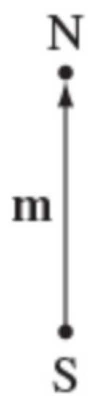
P112A:Electromagnetic Theory

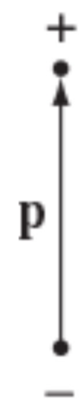
Magnetic Fields in Matter

(Chapter 6)

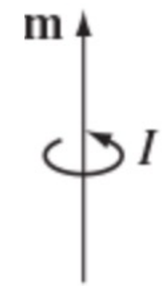
Dipole models



Magnetic dipole (Gilbert model)



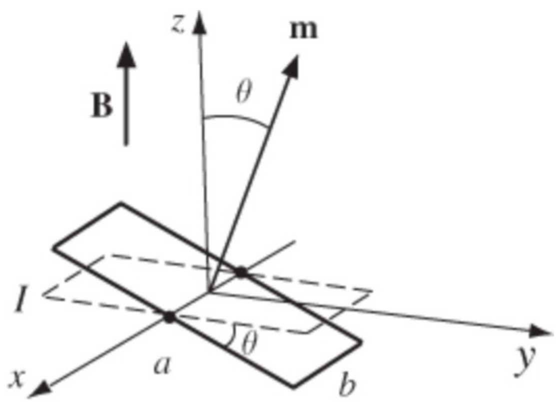
Electric dipole

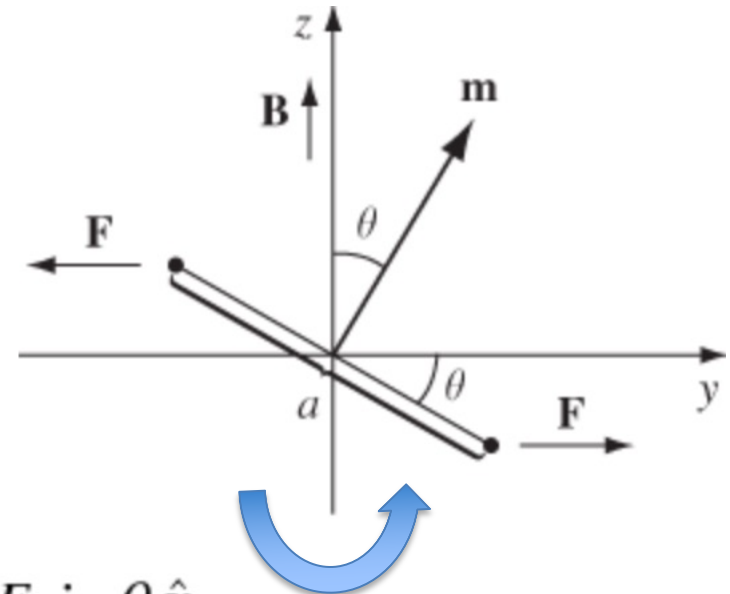


Magnetic dipole (Ampère model)

Torques and Forces on Magnetic Dipoles

Torque on a magnetic dipole (current loop):





$$F = IbB, \qquad \mathbf{N} = aF\sin\theta\,\hat{\mathbf{x}}.$$

$$\mathbf{N} = IabB\sin\theta\,\hat{\mathbf{x}} = mB\sin\theta\,\hat{\mathbf{x}}$$

$$\boxed{\mathbf{N} = \mathbf{m}\times\mathbf{B},}$$

Torque tends to cause dipole **m** to rotate towards **B**

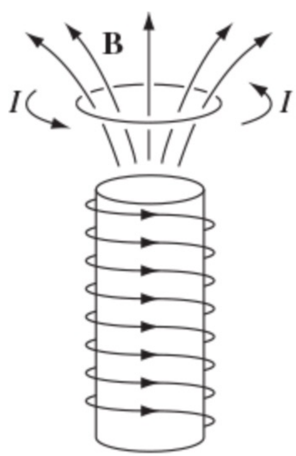
$$m = Iab$$

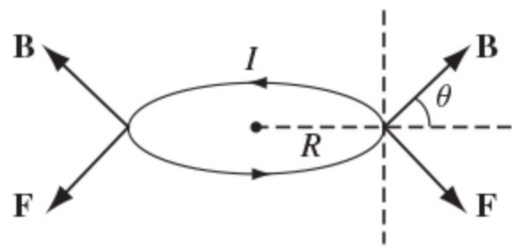
Torques and Forces on Magnetic Dipoles

In a **uniform** B field, the net force on current loop is zero:

$$\mathbf{F} = I\oint (d\mathbf{l} \times \mathbf{B}) = I\left(\oint d\mathbf{l}\right) \times \mathbf{B} = \mathbf{0};$$

Circular wire ring of radius , carrying a current, suspended above a short solenoid in the “fringing” region, B field is nonuniform:





Net force:

$$F = 2\pi IRB\cos\theta.$$

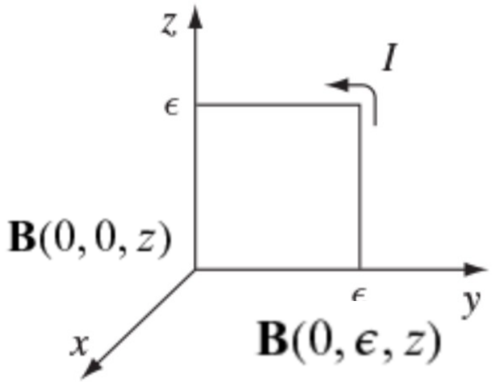
For an infinitesimal loop, with dipole moment **m**, in a B field:

Net force $\boxed{\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})}$

Proof (Problem 6.4)

Assume the dipole is an infinitesimal square, of side (if it's not, chop it up into squares, and apply the argument to each one). Choose axes as shown in Fig, and calculate along each of the four sides. Expand in a Taylor series – on the right-hand side, for instance

$$\mathbf{B} = \mathbf{B}(0, \epsilon, z) \approx \mathbf{B}(0, 0, z) + \epsilon \frac{\partial \mathbf{B}}{\partial y}\Big|_{(0,0,z)}.$$



$$d\mathbf{F} = I\{(dy\,\hat{\mathbf{y}})\times\mathbf{B}(0,y,0) + (dz\,\hat{\mathbf{z}})\times\mathbf{B}(0,\epsilon,z) - (dy\,\hat{\mathbf{y}})\times\mathbf{B}(0,y,\epsilon) - (dz\,\hat{\mathbf{z}})\times\mathbf{B}(0,0,z)\}$$

$$= I\Big\{-(dy\,\hat{\mathbf{y}})\times\underbrace{[\mathbf{B}(0,y,\epsilon)-\mathbf{B}(0,y,0)]}_{\approx\,\epsilon\frac{\partial\mathbf{B}}{\partial z}} + (dz\,\hat{\mathbf{z}})\times\underbrace{[\mathbf{B}(0,\epsilon,z)-\mathbf{B}(0,0,z)]}_{\approx\,\epsilon\frac{\partial\mathbf{B}}{\partial y}}\Big\}$$

$$\Rightarrow I\epsilon^2\left\{\hat{\mathbf{z}}\times\frac{\partial\mathbf{B}}{\partial y} - \hat{\mathbf{y}}\times\frac{\partial\mathbf{B}}{\partial z}\right\}. \quad \left[\text{Note that } \int dy\,\frac{\partial\mathbf{B}}{\partial z}\Big|_{0,y,0} \approx \epsilon\,\frac{\partial\mathbf{B}}{\partial z}\Big|_{0,0,0} \text{ and } \int dz\,\frac{\partial\mathbf{B}}{\partial y}\Big|_{0,0,z} \approx \epsilon\,\frac{\partial\mathbf{B}}{\partial y}\Big|_{0,0,0}.\right]$$

$$\mathbf{F} = m\left\{\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 0 & 0 & 1\\ \frac{\partial B_x}{\partial y} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial y}\end{vmatrix} - \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 0 & 1 & 0\\ \frac{\partial B_x}{\partial z} & \frac{\partial B_y}{\partial z} & \frac{\partial B_z}{\partial z}\end{vmatrix}\right\} = m\left\{\hat{\mathbf{y}}\frac{\partial B_x}{\partial y} - \hat{\mathbf{x}}\frac{\partial B_y}{\partial y} - \hat{\mathbf{x}}\frac{\partial B_z}{\partial z} + \hat{\mathbf{z}}\frac{\partial B_x}{\partial z}\right\}$$

$$= m\left[\hat{\mathbf{x}}\frac{\partial B_x}{\partial x} + \hat{\mathbf{y}}\frac{\partial B_x}{\partial y} + \hat{\mathbf{z}}\frac{\partial B_x}{\partial z}\right] \quad \left(\text{using } \boldsymbol{\nabla}\cdot\mathbf{B} = 0 \text{ to write } \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = -\frac{\partial B_x}{\partial x}\right).$$

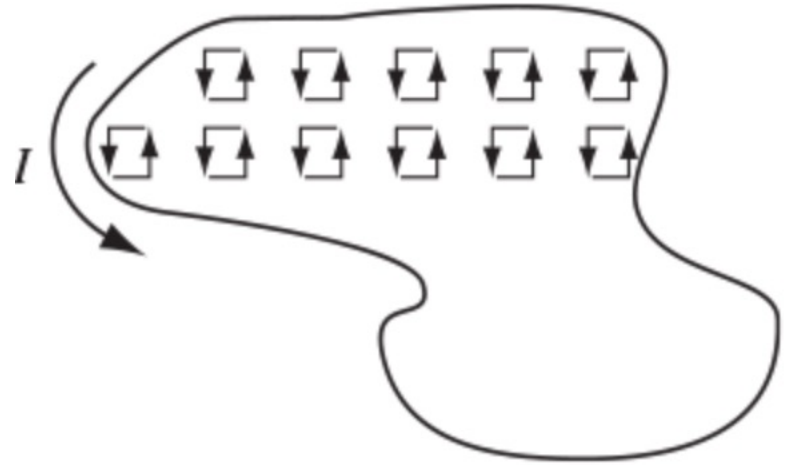
But $\mathbf{m}\cdot\mathbf{B} = mB_x$ (since $\mathbf{m} = m\hat{\mathbf{x}}$, here), so $\boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}) = m\boldsymbol{\nabla}(B_x) = m\left(\frac{\partial B_x}{\partial x}\hat{\mathbf{x}} + \frac{\partial B_x}{\partial y}\hat{\mathbf{y}} + \frac{\partial B_x}{\partial z}\hat{\mathbf{z}}\right)$.

Therefore $\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B})$.   qed

Torques and Forces on Magnetic Dipoles

$$\boxed{\mathbf{N} = \mathbf{m} \times \mathbf{B},} \qquad \boxed{\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})}$$

Derivation for the square loop gives the general result, Since any current loop could be built up from infinitesimal rectangles, with all the "internal" sides canceling:



Diamagnets, Paramagnets, Ferromagnets

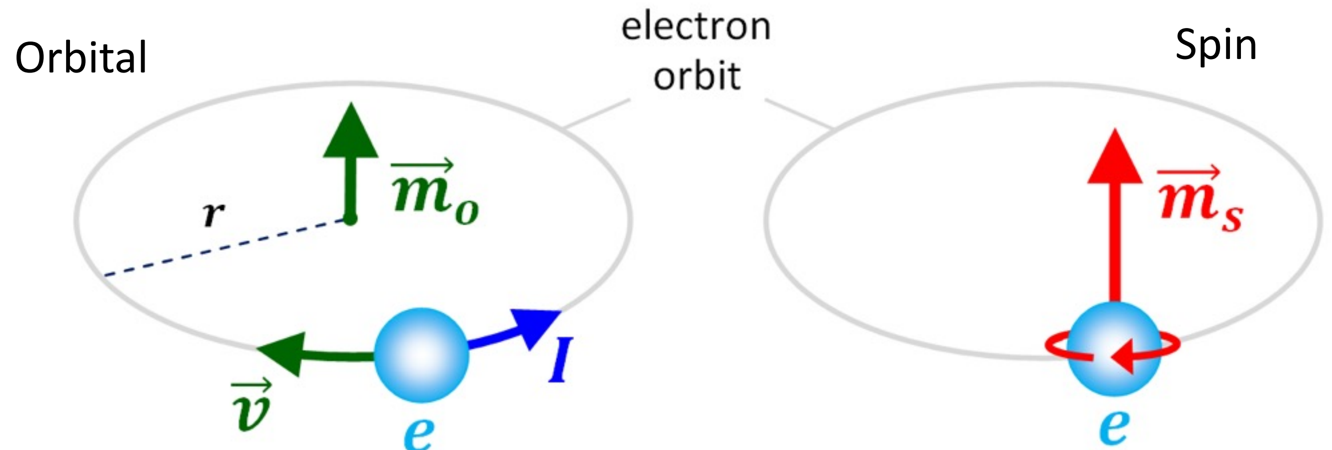
- Atoms can be considered as magnetic dipoles, when apply magnetic field, dipoles aligned, matter becomes **magnetized**

- Dipole moments direction parallel to field: **Paramagnets**

- Dipole moments direction opposite to field: **Diamagnets**

- Retain magnetization even after field removed: **Ferromagnets**

Atom magnetic dipole moment



http://www.e-magnetica.pl/

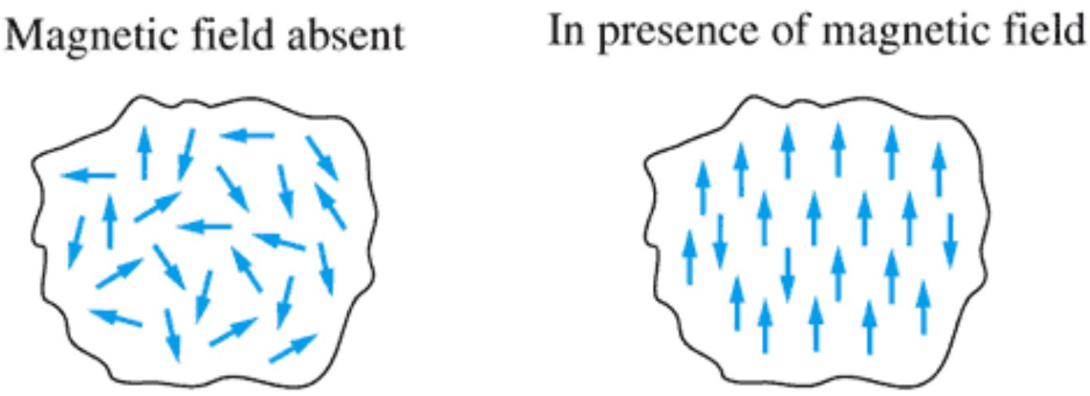
The dipole is atomic current loops.

magnetic moment = magnetic dipole moment

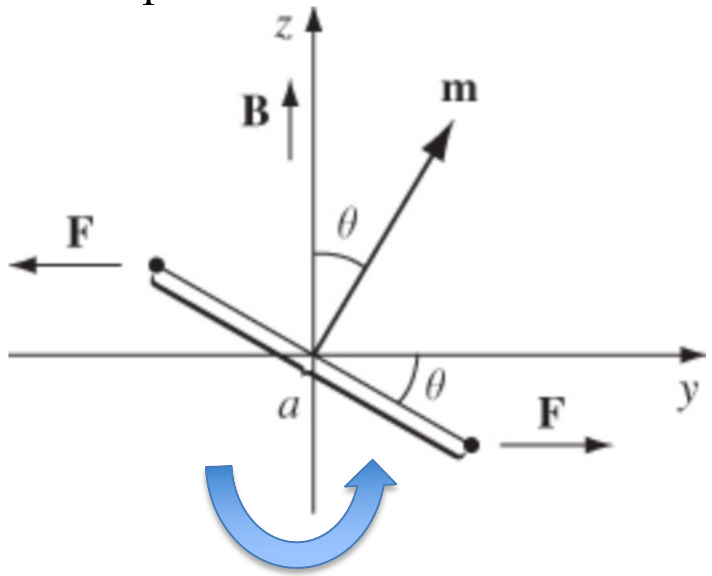
- Electron orbital magnetic moment: from electron motion around the nucleus
- Electron spin magnetic moment: from spin of electrons
- The magnetic moment of an atom is primarily determined by the electrons: neutrons and protons (nucleons) have only spins with much smaller magnetic moments due to their large mass.

Paramagnetism

- Thermal motion makes orientation of atom dipoles random, no macroscopic magnetic moment
- External **B** field aligns the magnetic moments of atom dipoles in its direction, matter appear to have macroscopic magnetic moment
- Involves the magnetic dipoles associated with the spins of unpaired electrons.



Paramagnetism



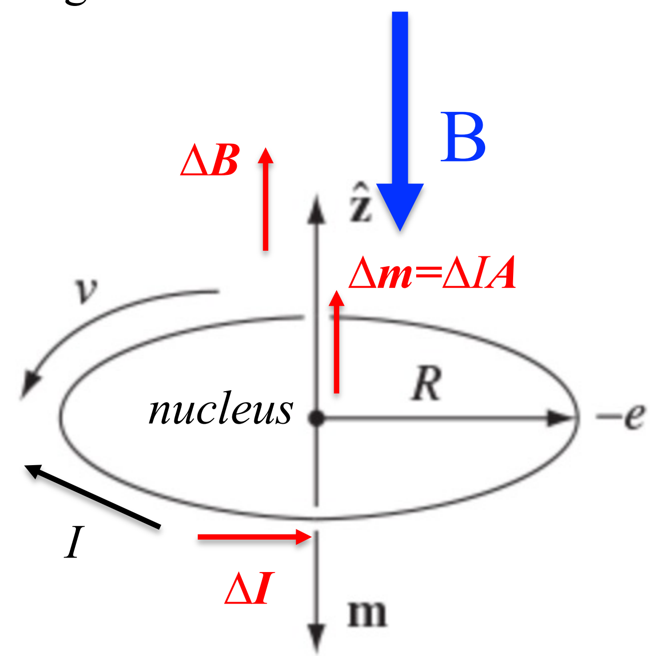
$$\boxed{\mathbf{N} = \mathbf{m} \times \mathbf{B},}$$

Torque tend to causes dipole **m** to rotate towards **B**

Diamagnetism

- **Diamagnetism:** Property of materials that create induced magnetic field and dipole moment opposite to externally field
- Involves the magnetic dipoles associated with the electron orbital motion.
- Caused by induced orbital dipole moments opposite to external field
- Typically much weaker than paramagnetism
- Typically observed in atoms with even numbers of electrons, where paramagnetism is usually absent, as dipole moments from electron spins cancel out.

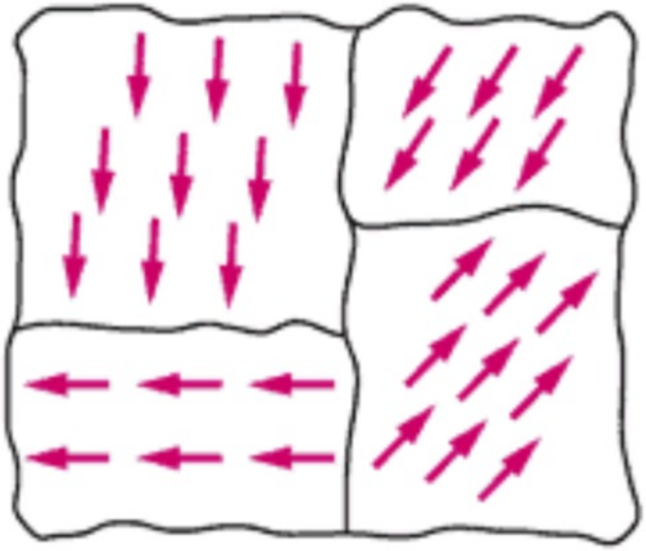
Effect of a Magnetic Field on Atomic Orbits



- When B turned on, induce a current $\Delta \boldsymbol{I}$
- $\Delta \boldsymbol{I}$ produces ΔB and Δm in opposite direction to resist B (Lenz's Law,)
- →Diamagnetism

Ferromagnetism

- Like paramagnetism, ferromagnetism involves the magnetic dipoles associated with the spins of unpaired electrons.
- In a ferromagnet, each dipole (from the spin of the unpaired electron) tends to point in the same direction as its neighbors in the small patch, called **domain**. This alignment is due to a quantum effect known as the exchange interaction
- After being magnetized by external magnetic field, thermal motion on atoms cannot randomize directions of domain dipoles → retaining magnetization after the field is removed



domains

Magnetization

In the presence of a magnetic field, matter becomes magnetized, material contains many tiny dipoles, with a net alignment along some direction:

- Dipole direction parallel to field: **Paramagnets**
- Dipole direction direction opposite to field: **Diamagnets**
- Retain magnetization even after the field is removed: **Ferromagnets**

Magnetization express the density of permanent or induced magnetic dipole moments in magnetized material:

$$\mathbf{M} \equiv \textit{magnetic dipole moment per unit volume.}$$

$$\boxed{\mathbf{M} = \frac{\sum \mathbf{m}_i}{V}}$$

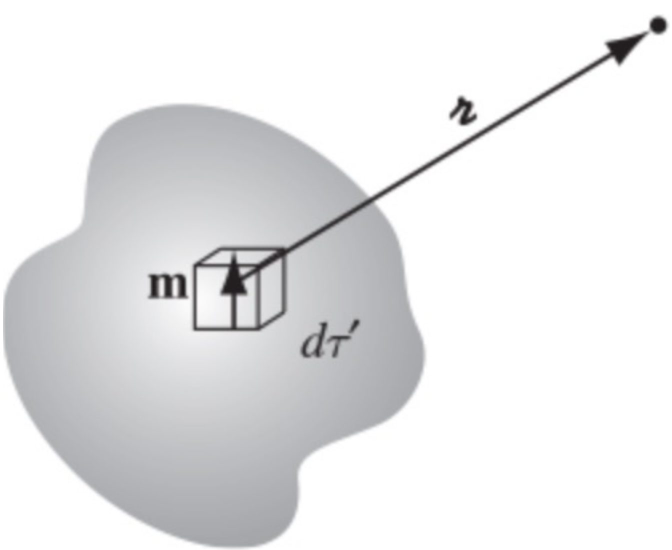
M plays a role analogous to the polarization **P** in electrostatic

Field of a Magnetized Object

Each small volume $\quad \mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\dfrac{\mathbf{m}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$

Consider the macroscopic field,

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'}$$



Bound Currents

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'}$$

$$\nabla'\frac{1}{\mathscr{r}} = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$ Proof see chapter 4

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \left[\mathbf{M}(\mathbf{r}') \times \left(\nabla'\frac{1}{\mathscr{r}}\right)\right] d\tau'$$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{1}{\mathscr{r}}[\nabla' \times \mathbf{M}(\mathbf{r}')]\, d\tau' + \frac{\mu_0}{4\pi}\oint \frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}') \times d\mathbf{a}']$$

Bound volume current density:

$$\boxed{\mathbf{J}_b = \nabla \times \mathbf{M},}$$

Bound surface current density:

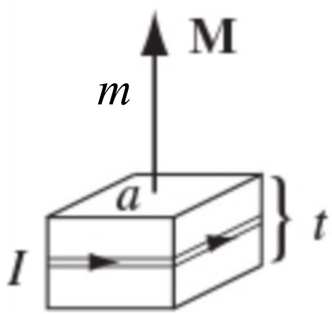
$$\boxed{\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}},}$$

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\, d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\, da'}$$

Physical Interpretation of Bound Currents

Uniformly magnetized material

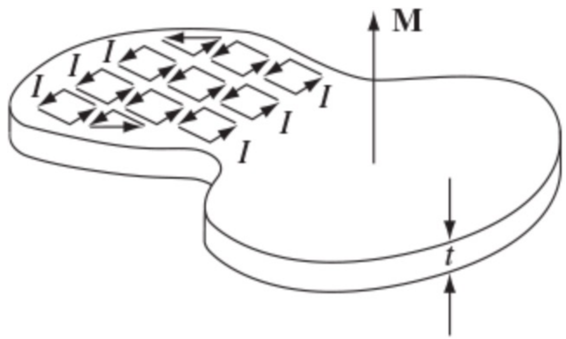
Dipoles represented by tiny current loops *I*



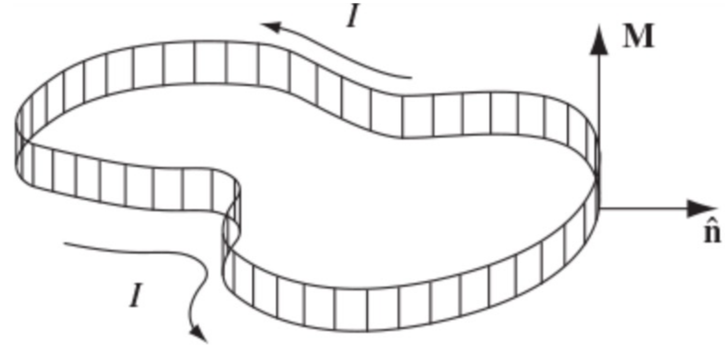
$$m = \mathrm{M}at = Ia$$

$$I = \mathrm{M}t$$

All the "internal" currents cancel



At surface no adjacent loop to do canceling → net current on surface = *I*



Surface bound current density

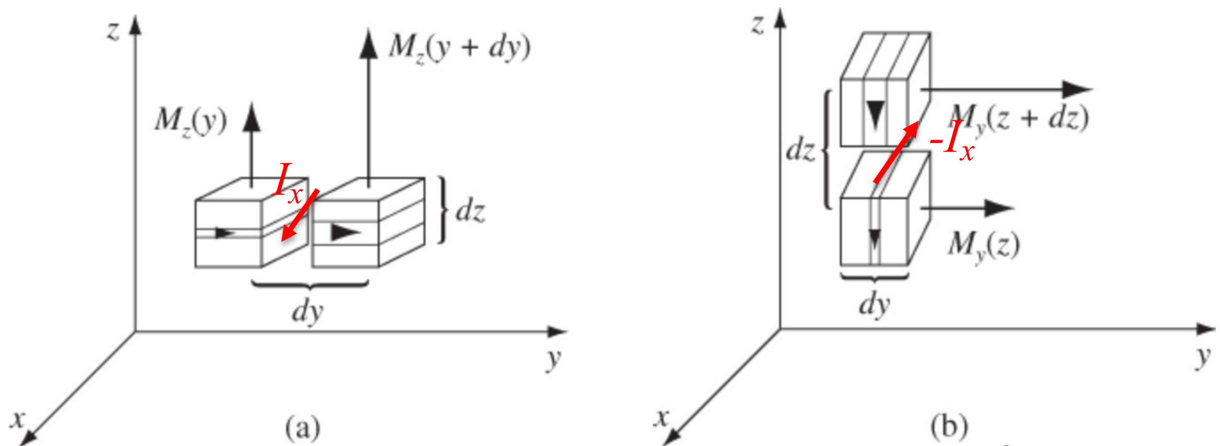
$$\mathrm{K}_b = I/t = \mathrm{M}$$

K_b direction perpendicular to M and surface normal

$$\boxed{\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}},}$$

Physical Interpretation of Bound Currents

When magnetization is nonuniform, the internal currents no longer cancel



M_z contribution to net I_x

$$I_{x\,Mz} = [\mathrm{M_z(y + dy) - M_z(y)]dz} = \frac{\partial M_z}{\partial y} dz \qquad (J_b)_{x\,Mz} = \frac{I_{x\,Mz}}{dz} = \frac{\frac{\partial M_z}{\partial y} dz}{dz} = \frac{\partial M_z}{\partial y}$$

M_y contribution to net I_x

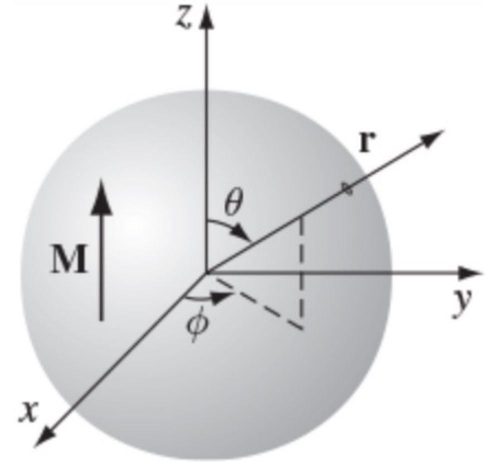
$$I_{x\,My} = -[\mathrm{M_y(z + dz) - M_z(z)]dy} = -\frac{\partial M_y}{\partial z} dy \qquad (J_b)_{x\,My} = \frac{I_{x\,My}}{dy} = \frac{-\frac{\partial M_y}{\partial z} dy}{dy} = \frac{\partial M_y}{\partial z}$$

Bound volume current density:

$$(J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$$

$$\boxed{J_b = \nabla \times \mathbf{M}}$$

Sphere (Example 6.1)



Find the magnetic field of a uniformly magnetized sphere.

Choosing the z-axis along the direction of $\mathbf{M}$

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \mathbf{0}, \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M \sin\theta\, \hat{\boldsymbol{\phi}}.$$

Now, a rotating spherical shell, of uniform surface charge σ, corresponds to a surface current density

$$\mathbf{K} = \sigma \mathbf{v} = \sigma \omega R \sin\theta\, \hat{\boldsymbol{\phi}}.$$

$$\sigma R \boldsymbol{\omega} \rightarrow \mathbf{M}.$$

Example 5.11, inside sphere shell:

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\, \hat{\mathbf{r}} - \sin\theta\, \hat{\boldsymbol{\theta}}) = \frac{2}{3}\mu_0 \sigma R \omega \hat{\mathbf{z}} = \frac{2}{3}\mu_0 \sigma R \boldsymbol{\omega}.$$

→ Inside $\mathbf{B} = \frac{2}{3}\mu_0 \mathbf{M},$

Outside B_{dip} from perfect dipole: $\mathbf{m} = \frac{4}{3}\pi R^3 \mathbf{M}$

Auxiliary Field **H**

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b \qquad \mathbf{J}_b = \nabla \times \mathbf{M}$$

Ampere's Law

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\nabla \times \mathbf{M})$$

$$\nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) = \mathbf{J}_f$$

Define Auxiliary Field **H**:

$$\boxed{\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}}$$

Amperes' Law in terms of **H**, only free charge involved

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}_f} \qquad \boxed{\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}}$$

Linear Media

For paramagnets and diamagnets there are the linear relations

$$\boxed{\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}}$$

$$\boxed{\mathbf{M} = \chi_m \mathbf{H}}$$ *Magnetic susceptibility* χ_m *Defined by H, not B,*

$$\boxed{\mathbf{B} = \mu \mathbf{H}}$$ *Permeability* μ

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}$$

Permeability of free space μ_o

$$\boxed{\mu \equiv \mu_0(1 + \chi_m)}$$

H~B, D~E

$$\boxed{\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}}$$

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$$

H and ***E*** are easier to measure than **B** and **D:**

- In lab, free current J_f are voltage V can be directly measured, so **H** and **E** can be determined from:

$$\nabla \times \mathbf{H} = \mathbf{J}_f \qquad \vec{E} = -\nabla V$$

- It is difficult to direct measure ρ_f and determined D with $\nabla \cdot \vec{D} = \rho_f$

Boundary conditions

$$B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = 0 \qquad \mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}) \qquad \mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

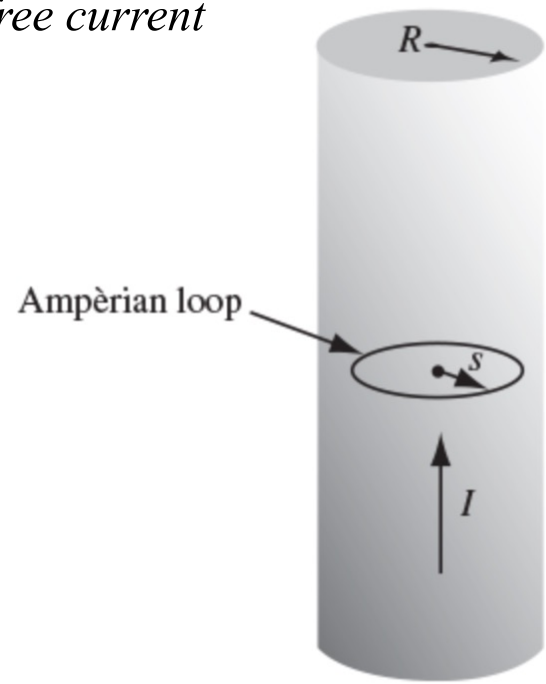
$$\boxed{H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}})}$$

$$\boxed{\mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}}}$$

Copper rod (Example 6.2)

A long copper rod of radius **R** carries a uniformly distributed (free) current **I** . Find **H** inside and outside the rod.

*Do note need to know bound currents to compute **H** from free current*



$$\boxed{\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{enc}}}$$

$$H(2\pi s) = I_{f_{enc}} = I\frac{\pi s^2}{\pi R^2},$$

Inside: $$\mathbf{H} = \frac{I}{2\pi R^2} s\hat{\boldsymbol{\phi}} \quad (s \leq R).$$

Outside: $$\mathbf{H} = \frac{I}{2\pi s}\hat{\boldsymbol{\phi}} \quad (s \geq R).$$

$$\boxed{\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}}$$

Outside: M = 0

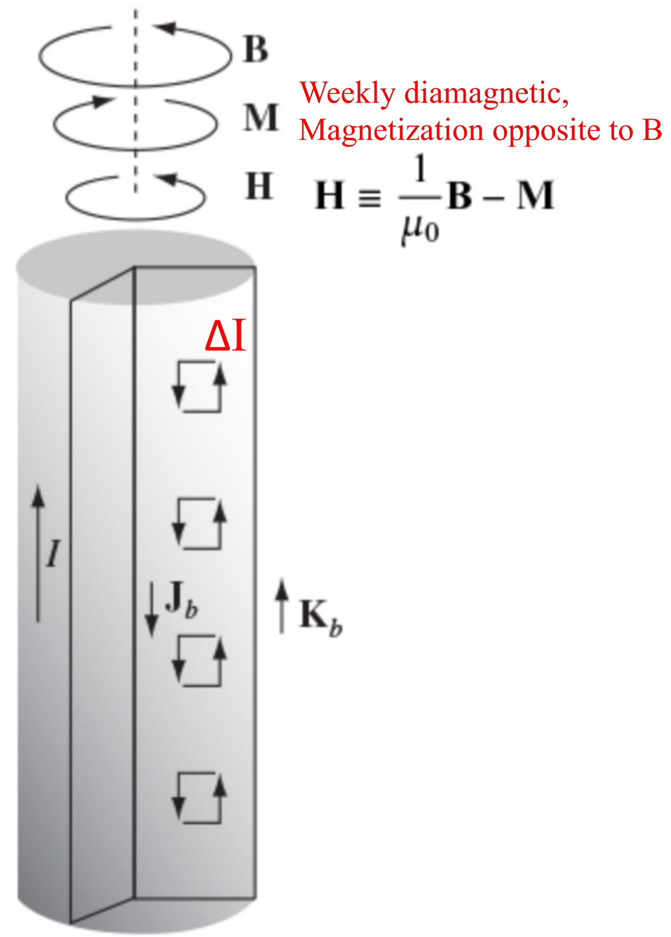
$$\mathbf{B} = \mu_0\mathbf{H} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} \quad (s \geq R),$$ same as for a nonmagnetized wire Ex5.7

Inside: since don't know M, cannot compute B

Copper rod (Example 6.2)

Bound currents

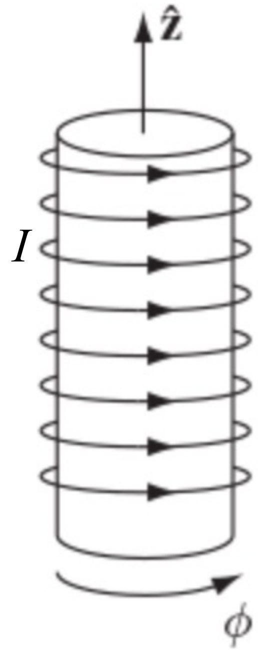
- Copper is weakly diamagnetic, so the dipoles will line up opposite to the field from I.
- This results in a bound volume current $\mathbf{J_b}$ running antiparallel to I, within the wire, and a surface bound current $\mathbf{K_b}$ parallel to I along the surface.



Solenoid with core (Example 6.3)

An infinite solenoid (n turns per unit length, current) is filled with linear material of susceptibility χ_m. Find the magnetic field inside the solenoid.

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{enc}}}$$



$$\mathbf{H} = nI\hat{\mathbf{z}}$$

$$\boxed{\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}}$$

$$\mathbf{B} = \mu_0(1 + \chi_m)nI\hat{\mathbf{z}}.$$

Paramagnetic medium $\chi_m>0$, B enhanced
Diamagnetic medium $\chi_m<0$, B reduced

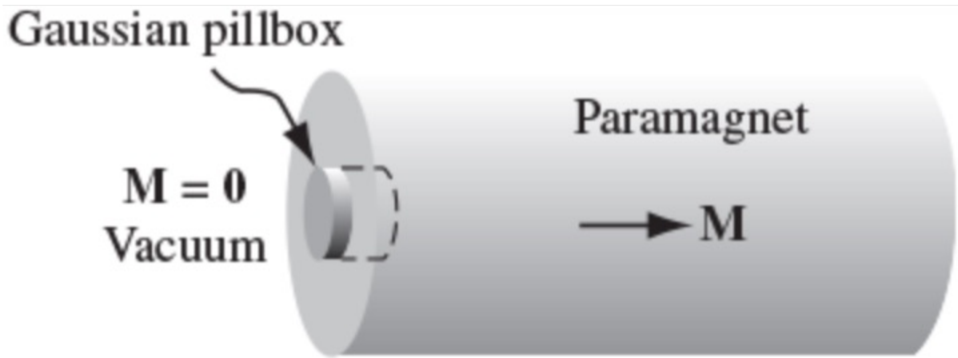
Bound surface current:

$$\boxed{\mathbf{M} = \chi_m\mathbf{H}} \qquad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \chi_m(\mathbf{H} \times \hat{\mathbf{n}}) = \chi_m nI\hat{\boldsymbol{\phi}}$$

Paramagnetic medium $\chi_m>0$, same direction of I
Diamagnetic medium $\chi_m<0$, opposite direction of I

Divergence of $\mathbf{H}$

$\mathbf{J}_b \sim \mathbf{J}_f$ in homogeneous linear material



At surfaces between materials of different susceptibility:

$$\oint \mathbf{M} \cdot d\mathbf{a} \neq 0 \rightarrow \int \nabla \cdot \mathbf{M}\, d\tau \neq 0$$

$$\mathbf{M} = \chi_m \mathbf{H}$$

$$\boxed{\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}}$$

$$\boxed{\nabla \cdot \mathbf{H} \neq 0}$$

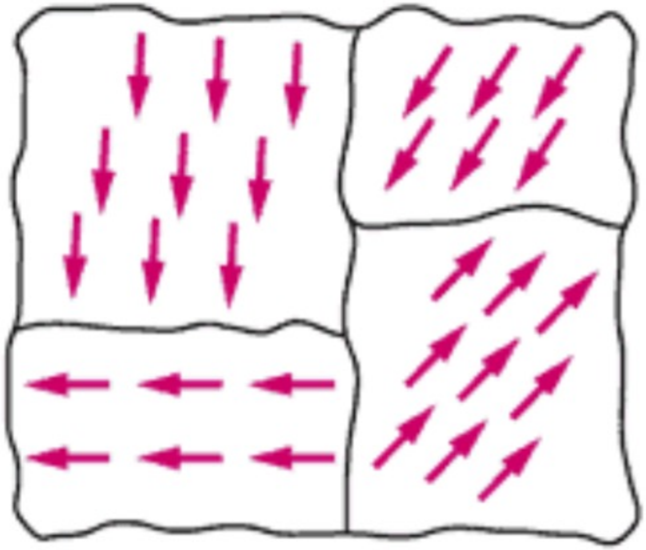
no vector potential for **H,** no "Biot–Savart" law for **H** in terms of $\mathbf{J}_f$,

Volume bound current density in a homogeneous linear material is proportional to the free current density:

$$\boxed{\mathbf{J}_b = \nabla \times \mathbf{M} = \nabla \times (\chi_m \mathbf{H}) = \chi_m \mathbf{J}_f}$$

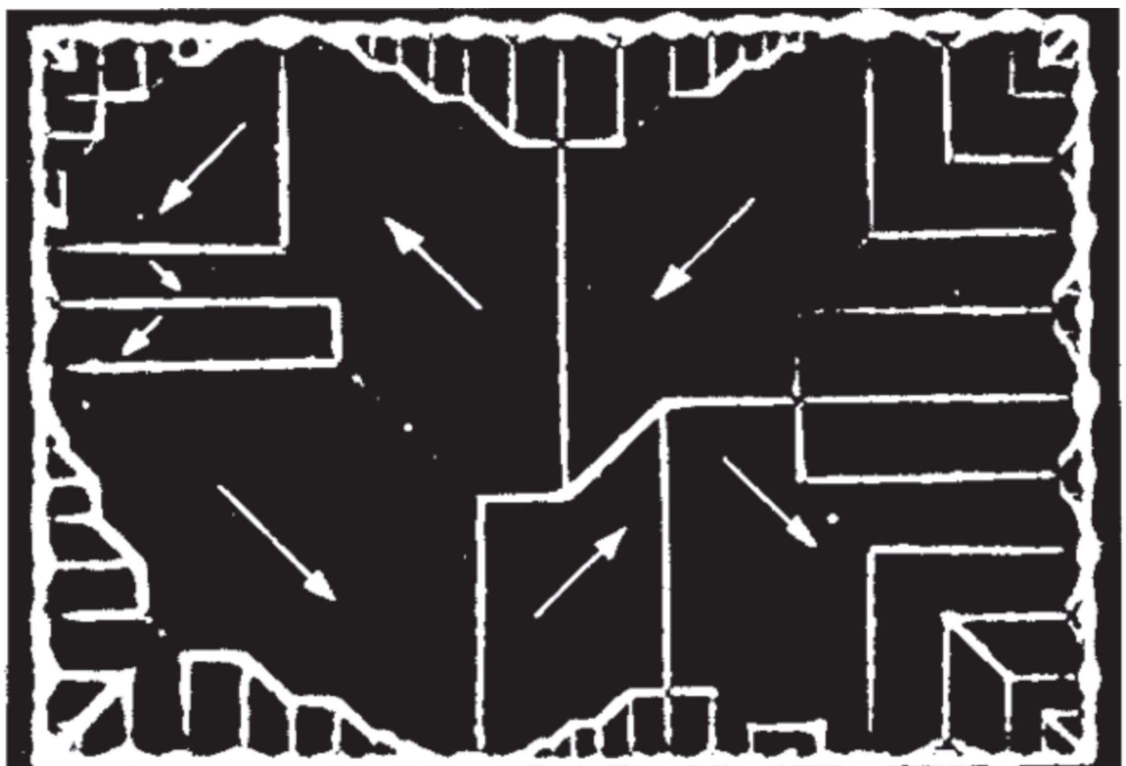
Ferromagnetism

- Like paramagnetism, ferromagnetism involves the magnetic dipoles associated with the spins of unpaired electrons.
- In a ferromagnet, each dipole (from spin of unpaired electron) tends to point in the same direction as its neighbors in small patches, called **domains**. This alignment is due to a quantum effect known as the exchange interaction
- After being magnetized by external field, thermal motion on atoms cannot randomize directions of domain dipoles → retaining magnetization after the field removed
- Ferromagnets are non-linear: require no external fields to sustain the magnetization



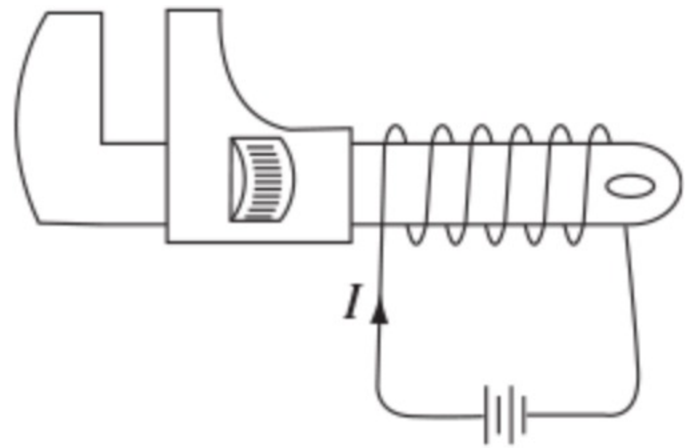
Make Permanent Magnet

- Put iron (ferromagnet) into a strong magnetic field, domains with an **M** in a similar direction of **H** grow. Saturation is reached when only these domains have survived
- This process is not entirely reversible: when the field is switched off, part of domains remain in the magnetized direction.

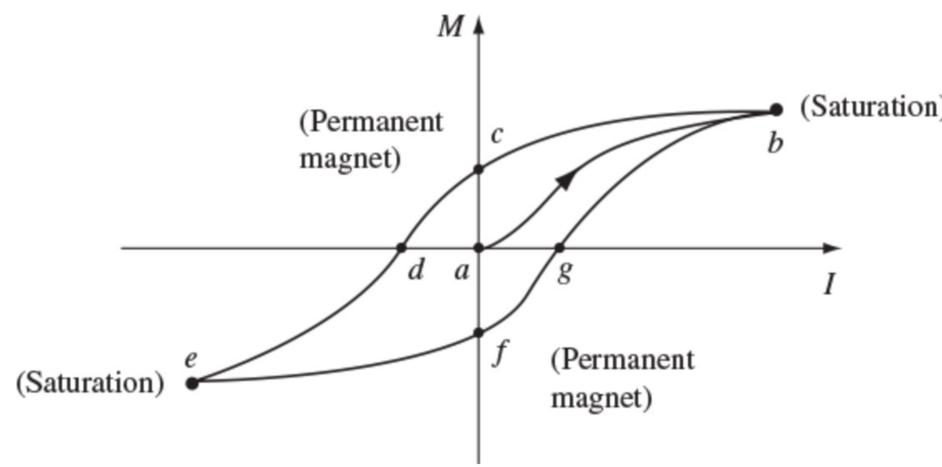


Hysteresis loop

- Unmagnetized iron before apply I (H) (point *a)*
- Increase I (H) to magnetize iron until reach saturation, M>0, points to left (*a*→*b*)
- *Decrease I (H) to 0,* part of domains remains remain magnetized (b→c)
- Increase I (*H*) negatively, M drops down to zero then magnetized in opposite direction (M<0) until reach saturation (**c→d→e**)
- Increase I (*H*) positively, part of domains remain magnetized to right direction (M<0) until point **g** (M=0), keep increasing *I*, magnetized to left direction again until reach saturation (**b**)
- This path is called a **hysteresis loop**. Notice that the magnetization of the wrench depends not only on the applied field ***H*** (that is, on ***I***), but also on its previous magnetic history.



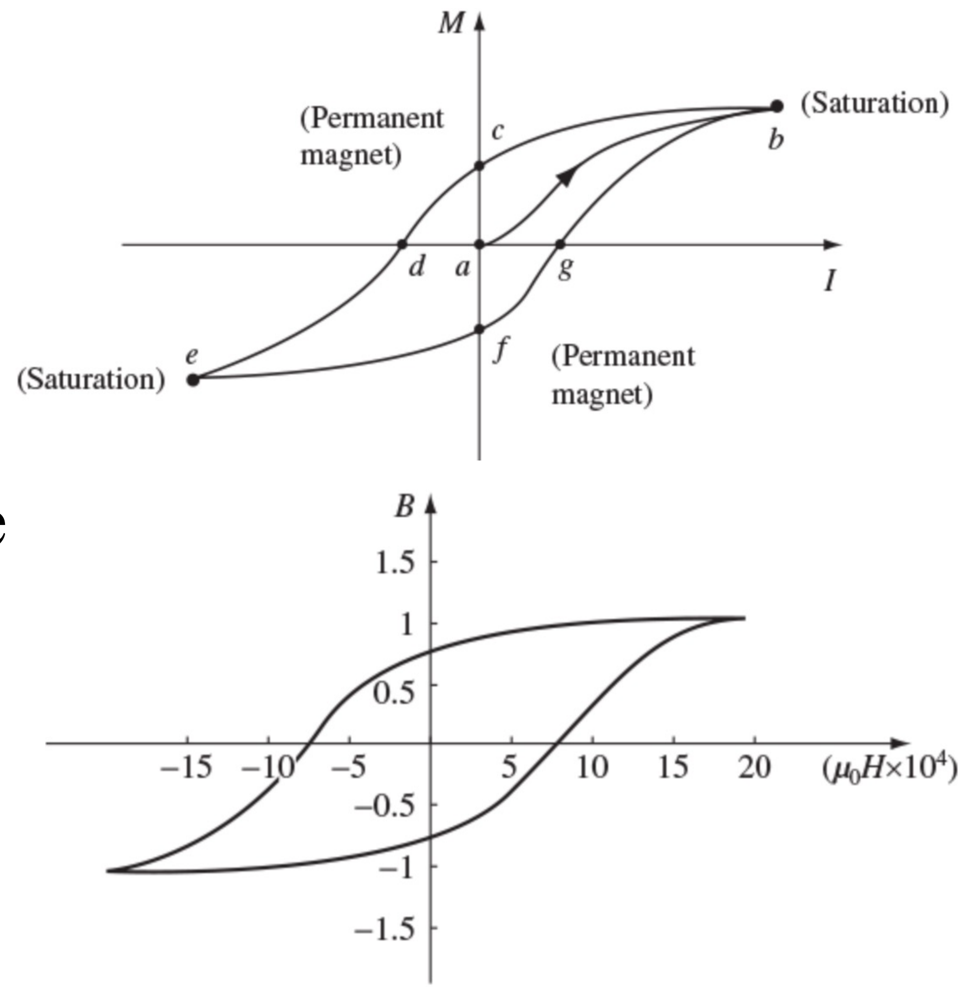
$$\boxed{\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{enc}}} \longrightarrow \mathbf{H} \propto \mathbf{I}$$



Hysteresis loop

Hysteresis loop

- Customary to draw hysteresis loops as plots of B against H:
  - $B = \mu_0(M+H)$, in practice $M>>H$, so use $B \approx \mu_0 M$ for y-axis
  - $H \propto I$, to make units consistent (teslas) use $\mu_0 H$ for x-axis



Hysteresis loop