# P112A:Electromagnetic Theory

# Vector Analysis
# (Chapter 1)

**Physics 112A Approximate Schedule, Winter 2024**

| Week | Reading Assignment | Homework "()" indicate problem numbers in 4th edition & Due date | Example & Practices | Concepts |
|---|---|---|---|---|
| Week 1 | Ch1 | HW1, 1/17<br>Ch1: 41, 50, 54, 55, 63 | Ch1: 37, 38, 39, 40, 42 | Vector field<br>Curvilinear coordinates |
| Week 2 | Ch2 | HW2, 1/24<br>Ch2: 7, 12(11), 17(16), 23(22) | Ch2: 12, 14, 21, 23, 24, 28 | Electric field<br>Electrostatic potential |
| Week 3 | Ch2 | HW3, 1/31<br>Ch2: 35(34), 39 (38), 43(42), 44 (43) | Ch2: 25, 36, 39, 44, 47, 48, 50 | Boundary conditions<br>Conductors |
| Week 4 | Ch3 | HW4, 2/7<br>Ch3: 10(9), 11(10), 12(11), 16(14), 18(16) | Ch3: 3, 8, 14, 15, 21, 24 | Method of images<br>Laplace's Equation |
| Week 5 | Ch3 | HW5, 2/14<br>Ch3: 20(18), 21 (19), 32(30), 34(32), 37(35) | Ch3: 34, 36, 43 | Separation of variables<br>Multipole expansion |
| Week 6 | Midterm<br>Ch4 | HW6, 2/21<br>Ch4: 8, 10 | Ch4: 7, 18, 19, 20, 26, 35 | Midterm, 2/13 Tuesday, Ch1-3<br>Polarization |
| Week 7 | Ch4<br>Ch5 | HW7, 2/28<br>Ch4: 12, 15, 20 21, 24<br>Ch5: 3, 6 | Ch4: 22, 33, 34<br>Ch5: 7, 10, 11 | Electric Displacement, Linear dielectric<br>Lorentz force |
| Week 8 | Ch5 | HW8, 3/6<br>Ch5: 12, 13,14 | Ch5: 15 | Ampere's Law<br>Vector potential |
| Week 9<br>3/ | Ch5 | HW9, 3/13<br>Ch5: 25(23), 26(24), 27(25), 28(26), 37(35), 39(37), 44(41) | Ch5: 27, 34, 35, 36, 37, 57 | Vector potential<br>Multipole expansion |
| Week 10 | Ch6 | HW10, 3/20<br>Ch6: 8, 10, 12, 16 | Ch6: 4, 7, 15, 17 | Magnetization<br>Ampere's Law in magnetized materials |
| Week 11<br>Final | | | | Final, 3/21 Thursday 8:00-10:00am<br>Ch1-6 |

Read textbook before class

Discussion/Quiz on Friday

# Electricity and Magnetism

- Branch of physics that studies how charges and currents interact and the resulting electromagnetic phenomena
- In P112, Solve for Maxwell's equations:

| Integral form (7D) | | Differential form |
|---|---|---|
| $\oint \vec{E} \cdot d\vec{A} = \dfrac{Q_{\text{encl}}}{\epsilon_0}$ | (Gauss's law for $\vec{E}$) | $\nabla \bullet \mathbf{E} = \rho/\varepsilon_o$ |
| $\oint \vec{B} \cdot d\vec{A} = 0$ | (Gauss's law for $\vec{B}$) | $\nabla \bullet \mathbf{B} = 0$ |
| $\oint \vec{B} \cdot d\vec{l} = \mu_0 \left( i_C + \epsilon_0 \dfrac{d\Phi_E}{dt} \right)_{\text{encl}}$ | (Ampere's law) | $\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_o \varepsilon_o \, d\mathbf{E}/dt$ |
| $\oint \vec{E} \cdot d\vec{l} = -\dfrac{d\Phi_B}{dt}$ | (Faraday's law) | $\nabla \times \mathbf{E} = -\, d\mathbf{B}/dt$ |

Area vector

Displacement current

*E field from charges is radial:*

$$\nabla \times \mathbf{E} = 0$$

*Forces on charge:*

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

*Current Density:*

$$\nabla \bullet \mathbf{J} = -d\rho/dt$$

# Math

- Vector algebra
  - Dot product: •, cross-product: X, triple products
- Vector calculus
  - Gradient: $\nabla F$, Divergence: $\nabla \cdot F$, Curl: $\nabla \times F$
  - Integral calculus: Gradient theorem, Divergence theorem, Curls theorem
- Coordinate systems
  - Cartesian, Curvilinear: polar, cylindrical, and spherical
- Dirac Delta function (1D and 3D)
- Vector Fields
  - Scalar and vector potentials

# Vector

- Vector fields are physical quantities, independent of the coordinate system
- Vector equations have same form in any coordinate system, e.g, $\boldsymbol{F} = Q(\boldsymbol{E}+\boldsymbol{v}x\boldsymbol{B})$ Lorentz Force Law

Vector notation in our class: **Bold** or error: $\boldsymbol{F}$ or $\vec{F}$

# Vector Operators in Cartesian Coordinates

Two vectors ***F*** *and* ***G*: *F*** = $F_x\,\hat{x} + F_y\,\hat{y} + F_z\,\hat{y}$ , ***G***= $G_x\,\hat{x} + G_y\,\hat{y} + G_z\,\hat{y}$

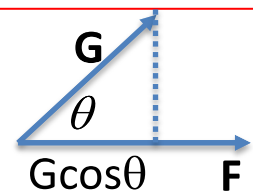
*$\hat{x}, \hat{y}, \hat{z}$: unit vectors in x, y and z directions*

## 1. Dot Product

$\mathbf{F}\cdot\mathbf{G} = FG\cos\theta$    *$\theta$: angle between **F** and **G***

$\mathbf{F}\cdot\mathbf{G} = F_xG_x + F_yG_y + F_zG_z$

*Geometrically, it is projection of one vector on the other*

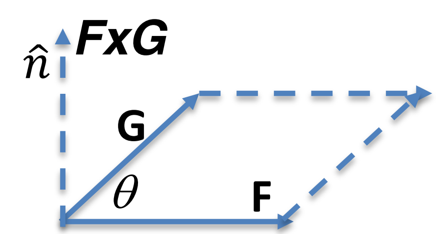



## 2. Cross Product

$\mathbf{F}\times\mathbf{G} = FG\sin\theta\,\hat{n}$

$$\mathbf{F}\times\mathbf{G} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix}$$

determinant



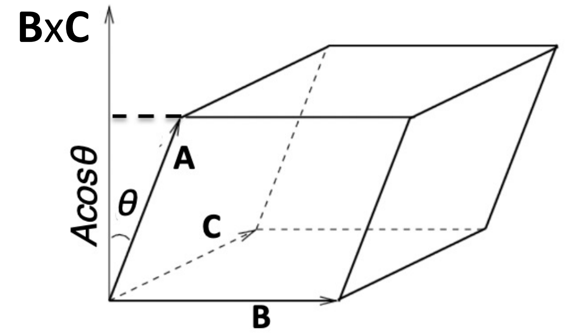

*$\theta$: angle between **F** and **G***
*$\hat{n}$ : unit vector direction determined by right hand rule*

*Geometrically, it is area of Parallelogram formed by **F** and **G***

$$\mathbf{F}\times\mathbf{G} = (F_yG_z - F_zG_y)\hat{x} - (F_xG_z - F_zG_x)\hat{y} + (F_xG_y - F_yG_x)\hat{z}$$

# Triple Products

- *Scalar triple product is volume of parallelepiped since* ***B*x*C*** *is area of base, and Acosθ is height:*




$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \begin{bmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{bmatrix}$$

- *Note that dot and cross can be interchanged:*

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}$$

# Triple Products

- *Vector triple product*

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\,(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}\,(\mathbf{A}\cdot\mathbf{B})$$

- *If B parallel to C, or A is perpendicular to B and C*

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = 0$$

- *Vector triple products are not associative and* Commutative

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$$

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = -\,\mathbf{C}\times(\mathbf{A}\times\mathbf{B})$$

# First Derivatives - Gradient

**$\nabla$ operator - "Del"**

$$\nabla \equiv \hat{x}\,\partial/\partial x + \hat{y}\,\partial/\partial y + \hat{z}\,\partial/\partial z$$

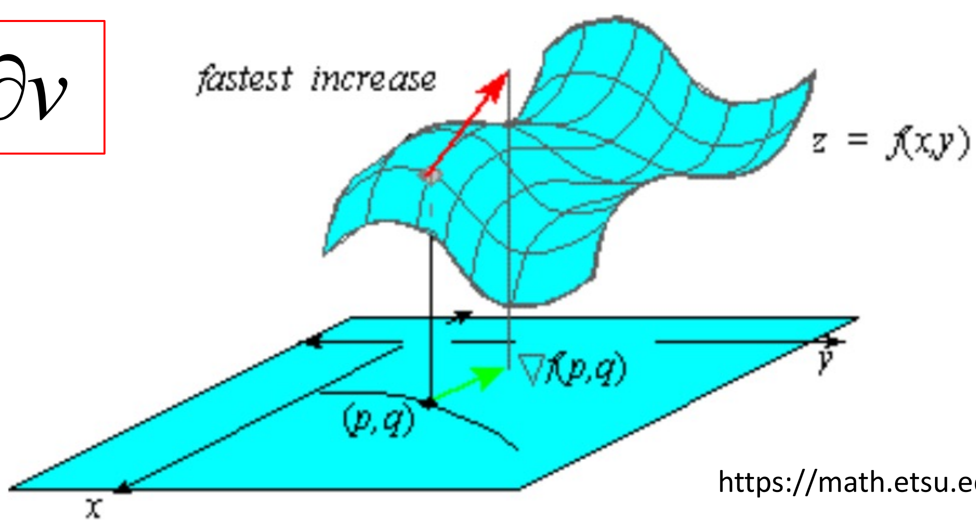
- Gradient of scalar function *f(x,y,z)*:

$$\boxed{\nabla f = \hat{x}\partial f/\partial x + \hat{y}\,\partial f/\partial y + \hat{z}\partial f/\partial z}$$

Or *grad f*

- Gradient of *f* is a vector, gives the direction and the rate of fastest increase of f at a given point
- Changing rate of *f* respect to direction ***v***:

$$\boxed{\nabla f \cdot \hat{v} = \partial f/\partial v}$$




Proof see A.3

https://math.etsu.edu/

# Divergence

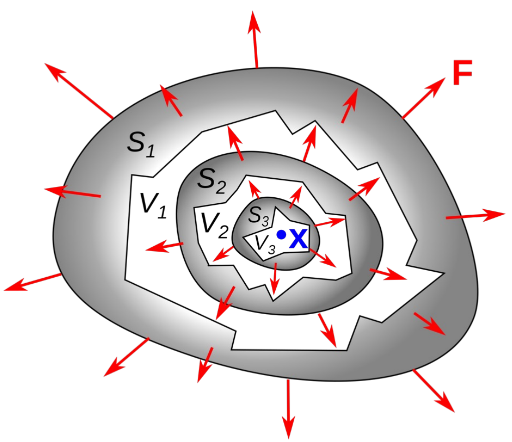
- Divergence of vector function ***F***

$$\boxed{\boldsymbol{\nabla \bullet F} = \partial F_x/\partial x + \partial F_y/\partial y + \partial F_z/\partial z,}\ or\ div\boldsymbol{F}$$

- Divergence at point **X** is the ratio of the flux of **F** out of the closed surface of a volume **V** enclosing point **X** to the volume of V

$$\boxed{\nabla \cdot F = \lim_{V \to 0} \frac{\Phi(S)}{V}}$$

$$\Phi(S) = \oiint_{S(V)} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$




Proof see A.4

# Curl

- Curl of vector function ***F***

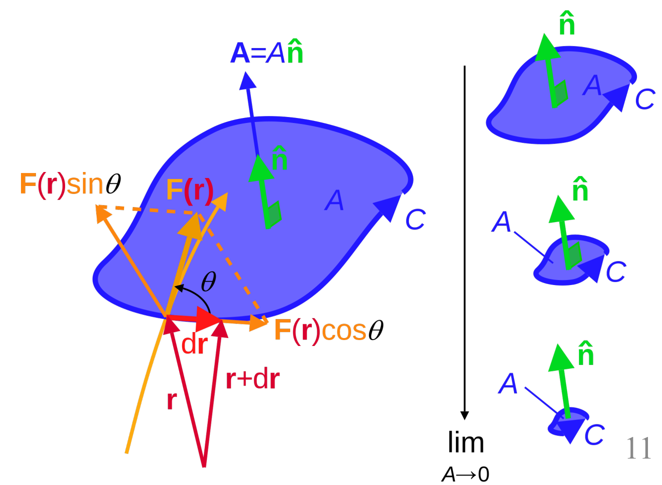
$$\nabla x \boldsymbol{F} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{bmatrix}$$

Or **curl *F***

$$\boxed{\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z}}$$

- Curl of **F** is a vector, projection of the curl of **F** at point $p$ onto axis $\hat{u} = \hat{x}, \hat{y}, \hat{z}$ is a closed line integral in a plane orthogonal to $\hat{u}$ divided by the area enclosed when A → 0, **A** oriented via the right-hand rule.

$$\boxed{(\nabla \times \mathbf{F})(p) \cdot \hat{\mathbf{u}} \overset{\text{def}}{=} \lim_{A \to 0} \frac{1}{|A|} \oint_C \mathbf{F} \cdot d\mathbf{r}}$$




Proof see A.5

# 2nd Derivatives - Laplacian

Laplace operator, Laplacian: $\nabla^2 = \nabla \bullet \nabla$

- Laplacian of scalar function f = div of $\nabla f$

$$\boxed{\nabla^2 f = \nabla \bullet (\nabla f) = \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2}$$

Laplace operator: $\Delta f = \nabla^2 f$

- Laplacian of vector function $\vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$:

$$\nabla^2 \vec{v} = \nabla^2 v_x \hat{x} + \nabla^2 v_y \hat{y} + \nabla^2 v_z \hat{z}$$

# Other 2nd Derivatives

- Curl of Grad

  $\nabla \times (\nabla f) = 0$

- Div of curl

  $\nabla \bullet (\nabla \times \mathbf{A}) = 0$

- Curl of curl

  $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \bullet \mathbf{A}) - \nabla^2 \mathbf{A}$

# Vector Identities

**Triple Products**

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

**Product Rules**

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

**Second Derivatives**

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = \mathbf{0}$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

- Prove by writing out in cartesian coordinates
- Provide in midterm and final :)

# Integral Calculus

- Gradient Theorem

$$\int_a^b (\nabla f)\cdot dl = f(b) - f(a)$$

- Fundamental Theorem for Divergence (Gauss's Thm)

$$\int_{Vol}(\nabla \cdot \boldsymbol{F})\, d\tau = \int_{Sur} \boldsymbol{F}\cdot \boldsymbol{da}$$

(over closed surface)


- Fundamental Theorem for Curl (Stoke's Thm)

$$\int_{Sur}(\nabla \times \boldsymbol{F})\cdot \boldsymbol{da} = \int_{loop} \boldsymbol{F}\cdot \boldsymbol{dl}$$


Proof see Appendix A

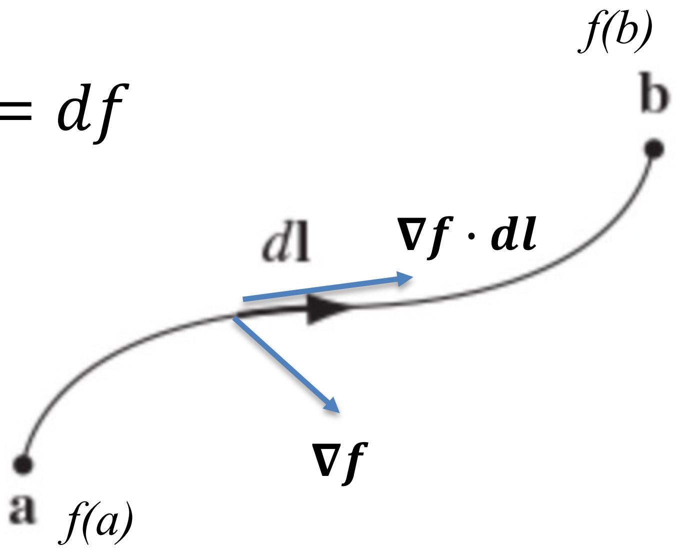
# Gradient Theorem

$$\boldsymbol{dl} = dx\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}} + dz\hat{\boldsymbol{z}}$$

$$\boldsymbol{\nabla f} = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$

$$\boldsymbol{\nabla f} \cdot dl = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

$$\boxed{\int_a^b \nabla f \cdot dl = \int_a^b df = f(b) - f(a)}$$

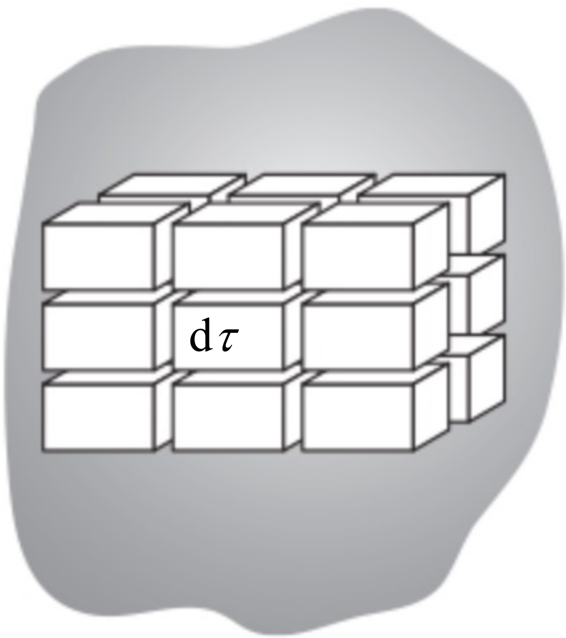


# Divergence Theorem (Gauss's)

$$\nabla \cdot F = \lim_{V \to 0} \frac{\Phi(S)}{V}$$


for each $d\tau$

$$(\boldsymbol{\nabla} \cdot \boldsymbol{F}) d\tau = \oint \boldsymbol{F} \cdot \boldsymbol{da}$$

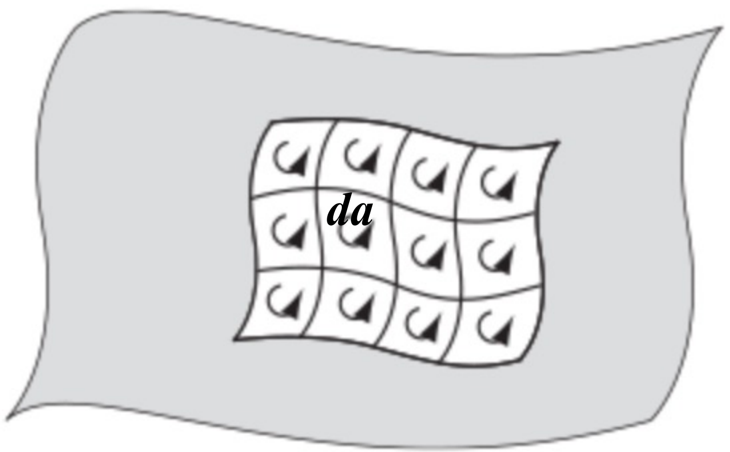



Internal contributions cancel in pairs for each internal boundary of two adjacent $d\tau$, since ***da*** always opposite

$$\boxed{\int_{Vol} (\nabla \bullet \boldsymbol{F}) d\tau = \int_{Sur} \boldsymbol{F} \bullet \boldsymbol{da}}$$

# Curl Theorem (Stoke's)

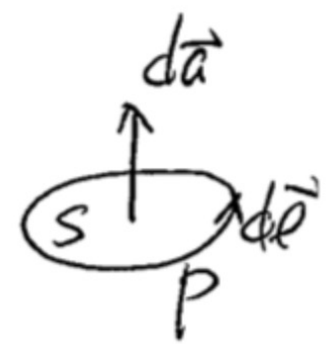
$$(\nabla \times \mathbf{F})(p) \cdot \hat{\mathbf{u}} \overset{\text{def}}{=} \lim_{A \to 0} \frac{1}{|A|} \oint_C \mathbf{F} \cdot d\mathbf{r}$$




for each ***da***

$$(\boldsymbol{\nabla}\times\boldsymbol{F}) \cdot \boldsymbol{da} = \oint \boldsymbol{F} \cdot \boldsymbol{d}l$$

Internal contributions cancel in pairs, because every internal line is the edge of two adjacent loops running in opposite directions




$$\boxed{\int_{Sur} (\nabla \times \boldsymbol{F}) \bullet \boldsymbol{da} = \int_{loop} \boldsymbol{F} \bullet \boldsymbol{dl}}$$

# Electricity and Magnetism

- Branch of physics that studies how charges and currents interact and the resulting electromagnetic phenomena
- In P112, Solve for Maxwell's equations:

Integral form (7D)

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0} \quad \text{(Gauss's law for } \vec{E}\text{)}$$

Area vector

$$\oint \vec{B} \cdot d\vec{A} = 0 \quad \text{(Gauss's law for } \vec{B}\text{)}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left( i_C + \epsilon_0 \frac{d\Phi_E}{dt} \right)_{encl} \quad \text{(Ampere's law)}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law)}$$

Differential form

$$\nabla \bullet \mathbf{E} = \rho/\varepsilon_o$$

$$\nabla \bullet \mathbf{B} = 0$$

Displacement current

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_o \varepsilon_o \, d\mathbf{E}/dt$$

$$\nabla \times \mathbf{E} = -\, d\mathbf{B}/dt$$

*E field from charges is radial:*

$$\nabla \times \mathbf{E} = 0$$

*Forces on charge:*

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

*Current Density:*

$$\nabla \bullet \mathbf{J} = -d\rho/dt$$

# Convert Maxwell’s Equations from Integral Form into Differential Form

## Gauss' Law

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\epsilon_0}$$

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \int_V \frac{\rho}{\epsilon_0} \mathrm{d}V$$

Invoke divergence theorem:

$$\int_V (\nabla \cdot \mathbf{E}) \mathrm{d}V = \int_V \frac{\rho}{\epsilon_0} \mathrm{d}V$$

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

## Gauss' Law for Magnetism

$$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$$

divergence theorem:

$$\int_V (\nabla \cdot \mathbf{B}) \mathrm{d}V = 0$$

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

# Convert Maxwell's Equations from Integral Form into Differential Form

## Faraday's Law

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$

**Stokes' theorem:**

$$\int_S (\nabla \times \mathbf{E}) \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

## Ampere's Law

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \mu_0 \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}$$

**Stokes' theorem:**

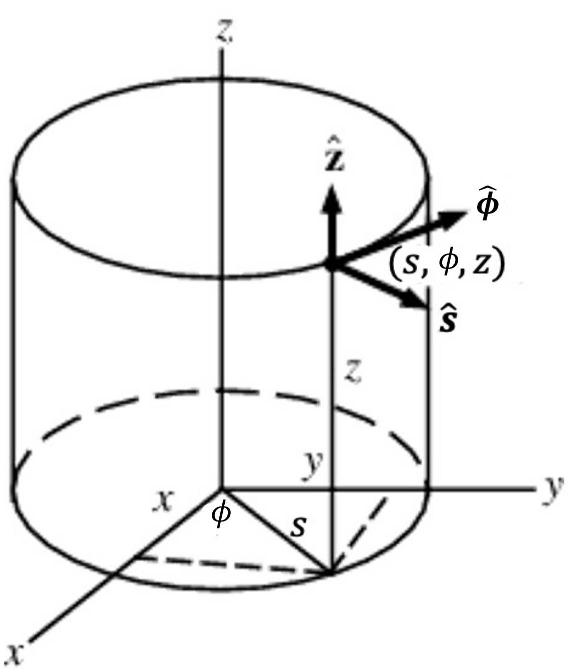
$$\int_S (\nabla \times \mathbf{B}) \cdot \mathrm{d}\mathbf{S} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \mu_0 \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}$$

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}}$$

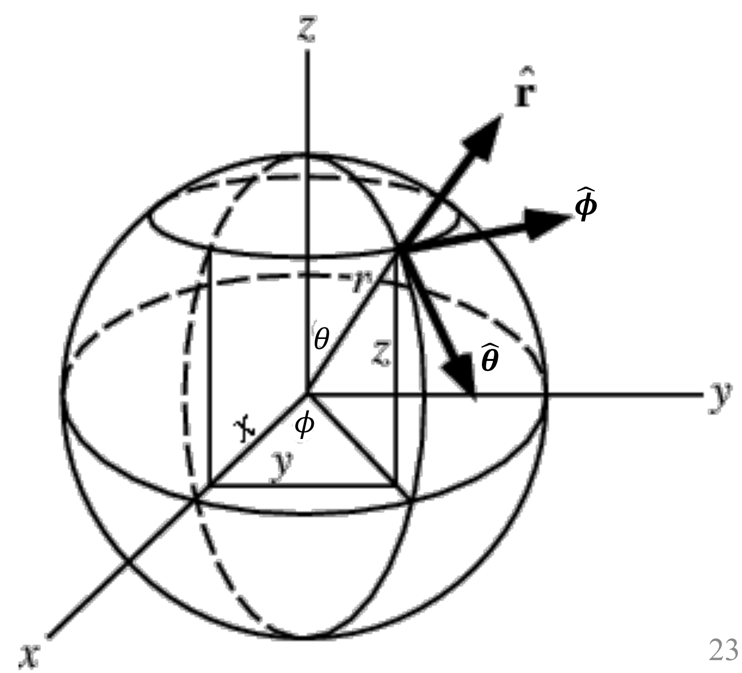
# Curvilinear Coordinates

- Vector identities were developed irrespective of coordinate system
- Three coordinate systems are commonly used: Cartesian, Cylindrical, and Spherical (in 2D, polar system, which is special case of cylindrical)

**Cylindrical coordinate**




**Spherical coordinate**



# Curvilinear – Cartesian Transformation

**Spherical**

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

**Cylindrical**

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

# Curvilinear Expressions of Vector Derivatives

**Cartesian** $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}},\ d\tau = dx\,dy\,dz$

*Gradient:* $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

*Divergence:* $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

*Curl:* $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

*Laplacian:* $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

**Cylindrical** $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}},\ d\tau = s\,ds\,d\phi\,dz$

*Gradient:* $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

*Divergence:* $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

*Curl:* $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

*Laplacian:* $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

**Spherical** $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}},\ d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

*Gradient:* $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

*Divergence:* $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

*Curl:* $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

*Laplacian:* $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Proof see Appendix A

# Dirac Delta Function

- Want to describe point charge Q as special case of an expression for charge density:

$$\rho(0) = Q\delta(x)$$

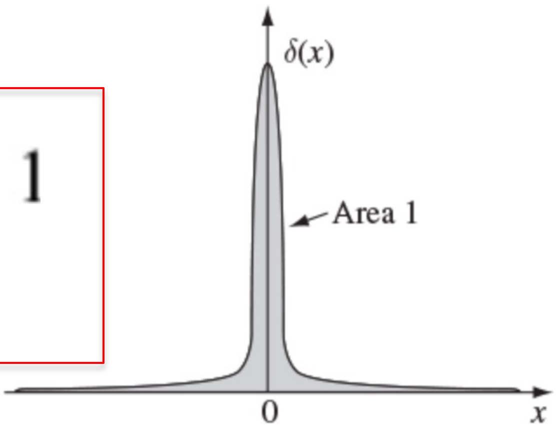
$$\textit{where:}\quad \delta(x) = 0 \text{ if } x \neq 0$$

$$= \infty \text{ if } x = 0$$

- Also need $\int \delta(x)dx = 1$
  - for any limits that include $x=0$, so total charge is Q

# 1D Dirac Delta Function

$$\boxed{\delta(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ \infty & \text{if } x = 0, \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)\, dx = 1}$$




If *f(x)* is ordinary (continuous) function:

$$\boxed{f(x)\delta(x) = f(0)\delta(x)}$$

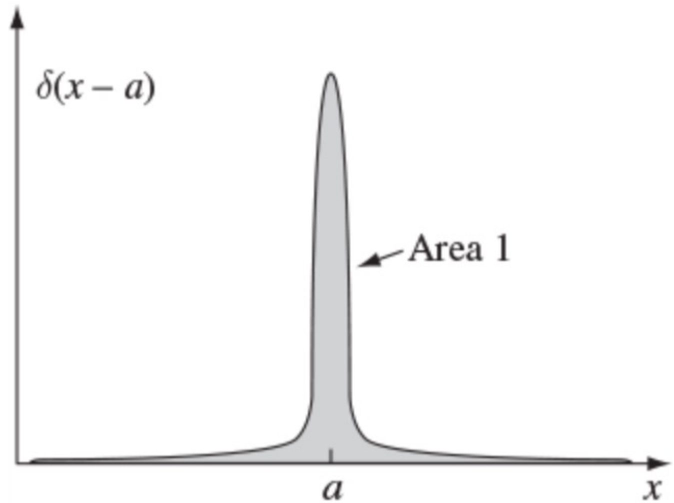
$$\boxed{\int_{-\infty}^{\infty} f(x)\delta(x)\, dx = f(0) \int_{-\infty}^{\infty} \delta(x)\, dx = f(0)}$$

# 1D Dirac Delta Function

Can shift the spike from $x$=0 to $x$=$a$

$$\delta(x-a) = \begin{cases} 0 & \text{if } x \neq a, \\ \infty & \text{if } x = a, \end{cases} \quad \text{with} \int_{-\infty}^{\infty} \delta(x-a)\,dx = 1$$

*Typically, the limits of integration are from x= -∞ to x= +∞, but will give 1 for any limits that include x=a.*




If *f(x)* is ordinary (continuous) function:

$$f(x)\delta(x-a) = f(a)\delta(x-a) \qquad \int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$$

For delta functions $D_1(x)$ and $D_2(x)$

if $\int_{-\infty}^{\infty} f(x)D_1(x)\,dx = \int_{-\infty}^{\infty} f(x)D_2(x)\,dx$ then $D_1(x)$=$D_2(x)$

# 3D Dirac Delta Function

Generalize the delta function to three dimensions:

$$\delta^3(\mathbf{r}) = \delta(x)\,\delta(y)\,\delta(z)$$

$$\int_{\text{all space}} \delta^3(\mathbf{r})\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = 1$$

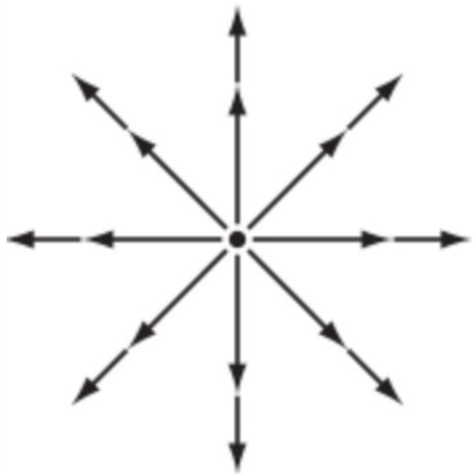
If $f(\mathbf{r})$ is ordinary (continuous) function:

$$\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{a})\,d\tau = f(\mathbf{a})$$

# Dirac delta function: $\nabla \cdot \frac{\hat{\boldsymbol{r}}}{\mathrm{r}^2}$

$$\boxed{\nabla \cdot \frac{\hat{\boldsymbol{r}}}{\mathrm{r}^2} = 4\pi\delta^3(\boldsymbol{r})}$$

F(r)= $\hat{\boldsymbol{r}}/r^2$



$r \neq 0$ $\quad \nabla \cdot \frac{\hat{\boldsymbol{r}}}{\mathrm{r}^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{1}{r^2}\right)\right] = 0$

Spherical *Divergence:* $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

Consider a sphere around r=0

$r=0$ $\quad \nabla \cdot \frac{\hat{\boldsymbol{r}}}{\mathrm{r}^2} = \lim_{volume \to 0} \frac{Total\ Flux}{Volume} = \lim_{r \to 0} \frac{4\pi r^2\left(\frac{1}{r^2}\right)}{\frac{4}{3}\pi r^3} = \lim_{r \to 0} \frac{4\pi}{\frac{4}{3}\pi r^3} = +\infty$

$$\iiint \nabla \cdot \frac{\hat{r}}{\mathrm{r}^2} d^3 r = Total\ flux = \oiint \frac{\hat{\boldsymbol{r}}}{\mathrm{r}^2} \cdot d\boldsymbol{a} = 4\pi$$

# Dirac delta function: $\nabla^2 \frac{1}{r}$

$$\boxed{\nabla^2 \frac{1}{r} = -4\pi\delta^3(\boldsymbol{r})}$$

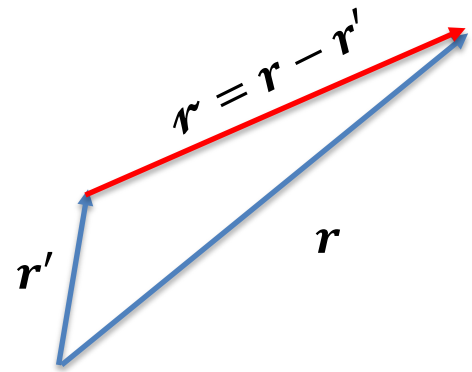
$$\boxed{\nabla(1/r) = \hat{\boldsymbol{r}}[\partial/\partial r](1/r) = \hat{\boldsymbol{r}}\,(-1/r^2) = -\hat{\boldsymbol{r}}/r^2}$$

Spherical *Gradient:* $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

$$\nabla^2(1/r) = \nabla\cdot\nabla(1/r) = \nabla\cdot(-\hat{\boldsymbol{r}}/r^2) = -4\pi\delta^3(\boldsymbol{r})$$

# Dirac delta function

More general:




Define: $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$

$$\boxed{\boldsymbol{\nabla} \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 4\pi\delta^3(\boldsymbol{\mathscr{r}})}$$

$$\boxed{\boldsymbol{\nabla}^2 \frac{1}{\mathscr{r}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}})}$$

# Derivative of Dirac Delta Function

$$\delta[\delta(x)]/dx = -\delta(x)/x$$

Integration by parts: $\int u\,dv = uv - \int v\,du.$ u = $xf(x)$, $v = \delta(x)$

$$\int xf(x)[d\delta(x)/dx]dx = \int xf(x)d\delta(x) = xf(x)\delta(x) - \int\delta(x)d[xf(x)]$$
$$=xf(x)\delta(x) - \int\delta(x)(f(x)dx+xdf(x))$$
$$=xf(x)\delta(x) - \int\delta(x)f(x)dx - \int\delta(x)xf(x)'dx$$
$$=x*f(x)\delta(x) - f(0) - 0*f(0)'=-f(0) = -\int f(x)\delta(x)dx$$

*Red term is zero evaluated at x= -∞ to x= +∞*

$$xf(x)[d\delta(x)/dx]dx = -f(x)\delta(x)$$

$$xd\delta(x)/dx = -\delta(x)$$

$$\boxed{d\delta(x)/dx = -\delta(x)/x}$$

**Example 1.15.**

Show that

k>0:

$$\delta(kx) = \frac{1}{|k|}\delta(x),$$

where $k$ is any (nonzero) constant. (In particular, $\delta(-x) = \delta(x)$: it's an *even* function.)

set $y=kx$ Integration by substitution

$$k>0: \int f(x)\delta(kx)dx = \int_{y=-\infty \to +\infty} f(y/k)\delta(y)(dy/k) = \int f(y/k)\delta(y)(dy/k) = f(0)/k$$

$$k<0: \int f(x)\delta(kx)dx = \int_{y=+\infty \to -\infty} f(y/k)\delta(y)(dy/k) = -\int f(y/k)\delta(y)(dy/k) = -f(0)/k$$

$$\int f(x)\delta(kx)dx = f(0)/\,|k|$$

For delta functions $D_1(x)$ and $D_2(x)$

if $\int_{-\infty}^{\infty} f(x)D_1(x)\,dx = \int_{-\infty}^{\infty} f(x)D_2(x)\,dx$ then $D_1(x)=D_2(x)$

$$\boxed{\delta(kx) = \frac{1}{|k|}\delta(x)}$$

# Helmholtz Theorem of Vector Fields

For a vector field **F** (**E** or **B**), if you are given Div F, Curl F and boundary condition

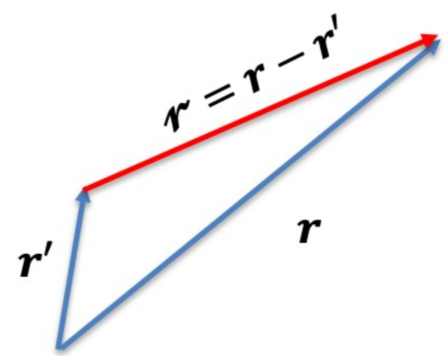
- $\nabla \bullet \boldsymbol{F} = D$
- $\nabla \times \boldsymbol{F} = \boldsymbol{C}$ ($\nabla \bullet C = 0$ because $\nabla \bullet (\nabla \times F) = 0$)
- $F = 0$ at $r = \infty$ or $F$ values defined on boundary (boundary condition)

Then vector field $F$ is uniquely determined:

V, scalar potential

$$\mathbf{F}(\mathbf{r}) = \nabla \boxed{\left( \frac{-1}{4\pi} \int \frac{\nabla' \cdot \mathbf{F}(\mathbf{r}')}{\mathscr{r}} \, d\tau' \right)} + \nabla \times \boxed{\left( \frac{1}{4\pi} \int \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{\mathscr{r}} \, d\tau' \right)}. \qquad \text{(B.10)}$$

***A***, vector potential



# Scalar Potential

- *If* $\nabla \times \boldsymbol{F} = 0$ (ie, curl-less field), then $\boldsymbol{F} = -\nabla V$, where $V$ is called the scalar potential

(10) $\nabla \times (\nabla f) = \mathbf{0}$

- The following are equivalent conditions (divergence theorem)
  *1.* $\nabla \times \boldsymbol{F} = 0$, everywhere in space
  *2.* $\int F \cdot dl$ is independent of path
  *3.* $\oint F \cdot dl = 0$ for a closed loop
- When boundary condition, $\boldsymbol{F}$ is unique but $\boldsymbol{V}$ is not unique since a constant can be added

For example, in electrostatics, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ and $\nabla \times \mathbf{E} = \mathbf{0}$, so

$$\mathbf{E}(\mathbf{r}) = -\nabla\left(\frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'\right) = -\nabla V$$

# Vector Potential

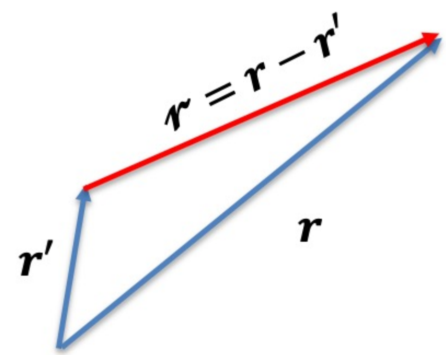
- If $\nabla \cdot F = 0$ (ie, divergence-less everywhere), then $F = \nabla \times A$, where **A** is a vector potential

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

- The following are equivalent conditions (curl theorem)

1. $\nabla \cdot F = 0$
2. $\int F \cdot da$ is independent of surface shape, for any given boundary line on the surface
3. $\int F \cdot da = 0$ for any closed surface

- $A$ is not unique. Gradient of any scalar function f can be added to $A$ because $\nabla \times (\nabla f) = 0$




in magnetostatics $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, so

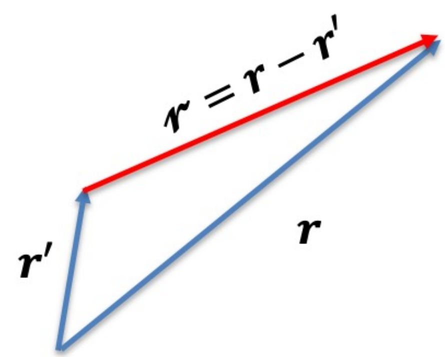
$$\boxed{\mathbf{B}(\mathbf{r}) = \nabla \times \left( \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} \, d\tau' \right) = \nabla \times \mathbf{A}}$$

# Helmholtz Decomposition

Any differentiable vector function **F** that goes to zero faster than 1/r as r→∞ can be expressed as (gradient of a scalar) + (curl of vector):

$$\boxed{\mathbf{F} = -\nabla V + \nabla \mathrm{x} A \text{ (aways true)}}$$

$$\mathbf{F}(\mathbf{r}) = \nabla\overbrace{\left(\frac{-1}{4\pi}\int \frac{\nabla' \cdot \mathbf{F}(\mathbf{r}')}{\mathscr{r}}\,d\tau'\right)}^{V} + \nabla \times \underbrace{\left(\frac{1}{4\pi}\int \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{\mathscr{r}}\,d\tau'\right)}_{A}. \tag{B.10}$$




For example, in electrostatics, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ and $\nabla \times \mathbf{E} = \mathbf{0}$, so

$$\mathbf{E}(\mathbf{r}) = -\nabla\left(\frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'\right) = -\nabla V \tag{B.11}$$

(where $V$ is the scalar potential), while in magnetostatics $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, so

$$\mathbf{B}(\mathbf{r}) = \nabla \times \left(\frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'\right) = \nabla \times \mathbf{A} \tag{B.12}$$

(where $\mathbf{A}$ is the vector potential).