Geometric group theory
Lecture 10

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1 Boundaries of hyperbolic groups

Last time, we proved that if G is a group acting properly discontinuously by isometries on a proper hyperbolic space X, then the induced action of G on ∂X is a convergence action. For a hyperbolic group we see more is true.

Remark 1.1. It is open whether the converse of the above statement holds. That is, given a convergence group action of G on compact metrisable M, whether there is a proper hyperbolic space X admitting a properly discontinuous action by isometries such that ∂X is G-equivariantly homeomorphic to M.

It is true, by work of Sun, that convergence groups admit reasonably nice actions on hyperbolic spaces. However, the boundaries of these spaces do not recover the original compactum, and the actions are generally not properly discontinuous.

The classification of convergence group elements gives a somewhat more straightforward proof of the classification of isometries of a proper hyperbolic space.

Theorem 1.2. *Every isometry of a proper hyperbolic space is elliptic, parabolic, or loxodromic.*

Proof. Let X be a proper hyperbolic metric space, $g \in \mathrm{Isom}(X)$. If g does not have bounded orbits, then it must be an element of infinite order, and so $\langle g \rangle$ acts properly discontinuously on X. Hence $\langle g \rangle$ is a convergence group on ∂X. Every infinite order element of $\langle g \rangle$ is parabolic or loxodromic in the sense of a convergence group on ∂X, which coincides with the definitions for isometries of a hyperbolic metric space. $\square$

Of course, a hyperbolic group acts as a convergence group on ∂G, but more is true. We first need the following observation.

Exercise 1.3. Let X be a proper hyperbolic metric space admitting a cocompact group action by isometries. If X is unbounded, then ∂X contains at least two points.

(Hint: use the fact that there is no non-trivial group action on the ray $[0, \infty)$ by isometries.)

Theorem 1.4. *Let G be a group acting geometrically on a proper hyperbolic metric space X. Then G is a uniform convergence group on ∂X.*

Proof (sketch): Let X be a δ-hyperbolic metric space that X acts on geometrically. In the former case ∂X is empty and there is nothing to prove. Suppose that G is infinite, then. We know that G acts as a convergence group on ∂X, since the action of G on X is properly discontinuous. We need to show that every point of G is a conical limit point.

Let $p \in \partial X$ be an arbitrary point. By the previous exercise, ∂X contains at least two points, so let $a \in \partial X - \{p\}$. Fix some $x \in X$ as a basepoint and let $B \subseteq X$ be a compact set with $x \in B$ and $G \cdot B = X$. Let ℓ be a geodesic line through x with endpoints p and a. Take (g_i) to be the sequence of elements of G for which $g_i B$ meets ℓ.

As ∂X is compact, we may pass to a subsequence of (g_i) such that $g_i p$ converges to a point $b \in \partial X$. We claim that the sequence (g_i) and the points a and b serve as witnesses to the fact that p is a conical limit point. For every $i \geq 0$, picking $y \in \ell$ far enough from x in the direction of p ensures that x is between $g_i x$ and $g_i y$. That is to say, there is a sequence $y_j \to p$ such that $\langle g_i x, g_i y_j \rangle_x$ is uniformly bounded for $j \geq i$. Since $g_i y_j \to g_i p$ as $j \to \infty$, there is a neighbourhood of a excluding $g_i p$ for all i. It follows that $b = \lim g_i p \neq a$.

We will take for granted that $g_i a \to a$, as this is easier to prove. Let $q \neq p, a$, and let $r = \langle a, q \rangle_x < \infty$. It follows from hyperbolicity that there is $z \in [a, q]$ such that $\mathrm{d}(x, z) \leq r + 2\delta$. But then $z_i = g_i z$ is a point on a geodesic $[g_i a, g_i q]$ with $\mathrm{d}(g_i x, g_i z) \leq r + 2\delta$. As $g_i x \in g_i B$ is uniformly close to the geodesic ℓ, it follows that $\langle g_i x, g_i q \rangle_x$ is roughly equal to $\mathrm{d}(x, g_i x)$. This latter quantity tends to infinity as $i \to \infty$, so that $g_i q$ lies in any neighbourhood of $a = \lim g_i x$ for sufficiently large i. That is, $g_i q \to a$ as required. $\square$

Definition 1.5. Let G be a hyperbolic group. Suppose that G acts geometrically on a proper hyperbolic space X. The *boundary* ∂G of G is the space ∂X.

The boundary exists for any hyperbolic group: one can take the Cayley graph with respect to any finite generating set as the space X. Moreover, ∂G is well-defined up to homeomorphism for a given hyperbolic group G. Indeed, the Milnor–Schwarz lemma tells us that if X and Y are two proper hyperbolic spaces admitting geometric actions by G with X hyperbolic, there is a G-invariant quasi-isometry $X \to Y$. Since quasi-isometries of proper hyperbolic spaces induce homeomorphisms of the boundary, there is a G-equivariant homeomorphism $\partial X \to \partial Y$ whenever this is the case.

The previous theorem does have a converse, due to Bowditch, so that we have a completely dynamical reformulation of hyperbolic groups. However, it is much more difficult to prove, so we only give a very rough sketch here.

Theorem 1.6. *Let G be a uniform convergence group of compact metrisable space M. Then G is a hyperbolic group and M is G-equivariantly homeomorphic to ∂G.*

Proof idea: Recall that we mentioned that a hyperbolic metric space X can be recovered up to quasi-isometry from a quasi-conformal structure on its boundary ∂X. The idea is as follows: analysing the action of G on M, we can equip M with a 'system of G-invariant annuli', that encode the structure of the group action. This is the key place where the uniformity of the convergence action is used, which ensures that such a system exists around each point.

A system of annuli is essentially a quasi-conformal structure on M, and we can use it to construct a version of a cross-ratio on M. Now, a cross-ratio on M induces a G-invariant *quasi-metric* on the space of triples $\Theta(M)$, which one can show is hyperbolic. One can then upgrade this quasi-metric to an actual G-invariant metric d on $\Theta(M)$. Since G is a uniform convergence group on M, it acts geometrically on the hyperbolic metric space $(\Theta(M), \mathrm{d})$. Thus $\partial\Theta(M) \cong M$ is G-equivariantly homeomorphic to ∂G by the observation before the theorem. □

Using the machinery we developed in the previous section, we are now able to deduce many strong facts about hyperbolic groups and their subgroups. First, we see that we have a more straightforward proof of the fact that every infinite order element of a hyperbolic group is loxodromic.

Theorem 1.7. *Let G be a hyperbolic group. Then every infinite order element of G is loxodromic.*

Proof. The group G acts as a uniform convergence group on ∂G. Since the action is uniform, ∂G contains only conical limit points. However, a conical limit point cannot be a parabolic fixed point. Therefore G contains no parabolic elements. To conclude, recall that every infinite order element of G is parabolic or loxodromic. □

We can also prove that hyperbolic groups contain no infinite torsion subgroups, as promised earlier. We state a simple lemma beforehand.

Lemma 1.8. *Let H be an infinite subgroup of a uniform convergence group G. Then ΛH contains at least two points.*

Proof. As H is infinite, ΛH is non-empty. Suppose that ΛH contains only a single point p. Necessarily, H fixes p, which is a conical limit point of G. But any infinite set fixing a conical limit point contains a loxodromic element. It follows that ΛH contains at least two points, the poles of this loxodromic. □

Theorem 1.9. *Let G be a hyperbolic group. Then G contains no infinite torsion subgroups.*

Proof. Let G be a hyperbolic group, so G is a uniform convergence group on ∂G. Let $H \leqslant G$ be an infinite subgroup, so that $\Lambda H \subseteq \partial G$ is non-empty. By Lemma 1.8, ΛH contains at least two points. We saw that H contains a loxodromic element (which generates a finite index subgroup of H) when ΛH contains two points. Now if ΛH contains at least three points, then H is non-elementary and so contains a non-abelian free subgroup by the Tits alternative for convergence groups. □

Another consequence of the above argument is a strong version of Tits' subgroup dichotomy for hyperbolic groups. Note that there are now several examples of non-linear hyperbolic groups, so the setting is really different to the original theorem.

Theorem 1.10 (Tits' alternative). *Let G be a hyperbolic group, $H \leqslant G$ a subgroup. Then H is either virtually cyclic, or H contains a non-abelian free subgroup.*

2 Topology of boundaries

We saw in a previous lecture that any compact metrisable space arises as the boundary of a proper hyperbolic metric space. In contrast, we have already seen that even the cardinalities of boundaries of hyperbolic groups are quite restricted, as the group action requires the boundary to have some sort of uniform symmetry.

The topology of boundaries is, indeed, also very constrained, to the extent that in (very) low dimensions, one can in fact classify exactly the spaces that arise as boundaries. A majority of the work lies in some deep results about the topology of the boundary of hyperbolic groups: they are always locally connected when they are connected, and they contain no global cut points. The classification of one-dimensional continua then gives the following.

Theorem 2.1. *Let G be a nonelementary hyperbolic group. If ∂G has dimension 0, then ∂G is a Cantor space. If ∂G is connected and has dimension 1, it is either a circle S^1, a Sierpiński carpet, or a Menger sponge.*

Note that these are all very regular spaces. Indeed, the circle is the only compact manifold in dimension one, the Sierpiński carpet is the universal plane curve, and the Menger sponge is the universal curve.

More can be said about the groups with these low-dimensional boundaries. Free groups act geometrically on trees whose boundaries are Cantor sets. Deep structural results on splittings of groups of Stallings and Dunwoody actually imply the following.

Theorem 2.2. *Let G be a hyperbolic group with ∂G a Cantor set. Then G contains a finite index non-abelian free subgroup.*

Similarly, we have seen that fundamental groups of hyperbolic surfaces are natural examples of hyperbolic groups with circle boundary. In fact, these turn out to be the only such groups, up to finite index.

Theorem 2.3. *Let G be a hyperbolic group with $\partial G \cong S^1$. Then G acts geometrically on the hyperbolic plane $\mathbb{H}^2$.*

This theorem combines the work of many authors, most notably Tukia and Gabai, and was also independetly proven by Casson and Jungreis. In reality, the theorem is a little stronger: it says that any convergence group action on the circle S^1 is conjugate in $\mathrm{Homeo}(S^1)$ to one induced by a properly discontinuous action by isometries on the hyperbolic plane $\mathbb{H}^2$. Again, the proof is long and complicated, but we give a very brief overview.

Proof idea: Let G be a convergence group on S^1. We extend this action to a convergence group action on the disc D^2 by cutting up the disc using axes of loxodromic elements.

If G contains a simple axis (that is, an axis disjoint from all its conjugates, then these axes cut the disc up into disjoint pieces that all meet the boundary circle, and one can extend the action in a canonical way to the interior of the disc. Tukia showed that this covers many cases, and the work of Gabai covers the rest of the cases. Once one has this, we can pick a G-equivariant triangulation of the disc D^2, which we can modify to resemble $\mathbb{H}^2$ by applying a conformal transformation of the disc. This is tantamount to conjugating the original action by a homeomorphism of the circle. $\square$

Embedding a Cantor set in a circle, one can actually deduce Theorem 2.2 from the above. Indeed, the action of the group on the Cantor set can be extended to an action on the circle, and the original group will act cocompactly on the convex hull of this embedded Cantor set. The quotient is (up to passing to a finite cover) a surface with boundary, which retracts onto a graph. Thus these groups have finite index free subgroups.

Moving one dimension up, the analysis becomes apparently much more difficult. For instance, the following still remains open after almost forty years.

Conjecture 2.4 (Cannon conjecture)**.** *Let G be a hyperbolic group with $\partial G \cong S^2$. Then G acts geometrically on $\mathbb{H}^3$.*

Similarly to how the classification of groups with S^1 boundary shows that groups with Cantor set boundary acts geometrically on a convex subset of $\mathbb{H}^2$, a resolution to the above conjecture would allow a description of the groups with Sierpiński carpet boundary as those acting geometrically on a convex subset of $\mathbb{H}^3$. Indeed, a Sierpiński carpet boundary embeds into a sphere, with the holes as round circles. There are finitely many orbits of these circles under the action of the group, and their stabilisers are well-behaved hyperbolic subgroups with circle boundary (and so, they act geometrically on $\mathbb{H}^2$). 'Doubling' the group along representatives of these finitely many conjugacy classes of stabiliser subgroups yields a hyperbolic group containing the original, and whose boundary is the entire sphere. Then, if the Cannon conjecture were true, we could obtain an action of the original group on a convex subset of $\mathbb{H}^3$ from this.

In even higher dimensions, the versions of this geometric conjecture are also open. However, in high enough dimensions, techniques from surgery theory allow a resolution to a sort of topological analogue.

Theorem 2.5 (Bartels–Lück–Weinberger)**.** *Let G be a torsion-free hyperbolic group with $\partial G \cong S^{n-1}$, where $n \geq 6$. Then $G \cong \pi_1 M$, where M is an aspherical n-manifold with $\widetilde{M} \cong \mathbb{R}^n$.*

To conclude, we state another result that emphasises the regularity of boundaries of hyperbolic groups. It tells us that any boundary that contains a subset homeomorphic to $\mathbb{R}^n$ is actually an n-sphere. This rules out any non-sphere manifolds arising as boundaries of hyperbolic groups. Note that the same argument actually works for the limit set of any non-elementary convergence group.

Theorem 2.6. *Let G be a hyperbolic group and suppose that ∂G contains a subset homeomorphic to an open subset of $\mathbb{R}^n$. Then ∂G is homeomorphic to the n-sphere S^n.*

Proof. The idea is to use the convergence property and a well-chosen loxodromic to show that ∂G is the union of two open n-balls glued along their boundary. We will need the generalised Schoenflies theorem, which states that a bicollared topologically embedded sphere in $\mathbb{R}^n$ separates it into two components, one bounded and one unbounded. This is the higher dimensional version of the Jordan curve theorem.

Let $U \subseteq \partial G$ be a subset homeomorphic to $\mathbb{R}^n$. By the density of loxodromic fixed point pairs, there is a loxodromic element $g \in G$ with $P_g, N_g \in U$. Let U_+ and U_- disjoint open neighbourhoods of P_g and N_g respectively, and let $f_+ : \mathbb{R}^n \to U_+$ and $f_- : \mathbb{R}^n \to U_-$ be homeomorphisms such that $f_+(0) = P_g$ and $f_-(0) = N_g$. Let B_1 and B_2 be the open balls of radius 1 and 2 in $\mathbb{R}^n$ respectively. We define $V_+ = f_+(B_2)$ and $V_- = f_-(B_1)$, which are neighbourhoods of P_g and N_g respectively. Since (g^m) is a convergence sequence, there is $m \geq 0$ such that

$$g^m(\partial G - V_-) \subseteq V_+.$$

Let $S = f_-(\partial B_1)$, so that S is a sphere that is the topological boundary of V_- in ∂G. There is some $\varepsilon > 0$ such that $S \times [-\varepsilon, \varepsilon]$ embeds in ∂G also, so S is a bicollared sphere. Of course, $S \subseteq \partial G - V_-$, so that $g^m S \subseteq V_+$. Now $S' = f_+^{-1}(g^m S) \subseteq B_2$ is a bicollared sphere in $\mathbb{R}^n$, and so by the generalised Schoenflies theorem separates $\mathbb{R}^n$ into two components, one bounded and homeomorphic to an open n-ball D^n, the other unbounded and homeomorphic to the complement of a closed n-ball in $\mathbb{R}^n$. Since $\partial G - V_-$ is compact (as a closed set in a compact Hausdorff space) whose topological boundary is S, its image under $f_+^{-1} g^m$ is exactly the closure of the bounded component of $\mathbb{R}^n - S'$.

Therefore $\partial G - V_-$ is homeomorphic to a closed n-ball. By construction, V_- is homeomorphic to an open n-ball. The two sets share a topological boundary, which is the $(n-1)$-sphere S. Thus $\partial G = D^n \cup_{S^{n-1}} D_n \cong S^n$, as required. $\square$