Problem 1.41. Compute the gradient and Laplacian of the scalar function $T = r(\cos\theta + \sin\theta\cos\phi)$. Check the Laplacian by converting T to Cartesian coordinates and using Eq. 1.42. Test the gradient theorem for this function, using the path shown in Fig. 1.41, from (0, 0, 0) to (0, 0, 2).

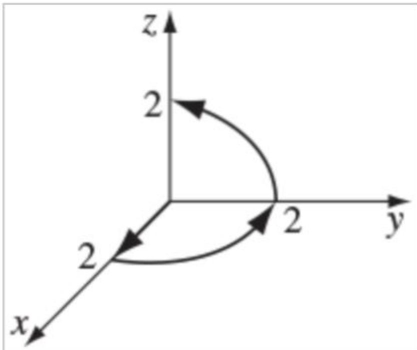

Fig. 1.41

Fig. 1.41

Problem 1.50.

(**a**) Let $\mathbf{F}_1 = x^2\hat{\mathbf{z}}$ and $\mathbf{F}_2 = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Calculate the divergence and curl of $\mathbf{F}_1$ and $\mathbf{F}_2$. Which one can be written as the gradient of a scalar? Find a scalar potential that does the job. Which one can be written as the curl of a vector? Find a suitable vector potential.

(**b**) Show that $\mathbf{F}_3 = yz\hat{\mathbf{x}} + zx\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$ can be written both as the gradient of a scalar and as the curl of a vector. Find scalar and vector potentials for this function.

Problem 1.54. Check the divergence theorem for the function

$$\mathbf{v} = r^2 \cos\theta\, \hat{\mathbf{r}} + r^2 \cos\phi\, \hat{\boldsymbol{\theta}} - r^2 \cos\theta \sin\phi\, \hat{\boldsymbol{\phi}},$$

using as your volume one octant of the sphere of radius R (Fig. 1.48). Make sure you include the *entire* surface. [*Answer:* $\pi R^4/4$.]

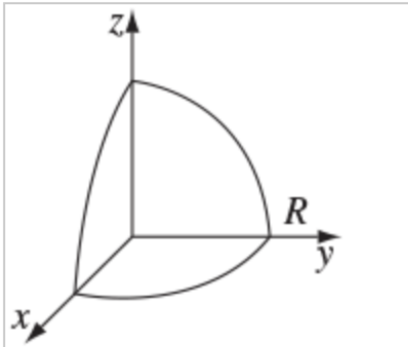

Fig. 1.48

Fig. 1.48

Problem 1.55. Check Stokes' theorem using the function $\mathbf{v} = ay\hat{\mathbf{x}} + bx\hat{\mathbf{y}}$ (a and b are constants) and the circular path of radius R, centered at the origin in the xy plane. [*Answer:* $\pi R^2(b - a)$.]

• **Problem 1.63.**

(a) Find the divergence of the function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r}.$$

First compute it directly, as in Eq. 1.84. Test your result using the divergence theorem, as in Eq. 1.85. Is there a delta function at the origin, as there was for $\hat{\mathbf{r}}/r^2$? What is the general formula for the divergence of $r^n\hat{\mathbf{r}}$? [*Answer:* $\nabla \cdot (r^n\hat{\mathbf{r}}) = (n+2)r^{n-1}$, unless $n = -2$, in which case it is $4\pi\delta^3(\mathbf{r})$; for $n < -2$, the divergence is ill defined at the origin.]

(b) Find the *curl* of $r^n\hat{\mathbf{r}}$. Test your conclusion using Prob. 1.61b. [*Answer:* $\nabla \times (r^n\hat{\mathbf{r}}) = \mathbf{0}$.]