Homework 10

Physics 112A

Problem 6.8 A very long circular cylinder of radius R carries a magnetization $M = ks^2\hat{\phi}$, where k is a constant. Find the magnetic field due to M, for points inside and outside the cylinder (and far from the ends).

$$\begin{aligned}
J_b &= \nabla \times M\hat{\phi} \\
&= \frac{1}{s}\frac{\partial}{\partial s}[sks^2]\hat{z} \\
&= 3ks\hat{z} \\
K_b &= M\hat{\phi} \times \hat{n} \\
&= -ks^2\hat{z} \\
&= -kR^2\hat{z}
\end{aligned}$$

When $s < R$:

$$\begin{aligned}
\int B \cdot dl &= \mu_0 \int_0^s J_b da \\
B 2\pi s &= \mu_0 \int_0^s 3ks2\pi s ds \\
B &= \frac{3\mu_0 k}{s}[\frac{1}{3}s^3]_0^s \\
&= \boxed{\mu_0 ks^2\hat{\phi}}
\end{aligned}$$

When $s > R$:

$$\begin{aligned}
\int B \cdot dl &= \mu_0(\int_0^R J_b da + \int_0^R K_b dl) \\
B 2\pi s &= \mu_0(\int_0^R 3ks2\pi s ds + \int_0^R -kR^2 2\pi ds) \\
&= \mu_0(2\pi kR^3 - 2\pi kR^2[s]_0^R) \\
B &= \boxed{0}
\end{aligned}$$

Problem 6.10 An iron rod of length L and square cross section (side length a) is given a uniform longitudinal magnetization M, and then bent around into a circle with a narrow gap (width w). Find the magnetic field at the center of the gap, assuming $w << a << L$. [Hint: Treat it as the superposition of a complete torus plus a square loop with revered current.]

$$\begin{aligned} J_b &= \nabla \times M\hat{\phi} \\ &= 0 \\ K_b &= M\hat{\phi} \times \hat{n} \\ &= M\hat{z} \end{aligned}$$

For a complete torus:

$$\begin{aligned} \int B \cdot dl &= \mu_0 \int K_b dl \\ B2\pi s &= \mu_0 M 2\pi s \\ B &= \mu_0 M\hat{\phi} \end{aligned}$$

For a square loop:

$$\begin{aligned} \int B \cdot dl &= \mu_0 \int -K_b dl \\ B\frac{a}{2} &= -\mu_0 K_b w \\ B &= -2\mu_0 M w \frac{1}{a}\hat{\phi} \end{aligned}$$

Problem 6.12 An infinitely long cylinder, of radius R, carries a "frozen-in" magnetization, parallel to the axis,

$$M = ks\hat{z}$$

where k is a constant and s is the distance from the axis; there is no free current anywhere. Find the magnetic field inside and outside the cylinder by two different methods:

(a) Locate all the bound currents, and calculate the field they produce.

$$\begin{aligned} J_b &= \nabla \times M\hat{z} \\ &= -\frac{\partial}{\partial s} ks\hat{\phi} \\ &= -k\hat{\phi} \\ K_b &= M\hat{z} \times \hat{n} \\ &= ks\hat{\phi} \\ &= kR\hat{\phi} \end{aligned}$$

When $s < R$:

$$\int B \cdot dl = \mu_0(\int J_b da + \int K_b dl)$$
$$Bl = \mu_0(-kl[R - s] + klR)$$
$$B = \boxed{\mu_0 ks\hat{z}}$$

When $s > R$:

$$\int B \cdot dl = \mu_0(\int J_b da + \int K_b dl)$$
$$B = \boxed{0}$$

(b) Use Ampere's law to find H, and then get B.
$H = 0$ since there is no free current anywhere.
When $s < R$:

$$B = \mu_0 M$$
$$= \boxed{\mu_0 ks\hat{z}}$$

When $s < R$:

$$M = 0$$
$$B = \boxed{0}$$

Problem 6.16 A coaxial cable consists of two very long cylindrical tubes, separated by linear insulating material of magnetic susceptibility χ_m. A current I flows down the inner conductor and returns along the outer one; in each case, the current distributes itself uniformly over the surface. Find the magnetic field in the region between the tubes. As a check, calculate the magnetization and the bound currents, and confirm that (together, of course, with the free currents) they generate the correct field.

$$\int H \cdot dl = I$$
$$H = \frac{I}{2\pi s}$$
$$B = \mu H$$
$$= \boxed{\mu_0(1 + \chi_m)\frac{I}{2\pi s}}$$

Check:

$$
\begin{aligned}
M &= \chi_m H \\
&= \chi_m I \frac{1}{2\pi s} \hat{\phi} \\
J_b &= \nabla \times M\hat{\phi} \\
&= \frac{1}{s} \frac{\partial}{\partial s} [s \chi_m I \frac{1}{2\pi s}] \hat{z} \\
&= 0 \\
K_b &= M\hat{\phi} \times \hat{n} \\
&= \chi_m I \frac{1}{2\pi s} \hat{z} \\
&= \chi_m I \frac{1}{2\pi a} \hat{z} \\
\int B \cdot dl &= \mu_0 (I + \int K_b dl) \\
B 2\pi s &= \mu_0 (I + \chi_m I \frac{1}{2\pi a} 2\pi a) \\
B &= \mu_0 I (1 + \chi_m) \frac{1}{2\pi s}
\end{aligned}
$$