Geometric group theory
Lecture 11

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1 Algorithms and computability

Finite presentations are inherently a tool for practical computation and combinatorial manipulation. Accordingly, the most basic questions one can ask about them are related to computability. The first three such questions were originally posed by Dehn in the early twentieth century.

We will not concern ourselves with a precise notion of computability here: consider the truth of a statement *decidable* if there is an algorithm (that is, a sequence of operations) one can perform which, after finitely many steps, will return that the statement is either true or false. The *word* and *conjugacy problems* ask whether presentations can allow us to meaningfully distinguish elements of the groups they define.

Definition 1.1. Let $P = \langle S \mid R \rangle$ be a finite presentation. We say the *word problem* is solvable in P if, given any two words w and v in S, the statement that w and v represent the same element is decidable. Similarly, we say that the *conjugacy problem* is solvable in P if the statement that w and v represent conjugate elements is always decidable.

It is straightforward, by applying Tietze transformations, to see that having solvable word problem is really a property of the group that a given finite presentation defines, and is independent of the choice of finite presentation for that group. Hence we may rightly call solvability of these problems group properties. Of course, the word problem is a special case of the conjugacy problem, since the conjugacy class of the identity contains only the identity.

Remark 1.2. The word problem is always *semi-decidable*: there is an algorithm such that, if w and v *do* represent the same element, will eventually halt and return true. Namely, one can in a naïve way enumerate all words representing the identity in a finite presentation, and observe that if w and v represent the same element, the word $w^{-1}v$ will appear somewhere in this list. Of course, this list is infinite, so the algorithm will never terminate if w and v represent different elements.

Exercise 1.3. Show that a finitely presented simple group has solvable word problem.

Typically harder, though not as obviously so, is the problem of distinguishing one finite presentation from another.

Definition 1.4. Let $\mathcal{C}$ be a class of groups. We say the *isomorphism problem* is solvable in $\mathcal{C}$ if, for any two finite presentations P and Q of groups in $\mathcal{C}$, the statement that P and Q are presentations of isomorphic groups is decidable.

If one could solve the isomorphism problem over the class of all finitely presented groups, then certainly one has a solution to the conjugacy (and hence, also word) problem, as one can decide whether the presentation obtained by adjoining the desired conjugacy relation gives a different group or not.

A landmark result, first achieved by Novikov and, independently, Boone in the 1950s, provides a decidedly negative answer to all of these problems.

Theorem 1.5 (Novikov–Boone)**.** *There is a finitely presented group with unsolvable word problem.*

The proof of the above is quite difficult, and has deep ties to mathematical logic. Before moving back to hyperbolic groups, we mention a particularly striking result obtained independently by Adian and Rabin, around the same time as the Boone–Novikov result above. They showed that determining whether finite presentation has essentially any interesting property is undecidable.

We say that a group property P is a *Markov property* if there exists a finitely presented group with P, and also a finitely presented group that is not a subgroup of any finitely presented group with P. Among such properties are being finite, having solvable word problem, and being hyperbolic.

Theorem 1.6 (Adian–Rabin)**.** *Let P be a Markov property. It is undecidable whether any given finite presentation defines a group that has P.*

As a contrast to the rather unpleasant situation one finds oneself in for finitely presented groups in general, hyperbolic groups have excellent computability properties. In fact, all three of the above problems are solvable within the class of hyperbolic groups. In the remainder of this section, we present a solution to the word problem. The conjugacy problem also admits a solution that is not much more difficult, while the isomorphism problem requires a large number of advanced tools to solve.

Definition 1.7 (Dehn presentation)**.** A finite presentation P is called a *Dehn presentation* if any word in the presentation which represents the identity contains more than half of a relator.

Dehn presentations are so named because they exhibit the key property possessed by the standard presentation of higher genus surface groups which was used by Dehn to solve the world problem in such groups. Indeed, having a Dehn presentation yields a very easy solution to the word problem.

Lemma 1.8. *The word problem is solvable in a group with a Dehn presentation.*

Proof. Let $\langle S \mid R \rangle$ be a Dehn presentation for a group G, and let w be a word in S. We may reduce the word w as follows: if there is a relator $r = r_1 r_2 \in R$ with $\ell(r_1) < \ell(r_2)$ and w contains r_2 as a subword, then let w' be the word obtained from w by replacing this instance of r_2 with r_1^{-1}. By construction $\ell(w') < \ell(w)$, and w' represents the same element of G as w.

Applying such a reduction finitely many times, we obtain a word w'' representing the same element of G as w, none of whose subwords are more than half of a relator. Since the presentation was a Dehn presentation, w'' represents the identity if and only if it is the empty word, deciding the problem. □

The solution to the word problem in hyperbolic groups is a consequence of the fact that every hyperbolic group has a Dehn presentation. This not only shows that hyperbolic groups are finitely presentable, but finitely presentable in a very effective way.

Theorem 1.9. *Let G be a hyperbolic group. Then G has a Dehn presentation.*

Proof. Let S be a finite generating set for G, so that $X = \Gamma(G, S)$ is δ-hyperbolic. Let $k \geq 0, \lambda \geq 1, c \geq 0$ be constants such that every k-local geodesic is (λ, c)-quasigeodesic, which exist since X is hyperbolic. Let R be the set of cyclically reduced words in S with length at most $\max\{2k, \lambda c\}$. We will show that $\langle S \mid R \rangle$ is a Dehn presentation for G.

Suppose that w is a word representing the identity in G. We may cyclically reduce w, if it is not already cyclically reduced. Let p be the loop in X based at the identity, obtained by following the edges corresponding to the letters of w. Consider first the case that that p is a k-local geodesic. Then p is a (λ, c)-quasigeodesic with length $\ell(w)$. Since the distance between the endpoints of p is zero, it follows that $\ell(w) \leq \lambda c$. Hence $w \in R$.

Now if p is not a k-local geodesic, then w contains a minimal subword v of length at most k whose corresponding subpath q of p is not geodesic. Let q' be a geodesic in X with the same endpoints as q, and let u be the word corresponding to q'. Of course, q' also has length at most k, As v was minimal, q and q' have no overlap, so vu^{-1} is cyclically reduced. Moreover as the concatenation of q and q' is a loop in X, the word vu^{-1} represents the identity. Finally, uv^{-1} has length at most $\ell(q) + \ell(q') \leq 2k$, so $vu^{-1} \in R$. Moreover, $\ell(u) < \ell(v)$ since q was not a geodesic, so that w contains more than half of a word in R as required. Thus, if $\langle S \mid R \rangle$ is a presentation, it is a Dehn presentation.

To conclude, observe that if $w = w_1 v w_2$ and $r = vu^{-1} \in R$, we have

$$\begin{aligned} w &= w_1 v u^{-1} u w_2 \\ &= w_1 v u^{-1} w_1^{-1} w_1 u w_2 \\ &= (w_1 r w_1^{-1})(w_1 u w_2). \end{aligned}$$

If, further, $\ell(u) < \ell(v)$, then $w_1 u w_2$ is a strictly shorter word than w. By a finite induction, then, every word representing the identity in G may be written as a product of conjugates of elements of R. Hence $\langle S \mid R \rangle$ defines a genuine presentation of G. □

Corollary 1.10. *The word problem is solvable in hyperbolic groups.*

Remark 1.11. As it turns out, having a Dehn presentation is equivalent to hyperbolicity, though we do not prove it here. Thus, computability is in some ways intrinsically tied to hyperbolicity.

Exercise 1.12. Solve the conjugacy problem in hyperbolic groups.

(Hint: Let w and v be cyclically reduced words, and suppose that they are conjugate by u, so $uwu^{-1} = v$. Take u to be such a conjugator with minimal length. If $\ell(u)$ is much larger than $\ell(w)$ and $\ell(v)$, the geodesic rectangle in a Cayley graph with sides labelled by w, u^{-1}, v, u is very long and thin. Use this to bound the length of such u, and hence effectively decide whether two words are conjugate.)

2 Small cancellation theory

One of the original motivations driving the development of the theory of hyperbolic groups was the so-called *small cancellation theory*, a collection of results and techniques that had been taking shape over several decades beforehand. Small cancellation theory is part of combinatorial group theory, the study of groups by their presentations. Though the ideas are somewhat geometric in nature, the mathematics is mostly combinatorial; hyperbolicity puts many of the results of the theory on a coherent geometric framework.

The main theme of small cancellation theory is the analysis of presentations where the relators do not have significant overlap (hence the 'small cancellation'). There are many different 'small cancellation conditions'; here, we state just one of the most common and important among them.

Definition 2.1 (Piece)**.** Let $\langle S \mid R \rangle$ be a group presentation, and suppose that each relator $r \in R$ is cyclically reduced. Suppose further that R is *symmetrised*, so that R is closed under cyclic permutations and inverses of words.

A non-trivial word w in S is called *piece* of the presentation if w is a maximal common prefix for two distinct relators in R.

Definition 2.2 ($C'(\lambda)$ small cancellation condition)**.** Let $\langle S \mid R \rangle$ be a group presentation, $\lambda > 0$. We say that the presentation satisfies the $C'(\lambda)$ *condition* if, whenever w is a piece of a relator $r \in R$, we have $\ell(w) < \lambda \ell(r)$.

Many (but far from all!) facts about groups satisfying the above classical small cancellation condition have been subsumed by hyperbolic groups.

Theorem 2.3. *Let $G = \langle S \mid R \rangle$ be a finitely presented group, with the presentation satisfying $C'(\lambda)$ for $\lambda \leq \frac{1}{6}$. Then G is a hyperbolic group.*

We do not give a proof of the above theorem, as it would be too much of a diversion. One should think of the $\frac{1}{6}$ appearing in the statement with regards to classical geometry. The tightest tiling of the Euclidean plane by regular polygons is that by triangles, with at most six triangles touching each point; if one wants to put more triangles in, one has to turn to the hyperbolic plane. Classical small cancellation theory inherently deals with planar diagrams and their geometry, so this link is in some senses very explicit.

We now turn to a remarkable construction due to Rips, which allows one to generate hyperbolic groups with many pathological properties.

Theorem 2.4 (Rips' construction)**.** *Let Q be a finitely presented group. There is a hyperbolic group G with 2-generated normal subgroup $N = \langle x, y\rangle \lhd G$ such that $G/N \cong Q$.*

We will give a proof in the next lecture, and for now we state a couple of applications. We say that a finite presentation has *solvable membership problem* if there is an algorithm that can determine whether a given word represents an element of a given finitely generated subgroup. Again, of course, this is independent of the choice of finite presentation for a given group. The word problem is a special case of this.

Corollary 2.5. *There is a hyperbolic group with unsolvable membership problem.*

Proof. Let Q be a finitely presented group with unsolvable word problem. Applying the Rips construction to Q, we obtain a hyperbolic group G with 2-generated normal subgroup N such that $G/N \cong Q$. If the membership problem were solvable in G, then there would be an algorithm for deciding whether a word in a generating set for G represents an element of N. But this yields a solution to the word problem in Q, for an element in Q is non-trivial if and only if it has a lift in G that is not in N. □

Corollary 2.6. *There is a hyperbolic group with a finitely generated subgroup that is not finitely presentable.*

Proof. Let Q be a finitely presented group containing a finitely generated but not finitely presentable subgroup P, for instance a free group of rank 2. Applying the Rips construction to Q, we obtain a hyperbolic group G with Q as a quotient and finitely generated kernel N. Then the preimage K of P in G under the quotient $G \to Q$ is generated by preimages of generators of P together with N. As N is finitely generated, K is also finitely generated. However, since N is normal, adding the generators of N as relators to any presentation of K yields a presentation of P. Hence K cannot be finitely presented, as P is not finitely presented. □

Note that it follows from more advanced methods that N is not finitely presentable whenever Q is infinite: one can show that G has *cohomological dimension* 2, and among such groups any finitely presented normal subgroup is either finite, free, or has finite index in G. We will see the kernel of the Rips construction is infinite and never free, and if Q is infinite, it must have infinite index.