NP Completeness

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https://github.com/lawliet89/np-complete

Computation Complexity Theory

- More than Big Oh Notation
- Classifying computational problems according to their inherent difficulty into different *classes*
- Relation between the different *classes*

Journey

1. Computation Resources
2. Decision Problems
3. Modelling via Turing Machines
4. P
5. Non-determinism
6. NP
7. Reduction
8. NP-Complete
9. Example proof
10. P = NP?

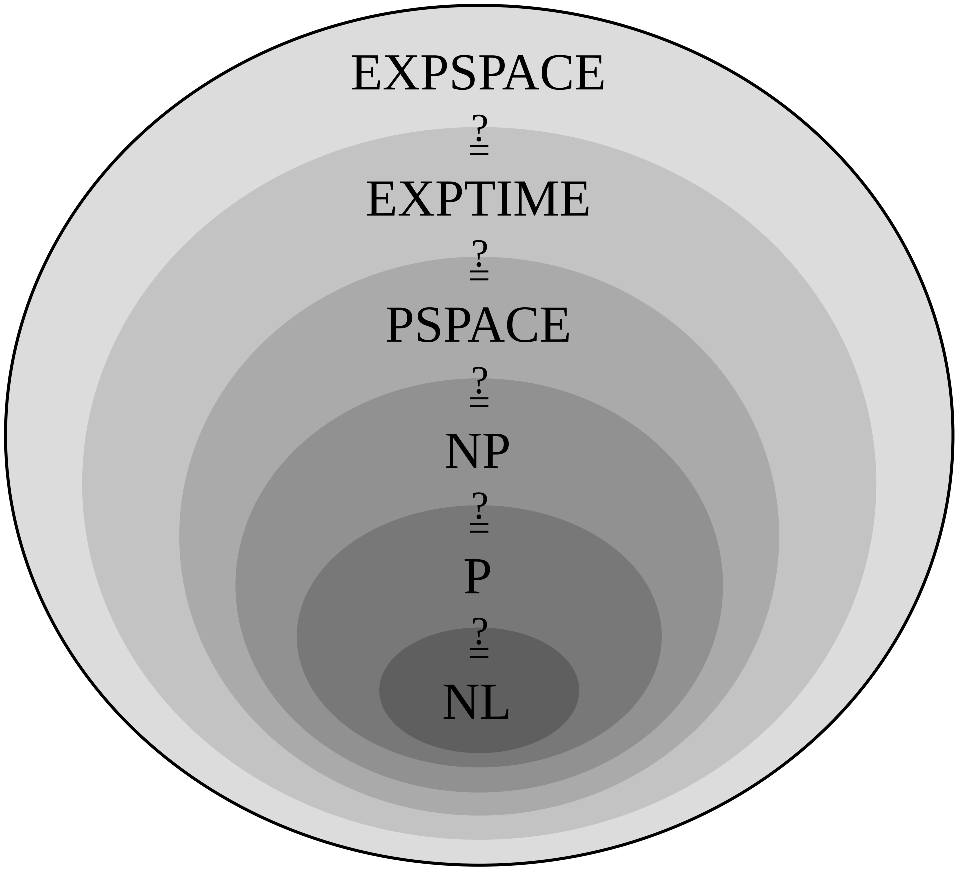
Computation Resources

- Time: Time taken by algorithm to solve. i.e. $\mathrm{TIME}(f(n))$
- Space: Space needed to solve. Includes inputs, outputs, and intermediates. i.e. $\mathrm{SPACE}(f(n))$

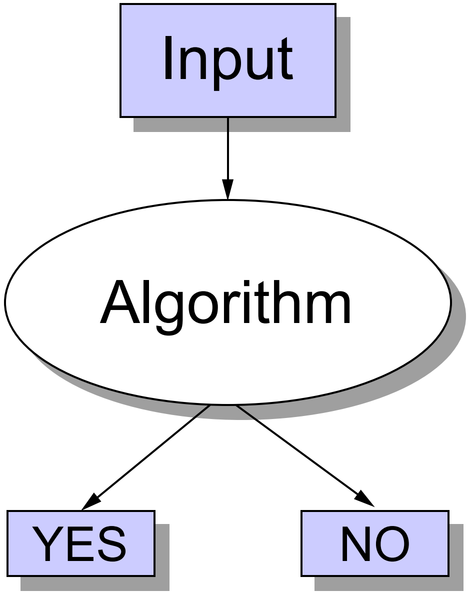
Hierarchy Theorem

- More space/time → Solve more (harder) problems.
- Intuitively, you need $f(n)$ time to be able to work with $f(n)$ space at least.
- But the reverse relation between space and time is less intuitive.

Complexity Hierarchy



Decision Problems



Reachability

`RCH`: Given a directed graph `G` and nodes `x`, `y`, is there a path from `x` to `y`?

Breadth First Search RCH Algorithm

Let `S` be a set of nodes to process. There are `n` nodes in the graph.
Initially, $S = \{x\}$.

At each stage, for some $z \in S$:

- remove `z` from `S`
- mark `z`
- Find all unmarked "successors" of `z` and add them to `S`

Until `y` is found or `S` is empty.

Worst case: each edge is examined once. There are at most n^2 edges.
So $O(n^2)$.

Decidable vs Undecidable

`RCH`: Decidable

`HALT`: Given the description of an *arbitrary* program and a finite input, decide whether the program finishes running or will run forever.

This is undecidable.

`HALT`

Assume an imaginary solver $H(p, x)$ which decides that if some program p will halt for some input x. (i.e. a black box that solves `HALT`)

Then, we construct program `P`:

```
program P(y):
  if H(y,y) = halt then
    loop forever
  else:
    return
```

- If $H(P, P)$ = `halt` then $P(P)$ runs forever.
- If $H(P, P)$ = `loop` then $P(P)$ halts.

H always gives the wrong answer. Generalized H cannot exst.

Tractable vs Intractable

- Tractable problems can be solved in a *feasible* or *practical* amount of time

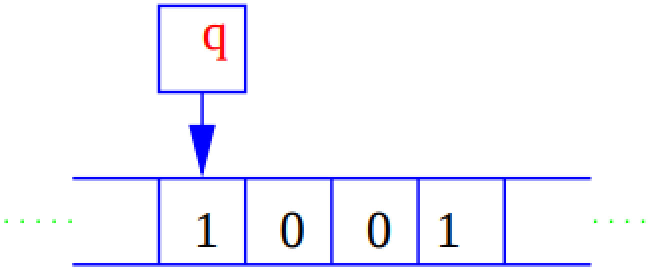
Cook-Karp Thesis: Tractable = polynomial time (P)

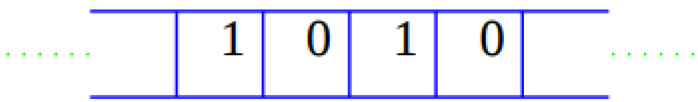
but... is n^{100} or $2^{n/100}$ more practical?

Deterministic Turing Machine (DTM)

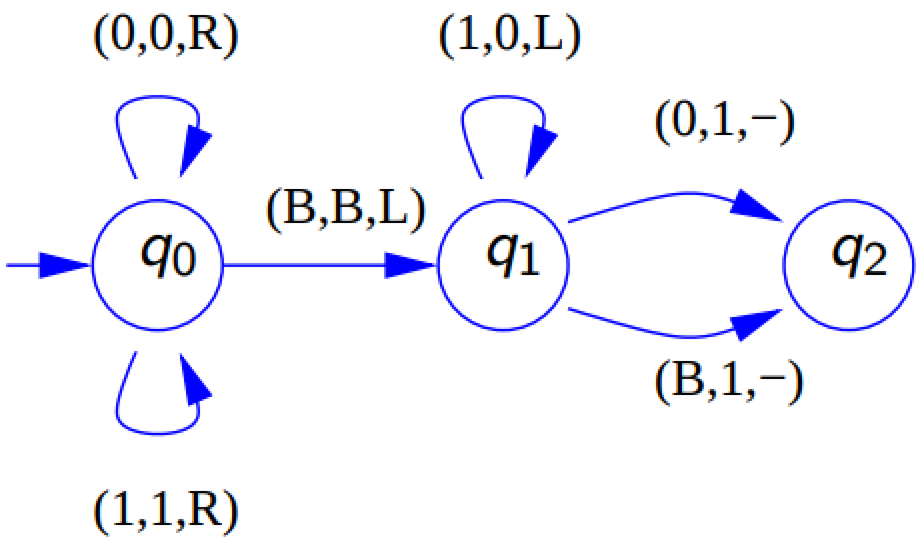
- A model of computation used in reasoning about complexity.
- Read/Write head over a tape. Can move right or left. The head stores the current *state* of the machine
- Symbols on tape
- A table of instructions where given the current state, and the symbol on the tape, decide whether to write to the tape, move left/right, or transition into a new state.

Turing Machine - Binary Successor





State Machine:



Further Modelling

- Use multiple tapes. 1-tape DTM is slower than a k-tape DTM by a polynomial factor.
- Random Access Machines (RAM): DTM is slower than RAM by $O(n^3)$.
- Multi tapes DTM are easy to reason. It's reasonable to use them to model computation.

Invariance Thesis

(similar to Church-Turing Thesis)

All *reasonable* sequential models of computation have the same time complexity as Deterministic Turing Machines (DTM) up to a polynomial.

e.g. RAMs, 1-tape DTM, k-tape DTM

Problems

- A DTM gives a "yes" or "no" answer to a decision problem,
- It is **deciding** the problem.

e.g. A DTM M decides that `2` is the next integer after `1`, but `0` is not.

If M decides a size n problem in $f(n)$ time, then

$$L \in \mathrm{TIME}(f(n))$$

(Formal) Languages

Formally, a language L is a set of strings over a given alphabet Σ, where B is the blank symbol.

Then let $L \subseteq (\Sigma - \{B\})^*$ and M be a DTM with alphabet Σ such that for any $w \in (\Sigma - \{B\})^*$:

- if $w \in L$ then $M(w)$ terminates with `yes`
- if $w \notin L$ then $M(w)$ terminates with `no`

Then we can say M **decides** L and L is recursive because it is decided by some DTM.

Then iff for any length $n = |w|$ of $w \in (\Sigma - \{B\})^*$, M operates within the time bound $f(n)$, we say

$$L \in \mathrm{TIME}(f(n))$$

Polynomial Time (P)

$$P = \bigcup_k TIME(n^k)$$

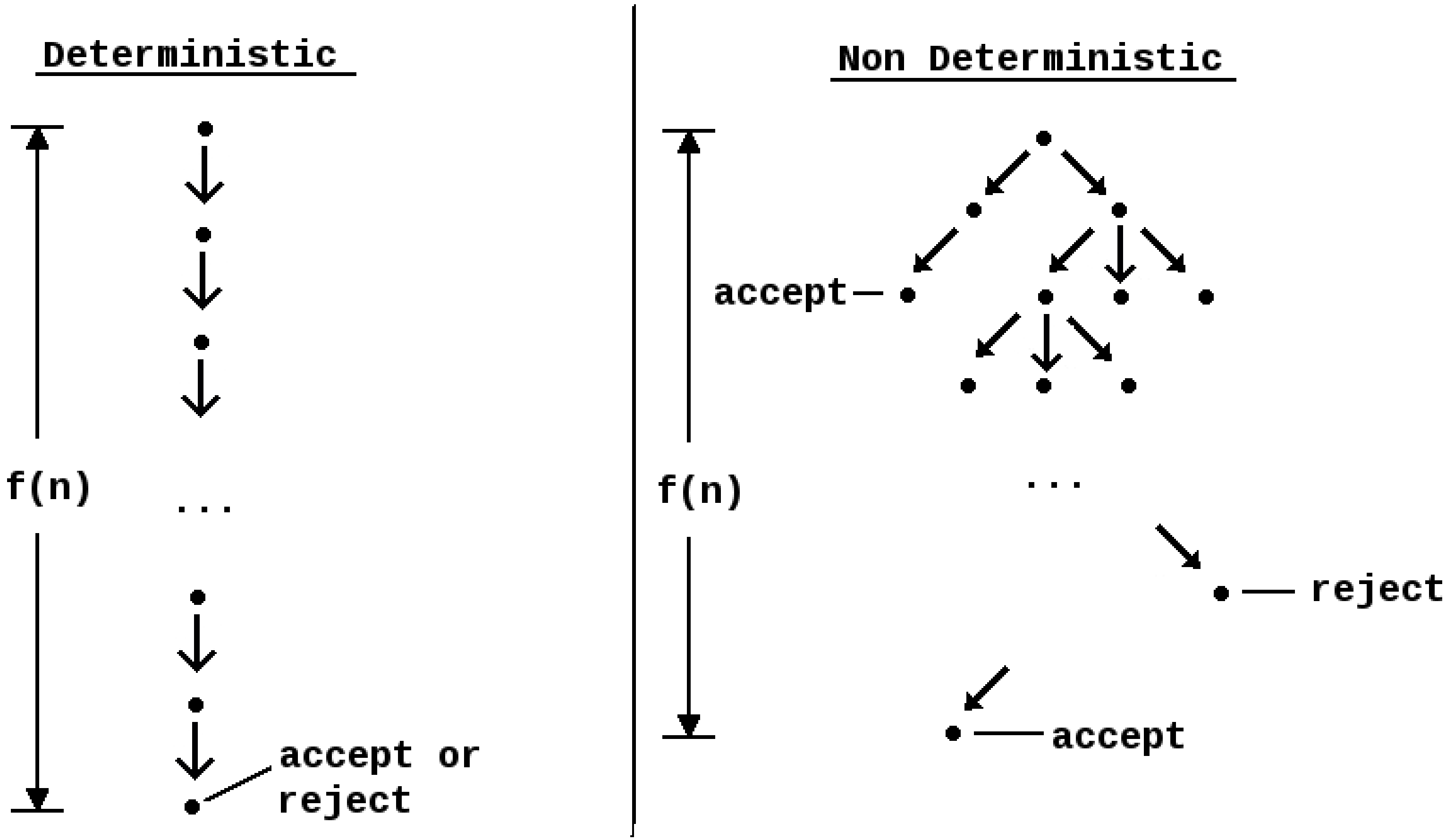
That is P is the set of decision problems which can be decided by a DTM in polynomial time for all inputs.

Known Problems in P

- Greatest Common Denominator (GCD)
- PRIMES - whether a number is prime (https://www.cse.iitk.ac.in/users/manindra/algebra/primality_v6.pdf)

Determinism vs Non-Determinism

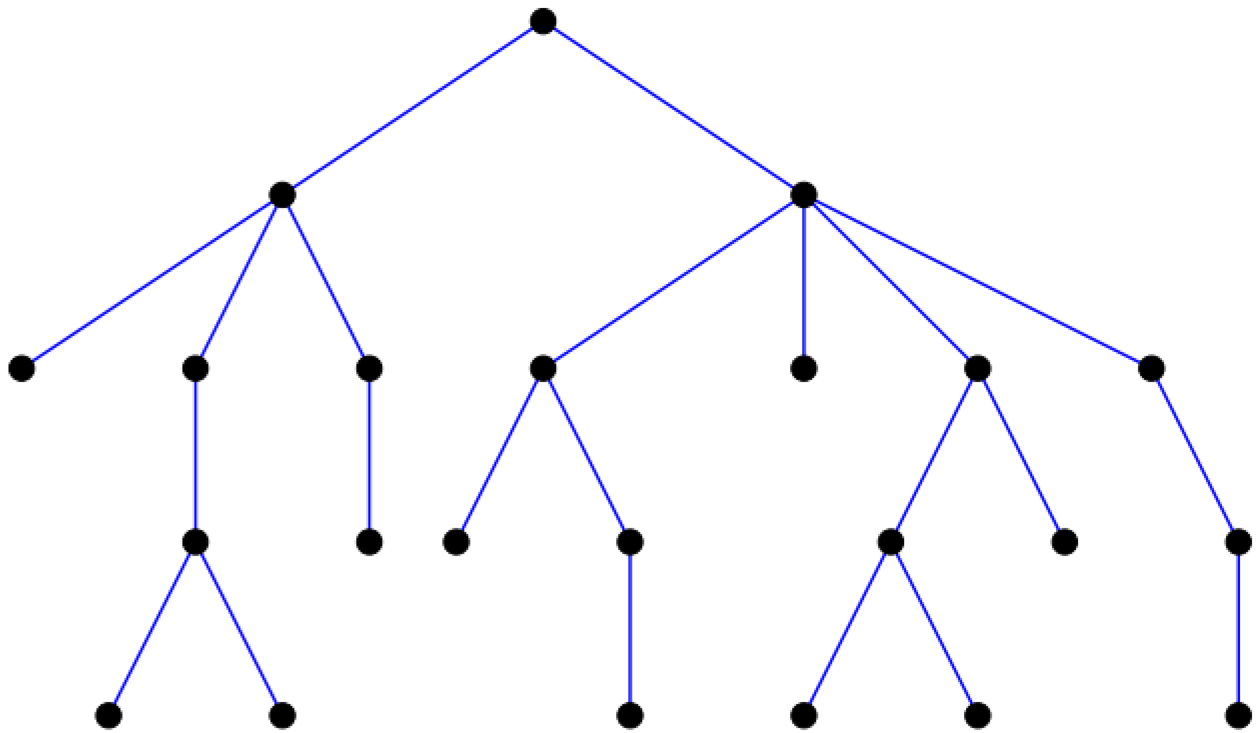
For the same input, $f(n)$ is the time taken to solve the input size n.



Notation for non-deterministic runtime: $\mathrm{NTIME}(f(n))$

Non-deterministic Turing Machines (NDTM)

A NDTM M has many possible computation for some input w that is chosen at random.



$M(w)$ represents the tree, with some depth.

NDTM is not practical.

NDTM Modelling

For the same input of size n to a problem L, if over multiple runs of the NDTM M

- It might say "yes" sometimes
- It might say "no" sometimes
- But it must always terminate within time $f(n)$

Then M decides L and

$$L \in \mathrm{NTIME}(f(n))$$

How is this useful?

- Easier to come up with a non-deterministic algorithm to some problem. (e.g. algorithm to "guess" a path from node `x` to `y`)
- Then we can try to simulate this with DTM.
- Define complexity classes with NDTM and their relationship with DTM complexity classes.

Simulating a NDTM

- A NDTM can be simulated by a DTM for a problem of size n in time $O(C^{f(n)})$ for some constant $C > 1$.

We don't know if the simulation can be improved. (i.e. We don't know if $P = NP$.)

(Formal) Nondeterministic Acceptance and Time

For some language L, if M returns `yes` on some input w, then we say M accetps L. (i.e. it can reject, or never terminate).

A NDTM M *operates within time* $f(n)$ if $M(w)$ has depth $\leq f(|w|)$.

Then we can say NDTM M *decides* a language L within time $f(n)$ if

- M *operates within time* $f(n)
- M accepts L

$L \in NTIME(f(n))$ iff L is decided by some NDTM operating within time $f(n)$.

$$L \in NTIME(f(n))$$

Non-deterministic Polynomial Time (NP)

$$NP = \bigcup_k NTIME(n^k)$$

That is NP is the set of decision problems that can be decided by a NDTM in polynomial time for all inputs.

More usefully: you can "guess" a solution to a NP problem in polynomial time and *verify* that it is a solution in polynomial time.

NP Problems

- All problems in P
- Integer factorization (most likely in NP)
- Graph Isomorphism: Given two graphs, is there a mapping of nodes from one graph to the other so that all the nodes still share the neighbours?
- `SAT`: Given a set of logic clauses with variables, is there an assignment of boolean values to the variables that will satisfy the clauses?
- Hamiltonian Path (`HP`): Given an undirected graph, is there a path visiting each node exactly once?
- Decision version of Travelling Salesman (`TSP(D)`): Is there a route visitng all cities with total distance less than some k?

Reduction

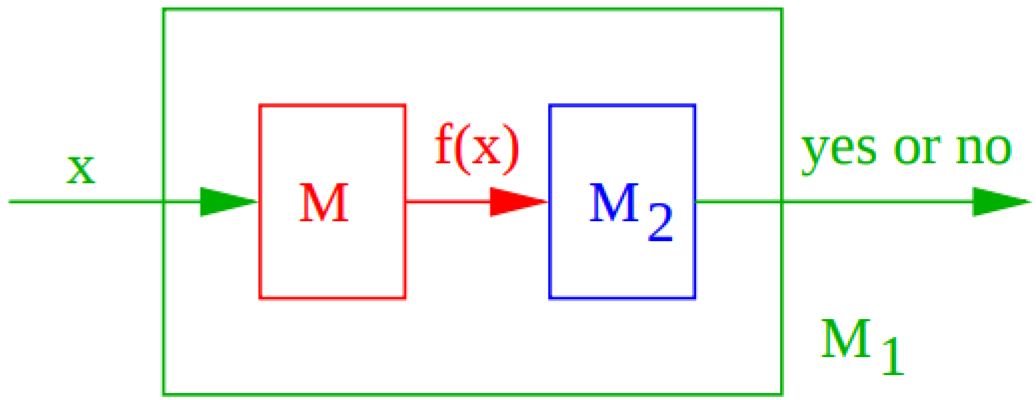
- Algorithm to transform one problem into another
- If we transform L_1 to L_2, we write $L_1 \leq L_2$
- L_2 is at least as *hard* as L_1
- If we can efficiently solve L_2, we can efficiently solve L_1.
- Karp reduction: this transformation algorithm must be in P.

(Formal) Karp Reduction

L_1 is Karp reducible to L_2 ($L_1 \leq L_2$) if there is a map f such that

- $x \in L_1$ iff $f(x) \in L_2$ and
- f is in P

Let M_1 decide L_1, M_2 decide L_2 and M computer $f(x)$. Then



Properties of Reduction

If $L_1 \leq L_2$

- if L_2 is in P, then L_1 in P
- if L_2 is in NP, then L_1 in NP
- $\leq$ is transitive

NP-complete (NPC)

NP-hard: "at least as hard as the hardest problems in NP"

- L is NP-hard if for some L' in NP, $L' \leq L \, \forall L \in NP$.

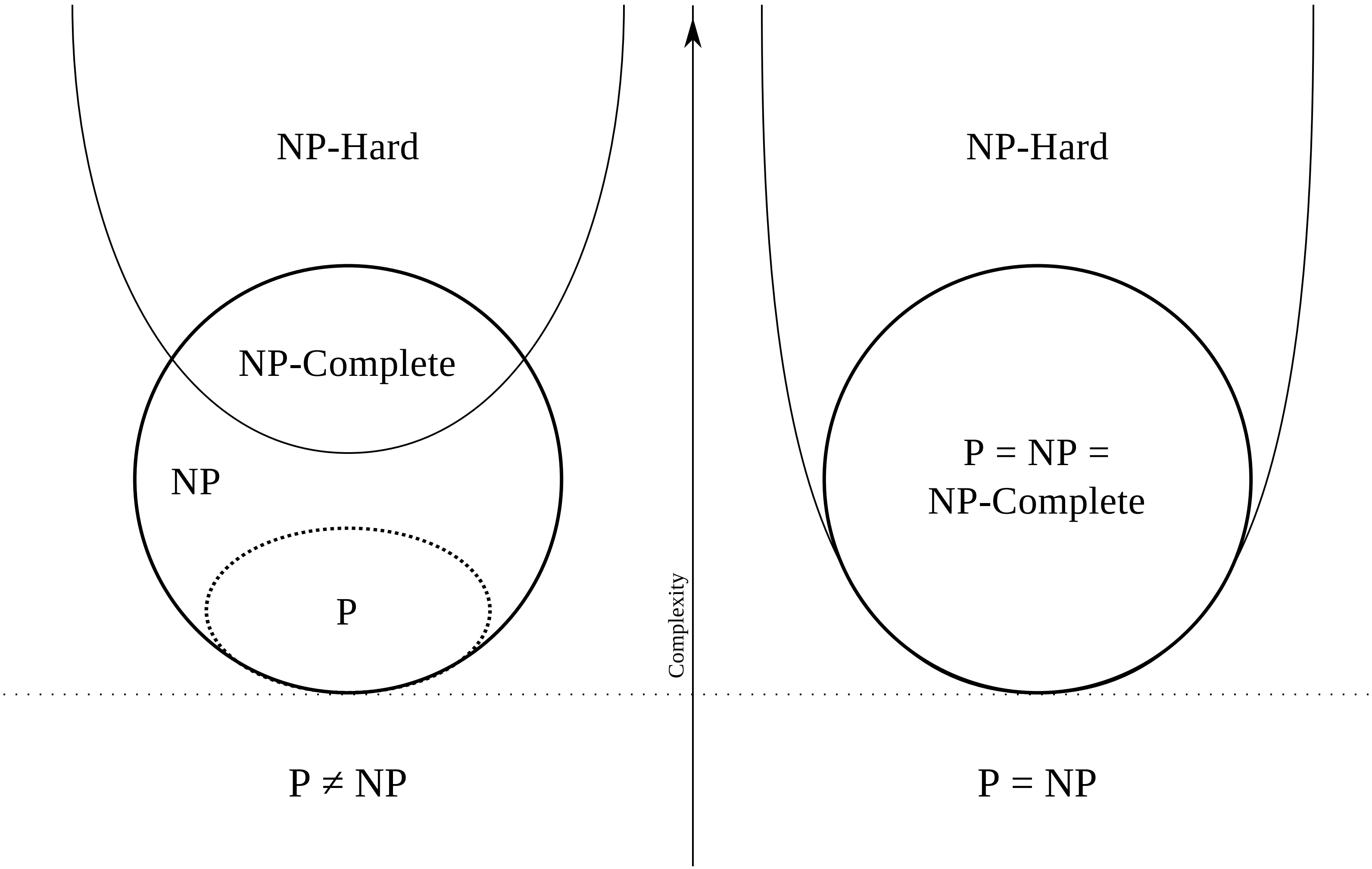
Then

L is NP-complete if

- L is in NP
- L is NP-hard

That is NPC problems are the "hardest" in NP

NPC Relation

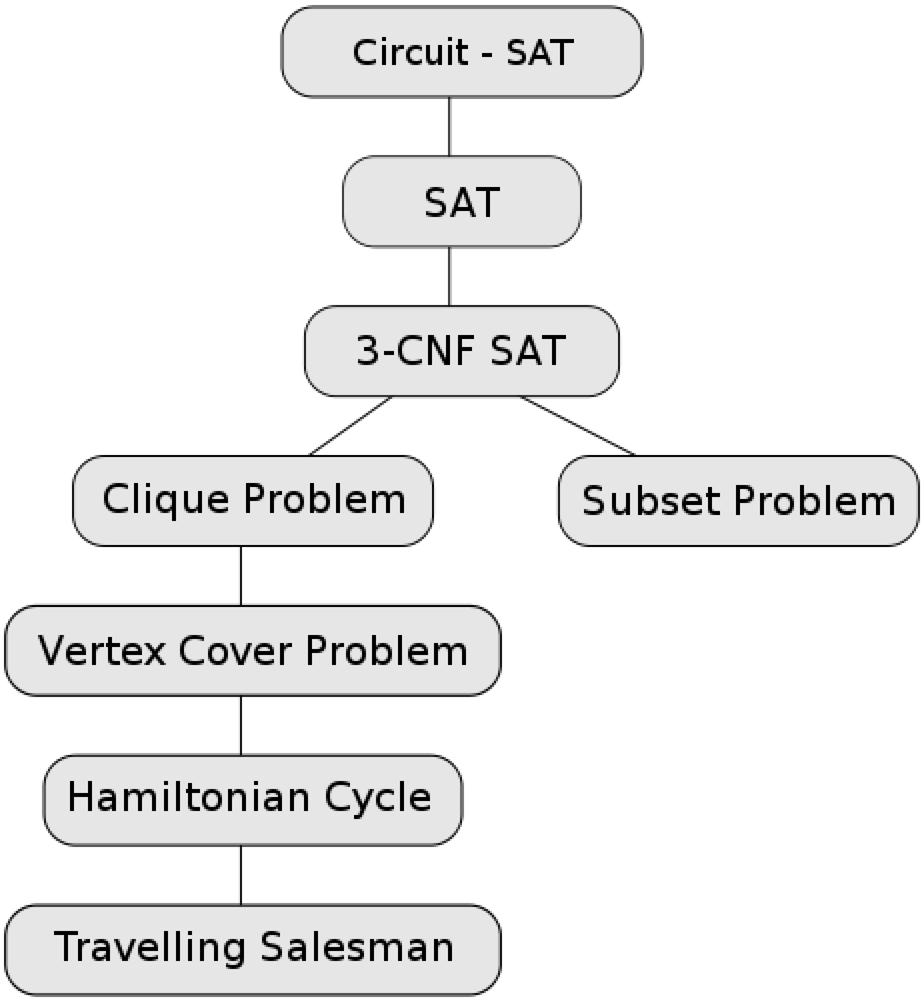


NPC Problems

- `SAT`: Proven by Cook-Levin Theorem
- Karp's 21 NP-Complete problems
- `TSP(D)`
- Knapsack problem: Can a value of at least V be achieved without exceeding the weight W?
- `HP`
- K-Graph colouring

Huge list of NPC problems

NPC Reduction



Prove that Vertex Cover is NP-Complete

`VERTEX COVER`: Given an undirected graph G, is there a set of vertices in G smaller than some size k such that all the edges in G have at least one end in the set of vertices?

Strategy:

- Prove `VERTEX COVER` is in NP
- Reduce a known NP-Complete problem to `VERTEX COVER`

Handwaving `VERTEX COVER` is in NP

- NP: Can guess a set of vertices and then check by examining at most n^2 edges.

`3SAT`:

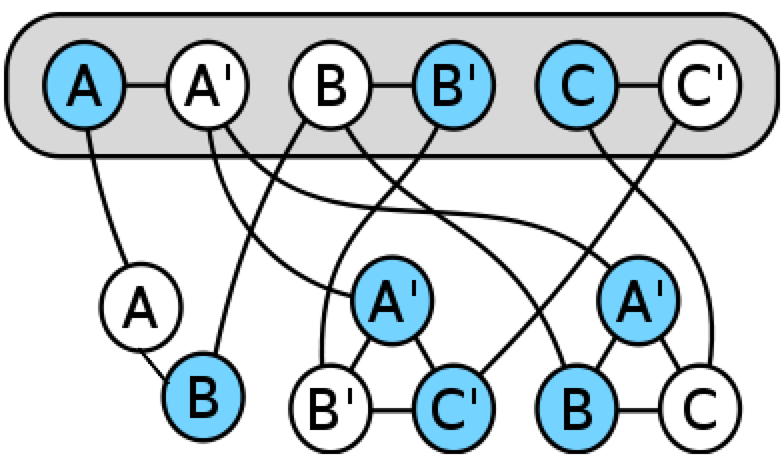
Consider a set of boolean variables, and a set of logical clauses where each clause has exactly three variables. Is there an assignment of values to the variables to satisfy the clauses?

One of [Karp's 21 NP-Complete problems]: Known NP-Complete problem

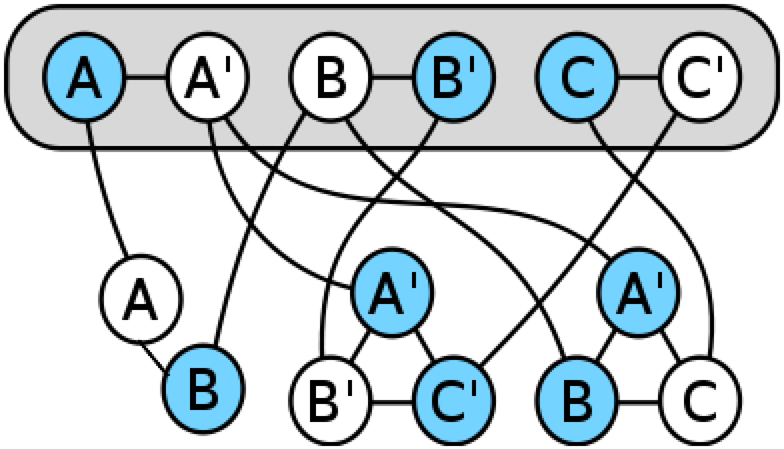
Reduce `3SAT` to `VERTEX COVER`

Consider the clauses: (A ∨ A ∨ B) ∧ (¬A ∨ ¬B ∨ ¬C) ∧ (¬A ∨ B ∨ C)

- Nodes for each variable and its complement, joined by an edge
- A "triangle" of variable nodes for each clause
- Add edges between variables of themselves
- For l variables with m clauses, vertex cover of at most $l + 2m$



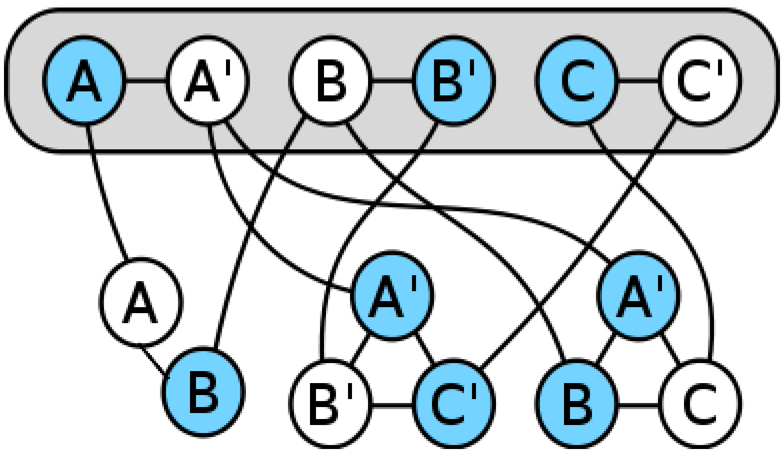
Vertex Cover



Why $\leq l + 2m$?

- l: Either the true/false for each variable: covers the edges between them
- Also cover the edges between the variable and their counterparts in the "triangles"
- Then for each "triangle", at most 2 more edges to cover the internal edges and the edges between the "incorrect" variables and the triangles

Solution to `3SAT`



The solution are the cover vertices that are part of the "variable pairs".

Reduction Algorithm

- $2l + 3m$ nodes graph construction
- Can be done in P-time

Thus, `VERTEX COVER` is NP-complete.

P = NP?

- P problems can be solved efficiently.
- NP problems have no known efficient algorithms to solve.

Consequences in Public Key Cryptrography

- Depend on "difficulty" of problems like Discrete logarithm and integer factorization that are known to be neither in P nor NP-complete to create "one-way" functions.
- If $P = NP$, then we can find an effective solution to some NP-complete problem and reduce the rest to that problem to solve them.
- If $P \neq NP$ then we can show that one-way functions exist.

Consequences in Operation

- If $P = NP$ Efficient solution to Integer Linear Programming (for optimization under constraints) and Travelling Salesman