# Geometric group theory
# Lecture 12

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## 1 The Rips construction

We prove the statement from the end of the previous lecture.

**Theorem 1.1.** *Let $Q$ be a finitely presented group. There exists a hyperbolic group $G$ with 2-generated normal subgroup $N = \langle x, y \rangle \lhd G$ such that $G/N \cong Q$.*

*Proof.* Let $\langle S \,|\, R \rangle$ be a finite presentation for $Q$, with finite sets $S = \{s_1, \ldots, s_n\}$ and $R = \{r_1, \ldots, r_m\}$. For simplicity, suppose that $S$ is symmetric. We write $S' = S \cup \{x, y\}$, where $x$ and $y$ are two additional letters. For each $i = 1, \ldots, n$, define the words in $S'$:

$$t_{i,x} = s_i x s_i^{-1} x y^{a_i} x y^{a_i+1} \ldots x y^{a_i'}$$
$$t_{i,y} = s_i y s_i^{-1} x y^{b_i} x y^{b_i+1} \ldots x y^{b_i'},$$

and for $i = 1, \ldots, m$:

$$r_i' = r_i x y^{c_i} x y^{c_i+1} \ldots x y^{c_i'},$$

where $a_i < a_i' < b_i < b_i' < c_i < c_i'$. Now let

$$R' = \{r_1', \ldots, r_m', t_{1,x}, \ldots, t_{n,x}, t_{1,y}, \ldots, t_{n,y}\}.$$

We define the group $G$ via the presentation $\langle S' \,|\, R' \rangle$.

We first verify that $G$ is hyperbolic. In fact, we will show that the presentation given is a $C'(1/6)$ presentation, and therefore hyperbolic. By the choice of the integers above, the noise words that suffix each relator in $R'$ contain no pieces. Hence any piece of a relator in $R'$ is a subword of some $r \in R$. Hence, choosing our integers such that $a_i' - a_i \geq 100 \max\{\ell(r) \,|\, r \in R\}$ and $\min\{b_i' - b_i, c_i' - c_i\} \geq 100$ ensures that the presentation is $C'(1/6)$.

Next, we show that $N = \langle x, y \rangle$ is a normal subgroup of $G$. Indeed, the relations $t_{i,x}$ and $t_{i,y}$ guarantee that the conjugate of $x$ or $y$ by any of the generators $s \in S$ lies in $N$. Since $G$ is generated by $S$ and $\{x, y\}$, it follows that $N$ is normal in $G$. Lastly, $G/N$ has the presentation $\langle S' \,|\, R' \cup \{x, y\} \rangle$. Applying Tietze transformations, one immediately sees that this recovers $Q = \langle S \,|\, R \rangle$, as required. $\square$

## 2 Quasiconvex subgroups

The Rips construction illustrates that general finitely generated subgroups of hyperbolic groups may be quite poorly behaved. We will touch on a class of subgroups whose intrinsic geometry somehow respects that of their ambient group, and are as a result much better behaved. Recall that a subspace of a metric space is *quasiconvex* if any geodesic with endpoints in the subspace is contained in a uniform neighbourhood of that subset.

**Definition 2.1.** Let $G$ be a group with finite generating set $S$. We say that $H \leqslant G$ is *quasiconvex* if there is $\sigma \geq 0$ such that $H$ is $\sigma$-quasiconvex as a subspace of $\Gamma(G, S)$.

A priori quasiconvexity of a particular subgroup is dependent on the choice of generating set of the ambient group.

**Exercise 2.2.** Find a finitely generated group $G$ and finitely generated subgroup $H \leqslant G$ such that $H$ is quasiconvex with respect to one generating set but not another.

Note that we could equivalently define quasiconvexity in terms of actions: given $G$ acting on $X$ geometrically, we say $H \leqslant G$ is quasiconvex if there is a $H$-invariant subspace $Y \subseteq X$ such that $H$ acts on $Y$ geometrically. This definition is equivalent to the previous by the Milnor–Schwarz lemma. Similarly to how the previous definition depended on the choice of generating set, this one depends on the choice of action.

For a hyperbolic group, quasiconvex subgroups coincide exactly with the finitely generated quasi-isometrically embedded subgroups. As a consequence, being quasiconvex is independent of choice of generating set, as a change of finite generating set gives a quasi-isometry.

**Lemma 2.3.** *Let $G$ be a hyperbolic group with finite generating set $S$, and $H \leqslant G$ a subgroup with finite generating set $T$. Then $H$ is quasiconvex in $G$ if and only if the inclusion map $\iota\colon (H, \mathrm{d}_T) \to (G, \mathrm{d}_S)$ is a quasi-isometric embedding.*

*Proof.* Quasi-isometrically embedded subspaces of hyperbolic metric spaces are quasiconvex, giving the backwards implication. For the forwards direction, any coarsely connected quasiconvex subspace of a metric space is quasi-isometrically embedded, with respect to the Rips metric. Since finitely generated subgroups are coarsely connected, the inclusion map in the statement is a quasi-isometric embedding. $\square$

We also saw earlier in the course that quasi-isometrically embedded subsets of a hyperbolic space are again hyperbolic. The above thus yields:

**Corollary 2.4.** *Quasiconvex subgroups of hyperbolic groups are hyperbolic.*

**Exercise 2.5.** Show that every finitely generated subgroup of a free group of finite rank is quasiconvex.

**Exercise 2.6.** Show that being quasiconvex passes to finite index subgroups and overgroups.

An important feature of quasiconvexity is their intersection closure.

**Proposition 2.7.** *Let $G$ be a hyperbolic group. If $H$ and $K$ are quasiconvex subgroups of $G$, then so is $H \cap K$.*

*Proof.* Let $S$ be a finite generating set for $G$ and $\sigma \geq 0$ be a quasiconvexity constant for $H$ and $K$. We will show that for any $r \geq 0$ there is $R \geq 0$ such that

$$N_r(H) \cap N_r(K) \subseteq N_R(H \cap K).$$

Suppose otherwise, so that for each $n \in \mathbb{N}$ there is an element $x_n \in N_r(H) \cap N_r(K)$ with $\mathrm{d}_S(x_n, H \cap K) \geq n$. By definition, there are $y_n, z_n \in G$ for every $n$ with the property that $|y_n|_S, |z_n|_S \leq r$, and $y_n x_n = h_n \in H$ and $z_n x_n = k_n \in K$. Rearranging, we have that $x_n = y_n^{-1} h_n = z_n^{-1} k_n$.

Since there are only finitely many $g \in G$ with $|g|_S \leq r$, we may pass to a subsequence for which $y_n = y$ and $z_n = z$ are constant. Hence we have that $h_n k_n^{-1} = yz^{-1}$ for all $n$ In particular, $h_n k_n^{-1} = h_1 k_1^{-1}$, and so $h_1^{-1} h_n = k_1^{-1} k_n \in H \cap K$ for all $n$. But then $x_1^{-1} x_n = h_1^{-1} y y^{-1} h_n = h_1^{-1} h_n \in H \cap K$, so that $x_n \in x_1(H \cap K)$. This means that $\mathrm{d}_S(x_n, H \cap K) \leq |x_1|_S$, a contradiction for large enough $n$. This proves the claim.

Now the claim gives us $\Sigma \geq 0$ such that $N_\sigma(H) \cap N_\sigma(K) \subseteq N_\Sigma(H \cap K)$. That $H \cap K$ is $\Sigma$-quasiconvex as a subset of $\Gamma(G, S)$ then follows immediately from the fact that any geodesic with endpoints in $H \cap K$ lies in $N_\sigma(H)$ and $N_\sigma(K)$. □

Quasiconvexity also generally seems to be at odds with normality; we sketch a proof of the following.

**Proposition 2.8.** *Let $G$ be a hyperbolic group, $H \leqslant G$ a quasiconvex subgroup. If $H$ is normal in $G$, then $H$ is either finite or has finite index in $G$.*

*Proof sketch:* Suppose that $H$ is infinite. Then $\Lambda H$ is a closed non-empty subset of $\partial G$. As $H$ is normal, for any $x \in \partial G$ and $g \in G$ we have $g \cdot Hx = H(gx)$, so that $\Lambda H$ is $G$-invariant. But since the action of $G$ on $\partial G$ is minimal, it must be that $\partial G = \Lambda H$. Thus $H$ has finite index in $G$. □

We note that the finite normal subgroups of a hyperbolic group $G$ all arise in a somewhat trivial way.

**Lemma 2.9.** *Let $G$ be a hyperbolic group, and let $K \lhd G$ be the kernel of the action of $G$ on $\partial G$. Then every finite normal subgroup of $G$ is contained in $K$.*

*Proof.* Let $F \lhd G$ be a finite normal subgroup. The quotient map $q \colon G \to G' = G/F$ is a quasi-isometry, so induces an equivariant homeomorphism of boundaries $\partial G \to \partial G'$. Now $F$ acts trivially on $\partial G'$, so it must act trivially on $\partial G$. Hence $F \subseteq K$. □

**Exercise 2.10.** Let $G$ be a hyperbolic group. Show that if $H \leqslant G$ is quasiconvex, then $H$ has finite index in its centraliser $C_G(H)$.

## 3 Topological properties

Hyperbolic groups also have properties that make them particularly well-behaved from the perspective of algebraic topology. Recall that, for a (discrete) group $G$, a *classifying space* is a space $BG$ for which $\pi_1(BG) = G$ and $\pi_n(BG) = 0$ for all $n \geq 2$. We will show that torsion-free hyperbolic groups have finite classifying spaces. We will make use of the following construction, which turns metric spaces into simplicial complexes, whose simplices in some sence coarsely approximate balls in the metric space.

**Definition 3.1** (Rips complex)**.** Let $X$ be a metric space, $r \geq 0$. The *Rips complex* on $X$ with parameter $r$ is the complex $P_r(X)$ whose vertex set is $X$ and with an $n$-simplex for every $(n+1)$-tuple $Y = \{x_0, \ldots, x_n\}$ with $\operatorname{diam}(Y) \leq r$.

The Rips complex, taken with a suitably large parameter, is always contractible for a hyperbolic metric space. It will therefore serve as a candidate for the universal cover $EG$ of our classifying space.

**Proposition 3.2.** *Let $X$ be a $\delta$-hyperbolic space, $X' \subseteq X$ a subspace with $X = N_1(X')$. If $r \geq 4\delta + 2$ then $P_r(X')$ is contractible.*

*Proof.* Let $r \geq 4\delta + 2$. By Whitehead's theorem, it suffices to show that the homotopy groups of $Y = P_r(X')$ are trivial. Pick a basepoint $x \in X'$ and suppose that $S^n \to Y$ is a continuous map of a sphere into $Y$ based at $x$. Since $S^n$ is compact, the image of this map lies in a finite subcomplex of $Y$. To show that the map is null-homotopic, we will show that every finite subcomplex of $Y$ is contractible.

Let $L \subseteq Y$ be a finite subcomplex. We will homotope $L$ to a strictly smaller complex $l'$ by moving vertices of $L$ towards the basepoint $x$. Repeating such a process finitely many times, $L$ is homotopic to a finite subcomplex of $Y$ in which every vertex is a distance of at most $\frac{1}{2}r$ from $x$. Such a subcomplex is contained in a face of a single simplex of $Y$, and is thus contractible.

Suppose that these is a vertex $v \in L$ with $\mathrm{d}_X(x, v) > \frac{1}{2}r$; we may take $v$ attaining a maximal such distance. Let $z$ be a point on a geodesic $[x, v] \subseteq X$ with $\mathrm{d}_X(z, v) = \frac{1}{2}r$ and a point $v' \in X'$ with $\mathrm{d}_X(z, v') \leq 1$. We will show that if $u \in L$ is a vertex with $\mathrm{d}_X(u, v) \leq r$, then $\mathrm{d}_X(u, v') \leq r$ also. This implies that if $(v, x_1, \ldots, x_n)$ is a simplex $Y$, then so is $(v', x_1, \ldots, x_n)$. Let $u \in L$ be a vertex with $\mathrm{d}_X(u, v) \leq r$. By the four-point inequality,

$$\mathrm{d}_X(u, v') + \mathrm{d}_X(x, v) \leq \max\{\mathrm{d}_X(u, v) + \mathrm{d}_X(x, v'), \mathrm{d}_X(u, x) + \mathrm{d}_X(v, v')\} + 2\delta.$$

In either case, one can use the defining inequalities from the previous paragraph to show that $\mathrm{d}_X(u, v') \leq \frac{1}{2}r + 2\delta + 1$. Now since $r \geq 4\delta + 2$, this implies $\mathrm{d}_X(u, v') \leq r$, proving the claim. Thus the subcomplex $L'$ obtained by replacing every simplex with $v$ as a vertex with $v'$ is well-defined, and homotopic to $L$ via the obvious affine maps in $(v, v', x_1, \ldots, x_n)$. □

It is straightforward to apply the above to a hyperbolic group via its Cayley graph.

**Theorem 3.3.** *Let $G$ be a hyperbolic group. There is a simplicial complex $P$ such that:*

1. *$P$ is finite dimensional, contractible, and locally finite;*
2. *$G$ acts on $P$ simplicially, cocompactly, with finite cell stabilisers;*
3. *the action is free and transitive on the vertex set of $P$*

*Hence if $G$ is torsion-free, then $P/G$ is a finite classifying space for $G$.*

*Proof.* Take $X$ to be a Cayley graph of $G$ with respect to some finite generating set $S$. Then $X$ is $\delta$-hyperbolic for some $\delta$, and $X = N_1(G)$, where $G$ is viewed as the vertex set of $X$. Pick $r = 4\delta + 2$, so that by Proposition 3.2, $P = P_r(G)$ is contractible.

Given $g \in G$ there are at most $2|S|^r$ elements $h \in G$ with $\mathrm{d}_X(g, h) \leq r$, so that $P$ is necessarily finite dimensional and locally finite. The vertex set of $P$ is exactly $G$, which $G$ acts on by (left) translation. This action is free and transitive, and extends to $P$ by linearly interpolating across simplices. The action on $P$ is necessarily simplicial, as it preserves adjacency in $X$. Finally, if $\sigma = (x_1, \ldots, x_n)$ is a simplex of $P$, then $G$ permutes $x_1, \ldots, x_n$ faithfully. Hence the stabiliser of $\sigma$ has order at most $n! = |S_n|$. □

Note that the above theorem tells us that $P$ is what is known as a *classifying space for proper actions* for any hyperbolic group $G$, often denoted $\underline{E}G$, which is like an $EG$ but with finite cell stabilisers. It just so happens that when $G$ is torsion free, there are no finite subgroups, so that any $\underline{E}G$ as actually an $EG$.