Homework 8

Physics 112A

Problem 5.12 Use the result of Ex. 5.6 to calculate the magnetic field at the center of a uniformly charged spherical shell, of radius R and total charge Q, spinning at constant angular velocity ω

$$B(z) = \frac{\mu_0 I}{4\pi}\frac{cos\theta}{r^2}2\pi R = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{\frac{3}{2}}}$$

θ is from the center of the sphere instead of from the ring, so $cos\theta \to sin\theta$.

$$\begin{aligned}
R &\to Rsin\theta \\
dI &= KRd\theta \\
&= \sigma v R d\theta \\
&= \frac{Q}{4\pi R^2}Rsin\theta\omega R d\theta \\
&= \frac{Q\omega}{4\pi}sin\theta d\theta \\
dB &= \frac{2\pi\mu_0}{4\pi}\frac{Rsin^2\theta}{R^2}dI \\
&= \frac{\mu_0}{2R}sin^2\theta\frac{Q\omega}{4\pi}sin\theta d\theta \\
&= \frac{Q\omega\mu_0}{8\pi R}\int_0^{\pi} sin^3\theta d\theta \\
&= \frac{Q\omega\mu_0}{8\pi R}[\frac{1}{3}cos^3\theta - cos\theta]_0^{\pi}] \\
&= \boxed{\frac{Q\omega\mu_0}{6\pi R}}
\end{aligned}$$

Problem 5.13 Suppose you have two infinite straight-line charges λ, a distance d apart, moving along at a constant speed v. How great would v have to be in order for the magnetic attraction to balance the electrical repulsion ? Work out the actual number. Is this a reasonable sort of speed ?

$$E = \frac{\lambda L}{2\pi\epsilon_0 d^2}$$
$$B = \frac{\mu_0 I L}{4\pi d^2}$$
$$F_C = -F_L$$
$$\frac{1}{4\pi\epsilon_0}\frac{(\lambda L)^2}{d^2} = -\lambda L(\frac{\lambda L}{2\pi\epsilon_0 d^2} + v\frac{\mu_0 I L}{4\pi d^2})$$
$$\frac{1}{4\epsilon_0} = -\frac{1}{2\epsilon_0} - \frac{\mu_0 v^2}{4}$$
$$1 = -2 - \mu_0\epsilon_0 v^2$$
$$v^2 = \frac{1}{\mu_0\epsilon_0}$$
$$v = \boxed{\frac{1}{\sqrt{\mu_0\epsilon_0}}}$$

The speed of light is also $\frac{1}{\sqrt{\mu_0\epsilon_0}}$, so this is not possible.

Problem 5.14 A steady current I flows down a long cylindrical wire of radius a. Find the magnetic field, both inside and outside the wire if:

(a) the current is uniformly distributed over the surface of the wire.

When $s < a$:

$$\int B \cdot dl = \mu_0 I$$
$$B 2\pi s = 0$$
$$B = \boxed{0}$$

When $s > a$:

$$\int B \cdot dl = \mu_0 I$$
$$B 2\pi s = \mu_0 I$$
$$B = \boxed{\frac{\mu_0 I}{2\pi s}\hat{\phi}}$$

(b) the current is distributed in such a way that J is proportional to s, the distance from the axis.

$$
\begin{aligned}
J(s) &= ks \\
I &= \int_0^a ks\pi s ds \\
&= \frac{1}{3}k\pi[s^3]_0^a \\
&= \frac{1}{3}k\pi a^3 \\
k &= \frac{3I}{a^3\pi}
\end{aligned}
$$

When $s < a$:

$$
\begin{aligned}
I_{\text{enc}} &= \int_0^s \frac{3I}{a^3\pi} s\pi s ds \\
&= \frac{3I}{a^3}\frac{1}{3}[s^3]_0^s \\
&= I\frac{s^3}{a^3} \\
\int B \cdot dl &= \mu_0 I \\
B2\pi s &= \mu_0 I \frac{s^3}{a^3} \\
B &= \boxed{\frac{\mu_0 I}{2\pi}\frac{s^2}{a^3}\hat{\phi}}
\end{aligned}
$$

When $s > a$:

$$
\begin{aligned}
\int B \cdot dl &= \mu_0 I_{\text{enc}} \\
B2\pi s &= \mu_0 I \\
B &= \boxed{\frac{\mu_0 I}{2\pi s}\hat{\phi}}
\end{aligned}
$$

Problem 5.25 Find the magnetic vector potential of a finite segment of straight wire carrying a current I. [Put the wire on the z-axis, from z_1 to z_2, and use Eq. 5.66.] Check that your answer is consistent with Eq. 5.37.

Eq. 5.66:

$$
A = \frac{\mu_0 I}{4\pi}\int \frac{dl}{r}
$$

Using cylindrical coordinates:

$$
\begin{aligned}
r^2 &= z^2 + s^2 \\
A &= \frac{\mu_0 I}{4\pi} \int_{z_1}^{z_2} \frac{dz}{\sqrt{z^2+s^2}} \hat{z} \\
&= \frac{\mu_0 I}{4\pi} \int_{z_1}^{z_2} [ln(z + \sqrt{s^2+z^2})]_{z_1}^{z_2} \\
&= \boxed{\frac{\mu_0 I}{4\pi} ln[\frac{z_2 + \sqrt{s^2+z_2^2}}{z_1+\sqrt{s^2+z_1^2}}]\hat{z}}
\end{aligned}
$$

Checking consistency:

$$
\begin{aligned}
B &= \nabla \times A \\
&= -\frac{\partial}{\partial s} A_z \hat{\phi} \\
&= -\frac{\mu_0 I}{4\pi} \frac{\partial}{\partial s}[ln[\frac{z_2+\sqrt{s^2+z_2^2}}{z_1+\sqrt{s^2+z_1^2}}]] \\
&= -\frac{\mu_0 I}{4\pi}[\frac{s}{z_2+\sqrt{s^2+z_2^2}} \frac{1}{\sqrt{s^2+z_2^2}} - \frac{s}{z_2+\sqrt{s^2+z_1^2}} \frac{1}{\sqrt{s^2+z_1^2}}] \\
&= -\frac{\mu_0 I s}{4\pi}[\frac{z_2-\sqrt{s^2+z_2^2}}{z_2^2-(s^2+z_2^2)} \frac{1}{\sqrt{s^2+z_2^2}} - \frac{z_1-\sqrt{s^2+z_1^2}}{z_1^2-(s^2+z_1^2)} \frac{1}{\sqrt{s^2+z_1^2}}] \\
&= -\frac{\mu_0 I}{4\pi s}[\frac{z_2-\sqrt{s^2+z_2^2}}{\sqrt{s^2+z_2^2}} - \frac{z_1-\sqrt{s^2+z_1^2}}{\sqrt{s^2+z_1^2}}] \\
&= -\frac{\mu_0 I}{4\pi s}[\frac{z_2}{\sqrt{s^2+z_2^2}} - \frac{z_1}{\sqrt{s^2+z_1^2}}] \\
&= \boxed{-\frac{\mu_0 I}{4\pi s}[sin\theta_2 - sin\theta_1]\hat{\phi}}
\end{aligned}
$$

Problem 5.26

(a) What current density would produce the vector potential $A = k\hat{\phi}$ (where k is a constant), in cylindrical coordinates ?

$$
\begin{aligned}
B &= \nabla \times A \\
&= \frac{1}{s}\frac{\partial}{\partial s}[sA_\phi]\hat{z} \\
&= \frac{k}{s}\hat{z} \\
J &= \frac{1}{\mu_0}\nabla \times B \\
&= -\frac{1}{\mu_0}\frac{\partial}{\partial s}B_z\hat{\phi} \\
&= \boxed{\frac{k}{\mu_0 s^2}\hat{\phi}}
\end{aligned}
$$

(b) Consider an azimuthally symmetric magnetic field; it points in the z direction, and its magnitude is a function only of s. Check that

$$
A = A(s)\hat{\phi} \text{ where } A(s) = \frac{1}{s}\int_0^s B(s')s'ds'
$$

by calculating its divergence and curl. (This generalizes Ex 5.12.)

$$
\begin{aligned}
\nabla \cdot A &= \nabla \cdot \frac{1}{s}\int_0^s B(s')s'ds'\hat{\phi} \\
&= \frac{1}{s}\frac{\partial}{\partial \phi}[\frac{1}{s}\int_0^s B(s')s'ds'] \\
&= \boxed{0} \\
\nabla \times A &= \nabla \times \frac{1}{s}\int_0^s B(s')s'ds'\hat{\phi} \\
&= \frac{1}{s}\frac{\partial}{\partial s}[s\frac{1}{s}\int_0^s B(s')s'ds']\hat{z} \\
&= \frac{1}{s}B(s)s \\
&= \boxed{B(s)}
\end{aligned}
$$

Problem 5.27 If B is uniform, show that $A(r) = -\frac{1}{2}(r \times B)$ works. That is, check that $\nabla \cdot A = 0$ and $\nabla \times A = B$. Is this result uique, or are there other functions with the same divergence and curl ?

$$
\begin{aligned}
\nabla \cdot A &= -\frac{1}{2}\nabla \times (r \times B) \\
&= -\frac{1}{2}[B \cdot (\nabla \times r) - r \cdot (\nabla \times B)] \\
&= \boxed{0} \\
\nabla \times A &= -\frac{1}{2}\nabla \times (r \times B) \\
&= -\frac{1}{2}[(B \cdot \nabla)r - (r \cdot \nabla)B + r(\nabla \cdot B) - B(\nabla \cdot r)] \\
&= -\frac{1}{2}[(B_x \frac{\partial r_x}{\partial x} + B_y \frac{\partial r_y}{\partial y} + B_z \frac{\partial r_z}{\partial z}) - B(3)] \\
&= -\frac{1}{2}[B - 3B] \\
&= \boxed{B}
\end{aligned}
$$

From Problem 5.26.b:

$$
\begin{aligned}
A &= \frac{B}{s}\int_0^s s' ds' \hat{\phi} \\
&= \boxed{\frac{1}{2}Bs\hat{\phi}} \\
\nabla \cdot A &= \frac{1}{s}\frac{\partial}{\partial \phi}\frac{1}{2}Bs \\
&= 0 \\
\nabla \times A &= \frac{1}{s}\frac{\partial}{\partial s}[s\frac{1}{2}Bs] \\
&= B
\end{aligned}
$$

Problem 5.28

(a) By whatever means you can think of (short of looking it up), find the vector potential a distance s from an infinite straight wire carrying a current I. Check that $\nabla \cdot A = 0$ and $\nabla \times A = B$.

$$
\begin{aligned}
\int B \cdot dl &= \mu_0 I \\
B &= \frac{\mu_0 I}{2\pi s}\hat{\phi} \\
\nabla \times A(s)\hat{z} &= \frac{\mu_0 I}{2\pi s}\hat{\phi} \\
-\frac{\partial}{\partial s}[A(s)] &= \frac{\mu_0 I}{2\pi s}\hat{\phi} \\
A(s) &= -\int_a^s \frac{\mu_0 I}{2\pi s} ds \\
&= \boxed{-\frac{\mu_0 I}{2\pi} ln[\frac{s}{a}]\hat{z}} \\
\nabla \cdot A &= \frac{\partial}{\partial z}[-\frac{\mu_0 I}{2\pi} ln(\frac{s}{a})] \\
&= 0 \\
\nabla \times A &= -\frac{\partial}{\partial s}[-\frac{\mu_0 I}{2\pi} ln(\frac{s}{a})] \\
&= \frac{\mu_0 I}{2\pi s}\hat{\phi} \\
&= B
\end{aligned}
$$

(b) Find the magnetic potential inside the wire, if it has radius R and the current is uniformly distributed.

$$
\begin{aligned}
\int B \cdot dl &= \mu_0 I \\
B 2\pi s &= \mu_0 J \pi s^2 \\
B &= \frac{1}{2}\mu_0 J s\hat{\phi} \\
-\frac{\partial}{\partial s}[A(s)] &= \frac{1}{2}\mu_0 J s\hat{\phi} \\
A(s) &= -\frac{1}{2}\mu_0 \frac{I}{\pi R^2}\int_a^s s ds \\
&= \boxed{-\frac{\mu_0 I}{4\pi R^2}(s^2 - a^2)\hat{z}}
\end{aligned}
$$