P112A:Electromagnetic Theory

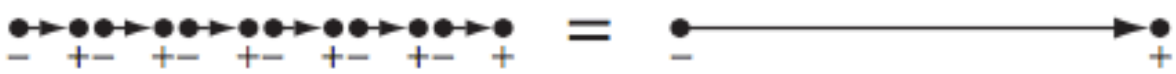
Electric Fields in Matter
(Chapter 4)

Physical Interpretation of Bound Charges

$$\boxed{\sigma_b \equiv \vec{P} \bullet \hat{n}}$$

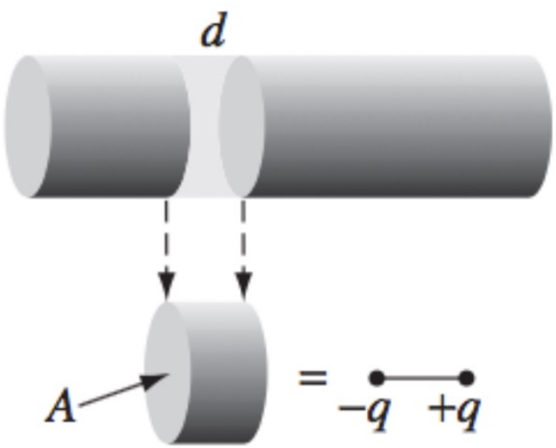
$$\boxed{\rho_b \equiv -\nabla \bullet \vec{P}}$$

- Bound charges are real:



- The middle combinations cancel to zero charge, but the edges have net charges forming the dipole moment
- Call it “bound” because it cannot be removed from the dielectric

How much charge is bound?

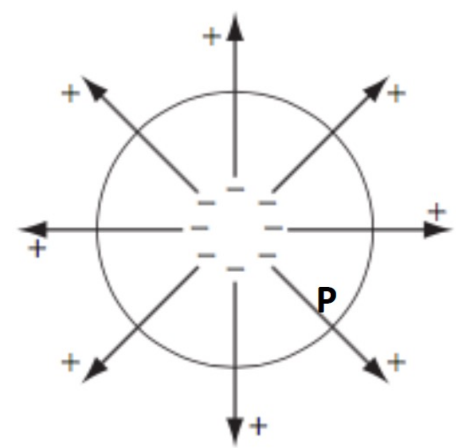


Dipole moment of tiny chunk is p=P(Ad)
Also, dipole moment is p=qd, so charge at disk end is qd=P(Ad) so q=PA and therefore σ_b=q/A=P.
(doesn't really matter the length of d, get the same q on the end)

Repeat the argument as on the left, but now cut the tube obliquely.
Same charge over the surface, but the area is larger, A/cos(θ), so charge density is less.
σ_b=qcosθ)/A=Pcosθ=**P•n**

Volume charge from Divergent *P*

- Positive charge is pushed out beyond the surface, leaving only negative charge inside. → volume charge
- Surface charge: $dq = \sigma_b dA = \mathbf{P} \bullet \hat{n} da = \mathbf{P} \bullet \mathbf{da}$
- Total charge leaving the surface must be equal and opposite in sign to the charge remaining in the volume:



$$\int_{vol} \rho_b d\tau = -\oint_{surf} \vec{P} \bullet d\vec{a} = -\int_{vol} (\nabla \bullet P) d\tau$$

Electric Displacement, *D*

E field created by both bound and free charge, introduce Electric Displacement, D to re-write Gauss's law in terms of free charge and polarization in dielectric:

$$\rho = \varepsilon_0 \left(\nabla \bullet \vec{E} \right) \quad Gauss\,Law$$

$$\rho = \rho_f + \rho_b$$

In dielectric, only consider volume change

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \bullet \vec{P} = \varepsilon_0 \left(\nabla \bullet \vec{E} \right)$$

$$\rho_f = \varepsilon_0 \left(\nabla \bullet \vec{E} \right) + \nabla \bullet \vec{P} = \nabla \bullet \left(\varepsilon_0 \vec{E} + \vec{P} \right)$$

$$Let \quad \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

D is called Displacement Field

$$\boxed{\rho_f = \nabla \bullet \vec{D}}$$

This equation rewrite's Gauss's Law. It requires free charge, not free and bound. The free charge is typically provided in the problem. Bound charge is not known

Gauss's Law with D

$$\rho = \varepsilon_0 \left(\nabla \bullet \vec{E} \right) \quad Gauss\,Law\ for\ E \qquad \boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$$

$$\boxed{\rho_f = \nabla \bullet \vec{D}}$$

Gauss's Law for D, differential form

$$\int_{vol} \rho_f d\tau = Q_{free}^{encl}$$

$$\int_{Vol} \left(\nabla \bullet \vec{D} \right) d\tau = \oint_{Surf} \vec{D} \bullet d\vec{a}$$

Divergence theorem

$$\boxed{\oint_{Surf} \vec{D} \bullet d\vec{a} = Q_{free}^{encl}} \quad Gauss\,Law\ for\ D$$

Integral form

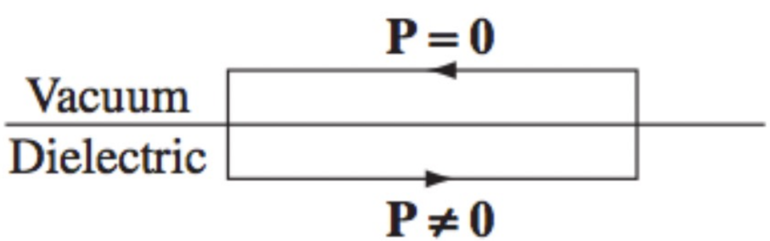
Compute with electric displacement $\mathbf{D}=\varepsilon_0 E+\mathbf{P}$ *and free charge* Q_{free},
Do not need to compute bound change

Curl of *D*

- ∇xE=0 was a consequence of the radial dependence of E from a point charge, **D** can have non-zero curl

$$\nabla \times \vec{D} = \nabla \times (\varepsilon_0 \vec{E} + \vec{P}) = \varepsilon_0 \cancelto{0}{\nabla \times \vec{E}} + \nabla \times \vec{P}$$

Consider boundary of dielectric, **P** = α**E** in Dielectric, **P**=0 outside:



$$\oint \vec{P} \bullet d\ell \neq 0 \xrightarrow{\text{Stoke's Theorem}} \nabla \times \vec{D} = \nabla \times \vec{P} \neq 0$$

In homogeneous dielectric material

- Inside homogeneous dielectric material, divergence and curl of **P**=0:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\nabla \times \vec{D} = \nabla \times (\varepsilon_0 \vec{E} + \vec{P}) = \varepsilon_0 \underbrace{\nabla \times \vec{E}}_{0} + \nabla \times \vec{P}$$

$$\nabla \times \vec{D} = 0$$

If curl of P is zero, then curl of D is zero

$$\nabla \bullet \vec{D} = \rho_f$$

Always true

$$\vec{D} = \varepsilon_0 \vec{E}_{vac}$$

D is found from free charge, ignoring dielectric material. **D** is simply related to vacuum **E** because **P**=0 in vacuum

vacuum permittivity

$$\vec{D} = \varepsilon \vec{E}$$

True if P linearly proportional to E (see next transparency)

permittivity

$$\vec{E} = \frac{\varepsilon_0 \vec{E}_{vac}}{\varepsilon} = \frac{\vec{E}_{vac}}{\varepsilon_r}$$

This equation says that E in dielectric is the same as E in vacuum, divided by the dielectric constant.

Relative permittivity

Linear Dielectrics

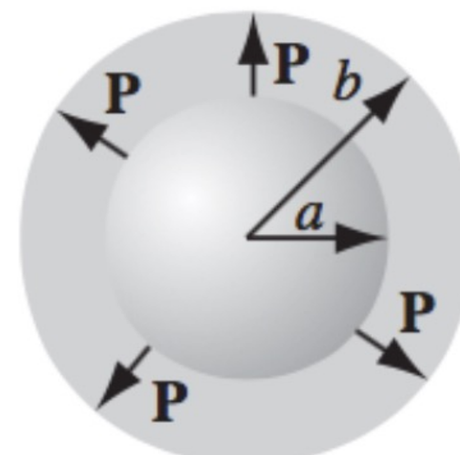
- Basic assumption: $\boldsymbol{P} \sim \boldsymbol{E}$, $\boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}$
- $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = \varepsilon_0 (1 + \chi_e) \boldsymbol{E} = \varepsilon_0 (\varepsilon_r) \boldsymbol{E} = \varepsilon \boldsymbol{E}$
  - ε_0 : electric permittivity of free space (electric constant, vacuum permittivity)
  - χ_e : electric susceptibility
  - ε: Permittivity
  - $\varepsilon_r = 1 + \chi_e$ Relative permittivity, Dielectric constant
  - $\varepsilon = \varepsilon_0 (1 + \chi_e) = \varepsilon_0 (\varepsilon_r)$
  - *Dielectric constant:* κ *in 7D,* ε_r *in 112A,* $\kappa = \varepsilon_r = \varepsilon / \varepsilon_0$
- Can not directly compute P from $P = \varepsilon_0 \chi_e E$, because P also contributes to E
- Strategy: Compute D first, then calculate P

Problem 4.15

- A thick spherical shell is made of dielectric material with frozen in polarization

$$\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$$

- There is no free charge
- Find **E** in all regions
- Use 2 methods:
  – Bound charge and Gauss's Law
  – Find D with Gauss's Law, then get E

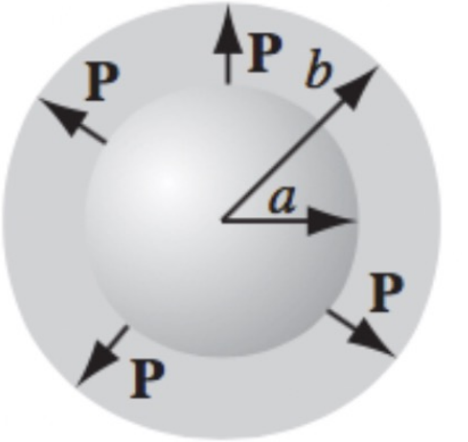


Bound charge and Gauss's Law

$$\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\rho_b = -\nabla \bullet \vec{P}(\vec{r}) = \frac{-1}{r^2}\frac{\partial}{\partial r} r^2\hat{r} \bullet \frac{k}{r}\hat{r} = \frac{-k}{r^2}$$



$$\sigma_b = \vec{P}(\vec{r}) \bullet \hat{n} = \frac{k}{r}\hat{r} \bullet \hat{r}\ \Big|_{r=b} = +\frac{k}{b}$$ r=b

$$\sigma_b = \vec{P}(\vec{r}) \bullet \hat{n} = \frac{k}{r}\hat{r} \bullet (-\hat{r})\ \Big|_{r=a} = \frac{-k}{a}$$ r=a

$$\boxed{Q_{enc} = 0 \qquad r < a}$$

$$Q_{enc} = \frac{-k}{a}\left(4\pi a^2\right) + \int \rho_b d\tau = -4\pi ak + \int_a^r \frac{-k}{r^2}(4\pi r^2 dr)$$

$$Q_{enc} = -4\pi ak - \left[4\pi k(r-a)\right] = -4\pi kr \qquad a < r < b$$

$$Q_{enc=} \frac{-k}{a}\left(4\pi a^2\right) + \frac{k}{b}\left(4\pi b^2\right) + \int_a^b \frac{-k}{r^2}(4\pi r^2 dr) = 0 \qquad r > b$$

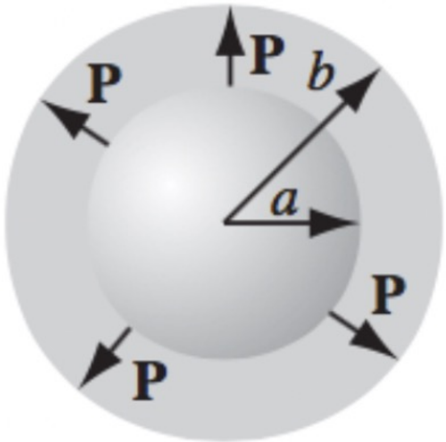
$$E = 0 \qquad r < a$$

$$E = 0 \qquad r > b$$

$$E = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} = \frac{-4\pi kr}{4\pi\varepsilon_0 r^2} = \frac{-k}{\varepsilon_0 r}\hat{r} \qquad a < r < b$$

Find D with Gauss's Law, then get E

Free charge is 0 everywhere
P is only non-zero for a<r<b



$$\oint \vec{D} \bullet d\vec{a} = Q_{free} = 0$$

$$\vec{D} = 0$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = 0$$

$$\vec{E} = -\vec{P} / \varepsilon_0$$

$$\vec{E} = -\frac{k\hat{r}}{r}\frac{1}{\varepsilon_0} = \frac{-k}{\varepsilon_0 r}\hat{r} \qquad a < r < b$$

Example 4.5

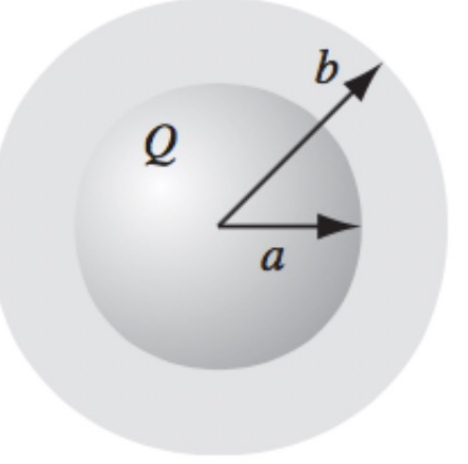
- A metal sphere of radius *a* caries a charge Q. It is surrounded to radius ***b*** by a dielectric material of permittivity ε. Find potential at center.

$$\boxed{\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}=\varepsilon_0(1+\chi_e)\,\mathbf{E}=\varepsilon_0(\varepsilon_r)\mathbf{E}=\varepsilon\mathbf{E}}$$

$$\boxed{\oint \vec{D}\bullet d\vec{a}=Q_{free}}$$

Find E everywhere and do the integration over path to get V

Note: Inside metal sphere, r<a: **E**=0, **P**=0, so **D**=ε_0**E**+**P**=0



$$\vec{D}(\vec{r})=\frac{Q}{4\pi r^2}\hat{r} \qquad r>a$$

so

$$\vec{E}(r)=\vec{D}/\varepsilon \qquad a<r<b$$

$$\vec{E}(r)=\vec{D}/\varepsilon_0 \qquad r>b$$

$$V_0-V_\infty=V_0=-\int_\infty^0 \vec{E}\bullet d\vec{\ell}=-\int_\infty^0 E(r)\hat{r}\bullet dr\hat{r}=-\int_\infty^0 E(r)dr$$

$$V_0=-\int_\infty^b\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)dr-\int_b^a\left(\frac{Q}{4\pi\varepsilon\, r^2}\right)dr-\int_a^0(0)\,dr=\frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b}+\frac{1}{\varepsilon a}-\frac{1}{\varepsilon b}\right)$$

Example 4.5

$$\vec{D}(\vec{r}) = \frac{Q}{4\pi r^2}\hat{r} \qquad r > a$$

so

$$\vec{E}(r) = \vec{D}/\varepsilon \qquad a < r < b$$

$$\vec{E}(r) = \vec{D}/\varepsilon_0 \qquad r > b$$

Once have E in the dielectric, can compute P, and then σ_b and ρ_b.

in dielectric, a<=r<=b

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 \chi_e \vec{D}/\varepsilon = \frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon r^2}\hat{r}$$

Direction of boundary surface points away from interior of dielectric

$$\rho_b = -\nabla \cdot \vec{P} = 0$$

$$\sigma_b = \vec{P}\cdot\hat{n}\ \Big|_{r=b} = \frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon r^2}\hat{r}\cdot(\hat{r})\ \Big|_{r=b} = \frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon b^2}$$

Outer surface

$$\sigma_b = \vec{P}\cdot\hat{n}\ \Big|_{r=a} = \frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon r^2}\hat{r}\cdot(-\hat{r})\ \Big|_{r=a} = -\frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon a^2}$$

inner surface

Boundary Conditions

$$\boxed{\boldsymbol{D}=\varepsilon_0\boldsymbol{E}+\boldsymbol{P}= \varepsilon_0(1+\chi_e)\ \boldsymbol{E} = \varepsilon_0(\varepsilon_r)\boldsymbol{E}= \varepsilon\boldsymbol{E}}$$

$$\boxed{\oint \vec{D}\bullet d\vec{a} = Q_{free}}$$

Boundary conditions for $E \rightarrow D$.

$$D_{above}^{perp} - D_{below}^{perp} = \sigma_f$$

$$\mathbf{E}_{\text{above}}^{\|} = \mathbf{E}_{\text{below}}^{\|} \longrightarrow D_{above}^{paral} - D_{below}^{paral} = P_{above}^{paral} - P_{below}^{paral}$$

$$\boxed{\epsilon_{\text{above}} E_{\text{above}}^{\perp} - \epsilon_{\text{below}} E_{\text{below}}^{\perp} = \sigma_f}$$

$$\boxed{\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f}$$

Laplace's Equation with Linear Dielectrics

$$\boxed{\boldsymbol{D}=\varepsilon_0\boldsymbol{E}+\boldsymbol{P}=\varepsilon_0(1+\chi_e)\,\boldsymbol{E}=\varepsilon_0(\varepsilon_r)\boldsymbol{E}=\varepsilon\boldsymbol{E}}$$

$$\boxed{\rho_f=\nabla\cdot\vec{D}}$$

- In a homogeneous isotropic linear dielectric, the bound charge density ρ_b is proportional to free charge density ρ_f.

$$\rho_b=-\nabla\cdot\vec{P}(\vec{r})=-\nabla\cdot(\varepsilon_0\chi_e\vec{E})=-\nabla\cdot(\varepsilon_0\chi_e\vec{D}/\varepsilon)=\frac{-\varepsilon_0\chi_e}{\varepsilon}\left(\nabla\cdot\vec{D}\right)$$

$$=\frac{-\varepsilon_0\chi_e}{\varepsilon}\rho_f=\frac{-\varepsilon_0\chi_e}{\varepsilon_0(1+\chi_e)}\rho_f=\frac{-\chi_e}{(1+\chi_e)}\rho_f$$

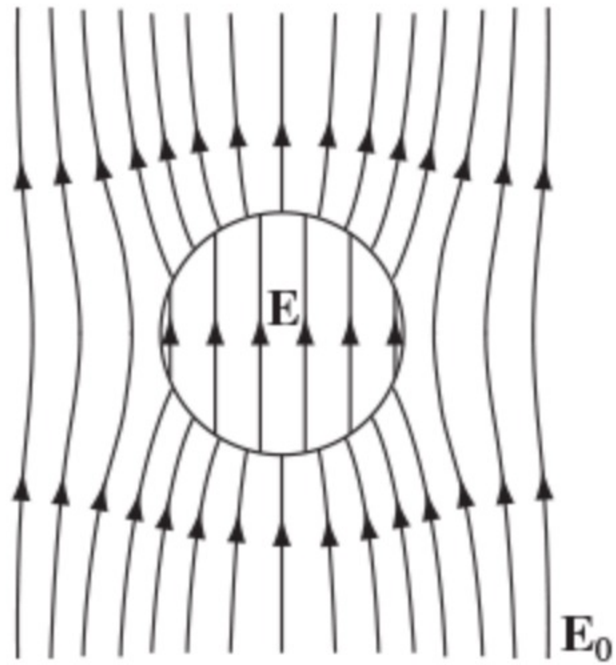
$$\varepsilon_{above}E_{above}^{perp}-\varepsilon_{below}E_{below}^{perp}=\sigma_f$$

$$\boxed{\begin{aligned}&\varepsilon_{above}\frac{\partial V_{above}}{\partial r}-\varepsilon_{below}\frac{\partial V_{below}}{\partial r}=-\sigma_f\\&V_{above}=V_{below}\end{aligned}}$$

Boundary condition for potential in dielectric materials

Example 4.7: Laplace+dielectric

A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field. Find the electric field inside the sphere.



(0) $r<R$, V_{in} = finite

(i) $V_{\text{in}} = V_{\text{out}}$, at $r = R$,

(ii) $\epsilon \frac{\partial V_{\text{in}}}{\partial r} = \epsilon_0 \frac{\partial V_{\text{out}}}{\partial r}$, at $r = R$,

(iii) $V_{\text{out}} \to -E_0 r \cos\theta$, for $r \gg R$.

$$\varepsilon_{above}\frac{\partial V_{above}}{\partial r} - \varepsilon_{below}\frac{\partial V_{below}}{\partial r} = -\sigma_f$$

No free charge on surface, and no dielectric material outside of sphere, so $\varepsilon_{above}=\varepsilon_0$

(i) $V_{\text{in}} = V_{\text{out}}, \quad \text{at } r = R,$

(ii) $\epsilon \dfrac{\partial V_{\text{in}}}{\partial r} = \epsilon_0 \dfrac{\partial V_{\text{out}}}{\partial r}, \quad \text{at } r = R,$

(iii) $V_{\text{out}} \to -E_0 r \cos\theta, \quad \text{for } r \gg R$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + B_\ell r^{-(\ell+1)}) P_\ell(\cos\theta)$$

General Solution for spherical coordinates

$$\begin{cases} V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & r \le R \\ V_{out}(r,\theta) = -E_0 r \cos\theta + \sum_{\ell=0}^{\infty} B_\ell r^{-(\ell+1)} P_\ell(\cos\theta) & r \ge R \end{cases}$$

BC (0) r=0, V = finite

B.C. (iii) A1 out = -E_0, other Al out = 0

$$\text{B.C.(i)}: A_\ell R^\ell P_\ell = -E_0 R \cos\theta + B_\ell R^{-(\ell+1)} P_\ell$$

$$\Rightarrow \begin{cases} A_1 R = -E_0 R + B_1 R^{-2} & \ell = 1 \\ A_\ell R^\ell = B_\ell R^{-(\ell+1)} & \ell \ne 1 \end{cases}$$

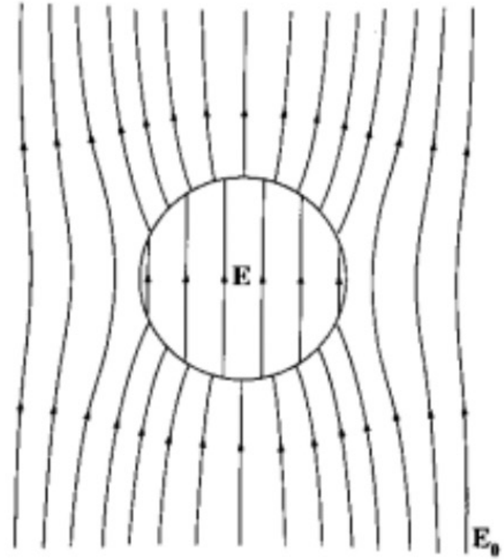
$$\text{B.C.(ii)}: \varepsilon_r \ell A_\ell R^{\ell-1} P_\ell = -E_0 \cos\theta - (\ell+1) B_\ell R^{-(\ell+2)} P_\ell$$

$$\Rightarrow \begin{cases} \varepsilon_r A_1 = -E_0 - 2B_1 R^{-3} & \ell = 1 \\ \varepsilon_r \ell A_\ell R^{\ell-1} = -(\ell+1) B_\ell R^{-(\ell+2)} & \ell \ne 1 \end{cases}$$

$$\begin{cases} A_1 R = -E_0 R + B_1 R^{-2} & \ell = 1 \\ A_\ell R^\ell = B_\ell R^{-(\ell+1)} & \ell \neq 1 \end{cases} \qquad \begin{cases} \varepsilon_r A_1 = -E_0 - 2B_1 R^{-3} & \ell = 1 \\ \varepsilon_r \ell A_\ell R^{\ell-1} = -(\ell+1) B_\ell R^{-(\ell+2)} & \ell \neq 1 \end{cases}$$

2 equations, 2 unknowns for each l

$$\Rightarrow \begin{cases} A_1 = -\dfrac{3E_0}{\varepsilon_r + 2}; B_1 = \dfrac{\varepsilon_r - 1}{\varepsilon_r + 2} R^3 & \ell = 1 \\ A_\ell = B_\ell = 0 & \ell \neq 1 \end{cases}$$



$$\begin{cases} V_{in}(r,\theta) = -\dfrac{3E_0}{\varepsilon_r + 2} r\cos\theta \\ V_{out}(r,\theta) = -E_0 r\cos\theta + (\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}) R^3 E_0 r^{-2}\cos\theta \end{cases}$$

With V everywhere, you can compute E everywhere. For example, E inside sphere is constant in direction and mag.

$$V_{in} = -\frac{3E_0}{\varepsilon_r + 2} r\cos\theta = -\frac{3E_0}{\varepsilon_r + 2} z$$

$$\vec{E}_{in} = -\nabla V_{in} = -\frac{\partial}{\partial z}\left(-\frac{3E_0}{\varepsilon_r + 2} z\right)\hat{z} = \frac{3E_0}{\varepsilon_r + 2}\hat{z}$$