Geometric group theory
Lecture 4

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1 The Gromov boundary

In this subsection we introduce a natural bordification of hyperbolic metric spaces. Intuitively, this is the space of 'endpoints' of geodesic rays in the space. The study of boundaries of hyperbolic metric spaces is not only useful – as we shall later see – to the theory of groups acting on hyperbolic spaces, but also plays an essential role in the understanding the metric geometry of these spaces.

Definition 1.1 (Gromov boundary)**.** Let X be a hyperbolic metric space. We say two geodesic rays are *asymptotic* if their images are a finite Hausdorff distance apart. The *Gromov boundary* of X is the set ∂X of equivalence classes of geodesic rays in X, up to the relation of being asymptotic.

Exercise 1.2. Suppose that X is a proper hyperbolic metric space. Show that for any $x \in X$ and $p \in \partial X$, there is a geodesic ray based at x whose endpoint is p. Show that for any $p, q \in \partial X$, there is a bi-infinite geodesic line whose endpoints are p and q.

(Hint: Construct a sequence of approximating geodesic segments, then use the Arzelà–Ascoli theorem and properness to conclude.)

It follows from the above exercise that for a proper hyperbolic space X, the set ∂X is in bijection with the classes or rays based at any particular given point $x \in X$. We will find it convenient to formulate some things regarding the boundary using this, though we must check independence from the choice of basepoint whenever we do so.

Many constructions and statements we present here will hold true for general hyperbolic spaces, but it turns out that if one does not assume that the space is proper, then many additional technicalities arise. For instance, in the above exercise, one would need to replace 'geodesic' with '$(1, 20\delta)$-quasigeodesic'. Since we will only ever be working with proper spaces in practice, we will usually include this assumption.

The following exercise demonstrates one such complication that comes with not assuming properness, even in the simple setting of trees.

Exercise 1.3. Show that if X is an unbounded proper hyperbolic space, then ∂X is non-empty. To contrast, construct an unbounded tree T for which ∂T is empty.

The Gromov boundary carries a natural topology, wherein we declare that two points are close if they have representative geodesics that stay close for a long time.

Definition 1.4 (Topology on the boundary)**.** Let X be a proper hyperbolic metric space and fix a point $x \in X$. For $p \in \partial X$ and $r > 0$, define the set

$$U(p,r) = \left\{ q \in \partial X \;\middle|\; \begin{array}{r} \gamma, \xi \colon [0,\infty) \to X \text{ are geodesic rays with } \gamma(0) = \xi(0) = x, \\ p = [\gamma], q = [\xi], \text{ and } \liminf_{t\to\infty} \langle \gamma(t), \xi(t) \rangle_x \geq r \end{array} \right\}.$$

The sets $U(p,r)$ form a basis of neighbourhoods of p for a topology on ∂X.

Exercise 1.5. Very that the collection of sets $U(p,r)$ actually define a neighbourhood basis for a topology on ∂X. That is, if $p, p' \in \partial X$ and $r, r' \geq 0$, then there is $q \in \partial X$ and $s \geq 0$ such that

$$U(q,s) \subseteq U(p,r) \cap U(p',r').$$

Further, show that this topology is independent of the choice of basepoint.

Example 1.6.

- If T is a tree with valence at least 3 at every vertex, then ∂T is a Cantor set.
- The boundary of $\mathbb{H}^n$ is the sphere S^{n-1}.

Exercise 1.7. Pick some basepoint x in a proper hyperbolic metric space X. We say a sequence $(x_i)_{i\in\mathbb{N}}$ *converges at infinity* if $\liminf_{i,j\to\infty} \langle x_i, x_j \rangle_x = \infty$. Two such sequences $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are said to be *asymptotic* if $\liminf_{i,j\to\infty} \langle x_i, y_j \rangle_x = \infty$. Show that the set of equivalence classes of asymptotic sequences is in bijection with ∂X. For a sequence (x_i) converging at infinity, we write that $x_i \to q \in \partial X$ if q is the image of the class (x_i) represents under this bijection.

Further, we equip this set with a topology generated by the basis of neighbourhoods

$$V(p,r) = \left\{ q \in \partial X \;\middle|\; \begin{array}{l} (x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}} \text{ are sequences converging at infinity} \\ \text{with } x_i \to p, y_i \to q, \text{ and } \liminf_{i,j\to\infty} \langle x_i, y_j \rangle_x \geq r \end{array} \right\}.$$

Show that this topology coincides with that of Definition 1.4.

Definition 1.8 (Gromov product at infinity)**.** Let X be a proper hyperbolic metric space and $x \in X$. Given $p, q \in X \cup \partial X$, we write

$$\langle p, q \rangle_x = \sup \liminf_{i,j\to\infty} \langle x_i, y_j \rangle_x$$

where the supremum is taken over all sequences (x_i) and (y_j) in X converging to p and to q respectively.

Exercise 1.9. Let X be a proper δ-hyperbolic metric space, $x \in X$ and $p, q \in X \cup \partial X$. Show that if $\langle p, q \rangle_x \leq r$, there is a point z on any geodesic line $[p,q]$ with $\mathrm{d}(x,z) \leq r + 2\delta$.

Boundaries of hyperbolic spaces admit metrics the induce the natural topology above. In particular, they are metrisable spaces, and so inherit all the wonderful properties of metric spaces (for example, they are regular, paracompact, and so on).

Definition 1.10 (Visual metric)**.** Let X be a proper hyperbolic metric space and let $a > 1$. Pick a basepoint $x \in X$ and for any $p, q \in \partial X$, choose a bi-infinite geodesic in X joining p and q. Call $l_x(p,q)$ the distance of the geodesic to the point x.

A metric d on ∂X is called a *visual metric with parameter a* if there is a constant C such that

$$\frac{1}{C} a^{-l_x(p,q)} \leq \mathrm{d}(p,q) \leq C a^{-l_x(p,q)}.$$

It is not very hard to construct explicit visual metrics on boundaries of hyperbolic spaces, but we will their existence for granted here. Note that visual metrics are not canonical: there may be many depending on the choice of parameter. However, all visual metrics on a proper hyperbolic metric space are *quasi-symmetric*. That is to say, there is a self-homeomorphism of the space mapping one metric to the other, which preserves annuli in the space in a uniform way. A *quasi-conformal map* is a map that is quasi-symmetric and has quasi-symmetric inverse. The metric structure of the boundary up to quasi-conformal transformations is a canonical invariant of the space. We will not give precise definitions or pursue these notions further here.

Proposition 1.11. *Let X be a proper hyperbolic metric space. Then ∂X is compact.*

Proof. Observe that the basis of neighbourhoods $U(p,r)$ define the same topology as in Definition 1.4 if one takes $r > 0$ ranging over the rational numbers. Thus ∂X is first countable. Since ∂X is metrisable, this means that compactness is equivalent to sequential compactness.

Sequential compactness is a quick consequence of properness and the Arzelà–Ascoli theorem. We may take a sequence of points (p_i) in ∂X and geodesic rays (γ_i) representing these points, and the aforementioned theorem tells us that some subsequence (γ_{n_i}) of these converges to a ray γ. It follows that $a = [\gamma]$ is a limit of the subsequence (p_{n_i}). □

One can extend the topology in Definition 1.4 to include points within the space, by allowing the both geodesic rays and segments in the definition. This gives a topology on $X \cup \partial X$ for which the subspace topology on X agrees with the topology induced by the metric on X. Indeed, essentially the same argument as the previous lemma shows that $X \cup \partial X$ is compact with this topology, so that ∂X can be thought of as a compactification of X.

Proposition 1.12. *Let X be a proper hyperbolic space. Then $X \cup \partial X$ is compact.*

There is in fact the maximal possible diversity among the spaces that can be realised as boundaries of hyperbolic spaces.

Proposition 1.13. *For any compact metrisable space M, there is a proper hyperbolic metric space X with $\partial X \cong M$.*

Proof. Every regular second countable Hausdorff space is homeomorphic to a subspace of the Hilbert cube $C = \prod_{n \in \mathbb{N}} [0, \frac{1}{n}]$ by the Urysohn metrisation theorem. Certainly, M satisfies this criterion as it is a compact metric space. Further, C can be embedded into the unit sphere of a separable Banach space: C already naturally lies in the unit ball of a separable Banach space $\mathcal{H}'$, so project it to the upper hemisphere of the unit ball in $\mathcal{H} = \mathcal{H}' \times \mathbb{R}$. Thus there is a topological embedding $\iota\colon M \to \mathcal{S}$, where $\mathcal{S}$ is the unit sphere of a separable Banach space $\mathcal{H}$.

We equip the open unit ball $\mathcal{B}$ with with the *Cayley-Klein metric* d, which is defined as follows. For any $x, y \in \mathcal{B}$, let $p, q \in \mathcal{S}$ be the points of $\mathcal{S}$ meeting the line $L = \{\lambda p + (1-\lambda)q \mid \lambda \in \mathbb{R}\}$. The metric is defined as

$$\mathrm{d}(x,y) = \frac{1}{2}\left|\log \frac{\|x-q\|\|y-p\|}{\|x-p\|\|y-q\|}\right|.$$

Note that the quantity in the logarithm is the *cross-ratio*, which may be familiar from projective geometry, and plays a large role in hyperbolic geometry. This metric makes $\mathcal{B}$ a model for an infinite dimensional hyperbolic space, and $\mathcal{S}$ is its ideal boundary.

One can verify that the geodesics with respect to this metric are exactly intersections of lines (that is, one-dimensional affine linear subspaces) of $\mathcal{H}$ with $\mathcal{B}$. It follows that the convex hull of a subset of a subset of $\mathcal{B}$ with respect to the metric d is exactly the convex hull with respect to $\mathcal{H}$, the set of all linear combinations of points in the subset.

Let $X \subseteq \mathcal{B}$ be the intersection of the convex hull of $\iota(M)$ with $\mathcal{B}$. Now X is a convex subset of a hyperbolic metric space, so is itself a hyperbolic metric space. Essentially by definition, $\partial X = \iota(M) \cong M$. It remains to verify that X is proper.

The convex hull (in the linear sense) of a compact subset of a Banach space is compact, so that $X \cup \iota(M)$ is compact. As $\iota(M)$ is closed in this set, this implies that X is locally compact. Finally, $\mathcal{B}$ is complete with respect to the Cayley-Klein metric, and X is convex (therefore closed) in $\mathcal{B}$, so X is also complete. The Hopf-Rinow theorem now tells us that X is a proper metric space. □

There are more general constructions, often called *hyperbolic cones* which allows one to realise any complete and bounded metric space as the boundary of a hyperbolic metric space: see Chapter 6 of Buyalo–Schroeder.

Proposition 1.14. *Let X and Y be proper hyperbolic metric spaces. If $f\colon X \to Y$ is a quasi-isometric embedding, then there is an induced map $\partial f\colon \partial X \to \partial Y$ which is a topological embedding. If, further, f is a quasi-isometry, then ∂f is a homeomorphism.*

Proof. Let $\lambda \geq 1$ and $c \geq 0$ be quasi-isometry constants for f. We will define ∂f by pushing forward a geodesic representative of each point in ∂X. Let $p \in \partial X$ and let $\gamma\colon [0,\infty) \to X$ be a geodesic ray based at a point $x \in X$ that tends to p. By definition, the path $f \circ \gamma$ is a (λ, c)-quasigeodesic ray in Y. By the Morse Lemma, there is a constant $M \geq 0$ such that for each $t \in (0,\infty)$, the segment $f \circ \gamma|_{[0,t]}$ is a Hausdorff distance of at most M from a geodesic ξ_t with the same endpoints. Moreover, as f is a quasi-isometry, $\mathrm{d}(f(y), f(\gamma(t))) \to \infty$ as $t \to \infty$.

By properness and the Arzelà–Ascoli theorem, the sequence of paths $(\xi_n)_{n\in\mathbb{N}}$ has a subsequence converging to a geodesic ray $\xi\colon [0,\infty) \to Y$ based at $y = f(x)$, with endpoint $q \in \partial Y$. We define $\partial f(p) = q$. Since any two geodesics define the same point in ∂X if and only if they lie within a finite Hausdorff distance of one another, the same is true of their images under f. It follows that the map ∂f is well-defined and injective.

Let us show that ∂f is continuous. Let $q \in \operatorname{im}(\partial f)$ and $s > 0$, and let δ be a hyperbolicity constant for X and Y. We need to exhibit $r > 0$ such that $\partial f(U(p, r)) \subseteq U(q, s)$, where $\partial f(p) = q$. Take

$$r = (s + M + \frac{1}{2}\lambda\delta + \frac{1}{2}c)\lambda + c + M + 1$$

and let $p' \in U(p, r)$. Let γ and γ' be geodesic rays based at a point $x \in X$ tending to p and p' respectively. By hyperbolicity and the definition of $U(p, r)$, we have $\mathrm{d}_X(\gamma(t), \gamma'(t)) \leq \delta$ for $t \leq r$. This gives us

$$\mathrm{d}_Y(f\gamma(t), f\gamma'(t)) \leq \lambda\delta + c \quad \text{for } t \leq r \tag{1.1}$$

Now $f \circ \gamma$ and $f \circ \gamma'$ are (λ, c)-quasigeodesic rays based at $y = f(x) \in Y$. By the Morse Lemma and arguments similar to above, therefore, there is $M \geq 0$ such that they are M-close to geodesics ξ and ξ' based at y and tending to $q = \partial f(p)$ and $q' = \partial f(p')$ respectively. It then follows from (1.1) then

$$\mathrm{d}_Y(\xi(t), \xi'(t)) \leq 2M + \lambda\delta + c \quad \text{for } t \leq \frac{1}{\lambda}r - c - M.$$

We can use this to bound the inner product from below:

$$\begin{aligned} 2\langle \xi(t), \xi'(t)\rangle_y &= \mathrm{d}(\xi(t), y) + \mathrm{d}(\xi'(t), y) - \mathrm{d}(\xi(t), \xi'(t)) \\ &\geq 2t - 2M - \lambda\delta - c \end{aligned}$$

for $t \leq \frac{1}{\lambda}r - c - M$. In particular, by choice of r, taking $t = \frac{1}{\lambda}r - c - M > 0$ shows us that $\langle \xi(t), \xi'(t)\rangle_y \geq s$. Observing that the inner product is monotone increasing in both factors along geodesics shows that $q' \in U(q, s)$ as required.

When f is a quasi-isometry, it has a quasi-inverse $g\colon Y \to X$ which satisfies $\mathrm{d}_\infty(g \circ f, \mathrm{id}_X) < \infty$. It follows from the functoriality properties of the next exercise that

$$\partial g \circ \partial f = \partial(g \circ f) = \partial \, \mathrm{id}_X = \mathrm{id}_{\partial X},$$

so that necessarily ∂f is a surjective map. □

The homeomorphism type of the boundary can serve as a useful quasi-isometry invariant. For example, one can distinguish the real hyperbolic spaces $\mathbb{H}^n$ and $\mathbb{H}^m$ from one another up to quasi-isometry when $n \neq m$, as their boundaries are spheres of different dimensions.

Note that the boundary is not a complete invariant: real hyperbolic $2n$-space $\mathbb{H}^{2n}$ and complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$ (the unique simply connected Hermitian manifold with constant negative holomorphic sectional curvature) are not quasi-isometric, though both have Gromov boundary S^{2n-1}. The *quasi-conformal structure* of the boundary of a hyperbolic space, a finer structure than its topological type, is actually enough to recover the hyperbolic space up to quasi-isometry, though we will not prove this here.

Exercise 1.15. Let X, Y, and Z be proper hyperbolic metric spaces. Show that the operator ∂ satisfies the following properties:

1. If $f, g\colon X \to Y$ are quasi-isometric embeddings with $\mathrm{d}_\infty(f, g) < \infty$, then $\partial f = \partial g$;
2. If $f\colon X \to Y$ and $g\colon Y \to Z$ are quasi-isometric embeddings, then $\partial(g \circ f) = \partial g \circ \partial f$;

In other words, ∂ is a functor from the category of proper hyperbolic metric spaces (with quasi-isometric embeddings as morphisms) to the category of compact metrisable spaces (with quasi-symmetric embeddings as morphisms).

2 Isometries of hyperbolic metric spaces

In this section we will classify isometries of hyperbolic metric spaces. We will see that a similar trichotomy holds as for those of classical hyperbolic spaces. Below is a largely elementary proof, stolen from the book of Coornaert–Delzant–Papadopoulos. There will be an arguably simpler proof of this fact later on, using much more advanced machinery; the proof here is non-examinable.

Definition 2.1. Let X be a hyperbolic metric space, and $g \in \mathrm{Isom}(X)$ an isometry. We say that g is *elliptic* if the set $\{g^n x\}_{n\in\mathbb{N}}$ is bounded for any $x \in X$, that g is *parabolic* if g has exactly one fixed point in ∂X, and that g is *loxodromic* if it has exactly two fixed points in ∂X.

Exercise 2.2. Let X be a proper hyperbolic metric space. Show that $g \in \mathrm{Isom}(X)$ is loxodromic if and only if the orbit map $n \mapsto g^n x$ is a quasi-isometry $\mathbb{Z} \to X$ for any $x \in X$.

(Hint: Consider the geodesic lines between the two fixed points of g. The set of these lines is preserved by g: show that g acts as a sort of translation along these lines. Picking a point on these lines realises the orbit map as a quasi-isometry (where the constants are related to the translation length).

Theorem 2.3. *Every isometry of a hyperbolic metric space is either elliptic, parabolic, or loxodromic.*

To show this, we will require the following two lemmas, which give us criteria for certain isometries of hyperbolic metric spaces to be loxodromic.

Lemma 2.4. *Let X be a δ-hyperbolic metric space, $g \in \mathrm{Isom}(X)$ an isometry. If there is $x \in X$ with*

$$\mathrm{d}(gx, x) > 2\langle g^2 x, x\rangle_{gx} + 2\delta,$$

then g is a loxodromic isometry.

Proof. For each $n \in \mathbb{N}$, we write $d_n = \mathrm{d}(g^n x, x)$. We may thus rewrite the lemma hypothesis as

$$d_2 \geq d_1 + 2\delta + \varepsilon \tag{2.1}$$

where $\varepsilon > 0$. We will show by induction that $d_n \geq d_{n-1} + \varepsilon$ for $n \in \mathbb{N}$. For the base case that $n = 1$ observe that by the triangle inequality, we have $d_2 \leq 2d_1$. Combining this with (2.1) implies $d_1 \geq \varepsilon$.

Of course, the case $n = 2$ follows from (2.1). Let $n \geq 3$ and suppose that $d_n \geq d_{n-1} + \varepsilon$. We apply the four-point condition to the points x, gx, g^2x, and $g^{n+1}x$, giving that $d_2 + d_n \leq \max\{d_1 + d_{n+1}, d_1 + d_{n-1}\} + 2\delta$. Rearranging, we have

$$d_2 + d_n - d_1 - 2\delta \leq \max\{d_{n+1}, d_{n-1}\}.$$

Rearranging the equation (2.1) reduces the above to

$$d_n + \varepsilon \leq \max\{d_{n+1}, d_{n-1}\}.$$

By the induction hypothesis, $d_{n-1} + \varepsilon \leq d_n$, so we are must in fact have $d + n + \varepsilon \leq d_{n+1}$, as required. It follows immediately that $d_n \geq n\varepsilon$. Now

$$\varepsilon|n - m| \leq \mathrm{d}(g^n x, g^m x) \leq d_1|n - m|$$

so that the map $n \to g^n x$ is a quasi-isometry, completing the lemma. □

Lemma 2.5. *Let X be a δ-hyperbolic metric space, and $g, h \in \mathrm{Isom}(X)$ non-loxodromic isometries. If there is $x \in X$ such that*

$$\mathrm{d}(gx, x) > 2\langle gx, hx\rangle_x + 6\delta \quad \text{and} \quad \mathrm{d}(hx, x) > 2\langle gx, hx\rangle_x + 6\delta$$

then gh and hg are loxodromic isometries.

Proof. Let $x \in X$ be as in the lemma statement. We will attempt to find a bound on the Gromov product $\langle x, ghghx\rangle_{ghx}$, so that we can apply Lemma 2.4 to the isometry gh. The proof is quite technical and involves a lot of inequalities, but the idea is simply that one can transfer the condition on the bounded inner products in the hypothesis along a polygon in X to get the required bound. The fact that X is hyperbolic is what allows one to do this without losing too much length, essentially.

Since g and h are not loxodromic, Lemma 2.4 tells us that

$$\mathrm{d}(g^2x, x) \leq \mathrm{d}(gx, x) + 2\delta \quad \text{and} \quad \mathrm{d}(h^2x, x) \leq \mathrm{d}(hx, x) + 2\delta. \tag{2.2}$$

Moreover, rearranging the lemma hypotheses, we obtain

$$\mathrm{d}(gx, hx) \geq \mathrm{d}(gx, x) + 6\delta + \varepsilon \quad \text{and} \quad \mathrm{d}(gx, hx) \geq \mathrm{d}(hx, x) + 6\delta + \varepsilon \tag{2.3}$$

for some $\varepsilon > 0$. Now applying the four-point condition to x, gx, g^2x, and ghx and simplifying, we obtain

$$\mathrm{d}(gx, x) + \mathrm{d}(gx, hx) \leq \max\{\mathrm{d}(g^2x, x) + \mathrm{d}(hx, x), \mathrm{d}(gx, x) + \mathrm{d}(ghx, x)\} + 2\delta.$$

But (2.2) and (2.3) imply that $\mathrm{d}(gx, x) + \mathrm{d}(gx, hx) \geq \mathrm{d}(g^2x, x) + \mathrm{d}(hx, x) + 4\delta + \varepsilon$. As $\varepsilon > 0$, we must have that $\mathrm{d}(gx, x) + \mathrm{d}(gx, hx) \leq \mathrm{d}(gx, x) + \mathrm{d}(ghx, x) + 2\delta$. A symmetrical argument applies for hg, considering the four points x, hx, h^2x, and hgx. Simplifying and applying (2.3), we get the equations

$$\begin{aligned} \mathrm{d}(gx, x) + 4\delta + \varepsilon \leq \mathrm{d}(ghx, x) \quad &\text{and} \quad \mathrm{d}(gx, x) + 4\delta + \varepsilon \leq \mathrm{d}(hgx, x), \\ \mathrm{d}(hx, x) + 4\delta + \varepsilon \leq \mathrm{d}(ghx, x) \quad &\text{and} \quad \mathrm{d}(hx, x) + 4\delta + \varepsilon \leq \mathrm{d}(hgx, x). \end{aligned} \tag{2.4}$$

Now applying the four-point condition to the points x, gx, ghx, and $ghgx$ gives similarly

$$\mathrm{d}(ghx, x) + \mathrm{d}(hgx, x) \le \max\{2\,\mathrm{d}(gx, x), \mathrm{d}(x, ghgx) + \mathrm{d}(hx, x)\} + 2\delta.$$

Now the first line of (2.4) shows that $\mathrm{d}(ghx, x) + \mathrm{d}(hgx, x) > 2\,\mathrm{d}(gx, x)$, so the first term in the above maximum is redundant. Two applications of (2.4) then tell us that we have

$$\mathrm{d}(gx, x) + 6\delta + \varepsilon \le \mathrm{d}(ghgx, x) \quad \text{and} \quad \mathrm{d}(hx, x) + 6\delta + \varepsilon \le \mathrm{d}(ghgx, x). \tag{2.5}$$

Finally, we apply the four-point condition to the points $x, ghx, ghgx$, and $ghghx$. This gives us

$$\mathrm{d}(ghx, x) + \mathrm{d}(ghgx, x) \le \max\{\mathrm{d}(ghx, x) + \mathrm{d}(hx, x), \mathrm{d}(ghghx, x) + \mathrm{d}(gx, x)\} + 2\delta.$$

Similarly to before, (2.5) rules out the first term in the maximum. Applying (2.5) to the remaining inequality gives us

$$\mathrm{d}(ghx, x) + 6\delta + \varepsilon \le \mathrm{d}(ghghx, x),$$

which one straightforwardly rearranges to see that $\mathrm{d}(ghx, x) > \langle ghghx, x\rangle_{ghx}$. We now apply Lemma 2.4 to conclude that gh is loxodromic. A symmetrical argument concludes the same about hg. □

Proof of Theorem 2.3. Let $g \in \mathrm{Isom}(X)$ be an isometry that is neither elliptic, parabolic, nor loxodromic, and let $x \in X$. As g is not elliptic, the orbit $\{g^n x\}_{n\in\mathbb{N}}$ is unbounded. Since $X \cup \partial X$ is compact by Proposition 1.12 and g is not parabolic, there are subsequences $(g^{n_i}x)$ and $(g^{m_i}x)$ converging to distinct points $a, b \in \partial X$.

By the definition of the topology on ∂X, there is some $r \ge 0$ and $N \in \mathbb{N}$ such that

$$\langle g^{n_i}x, g^{m_i}x\rangle_x \le r$$

for all $i \ge N$. Now since g has unbounded orbits, there is $N' \in \mathbb{N}$ such that $\mathrm{d}(g^n x, x) \ge 2r + 6\delta$ for $n \ge N'$. Choosing $i \ge \max\{N', n_N, m_N\}$ allows us to apply Lemma 2.5 to see that $g^{n_i+m_i}$ is loxodromic. This contradicts the fact that g is not loxodromic. □