Geometric group theory
Lecture 13

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1 Free constructions

There are some natural constructions in the category of groups that allow one to build larger groups out of smaller ones. Of course, among these are things like direct products (which is the categorical product) and, more generally, group extensions. We are interested here in the more free constructions of this variety. The most basic of these operations is the free product.

Definition 1.1 (Free product)**.** Let G and H be groups. The *free product* $G * H$ of G and H is the coproduct of G and H. That is, for any group K and any homomorphisms $\varphi\colon G \to K, \psi\colon H \to K$, there is a unique homomorphism $f\colon G * H \to K$ such that the following commute:

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\varphi"'] & G * H \arrow[d, dashed, "f"] & H \arrow[l] \arrow[dl, "\psi"] \\ & K & \end{tikzcd}$$

Similarly to free groups, it is clean to define a free product in terms of a universal property, but it is often useful to have a model to work with (and to show that a coproduct actually exists!). We say a word in $G \sqcup H$ is *reduced* if it contains no consecutive pairs of the form gg' with $g, g' \in G$ or hh' with $h, h' \in H$. That is, it strictly alternates between letters in G and letters in H. There is an obvious reduction relation, and we can verify that the group of equivalence classes of reduced words in $G \sqcup H$ (with the operation of concatenation of representatives) is in fact the free product $G * H$.

Exercise 1.2. Show that if $G = \langle S \mid Q \rangle$ and $H = \langle T \mid R \rangle$ are presentations, then $G * H$ has the presentation $\langle S \cup T \mid Q \cup R \rangle$.

It can be useful to consider these free constructions in the context of topological spaces. The Seifert–van Kampen tells us that the fundamental group of the wedge of two locally contractible spaces is a free product of the fundamental groups of the two spaces. Of course, the wedge and the free product are both categorical coproducts (in the category of pointed topological spaces and groups respectively), and the π_1 map is functorial. More generally, this theorem tells us that if we glue two spaces together along some open path-connected subspace, the fundamental group is a *pushout* of the corresponding fundamental groups. This brings us to the more general form of the free product.

Definition 1.3 (Amalgamated free product)**.** Let G, H, and K be groups, and suppose that $\varphi\colon K \to G, \psi\colon K \to H$ are injective homomorphisms. The *free product of* G *and* H *amalgamated over* K is the pushout of φ and ψ. Namely, it is the group $G *_K H$ such that for any group homomorphisms $G \to L$ and $H \to L$, there is a unique homomorphism $G *_K H \to L$ such that the following diagram commutes:

$$\begin{tikzcd}
K \arrow[r, "\varphi"] \arrow[d, "\psi"'] & G \arrow[d] \arrow[ddr, bend left] & \\
H \arrow[r] \arrow[drr, bend right] & G *_K H \arrow[dr, dashed] & \\
& & L
\end{tikzcd}$$

In a slight abuse of notation, we usually suppress mention of the homomorphisms φ and ψ entirely, and treat K as a common subgroup of both G and H. Again, amalgamated free products have an obvious presentation.

Exercise 1.4. In the notation of the above definition, show that $G *_K H$ has the presentation of $G * H$ with the added relations that $\varphi(k) = \phi(k)$ for each $k \in K$.

Show that if G and H are finitely presented and K is finitely generated, then $G *_K H$ is finitely presented.

Like in a free product, elements in amalgamated free products can also be written in a unique minimal way as reduced words in G and H, though their description is a little more involved. We call such expressions *normal forms* for elements. That such a normal form exists once one fixes a transversal of the amalgamating subgroup in each of the factors is a consequence of the following theorem. Note that in a free product, the amalgamating subgroup is trivial, so that the transversals comprise the entirety of each factors, and hence every reduced word is already a normal form.

Theorem 1.5. *Let* G *and* H *be groups,* $K \leqslant G, H$ *a common subgroup. Let* $a_1, \ldots, a_n \in G *_K H$ *that alternate between images of either* G *or* H*. If* $a_1 \ldots a_n = 1$*, then either*

1. $n = 1$ *and* $a_1 = 1$*;*
2. $n > 1$ *and there is* $i = 1, \ldots, n$ *such that* a_i *is in the image of* K*.*

The other main free construction of importance to infinite groups is known as the *HNN extension*, named after its inventors Graham Higman and Bernard and Hannah Neumann. It is a little harder to describe this construction with a universal property (it is a homotopy colimit), so we give the traditional presentation-based definition.

Definition 1.6 (HNN extension)**.** Let G be a group, $H \leqslant G$ be a subgroup, and $\varphi\colon H \to G$ an injective homomorphism. The *HNN* extension $G*_\varphi$ is the group with the presentation

$$\langle G, t \,|\, tht^{-1} = \varphi(h) \text{ for all } h \in H\rangle.$$

We call the subgroups H and $\varphi(H)$ the *associated subgroups* of the HNN extension, G the *base* of the extension, and t is called the *stable letter*.

Of course, an HNN extension is in general not an actual group extension. Again in analogy with spaces, HNN extensions correspond to fundamental groups of *partial mapping tori* — spaces one obtains by gluing pieces of another space to itself, along a cylinder say. This goes some way to explaining why there is no simple universal property for this construction, since there is no interval object in the category of groups. There are also normal forms for elements in an HNN extension, similarly to amalgamated free products. This is a consequence of the following.

Theorem 1.7 (Britton's lemma)**.** *Let $G*_\varphi$ be an HNN extension of a group G with associated subgroups H and $\varphi(H)$, with stable letter t. Let $g_0, \ldots, g_n \in G$ and $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$. Suppose that $g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \ldots t^{\varepsilon_n} g_n = 1$. Then either*

1. *$n = 0$ and $g_0 = 1$;*
2. *$n > 0$ and for some $1 \leq i \leq n-1$, we have $\varepsilon_i = -\varepsilon_{i+1}$ and either $g_i \in H$, if $\varepsilon_i = 1$, or otherwise $g_i \in \varphi(H)$, if $\varepsilon_i = -1$.*

Example 1.8. Let G be a group, and $\varphi\colon G \to G$ an automorphism. Then the HNN extension $G*_\varphi$ with associated subgroups G and $\varphi(G) \cong G$ is exactly the semi-direct product $G \rtimes_\varphi \mathbb{Z}$, where $\mathbb{Z}$ is the infinite cyclic subgroup generated by the stable letter.

Example 1.9. The Baumslag–Solitar group $\mathrm{BS}(m, n)$ is an HNN extension $\mathbb{Z}*_\varphi$, where φ is an isomorphism of the subgroups $m\mathbb{Z}$ and $n\mathbb{Z}$ of $\mathbb{Z}$.

It is a straightforward consequence of Britton's lemma that the natural inclusion of the base group into an HNN extension is an embedding. As such, HNN extensions are a particularly useful tool for building embeddings of groups. For instance, they play a significant role in the proof of Higman's theorem, which states that a finitely generated group is recursively presented if and only if it embeds into a finitely presented group. We give an simpler example of such an application.

Theorem 1.10. *Every countable group embeds in a group generated by two elements.*

Proof. Let $C = \{c_n \,|\, n \geq 0\}$ be a countable group, and let $F = C * \langle a, b\rangle$ be the free product of C with the free group on a and b. For simplicity, we assume that $c_0 = 1$ is the identity in C. Now the set $\{b^i ab^{-i} \,|\, i \geq 0\}$ freely generates an infinite rank free group H in $\langle a, b\rangle$, and similarly $\{c_i a^i ba^{-i} \,|\, i \geq 0\}$ also freely generates an infinite rank free group K in C. Take $G = F*_\varphi$ to be an HNN extension of F, where $\varphi\colon H \to K$ is such that $\varphi(b^i ab^{-i}) = c_i a^i ba^{-i}$ for each $i \geq 0$. Of course, C is embedded in F, which is in turn embedded in G, so C embeds in G. Moreover, G has the presentation

$$G = \langle F, t \,|\, tat^{-1} = b, tb^i ab^{-i} t^{-1} = c_i a^i ba^{-i}, i \geq 1\rangle,$$

from which it can be seen that a and t form a generating set. $\square$

The above allows one to equip any countable group with a proper metric, as a subspace of a 2-generated group it embeds in with respect to its word metric. It should not be immediately obvious that this is possible, but that it is allows one to study the coarse geometry of countable groups.

Exercise 1.11. Use Britton's lemma to show that every finite order element of an HNN extension $G*_\varphi$ is conjugate into G.

2 Decompositions of groups

We will see that the free constructions above play a crucial role in understanding the algebraic structure of infinite groups. First, we will need a definition.

Definition 2.1 (Ends of a space)**.** Let X be a topological space, and $K_1 \subseteq K_2 \subseteq \dots$ a sequence of nested compact subsets, the union of whose interiors covers X. An *end* of X is a nested sequence $U_1 \supseteq U_2 \supseteq \dots$, with each U_i a connected component of $X - K_i$.

If G is group with finite generating set S, then $e(G)$ is the number of ends of the space $\Gamma(G, S)$.

The ends of space are straightforwardly seen to be independent of the choice of exhaustion (K_i). As a consequence, the space of ends is a quasi-isometry invariant among proper metric spaces. It follows also that the ends of a group do not depend on the choice of generating set. For a hyperbolic group G, the ends are exactly the connected components of the boundary ∂G. The following may be reminiscent of a similar fact we saw for the cardinalities of convergence groups.

Exercise 2.2. Show that a finitely generated group has $0, 1, 2$, or infinitely many ends.

Ends of groups were introduced independently by Freudenthal and Hopf. It is obvious that the finitely generated groups with zero ends are exactly the finite groups, as the Cayley graph of every finitely generated infinite group contains is unbounded. Freudenthal and Hopf both also obtained the following characterisation of the two-ended groups.

Theorem 2.3. *Let G be a group with two ends. Then G contains an infinite cyclic group of finite index.*

The above admits a great variety of different proofs. While most are not especially difficult or long, they are also not particularly easy or short, so we omit the proof here. In the 60s, Stallings obtained the following striking result, which can be interpreted as one of the first major theorems in geometric group theory.

Theorem 2.4 (Stallings)**.** *A finitely generated group G has infinitely many ends if and only if G is an amalgamated free product $G = H *_K H'$ where $K \neq H, H'$ is a finite subgroup, or an HNN extension $G = H*_\varphi$ with finite associated subgroups.*