Geometric group theory
Lecture 2

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1 Geometric group actions

Definition 1.1 (Quasi-geodesics)**.** Let $\lambda \geq 1$ and $c \geq 0$. A (λ, c)-*quasi-geodesic* in X is a (λ, c)-quasi-isometric embedding of a closed interval $I \subseteq \mathbb{R}$ into X. The *endpoints* of a quasi-geodesic are the images of the endpoints of the interval, if the interval is bounded.

We call a $(1, 0)$-quasi-geodesic a *geodesic*. A space X is called a *geodesic space* if every pair of points can be joined by a geodesic. Given two points $x, y \in X$ we will often denote by $[x, y]$ a choice of geodesic whose endpoints are x and y.

Remark 1.2. A $(\lambda, 0)$-quasi-geodesic is necessarily continuous. In particular, geodesics are continuous.

Example 1.3. The plane $\mathbb{R}^2$ is a geodesic space, but $\mathbb{R}^2 - \{0\}$ is not.

Quasi-geodesics need not be continuous in general. The following lemma allows us to restrict our attention to continuous quasi-geodesics in many scenarios, without losing very much. We only sketch a very rough idea here and leave the details to the reader, though those less interested in this rather dull technical exercise may also find a proof in Bridson–Haefliger, Lemma III.H.1.11.

Lemma 1.4. *Let X be a geodesic space, $\lambda \geq 1$, and $c \geq 0$. There is a constant $c' = c'(\lambda, c) \geq 0$ such that the following is true.*

Let $\gamma \colon I \to X$ be a (λ, c)-quasi-geodesic in X. Then there is a continuous (λ, c')-quasi-geodesic $\gamma' \colon J \to X$ with the same endpoints as γ, such that the images of γ and γ' are a Hausdorff distance of at most c' from one another.

Proof. Partition I along its integer points, and construct γ' by concatenating geodesics joining the images of this partition in X. As γ is (λ, c)-quasi-geodesic, the length of each of these segments is at most $\lambda + c$, so the second claim holds as long as we choose c' greater than this. For the first and last claims, we can use the quasi-geodesicity of γ and compare the paths by their images of the integer partition. $\square$

Recall that the Heine–Borel theorem tells us that closed and bounded subsets of finite dimensional Euclidean spaces are compact. Many other spaces we will consider have this important property, such as locally finite graphs. We view this as a sort of finiteness property; non-proper spaces include things such as Banach spaces of infinite dimension and graphs with infinite valence at a vertex. We will give a name to metric spaces satisfying this property more generally.

Definition 1.5 (Proper space)**.** A metric space X is called *proper* if each of its closed and bounded sets are compact.

Exercise 1.6. Let X be a proper metric space, and suppose $G \leqslant \mathrm{Isom}(X)$ is a subgroup of isometries. Show that the action of G on X is properly discontinuous if and only if G is a discrete subgroup of $\mathrm{Isom}(X)$, equipped with the compact-open topology.

Exercise 1.7. A *length space* is a metric space where any two points can be joined by a rectifiable path, and the distance between two points coincides with the infimum of the lengths of all such paths. Prove the *Hopf–Rinow theorem*: every complete and locally compact length space is proper and geodesic.

Perhaps the most important observation we will make is the following, which is sometimes referred to as the *fundamental lemma of geometric group theory.*

Proposition 1.8 (Milnor–Schwarz Lemma)**.** *Let X be a proper geodesic metric space, and suppose that G acts on X cocompactly by isometries. Then there is a generating set S of G such that the orbit map*

$$G \to X, g \to g \cdot x$$

is a quasi-isometry for any $x \in X$. Moreover, if the action is properly discontinuous, then S is finite.

Proof. Let $x \in X$ be an arbitrary point. As X is proper, it is locally compact. Therefore cocompactness of the action is equivalent to the existence of compact $B \subseteq X$ with $x \in B$ such that $G \cdot B = X$. As B is compact, it is a bounded set: let R be the diameter of B and define

$$S = \{s \in G \mid \mathrm{d}_X(x, s \cdot x) \leq 3R\} - \{1\},$$

We first show that S is a generating set for G. Let $g \in G$ be an element and write $d = \mathrm{d}_X(x, g \cdot x)$. As X is a geodesic space, there is a geodesic $\gamma\colon [0, d] \to X$ between x and $g \cdot x$. We may choose a partition $0 = t_0 < \cdots < t_n = d$ of $[0, d]$ such that $t_i - t_{i-1} = R$ for each $i = 1, \ldots, n-1$ and $t_n - t_{n-1} \leq R$. It follows that

$$n \leq \frac{1}{R}\mathrm{d}_X(x, g \cdot x) + 1. \tag{1.1}$$

As $G \cdot B = X$, there is $g_i \in G$ such that $\gamma(t_i) \in g_i B$ for each $i = 0, \ldots, n$. We may take $g_0 = 1$ and $g_n = g$.

For each $i = 1, \ldots, n,$, we have

$$\begin{aligned} \mathrm{d}_X(g_{i-1} \cdot x, g_i \cdot x) &\leq \mathrm{d}_X(g_{i-1} \cdot x, \gamma(t_{i-1})) + \mathrm{d}_X(\gamma(t_{i-1}), \gamma(t_i)) + \mathrm{d}_X(\gamma(t_i), g_i \cdot x) \\ &\leq R + R + R = 3R \end{aligned}$$

so that $\mathrm{d}_X(x, g_{i-1}^{-1}g_i \cdot x) \leq 3R$. It follows by definition that $g_{i-1}^{-1}g_i \in S$ for each $i = 1, \ldots, n$. By a finite induction it follows that $g = g_n \in \langle S \rangle$. As g was arbitrary, S generates G.

At this point we remark that since X is proper, then the ball of radius $3R$ about x is compact. Hence, if the action of G is properly discontinuous, then the set S is finite.

We now show that the map in the statement is a quasi-isometry. Since the map is G-equivariant, we need only bound $\mathrm{d}_X(x, g \cdot x)$ from above and below by linear functions of $\mathrm{d}_S(1, g)$. From the above, g can be written as a word of length n in S, namely

$$g = g_1(g_1^{-1}g_2)(g_2^{-1}g_3)\ldots(g_{n-2}^{-1}g_{n-1})(g_{n-1}^{-1}g_n).$$

Hence $\mathrm{d}_S(1, g) \leq n$. By (1.1), this implies $\mathrm{d}_S(1, g) \leq \frac{1}{R}\mathrm{d}_X(x, g \cdot x) + 1$. Moreover, if $w = s_1 \ldots s_n$ is a word of minimal length representing g. Then

$$\begin{aligned}
\mathrm{d}_X(x, g \cdot x) &\leq \mathrm{d}_X(x, s_1 x) + \cdots + \mathrm{d}_X(s_1 \ldots s_{n-1} \cdot x, g \cdot x) \\
&\leq \sum_{i=1}^{n} \mathrm{d}(x, s_i \cdot x) \\
&\leq 3Rn = 3R\,\mathrm{d}_S(1, g)
\end{aligned}$$

where the first inequality holds by the triangle inequality, the second by the fact G acts by isometries, and the third by the definition of S. ☐

Recall that two groups are *commensurate* if they contain isomorphic subgroups of finite index. We have a basic consequence of the Milnor–Schwarz lemma

Lemma 1.9. *Finitely generated commensurate groups are quasi-isometric.*

Proof. It is enough to show that a group is quasi-isometric to any of its finite index subgroups. Let $H \leqslant_f G$ and let S be a finite generating set for G. As G acts properly discontinuously by isometries on $\Gamma(G, S)$, so does H. Moreover, every point of $\Gamma(G, S)$ is at most $[G : H]$ from H, so the action is cocompact. Hence by the Milnor–Schwarz lemma, H has a finite generating set T for which $\Gamma(H, T)$ is quasi-isometric to $\Gamma(G, S)$. ☐

Remark 1.10. Albert Schwarz, whose name appears as the second component of above named result, is a Russian-born mathematician who, after beginning in topology, spent a majority of his career working on mathematical physics. The name Schwarz is a German-Jewish name, and was transliterated to Russian as Шварц. Many sources still cite this result as the 'Švarc–Milnor' or 'Milnor–Švarc' lemma, owing to a curious decision by the AMS in the 1950s to re-transliterate Шварц as Švarc. Amusingly, Schwarz later moved to the United States, where he goes by the original spelling of his family name.

Remark 1.11. The main initial motivation for considering quasi-isometries comes from differential geometry; they fundamentally clarify the relationship between continuous structures and certain discrete objects approximating them. In particular, Schwarz and Milnor were interested in relating volume growth in universal covers of Riemannian manifolds to some notion of growth in their fundamental groups. That these rates are the same for compact manifolds, up to a suitable equivalence relation, is a straightforward consequence of the Milnor–Schwarz lemma.

We conclude this section with the statement of a major theorem of Gromov. The proof is well beyond the scope of this course, but it is a strong indicator that one can recover a remarkable amount of algebraic information from asymptotic geometric data. We will not give a precise definition here, but the *growth rate* of a finitely group is the rate of growth of the function $r \mapsto |B(1, r)|$, where the ball is taken the group with respect to some word metric for a finite generating set. It is not difficult to see that this is a quasi-isometry invariant.

Theorem 1.12. *Let G be a finitely generated group. Then G has a finite index nilpotent subgroup if and only if it has polynomial growth.*

2 Negative curvature in spaces

There are many notions of curvature in spaces. To do geometric group theory, we are interested in formulating a notion that applies to metric spaces in general. This approach is informed by more classical notions of negative curvature in manifolds. Given a Riemannian manifold, the curvature may be formalised using *sectional curvature*: given two linearly independent vectors in a tangent space to a point, one calculates the Gaussian curvature of the surface with tangent plane equal to the span of these vectors.

Already, the topology of complete manifolds with everywhere non-positive sectional curvature is tied to group theory – the Cartan–Hadamard theorem tells us that the universal cover of such a manifold of dimension n is homeomorphic to $\mathbb{R}^n$. It follows that these manifolds are aspherical, and so their algebraic topology is largely determined by their fundamental groups. The most basic examples of manifolds of non-positive curvature are those with constant negative curvature: these are *hyperbolic manifolds*. The study of hyperbolic manifolds is incredibly vast and incredibly rich; we here give a brief overview of the basics in low dimensions, to give some intuition and motivation for the more abstract, metric, and combinatorial notions that will be the focus of most of this course.

2.1 Hyperbolic geometry

History. Hyperbolic geometry is the geometry of space with a constant negative curvature, and can be thought about in contrast to the geometry of space with zero curvature (i.e. Euclidean geometry) and constant positive curvature (i.e. spherical geometry). The development of hyperbolic geometry has a storied history, and was born out of an almost two-millennia-long attempt to reconcile a difficult tension in Euclid's classical axiomatisation of geometry.

The core point of contention was the nature of Euclid's fifth axiom, called the 'parallel postulate'. Contrary to the other four axioms (e.g. there exists a straight line between any two points, all right angles are equal), the parallel postulate is much more complicated, stating 'if two lines meet a third line, then the two lines will meet on the side of the third line for which the angle sum is less than the sum of two right angles'.

Coupled with the fact that more than half of the propositions in the first book of the Elements do not invoke the parallel postulate, it was widely believed that it should follow from the other axioms. Out of many attempts to prove this, it was gradually realised that the rejection of this axiom actually entails a consistent and robust geometry, often called 'absolute geometry', and that the truth of the parallel postulate in a particular model of geometry is independent of the other axioms. There are in fact only two models of absolute geometry, and they are exactly Euclidean and hyperbolic geometry, with the latter obtained by taking a negation of the parallel postulate instead of the postulate itself.

The Poincaré ball. We will write $\mathbb{H}^n$ for hyperbolic space of dimension n. This is, the unique simply connected Riemannian manifold of constant negative curvature -1. As a model for this space, we will take the open unit ball in Euclidean space $\mathbb{R}^n$, equipped with the metric

$$\mathrm{d}s^2 = \frac{4\|\mathrm{d}\mathbf{x}\|^2}{(1-\|\mathbf{x}\|^2)^2}.$$

This model is called the *Poincaré ball model* for hyperbolic space. There is no isometric embedding of $\mathbb{H}^n$ into Euclidean space of any dimension (unlike, say, a sphere with its intrinsic metric), so any such model must be far from distance-preserving. In fact, this model is *conformal* – it preserves angles – but it is easy to see that distances between points are heavily distorted from their Euclidean counterparts. Other common models are the hyperboloid model and the half space model; each comes with its own advantages and drawbacks.

Hyperbolic space has a natural bordification $\partial\mathbb{H}^n$, which we call the *space at infinity* or simply the *boundary*. From the ball model, this boundary is clear to see as the boundary sphere $\partial\mathbb{H}^n \cong S^{n-1}$. The geodesics in this space are given by diametrical lines and arcs of circles that are perpendicular to the boundary $\partial\mathbb{H}^n$. The group of isometries of $\mathbb{H}^n$ in this model is the Lie group $\mathrm{SO(n,1)}$ of special orthogonal matrices of signature $(n,1)$. The space $\mathbb{H}^n$ is *homogenous* and *isotropic* – its group of isometries acts transitively on the space, and transitively on the tangent space at any given point.

Exercise 2.1. Verify using the path integral formula

$$\ell(\gamma) = \int_{t\in I} \frac{2|\gamma'(t)|}{1-\|\gamma(t)\|^2}\,\mathrm{d}t$$

for a path $\gamma\colon I \to \mathbb{H}^n$, that the geodesics in $\mathbb{H}^n$ are as described above.

Hyperbolic geometry has some interesting features that distinguish it from Euclidean geometry. The easiest to see of these is the non-parallelism of geodesics described in the previous section: this can be seen by the description of geodesics as above. In fact, geodesics will always diverge from each other rather quickly, in one direction or another. There is also the phenomenon of *ultra-parallelism*, where geodesics can share one endpoint in $\partial\mathbb{H}^n$. Such geodesics will stay a bounded distance from one another as they approach one point at infinity, and diverge in the other direction.

A second key feature of hyperbolic geometry is the uniform thinness of polyhedra. In Euclidean geometry, due to the existence of homothety, there are triangles of arbitrarily large area and arbitrarily large incircles. In hyperbolic geometry, this behaviour is forbidden. Let M be a compact Riemannian surface and recall the Gauss-Bonnet formula from differential geometry:

$$\iint_M K \mathrm{d}A + \int_{\partial M} k_g \mathrm{d}s = 2\pi\chi(M),$$

where K is the Gaussian curvature of M and k_g is the geodesic curvature of ∂M. With a smoothing argument, one can apply this formula to calculate the area of triangles. For an isometrically embedded triangle T in hyperbolic space, $K \equiv 1$, $\chi(T) = 1$, and contribution of the integral involving the boundary corresponds exactly to the sum of the external angles. Hence

$$\text{area}(T) = \pi - (\alpha + \beta + \gamma),$$

where α, β, and γ are the internal angles of T. The area of a hyperbolic triangle is thus always bounded from above by π. It follows, for instance, that there is a uniform bound on the radius of an incircle in a hyperbolic triangle; they are all thin.

Exercise 2.2. Show explicitly that $\mathbb{H}^2$, and thus $\mathbb{H}^n$, has triangles that are uniformly thin in the above sense, with the constant $\frac{1}{2}\log 3$ as the bound on radii. (Hint: the worst you could do is an *ideal triangle*, one whose vertices lie on the boundary circle of $\mathbb{H}^2$.)

Another clear consequence of the above formula is the fact that the angle sum in a hyperbolic triangle is always *less* than π, contrasting the Euclidean case, where it is equal to π. In fact, the angle sum decreases proportionally to the area, with ideal triangles having the largest area and angle sum of zero.