P112A:Electromagnetic Theory

Electric Fields in Matter
(Chapter 4)

Electrical properties of solid Materials

- Charges in conductors and insulators (or dielectrics)
  - Conductors: Substances with an unlimited supply of electrons which are free to move throughout the material (but not off it)
  - Dielectrics: all charges are a8ached to specific atoms or molecules. All they can do is slosh around a bit within the atom or molecule

Dielectrics

- While the motion is limited, the cumulative effects account for the observed changes in E-fields within materials
- There are two principle ways for external electric fields to distort the charge distribution of an atom in a dielectric material: **stretching** and **rotating**

– While neutral atoms do not typically have a dipole moment, many molecules do.

– However, most materials with dipole molecules are randomly oriented so no net dipole moment in the absence of an external E-field.

Inducing a dipole by stretching

- If a neutral atom is placed in an external E field, the positive nucleus is pushed away, while the surrounding electron cloud is pulled in opposite direction.
- The displaced + and - charges set up an E-field in the direction opposite to the external field. The larger the displacement, the larger the induced E-field. Equilibrium is reached when the force from the external E is equal and opposite to the force from the induced E field.
- The displaced charge forms a dipole, that is described by a dipole moment, $\boldsymbol{p}$. For simple atoms, $\boldsymbol{p}$ is same direction as E, and the magnitude of the effect is proportional to E: $\boldsymbol{p}=\alpha\boldsymbol{E}$, where α is called the atomic polarizability.

Example 4.1: Model of polarized atom

- Assume electrons surrounding an atom uniformly fill a sphere of radius a. In the presence of an external $\boldsymbol{E}$ pointing to the right, the nucleus shifted is shifted right, and electron cloud to the left. The shift is small compared to "a" so the cloud remains spherical. The shift from the center of the electron cloud is called "d". Net force on nuclear charge is 0, so the value of $\boldsymbol{d}$ is determined when $\mathbf{E_e}$ from electron cloud acting on nuclear charge is the same as external $\mathbf{E}$.

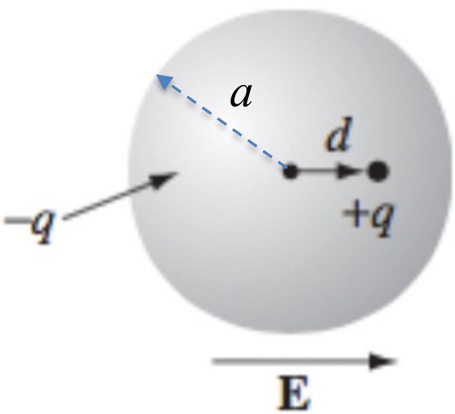
Use Gauss's Law to compute $\mathbf{E_e}$ from uniform density of electric charge as a function of radius d:

$$Q_{enc} = \frac{4}{3}\pi d^3 \rho \qquad |E_e| = \frac{1}{\varepsilon_0} \bullet \frac{Q_{enc}}{4\pi d^2} \qquad \rho = -q/(4/3\pi a^3)$$

$$\left|\vec{E}_e\right| = \frac{\rho d}{3\varepsilon_0} = \frac{qd}{4\pi\varepsilon_0 a^3}; \quad p = qd$$

$$E = \left|\vec{E}_e\right| = \frac{qd}{4\pi\varepsilon_0 a^3} = \frac{p}{4\pi\varepsilon_0 a^3}$$

$$p = \left(4\pi\varepsilon_0 a^3\right)E = \left[3\varepsilon_0(volume)\right]E$$



p=αE, atomic polarizability α = 3ε₀•Volume

Model of polarized atom

- Model predicts magnitude of α is proportional to Volume of atom and ε_0. Lets test it.
- Consider hydrogen: $r \sim 0.5\times10^{-10}$ m
- Vol~ $4(0.5\times10^{-10}\text{ m})^3 = 0.5\times10^{-30}\text{ m}^3$.
- $\alpha/\varepsilon_0 = 3*\text{Vol} = 1.5\times10^{-30}\text{ m}^3$ (predicted)
- $\alpha/\varepsilon_0 = 4\pi(0.67)\times10^{-30} = 8\times10^{-30}\text{ m}^3$ (measured)

Not too bad for 7D, 7E physics model

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.67	0.396	24.1	1.64	43.4	59.4

TABLE 4.1 Atomic Polarizabilities ($\alpha/4\pi\epsilon_0$, in units of 10^{-30} m^3). *Data from: Handbook of Chemistry and Physics*, 91st ed. (Boca Raton: CRC Press, 2010).

Values, for the most part, increase with volume of atom

Is model reasonable? How much displacement are we talking about?

- Suppose hydrogen is placed between 2 metal plates, $1mm=10^{-3}m$ apart, connected to a 1 kV power supply. What is d, the separation between + nucleus and center of electron cloud?

$|\mathbf{E}_{ext}|=V/d= 10^3/10^{-3} = 10^6$ V/m

$|\mathbf{p}|=dq=\alpha|E_{ext}|$

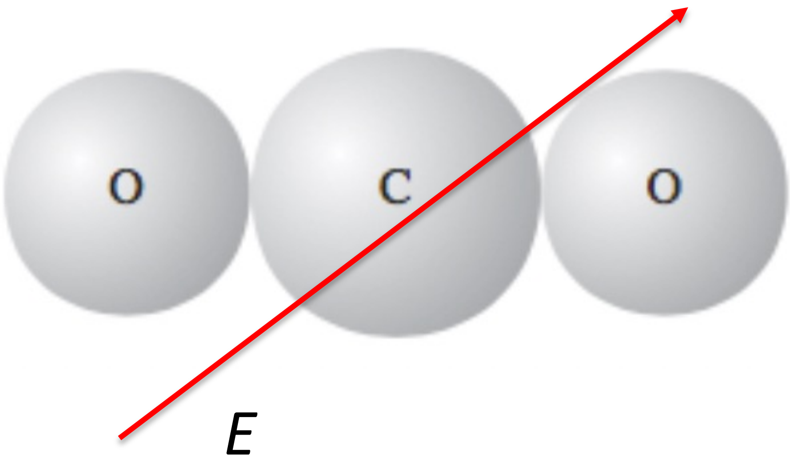
$d=\alpha|E_{ext}|/q = 4\pi\varepsilon_0(0.667x10^{-30})[10^6]/1.6x10^{-19}C = 12(2/3x10^{-30})(9x10^{-12})$
$[10^6]/2x10^{-19}= 36x10^{-17}$ m

d =0.4x10^-15 m, which is 10^{-5} of the atomic radius $\sim 10^{-10}$ m, so displacements are really tiny for most lab situations. Seems reasonable to assume the electron cloud does not change much → our model is reasonable

Polarizability of Molecules

- For molecules, frequently, they polarize more readily in one direction than another.
- For example, CO_2, polarizability is larger along the axis than perpendicular to it :

$$\mathbf{p} = \alpha_{\perp}\mathbf{E}_{\perp} + \alpha_{\parallel}\mathbf{E}_{\parallel}.$$



The direction of ***p*** *may not be the same direction as the External E field*

Asymmetric Molecules

- CO_2 is linear, and relatively symmetric. But most molecules are not. A more complicated geometry requires:

$$\left.\begin{aligned} p_x &= \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \\ p_y &= \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z \\ p_z &= \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z \end{aligned}\right\}$$

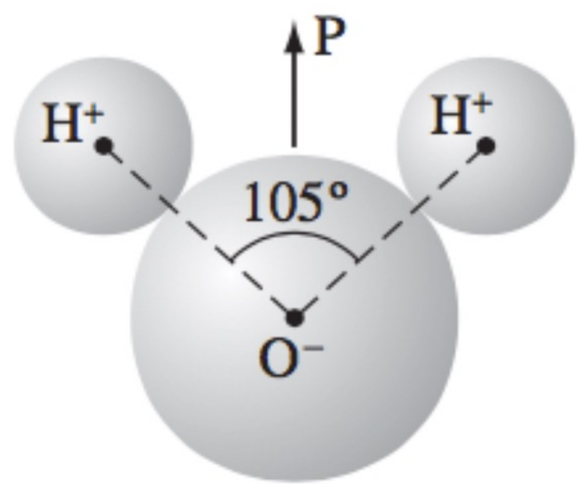
*Can be written in terms of a polarizability matrix, or Tensor. Here **p** is induced by external **E***

*One needs to measure 9 numbers, but the numbers vary with choice of axis. If it is possible to predict direction of "**principle axis**", then only need to measure 3 numbers because all off-diagonal terms vanish.*

Polar molecules

- A neutral atom has no dipole moment to start with, *p* was induced by the applied E field. However, some molecules have a built-in, permanent dipole moment.

Water molecule has $|p|=6\text{x}10^{-30}$ C*m, which is atypically large. In a liquid or gas, the orientations are random so the net dipole moment is 0. Not so obvious that solid should have no net dipole moment.



Torque on permanent dipole moment

- In a uniform field, the force on the positive charge, $F_+=qE$, cancels the force on the negative side. However, there will be a torque, **N**. (**τ** in 7D)

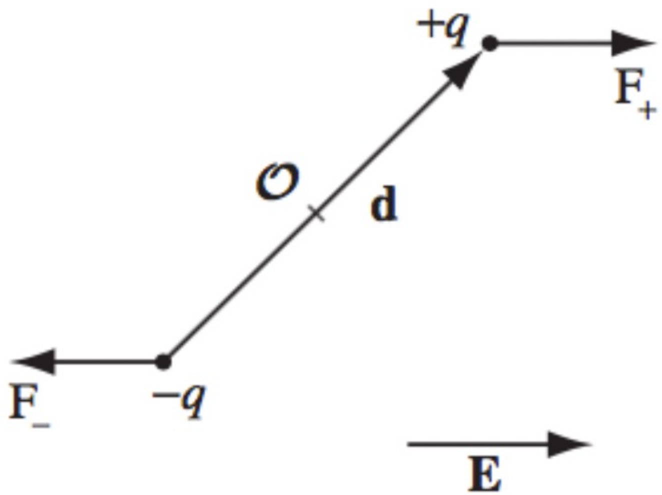
In uniform external E field, net force = 0

Torque:

$$\vec{N} = (\vec{r}_+ \times \vec{F}_+) + (\vec{r}_- \times \vec{F}_-)$$

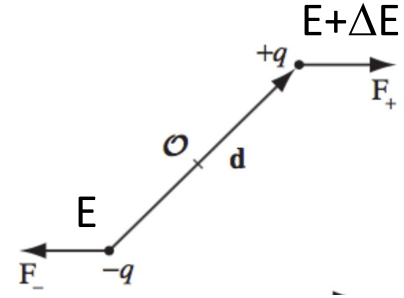
$$\vec{N} = \left(\left[\frac{\vec{d}}{2}\right] \times q\vec{E}\right) + \left(\left[\frac{-\vec{d}}{2}\right] \times \left[-q\vec{E}\right]\right)$$

$$\vec{N} = q\vec{d} \times \vec{E} = \vec{p} \times \vec{E}$$



Non-uniform external E Field

- In non-uniform external $\boldsymbol{E}$, the forces do not necessarily cancel. Of course, $\boldsymbol{E}$ must change rather abruptly over the distance of a molecule for this to be significant. So this is usually not considered in dielectrics.
- Nevertheless, the magnitude of the force is worth thinking about



$$\vec{F} = \vec{F}_+ - \vec{F}_- = q(\vec{E}_+ - \vec{E}_-) = q\Delta\vec{E} = q\left(\Delta E_x\hat{x} + \Delta E_y\hat{y} + \Delta E_z\hat{z}\right)$$

$$(1.34) \quad dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz$$

$$(1.35) \quad dT = \left[\left(\frac{\partial T}{\partial x}\right)\hat{x} + \left(\frac{\partial T}{\partial y}\right)\hat{y} + \left(\frac{\partial T}{\partial z}\right)\hat{z}\right] \bullet \left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$$

$$dT = \nabla T \bullet d\vec{\ell} \Rightarrow \Delta T = \nabla T \bullet \Delta\vec{\ell}$$

Change from differential distance to finite difference

$$\Delta E_x = (\nabla E_x) \bullet \vec{d} = \vec{d} \bullet (\nabla E_x)$$

Dot products commute

$$\Delta E_y = \vec{d} \bullet (\nabla E_y); \quad \Delta E_z = \vec{d} \bullet (\nabla E_z)$$

$$\vec{F} = q\left(\Delta E_x\hat{x} + \Delta E_y\hat{y} + \Delta E_z\hat{z}\right) = q\left(\left[\vec{d} \bullet (\nabla E_x)\right]\hat{x} + \left[\vec{d} \bullet (\nabla E_y)\right]\hat{y} + \left[\vec{d} \bullet (\nabla E_z)\right]\hat{z}\right)$$

$$\vec{F} = q(\vec{d} \bullet \nabla)\vec{E}$$

Gives same thing as previous eqn, check with Cartesian coordinates

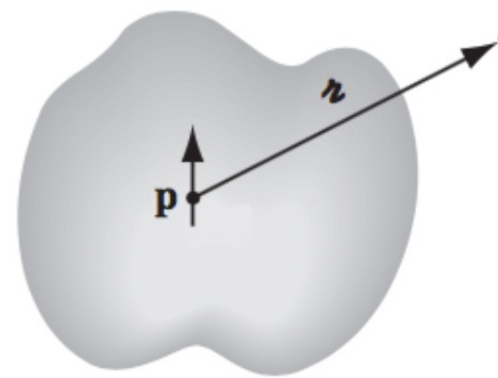
$$\boxed{\vec{F} = (\vec{p} \bullet \nabla)\vec{E}}$$

Dipole moment per volume, P

- To summarize. If dielectric material placed in external E field.
  - Neutral atoms: Induce dipoles, pointing in same direction as **E** (stretching)
  - Polar molecules: experience a torque, tending to line up dipole moment with **E** (rotating)
- So a lot of dipoles in material point along direction of field, so material becomes polarized.
- Dipole moment per unit volume, **P**, is called the polarization.
- Lets no longer worry about what causes **P** (rotating, stretching or some combination), but lets examine the **E** created by a chunk of polarized material.
- Eventually we will want to combine the **E** created by **P** AND the original **E** that created **P** in the first place.

Field of a Polarized Object

- Suppose we have a piece of polarized material with polarization P. What is the field produced by this object?
- For a single dipole chunk, d*p*, within a material characterized by P, the potential is

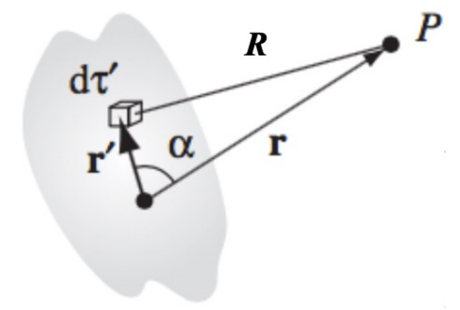


$$dV(r) = \frac{1}{4\pi\varepsilon_0}\frac{(d\vec{p})\bullet\hat{R}}{R^2}$$

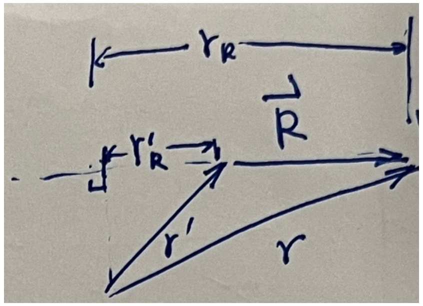
$$where \quad d\vec{p} = \vec{P}d\tau$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{(d\vec{p})\bullet\hat{R}}{R^2} = \frac{1}{4\pi\varepsilon_0}\int\frac{(\vec{P}(\vec{r}')\bullet\hat{R}}{R^2}d\tau'$$

Field of a Polarized Object



$$\boxed{\begin{aligned}\nabla'(1/R) &= \frac{\partial\left(\frac{1}{R}\right)}{\partial r_R'}\hat{R} = \frac{\partial\left(\frac{1}{|r-r'|}\right)}{\partial r_R'}\hat{R} = \frac{\partial\left(\frac{1}{r_R-r_R'}\right)}{\partial r_R'}\hat{R} \\ &= -\frac{1}{(r_R-r_R')^2}\frac{\partial(r_R-r_R')}{\partial r_R'}\hat{R} = \frac{\hat{R}}{R^2}\end{aligned}}$$



$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{P}(\vec{r}')\bullet\hat{R}}{R^2}d\tau' = \frac{1}{4\pi\varepsilon_0}\int \left(\vec{P}(\vec{r}')\bullet\nabla'\left(\frac{1}{R}\right)d\tau'\right.$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\int_{Vol}\nabla'\bullet\left(\frac{\vec{P}}{R}\right)d\tau' - \int_{Vol}\left(\frac{1}{R}\right)\left(\nabla'\bullet\vec{P}\right)d\tau'\right]$$

Product Rule 5

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

$$= \frac{1}{4\pi\varepsilon_0}\left[\int_{surf}\left(\frac{\vec{P}}{R}\right)\bullet da' - \int_{Vol}\left(\frac{1}{R}\right)\left(\nabla'\bullet\vec{P}\right)d\tau'\right]$$

Divergence Thm

Surface and Volume Bound Charges

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left[\int_{surf}\left(\frac{\vec{P}}{R}\right)\bullet da' - \int_{Vol}\left(\frac{1}{R}\right)\left(\nabla'\bullet\vec{P}\right)d\tau'\right]$$

The first term looks like the potential from the **surface charge density**

$$\sigma_b \equiv \vec{P}\bullet\hat{n}$$

The second term looks like the potential of a **volume charge density**

$$\rho_b \equiv -\nabla\bullet\vec{P}$$

Instead of integrating contributions from all **dipoles** in dielectric, fields can be computed from these **bound charges**:

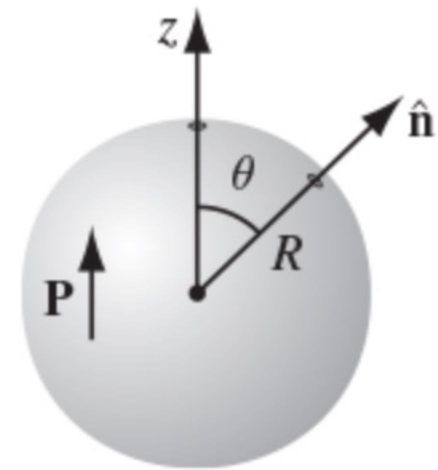
$$\boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\left[\int_{surf}\left(\frac{\sigma_b}{R}\right)da' + \int_{Vol}\left(\frac{\rho_b}{R}\right)d\tau'\right]}$$

Example 4.2

Find E field produced by uniformly polarized, **P**, dielectric sphere of radius R. Choose z-axis to correspond to direction of polarization. Normal to spherical surface is radial direction, **r**.

$$\sigma_b \equiv \vec{P} \bullet \hat{n} = P\hat{z} \bullet \hat{r} = P\cos\theta$$

$$\rho_b \equiv -\nabla \bullet \vec{P} = -\nabla \bullet P\hat{z} = 0$$



Bound charge only on surface, solved in Example 3.9 with variable separation method:

$$\sigma_0(\theta) = k\cos\theta = kP_1(\cos\theta),$$

$$V(r,\theta) = \frac{k}{3\epsilon_0} r\cos\theta \quad (r \le R),$$

$$V(r,\theta) = \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta \quad (r \ge R).$$

$$\xrightarrow{k=P}$$

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r\cos\theta & \text{for } r \le R, \\ \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta & \text{for } r \ge R. \end{cases}$$

Example 4.2

$$V(r,\theta)=\frac{r\,\mathrm{P}\cos\theta}{3\varepsilon_0}\qquad r<R$$

$$V_{\text{dip}}(r,\theta)=\frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2}=\frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

$$V(r,\theta)=\frac{\mathrm{P}\,R^3\cos\theta}{3\varepsilon_0 r^2}\qquad r>R$$

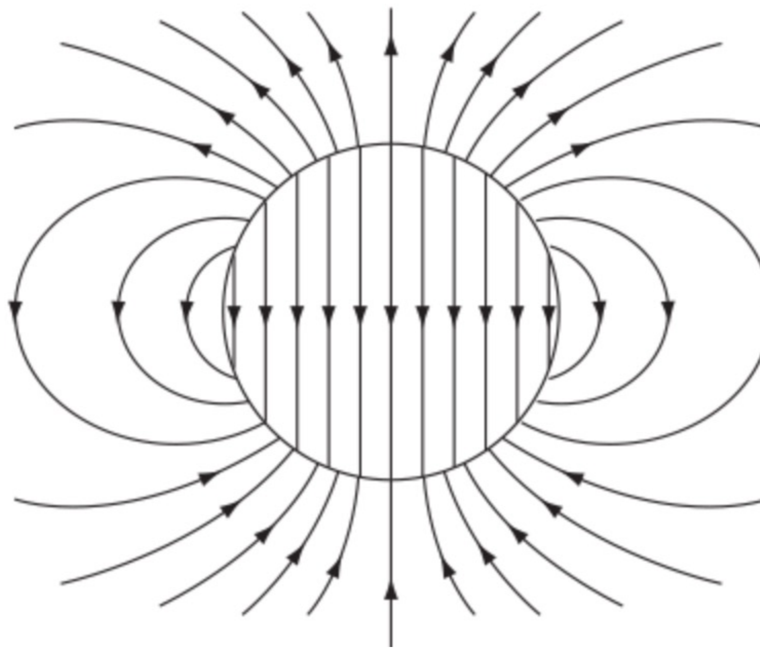
Uniformly polarized sphere produces the same potential outside as a ideal dipole, p=P(volume of sphere)*

rcos(θ) =z, when r<=R:

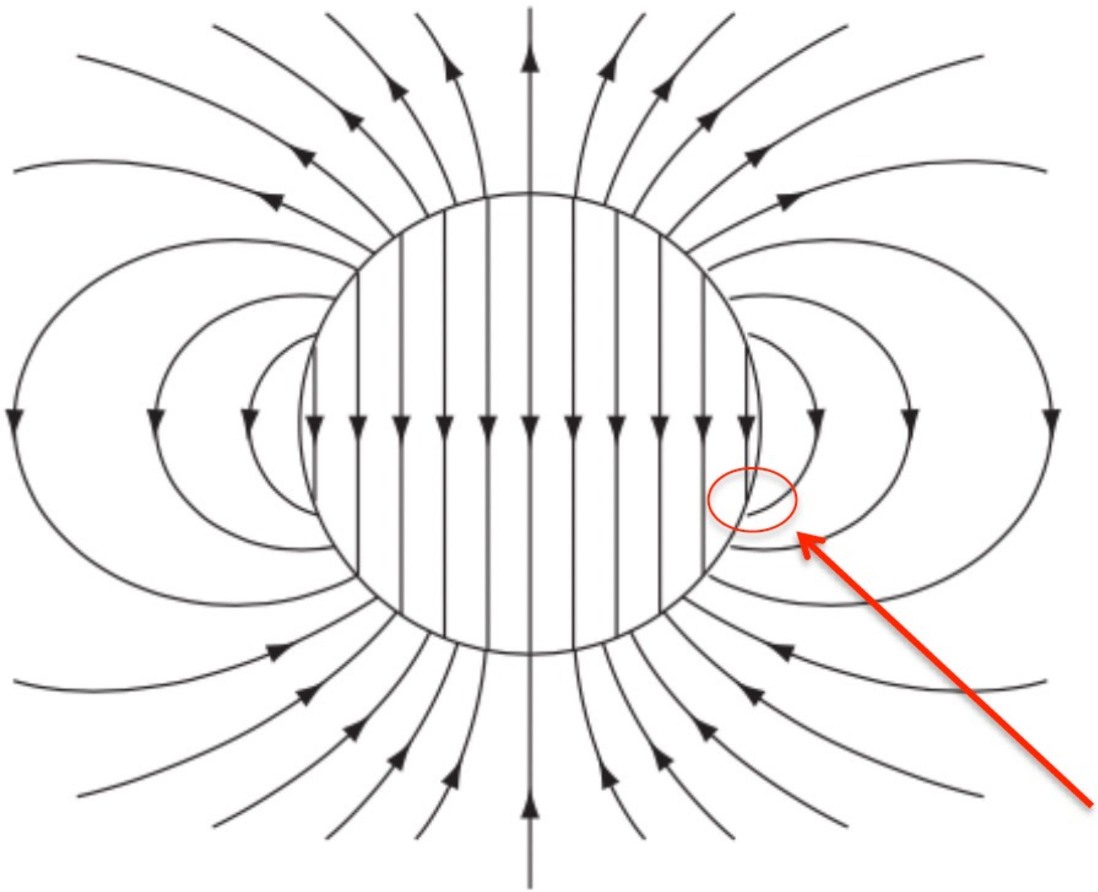
$$V(z)=\frac{r\,\mathrm{P}\cos\theta}{3\varepsilon_0}=\frac{\mathrm{P}\,z}{3\varepsilon_0}$$

$$\vec{E}=-\nabla V(z)=-\frac{\partial}{\partial z}\left(\frac{\mathrm{P}\,z}{3\varepsilon_0}\right)\hat{z}=-\frac{\mathrm{P}}{3\varepsilon_0}\hat{z}$$

E is uniform within a dielectric sphere of constant polarization



Graphic of E field



Field lines are not continuous because of the bound surface charge created by the dipoles (or polarized material)

We learned that there is a discontinuity of the E field

$\mathbf{E}_{above}-\mathbf{E}_{below}=(\sigma/\varepsilon_0)\mathbf{n}$

Since not a conductor, you can have E tangent to spherical surface. But it does have to be continuous across boundary.