Lecture 3: Groups

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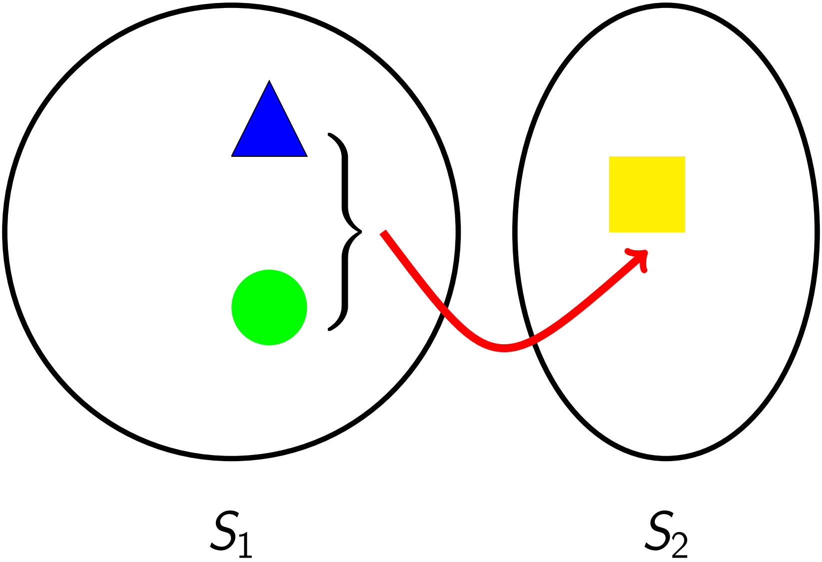
Multiplication on Sets



Definition

A set of elements $\mathcal{G} = \{a, b, c, \ldots\}$ with a multiplication (composition) $a \cdot b$ is a group iff

(i) $c = a \cdot b \in \mathcal{G}$ (closure), and c is unique

(ii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity)

(iii) $\forall a \in \mathcal{G}$ $\exists$ identity element $e \in \mathcal{G}$, i.e. $a \cdot e = a$

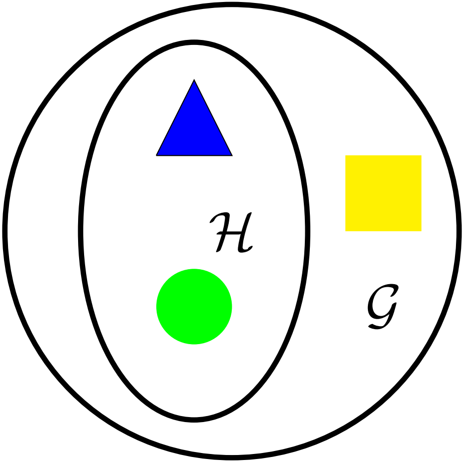
(iv) $\forall a \in \mathcal{G}$ $\exists$ inverse a^{-1}, i.e., $a \cdot a^{-1} = e$.

- A group $\mathcal{G} = \{a, b, c, \ldots\}$ is Abelian (commutative) iff $a \cdot b = b \cdot a$ (commutativity).
- The order ord$\mathcal{G}$ is the number of group elements.
- The order of $a \in \mathcal{G}$ is the smallest integer m such that $a^m = e$.

Examples

- Integers $\mathbb{Z}$ with addition
- Invariance (symmetry) operations
- Matrix groups
- Permutations

Subgroups



A group $\mathcal{H}$ is a subgroup of $\mathcal{G}$ iff $\mathcal{H} \subset \mathcal{G}$ and $\mathcal{G}$ and $\mathcal{H}$ share the same multiplication.

- ord$\mathcal{H} \leq$ ord$\mathcal{G}$

Classes

The set K_a is a (conjugacy) class of $a \in \mathcal{G}$ iff

$$K_a = \{b \cdot a \cdot b^{-1} | b \in G\}.$$

- Each group element belongs to some class.
- A group is Abelian iff each element has its own class.
- Conjugate elements, i.e., elements of the same class, are of the same order.
- The K_a generally do not form a group!