Vector Derivatives

Cartesian $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}, \ d\tau = dx\,dy\,dz$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial x}\hat{\mathbf{x}} + \dfrac{\partial t}{\partial y}\hat{\mathbf{y}} + \dfrac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\quad \nabla \cdot \mathbf{v} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}$

Curl: $\quad \nabla \times \mathbf{v} = \left(\dfrac{\partial v_z}{\partial y} - \dfrac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\dfrac{\partial v_x}{\partial z} - \dfrac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian: $\quad \nabla^2 t = \dfrac{\partial^2 t}{\partial x^2} + \dfrac{\partial^2 t}{\partial y^2} + \dfrac{\partial^2 t}{\partial z^2}$

Spherical $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}, \ d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial r}\hat{\mathbf{r}} + \dfrac{1}{r}\dfrac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \dfrac{1}{r\sin\theta}\dfrac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence: $\quad \nabla \cdot \mathbf{v} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 v_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(\sin\theta\,v_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial v_\phi}{\partial \phi}$

Curl: $\quad \nabla \times \mathbf{v} = \dfrac{1}{r\sin\theta}\left[\dfrac{\partial}{\partial \theta}(\sin\theta\,v_\phi) - \dfrac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$

$\qquad\qquad + \dfrac{1}{r}\left[\dfrac{1}{\sin\theta}\dfrac{\partial v_r}{\partial \phi} - \dfrac{\partial}{\partial r}(rv_\phi)\right]\hat{\boldsymbol{\theta}} + \dfrac{1}{r}\left[\dfrac{\partial}{\partial r}(rv_\theta) - \dfrac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$

Laplacian: $\quad \nabla^2 t = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial t}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial t}{\partial \theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 t}{\partial \phi^2}$

Cylindrical $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}, \ d\tau = s\,ds\,d\phi\,dz$

Gradient: $\quad \nabla t = \dfrac{\partial t}{\partial s}\hat{\mathbf{s}} + \dfrac{1}{s}\dfrac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \dfrac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence: $\quad \nabla \cdot \mathbf{v} = \dfrac{1}{s}\dfrac{\partial}{\partial s}(sv_s) + \dfrac{1}{s}\dfrac{\partial v_\phi}{\partial \phi} + \dfrac{\partial v_z}{\partial z}$

Curl: $\quad \nabla \times \mathbf{v} = \left[\dfrac{1}{s}\dfrac{\partial v_z}{\partial \phi} - \dfrac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\dfrac{\partial v_s}{\partial z} - \dfrac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}}$

$\qquad\qquad + \dfrac{1}{s}\left[\dfrac{\partial}{\partial s}(sv_\phi) - \dfrac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$

Laplacian: $\quad \nabla^2 t = \dfrac{1}{s}\dfrac{\partial}{\partial s}\left(s\dfrac{\partial t}{\partial s}\right) + \dfrac{1}{s^2}\dfrac{\partial^2 t}{\partial \phi^2} + \dfrac{\partial^2 t}{\partial z^2}$

Vector Identities

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

Fundamental Theorems

Gradient Theorem:	$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$
Divergence Theorem:	$\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$
Curl Theorem:	$\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Spherical and Cylindrical Coordinates

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Basic Equations of Electrodynamics

Maxwell's Equations

In general:

$$\begin{cases} \boldsymbol{\nabla} \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho \\[2mm] \boldsymbol{\nabla} \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2mm] \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \\[2mm] \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f \\[2mm] \boldsymbol{\nabla} \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2mm] \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \\[2mm] \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\[2mm] \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\[2mm] \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \dfrac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

Lorentz Force Law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy: $\quad U = \dfrac{1}{2}\displaystyle\int \left(\epsilon_0 E^2 + \dfrac{1}{\mu_0}B^2 \right) d\tau$

Momentum: $\quad \mathbf{P} = \epsilon_0 \displaystyle\int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector: $\quad \mathbf{S} = \dfrac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula: $\quad P = \dfrac{\mu_0}{6\pi c}q^2 a^2$

Orthogonality of sine functions

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)dy = 0 \quad if \quad n \neq n'$$

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)dy = a/2 \quad if \quad n = n'$$

Legendre Polynomials

$$\int_{-1}^{1} P_\ell(\cos\theta)P_{\ell'}(\cos\theta)d(\cos\theta) = 2/(2\ell+1) \quad if \quad \ell = \ell'$$

$$\int_{-1}^{1} P_\ell(\cos\theta)P_{\ell'}(\cos\theta)d(\cos\theta) = 0 \quad if \quad \ell \neq \ell'$$

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= (3x^2 - 1)/2 \\
P_3(x) &= (5x^3 - 3x)/2 \\
P_4(x) &= (35x^4 - 30x^2 + 3)/8 \\
P_5(x) &= (63x^5 - 70x^3 + 15x)/8
\end{aligned}
$$