Geometric group theory
Lecture 8

Lawk Mineh

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## 1 Ping-pong and the Tits' alternative

The following criterion allows us to build free groups of a given rank in a convergence group, given certain loxodromic elements. The general idea is to find some elements that bat some disjoint subsets of the space back and forth (hence the 'ping-pong'), and these elements necessarily generate a free group as a result. Ping-pong type arguments apply in many settings, and were initially utilised by Jacques Tits, who used them to prove that every subgroup of a finitely generated linear group either has a finite index solvable subgroup, or contains a non-abelian free group. We will see shortly that a sort of analogue of this theorem holds for convergence groups.

**Proposition 1.1** (Ping-pong lemma). *Let $G$ be a convergence group on compact metrisable $M$. If $g_1, \dots, g_n \in G$ are loxodromic elements with disjoint fixed point sets in $M$, then there are $m_1, \dots, m_n \in \mathbb{Z}$ such that $g_1^{m_1}, \dots g_n^{m_n}$ generate a free group of rank $n$.*

*Proof.* For convenience we treat the case that $n = 2$; the general argument is virtually identical. Let $g$ and $h$ be loxodromic elements of $G$ with disjoint fixed point sets. Pick neighbourhoods $U_+, U_-, V_+$, and $V_-$ of $\{P_g, N_g\}$ and $\{P_h, N_h\}$ respectively, whose closures are disjoint. Since $(g^n)$ and $(h^n)$ are convergence sequences, we may replace $g$ and $h$ be powers such that

$$\begin{aligned} g(M - U_-) \subset U_+ \quad &\text{and} \quad g^{-1}(M - U_+) \subset U_-, \\ h(M - V_-) \subset V_+ \quad &\text{and} \quad h^{-1}(M - V_+) \subset V_-. \end{aligned} \tag{1.1}$$

Let $F = F(g, h)$ be the free group generated by $g$ and $h$ and consider the natural homomorphism $\varphi \colon F \to \langle g, h \rangle \leqslant G$. We must show that $\varphi$ is injective.

For every word $w$ in $A = \{g, h, g^{-1}, h^{-1}\}$, we inductively define the set $X(w) \subseteq M$ as follows. Call $U(g) = U_+, U(g^{-1}) = U_-, U(h) = V_+, U(h^{-1}) = V_-$. We define

$$X(a) = \varphi(a) \cdot \bigcup_{b \neq a^{-1}} U(b)$$

for any letter $a \in A$. Finally, we inductively define $X(aw) = \varphi(a) \cdot X(w)$ for any letter $a \in A$ and word $w$ in $A$. Now let $w$ be a reduced word in $A$ with $\varphi(w) = 1$. If $w$ is non-empty, then $X(w) \subseteq U(a)$ by (1.1), where $a$ is the first letter of $w$. But this means that $\varphi(w)$ acts non-trivially on the sets $U(b)$, where $b \neq a^{-1}$, contradicting that $\varphi(w) = 1$. Hence $w$ is the empty word, and $\varphi$ is injective as required. $\square$

We show that this is in fact a generic situation in a non-elementary convergence group, in the sense that loxodromic fixed point pairs are dense in the limit set. As a preliminary, we first prove a one-sided version of this.

**Lemma 1.2.** *Let $G$ be a convergence group on $M$ such that $\Lambda G$ has at least two points. Then for every open subset $U \subseteq M$ with non-empty intersection with $\Lambda G$, there is a loxodromic element of $G$ with a fixed point in $U$.*

*Proof.* Let $U \subseteq M$ be an open subset, and pick limit points $a \in U$ and $a' \neq a$. Since $a, a' \in \Lambda G$, there are convergence sequences $(g_i)$ and $(h_i)$ whose attracting points are $a$ and $a'$ respectively. Let $b$ and $b'$ be the repelling points of $(g_i)$ and $(h_i)$. Let $V$ be a neighbourhood of $a'$ whose closure is disjoint from that of $U$, and $b, b' \notin V$ (if $b, b' \neq a'$).

If $a \neq b$, then $g_i$ is loxodromic with one fixed point in $U$ for sufficiently large $i$. Similarly, if $a' \neq b'$, then $h_i$ is loxodromic with a fixed point in $V$ for sufficiently large $i$. Since $(g_j)$ converges locally uniformly on $M - \{b\}$ to the constant function on $a$, for sufficiently large $j$, the element $g_j h_i g_j^{-1}$ is a loxodromic with fixed point in $g_j a' \in U$.

It thus remains to consider only the case that $a = b$ and $a' = b'$. Then $h_i \overline{U} \subseteq V$ for sufficiently large $i$, and likewise $g_j \overline{V} \subseteq U$ for sufficiently large $j$. Hence $g_j h_i \overline{U} \subseteq U$, whence $g_j h_i$ is loxodromic with a fixed point in $U$. $\square$

**Theorem 1.3.** *Let $G$ be a convergence group on $M$. For any disjoint open subsets $U$ and $V$ that have non-empty intersection with $\Lambda G$, there is a loxodromic element $g \in G$ with $P_g \in U$ and $N_g \in V$.*

*Proof.* By Lemma 1.2, there are loxodromic elements $g$ and $h$ with $P_g \in U$ and $P_h \in V$. If $N_g = P_h$ or $N_h = P_g$, then the fixed points of $g$ and $h$ coincide, and so $g$ is a loxodromic with endpoints in $U$ and $V$. Suppose otherwise then, that the fixed points of $g$ and $h$ are distinct from one another. Then by a lemma from the last lecture, $k = g^n h^{-n}$ yields a loxodromic element with $P_k \in U$ and $N_k \in V$, for large enough $n$. $\square$

The density of loxodromic fixed point pairs gives us the aforementioned dichotomy for subgroups of a convergence group.

**Corollary 1.4** (Tits' alternative)**.** *Let $G$ be a convergence group, $H \leqslant G$ a subgroup. If $H$ is non-elementary, it contains a non-abelian free subgroup.*

*Proof.* Since $H$ is nonelementary, $\Lambda H \subseteq \Lambda G$ is uncountable. In particular, $\Lambda H$ contains at least four points. Picking disjoint neighbourhoods of these, we can apply Theorem 1.3 to find a pair of loxodromic elements with distinct poles in $\Lambda H$. The theorem now follows from Proposition 1.1. $\square$

## 2 Uniform convergence groups

We now analyse the behaviour of limit points in the limit set of a convergence group in depth. A particularly important class of limit points are the *conical limit points*, which are points that are well-approximated by orbits of the group.

**Definition 2.1** (Conical limit point)**.** Let $G$ be a convergence group on compact metrisable space $M$. A point $p \in M$ is called a *conical limit point* if there are distinct points $a, b \in M$ and a sequence $(g_i)$ in $G$ such that $g_i p \to b$ and $g_i q \to a$ for all $q \in M - \{p\}$.

Replacing $(g_i)$ in the above with a convergence subsequence and taking inverses, it is immediate that a conical limit point of a convergence group $G$ is in fact a limit point of $G$ (that is, it is contained in $\Lambda G$). The origin of 'conical' in the name above is in reference to a characterisation of such points in classical hyperbolic geometry. Suppose that $p \in \partial \mathbb{H}^n$ is a point in the boundary of $\mathbb{H}^n$, and $G \leqslant \mathrm{Isom}(\mathbb{H}^n)$ is a discrete group of isometries. A neighbourhood of a line in $\mathbb{H}^n$ tending to $p$ is a exactly a cone in the upper half space model, and $p$ is a conical limit point of $G$ if and only if there is an infinite $G$-orbit contained in such a cone.

**Definition 2.2** (Uniform convergence group)**.** A convergence group on compact metrisable space $M$ is called *uniform* if every point of $M$ is a conical limit point.

Of course, a uniform convergence group is necessarily minimal. The power of the above definition lies in the fact that the dynamics of conical limit point is very constrained. That is, one has strong control over the types of elements and subgroups that fix conical limit points.

**Example 2.3.** If $a$ and $b$ are fixed points of a loxodromic element $g$, then the sequences $(g^n)$ and $(g^{-n})$ are witnesses to the fact that $a$ and $b$ are conical limit points.

On the other hand, parabolic fixed points can never be conical limit points.

**Lemma 2.4.** *Let $G$ be a convergence group on $M$. If $p$ is a parabolic fixed point, then it is not a conical limit point.*

*In particular, a uniform convergence group contains no parabolic elements.*

*Proof.* Suppose that $p \in M$ is both a parabolic fixed point and a conical limit point. Thus there is a parabolic element $g \in G$ with fixed point $p$ and also a sequence $(h_n)$ and distinct points $a, b \in M$ with $h_n p \to b$ and $h_n q \to a$ for $q \neq p$. We may suppose without loss of generality that $h_n p \neq a$ for any $n$, by deleting finitely many terms in the sequence.

Fix some $n \in \mathbb{N}$ and consider the sequence $(h_i^{-1} h_n)_{i \in \mathbb{N}}$. By definition, we have that $h_i^{-1} h_n q \to p$ for $q \neq h_n^{-1} a$. Passing to a convergence subsequence, $(h_i^{-1} h_n)$ has attracting point $p$ and repelling point $h_n^{-1} a \neq p$. Hence the elements $h_i^{-1} h_n$ are eventually loxodromic.

Consider the sequence of elements $k_n = h_n g h_n^{-1}$ for $n \in \mathbb{N}$. Since $k_n$ is a conjugate of a parabolic element, it is also parabolic. We claim that $k_m$ and $k_n$ are distinct for any $n$ and sufficiently large $m$ (and so the sequence has infinitely many distinct terms).

Indeed, suppose otherwise, that $k_m = k_n$. This implies that $h_m g h_m^{-1} = h_n g h_n^{-1}$ or, to rephrase, that $g$ commutes with $h_m^{-1} h_n$. However, for sufficiently large $m$, we saw that $h_m^{-1} h_n$ is loxodromic, and loxodromic fixed points cannot be parabolic fixed points, a contradiction. Hence we may pass to a convergence subsequence of $(k_n)$.

Now since $g$ fixes $p$, we have $k_n(h_n p) = h_n g p = h_n p \to b$. On the other hand, for any $q \neq p$, we have $h_n g q \to a$ and so $k_n(h_n q) = h_n g q \to a$. It follows that $a$ and $b$ are attracting and repelling points for $k_n$ and so $k_n$ is loxodromic for large enough $n$. However, then $k_n$ and $g$ share a single fixed point, which is a contradiction. $\square$

The following emphasises that conical points, when they actually are fixed points, behave like loxodromic fixed points. Indeed, one can think of conical limit points as points in $M$ that want to be loxodromic fixed points. We will see later that non-elementary convergence groups are countable, and as such there can only be countably many loxodromic fixed points, while their limit sets are uncountable.

**Proposition 2.5.** *Let $G$ be a convergence group on $M$, and suppose that $p \in M$ is fixed by infinitely many elements $(h_j)$ of $G$. If $p$ is a conical limit point, then $h_j$ is loxodromic for some $j \in \mathbb{N}$.*

*Proof.* Let $a \neq b$ be points and $(g_i)$ a sequence as in the definition of conical limit point, so $g_i p \to b$ and $g_i q \to a$ for $q \neq p$. We may pass to a convergence subsequence of $(g_i)$; necessarily $p$ is its repelling point and $a$ is its attracting point.

Suppose first that for some $j \in \mathbb{N}$, there are infinitely many distinct conjugates $k_i = g_i h_j g_i^{-1}$. Thus we may pass to a convergence subsequence of $(k_i)$. Fix some $q \neq p$, and define $p_i = g_i p$ and $q_i = g_i q$. We have that $k_i p_i = p_i \to b, q_i \to a$, and $k_i q_i = g_i h_j q \to a$, as $h_j q \neq p$. It follows that $a$ and $b$ are the attracting and repelling points of $(k_i)$. As $a \neq b$, the terms of this sequence are eventually loxodromic. Since $h_j$ is conjugate to $k_i$, this implies that $h_j$ is in fact loxodromic.

On the other hand, suppose that for each $j$, there are finitely many distinct conjugates $g_i h_j g_i^{-1}$. We may thus pass to a subsequence of $(g_i)$ such that for each $j \in \mathbb{N}$, there is $i(j) \in \mathbb{N}$, such that for $i \geq i(j)$, the sequence $(g_i h_j g_i^{-1})$ is constant. Define $\varphi \colon H \to G$ as $\varphi(h_j) = g_{i(j)} h_j g_{i(j)}^{-1}$, where $H = \{h_j \mid j \in \mathbb{N}\}$. It is straightforward to check that $\varphi$ is injective.

Now let $j \in \mathbb{N}$. Observe that for $i \geq i(j)$, we have $\varphi(h_j) p_i = g_i p$, so that $\varphi(h_j) p_i \to b$. Similarly, $\varphi(h_j) q_i \to a$. Hence both $a$ and $b$ are fixed by each $\varphi(h_j)$. There is thus a convergence subsequence of $(\varphi(h_j))$ with attracting and repelling points $a \neq b$. This, in turn, shows that the elements $\varphi(h_j)$ are eventually loxodromic. Now $h_j$ is conjugate to $\varphi(h_j)$, this shows $h_j$ is loxodromic for sufficiently large $j$. $\square$