Geometric group theory
Lecture 7

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1 Convergence groups

We have already seen that boundaries provide a useful invariant for the coarse geometric features of a hyperbolic space. When it comes to groups, it turns out that one can further study the dynamical properties of the action of the group on its boundary to recover algebraic information about the group. In this section we will build a framework for understanding hyperbolic groups through their actions on boundary spaces.

If G is a group acting by isometries on a hyperbolic space X, every element of G induces a homeomorphism of its boundary ∂X. This gives us a representation $G \to \mathrm{Homeo}(\partial X)$. To describe what sort of action this is, we will take a step outwards in terms of the level of abstraction, and consider convergence groups: groups that act by homeomorphisms on arbitrary metrisable compacta. The motivation for the following definition really comes from the action of subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ acting on the boundary sphere of $\mathbb{H}^n$. Indeed, the setting of subgroups of $\mathrm{Homeo}(S^n)$ arising from isometry groups of hyperbolic space is the origin of the notion of a convergence group.

Definition 1.1 (Convergence sequence)**.** Let M be a compact metrisable space, and G a group acting on M by homeomorphisms. An sequence of elements $(g_n)_{n\in\mathbb{N}}$ of G is called a *convergence sequence* there are points $a, b \in M$ such that g_n converges locally uniformly on $M - \{b\}$ to the constant function on a.

We call the points a and b the *attracting* and *repelling* points for the sequence (g_n). Note that a and b need not be distinct.

Exercise 1.2. Show that if (g_i) is a convergence sequence with attracting point a and repelling point b, then (g_i^{-1}) is a convergence sequence will attracting point b and repelling point a.

Definition 1.3 (Convergence group)**.** Let M be a compact metrisable space, and G a group acting on M by homeomorphisms. We say that G is a *convergence group on M* if every infinite sequence of distinct elements has a convergence subsequence.

The *limit set* of G is the set ΛG of accumulation points of G-orbits of points in M. We say that the action is *minimal* if $\Lambda G = M$. Further the action is *elementary* if ΛG has at most two points, and is *non-elementary* otherwise.

Example 1.4.

- A finite group is a convergence group on the empty set, and every group is a convergence group on a point or two points.
- Let G be a group acting properly discontinuously on a hyperbolic space X. We will see later that G is a convergence group on ∂X.

Exercise 1.5. Suppose G is a convergence group on compact metrisable space M. Show that if G is non-elementary, then ΛG is uncountable.

Remark 1.6. It is immediate from the definition of a convergence group that the map $G \to \text{Homeo}(M)$ has finite kernel, when G is a non-elementary convergence group on M.

Remark 1.7. The above definition is sometimes referred to as a *discrete convergence group* in the literature. A general convergence sequence (g_i) is a sequence that is either a convergence sequence in the above sense, or otherwise converges uniformly to a homeomorphism $g \in \text{Homeo}(M)$; a general convergence group is a group that acts on a compact metrisable space and every infinite sequence has a general convergence subsequence. This notion is of interest as it allows one to study non-discrete group actions on compact spaces, but here we will only be interested in discrete groups.

We will engage in an analysis of the elements of convergence groups by studying fixed points and limit points on the boundary. Firstly, as with isometries of hyperbolic spaces, elements of convergence groups fall into a familiar trichotomy.

Definition 1.8 (Elements of convergence groups)**.** Let G be a convergence group on compact metrisable space M, and let $g \in G$ be an element. Then we say g is:

(i) *elliptic* if it has finite order;
(ii) *parabolic* if it is of infinite order and has exactly one fixed point in M; or
(iii) *loxodromic* if it is of infinite order and has exactly two fixed points in M.

Lemma 1.9. *Every element of a convergence group is either elliptic, parabolic, or loxodromic.*

Proof. Let G be a convergence group on M and $g \in G$ an infinite order element. Then there is some sequence (g^{n_i}) that is a convergence sequence with attracting point a and repelling point b. Now $g^{n_i}(gp) = gg^{n_i}p \to ga$ uniformly away from $g^{-1}b$, so that ga is also an attracting point for (g^{n_i}). Hence $ga = a$, so g has a fixed point in M. Moreover, since the sequence convergence to a constant function on a locally uniformly outside $M - \{b\}$, the only possible fixed points of g are a and b. Therefore any infinite order element is either parabolic or loxodromic. $\square$

We have a dynamical criterion for being a loxodromic element. It allows us to reduce the rather exact property of having fixed points to a topological nesting property that is easier to verify.

Lemma 1.10. *Let G be a convergence group on M and $g \in G$ an element. If there is a proper open subset $U \subseteq M$ such that $g\overline{U} \subseteq U$, then (g^i) is a convergence sequence with attracting point in U and repelling point in $M - \overline{U}$. Moreover, g is a loxodromic element.*

Proof. Consider the sets $A = \bigcap g^n \overline{U}$ and $B = \bigcap g^{-n}(M - U)$. It is immediate from the definition that A and B are fixed by g. By definition, A and B are disjoint. We show that A and B both consist of singletons, which proves the lemma.

As G is a convergence group, there is a sequence (n_i) such that (g^{n_i}) is a convergence sequence with attracting point a and repelling point b. Of course, we must have $a \in A$ and $b \in B$. Suppose that $c \in B$ with $c \neq b$. Then $g^{n_i} c \in U$ for sufficiently large i. However, B is disjoint from U, so this is a contradiction. Hence $B = \{b\}$ and by a symmetrical argument with (g^{-n_i}) shows $A = \{a\}$. $\square$

Exercise 1.11. Use the previous lemma to show that if (g_i) is a convergence sequence with distinct attracting and repelling points, the elements g_i are eventually loxodromic.

Lemma 1.12. *Let G be a convergence group on M, and let $g \in G$ be an infinite order element. Then (g^i) is a convergence sequence whose attracting and repelling points coincide with the fixed points of g.*

Proof. Let (g^{n_i}) be a convergence subsequence of (g^i) with attracting point a and repelling point b. As in the proof of Lemma 1.9, a and b are necessarily fixed points of g. Suppose g is parabolic and that (g^i) is not a convergence sequence. Then there is a sequence of points $p_i \in M$ such that $p_i \to p \neq a$, and $g^{n_i} p_i \to q \neq a$, after possibly passing to a subsequence of (g^{n_i}). But then (g^{n_i}) has attracting point a and repelling point $p \neq a$. This implies that p and a are distinct fixed points of g, which contradicts the fact that g is parabolic.

Now suppose g is loxodromic. Then pick some neighbourhood U of a with $b \notin \overline{U}$. The set $\overline{U}$ is a compact subset not meeting b. As (g^{n_i}) is a convergence sequence, there is some n_i such that $g^{n_i} \overline{U} \subseteq U$. Now by Lemma 1.10, the element $h = g^{n_i}$ is loxodromic. Moreover, (h^j) is a convergence sequence with attracting point a and repelling point b. It follows that (g^i) is a convergence sequence also, since h is a power of g. $\square$

In light of the above, it makes sense to give special name to the fixed points corresponding to an element of a convergence group.

Definition 1.13 (Poles)**.** Let G be a convergence group on M and $g \in G$ an infinite order element. We write P_g (respectively, N_g) for the attracting (respectively, repelling) point of the convergence sequence (g^i), and we call it the *positive* (respectively, *negative*) *pole of g*.

Of course, for a parabolic element, the positive and negative poles are the same point.

In hyperbolic space, if two axes share a single point at infinity, then translations along those axes do not generate a discrete subgroup of isometries. This behaviour is reflected in the discrete nature of convergence groups (cf. Remark 1.7), in the form that two elements cannot share one pole without sharing the other.

Lemma 1.14. *Let G be a convergence group on M, and $g, h \in G$ infinite order elements. Then the fixed point sets of g and h in M are either disjoint or coincide.*

Proof. If both g and h are parabolic, the statement is trivial. Suppose that both g and h are loxodromic, and suppose that $P_g = P_h$ while $N_g \neq N_h$. Let U be a neighbourhood of P_g such that $N_g, N_h \notin \overline{U}$. By Lemma 1.12, (g^i) and (h^i) are convergence sequences with attracting point P_g and repelling points N_g and N_h respectively. Then there are $i, j \in \mathbb{N}$ such that $g^i\overline{U}$ and $h^j\overline{U}$ are contained in U. For conveience, we relabel so that $g = g^i$ and $h = h^j$.

Define $F = \overline{U} - gU \neq \emptyset$, and note that the sets g^iF cover $U - \{P_g\}$. Let $p \in F$ be a point. Then for each i, there is n_i such that $h^ip \in g^{n_i}F$. Necessarily, $n_i \to \infty$ as $i \to \infty$. Let $k_i = g^{-n_i}h^i$, so that k_i fixes P_g, and $k_iN_h = g^{-n_i}N_h \to N_g$. This implies that there are infinitely many distinct k_i, since $N_g \neq N_h$. Let (k_{m_i}) be a convergence subsequence. By the above, the attracting and repelling points of (k_{m_i}) must be among P_g, N_g, and N_h. However, $k_i(p) \in F$ for all i and $P_g, N_g, N_h \notin \overline{F}$, a contradiction.

Finally, if g is loxodromic and h is parabolic, then g and hgh^{-1} are loxodromic elements with one shared fixed point, which we have just shown to be impossible. □

Exercise 1.15. Show that if g is a loxodromic element of non-elementary convergence group G, then the stabiliser of $\mathrm{fix}(g)$ contains $\langle g \rangle$ has a finite index subgroup.

Exercise 1.16. Suppose that G is a non-elementary convergence group, $H \leqslant G$ a subgroup. Show that if ΛH consists of two points, then H is contains a loxodromic element g such that $\langle g \rangle$ has finite index in H.

The following gives us a way to construct a loxodromic with prescribed fixed points out of two others. We will see some powerful algebraic consequences of this fact in the next lecture.

Lemma 1.17. *Let G be a convergence group on M, and suppose that $g, h \in G$ are loxodromic elements with disjoint fixed point sets. Let U and V be neighbourhoods of P_g and P_h respectively. Then there is $n \geq 1$ such that $k = g^nh^{-n}$ is loxodromic with $P_k \in U$ and $N_k \in V$.*

Proof. Let U_+, U_-, V_+, and V_- be neighbourhoods of P_g, N_g, P_h, and N_h respectively, whose closures are disjoint. We may suppose that $U_+ \subseteq U$ and $V_+ \subseteq V$. For i sufficiently large, we have the inclusions

$$g^i\overline{M - U_-} \subseteq U_+,\ g^{-i}\overline{M - U_+} \subseteq U_-,\ h^i\overline{M - V_-} \subseteq V_+,\ \text{and}\ \ h^{-i}\overline{M - V_+} \subseteq V_-. \tag{1.1}$$

Let $k = g^ih^{-i}$ and observe that (1.1) implies that

$$k\overline{U_+} \subseteq g^ih^{-i}\overline{M - V_+} \subseteq g^iV_- \subseteq g^i\overline{M - U_-} \subseteq U_+.$$

Similarly, $k^{-1}\overline{V_+} \subseteq V_+$. Applying Lemma 1.10, we see that k is a loxodromic element whose fixed points lie in $U_+ \subseteq U$ and $V_+ \subseteq V$, as required. □