Lecture 7: Irreducible Representations

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Direct Sums

- Direct sum of two vector spaces V_1, V_2:
$$V_1 \oplus V_2 = (v_1, v_2) | v_1 \in V_1, v_2 \in V_2$$
- Example: $\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$
- Often called "direct product", but the direct (Cartesian) product ($\times$) of sets does not imply linear structure. Do not confuse with outer (tensor) product!
- (v_1, v_2) is also written $v_1 \oplus v_2$ and represented by "vector of vectors" $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$
- Direct sum of linear operators $\mathbf{P}_1 : V_1 \to V_1$ and $\mathbf{P}_2 : V_2 \to V_2$: $\mathbf{P}_1 \oplus \mathbf{P}_2 : V_1 \oplus V_2 \to V_1 \oplus V_2$ such that
$$\mathbf{P}_1 \oplus \mathbf{P}_2 (v_1, v_2) = (\mathbf{P}_1 v_1, \mathbf{P}_2 v_2).$$
- Matrix notation:
$$\mathbf{P}_1 \oplus \mathbf{P}_2 = \begin{pmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{pmatrix}$$

Irreducible Representations

- Consider group $\mathcal{G} = \{a, b, \ldots\}$
- Direct sum of two representations $\Gamma_1 = \{\mathbf{P}_1(a), \mathbf{P}_1(b), \ldots\}$, $\Gamma_2 = \{\mathbf{P}_2(a), \mathbf{P}_2(b), \ldots\}$ of $\mathcal{G}$:

$$\Gamma_1 \oplus \Gamma_2 = \{\mathbf{P}_1(a) \oplus \mathbf{P}_2(a), \mathbf{P}_1(b) \oplus \mathbf{P}_2(b), \ldots\}$$

- A representation that can be written as ("decomposed/subduced into") a direct sum of two or more nontrivial representations of $\mathcal{G}$ is reducible.
- An irreducible representation (IRREP) is not reducible.

Properties of IRREPs

- Number of IRREPs = number of classes. Finite for finite groups!
- Every reducible representation Γ can be decomposed into IRREPs:

$$\Gamma = \alpha \oplus \beta \oplus \cdots$$

  The same IRREP may appear multiple times (“inequivalent” IRREPs)
- Similarly, the underlying representation space is direct sum of IRREP spaces invariant under a given IRREP:

$$S_\Gamma = S_\alpha \oplus S_\beta \oplus \cdots$$

- Dimension $\dim(S_\alpha) = n_\alpha d_\alpha$, with “IRREP dimension” d_α

$$\dim(S_\Gamma) = n_\alpha d_\alpha + n_\alpha d_\beta + \cdots$$

- Multiplicity of an IRREP α in REP Γ:

$$n_\alpha = \frac{1}{g} \sum_{a \in G} \chi^*_\alpha(a) \chi_\Gamma(a)$$

- IRREP characters of point groups available in character tables

Example: C_{2v}

- Character table:

C_{2v}	E	C_2	σ_v	$\sigma_{v'}$	
A_1	1	1	1	1	$z; x^2; y^2; z^2$
A_2	1	1	-1	-1	$R_z; xy$
B_1	1	-1	1	-1	$y; R_x; yz$
B_2	1	-1	-1	1	$x; R_y; xz$

- Decomposition of $\Gamma_{\mathbf{r}}$ into IRREPs:

	E	C_2	σ_v	$\sigma_{v'}$
$\chi_{\Gamma_{\mathbf{r}}}$	3	-1	1	1
$*\chi_{A_1}$	3	1	1	1
n_{A_1}	$\frac{1}{4}(3-1+1+1)=1$			

- Analogous: $n_{A_2}=0$, $n_{B_1}=n_{B_2}=1$.