10}\ in\left(2x^2 - \frac{3}{x}\right)^{11}$

iii. $x^{12}\ in\left(\frac{x^2}{3} - \frac{3}{x^3}\right)^6$

Solution

i. $x^{12}\ in\left(2x+3\right)^{15}$

$\left(2x+3\right)^{15}$, $\Rightarrow n = 15,\ x = 2x,\ a = 3,$

$T_{r+1} = {}^nC_r x^{n-r} a^r$

$= {}^{15}C_r \left(2x\right)^{15-r} \left(3\right)^r$

$= {}^{15}C_r \left(2\right)^{15-r} \left(x\right)^{15-r} \left(3\right)^r \qquad (1)$

This term will contain x^{12}, if $x^{15-r} = x^{12}, \Rightarrow 15 - r = 12, \Rightarrow r = 3$

So that $(3+1)^{th}$ i.e. 4^{th} term contains x^{12}

Putting $r = 3$ into eqn (1) we get

$T_{3+1} = {}^{15}C_3 \left(2\right)^{15-3} \left(x\right)^{15-3} \left(3\right)^3$

$T_4 = 50319360x^{12}$

Hence the coefficient of x^{12} *is* 50319360

ii. The coefficient of x^{10} *in* $\left(2x^2 - \frac{3}{x}\right)^{11}$

$\left(2x^2 - \frac{3}{x}\right)^{11}, \Rightarrow n = 11,\ x = 2x^2,\ a = -\frac{3}{x},$

$T_{r+1} = {}^{n}C_r x^{n-r} a^r$

$= {}^{11}C_r \left(2x^2\right)^{11-r} \left(\frac{3}{x}\right)^r$

$= (-1)^r\, {}^{11}C_r \left(2\right)^{11-r} \left(x^2\right)^{11-r} \left(x^{-1}\right)^r \left(3\right)^r$

$= (-1)^r\, {}^{11}C_r \left(2\right)^{11-r} \left(x\right)^{22-2r} \left(x\right)^{-r} \left(3\right)^r$

$= (-1)^r\, {}^{11}C_r \left(2\right)^{11-r} \left(x\right)^{22-3r} \left(3\right)^r \qquad (1)$

This term will contain x^{10}, if $x^{22-3r} = x^{10}, \Rightarrow 22 - 3r = 10, \Rightarrow r = 4$

So that $(4+1)^{th}$ i.e. 5^{th} term contains x^{10}

Putting $r = 4$ into eqn (1) we get

$T_{4+1} = (-1)^4\, {}^{11}C_4\, (2)^{11-4} (x)^{22-3(4)} (3)^4$

$T_5 = 50319360x^{10}$

Hence the coefficient of x^{10} *is* 50319360

iii. The coefficient of x^2 *in* $\left(\frac{x^2}{3} - \frac{2}{x^3}\right)^6$

$\left(\frac{x^2}{3} - \frac{2}{x^3}\right)^6$, $\Rightarrow n = 6,\ x = \frac{x^2}{3},\ a = -\frac{2}{x^3},$

$T_{r+1} = {}^nC_r x^{n-r} a^r$

$= {}^6C_r \left(\frac{x^2}{3}\right)^{6-r} \left(-\frac{2}{x^3}\right)^r$

$= (-1)^r\, {}^6C_r \left(x^2\right)^{6-r} \left(3^{-1}\right)^{6-r} (2)^r \left(x^{-3}\right)^r$

$= (-1)^r\, {}^6C_r\, (x)^{12-2r} (3)^{-6+r} (2)^r (x)^{-3r}$

$= (-2)^r\, {}^6C_r \left(\frac{1}{3}\right)^{6-r} (x)^{12-5r}$ (1)

This term will contain x^2, if $x^{12-5r} = x^2, \Rightarrow 12 - 5r = 2, \Rightarrow r = 2$

So that $(2+1)^{th}$ i.e. 3^{th} term contains x^2

Putting $r = 2$ into eqn (1) we get

$T_{2+1} = (-2)^2\, {}^6C_2 \left(\frac{1}{3}\right)^{6-2} (x)^{12-5(2)}$

$T_3 = \dfrac{20}{27} x^2$

Hence the coefficient of x^2 *is* $\dfrac{20}{27}$

THEORY OF QUADRATIC EQUATION

The solution of the general quadratic equation $ax^2 + bx + c = 0$ whereby a,b and c are real number is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (1)$$

$\Rightarrow$

$$x = \frac{-b \pm \sqrt{D}}{2a} \qquad (2)$$

Where D=$b^2 - 4ac$

The symbol D is called the discriminant of the equation and it enables us to investigate the dependence of the roots on the values of a,b and c.

Types of roots of quadratic equation

(1) If D is positive i.e D>0, it is square roots will be real number and we shall obtain two different real roots of the equation.

E.g $2x^2 - 3x - 1 = 0$

$\Rightarrow$

$a = 2, \quad b = 3, \quad c = -1$

$\therefore D = b^2 - 4ac$

$= (-3)^2 - 4(2)(-1)$

$= 9 + 8 = 17$

$\Rightarrow D = 17$

$\Rightarrow D = 0$

$\therefore$ We shall obtain two different real roots of the equation.

$$\therefore x = \frac{-b \pm \sqrt{D}}{2a}$$

i.e $x = \frac{-(-3) \pm \sqrt{17}}{4}$

$$= \frac{3 \pm \sqrt{17}}{4}$$

(2) If D is negative i.e D<0, it is square roots is that of a negative number and therefore the equation has complex roots.

e.g $2x^2 - x + 3 = 0$

$a = 2, b = -1, c = 3$

$\therefore D = b^2 - 4ac$

$= (-1)^2 - 4(2)(3)$
$= -23$

$D = -23$

$\rightarrow D<0$

$$\therefore x = \frac{-b \pm \sqrt{D}}{2a}$$

$$= \frac{-(-1) \pm \sqrt{-23}}{4}$$

$= \frac{1 \pm i\sqrt{3}}{4}$

So the equation has complex roots.

(3) If D=0, its square root will be real and both roots of the equation will have equal real roots and equal to $\frac{-b}{2a}$

e.g $4x^2 - 12x + 9 = 0$

$\Rightarrow D = b^2 - 4ac = (-12)^2 - 4(4)(9) = 0$

$\Rightarrow D = 0$

Therefore the equation will be real and equal $\frac{-b}{2a}$

i.e $x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-b}{2a} = \frac{-(-12)}{2 \times 4} = \frac{12}{8} = \frac{3}{2}$

Examples

(1) Find the value of M so that the equation $4x^2 - 8x + m = 0$ shall have equal roots.

Solution

For the equation to have equal roots D=0

$\Rightarrow b^2 - 4ac = 0$ ______________________________ (1)

But $a = 4, b = -8, c = m$ ______________________________ (2)

Substitute (2) in (1), we have

$(-8)^2 4(4)(m) = 0$
$64 - 16m = 0$
$64 = 16m$
$\Rightarrow m = 4$

So our quadratic equation will now be $4x^2 - 8x + 4 = 0$

So to set the roots we have

$$x = \frac{-b}{2a} + 0$$
$$x = \frac{-b}{2a}$$
$$x = \frac{-(8)}{2\times 4}$$
$$= \frac{8}{8} = 1$$

Factorizing we have $4x^2 - 4x - 4x + 4 = 0$

$$\Rightarrow 4x - (x-1) - 4(x-1) = 0$$
$$\Rightarrow (x-1)(4x-4) = 0$$

Either $x - 1 = 0$ or $4x - 4 = 0$

$\Rightarrow x = 1, x = 1$

Therefore quadratic equation have equal roots

(2) Find the values of P so that the equation $x^2 - (p-2)x + (2p+1) = 0$ shall have equal roots.

Solution

For the equation to have equal roots

D=0

$\therefore b^2 - 4ac = 0$, where

$a = 1, b = -(p-2), c = 2p+1$

$\Rightarrow (-(p-2))^2 - 4(1)(2p+1) = 0$
$\Rightarrow (p-2)^2 - 4(2p+1) = 0$
$\Rightarrow p^2 - 4p + 4 - 8p - 4 = 0$
$\Rightarrow p^2 - 12p = 0$
$\Rightarrow p(p-12) = 0$

$\Rightarrow p = 0$ or $p = 12$

So to set the root of $p = 0$

So if $p = 0$, our quadratic equation will be

$x^2 - (0-2)x + (2(0)+1) = 0$
$x^2 - (-2)x + 1 = 0$
$x^2 + 2x + 1 = 0$

Factorize

$x^2 + x + x + 1 = 0$
$\Rightarrow x(x+1) + 1(x+1) = 0$
$\Rightarrow (x+1)(x+1) = 0$

$\Rightarrow x = -1$ twice

If $p = 12$, our quadratic equation will be

$x^2 - (12-2)x + (2(12)+1) = 0$

$x^2 - 10x + 25 = 0$
$\Rightarrow (x-5)(x-5) = 0$

$\Rightarrow x = 5$ twice

(3) If the equation $x^2 - 3x + 25 = q(x-3)$ has equal roots, find the possible values of q

Solution

$x^2 - 3x + 1 = q(x-3)$

Rewriting the equation in standard form we have

$x^2 - 3x + 1 - q(x-3) = 0$

$$\Rightarrow x^2 - 3x + 1 - qx + 3q = 0$$
$$\Rightarrow x^2 - 3x - qx + 1 + 3q = 0$$
$$\Rightarrow x^2 - x(3+q) + (1+3q) = 0$$

Complete with $ax^2 + bx + c$

$\Rightarrow a = 1, b = -(3+q), c = 1 + 3q$

For the equation to here equal roots $D = 0$

$\Rightarrow b^2 - 4ac = 0,$

$$\Rightarrow \left(-(3+q)\right)^2 - 4(1)(1+3q) = 0$$
$$\Rightarrow (3+q)^2 - 4 - 12q = 0$$
$$\Rightarrow q^2 + 6q + 9 - 4 - 12q = 0$$

$$\Rightarrow q^2 - 6q + 5 = 0$$
$$\Rightarrow q^2 + q - q + 5 = 0$$
$$q(q-5) - 1(q-5) = 0$$
$$(q-5)(q-1) = 0$$

$\Rightarrow q = 5$ or 1

Exercise

(1) What is the value of a such that the equation $(3a+1)x^2+(a+2)x+1=0$? shall have equal roots.

(2) Find the values of $\underline{\lambda}$ for the roots of the equation $x^3-(3\lambda+1)x^2+(5p-1)x+p=1$ has equal roots.

(3) Find P if the equation $(7p+1)x^2+(5p-1)x+p=1$ has equal roots.

(4) Find a if the equation_ $(5a+1)x^2-8ax+3a=0$ has equal roots.

(5) Show that the equation $x^2-2px+p^2-q^2=0$ has real roots provide p ad q are both real.

(6) For what values of k does the equation $x^2-(4+k)x+9=0$ has equal roots.

SUM AND PRODUCT OF ROOTS OF QUADRATIC EQUATION

If the equation $ax^2+bx+c=0$ has roots α and β, then its equivalent to the equation.

$(x-\alpha)(x-\beta)=0$, as this gives $x=\alpha, x=\beta$ expanding $(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta$. For any quadratic equation $ax^2+bx+c=0$ with roots α and β then,

(1) The sum of the roots of quadratic equation is $\alpha+\beta=\frac{-b}{a}$

(2) The product of the roots of quadratic equation is $\alpha.\beta=\frac{c}{a}$

In general, the above formulas give rise to two problems

(1) Given a quadratic equation we can find the sum and product of the roots of the quadratic equation.

(2) Given the roots of a quadratic equation we can formulate or construct corresponding quadratic equation

Example

(1) Find the sum and product the following quadratic equation

(1) $2x^2+3x-1=0$

Solution

$2x^2+3x-1=0$

$\Rightarrow a=2, b=3, c=-1$

$\therefore$ Sum $\alpha+\beta=\frac{-b}{a}=\frac{-3}{2}$

Product $\alpha\beta=\frac{c}{a}=\frac{-1}{2}$

(2) $pt^2-qt-r=0$

Solution

$a=p, b=-q, c=-r$

$\therefore \alpha+\beta=\frac{-b}{a}=\frac{-(-q)}{p}=\frac{q}{p}$

$\alpha\beta=\frac{c}{a}=\frac{-r}{p}$

(3) $x^2+\sqrt{3x}+1$

Solution

$a = 1, b\sqrt{3}, c = 1$

$\therefore \alpha + \beta = \frac{-b}{a} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$

$\alpha\beta = \frac{c}{a} = \frac{1}{1} = 1$

Example 2

Construct equations whose roots are

(1) 3 and -2 (2) $\frac{3}{4}$ and $\frac{-1}{2}$ (3) $\sin^2\theta, \cos^2\theta$

Solution

(1) 3 and 2

$\Rightarrow \alpha = 3, \beta = -2$

$\therefore$ Sum of the roots $\alpha + \beta = 3 + (-2) = 1$ and product of the roots $\alpha\beta = (3)(-2) = -6$

$\therefore$ To construct the equation, we have

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$
$$\Rightarrow x^2 - (1)x + (-6) = 0$$
$$\Rightarrow x^2 - x - 6 = 0$$

(2) $\frac{3}{4}$ and $\frac{-1}{2}$

$$\Rightarrow \alpha = \frac{3}{4}, \beta = \frac{-1}{2}$$

$$\therefore \alpha + \beta = \frac{3}{4} + (\frac{-1}{2}) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

$$\alpha\beta = (\frac{3}{4})(\frac{-1}{2}) = \frac{-3}{8}$$

$$\Rightarrow x^2 - \frac{1}{4}x - \frac{3}{8} = 0$$

$$\Rightarrow 8x^2 - 2x - 3 = 0$$

(3) $\sin^2\theta, \cos^2\theta$

$$\Rightarrow \alpha = \sin^2\theta, \beta = \cos^2\theta$$

$$\therefore \alpha + \beta = \sin^2\theta + \cos^2\theta = 1$$

$$\alpha\beta = \left(\sin^2\theta\right)\left(\cos^2\theta\right) = \sin^2\theta\cos^2\theta$$

$$\Rightarrow x^2 - \left(\alpha + \beta\right)x + \alpha\beta\beta = 0$$

$$\Rightarrow x^2 - x + \sin^2\theta\cos^2\theta = 0$$

(4) Given that the roots of the equation $2x^2 + 2x - 3 = 0$ are α and β, find the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

Solution

Given $2x^2 + 2x - 3 = 0$

$$a = 2, b = 2, c = -3$$

$$\alpha + \beta = \frac{-b}{a} = \frac{-2}{2} = -1$$

$$\alpha\beta = \frac{c}{a} = \frac{-3}{2}$$

$\therefore$ Sum $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=-1/\frac{-3}{2}=\frac{2}{3}$

Product $\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)=\frac{1}{\alpha\beta}=1/\frac{-3}{2}=\frac{-2}{3}$

$\therefore x^2-\left(\frac{\alpha+\beta}{\alpha\beta}\right)x+\alpha\beta=0$

$\Rightarrow 3x^2-2x-2=0$

Symmetric properties

Knowing the values of $\alpha+\beta$ and $\alpha\beta$ for a given equation, we can calculate the values of other functions of α and β provided they are symmetric.

A symmetric function of α and β is one in which if α and β are interchanged, if the results is the same or is multiplied by -1 e.g $\alpha^2+\beta^2$ is a symmetric function as it becomes $\beta^2+\alpha^2$. Similarly $\alpha^2-\beta^2$ become $-(\beta^2-\alpha^2)$, so that $\alpha^2-\beta^2$ is also symmetric.

Example

If α and β are the roots of $x^2+3x-5=0$, calculate the values of the following;

(1) $\alpha+\beta$ (2) $\alpha\beta$ (3) $\alpha-\beta$ (4) $\alpha^2+\beta^2$ (5) $\alpha^3+\beta^3$ (6) $\frac{1}{\alpha}+\frac{1}{\beta}$ (7) $\alpha^3-\beta^3$ (8) $\alpha^3-\beta^2$ (9) $\frac{1}{\alpha+1}+\frac{1}{\beta+1}$ (10) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

Solutions

From the given equation $x^2+3x-5=0$, we know that $a=1, b=3, c=-5$

(1) $\alpha + \beta = \frac{-b}{a} = \frac{-3}{1} = -3$

(2) $\alpha\beta = \frac{c}{a} = \frac{-5}{1} = -5$

(3) $\alpha - \beta$ can not be found directly, so we use

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

$$\begin{aligned} &= (-3)^2 - 4(-5) \\ &= 9 + 20 \\ &= 29 \end{aligned}$$

$\therefore \alpha - \beta = \pm\sqrt{29}$ depending on whether $\alpha > \beta$ or $\beta > \alpha$

(4) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$\begin{aligned} &= (-3)^2 - 2(-5) \\ &= 9 + 10 \\ &= 19 \end{aligned}$$

(5) $\alpha^3 + \beta^3 = (\alpha + \beta)\left[(\alpha + \beta)^2 - 3\alpha\beta\right]$

$$\begin{aligned} &= (-3)\left[(-3)^2 - 3(-5)\right] \\ &= -3(9 + 15) \\ &= -3(24) \\ &= -72 \end{aligned}$$

(6) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-3}{-5} = \frac{3}{5}$

(7) $\alpha^3 - \beta^3 = (\alpha - \beta)\left[(\alpha + \beta)^2 - \alpha\beta\right]$

$$= +\sqrt{29}\left[(-3)^2 - (-5)\right]$$

$= \sqrt{29}(9+5)$

$= 75$

(8) $\alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta)$

$= (-3)(\sqrt{29})$

$= -16$

(9) $\frac{1}{\alpha+1} + \frac{1}{\beta+1} = \frac{\beta+1+\alpha+1}{(\alpha+1)(\beta+1)} = \frac{(\alpha+\beta)+2}{\alpha\beta+(\alpha+\beta)+1} = \frac{-3+2}{-5+(-3)+1} = \frac{-1}{-5-3+1} = \frac{-1}{-7} = \frac{1}{7}$

(10) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{19}{-5} = \frac{-19}{5}$

<u>Further example</u>

(1) If α, β are the roots of $2x^2 - 2x - 1 = 0$, form the equation whose roots are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$.

<u>Solution</u>

We use the form x^2 - (sum of the) x+ (product of roots)=0, from the given equation $a = 2, b = -2, c = -1$

$\therefore \alpha + \beta = \frac{-b}{a} = \frac{-2}{2} = -1$

$\alpha\beta = \frac{c}{a} = \frac{-1}{2}$

Now sum of new roots $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$

But $\alpha^2 + \beta = (\alpha+\beta)^2 - 2\alpha\beta$

$$=\frac{(-1)^2-2(\frac{-1}{2})}{-\frac{1}{2}}=\frac{1+1}{-\frac{1}{2}}=2.(\frac{-2}{1})=-4$$

And

Product of new roots $\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)=\frac{\alpha\beta}{\beta\alpha}=\frac{\alpha\beta}{\alpha\beta}=\frac{-\frac{1}{2}}{-\frac{1}{2}}=1$

So the new equation is x^2 -(sum of new roots) x (product of new roots)

$\rightarrow x^2+4x+1=0$

(2) Given that α,β are the roots of the equation $3x^2-x-5=0$, form the equation whose roots are $2\alpha-\frac{1}{\beta},2\beta-\frac{1}{\alpha}$

<u>Solution</u>

$3x^2-x-5=0, a=3, b=-1, c=-5$

$\therefore \alpha+\beta=-\frac{b}{a}=-\left(\frac{-1}{3}\right)=\frac{1}{3}, \alpha\beta=\frac{c}{a}=\frac{-5}{3}$

Sum of new roots $2\alpha-\frac{1}{\beta}+2\beta-\frac{1}{\alpha}$

$$= \frac{2\beta\alpha - 1}{\beta} + \frac{2\beta\alpha - 1}{\alpha}$$

$$= \frac{\alpha(2\alpha\beta - 1) + \beta(2\beta\alpha - 1)}{\alpha\beta}$$

$$= \frac{= 2\alpha^2\beta - \alpha + 2\beta^2\alpha - \beta}{\alpha\beta}$$

$$= \frac{2\alpha\beta(\alpha + \beta) - (\alpha + \beta)}{\alpha\beta}$$

$$= \frac{2\left(\frac{-5}{3}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{3}\right)}{\frac{-5}{3}}$$

$$= \frac{-10}{9} - \frac{1}{3} \times \frac{-3}{5}$$

$$= \frac{-10 - 3}{9} \times \frac{-3}{5} = \frac{-13}{9} \times \frac{-3}{5} = \frac{13}{15}$$

Product of new roots $\left(2\alpha - \frac{1}{\beta}\right)\left(2\beta - \frac{1}{\alpha}\right)$

$$= 4\alpha\beta - \frac{2\alpha}{\alpha} - \frac{2\beta}{\beta} + \frac{1}{\alpha\beta}$$

$$= 4\alpha\beta - 2 - 2 + \frac{1}{\alpha\beta}$$

$$= 4\alpha\beta - 4 + \frac{1}{\alpha\beta}$$

$$= 4\left(\frac{-5}{3}\right) - 4 + \frac{1}{5/3}$$

$$= \frac{-20}{3} - 4 - \frac{3}{5}$$

$$= -\frac{169}{15}$$

Therefore the new equation will be x^2 -(sum of new roots) x +(product of new roots)

$$x^2-\frac{13}{15}x-\frac{169}{15}=0$$
$$\Rightarrow 15x^2-13x-169=0$$

(3) One root of the equation $2x^2-x=c=0$ is double the roots find C

Solution

Given $2x^2-x=c=0$

$\Rightarrow a=2, b=1, c=c$

Let the roots be α and 2α.

Then the sum of the roots;

$$\alpha+2\alpha=-\frac{b}{a}$$

$$\Rightarrow 3\alpha=\frac{-(-1)}{2}=\frac{1}{2}$$

$$\Rightarrow 3\alpha=\frac{1}{2} \qquad \text{———————— (1)}$$

Product of the roots

$$(\alpha)(2\alpha)=\frac{c}{a}$$

$$2\alpha^2=\frac{c}{a} \qquad \text{———————— (2)}$$

From the equation (2)

$$4\alpha^2 = c$$

$$\alpha^2 = \frac{c}{4}$$

$$\alpha = \sqrt{\frac{c}{4}} = \frac{\sqrt{c}}{2}$$

$$\Rightarrow \alpha = \frac{\sqrt{c}}{2} \qquad \text{——————} \quad (3)$$

Substitute equation (3) in (1)

$$3\left(\frac{\sqrt{c}}{2}\right) = \frac{1}{2}$$

$$\frac{3}{2}\sqrt{c} = \frac{1}{2}$$

$$\sqrt{c} = \frac{1}{2} \times \frac{2}{3}$$ (multiply both sides by 2/3)

$$\sqrt{c} = \frac{2}{6}$$

$$\sqrt{c} = \frac{2}{6}$$ (square both sides)

$$c = \frac{4}{36}$$

$$c = \frac{1}{9}$$

COMPLEX NUMBER

Complex Numbers: An ordered pair of real number such as (x, y) is termed as a complex number if we write: $Z = (x, y)$ or $x + iy$, where $i = \sqrt{-1}$ then x is called real part, and y is imaginary part of the Complex Number Z.

$x = R_z$ or $R(Z)$ or $\text{Re}(z)$ and

$y = I_z$ or $I(z)$ or $\text{Im}(z)$

The Symbol (i)

Evidently the system of real numbers is not sufficient for all mathematical needs example there is no real number (Rational /Irrational) which satisfies $x^2 + 1 = 0$. Euler for the first time introduced the symbol (i) with property $i^2 = -1$ and Gauss introduced a number of $x + iy$ which satisfies every algebraic equation with real coefficient.

Powers of i

Since i represents $\sqrt{-1}$ it is clear that

(i) $i^2 = -1$ (II) $i^3 = i^2 \cdot i = 1 \cdot i = -i$ (iii) $i^4 = i^2 \cdot i^2 = 1$ (iv) $i^5 = i^4 \cdot i = i$

Note: every time a factor i^4 occurs, it can be replaced by the factor 1.

$i^9 = (i^4)^2 \cdot i = 1^2 \cdot i = i$

$i^{20} = (i^4)^5 = (1)^5 = 1$

$i^{30} = (i^4)^7 \cdot i^2 = (1)^7 \cdot (-1) = -1$

Try

a) i^{42} b) i^{12} c) i^{11}

Negative Integer Powers

Negative powers follow the reciprocal of i because $i^2 = -1$ by dividing both sides by i $\frac{i^2}{i} = \frac{-1}{i} = -i^{-1}$ so that $i^{-1} = -i$

i) $i^{-2} = (i^2)^{-1} = (-1)^{-1} = -1$

ii) $i^{-3} = (i^{-2})i^{-1} = (-1)(-i) = i$

iii) $i^{-4} = (i^{-2})^2 = (-1)^2 = 1$

Try

(i) i^{-8} (ii) i^{19} (iii) i^{-30}

Equality of Complex Number

Two Complex numbers (x, y) and (x', y') are equal iff $x = x'$, $y = y'$

Modulus of a Complex Number

If $z = x + iy$ be a complex number then its modulus (module) is denoted by

$|z| = |x + iy| = \sqrt{x^2 + y^2}$ Evidently

$|z| = 0 \Leftrightarrow x = 0, y = 0$

Complex numbers have many application in engineering and science. To use them, we must know how to carry out the usual arithmetical operations.

Given A Complex Numbers

$Z_1 = (x_1, y_1)$ $Z_2 = (x_2, y_2)$ $Z_3 = (x_3, y_3)$ we define the following operations:

1.Addition and Subtraction of Complex Number

Problem 1: If $Z_1 = (4+5i)$ and $Z_2 = (3-2i)$, find $Z_1 + Z_2$

Solution

$Z_1 + Z_2 = (4+5i) + (3-2i)$

Add the real parts together and imaginary parts together, and this gives

$7+3i$

Problem 2 If $Z_1 = 4+7i$ and $Z_2 = 2-5i$, then find $Z_1 + Z_2$ and $Z_1 - Z_2$

Solution

$Z_1 + Z_2 = (4+7i)+(2-5i)$

$=6+2i$

$Z_1 - Z_2 = (4+7i)-(2-5)$

$= 2+12i$

Problem 3 : If $Z_1 = 5+7i, Z_2 = 3-4i, Z_3 = 6-3i$ then find the value of $Z_1 + Z_2 - Z_3$.

Solution

$$Z_1 + Z_2 - Z_3 = (5+7i)+(3-4i)-(6-3i)$$

$$= 2+6i$$

Play Questions

If $Z_1 = 6+5i, Z_2 = 4-3i, Z_3 = 2-7i$. Find :

I) $Z_1 - Z_2 - Z_3$

II) $Z_1 - Z_2 + Z_3$

2. Multiplication of complex numbers

Problem 4: Given that $Z_1 = 3+4i$ $Z_2 = 2+5i$ find Z_1Z_2

Solution

$$(3+4i)(2+5i) = -14+23i$$

Problem 5: Given that $Z_1 = (4-5i), Z_2 = (3+2i)$ find Z_1Z_2

Solution

$$(4-5i)(3+2i)$$

$$= 22-7i$$

Play Questions

(1) Given that $Z_1 = (26-7i)$ $Z_2 = (1-2i)$ find Z_1Z_2

(2) $Z_1 = (3+4i) Z_2 = (2-5i)$ $Z_3 = (1-2i)$ find $Z_1Z_2Z_3$

Problem 6: Did the product of $(7-6i)$ *and* $(4+3i)$ gives purely

Solution

$(7-6i)(4+3i)=8i$ which is purely real

3. Division

Consider an equation $Z_1Z_2=Z'$ where $Z_1=(x_1,y_1), Z_2=(x_2,y_2)$ *and* $Z'=(x',y')$

Now $Z_1Z_2=(x_1x_2-y_1y_2, x_1y_2+x_2y_1)=Z'=(x',y')$ which gives

$$x_1x_2-y_1y_2=x'$$
$$x_1y_2+x_2y_1=y'$$

Solving $x_2=\frac{y_1y'-x_1x'}{x_1^2+y_1^2},\frac{x_1y'-x'y_1}{x_1^2+y_1^2}$ provided that $x_1^2+y_1^2\neq 0$ i.e $|Z_1|\neq 0$ thus we have a unique solution $Z_2=\frac{z'}{z'}$ is the quotient.

Conjugate Complex Numbers

If $Z=(a+ib)$ then $a-ib$ is said to be the conjugate of complex number Z and denoted by $\overline{Z}$

Evidently $\overline{Z_1+Z_2}=\overline{Z_1}+\overline{Z_2}, \overline{Z_1Z_2}=\overline{Z_1}+\overline{Z_2}$

$$Z\overline{Z}=(x+iy)(x-iy)=x^2+y^2=|z|^2$$

Problem 7. Multiply $(3-5i)$ by a suitable factor to give a product that is entirely real.

Solution

To obtain a real product, we can multiply (3-5i) by its conjugate

$(3-5i)(3+5i)=34$

Problem 8: Evaluate $\frac{7-4i}{4+3i}$

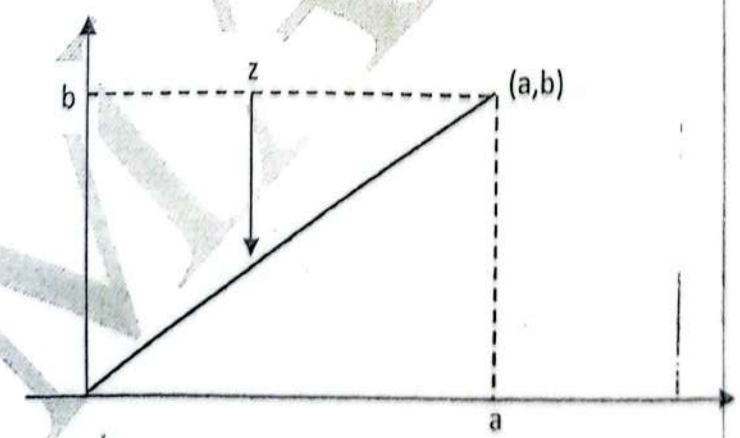
Solution

We multiply by the conjugate of the denominator

$$\frac{7-4i}{4+3i} \bullet \frac{4-3i}{4-3i} = \frac{28-21i-16i-12}{16+9} = \frac{16-37i}{25}$$

Graphical Representation of Complex Number (Argand Diagram)

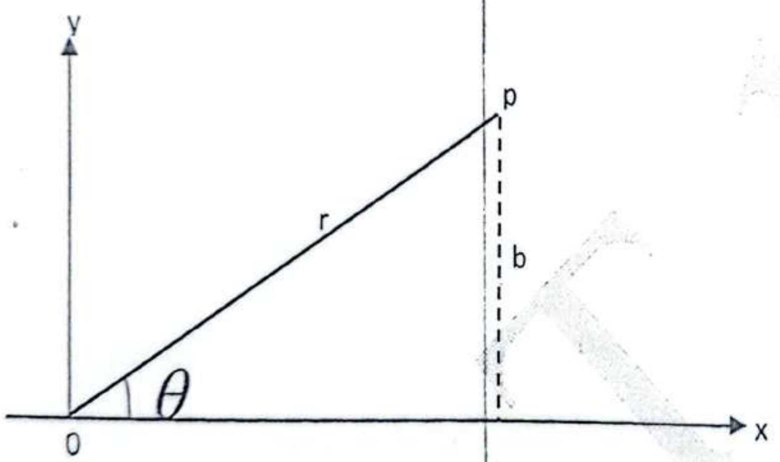
So far we have looked at the algebra of complex numbers, now we want to look at geometry. A real number can be graphically represented as a point on a line. The real line by using a Cartesian coordinate system, a pair of real numbers can be graphically represented by a point in the plane. French Mathematician Jean- Robert Argand devised a means of representing a complex number using the same Cartesian coordinates system.



This straight line is then the graphical representation of complex number Z= a+ib and the plane its plotted against is referred to as complex plane and the diagram is called the Argand diagram. The two axes are real and imaginary axes respectively but note that its real numbers that are plotted on both axes.

POLAR FORM OF A COMPLEX NUMBER

It is convenient sometimes to express a complex number a+ib in a different form. On an Argand diagram, let OP be a complex number $a + ib$, let r= length of the complex number and θ be the angle made with OX



Then $r^2 = a^2 + b^2$ such that $r = \sqrt{a^2 + b^2}$

$$\tan\theta = \frac{b}{a}$$

And $\theta = \tan^{-1}\frac{b}{a}$

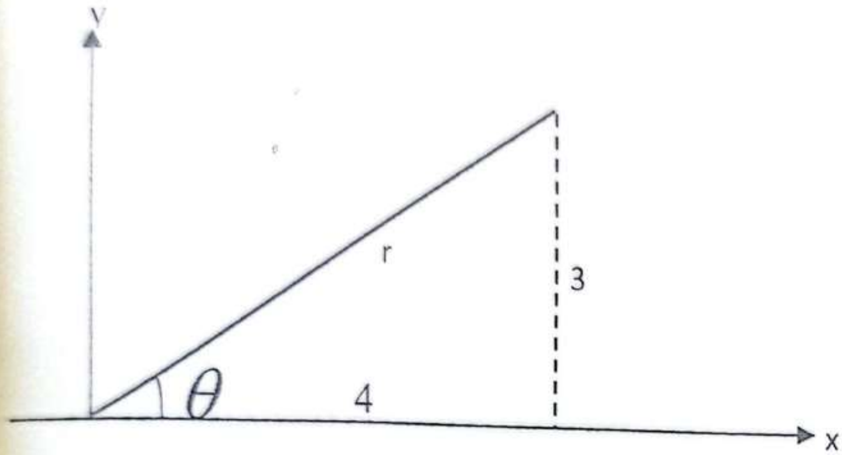
$$a = r\cos\theta, b = r\sin\theta$$
$$\sin ce\ Z = r\cos\theta + i\sin\theta$$
$$Z = r(\cos\theta + i\sin\theta)$$

This is called the polar form of the complex number a + bi, where

$r = \sqrt{a^2 + b^2}\ and\ \theta = \tan^{-1}\frac{b}{a}$

Problem 9: Express $Z = 4 + 3i$ in polar form

Solution



$r^2 = 4^2 + 3^2 = 25, r = 5$

$\tan\theta = \frac{3}{4}, \theta = 36.87$

$Z = a + ib = r(\cos\theta + i\sin\theta)$

$Z = 5(\cos 36.87 + i\sin 36.87)$

Problem 10: Find the polar form of complex number (2+3i)

Solution

$$Z = r(\cos\theta + i\sin\theta)$$
$$r^2 = 2^2 + 3^2 = 13$$
$$r = 3.606$$
$$\tan\theta = \frac{3}{2}, \theta = 56.3$$
$$Z = 3.606(\cos 56.3 + i\sin 56.3)$$

MAT 101 ELEMENTARY MATHEMATICS 1

SET THEORY

1.1 DEFINITION

A set can be defined as a well-defined collection of objects. The word "well-defined" emphasis that no matter how a set and its elements are described, it should be clear which elements belong to the set and which do not. The objects contained in the set are called **Elements** of the set.

A set is usually denoted by a capital letters and its elements are denoted by small letters. If x is an element of set S, then we write $x \in S$ read as "x belong to S". If x is not an element of S, then we write $x \notin S$ read as "x does belong to S"

Example 1.1.1

N = The set of all the Natural numbers

W = The set of days in the week

Z = The set of all positive integer

N = The set of Nigerian presidents since independence

1.2 NOTATIONS AND METHOD OF DESCRIBING SET

NOTATIONS

Mathematics as all other disciplines has it own basically accepted shorthand. In this course may come across them and some of which are:

$\in$ belong to

$\notin$ does not belong to

$\ni$ such that

$\exists$ there exist

$\exists !$ there exist a unique

$\forall$ for every member

$\Rightarrow$ implies

$\Leftrightarrow$ Iff if and only if

Some sets (or collections) are so basic and they have their own proprietary symbols. Five of these are listed below:

(i) $\mathbb{N}$: Set of Natural numbers

(ii) $\mathbb{Z}$: The ring of integers =(…-2,-1,0,1,2…)

$\mathbb{Z}^{+}$ = The set of positive integers

$\mathbb{Z}^{-}$ = The set of negative integers

(iii) $\mathbb{Q}$: The field of rational numbers = $\left(\frac{a}{b} i, a, b \in \mathbb{Z}, b \neq 0\right)$

(iv) $\mathbb{R}$: The field of real numbers

(iv) $\mathbb{C}$: The field complex numbers = $\{a+b : a, b \in \mathbb{R}\} (i^2 = -1)$

METHOD OF DESCRIBING SET

There are basically two ways of representing sets:

1. A set may be described by enclosing all the elements of the set in curly brackets {} separated by commas. This is called **Tabular representation or Listing Method**

   Example 1.2.1

   In the example above we have:

   $\mathbb{N} = \{1,2,3,4\ldots\}$

   W = {Sun, Mon, Tue,…Sun}

2. The second way is the **Set builder or Property form.** This form describes the elements of the set by referring to their common property, if all the

elements of a set satisfy a certain property p, and its only the elements of that satisfy this property then S may be written as:

S:{x: x satisfy p}

Symbolically, this is written as

S = {x: x…} or S = {x/x…}

Then colon(:) and (/) means "such that", using this type of notation the set in example 1.1.1 is described as

N = {x: x is a Natural number}

W = {x: x is a days in a week }

Z = {x: x is a positive integer}

N = {x: x is a Nigerian presidents since Independence}

Example 1.2.2

Describe the set of all positive integers in

(1) Tabular form

(2) Set builder form

Solution

(i) To write the set of all positive integers in tabular form, we list out all elements of positives integers

$\mathbb{Z}^+ = \{1,2,3,4\ldots n\}$

(ii) $\mathbb{Z}^+ = \{x/x$ is a positive integer$\}$

Example 1.2.3

List the elements of the sets

(i) B ={x: x is a prime factor of 28}

(ii) Positive integer less than 20 which are divisible by 5

Solution

(i) B ={x: x is a prime factor of 28}

Prime factor of 28 are 2 and 7

So B = {2,7}

(ii) Positive integers less than 20 divisible by 5

1,2,3,4,<u>5</u>,6,7,8,9,<u>10</u>,11,12,13,14,<u>15</u>,16,17,18,19

i.e $\mathbb{Z}^{+}$ = {5,10,15}

Example 1.2.4

Describe the following set using correct notation

(i) All solution to the equation $2x^2-5x-3=0$

(ii) All number of the form 2,3,5,7,11,13,17,19

Solution

The solution to the equation $2x^2-5x-3=0$ by factorization $2x^2-5x-3=0$

$$2x^2+x-6x-3$$
$$x(2x+1)-3(2x+1)$$
$$(x-3)(2x+1)$$

Either x =3 or x = -1/2

i.e x = {-1/2,3}

(ii) The numbers are prime numbers i.e $\mathbb{Z}$ = {x: x is a prime number}

1.3TYPES OF SET

1.3.1 UNIVERSAL SET: The set that contains all possible elements in a particular problem of interest is said to be the Universal Set. It is usually denoted by μ, unless otherwise specified.

It is a mother set in which all other set/every other set is its subset.

Example 1.3.1.1

μ= { All integers between 1-10}

{1,2,3,,4,5,6,7,8,9,10}

Example 1.3.1.2

μ= {x: x is a set of Natural numbers}

1.3.2 FINITE AND INFINTE SET

Some collection of object according to a well-defined common property can be large or small .

If the member of a set have a define number like the days of the week, we term this as finite, otherwise it's infinite like the set of Natural numbers or counting numbers. In other words, a set is said to be finite, if we could list all the elements in the set or if in counting the elements of the set the counting comes to an end, on the other hand if we couldn't list all the elements of the set we call such a set an infinite set.

Example 1.3.2.1

A = {Days in a week} A is a finite set

B = {2,3,4,6,…} is an infinite set. The dots indicates that the set continue without coming to an end.

1.3.3 EQUIVALENCE AND EQUALITY OF SET

Two sets A and B are said to be equivalence if they have equal member or elements that is if they have the same number of elements then $A \cong B$

Example 1.3.3.1

Let $A = \{1,2,3,4,5\}$

$B = \{a,b,c,d,e\}$

Then $A \cong B$(because they have equal number of elements) in equivalence set A and B every member of A is matched to one only one member of B and every elements in B is matched to one and only one member of A.

That is $A = \{1,2,3,4,5\}$

$B = \{a,b,c,d,e\}$

Two sets A and B are said to be equal if they have the same elements, irrespective of the order of arrangement.

Example 1.3.3.2

Let A = {1,2,3,4}

B = {1,2,3,2,4,4} then A=B

Because the element of A are 1,2,3, and 4 and the elements of B are just 1,2,3,4. Thus set A=B because they contain exactly the same elements.

Note: All equivalent set have the same number of elements but all equal set have the same members.

Example 1.3.3.3

D = {a,b,c,d,e} is equivalent to {1,2,3,4,5} but they are not equal because they do not have the same members

Example 1.3.3.4

The set {2,4,6,8} is equivalent to {Mon, Tue, Wed, Thur} but not equal

Example 1.3.3.5

The set {1,2,3,4} is equal to {2,3,4,1} and also equivalent. So all equal set are equivalent but all equivalent set are not necessarily equal.

Example 1.3.3.6

Let X = {x: 0<x<20, x is an integer divisible by 5}

Y = {5,10,15}

Are these sets equal?

Solution

X being an integer we list the number less than 20

1,2,3,4,<u>5</u>,6,7,8,9,<u>10</u>,11,12,13,14,<u>15</u>,16,17,18,19

X = {5,10,15} and

Y = {5,10,15} Hence x=y.

1.3.4 NULL SET OR EMPTY SET OR VOID SET

A null set is a set that contain no elements and it is denoted by a symbol {} or ∅

Example 1.3.4.1

A = {Student in university below 10 years}

So A = {} or ∅

This set is empty because no students are below 10 years in the university.

Example 1.3.4.2

A = {x: $7<x<11$, x is a prime number}

Solution

By listing the elements in A we have A ={8,9,10}. So, the only elements contained in set A are 8,9,and 10 and non of them is a prime number, so A = ∅.

1.3.5 SUBSETS AND SUPERSETS

If every elements of set A is also element of set B i.e. if $x \in A \Rightarrow x \in B$, then A is said to be subset of B. Symbolically, this is denoted by $A \subseteq B$ meaning A is a subset of B. in other words, set A is said to be subset of set B. if all the elements of set A are found in set B. Similarly, if A is a subset of B, thus is denoted by $B \supseteq A$.

Two sets A and B are equal i.e. A = B if $A \subseteq B$ and $B \supseteq A$

Example 1.3.5.1

Let G = {3,5,7,9} and let F = {1,3,5,7,8,9} then $G \subseteq F$ because elements in set G is also a member of set F

i.e. $G \subseteq F$ and $F \supseteq G$ meaning that G is a subset of F and F is a superset of G("G is contained in F" and "F contained in G")

REMARK

1. An empty or null set is a subset of every other set since the null set contain no elements , it can be contained in any set without changing the set.

2. Every set is a subset of itself because all the elements in the set are contained in the set itself.

1.3.6 PROPER SUBSET

Set A is said to be a proper subset of set B if there is at least one element in set B which is not in set A. And this is written as $A \subset B$ meaning A is a proper set of B otherwise, B is said to be an improper subset of A.

Example 1.3.6.1

Let $A = \{1,2,3,4,5,6\}$

$B = \{1,2,3,4,5,6,7,8,9,10\}$

Then $A \subset B$ because there is at least one element in B which is not in set A i.e. 7,8,9, and 10 are not in A .

1.3.7 CADINALITY OF A SET

The number of elements in a set is called the cardinality of the set. The symbol n(A) represents the cardinality of the set.

Example 1.3.7.1

The cardinality of the set $x = \{10,12,15,17\}$ is 4. Since the set x contains 4 elements . Thus $n(k) = 4$.

1.3.8 THE POWER SET

The power set is the family of all subsets of a given sets, say S denoted by 2^s where 5 is the number of elements in S.

For example, if a set A is finite with n elements, that is if $A = \{1,2,3,\ldots n\}$, the power set of A is 2^n. 2^n times the number of possible subsets in set A, where A is the number of elements contained in A.

Example 1.3.8.1

If $B = \{1,2,4\}$ and p(B)

Solution

The power set in the set 2^n, where $2^3$is the number of all possible subsets of B. Therefore, $2^3=8$, this shows that eight subset can be formed from set B, which originally contained three elements. Hence , the subset formed from B are:

{}, {1,2,4},{1,2},{1,4},{2,4},{1},{2},{4]

1.4 BASIC SET OPERATIONS

Set can be combined in number of different ways to produce another set. Here are representation of some basic set operations with their properties.

1.4.1 UNION OF SETS

The Union of set A and B is the set of all elements, which belongs to either A and B or both and is defined as:

$A \cup B = \{x \in A \cup x \in B\}$

Note: (i) if $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

(ii) If $\notin A \cup B \Leftrightarrow x \notin A$ or $x \notin B$

Example 1.4.1.1

If $A = \{2,4,6,7\}$

$B = \{1,3,5,8\}$

Then $A \cup B = \{1,2,3,4,5,6,7,8\}$

Example 1.4.1.2

If $A = \{1,2,4,3,6,10\}$

$B = \{1,2,3,4,5,6,7,8,9,10\}$

Then $A \cup B = \{1,2,3,4,5,6,7,8,9,10\}$

Note: (i) Elements are not repeated in a set

(iii) It is also clear that

$A \subseteq A \cup B$ and $B \subseteq A \cup B$

BASIC PROPERTY OF UNION

Let A, B and C sets, the following properties holds:

(i) $A \cup A = A$ {Idempotent property}

(ii) $A \cup B = B \cup A$ {Commutative property}

(iii) $(A \cup B) \cup C = A \cup (B \cup C)$ {Associative property}

(iv) $A \subseteq A \cup B$ and $B \subseteq A \cup B$

(v) If $A_1, A_2, \ldots A_n$ is an finite family sets, then their union is defined as $\bigcup_{i=1}^{n} A_i$

(vi) $A \cup \varnothing = A$ $\varnothing$ is an identity element

1.4.2 INTERSECTION OF SETS

The intersection of set A and B denoted by $A \cap B$ is the set of all those elements that belongs to both A and B respectively. It can be summarize using mathematical notation as:

$A \cap B = \{x / x \in A \cap x \in B\}$

(i) $X \in A \cap B \Leftrightarrow x \in A$, and $x \in B$

(ii) $A \cap B \subseteq A$ and $A \cap B \subseteq B$

Where $\cap$ represent "And"

Example 1.4.2.1

If $A = \{1,2,3,4,5,6,7,8\}$

$B = \{4,5,6,7,8,9,10\}$

Then $A \cap B = \{4,5,6,7,8\}$

Example 1.4.2.2

If $A = \{x / x = -2n, n \in \mathbb{N}\}$

$B = \{x / x = 2n, n \in \mathbb{N}\}$

$A \cap B = \{x / x = -2, n \in \mathbb{N}\} \cap \{x / x = 2n, n \in \mathbb{N}\}$

$= \{\ldots -6, -4, -2\} \cap \{2, 4, 6 \ldots\}$

$= \{\}$ or $\varnothing$

BASIC PROPERTIES OF INTERSECTION

Let A, B and C sets the following properties holds:

(i) $A \cap A = A$ {Idempotent property}

(ii) $A \cap B = B \cap A$ {Commutative property}

(iii) $(A \cap B) \cap C = A \cap (B \cap C)$ {Associative property}

(iv) $A \cap B \subseteq A$ and $A \cap B \subseteq B$

(v) $A \cap B = A$ iff $A \subseteq B$ and vice versa

(vi) If $A_1, A_{2\ldots}, A_n$ is a finite family of sets then their intersection is denoted by

$\bigcap_{i=1}^{n} A_i$ or in n extended form as $A_1 \cap A_2 \cap \ldots \cap A_n$

(vii) $A \cap \xi = A$ {A is an identity property}

1.4.3 DISJOINT SETS

When two sets A and B have no element in common, the two set are said to be disjoint set. I.e. if no elements of A is in B and no elements of B is in A then A and B are disjoint.

Example $Y = \{1,3,5,7\}$

$Z = \{2,4,6,8,10\}$

Solution

Then Y and Z are said to be disjoint set because they have no elements in common and their intersection is null see $Y \cap Z = \{\}$

1.4.4 DIFFERENCE OF SET

The difference of two set A and B denoted by "A-1" "A-b" or A/B read as A minus B A difference B is the set of all the elements in A which are not ion B. The difference of A and B may also be defined as :

$A\text{-}B = \{x : x \in A, x \notin B\}$

$B\text{-}A = \{x : x \in B, x \notin A\}$

Note: $A\text{-}B \neq B\text{-}A$

Example 1.4.4.1

Find the difference between the two sets

$A = \{1,2,3,4,5,6,8\}$

B ={2,1,3,5,7}

Solution

(i) A-B ={4,6,8}

(ii) B-A ={7}

1.4.5 COMPLIMENT OF THE YEAR

The compliment of a set A is the set of all the elements that do not belong to A but the element to the universal set U. Then is denoted by A^c or A^1

Example 1.4.5.1

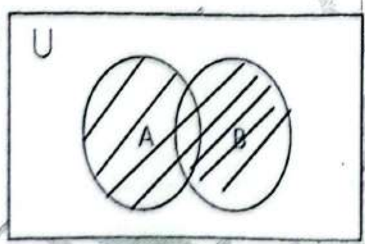
Let U= {1,2,3,4,5,6,7,8}

A = {1,3,5,7}

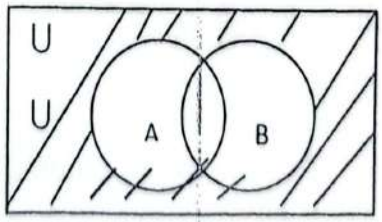
Then A^c={2,4,6,8}

1.5VENN DIAGRAM

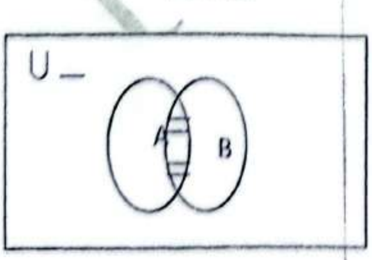
A Venn diagram is a pictorial representation of sets which clearly shows the relationship between them The universal set U is usually represented as a rectangle and the set as simple closed curves which are shown below.

(a)


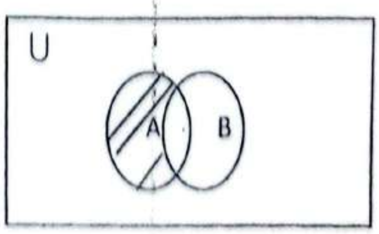
$A \cup B$

(b)


$(A \cup B)^c$

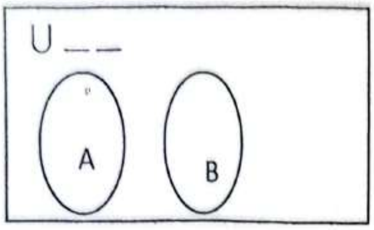
(c)


$A \cap B$

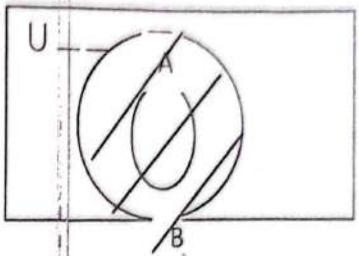
(d)


$A - B$

(e)



(f)



$A \cap B \neq \varnothing$ $\qquad$ $B \subset A$

1.5.1 SOME IMPORTANT RESULT ON NUMBER OF ELEMENTS IN SETS

If A,B, and C are finite set and $\cup$ be that finite universal set then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow$ A,B are disjoint

(iii) $A(A-B) = n(A) - n(A \cap B)$ i.e. $\cap(A-B) + \cap(A \cap B) = \cap(A)$

(iv) $\cap(A \Delta B)$ = number of elements which which belong to exactly A or B

$= n((A-B) \cup (B-A))$

$= n(A-B) + n(B-A)$ { :. A –B & B –A are disjoint

$= n(A) - n(A \cap B) - n(A \cap B) + n(B)$

$= n(A) + n(B) - 2n(A \cap B)$

(v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

(vi) Number of element in exactly two of the set A,B,C $= n(A \cap B) + n(B \cap C) + n((\cap A) - 3\cap(A \cap B \cap C))$

(vii) Number of elements in exactly one of the set A,B,C $= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

Example 1.5.1.1

In a survey of 400 students in a school, 100 were listed as smokers , 156 as gum chewers, 75 are both smokers and gum chewers. Find out how many students were neither smoking nor chewing gum.

Method 1

By using the formula

Let S = Smokers

G = Gum chews

$S \cap G$ = Both smokers and gum chewers

$\cap(S \cup G) = n(S) + n(G) - n(S \cap G)$

$= 100 + 156 - 75$

$= 181$

:. The number of students were neither smoking nor chewing gum = 400 -181 = 219

METHOD 2

By using Venn diagram

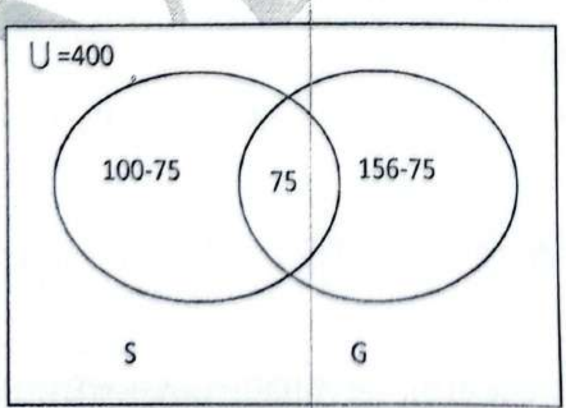
Let $\cup$ = Universal Set = Set of 400 students

S = Smokers

G = Gum chewers

$S \cap G$ = Both smokers and gum chewers

S



:. The numbers of students that neither smoking nor chewing gum = 400 – 181 = 219

Example 1.5.1.2

In a group of 800 people, 550 can speak Hausa and 450 can speak English. How many can speak both Hausa and English

Solution

Let H = Speaking Hausa

E = Speaking English

METHOD 1

By using the formula

$n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$\Rightarrow n(H \cap E) = n(H) + n(E) - n(H \cup E)$

$= 550 + 450 - 800$

$= 200$

METHOD 2

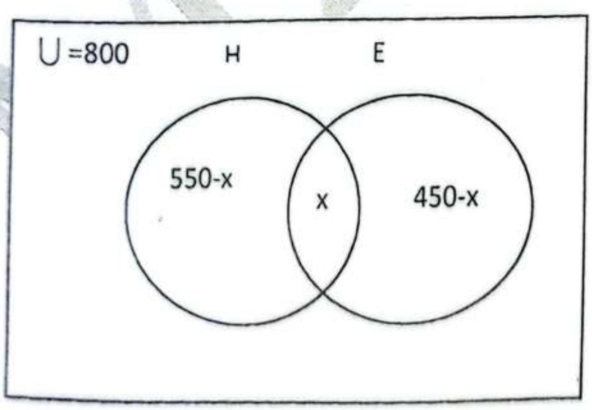
By using Venn diagram

Let U = Universal set

H = People that can speak Hausa

E = people that can speak English

$H \cap E$ = People that can speak both Hausa and English



Let $x = H \cap E$

$800 = 550 - x + x + 450 - x$

$800 = 550 + 450 - x$

$800 = 1000 - x$

$X = 1000 - 800$

$X = 200$

$\therefore H \cap E = 200$

Example 1.5.1.3

In a group of 50 people, 35 speak Hausa, 25 speak both English and Hausa and all the people speak at least one of the two language. How many people speak only English and not Hausa? How many people speak English?

Solution

METHOD 1

By using formula

Let H represent Hausa

E represent English

It is given that

$n(H \cap E) = 50$, $n(H) = 53$, $n(H \cap E) = 25$.

Now,

The number of people speaking English and not Hausa is

$n(E - H) = n(H \cap E) - n(H)$

$= 50 - 35$

$= 15$

The number of people speak English is

$n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$50 = 35 + n(E) - 25$

$50 = 35 - 25 + n(E)$

$n(E) = 50 - 10$

$n(E) = 40$

Hence, the number of people who speak English is 40.

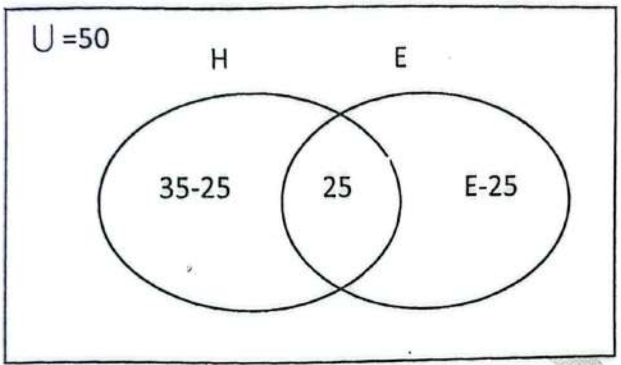
METHOD 2

By using Venn diagram

Let U = Universal set

H = People that speak Hausa

E = People that speak English



The number of people that speak English

$50 = 35 - 25 + 25 + E - 25$

$50 = 10 + E$

$E = 50 - 10$

$E = 40$

The number of people speaking English but not Hausa

$= n(E) - n(H \cap E)$ or $= n(E) - 25$

$= 40 - 25$ Where $n(E) = 40$

$= 15$ $= 40 - 15 = 15$.

Example 1.5.1.4

In an examination 950 students failed mathematics, 600 failed physics and another 600 failed chemistry, 450 failed in both mathematics and physics, 400 students failed in both mathematics and chemistry, 150 students failed in both physics and chemistry. The students that failed all the three subjects were just 75.

How many students took the Examination?

Solution

Let M = Mathematics = 950

P = Physics = 600

C = Chemistry = 600

$M \cap P = 450$

$M \cap C = 400$

$P \cap C = 150$

$M \cap P \cap C = 75$

METHOD 1

By using the formula

$n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(M \cap P) + n(M \cap C) - n(P \cap C) + n(M \cap P \cap C)$

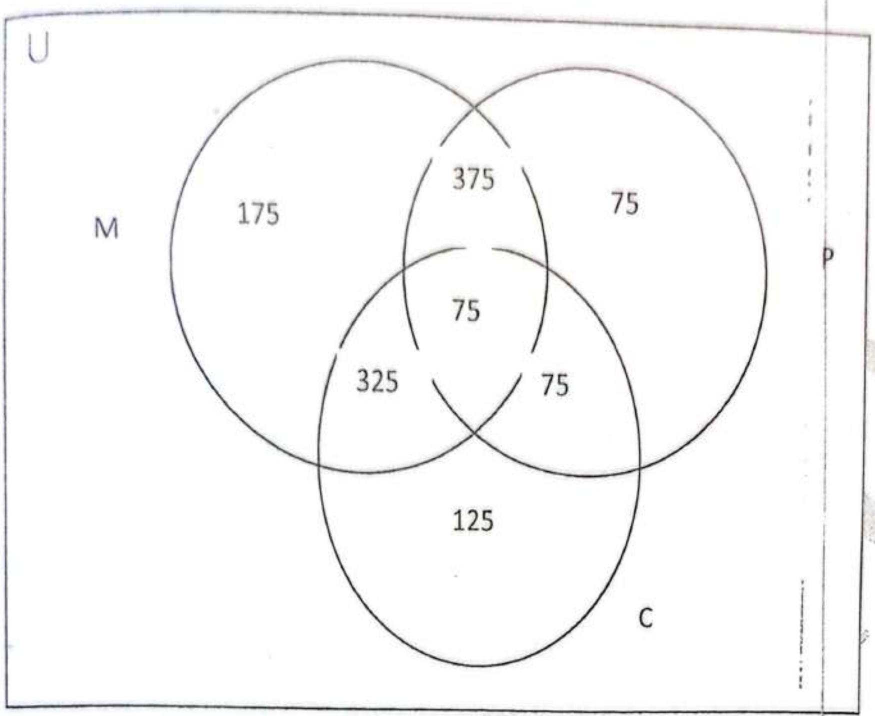
$= 950 + 600 + 600 - 450 - 400 - 150 + 75$

$= 2150 - 1000 + 75$

$= 1225$

METHOD 2

By using Venn diagram



$P \cap C = 150 - 75 = 75$

$M = 950 - 375 - 325 - 75 - 325 = 175$

$P = 600 - 375 - 75 - 75 = 75$

$C = 600 - 325 - 75 - 75 = 125$

This may also be completed in the way as follows

$175 + 375 + 75 + 325 + 125 + 75 + 75 = 1225$

Exercises

(1) Describe the following set in builder form

(a) $\{ -7,-5,-3,-1,1,3,5,7,9 \}$

(b) $\{10,11,12,13,14,15,\ldots\}$

(2) Let $U = \{x : x \in \mathbb{N}\}$ Find

(a) $A = \{x \,/\, x=2n,\ n \in \mathbb{N}\}$

(b) $B = \{x \,/\, 2<x<8,\ n \in \mathbb{N}\}$

(c) $C = \{x \,/\, x \text{ is an even number} < 9\}$

(3) Let $U = \{$English alphabets$\}$

(a) $A = \{$Vowels$\}$

(b) $B = \{$Letters in the word "Hard work"$\}$

(c) $C = \{$Letters in the phrase " I Like It "$\}$

Use the above set to prove the properties of union

(4) If $A = \{x: x = 2n, n \in \mathbb{N}\}$

$B = \{x : x = n, n \in \mathbb{N}\}$

Find $A \cap B$

(5) There are 40 students in a chemistry class, 60 students in a physics lab. Find the number of students which are either in physics or chemistry class.

(i) The total class meet at same hour

(ii) The two classes meet at different hour and 20 students are enrolled in both the subjects.

(6) In a town of 10,000 families it was found that 40% families speak Hausa 20% families speak Yoruba and 10% families speak Igbo. 5% families speak Hausa and Yoruba , 3% speak Yoruba and Igbo and 4% speak Hausa and Igbo. If 2% families speak all the three languages. Find the number of families which speak:

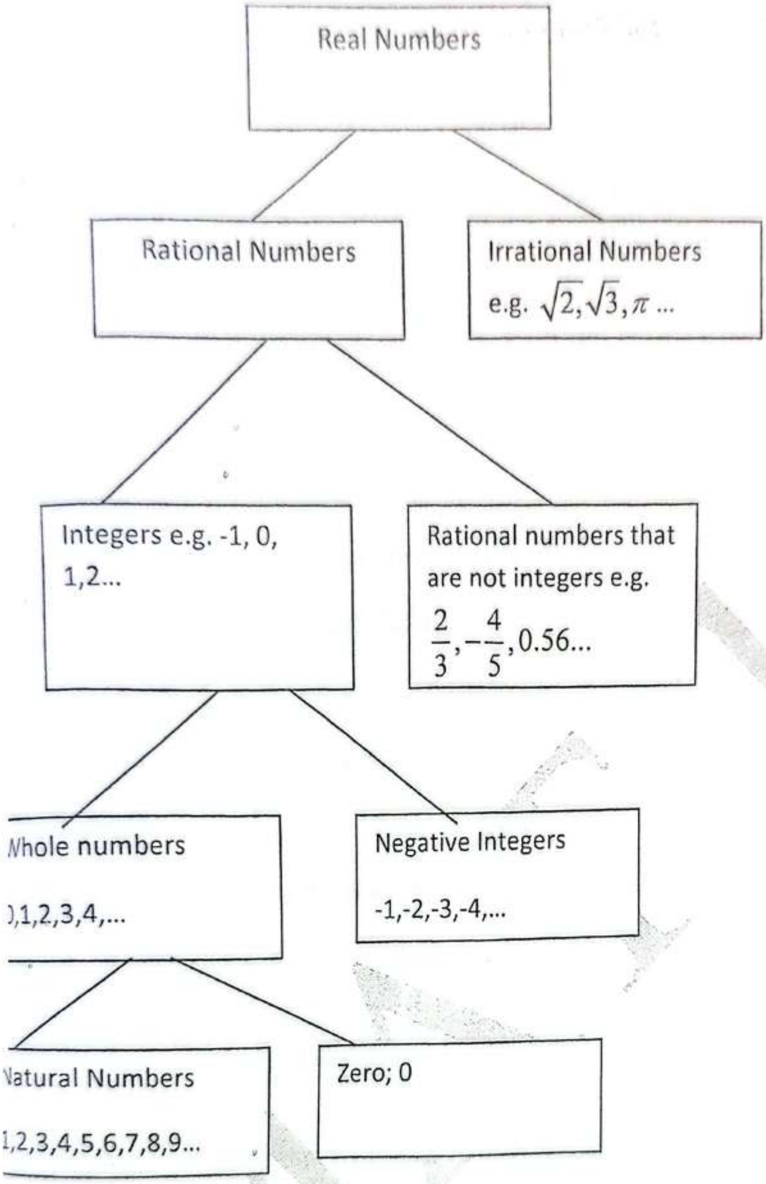
(i) Hausa Only

(ii) Yoruba Only

(iii) None of Hausa, Igbo, and Yoruba

2.0REAL NUMBERS

A real number is either a rational or irrational numbers. A real number is positive if it is greater than 0, negative if it is less than 0.



2.1 RATIONAL NUMBERS

The natural number are the number that can be written as the ratio of two integers. All rational numbers when written in their equivalent decimal form will have terminating or repeating decimal . in other hand, rational number are numbers that can be written as an integer divided by an integer (or a ratio of integers). Example $\frac{2}{3}, -\frac{4}{5}, 0.56...$

2.1IRRATIONAL NUMBERS

The irrational numbers are any real numbers that can not terminating be represented as the ratio of two integers, the numbers usually are imperfect roots, Pi is also an irrational numbers. Irrational when written in their equivalent decimal form have non-terminating and non-repeating decimals, that square root of a prime number is irrational. In other hand irrational number are numbers that cannot be be written as an integer divided by an integer. Example $\sqrt{2}, \sqrt{3}, \pi$

2.2INTEGERS

The integer consist of the natural numbers together with the negatives and 0. In the hand, integers are all the whole numbers and their additive inverse, fraction or decimal. Example -2,-1,0,1,2,3…

An integer is EVEN if it can be written in the form 2n, where n is an integer (if 2 is a factor)

An integer is ODD if it is can be written in the form 2n-1, where n is an integer (if 2 is not a factor).

Example : 2,4,6,… are even integers

1, 3, 5,… are odd integers

2.3WHOLE NUMBERS

The whole numbers are the natural numbers and zero. Example 0,1,2,3,4,5,6,…

2.4NATURAL NUMBER

The natural number are the numbers used for counting. Example 1,2,3,4,5,…

A natural number is a prime number if it is greater than 1 and it is not prime. Example 8,24,33

2.5PROPERTYOF REAL NUMBER

Let a,b, and c represent real number

(a) Closure property

The sum a+b and the product ab are unique real number.

Example:

(i) The sum of two real number is a real number $1+5 = 6$ and 6 is a real number.

(ii) The product of two real number is areal number $7 \times 3 = 21$, and 21 is areal number

(b) Commutative property

$a + b = b + a \qquad \forall\, a,b \in \mathbb{R}$

$a \,.\, b = b \,.\, a \qquad \forall\, a,b \in \mathbb{R}$

Example

(i) $2 + 3 = 3 + 2 = 5 \in \mathbb{R}$

(ii) $2 \times 3 = 3 \times 2 = 6 \in \mathbb{R}$

(c) Associative property

$(a + b) + c = a + (b + c)$

$(ab)c = a(bc)$

Example

$(1 + 2) + 3 = 1 + (2 + 3) = 6 \in \mathbb{R}$

$(1 \times 2) \times 3 = 1 \times (2 \times 3) = 6 \in \mathbb{R}$

(d) Distributive property

$a(b + c) = ab + ac$

$(a + b)c = ac + bc$

Example

$2(3 + 4) = 2 \times 3 + 2 \times 4$

$2(7) = 6 + 8$

$14 = 14 \in \mathbb{R}$

$(2 + 3)4 = 2 \times 4 + 3 \times 4$

$(5)4 = 8 + 12$

$20 = 20$ and $20 \in \mathbb{R}$

(e) Additive Identity Property

$a + 0 = 0 + a = a$

Example

(i) $2 + 0 = 0 + 2 = 2 \in \mathbb{R}$

(f) Multiplicative property

$a \times 1 = 1 \times a = a$

Example

$4 \times 1 = 1 \times 4 = 4 \in \mathbb{R}$

(g) Additive Inverse property

$a + (-a) = 0$

Example

$10 + (-10) = 0 \in \mathbb{R}$

(h) Multiplicative Inverse Property

$a \times \frac{1}{a} = 1$, $a \neq 0$

example

$20 \times \frac{1}{20} = 1 \in \mathbb{R}$

Play Questions III

9. Insert two arithmetic means between 4 and 18
10. Find the indicated sum for the given A.P

    b. $S_{14}: \frac{3}{10}, \frac{2}{5}, \frac{1}{2}$ b. $S_{12}: -5, -4\frac{5}{8}, -4\frac{1}{4}$

11. Given that 4, P,Q,13 are consecutive terms of an A.P, find the values of P and Q
12. The 6^{th} term of an A.P is -5 and the 10^{th} term is -21. Find the sum of the first thirty terms.
13. The 3^{rd} and 7^{th} terms of an A.P are -1 and 11 respectively, find the n^{th} term and the number of terms which must be added to get a sum of 430.
14. Which term of the sequence $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}...$ is the first negative term?
15. Which term of the sequence $8-6i, 7-4i, 6-2i,...$ is (i) purely real (ii) purely imaginary?
16. Find the tenth term and the n-th term of the following sequence: $\frac{1}{2}, 1, 2, 4, 8, ...$

Play Questions 2

6. Find the ninth term and the general term of the progression $\frac{1}{4}, -\frac{1}{2}, 1, -2$
7. Find the sum of the first 8 terms of the G.P with a = 5 and r = 2
8. The 1^{st} and 6^{th} terms of a G.P are 153 and 17/27. Find the sum of the first four terms.
9. If $T_2 = 35 \;\; and \;\; T_4 = 875$ respectively. Find the first and fifth terms.
10. Find the sum to infinity of the following series 4, 1, ¼,….

Play Questions

(3) Given that $Z_1 = (26-7i)$ $Z_2 = (1-2i)$ find $Z_1 Z_2$

(4) $Z_1 = (3+4i) Z_2 = (2-5i)$ $Z_3 = (1-2i)$ find $Z_1 Z_2 Z_3$

Play Questions

(7) Describe the following set in builder form

(c) { -7,-5,-3,-1,1,3,5,7,9 }

(d) {10,11,12,13,14,15,…}

(8) Let $\cup = \{x: x \in \mathbb{N}\}$ Find

(d) $A = \{x / x=2n, n \in \mathbb{N}\}$

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(9) Let $\cup = \{\text{English alphabets}\}$

(d) A = {Vowels}

(e) B = {Letters in the word "Hard work"}

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Use the above set to prove the properties of union

(10) If $A = \{x: x = 2n, n \in \mathbb{N}\}$

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