13

OSCILLATORY MOTION

EXERCISES

Section 13.1 Describing Oscillatory Motion

11. **INTERPRET** This problem deals with the two quantities that characterize periodic motion: period and frequency, which are related by Equation 13.1, $f = 1/T$.

DEVELOP The doctor counts 75 beats per minute. Convert this to number of beats per second to find the frequency in Hz, and use Equation 13.1 to find the corresponding period in s.

EVALUATE The frequency f in Hz is

$$f = (75 \text{ bpm})\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 1.25 \text{ Hz}$$

The period T is

$$T = 1/f = 1/(1.25 \text{ Hz}) = 0.8 \text{ s}$$

ASSESS Frequency and period are reciprocals, so their units are reciprocal as well. For frequency, the unit is inverse time $(\text{Hz} = \text{s}^{-1})$ and for the period, the unit is time (s).

12. **INTERPRET** The question here is about the oscillatory behavior of the violin string. Given the frequency of oscillation, we are asked to find the period.

DEVELOP The relationship between period and frequency is given by Equation 13.1, $T = 1/f$.

EVALUATE Using Equation 13.1, we obtain

$$T = \frac{1}{f} = \frac{1}{440 \text{ Hz}} = 2.27 \times 10^{-3} \text{ s}$$

ASSESS The period of oscillation is the inverse of the frequency. Note that the unit of frequency is the hertz; (1 Hz = 1 s^{-1}).

13. **INTERPRET** This problem gives us the frequency of an oscillating molecule from which we are asked to find the period of oscillation.

DEVELOP Use Equation 13.1, $f = 1/T$.

EVALUATE Inserting the given frequency into Equation 13.1, we find a period of

$$T = \frac{1}{f} = \frac{1}{8.66 \times 10^{13} \text{ Hz}} = 1.15 \times 10^{-14} \text{ s} = 11.5 \text{ fs}$$

ASSESS This is quite fast, although still measurable with modern ultrafast pulsed lasers.

14. **INTERPRET** This problem involves calculating the frequency and period given the number of oscillations in a given time.

DEVELOP Divide the number of oscillations by the time period to find the frequency, and then use Equation 13.1 $f = 1/T$ to find the period. There are 95 oscillations in 10 minutes, or 600 seconds.

EVALUATE **(a)** The period is $T = \dfrac{10 \text{ min}}{95} = 0.105 \text{ min} = 6.32 \text{ s}.$

(b) The frequency is

$$f = \frac{1}{T} = \frac{1}{6.32\ \text{s}} = 0.16\ \text{Hz}$$

ASSESS This is a relatively slow oscillation, as you might expect for a building-sized object.

15. **INTERPRET** We want to find the period of the hummingbird's wing flap.

DEVELOP From Equation 13.1, the period is the inverse of the frequency: $T = 1/f$.

EVALUATE The period of one hummingbird wing flap is:

$$T = \frac{1}{f} = \frac{1}{(45\ \text{Hz})} = 22\ \text{ms}$$

ASSESS Our eyes cannot follow the beating of a hummingbird's wings. That's because the human eye essentially processes information at a rate of about 30 times per second, or equivalently once every 33 ms.

Section 13.2 Simple Harmonic Motion

16. **INTERPRET** In this problem, a mass attached to a spring undergoes simple harmonic motion. Given the mass, the spring constant, and the amplitude, we are asked to compute the frequency and the period of oscillation, the maximum velocity, and the maximum force in the spring.

DEVELOP Given the spring constant k and the mass, the frequency and period of the oscillation may be obtained from Equations 13.8b and 13.8c:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$$

Using Equations 13.9 and 13.10, the maximum speed and the maximum force are:

$$v_{\max} = \omega A$$

$$F_{\max} = ma_{\max} = m\omega^2 A$$

EVALUATE **(a)** For a mass on a spring, Equation 13.7a gives the angular frequency as

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.6\ \text{N/m}}{0.16\ \text{kg}}} = 5.92\ \text{rad}\cdot\text{s}^{-1}$$

Using Equation 13.7b, the frequency of oscillation is

$$f = \frac{\omega}{2\pi} = \frac{5.9\ \text{rad}\cdot\text{s}^{-1}}{2\pi\ \text{rad}} = 0.94\ \text{Hz}$$

(b) The period of oscillation is the inverse of the frequency (see Equation 13.7c):

$$T = \frac{1}{f} = 1.1\ \text{s}$$

(c) From Equation 13.9, we find the maximum speed to be

$$v_{\max} = \omega A = \left(5.92\ \text{s}^{-1}\right)(0.22\ \text{m}) = 1.3\ \text{m/s}$$

(d) Similarly, using Equation 13.10, the maximum force in the spring is

$$F_{\max} = ma_{\max} = m\omega^2 A = (0.16\ \text{kg})\left(5.92\ \text{s}^{-1}\right)(0.22\ \text{m}) = 1.2\ \text{N}$$

ASSESS Note that the results are reported to two significant figures, as warranted by the precision of the data. The force in the spring can also be calculated by using Equation 13.2: $F = kx$. In fact,

$$F_{\max} = ma_{\max} = m\omega^2 A = kx_{\max} = kA = (5.92\ \text{N/m})(0.22\ \text{m}) = 1.2\ \text{N}$$

17. **INTERPRET** We can consider the car's suspension as a simple spring-mass system.

DEVELOP Equations 13.7b and 13.7c give the expressions for frequency and period in the case of simple harmonic motion:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ and } T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$$

Evaluate Using the expressions above and the mass and spring constant given, the frequency is:

$$f = \frac{1}{2\pi}\sqrt{\frac{26\text{ kN/m}}{2100\text{ kg}}} = 0.56\text{ Hz}$$

and the period is:

$$T = 2\pi\sqrt{\frac{2100\text{ kg}}{26\text{ kN/m}}} = 1.8\text{ s}$$

Assess If the car hits a bump, it will start to vibrate at this slow frequency. However, the shock absorbers will damp this motion, so the vibrations should die away after a few cycles.

18. **Interpret** In this problem, a mass attached to a spring undergoes simple harmonic motion. Given the maximum speed and the maximum acceleration of the mass, we want to find the angular frequency, the spring constant, and the amplitude of the oscillation.

Develop The maximum speed and the maximum acceleration of the mass can be obtained from Equations 13.9 and 13.10 by taking the maximum value of the trigonometric functions (i.e., 1). Taking the absolute value, the result is

$$v_{\text{max}} = \omega A \qquad a_{\text{max}} = \omega^2 A$$

From a_{max} and v_{max}, the angular frequency may be obtained from $\omega = a_{\text{max}}/v_{\text{max}}$. Once ω is known, the spring constant and the amplitude of the oscillation can be calculated using:

$$k = m\omega^2 \qquad A = \frac{a_{\text{max}}}{\omega^2} = a_{\text{max}}\left(\frac{v_{\text{max}}}{a_{\text{max}}}\right)^2 = \frac{v_{\text{max}}^2}{a_{\text{max}}}$$

Evaluate **(a)** The angular frequency is

$$\omega = \frac{a_{\text{max}}}{v_{\text{max}}} = \frac{18.6\text{ m/s}^2}{1.75\text{ m/s}} = 10.63\text{ s}^{-1} \approx 10.6\text{ s}^{-1}$$

(b) Similarly, the amplitude of the motion is

$$A = \frac{v_{\text{max}}^2}{a_{\text{max}}} = \frac{(1.75\text{ m/s})^2}{1.86\text{ m/s}^2} = 0.165\text{ m}$$

(b) The spring constant is $k = m\omega^2 = (0.342\text{ kg})(10.63\text{ s}^{-1})^2 = 38.6\text{ N/m}.$

Assess To check that our results are correct, let's compute the amplitude A using the value of ω found in **(a)**. In either way, we have

$$A = \frac{v_{\text{max}}}{\omega} = \frac{1.75\text{ m/s}}{10.63\text{ s}^{-1}} = \ 0.165\text{ m}$$

$$A = \frac{a_{\text{max}}}{\omega^2} = \frac{18.6\text{ m/s}^2}{(10.63\text{ s}^{-1})^2} = 0.165\text{ m}$$

The results indeed agree.

19. **Interpret** This problem requires us to find the angular frequency, period, and maximum acceleration of a simple harmonic oscillator, given its amplitude and maximum speed.

Develop From Equation 13.9, we see that the maximum speed is related to the amplitude and angular frequency by $v_{\text{max}} = \omega A$. Knowing the angular frequency, use Equation 13.1, $f = 1/T$, to find the period. Finally, from Equation 13.10, we see that the maximum acceleration is given by $a_{\text{max}} = \omega^2 A = \omega v_{\text{max}}$.

Evaluate **(a)** The angular frequency is $\omega = v_{\text{max}}/A = (4.6\text{ m/s})/(0.27\text{ m}) = 17\text{ rad/s}$.

(b) The period is $T = 1/f = 2\pi/\omega = 2\pi/(17\text{ rad/s}) = 0.37\text{ s}$.

(c) The maximum acceleration is $a_{\text{max}} = \omega v_{\text{max}} = (17.04\text{ rad/s})(4.6\text{ m/s}) = 78.2\text{ m/s}^2$.

Assess Although radians are dimensionless, it is sometimes useful to write them to remind ourselves that we are dealing with an angular quantity as opposed to a linear quantity.

Section 13.3 Applications of Simple Harmonic Motion

20. **INTERPRET** This problem involves the simple harmonic motion of a pendulum. We are asked to find the length of a simple pendulum so that its period is a given value.

DEVELOP The period and length of a simple pendulum (at the surface of the Earth) are related by Equation 13.15, $T = 2\pi\sqrt{L/g}$.

EVALUATE (a) Solving the expression above for the length L of the pendulum and inserting T = 0.20 ms, we find

$$L = g\left(\frac{T}{2\pi}\right)^2 = \left(9.8\ \text{m/s}^2\right)\left(\frac{0.20\ \text{s}}{2\pi}\right)^2 = 9.9\ \text{mm}$$

(b) For T = 5.0 s, we find

$$L = \left(9.8\ \text{m/s}^2\right)\left(\frac{5.0\ \text{s}}{2\pi}\right)^2 = 6.2\ \text{m}$$

b) For T = (2.0 min)(60 s/min) = 120 s, we find

$$L = \left(9.8\ \text{m/s}^2\right)\left(\frac{120\ \text{s}}{2\pi}\right)^2 = 3.6\ \text{km}$$

ASSESS The pendulum length is quadratic in the period, so doubling the period quadruples the pendulum length. Thus, making a pendulum with a period longer than a few seconds is not very practical on Earth.

21. **INTERPRET** This problem involves simple harmonic motion of the pendulum in a grandfather clock. We want to find the time interval between successive ticks.

DEVELOP The period of a simple pendulum is given by Equation 13.15:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

We note that the clock ticks twice during each period of oscillation, so the time between clicks is a half-period.

EVALUATE Using Equation 13.15, the time between ticks is

$$\Delta t = \frac{T}{2} = \pi\sqrt{\frac{L}{g}} = \pi\sqrt{\frac{1.35\ \text{m}}{9.81\ \text{m/s}^2}} = 1.17\ \text{s}$$

ASSESS One tick every 1.17 s seems reasonable. Note that if we increase the length of the pendulum, then the period and the time between ticks will also increase.

22. **INTERPRET** This problem involves finding the rotational inertia of a hollow sphere (volleyball) about an axis through its center and using that to find the torsional constant of the wire that suspends the sphere, given the frequency of oscillation.

DEVELOP The rotational inertia of a hollow sphere about an axis through its center is $2MR^2/3$ (see Table 10.2).Use this result in Equation 13.12 to find the torsional constant.

EVALUATE Solving Equation 13.12 for the torsional constant and inserting the given quantities, we find

$$\kappa = \omega^2 I = \frac{(2\pi f)^2 2MR^2}{3} = \frac{\left(2\pi\times 1.57\ \text{s}^{-1}\right)^2 2(0.280\ \text{kg})(0.105\ \text{m})^2}{3} = 0.200\ \text{N}\cdot\text{m/rad}.$$

ASSESS We can check that the units are correct by inspecting Equation 13.12. We find that

$$\text{rad/s} = \sqrt{\frac{\text{N}\cdot\text{m}\cdot\text{rad}^2}{\text{kg}\cdot\text{m}^2}} = \sqrt{\frac{\text{kg}\cdot\text{m}^2\text{s}^{-2}\cdot\text{rad}^2}{\text{kg}\cdot\text{m}^2}} = \text{rad/s}$$

So the units are correct.

23. **INTERPRET** The problem is about a physical pendulum—a meter stick suspended from one end. We want to know its period of oscillation.

DEVELOP The period of a physical pendulum can be obtained from Equation 13.13.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgL}}$$

The meter stick is a physical pendulum whose center of mass is $L = L_0/2 = 0.50$ m below the point of suspension. From Table 10.2, we see that the rotational inertia of a stick about one end is $I = mL_0^2/3$.

EVALUATE Using Equation 13.13, the period of the meter stick is

$$T = 2\pi\sqrt{\frac{I}{mgL}} = 2\pi\sqrt{\frac{mL_0^2/3}{mgL_0/2}} = 2\pi\sqrt{\frac{2L_0}{3g}} = 2\pi\sqrt{\frac{2(1.0\ \text{m})}{3(9.8\ \text{m/s}^2)}} = 1.6\ \text{s}$$

ASSESS A simple experiment can be carried out to verify that a period of 1.6 s (or roughly five complete oscillations in 8 seconds) for the meter stick is reasonable.

Section 13.4 Circular and Harmonic Motion

24. **INTERPRET** We look at one component of circular motion as simple harmonic motion and convert rpm to cycles/second. Our goal is to find the frequency in hertz and the angular frequency of the simple harmonic motion.

DEVELOP Convert revolutions per minute (rpms) to frequency (revolutions per second). For part (b), convert the frequency to an angular frequency using $f = \omega/(2\pi)$.

EVALUATE

(a) $f = (720\ \text{rpm})(1.00\ \text{min} / 60.0\ \text{s}) = 12.0\ \text{s}^{-1} = 12.0\ \text{Hz}.$

(b) The angular frequency is $\omega = 2\pi f = (2\pi\ \text{rad})(12.0\ \text{s}^{-1}) = 75.4\ \text{rad / s}.$

ASSESS Circular motion and simple harmonic motion are very closely related! That's why we use the symbol ω for both — it's the same for both.

25. **INTERPRET** This problem involves a body undergoing simple harmonic motion in two dimensions, with a different angular frequency in each dimension. We are to find how many oscillations it makes before returning to its initial position.

DEVELOP Let the angular frequencies be ω_x and ω_y, such that

$$\frac{\omega_x}{\omega_y} = \frac{5.75}{1} = \frac{23}{4}$$

Since $T = 2\pi/\omega$, the ratio of the periods for the *x*- and *y*-components of the motion is then equal to

$$\frac{T_x}{T_y} = \frac{2\pi/\omega_x}{2\pi/\omega_y} = \frac{\omega_y}{\omega_x} = \frac{4}{23}$$

EVALUATE The above equation gives $23\ T_x = 4T_y$, which means that twenty-three oscillations are completed in the *x*-direction and four are completed in the *y*-direction before the object returns to its initial position.

ASSESS Since $\omega_x = 5.75\ \omega_y > \omega_y$, we have $T_x = T_y / 5.75 < T_Y$. The smaller the period, the greater the number of oscillations completed in a given time interval.

Section 13.5 Energy in Simple Harmonic Motion

26. **INTERPRET** In this problem we have a mass undergoing a simple harmonic motion by oscillating on a spring. We are to find its oscillation amplitude given the total energy of the mass-spring system.

DEVELOP The total mechanical energy stored in the system is

$$\begin{aligned} E &= U + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}k\left(A\cos\omega t\right)^2 + \frac{1}{2}m\left(-A\omega\sin\omega t\right)^2 \\ &= \frac{1}{2}kA^2\cos^2\omega t + \frac{1}{2}m\omega^2A^2\sin^2\omega t = \frac{1}{2}kA^2\left(\cos^2\omega t + \sin^2\omega t\right) \\ &= \frac{1}{2}kA^2 \end{aligned}$$

where we have used Equations 13.8 and 13.9 for $x(t)$ and $v(t)$, with $\phi = 0$. The equation relates the total energy to the amplitude of the oscillation.

EVALUATE From the above expression for *E*, we find the amplitude to be

$$A=\sqrt{\frac{2E}{m\omega^2}}=\sqrt{\frac{2E}{m(2\pi f)^2}}=\sqrt{\frac{2(0.51\text{ J})}{(0.45\text{ kg})(2\pi\times 1.2\text{ Hz})^2}}=0.20\text{ m}$$

ASSESS The total energy E stored in the system is found to be proportional to A^2 and is independent of time as required by energy conservation, despite the fact that K and U vary with time.

27. **INTERPRET** This problem involves finding the maximum angular displacement and speed given the torsional constant, rotational inertia, and total energy.

DEVELOP Because only conservative forces (i.e., the torsional forces of the wire) act on the torsional oscillator, we can apply conservation of energy to relate its total energy to its oscillation parameters. Specifically, we choose to evaluate the total energy when the rotational speed is maximum, at which point the torsional potential energy is zero. Thus,

$$E_{\text{tot}}=\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2_{\max}$$

where $(d\theta/dt)^2_{\max}$ is the maximum angular speed (not to be confused with ω, the constant natural frequency of the oscillator). We can determine $(d\theta/dt)^2_{\max}$ by inspecting the equation of motion for a torsional oscillator (Equation 13.11). Because it is the same as that for a linear oscillator, with k replaced by κ, we can immediately write

$$\left(\frac{d\theta}{dt}\right)_{\max}=\omega A$$

where ω is the natural angular frequency of the oscillator and A is the maximum angular displacement (in radians). Inserting this into the expression for total energy and using Equation 13.12, $\omega=\sqrt{\kappa/\text{I}}$, we find

$$E_{\text{tot}}=\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2_{\max}=\frac{1}{2}I\omega^2A^2=\frac{1}{2}\kappa A^2$$

From these expressions, we can find both the maximum displacement A and the maximum angular speed $(d\theta/dt)^2_{\max}$.

EVALUATE The maximum angular displacement is

$$A=\pm\sqrt{\frac{2E_{\text{tot}}}{\kappa}}=\pm\sqrt{\frac{2(4.4\text{ J})}{3.4\text{ N}\cdot\text{m/rad}}}=\pm 1.6\text{ rad}=\pm 92^\circ$$

where the two signs indicate that the maximum angular displacement occurs in both the counterclockwise direction and the clockwise direction. The maximum angular speed is

$$\left(\frac{d\theta}{dt}\right)_{\max}=\pm\sqrt{\frac{2E_{\text{tot}}}{I}}=\pm\sqrt{\frac{2(4.4\text{ J})}{1.6\text{ kg}\cdot\text{m}^2}}=\pm 2.3\text{ s}^{-1}=\pm 15\text{ rad/s}$$

where the two signs indicate that this speed is attained in both the counterclockwise direction and the clockwise direction.

ASSESS Notice how conservation of total mechanical energy allowed us to evaluate the total mechanical energy where it was simplest to do so.

28. **INTERPRET** In this question, we want to determine the amount of energy contained within the oscillatory motion of a moving vehicle and calculate what percentage of the kinetic energy it makes up.

DEVELOP The relationship between the spring potential energy and amplitude of oscillation is given by Equation 7.4: $U=\frac{1}{2}kx^2$. Meanwhile, the kinetic energy is $K=\frac{1}{2}mv^2$. We can calculate both and determine the percentage of the car's kinetic energy contained within the vehicle's oscillations.

EVALUATE Taking the ratio of the spring potential energy to the vehicle's kinetic energy, we obtain

$$\frac{U}{K}=\frac{\frac{1}{2}kx^2}{\frac{1}{2}mv^2}=\left(\frac{\omega x}{v}\right)^2=\left(\frac{(2\pi)(1.1\text{ Hz})(0.15\text{ m})}{(25\text{ m/s})}\right)^2=0.0017$$

where we have used Equations 13.7a and 13.7b to express the spring constant in terms of the given frequency and converted the speed to SI units. We find that about 0.17% of the car's kinetic energy is in the oscillations.

ASSESS We find that the result is independent of the vehicle's mass.

Section 13.6 Damped Harmonic Motion and Section 13.7 and Resonance

29. **INTERPRET** The problem involves damped harmonic motion. We are interested in finding out how long it takes for the vibration amplitude of a piano string to drop to half its initial value.

DEVELOP The solution to damped harmonic motion is given by Equation 13.17:

$$x(t) = Ae^{-bt/2\mathrm{m}}\cos(\omega t + \phi)$$

Thus, the amplitude is half the initial value when $e^{-bt/2m} = 0.5.$

EVALUATE Taking the natural logarithms of both sides of the equation above gives $bt/2m = \ln 2.$ Thus, the time for the amplitude to be halved is

$$t = \frac{2m}{b}\ln 2 = \frac{\ln 2}{4.4\ \mathrm{s}^{-1}} = 0.16\ \mathrm{s}$$

ASSESS Since the amplitude drops by half in just 0.16 s, we conclude that the damping must be rather strong.

30. **INTERPRET** This problem involves driven and damped harmonic motion. Given the damping constant, we are asked to find the amplitude as a function of the driving frequency.

DEVELOP The amplitude at resonance $(\omega_\mathrm{d} = \omega_0)$ is $A_\mathrm{res} = F_\mathrm{d}/b\omega_0$ so Equation 13.19 can be rewritten as:

$$\frac{A}{A_\mathrm{res}} = \frac{A}{(F_\mathrm{d}/b\omega_0)} = \frac{(b\omega_0/m)}{\sqrt{(\omega_\mathrm{d}^2-\omega_0^2)^2 + b^2\omega_\mathrm{d}^2/m^2}} = \left[\left(\frac{m\omega_0}{b}\right)^2\left(\frac{\omega_\mathrm{d}^2}{\omega_0^2}-1\right)^2 + \frac{\omega_\mathrm{d}^2}{\omega_0^2}\right]^{-1/2}$$

Using the given quantities, we can find the ratios of the amplitudes for $\omega_\mathrm{d} = 1.1\,\omega_0$ and $\omega_\mathrm{d} = 0.9\,\omega_0$.

EVALUATE If $b/m = \omega_0 = 5,$ and $\omega_\mathrm{d} = 1.1\,\omega_0$ (10% above resonance), then

$A = A_\mathrm{res} = 1 = \sqrt{25(1.21-1)^2 + 1.21} = 65.8\%.$ For $\omega_\mathrm{d} = 0.9\,\omega_0$ (10% below resonance),

$A = A_\mathrm{res} = 1 = \sqrt{25(0.81-1)^2 + 0.81} = 76.4\%.$

ASSESS Notice that, for a given departure from resonance, the amplitude of the oscillation depends on whether the departure is positive or negative. The reason for this is that Equation 13.19 is not symmetric about the resonance (see Fig. 13.25).

31. **INTERPRET** This problem involves nondamped driven harmonic motion. We want to find the speed of the car that leads to a maximum vibration amplitude, which will occur when the frequency with which the car drives over the bumps equals the natural frequency of its suspension system.

DEVELOP The peak amplitude occurs at resonance, when $\omega_\mathrm{d} = \omega_0,$ so the car receives an impulse from the bumps once each period. The condition for resonance is, therefore, that the car travel the distance between bumps in one period:

$$L_0 = \frac{48\ \mathrm{m}}{vT_0}$$

EVALUATE Solving the resonance condition for the speed v gives

$$v = \frac{L_0}{T_0} = L_0 f_0 = (48\ \mathrm{m})(0.45\ \mathrm{Hz}) = 22\ \mathrm{m/s} = 78\ \mathrm{km/h}$$

where we have used Equation 13.1, $T = 1/f$.

ASSESS If the spacing between bumps increases, then the car speed must also go up to meet the rather unpleasant resonance condition!

EXAMPLE VARIATIONS

32. **INTERPRET** This is a problem involving simple harmonic motion, with a tuned mass damper making up the oscillating system. We're given the oscillation frequency, mass, and amplitude.

DEVELOP Equation 13.7c, $T = 1/f = 2\pi\sqrt{m/k}$, will give the spring constant. Equations 13.9 and 13.10 show that the maximum speed and acceleration are $v_{\text{max}} = \omega A$ and $a_{\text{max}} = \omega^2 A$, and we can get the angular frequency ω from the ordinary frequency using Equation 13.6: $\omega = 2\pi f$.

EVALUATE First we solve Equation 13.7c for the spring constant:

$$k = 4\pi^2 f^2 m = \left(4\pi^2\right)(6.85\,\text{Hz})(142\,\text{kg}) = 263\,\text{kN/m}$$

The angular frequency is $\omega = 2\pi f = 43.0\,\text{rad/s}$. Then we have $v_{\text{max}} = \omega A = \left(43.0\ \text{s}^{-1}\right)(0.0486\,\text{m}) = 2.09\,\text{m/s}$ and $a_{\text{max}} = \omega^2 A = \left(43.0\ \text{s}^{-1}\right)^2 (0.0486\,\text{m}) = 90.0\,\text{m/s}^2$.

ASSESS The mass of the damper is relatively light weight in comparison to the building for which it is reducing structural vibrations.

33. **INTERPRET** This is a problem involving simple harmonic motion, with a tuned mass damper making up the oscillating system. We're given the spring constant and period of oscillation.

DEVELOP Equation 13.7c, $T = 2\pi\sqrt{m/k}$, will give the mass of the damper.

EVALUATE Solve Equation 13.7c for the mass we obtain:

$$m = \frac{kT^2}{4\pi^2} = \frac{\left(0.288\times10^6\ \text{N/m}\right)(5.71\,\text{s})^2}{\left(4\pi^2\right)} = 238\,\text{Mg}$$

Where we have converted from kg to Mg after evaluating.

ASSESS The large spring constant makes sense given the huge mass involved.

34. **INTERPRET** This is a problem involving simple harmonic motion, with a large pendulum used as a tuned mass damper making up the oscillating system. We're given the length of the pendulum, along with the mass and diameter of the solid ball making up the bob.

DEVELOP Since we are treating this system as a simple pendulum, Equation 13.15, $T = 2\pi\sqrt{L/g}$, will give the oscillation period of the damping pendulum. If we neglect the mass of the cable, the length L is equal to the cable length plus the radius of the solid ball.

EVALUATE Evaluating Equation 13.15 we obtain:

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{\left(8.40\,\text{m} + \frac{1}{2}(5.49\,\text{m})\right)}{\left(9.8\,\text{m/s}^2\right)}} = 6.70\,\text{s}$$

ASSESS The period of oscillation is independent of the pendulum bob's mass.

35. **INTERPRET** We want to revisit the previous problem, involving a pendulum serving as a tuned mass damper, but treating it as a physical pendulum instead of a simple pendulum.

DEVELOP Since we are treating this system as a physical pendulum, Equation 13.13, $\omega = \sqrt{mgL_{\text{cm}}/I}$, will give the angular frequency of the damping pendulum, from which we can obtain the oscillation period using Equation 13.5, $T = 2\pi/\omega$. If we neglect the mass of the cable, the length L_{cm} is equal to the cable length plus the radius of the solid ball, which means we can use the parallel axis theorem along with the expression found in Table 10.2 for a solid sphere, to express the effective rotational.

EVALUATE We begin by expressing the rotational inertia of the pendulum bob

$$I = \frac{2}{5}m\left(\frac{D}{2}\right)^2 + m\left(L + \frac{D}{2}\right)^2$$

Then we write the expression for the period, using the value we obtain for the angular frequency using the effective rotational inertia

$$T = 2\pi\sqrt{\frac{\frac{D^2}{10}+\left(L+\frac{D}{2}\right)^2}{g\left(L+\frac{D}{2}\right)}}$$

Evaluating this for the given parameters given in Problem 34, we arrive at a period of 6.78 s.

ASSESS The period of oscillation is 80 ms longer when treating the system as a physical pendulum.

36. **INTERPRET** This problem involves the concept of energy conservation in simple harmonic motion. We're asked to find a speed, which is related to kinetic energy.

DEVELOP When the kinetic energy equals the potential energy, each must be half the total energy. When the energy is all kinetic, the oscillating mass has its maximum velocity, given by Equation 13.9 as $v_{\max} = \omega A$, meaning the total energy is equal to $\frac{1}{2}mv_{\max}^2 = \frac{1}{2}m\omega^2 A^2$. When the energies are split equally, the kinetic energy will be equal to $K = \frac{1}{2}mv^2 = \frac{1}{4}m\omega^2 A^2$. We then find that the speed at this point is given by

$$v = \frac{\omega A}{\sqrt{2}}$$

EVALUATE Evaluating the speed at the point where the kinetic energy and potential energies are equal using the values given we find

$$v = \frac{\omega A}{\sqrt{2}} = \frac{(2\pi)\left(0.377\ \text{s}^{-1}\right)(28.2\ \text{cm})}{\sqrt{2}} = 47.2\ \text{cm/s}$$

ASSESS Since the kinetic energy depends on the square of the speed, it's lower not by a factor of 2 but of $\sqrt{2}$.

37. **INTERPRET** This problem involves the concept of energy conservation in simple harmonic motion. We're asked to find the amplitude and maximum speed.

DEVELOP When the energy is all kinetic, the oscillating mass has its maximum velocity, given by Equation 13.9 as $v_{\max} = \omega A$, meaning if we have the amplitude and angular frequency we can determine the maximum velocity. We can find the amplitude by expressing the total energy as $U = \frac{1}{2}kA^2$, allowing us to write

$$A = \sqrt{\frac{2E}{k}}\ ;\ v_{\max} = \omega\sqrt{\frac{2U}{k}}$$

EVALUATE Evaluating the amplitude and maximum speed using the given values for the spring constant, angular frequency, and total energy we find

$$A = \sqrt{\frac{2U}{k}} = \sqrt{\frac{2(7.69\ \text{J})}{(63.7\ \text{N/m})}} = 49.1\ \text{cm}$$

$$v_{\max} = \omega\sqrt{\frac{2U}{k}} = \left(2.38\ \text{s}^{-1}\right)\sqrt{\frac{2(7.69\ \text{J})}{(63.7\ \text{N/m})}} = 1.17\ \text{m/s}$$

ASSESS From our expression we see that the smaller the spring constant, the higher the amplitude of oscillation for a given energy stored in the mass-spring system.

38. **INTERPRET** This problem involves the concept of energy conservation in simple harmonic motion. We're asked to find an angle and a speed at a moment in a pendulum's swing.

DEVELOP When the kinetic energy equals the potential energy, each must be half the total energy. When the energies are split equally, the kinetic energy will be equal to $K = \frac{1}{2}mv^2 = \frac{1}{2}mgh_{\max}$, where $h = L(1-\cos\theta)$ is the height of the pendulum relative to its equilibrium position. Given the maximum amplitude $\theta_{\max}$ of the swinging pendulum, we can determine the amplitude θ_{half} at this point noting:

$$mgh_{\text{half}} = \frac{1}{2}mgh_{\max}$$

$$\left(1-\cos\theta_{\text{half}}\right)=\frac{1}{2}\left(1-\cos\theta_{\max}\right)$$

$$\theta_{\text{half}}=\text{acos}\left(\frac{1}{2}\left(1+\cos\theta_{\max}\right)\right)$$

Given the period of the swinging pendulum, we can determine its length using Equation 13.15: $T=2\pi\sqrt{L/g}$. Knowing this we can find the speed at the halfway point using the conservation of energy equation, giving

$$v=\sqrt{gL\left(1-\cos\theta_{\max}\right)}$$

EVALUATE Evaluating the amplitude at the point where the kinetic energy and potential energy are equal using the values given we find $\theta_{\text{half}}=6.25°$. Using the given period, we find the length is equal to 1.71 m. From the expression for the speed, we find

$$v=\sqrt{gL\left(1-\cos\theta_{\max}\right)}=\sqrt{\left(9.8\text{ m/s}^2\right)\left(1.71\text{ m}\right)\left(1-\cos\left(8.85°\right)\right)}=44.6\text{ cm/s}$$

ASSESS Much like the case of a mass-spring system, this velocity is smaller than the maximum speed by a factor of $\sqrt{2}$.

39. **INTERPRET** This problem involves the concept of energy conservation in simple harmonic motion. We're asked to find expressions for the maximum speed and total energy of a swinging pendulum.

DEVELOP When the energy of the pendulum is all potential energy, we can express it as $U=mgh_{\max}$, where $h_{\max}=L\left(1-\cos\theta_{\max}\right)$ is the height of the pendulum relative to its equilibrium position when its amplitude is maximized. Given the period of the swinging pendulum, we can determine its length using Equation 13.15: $T=2\pi\sqrt{L/g}$. To find the maximum speed we can then equate the total energy to the kinetic energy the pendulum mass would have when at the bottom of its swing, and solve for v .

EVALUATE From Equation 13.15 we find the length of the pendulum in terms of its oscillation period is given by $L=gT^2/4\pi^2$, meaning we can express the total energy as

$$U=mgL\left(1-\cos\theta_{\max}\right)=mg^2T^2\left(1-\cos\theta_{\max}\right)/4\pi^2$$

and the maximum speed as

$$v=\sqrt{2gh_{\max}}=2g^2T^2\left(1-\cos\theta_{\max}\right)/4\pi^2=\left(gT/\pi\right)^2\sqrt{\left(1-\cos\theta_{\max}\right)/2}$$

or equivalently, $v=gT\sin\left(\theta_{\max}/2\right)/\pi$

ASSESS From the expression for the maximum speed we can see that the longer the period of oscillation, the higher the speed the pendulum will have when all the potential energy is converted into kinetic energy. This makes sense a longer period is caused by a larger vertical displacement, and thus a higher potential energy.

PROBLEMS

40. **INTERPRET** This problem involves the simple harmonic motion of the oxygen atoms in a carbon dioxide molecule. We are to interpret the chemical bonds as springs are given the frequency of oscillation of the oxygen atoms and their mass, from which we can calculate the spring constant.

DEVELOP The relationship between the oscillating frequency and the spring constant is given by Equation 13.7a, $\omega=\sqrt{k/m}$. Use the conversion factor in Appendix C to convert the mass of the oxygen atom from u to kg.

EVALUATE Solving for *k* and inserting the given quantities gives

$$\begin{aligned}\omega&=2\pi f=\sqrt{k/m}\\k&=\left(2\pi f\right)^2m\\&=\left(2\pi\times4.0\times10^{13}\text{ Hz}\right)^2\left(16\times1.66\times10^{-27}\text{ kg}\right)=1.7\times10^3\text{ N/m.}\end{aligned}$$

ASSESS Note that we converted from angular frequency to frequency because the data is given in frequency.

41. **INTERPRET** This problem is about simple harmonic motion of a pendulum. We want to know what its period is when it is treated as a simple pendulum or as a physical pendulum.

DEVELOP Draw a diagram of the situation (see figure below). The periods of a simple pendulum and a physical pendulum are given by

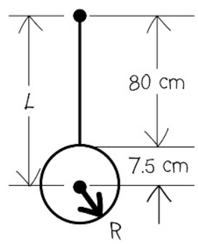
$$T_{\text{simple}} = 2\pi\sqrt{\frac{L}{g}}$$

$$T_{\text{phys}} = 2\pi\sqrt{\frac{I}{mgL}}$$

The rotational inertia I of the physical pendulum can be found using the parallel axis theorem (see Equation 10.17 with $d = L$ and $I_{\text{cm}} = 2MR^2/5$), which gives

$$I = \frac{2}{5}MR^2 + ML^2$$

Use these two expressions to compare the periods of the pendulum in the two cases.



EVALUATE When treated as a simple pendulum, the period is

$$T_{\text{simple}} = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{0.875\text{ m}}{9.81\text{ m/s}^2}} = 1.877\text{ s}$$

On the other hand, if we regard the pendulum as a physical pendulum with rotational inertia $I = \frac{2}{5}mR^2 + mL^2$, then its period becomes

$$T_{\text{phys}} = 2\pi\sqrt{\frac{I}{mgL}} = 2\pi\sqrt{\frac{\frac{2}{5}mR^2 + mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}\sqrt{1+\frac{2R^2}{5L^2}} = T_{\text{simple}}\sqrt{1+\frac{2R^2}{5L^2}}$$

$$= T_{\text{simple}}\sqrt{1+\frac{2}{5}\left(\frac{0.075\text{ m}}{0.875\text{ m}}\right)^2} = (1.00147)T_{\text{simple}} = 1.880\text{ s}$$

The fractional error is

$$\frac{T_{\text{phys}} - T_{\text{simple}}}{T_{\text{phys}}} = 0.147\%.$$

ASSESS The period found by assuming the pendulum to be simple is slightly less than if we treat it as a physical pendulum.

42. **INTERPRET** This problems asks for the resonant frequency of the human eye if we model it as a mass and the muscles that hold it as a spring.

DEVELOP The natural (or resonant) frequency is $\omega_0 = \sqrt{k/m}$.

EVALUATE In terms of oscillations per second, the resonant frequency of the human eye is

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2.5\text{ kN/m}}{7.5\text{ g}}} = 92\text{ Hz}$$

ASSESS You wouldn’t be able to shake your head this fast by yourself, but you might feel this much vibration in an extremely bumpy amusement ride, for example.

43. **INTERPRET** This problem involves simple harmonic motion. We are asked to find the time for which a mass is in contact with a spring given the initial speed of the mass and the parameters of the spring.

DEVELOP The spring is initially in its equilibrium position. The mass will compress the spring, then the spring will extend again to its equilibrium position, at which point the mass will depart to the left because the spring will begin to slow down (because it enters its extension phase). Thus, the mass is in contact with the spring for a half-period, so the time duration will be $T/2$, which we can find using Equation 13.7c. From Equation 13.9, which describes the motion of a mass-spring system, we see that the maximum velocity is related to the amplitude by $v_{\max} = \omega A$, from which we can find the maximum compression A of the spring, given that $v_{\max} = v_0$.

EVALUATE (a) The mass is in contact with the spring for a time t given by

$$t = \frac{T}{2} = \pi\sqrt{\frac{m}{k}}$$

(b) Using the fact that $\omega = 2\pi/T$, the maximum compression is

$$A = \frac{v_0}{\omega} = \frac{v_0 T}{2\pi} = v_0\sqrt{\frac{m}{k}}$$

ASSESS We could use conservation of total mechanical energy to find the maximum compression as well. The initial energy of the mass-spring system is $K = mv_0^2/2$ and the energy when the spring is at maximum compression is completely potential energy (because the velocity of the mass is zero at this point) and is given by $U = kA^2/2$. Equating the two gives

$$\frac{1}{2}mv_0^2 = \frac{1}{2}kA^2$$
$$A = \pm v_0\sqrt{\frac{m}{k}}$$

where the two signs indicate that the maximum displacement from equilibrium of the spring may be an extension or a compression.

44. **INTERPRET** We want to show that $x(t) = A\sin\omega t$ is a solution to the differential equation given in Equation 13.3, which we can do by simply substituting this expression into Equation 13.3 and verifying that the equality holds.

DEVELOP Equation 13.3, $m\,d^2x/dt^2 = -kx,$ describes a system undergoing simple harmonic motion in one dimension. To show that $x(t) = A\sin\omega t$ is a solution to the differential equation, we simply compute the second time derivative of x, and show that the left-hand side still equates with the right-hand side.

EVALUATE By differentiating $x(t) = A\sin\omega t$ with respect to t twice, we obtain

$$\frac{dx}{dt} = \frac{d}{dt}(A\sin\omega t) = \omega A\cos\omega t$$
$$\frac{d^2x}{dt^2} = \frac{d}{dt}(\omega A\cos\omega t) = -\omega^2 A\sin\omega t = -\omega^2 x$$

Substituting into Equation 13.3, we find $m(-\omega^2 x) = -kx,$ which is satisfied if $\omega^2 = k/m.$

ASSESS Equation 13.3 describes a system with periodic motion. Since the cosine function is also periodic, it can be used as a solution to the differential equation.

45. **INTERPRET** This problem involves the simple harmonic motion of a pendulum. We are asked to find the period of one cycle of the pendulum, and from that find the duration of the physics class.

DEVELOP From Equation 13.13, we know that the period is related to the pendulum parameters by

$$T = 2\pi\sqrt{\frac{I}{mgL}}$$

From Table 10.2, we find that the rotational inertia I for a thin rod rotating about its end is $I = ML^2/3$, so

$$T = 2\pi\sqrt{\frac{mL^2}{3mgL}} = 2\pi\sqrt{\frac{L}{3g}}$$

From this, we can find the duration of the class, assuming the student starts his pendulum the instant class starts.

EVALUATE We are told the pendulum completes 5974 cycles, so the class duration is

$$t = (5974)(2\pi)\sqrt{\frac{0.188\text{ m}}{3\left(9.8\text{ m/s}^2\right)}} = 3.0\times10^3\text{ s} = 50\text{ min}$$

ASSESS Adding the 10-minute break between classes to walk to the next class, the 50-min class time allows the classes to start on the hour, which seems quite reasonable.

46. **INTERPRET** In this problem we want to find the period of a pendulum that is inside a rocket that experiences various accelerating conditions. It may be helpful to think of an elevator instead of a rocket in this problem.

DEVELOP Consider a person standing in an elevator that is accelerating upward at the rate a, and apply Newton's second law to that person. The forces acting on the person are the upward normal force from the floor and the downward force due to gravity. Thus, Newton's second law gives us

$$F_{\text{net}} = n - mg = ma$$
$$n = m(g + a) = mg_{\text{eff}}$$

where $g_{\text{eff}} = g + a$ is the effective gravitational field exerted on all occupants of the elevator. Note that the sign of the elevator's acceleration is relative to gravity; for a downward-accelerating elevator, the sign of the acceleration would be negative and we would have $g_{\text{eff}} = g - a$. The pendulum in the accelerating rocket experiences the same effective gravitational acceleration, so we only need to replace g by g_{eff} in the equations of motion for a pendulum. Thus, from Equation 13.15, for small oscillations, we have

$$T = 2\pi\sqrt{\frac{L}{g_{\text{eff}}}}$$

EVALUATE **(a)** For the rocket at rest on its launch pad ($a = 0$), $g_{\text{eff}} = g$, and $T = 2\pi\sqrt{L/g}$.

(b) For $a = g/2$, $g_{\text{eff}} = g + g/2 = 3g/2$, and $T = 2\pi\sqrt{2L/(3g)}$.

(c) For $a = -g/2$, $g_{\text{eff}} = g - g/2 = g/2$, and $T = 2\pi\sqrt{2L/g}$.

(d) For free fall, the rocket accelerates downward at $a = -g$, so $g_{\text{eff}} = g - g = 0$ and $T \to \infty$, which means that the pendulum does not oscillate.

ASSESS That the pendulum does not oscillate in **(d)** is reasonable, because the mass on the pendulum is accelerating downward at the same rate as the rocket ship, so it is motionless with respect to its frame of reference. Therefore, no forces act on it in its frame of reference.

47. **INTERPRET** We will analyze a bacteria protein that undergoes simple harmonic motion.

DEVELOP The peak force occurs when the mass-spring system is maximally displaced from equilibrium: $F_{\text{peak}} = k|x_{\max}|$. The maximum displacement for simple harmonic motion is, by definition, the amplitude: $|x_{\max}| = A$. From this, we can find the spring constant, and from Equation 13.7b, the effective mass is $m = k/(2\pi f)^2$.

EVALUATE **(a)** The spring constant of the dynein-microtubule system is

$$k = \frac{F_{\text{peak}}}{A} = \frac{1.0\text{ pN}}{16\text{ nm}} = 0.0625\text{ pN/nm} = 63\text{ µN/m}$$

(b) Given the frequency of oscillation, the effective mass being oscillated is

$$m = \frac{k}{(2\pi f)^2} = \frac{63\text{ µN/m}}{(2\pi\cdot72\text{ Hz})^2} = 3.1\times10^{-10}\text{ kg}$$

ASSESS The effective mass is about 100 times that of a typical human cell. This seemingly large value may be because the system is moving through a viscous fluid that acts like additional mass.

48. **INTERPRET** The problem involves simple harmonic motion of the mass-spring system that oscillates vertically.
DEVELOP At the highest point, there is no spring force, since the spring is in its unstretched configuration. Therefore, the acceleration is just g (downward). This is also the maximum acceleration during the simple harmonic motion, since a_{max} occurs where the displacement is maximum. Thus, using Equation 13.10, we have $a_{max} = g = \omega^2 A.$ Once ω is known, the period of oscillation can be obtained by using Equation 13.7c.

EVALUATE The peak-to-peak displacement is $2A = 3.0$ cm. Thus, using Equation 13.7c, the period is

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{A}{g}} = 2\pi\sqrt{\frac{0.015\text{ m}}{9.8\text{ m/s}^2}} = 0.25\text{ s}$$

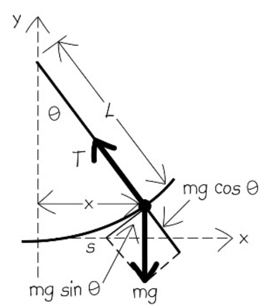
ASSESS Our result confirms that the greater the amplitude, the longer the period of oscillation.

49. **INTERPRET** This problem involves the simple harmonic motion of a pendulum. We are to derive the equations of motion by considering the linear forces and displacements of the bob, rather than the angular torques and displacements. This derivation will involve applying Newton's second law.

DEVELOP Draw a diagram of the situation (see figure below). The tangential component of Newton's second law, for a simple pendulum of mass m and length L, is $m\left(d^2s/dt^2\right) = -mg\sin\theta.$ The radial component guarantees that the motion follows a circular arc. The horizontal displacement is $x = L\sin\theta.$ For small displacements, $x \approx L\theta = s,$ so the equation of motion is approximately

$$m\frac{d^2s}{dt^2} \approx m\frac{d^2x}{dt^2} = -mg\frac{x}{L}$$

$$\frac{d^2x}{dt^2} = -\left(\frac{g}{L}\right)x$$



EVALUATE This equation is the same as Equation 13.3, with $k/m = g/L$. Thus, from Equation 13.7c, the period is $T = 2\pi/\omega = 2\pi\sqrt{m/k} = 2\pi\sqrt{L/g}$.

ASSESS Notice that this derivation assumes small displacements from equilibrium, where $\sin\theta \approx \theta$. For larger displacements, the period derived here is not valid.

50. **INTERPRET** In this problem we want to find the radius of the solid disk such that its vertical oscillation will have the same period as its torsional oscillation.
DEVELOP If the periods are the same, then the angular frequencies must also be the same. The angular frequencies for the vertical and torsional oscillations are given by Equations 13.7a and 13.12:

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{(vertical)}$$

$$\omega = \sqrt{\frac{\kappa}{I}} \qquad \text{(torsional)}$$

EVALUATE From Table 10.2, the rotational inertia of the solid disk is $I = mR^2/2.$ Equating the angular frequencies for vertical and torsional oscillations, we find

$$\frac{k}{m} = \frac{\kappa}{I} = \frac{\kappa}{mR^2/2}$$

$$R = \sqrt{\frac{2\kappa}{k}}$$

ASSESS The torsional constant κ has units of N·m, while the spring constant has units of N/m. Thus, the quantity $\sqrt{\kappa/k}$ has units of meters, as expected for the disk radius. The result makes sense because if we increase the disk radius, then the rotational inertia also increases. To keep the period the same would therefore require a greater torsional constant.

51. **INTERPRET** This problem involves simple harmonic motion of the torsional variety. We are to find the mass of the oscillating beam, given the change in its oscillating frequency when two masses are attached to its extremities.

DEVELOP The frequencies before and after the addition of the steelworkers are related by $f_2 = 0.81 f_1$. These frequencies can be related to the rotational inertias of the beam with and without the steelworkers through Equation 13.12, which gives

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I_1}}$$

$$f_2 = \frac{\omega_2}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I_2}}$$

Since the torsional constant κ does not change, the ratio of these expressions gives $f_2/f_1 = \sqrt{I_1/I_2}$. From Table 10.2, we know that the rotational inertia of a beam about its center is $I_1 = ML^2/12$, and adding two masses at the extremities of the beam changes the rotational inertia to $I_2 = I_1 + 2m(L/2)^2$, so we can solve for the mass M of the beam.

EVALUATE Solving for the mass M gives

$$0.81 = \frac{f_2}{f_1} = \sqrt{\frac{I_1}{I_2}} = \sqrt{\frac{ML^2/12}{ML^2/12 + 2m(L/2)^2}}$$

$$0.81 = \left(1 + \frac{6m}{M}\right)^{-1/2}$$

$$M = \frac{6m}{(0.81)^{-2} - 1} = \frac{6(83\text{ kg})}{(0.81)^{-2} - 1} = 950\text{ kg}$$

ASSESS If each steelworker has a mass that is half that of the beam the change in oscillating frequency would be $f_2/f_1 = 1/\sqrt{4} = 50\%$.

52. **INTERPRET** In this problem, we want to find the mass of the tire valve stem, given the period of oscillation of the bicycle wheel. We can apply the concepts of simple harmonic motion to this problem.

DEVELOP The bicycle wheel may be regarded as a physical pendulum, with rotational inertia $I = MR^2 + mR^2 = (M+m)R^2$ about its central axle, where $M = 600$ g is the mass of the wheel (thin ring) and m is the mass of the valve stem (a circumferential point mass). Because of the valve stem, the center of mass of the combined wheel-and-valve-stem system is not at the axis of rotation, so this system constitutes a pendulum oscillating about the wheel axle. The distance L of the oscillating center of mass from the axle is given by

$$(M+m)L = mR$$

$$L = \frac{mR}{M+m}$$

(this is just Equation 9.2, with origin at the center of the wheel, so $x_1 = 0$ for M, $x_2 = R$ for m, and $x_{\text{cm}} = L$). For small oscillations, the angular frequency is given by Equation 13.13, where the mass in the denominator is the total mass (we'll use $M_\text{T} = M + m$ as the total mass). Knowing the angular frequency or the period allows us to determine the mass m of the valve stem.

EVALUATE From Equations 13.7c and 13.13, the period of oscillation is given by

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{M_T gL}} 2\pi\sqrt{\frac{I}{(M+m)gL}} = 2\pi\sqrt{\frac{(M+m)R^2}{mgR}} = 2\pi\sqrt{\left(\frac{M}{m}+1\right)\frac{R}{g}}$$

Solving for m, we find

$$m = M\left[\frac{g}{R}\left(\frac{T}{2\pi}\right)^2 - 1\right]^{-1} = (600\text{ g})\left[\frac{9.8\text{ m/s}^2}{0.3\text{ m}}\left(\frac{14\text{ s}}{2\pi}\right)^2 - 1\right]^{-1} = 3.7\text{ g}$$

ASSESS A mass of 3.7 g is reasonable for the valve stem. Note that this value is much smaller than the mass of the wheel, as we expect.

53. **INTERPRET** The problem asks us to find several characteristics of a object's trajectory given its equation of motion. The object undergoes simple harmonic motion in both the x- and y-directions, with the motion in each direction being 90° out of phase (i.e., when the displacement is maximum in one direction it is at zero in the other direction).

DEVELOP The position of the object as a function of time is $\vec{r} = A\sin\omega t\hat{i} + A\cos\omega t\hat{j}$. Its position can therefore be found by calculating the magnitude $|\vec{r}|$. The velocity may be found by differentiating the position with respect to time, and the speed is the magnitude of the velocity, $|\vec{v}|$. Finally, the angular speed may be found by dividing the speed by the radius, since this object executes circular motion (i.e., the velocity is always perpendicular to the radial position vector).

EVALUATE **(a)** The object's distance from the origin is $|\vec{r}| = \sqrt{(A\sin\omega t)^2 + (A\cos\omega t)^2} = A$ a constant, so its path is a circle with radius A. **(b)** Differentiating $r(t)$, we find $\vec{v} = d\vec{r}/dt = \omega A\cos\omega t\,\hat{i} - \omega A\sin\omega t\,\hat{j}.$ **(c)** The object's speed is

$$|\vec{v}| = \sqrt{(\omega A\cos\omega t)^2 + (-\omega A\sin\omega t)^2} = \omega A,$$

also a constant. **(d)** From Equation 10.3, the angular speed is $v/r = \omega A/A = \omega$.

ASSESS Note that $\vec{v}\cdot\vec{r} = 0,$ as required for circular motion.

54. **INTERPRET** If an insect's muscles can truly "choose" the natural frequency of its wings, then we should be able to predict what effect a change in wing mass will have on the flapping frequency.

DEVELOP The natural frequency is related to the mass by $\omega = \sqrt{k/m}$.

EVALUATE If the mass is reduced by 15%, that is, $m_f = 0.85\,m_i$, then the final frequency will be

$$\omega_f = \sqrt{\frac{k}{m_f}} = \sqrt{\frac{k}{0.85m_i}} = 1.08\omega_i$$

This indicates that the frequency increases by 8%.

ASSESS The less massive an object is, the faster it will want to oscillate. Our answer is in agreement with that.

55. **INTERPRET** In this question we want to determine an expression for the period of oscillation of a physical pendulum made up of a hollow sphere suspended by a massless string.

DEVELOP In the small-amplitude approximation, the oscillatory motion of a pendulum can be treated as a differential equation in the angular displacement θ. Knowing this, one can arrive at Equation 13.13, $\omega = \sqrt{mgL/I}$, for the angular frequency of the pendulum. For a physical pendulum we want to express its rotational inertia I about a pivot point a distance L away from its center of mass. Since we know the object is a hollow sphere and the string attached to its surface is massless, we can use the expressions from Table 10.2 to find the rotational inertia. Knowing the angular frequency, we can then obtain the rotational period using Equation 13.5.

EVALUATE We are told the string attaches to the surface of the sphere, and has a length equal to its diameter D, which means the distance from the pivot to the center of mass is given by $L = \frac{3}{2}D$. Since the hollow sphere rotates about this pivot point a distance L away, we use the expression from Table 10.2, for its rotational inertia about the center, and the parallel axis theorem to find its effective rotational inertia is given by

$$I = \frac{2}{3}mR^2 + mL^2 = \frac{2}{3}m\left(\frac{D}{2}\right)^2 + m\left(\frac{3}{2}D\right)^2 = \frac{29}{12}mD^2$$

Plugging this into Equation 13.13, we calculate that the angular frequency is equal to

$$\omega = \sqrt{\frac{mgL}{I}} = \sqrt{\frac{mg\left(\frac{3}{2}D\right)}{\left(\frac{29}{12}mD^2\right)}} = \sqrt{\frac{18g}{29D}}$$

Which means the period of oscillation is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{29D}{18g}} = \frac{\pi}{3}\sqrt{\frac{58D}{g}}$$

ASSESS Just like in Example 13.4, we find that the result is independent of the object's mass.

56. **INTERPRET** This problem involves conservation of motion and Newton's second law ($F = ma$). To apply conservation of energy, we need to choose a zero for the potential energy, so we will choose Jane's starting position to have zero potential energy. Apply Newton's second law to find the maximum tension in the vine.

DEVELOP To find the maximum tensile force, apply Newton's second law to Jane as she is swinging. First, make a free-body diagram of her (see figure below, left side). Because Jane is executing circular motion, here acceleration is centripetal and is directed toward the center of the circle (i.e., along the vine). It is given by $a_c = v^2/R$ (see Equation 3.16). Using this result, Newton's second law gives

$$F_{\text{net}} = ma$$

$$T - mg\cos\theta = m\frac{v^2}{R}$$

$$T = m\frac{v^2}{R} + mg\cos\theta$$

Thus, we see that the maximum tension occurs when $\theta = 0$, which is when Jane is at the bottom of her trajectory. To find the value of this maximum tension, we need to calculate her speed at that point, which we can do using conservation of energy. The initial distance between Tarzan and Jane in Fig. 13.12 is $d = 8$ m. Jane's mass is $m = 60$ kg. Draw a sketch of the situation that shows how far down Jane will swing (see figure below, right side). The distance $x = d/2$, and we can use the Pythagorean theorem to find y and thus Δh. This gives

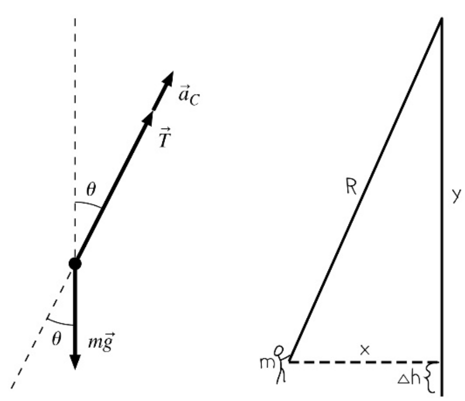
$$R^2 = x^2 + y^2 = x^2 + (R - \Delta h)^2$$

$$\Delta h = R - \sqrt{R^2 - x^2}$$

Because we chose her starting position to have zero potential energy, her initial total energy (potential plus kinetic) is zero. By conservation of total mechanical energy, her final total energy at the bottom of the swing must also be zero, so

$$0 = K_f + U_f = \frac{1}{2}mv^2 - mg\Delta h$$

$$v^2 = 2g\Delta h$$



EVALUATE The maximum tension in the vine occurs at $\theta = 0$, so

$$T_{\max} = m\frac{v^2}{R} + mg = m\frac{2g\Delta h}{R} + mg = mg\left[\frac{2}{R}\left(R - \sqrt{R^2 - (d/2)^2}\right) + 1\right] = mg\left(3 - \sqrt{4 - (d/R)^2}\right)$$

$$= (60\text{ kg})(9.8\text{ m/s}^2)\left[3 - \sqrt{4 - \left(\frac{8.0\text{ m}}{25\text{ m}}\right)^2}\right] = 600\text{ N}$$

Thus, the maximum tension is 600 N (to two significant figures) and it occurs at the bottom of Jane's trajectory.

ASSESS This force is only slightly greater than Jane's weight, since she is moving rather slowly. We also see that $d \le 2R$ for the radical to remain real, which makes sense because $2R$ is the maximum length she can travel horizontally. This extremum will give the maximum T_{max} value, which is $3mg$.

57. **INTERPRET** We are told how the period of the oscillating device varies with the mass of the animal placed in its cage. From this calibration curve, we can find the spring constant and the mass of the cage.

DEVELOP The square of the period is linear with the mass: $T^2 = \left(4\pi^2/k\right)m,$ from Equation 13.7c. Therefore, the slope of the given equation should be equal to $(4\pi^2/k).$ The value of the equation at $m = 0$ should correspond to the mass of the cage:

$$T^2\Big|_{m=0} = \left(4\pi^2/k\right)m_{cage}$$

EVALUATE **(a)** The slope in the given equation is $4.5\ s^2/kg,$ so the spring constant is

$$k = \frac{4\pi^2}{4.5\ s^2/kg} = 8.8\ kg/s^2 = 8.8\ N/m$$

(b) The given equation equals $3.3\ s^2$ for $m = 0,$ so the mass of the cage is

$$m_{cage} = \left(\frac{k}{4\pi^2}\right)T^2\Big|_{m=0} = \left(\frac{8.8\ kg/s^2}{4\pi^2}\right)3.3\ s^2 = 0.74\ kg$$

ASSESS When the cage is empty, the small mass measuring device (SMMD) oscillates with a period of 1.65 s. To lengthen the period by one second (i.e., $T^2 = 7.0\ s^2$), it would take an animal of mass 0.8 kg. This gives us a sense of how sensitive the SMMD is.

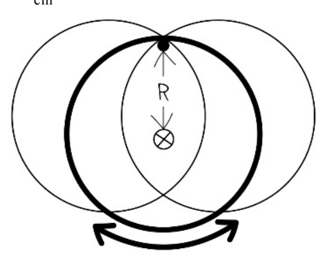
58. **INTERPRET** This problem is about simple harmonic motion of a hoop. We treat it as a physical pendulum, and solve for its period. We will need to invoke the parallel axis theorem to find the rotational inertia of the hoop.

DEVELOP From Equation 13.13, the period of a physical pendulum is

$$T_{phys} = 2\pi\sqrt{\frac{I}{mgL}}$$

where I is the rotational inertia of the hoop and $L = R$ in this case because the center of mass of the hoop is a distance R from the pivot point. To calculate I, we apply the parallel axis theorem, for which we draw a sketch of the situation (see figure below). From Table 10.2, we find the rotational inertia about the center of mass of the hoop is $I_{cm} = mR^2$. The hoop is rotating about a point a distance $d = R$ from its center, so the parallel axis theorem gives the rotational inertia about this point to be

$$I = I_{cm} + md^2 = mR^2 + mR^2 = 2mR^2$$



EVALUATE Inserting the result for the rotational inertia into Equation 13.13, we find the period to be

$$T_{phys} = 2\pi\sqrt{\frac{I}{mgL}} = 2\pi\sqrt{\frac{2mR^2}{mgR}} = 2\pi\sqrt{\frac{2R}{g}}$$

ASSESS The period of our oscillating hoop is the same as a simple pendulum of mass m and length $L = 2R$.

59. **INTERPRET** This problem involves the simple harmonic motion of a mass-spring system, with the caveat that the spring is composed of two springs with different spring constants and that are acting in opposite directions.

DEVELOP When the mass is at its equilibrium position, the springs are either (1) both extended, (2) both compressed, or (3) both in equilibrium because the forces applied by the springs must act opposite to each other (for the first two scenarios), or they must apply zero force (for the last scenario). If the mass is moved by an amount Dx to the right of the equilibrium position, the force of the first spring increases by $k_1\Delta x$ to the left, and the force of the second spring decreases by $k_2\Delta x$ to the right (which is also an increase to the left). Thus, the net force is $(k_1 + k_2)\Delta x$ to the left, which represents a restoring force (opposite to the displacement Δx to the right). The effective spring constant is therefore $k_{\text{eff}} = k_1 + k_2$.

EVALUATE Inserting the effective spring constant into Equation 13.7a, we find the frequency of oscillation to be

$$\omega = \sqrt{(k_1 + k_2)/m}$$

ASSESS If one spring constant is much larger than the other, then it will dominate, as we would expect.

60. **INTERPRET** In this question we want to determine how the spring constants of two different mass-spring systems compare, knowing their total energy and amplitude.

DEVELOP We are told that both systems have the same total energy, meaning $\frac{1}{2}k_{\text{A}}x_{\text{A}}{}^2 = \frac{1}{2}k_{\text{B}}x_{\text{B}}{}^2$. We also know that system A has twice the amplitude of system B, meaning $x_{\text{A}} = 2x_{\text{B}}$. We can plug this into our expression for the system energies and find the relationship between the two spring constants.

EVALUATE Plugging in the expression the systems' amplitudes gives

$$\frac{1}{2}k_{\text{A}}(2x_{\text{B}})^2 = \frac{1}{2}k_{\text{B}}x_{\text{B}}{}^2 \rightarrow k_{\text{B}} = 4k_{\text{A}}$$

ASSESS Since system A oscillates with a larger amplitude than system B, its spring constant is smaller and will impart a smaller restoring force. The larger spring constant of system B can store more energy with a smaller displacement due to the larger restoring force it can impart on the mass.

61. **INTERPRET** This problem involves simple harmonic motion and potential energy. For the potential energy, we will take the lowest point of the pendulum's trajectory as the zero of potential energy. We are asked to express the potential energy of a simple pendulum in the small-amplitude limit.

DEVELOP The potential energy of a simple pendulum or equation is $U = mgh = mgL(1 - \cos\theta)$. Use the small-angle formula to approximate the cosine.

EVALUATE For small angles, $\cos\theta \approx 1 - \theta^2/2$, so the potential energy becomes

$$U \approx mgL\left[1 - \left(1 - \frac{\theta^2}{2}\right)\right] = \frac{1}{2}mgL\theta^2$$

which is proportional to the angular displacement squared, as indicated in the problem statement.

ASSESS You can do a Taylor-series expansion of $\cos\theta$ to verify the small-angle formula for $\cos\theta$.

62. **INTERPRET** In this problem we want to recover the Newton's second law for simple harmonic motion by differentiating the expression for total mechanical energy.

DEVELOP Newton's second law applied to the mass-spring system gives $m\,d^2x/dt^2 = -kx$ (Equation 13.3). To recover this expression from the energy equation, we shall make use of $v = dx/dt$ and $dv/dt = d^2x/dt^2$.

EVALUATE Since E is a constant, we have

$$\begin{aligned}\frac{dE}{dt} = 0 = \frac{d}{dt}\left(\frac{1}{2}kx^2 + \frac{1}{2}mv^2\right) &= kx\frac{dx}{dt} + mv\frac{dv}{dt} = kxv + mv\frac{dv}{dt} \\ &= v\left(kx + m\frac{dv}{dt}\right) \\ &= v\left(kx + m\frac{d^2x}{dt^2}\right)\end{aligned}$$

The above expression implies that the term in brackets is zero, or $m\,d^2x/dt^2 = -kx$, which is Equation 13.3.

ASSESS Because there is no damping term in the equation of motion, the total energy in the system is conserved.

63. **INTERPRET** This problem involves simple harmonic motion, conservation of energy, and rotational inertia. We are asked to derive the equation of motion from energy considerations for the spring-cylinders system shown in Fig. 13.33.

DEVELOP With reference to Equation 10.20 (and the condition $v = \omega R$ for rolling without slipping), the kinetic energy of the system is

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}I_{\text{cm}}\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 = \frac{3}{4}Mv^2$$

The potential energy of the spring is $U = kx^2/2$. Combining these two, the total mechanical energy is

$$E = K + U = \frac{3}{4}Mv^2 + \tfrac{1}{2}kx^2$$
$$= \frac{3}{4}M\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}kx^2$$

$$U = \tfrac{3}{4}M\,(dx/dt)^2 + \tfrac{1}{2}kx^2.$$

where we have used $v = dx/dt$. Differentiate this expression to find an expression involving acceleration ($a = d^2x/dt^2$).

EVALUATE Differentiating, we find:

$$\frac{dE}{dt} = 0 = \frac{3}{4}M\cdot 2\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right) + \frac{1}{2}k\cdot 2x\left(\frac{dx}{dt}\right) \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{2k}{3M}x \equiv -\omega^2 x$$

where we have recognized the prefactor in the last term to be the angular frequency ω. Thus, we have

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

which is the same as the expression found in Problem 62, except that for this problem the angular frequency contains different factors, which reflects the difference in the geometry of the two problems.

ASSESS The energy method is particularly convenient for analyzing small oscillations, since complicated details of the forces can be avoided.

64. **INTERPRET** The motion of the mass sliding on the track is periodic. Given the vertical height y as a function of the position, we can determine its period of oscillation.

DEVELOP The potential energy, relative to the bottom of the track, is

$$U(x) = mgy = mgax^2$$

Comparing this to the expression for the potential energy in simple harmonic motion, $U = kx^2/2$, we identify the "spring constant" as $k = 2mga$.

EVALUATE Using Equation 13.7c, we find the period of the motion is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{2mga}} = \frac{2\pi}{\sqrt{2ga}}$$

ASSESS Indeed, we find that

$$F_x = -\frac{dU}{dx} = -2mgax = m\frac{d^2x}{dt^2}$$

represents simple harmonic motion with $\omega = \sqrt{2ga}$. The y-component of the motion, however, is not simple harmonic motion.

65. **INTERPRET** This problem involves underdamped harmonic oscillation. We are asked to find how many oscillations a mass-spring system will execute before the damping stops the motion.

DEVELOP From Equation 13.17, the time for the amplitude to decay to $1/e$ of its original value is $t = 2m/b$ while the period is $T = 2\pi\sqrt{m/k}$. Dividing the former by the latter gives the number of oscillations that occur before the amplitude is reduced to $1/e$ of its initial value.

EVALUATE Inserting the given quantities, we find

$$N = \frac{t}{T} = \frac{2m}{2\pi b}\sqrt{\frac{k}{m}} = \frac{\sqrt{km}}{\pi b} = \frac{\sqrt{(3.3\ \text{N/m})(0.34\ \text{kg})}}{\pi\left(8.4\times 10^{-3}\ \text{kg/s}\right)} = 40$$

complete oscillations occur.

ASSESS More oscillations will occur for a larger mass, or a stronger spring constant.

66. **INTERPRET** This problem is about an underdamped oscillator. We would like to show that the amplitude of oscillation has a maximum at some driving frequency less than the natural frequency of undamped motion.

DEVELOP The amplitude of the driven oscillation is given by Equation 13.19:

$$A=\frac{F_0/m}{\sqrt{\left(\omega_{\rm d}^2-\omega_0^2\right)^2+b^2\omega_{\rm d}^2/m^2}}$$

From the equation, we see that A is a maximum when the denominator of the right-hand side is a minimum. The condition for this is

$$\frac{d}{d\omega_{\rm d}}\left[\left(\omega_{\rm d}^2-\omega_0\right)^2+\frac{b^2\omega_{\rm d}^2}{m^2}\right]=2\left(\omega_d^2-\omega_0^2\right)(2\omega_{\rm d})+\frac{2\omega_{\rm d}b^2}{m^2}=2\omega_{\rm d}\left[2\left(\omega_{\rm d}^2-\omega_0^2\right)+\frac{b^2}{m^2}\right]=0$$

EVALUATE Since $\omega_{\rm d}\neq 0,$ the driving frequency that satisfies the above condition is

$$\omega_{\rm d}^*=\sqrt{\omega_0^2-b^2/2m^2}$$

Evidently, we have $\omega_{\rm d}^*<\omega_0$.

ASSESS Although the motion is underdamped for $b<2m\omega_0,$ the amplitude A has a maximum in the physical region $(\omega_d>0,\ b>0)$ only for $b<\sqrt{2}m\omega_0,$ and A has sharp resonance-type behavior for $b\ll 2m\omega_0$.

67. **INTERPRET** This problem involves the simple harmonic motion of a vertical mass-spring system. Given the spring parameters, we are to find the amplitude and period of the resulting motion.

DEVELOP The distance from the initial position of the mass on the unstretched spring to the equilibrium position, where the net force is zero, is just the amplitude of the oscillations, since the initial velocity for a dropped mass is zero. Because the net force at the equilibrium position is zero (by Newton's second law, $F_{\rm net}=ma$, the force must be zero because $a=0$ at equilibrium), we have

$$F_{\rm net}=0=kA-mg$$
$$kA=mg$$

from which we can find the amplitude A. To find the period, recall that the force due to gravity does not affect the period, only the equilibrium position. Thus, the period is given by Equation 13.7c.

EVALUATE **(a)** Solving the expression above for A gives
$A=mg/k=(0.55\text{ kg})\left(9.8\text{ m/s}^2\right)/(74\text{ N/m})=7.3\text{ cm}.$

(b) The period is $T=2\pi\sqrt{m/k}=2\pi\sqrt{(0.55\text{ kg})/(74\text{ N/m})}=0.54\text{ s}.$

ASSESS For part (a), we can also consider dropping the mass from the unstretched position (zero spring force), so the initial acceleration is at its maximum magnitude, which is just g, so $a_{\rm max}=\text{g}=\omega^2A=k/(mA),$ or $A=mg/k,$ as before.

68. **INTERPRET** The problem involves a physical pendulum—a meter stick suspended from a small hole at the 30-cm mark. We want to calculate its period of oscillation. Because the meter stick is not oscillating about its center of mass, we will need to invoke the parallel-axis theorem to find the rotational inertia of the meter stick.

DEVELOP The period of a physical pendulum can be obtained from Equation 13.13:

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{I}{mgL}}$$

In this problem, the meter stick $(l=1.0\text{ m})$ is the physical pendulum whose center of mass is at the 50-cm mark, or $L=50-30=20\text{ cm}=l/5\text{ cm}$ below the pivot point. From Table 10.2, we find that the rotational inertia of a stick about its center is $I_{\rm cm}=Ml^2/12,$ and we apply the parallel-axis theorem to find the rotational inertia about an axis $l/5=20\text{ cm}$ from the center of mass:

$$I=I_{\rm cm}+md^2=\frac{1}{12}ml^2+m\left(\frac{l}{5}\right)^2=\frac{37}{300}ml^2$$

EVALUATE Inserting the rotational inertia into the expression of the period gives

$$T = 2\pi\sqrt{\frac{I}{mgL}} = 2\pi\sqrt{\frac{\frac{37}{300}ml^2}{mg\left(\frac{l}{5}\right)}} = 2\pi\sqrt{\frac{37l}{60}} = 2\pi\sqrt{\frac{37(1\text{ m})}{60\left(9.8\text{ m/s}^2\right)}} = 1.58\text{ s}$$

ASSESS The result can be compared with that obtained in Exercise 23. Note that we were careful to distinguish the total length *l* of the rod from the effective length *L* of the physical pendulum (the distance from the pivot to the center of mass).

69. **INTERPRET** We're given a general equation for the potential energy of a particle and asked to find the frequency of its simple harmonic motion.

DEVELOP Recall that the force on a particle is related to the potential energy through Equation 7.8: $F = -dU/dx.$

EVALUATE The force on the particle as a function of distance is

$$F = -\frac{d}{dx}\left[ax^2\right] = -2ax$$

This has the form of a restoring force, as in Equation 13.2 with $k = 2a.$ So the particle undergoes simple harmonic motion with a frequency given by Equation 13.7b:

$$f = \frac{1}{2\pi}\sqrt{\frac{2a}{m}}$$

ASSESS What this tells us is that we have simple harmonic motion whenever the potential energy is proportional to the distance squared.

70. **INTERPRET** This problem involves the simple harmonic motion of a torsional pendulum. The pendulum consists of a rod with two identical (unknown) masses attached to the extremities, so we will need to find the rotational inertia of the ensemble. We are given the torsional constant of the wire and are asked to find the mass of the identical masses.

DEVELOP The period of a torsional pendulum is given by Equation 13.12, $\omega = 2\pi/T = \sqrt{\kappa/I}.$ The rotational inertia of the rod and two masses about an axis perpendicular to the rod and through its center is

$$I = \frac{1}{12}ML^2 + 2m\left(\frac{L}{2}\right)^2$$

which allows us to solve for the mass *m*.

EVALUATE Inserting the expression for the rotational inertia into the expression for the period and solving for the mass *m* gives

$$\frac{2\pi}{T} = \sqrt{\frac{\kappa}{I}} = \sqrt{\frac{\kappa}{ML^2/12 + mL^2/2}}$$

$$m = \frac{1}{2}\left[\left(\frac{T}{\pi L}\right)^2\kappa - \frac{M}{3}\right] = \frac{1}{2}\left[\left(\frac{5.6}{\pi(1.5\text{ m})}\right)^2(0.63\text{ N}\cdot\text{m}) - \frac{0.45\text{ kg}}{3}\right] = 0.37\text{ kg}$$

ASSESS From the final expression for the mass *m*, we see that a larger mass would correspond to a larger period, a smaller mass *M*, or a smaller rod length.

71. **INTERPRET** In this problem, we are given two mass-spring systems with the same mass but different frequencies, and we want to compare their energies and maximum accelerations.

DEVELOP As shown in Section 13.5, the energy of a mass-spring system is $E = m\omega^2A^2/2.$ Also, from Equation 13.10, the maximum acceleration is $a_{max} = \omega^2 A.$ Use the fact that $\omega_1 = 3\omega_2$ to find the ratio of the energies and the ratio of the accelerations.

EVALUATE **(a)** If *m* and *A* are the same but $\omega_1 = 3\omega_2$, we have

$$\frac{E_1}{E_2} = \frac{m\omega_1^2 A/2}{m\omega_2^2 A/2} = \left(\frac{\omega_1}{\omega_2}\right)^2 = (3)^2 = 9$$

$$E_1 = 9E_2$$

(b) Comparing the maximum accelerations of the two systems, we find

$$\frac{a_{\max,1}}{a_{\max,2}} = \frac{\omega_1^2 A}{\omega_2^2 A} = \left(\frac{\omega_1}{\omega_2}\right)^2 = (3)^2 = 9$$

$$a_{\max,1} = 9a_{\max,2}$$

Assess Both the energy and the maximum acceleration of the mass-spring system increase with ω^2. Tripling the frequency means E and $a_{\max}$ are multiplied by a factor of 9.

72. **Interpret** In this problem, we are given two mass-spring systems with the same mass and the same total energies, and we want to compare their frequencies and their maximum accelerations.

Develop From Fig. 13.19, we see that when the kinetic energy is maximum, the potential energy is minimum (which we set to zero without loss of generality), so we can write $E = K_{\max}$. From Equation 13.9, we see that

$$v_{\max}^2 = \omega^2 A^2$$

so the total energy may be expressed as

$$E = K_{\max} = \frac{1}{2}mv_{\max}^2 = \frac{1}{2}m\omega^2 A^2$$

Given that $A_1 = 4A_2$, we can find the ratio of their frequencies. From Equation 13.10, we see that the maximum acceleration is $a_{\max=} - \omega^2 A$; so, given that $A_1 = 4A_2$, we can find the ratio of their maximum accelerations.

Evaluate **(a)** If $A_1 = 7A_2$, then

$$E = \frac{1}{2}m\omega_1^2 A_1^2 = \frac{1}{2}m\omega_2^2 A_2^2$$

$$\frac{\omega_1}{\omega_2} = \frac{A_2}{A_1} = \frac{1}{4}$$

so $4\omega_1 = \omega_2$, or equivalently, $4f_1 = f_2$.

(b) The maximum accelerations are related by

$$a_{\max,1} = -\omega_1^2 A_1 = -\left(\frac{\omega_2}{4}\right)^2 (4A_2) = \frac{1}{4}\left(-\omega_2^2 A_2\right) = \frac{1}{4}a_{\max,2}$$

so $4a_{\max,1} = a_{\max,2}$.

Assess Thus, for the same total energy, if we multiply the amplitude by four, the frequency and the maximum acceleration are reduced by a factor of four. In other words, for the same energy, the frequency and acceleration are inversely proportional to the amplitude of the oscillation.

73. **Interpret** In this question we want to show that the equation of motion of an oscillating system satisfies Equation 13.18 (Newton's second law in the presence of a driving force), with an amplitude given by Equation 13.9.

Develop We want to directly substitute $x = A\cos(\omega_d t + \varphi)$ into Equation 13.8:

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_d \cos\omega_d t$$

and show that it holds when the amplitude is given by

$$A(\omega_d) = \frac{F_d}{m\sqrt{\left(\omega_d^{\,2} - \omega_0^{\,2}\right)^2 + b^2\omega_d^{\,2}/m^2}}$$

Evaluate We begin by taking the first and second derivatives of x with respect to time, plugging them into Equation 13.18, and simplifying.

$$\frac{dx}{dt} = -\omega_d A \sin(\omega_d t + \varphi); \; \frac{d^2x}{dt^2} = -\omega_d^2 A \cos(\omega_d t + \varphi)$$

$$m\left(-\omega_d^2 A\cos(\omega_d t + \varphi)\right) = -k\left(A\cos(\omega_d t + \varphi)\right) - b\left(-\omega_d A\sin(\omega_d t + \varphi)\right) + F_d \cos\omega_d t$$

$$A[m\left(\omega_0^2 - \omega_d^2\right)\cos(\omega_d t + \varphi) - (b\omega_d)\sin(\omega_d t + \varphi)] = F_d \cos\omega_d t$$

Looking at the left-hand side only, we can apply the angle-sum identity found in Appendix A to rewrite it as

$$A\left[m\left(\omega_0^2 - \omega_d^2\right)\left(\cos(\omega_d t)\cos(\varphi) - \sin(\omega_d t)\sin(\varphi)\right) - (b\omega_d)\left(\sin(\omega_d t)\cos(\varphi) + \cos(\omega_d t)\sin(\varphi)\right)\right]$$

$$[m\left(\omega_0^2 - \omega_d^2\right)\cos(\varphi) - (b\omega_d)\sin(\varphi)]\, A\cos(\omega_d t) \; - \; [m\left(\omega_0^2 - \omega_d^2\right)\sin(\varphi) + (b\omega_d)\cos(\varphi)]\, A\sin(\omega_d t)$$

Which if we then plug back into Equation 13.18, we can see that

$$[m\left(\omega_0^2 - \omega_d^2\right)\cos(\varphi) - (b\omega_d)\sin(\varphi) - F_d / A]\cos(\omega_d t) \; = \; [m\left(\omega_0^2 - \omega_d^2\right)\sin(\varphi) + (b\omega_d)\cos(\varphi)]\sin(\omega_d t)$$

Which will only hold if the coefficients of $\cos(\omega_d t)$ and $\sin(\omega_d t)$ are both equal to zero. That is:

$$m\left(\omega_0^2 - \omega_d^2\right)\sin(\varphi) + (b\omega_d)\cos(\varphi) = 0$$

$$m\left(\omega_0^2 - \omega_d^2\right)\cos(\varphi) - (b\omega_d)\sin(\varphi) = \frac{F_d}{A}$$

From the first equation we find that

$$\tan(\varphi) = \frac{-b\omega_d}{m\left(\omega_0^2 - \omega_d^2\right)}$$

Which allows us to express $\cos(\varphi)$ and $\sin(\varphi)$, using the Pythagorean identities: $\sec^2(\varphi) = 1 + \tan^2(\varphi)$ and $\csc^2(\varphi) = 1 + \cot^2(\varphi)$, respectively. Using these we find that:

$$\cos(\varphi) = \frac{1}{\sqrt{1 + \tan^2(\varphi)}} = \frac{m\left(\omega_0^2 - \omega_d^2\right)}{\sqrt{m^2\left(\omega_0^2 - \omega_d^2\right)^2 + b^2\omega_d^2}}$$

$$\sin(\varphi) = \frac{\tan(\varphi)}{\sqrt{1 + \tan^2(\varphi)}} = \frac{-b\omega_d}{\sqrt{m^2\left(\omega_0^2 - \omega_d^2\right)^2 + b^2\omega_d^2}}$$

Plugging these back into our second equation, we arrive at an expression for the amplitude:

$$\frac{m^2\left(\omega_0^2 - \omega_d^2\right)^2 + b^2\omega_d^2}{\sqrt{m^2\left(\omega_0^2 - \omega_d^2\right)^2 + b^2\omega_d^2}} = \frac{F_d}{A}$$

$$A = \frac{F_d}{\sqrt{m^2\left(\omega_0^2 - \omega_d^2\right)^2 + b^2\omega_d^2}}$$

After recognizing that the order of frequencies in the first term of the denominator doesn't affect the result since their difference is squared, and factoring the mass, we arrive at Equation 13.19

$$A(\omega_d) = \frac{F_d}{m\sqrt{\left(\omega_d^2 - \omega_0^2\right)^2 + b^2\omega_d^2 / m^2}}$$

ASSESS In the presence of a driving force, the amplitude will depend on various quantities describing the system, including its mass. When the system is driven at the same frequency as its natural frequency (at resonance), this expression will simplify to: $A(\omega_d = \omega_0) = F_d / b\omega_d$, where there no longer is any mass dependence.

74. **INTERPRET** This problem involves simple harmonic motion and Newton's second law $(F = ma)$, which we can apply to find the coefficient of static friction, given that the upper block begins to slip for an oscillation amplitude of 35 cm.

DEVELOP Make a sketch of the situation, showing the friction force and the direction of acceleration (see figure below). If the contact surface is horizontal, it is the frictional force that accelerates the upper block. Therefore, applying Newton's second law in the horizontal direction to the upper block gives

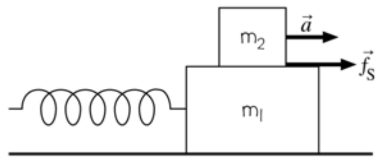
$$m_2 a(t) = f_s \le \mu_s n = \mu_s m_2 g$$

$$a(t) \le \mu_s g$$

where we have used $n = mg$ (from applying Newton's second law in the vertical direction). In simple harmonic motion, $a_{\max} = \omega^2 A$, so when the upper block begins to slip, $\omega^2 A = \mu_s g$. Using Equation 3.7a to eliminate ω gives

$$\frac{kA}{m} = \mu_s g$$

which allows us to solve for μ_s.



EVALUATE Solving for μ_s and inserting the given values gives

$$\mu_s = \frac{kA}{mg} = \frac{(8.7 \text{ N/m})(0.35 \text{ m})}{(0.60 \text{ kg})(9.8 \text{ m/s}^2)} = 0.52$$

ASSESS From the expression for μ_s, we see that larger-amplitude oscillations require a proportionally greater coefficient of static friction, as expected.

75. **INTERPRET** The rolling ball on the track executes simple harmonic motion. Given the vertical height y as a function of the position, we can determine its period of oscillation by applying the principle of conservation of total mechanical energy. To do so, we choose the bottom of the track to have zero potential energy.

DEVELOP The potential energy of the ball, relative to the bottom of the track, is

$$U(x) = mgy = mgax^2$$

and its kinetic energy from rolling (without slipping) is

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I_{\text{cm}}\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)^2 = \frac{7}{10}mv^2$$

where we have used $I_{\text{cm}} = 2mR^2/5$ from Table 10.2. Since the total mechanical energy, $E = U + K$ is constant, we have

$$0 = \frac{dE}{dt} = \frac{d}{dt}\left(mgax^2 + \frac{7}{10}mv^2\right) = 2mgax\frac{dx}{dt} + \frac{7}{5}mv\frac{dv}{dt}$$

If we assume that $v = \sqrt{(dx/dt)^2 + (dy/dt)^2} \approx dx/dt$ (i.e., the vertical displacement is small), then the above expression can be simplified to

$$0 = 2mgax\frac{dx}{dt} + \frac{7}{5}mv\frac{dv}{dt} = 2mgaxv + \frac{7}{5}mv\frac{d^2x}{dt^2} = mv\left(2gax + \frac{7}{5}\frac{d^2x}{dt^2}\right)$$

or

$$\frac{d^2x}{dt^2} = -\frac{10}{7}gax$$

This expression allows us to solve for the angular frequency, and hence the period.

EVALUATE Comparing the above expression with Equation 13.3 and with the help of Equation 13.7a, we see that the angular frequency of the oscillation is

$$\omega = \sqrt{\frac{10ga}{7}}$$

Therefore, the period is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{7}{10ga}}$.

ASSESS In Problem 64, the period of the sliding point mass is found to be $2\pi/\sqrt{2ga}$. Therefore, we see that the period for the rolling ball is longer, which is expected because increasing rotational inertia increases the period. Checking the units of the expression for T gives

$$\sqrt{\frac{1}{(\mathrm{m/s^2})(\mathrm{m^{-1}})}} = \mathrm{s}$$

as expected for a period.

76. **INTERPRET** In this question we want to explore what would happen if an object were dropped down a hole through Earth's center, connecting one side to the other.

DEVELOP We want to express Newton's second law for the given acceleration $g(r)$ experienced by the dropped object, and from it develop an equation analogous to Equation 13.3: $m\frac{d^2x}{dt^2} = -kx$. Once we have this expression, we can determine the analogous angular frequency and from it determine the period of simple harmonic motion.

EVALUATE Writing Newton's second law for the given acceleration we find

$$F = -mg(r) \rightarrow \; m\frac{d^2r}{dt^2} = -mg_0\left(\frac{r}{R_\mathrm{E}}\right)$$

$$\frac{d^2r}{dt^2} = -\frac{g_0}{R_\mathrm{E}}r$$

Where we have chosen the positive radial direction (away from the center of the Earth) to be positive, making the accelerating force negative and more massive the further away from the center the object is. The resultant equation is that of simple harmonic motion, and using the same treatment as in Section 13.2, we arrive at an analogous expression for the angular frequency $\omega = \sqrt{g_0 / R_\mathrm{E}}$. Thus, the period of simple harmonic motion is given by

$$T = 2\pi\sqrt{\frac{R_\mathrm{E}}{g_0}}$$

Substituting in the gravitational constant, $g_0 = 9.8\ \mathrm{m/s^2}$, found at Earth's surface and the radius of the Earth, $R_\mathrm{E} = 6.37\times10^6\ \mathrm{m}$, found in Appendix E we obtain a period of $T = 5{,}063\,\mathrm{s} = 84.4\ \mathrm{min}$. This is the same as the orbital period found with an orbital radius exactly equal to R_E , and close to the period for circular low Earth orbit which was found in Example 8.2.

ASSESS Much like we saw in Example 8.2, this result is independent of the object's mass, and instead relies on the effective gravitational force applied by the much more massive planet Earth.

77. **INTERPRET** This problem involves the simple harmonic motion of a physical pendulum. We are asked to find the pivot point that will result in the minimum period for the pendulum. Because the pendulum will not be pivoting about its center of mass, we will invoke the parallel axis theorem to find the rotational inertia of the pendulum about its new pivot point.

DEVELOP From Equation 13.13, we see that the period of a physical pendulum is

$$T = 2\pi\sqrt{\frac{I}{MgL}}$$

(see also Example 13.4). To find the rotational inertia, apply the parallel axis theorem, which gives

$$I = I_\mathrm{cm} + Md^2 = \frac{MR^2}{2} + ML^2$$

where $I_\mathrm{cm} = MR^2/2$ is found in Table 10.2, and $L = d$ is the distance from the center of mass to the new pivot point. Inserting this into the expression for the period gives

$$T = 2\pi\sqrt{\frac{\frac{1}{2}MR^2 + ML^2}{MgL}} = 2\pi\sqrt{\frac{L}{g} + \frac{R^2}{2gL}}$$

This will be a minimum when its derivative with respect to L is zero, or

$$\frac{d}{dL}\left(\frac{L}{g} + \frac{R^2}{2gL}\right) = 0$$

$$1 - \frac{R^2}{2L_{\min}^2} = 0$$

which we can solve to find $L_{\min}$.

EVALUATE Solving for $L_{\min}$, we find $L_{\min} = R/\sqrt{2}$.

ASSESS The fact that this represents a minimum can be seen either by calculating $d^2T/dL^2 > 0$, or by noting that $T \to \infty$ for $L \to 0$ and $L \to \infty$, and so has a minimum between.

78. **INTERPRET** In this question we want to estimate the period of oscillation of a variable star, given the expression of its simple harmonic motion.

DEVELOP We are told the simple harmonic motion of a variable star's radius can be expressed by

$$\frac{d^2(\delta R)}{dt^2} = -\frac{GM}{R_0^3}\delta R$$

Meaning its angular frequency is equal to $\sqrt{GM/R_0^3}$, and thus its period of oscillation is given by

$$T = 2\pi\sqrt{\frac{R_0^3}{GM}}$$

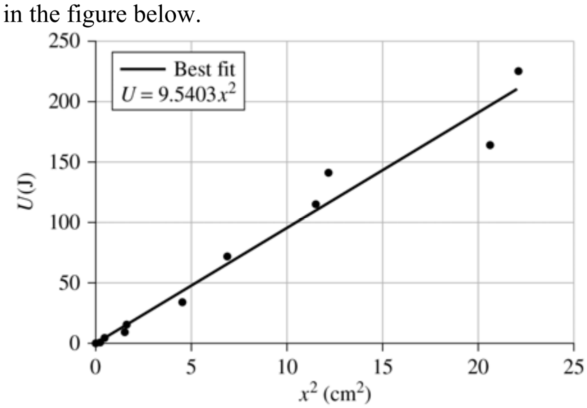
EVALUATE We are told the mass and radius of Delta Cephei are, respectively, 5 and 50 times those of the sun. Plugging in the values found in Appendix E for the Sun's mass, $M_S = 1.99 \times 10^{30}$ kg, and the Sun's radius, $R_S = 6.96 \times 10^8$ m, we find the period is approximately equal to

$$T = 2\pi\sqrt{\frac{(5R_S)^3}{G(50M_S)}} \cong 20\,\text{days}$$

ASSESS This answer differs from the actual period because of over oversimplified physics and because changes in the star's radius are too large for the assumption of a linear restoring force.

79. **INTERPRET** We are asked to compute an equation relating the potential energy of a steel beam to the square of the beam deflection. From this, we can estimate the resonant frequency of the beam.

DEVELOP In a spreadsheet or other software, we enter the values of x and then square them. We then plot the values of U vs. x^2, as shown in the figure below.



EVALUATE The best-fit line through the data points has a slope of 9.54, with units of J/cm^2. For an elastic force, the potential energy is related to the displacement by $U = \frac{1}{2}kx^2$ (Equation 7.4), so

$$k = 2\left(9.54\ \text{J}/\text{cm}^2\right) = 190.8\ \text{kN}/\text{m}$$

For the given mass, the resonant (natural) frequency is $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{190.8\ \text{kN}/\text{m}}{3320\ \text{kg}}} = 7.58\ \text{s}^{-1}$.

This can be expressed in terms of the ordinary frequency, $f = \omega/2\pi$, as 1.21 Hz.

ASSESS If you tapped on the beam, its dominant vibration mode would have a frequency of about 1 Hz (from $f = \omega/2\pi$). This is too low for humans to hear (our ears have an audible range from 20 Hz to 20 kHz). But we may hear some of the higher harmonics, which we will learn about in Chapter 14.

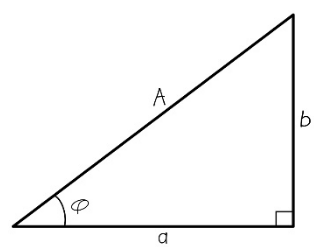
80. **INTERPRET** We are to show that $x(t) = a\cos(\omega t) - b\sin(\omega t)$ represents simple harmonic motion, or that it is equivalent to $x(t) = A\cos(\omega t + \phi)$, where $A = \sqrt{a^2 + b^2}$ and $\phi = \text{atan}(b/a)$. To do this, we use trigonometric identities.

DEVELOP Use the trigonometric identity $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, using $x(t) = A\cos(\omega t + \phi)$ as a starting point.

EVALUATE

$$\begin{aligned} x(t) = A\cos(\omega t + \phi) &= A(\cos\omega t\cos\phi - \sin\omega t\sin\phi) \\ &= (A\cos\phi)\cos\omega t - (A\sin\phi)\sin\omega t \end{aligned}$$

Now we draw an arbitrary right triangle, as shown in the figure below. We can see from the figure that $a = A\cos\phi$, $b = A\sin\phi$, $A = \sqrt{a^2 + b^2}$, and $\tan\phi = b/a$. Inserting these results into the expression above for $x(t)$ gives $x(t) = a\cos(\omega t) - b\sin(\omega t)$.



ASSESS This is as much a trigonometry problem as a physics problem—it allows us to see that simple harmonic motion can be expressed in many ways. You can also demonstrate that $x(t) = a\cos(\omega t) - b\sin(\omega t)$ represents simple harmonic motion by substituting it into Equation 13.3, which gives

$$m\frac{d^2}{dt^2}\left[a\cos(\omega t) - b\sin(\omega t)\right] = -\omega^2 m\left[a\cos(\omega t) - b\sin(\omega t)\right] = -\omega^2 m x(t)$$

which satisfies Equation 13.3 provided $\omega = \sqrt{k/m}$.

81. **INTERPRET** We use torsional oscillations of a bird feeder to determine the mass of the birds on the feeder. We will assume that the birds are point masses for this problem: knowing the rotational inertia of the feeder, the period, and the spring constant, we find the initial rotational inertia contributed by the birds, and thus the birds' masses.

DEVELOP We use the torsional constant $\kappa = 5.00\ \text{N}\cdot\text{m/rad}$ of the suspension wire, and $\omega = \sqrt{\kappa/I_{\text{tot}}}$. The period of oscillation is $f = \omega/(2\pi) = 2.6\ \text{Hz}$. From Table 10.2, the rotational inertia of the disk is $I = MR^2/2$, where M = 0.34 kg and R = 0.25 m. The rotational inertia of the birds, assuming they are approximately point masses, is $I_\text{b} = 2mR^2$. We will solve $\omega = \sqrt{\kappa/I_{\text{tot}}}$ for I_b and thus find m.

EVALUATE

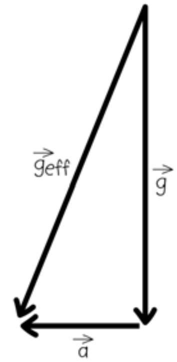
$$\omega = \sqrt{\frac{\kappa}{I_\text{b} + I}} = \sqrt{\frac{\kappa}{2mR^2 + \frac{1}{2}MR^2}}$$

$$m = \frac{\kappa}{2R^2\omega^2} - \frac{M}{4} = \frac{\kappa}{8\pi^2 R^2 f^2} - \frac{M}{4} = \frac{5.00\ \text{N}\cdot\text{m}/\text{rad}}{8\pi^2(0.25\ \text{m})^2(2.6\ \text{s}^{-1})^2} - \frac{0.34\ \text{kg}}{4} = 65\ \text{g}$$

ASSESS This is a reasonable mass for a songbird.

82. **INTERPRET** A pendulum at rest on Earth has a given frequency. The frequency becomes higher when accelerating. We use a vector sum to find the effective value of g in the simple pendulum equation and calculate the acceleration. We will use the simple pendulum approximation.

DEVELOP Start with a sketch, as shown in the figure below.



From the ratio of the initial period $T = 2\pi\sqrt{L/g_{\text{eff}}} = \dfrac{60 \text{ s}}{93 \text{ cycles}} = 0.645 \text{ s}$ and the final period $T' = 2\pi\sqrt{L/g_{\text{eff}}} = \dfrac{60 \text{ s}}{93 \text{ cycles}} = 0.645 \text{ s},$ we can calculate g_{eff}. Once we know g_{eff}, we can use $g_{\text{eff}}^2 = g^2 + a^2$ to find a.

EVALUATE Taking the ratio of the periods gives

$$\frac{T}{T'} = \sqrt{\frac{g_{\text{eff}}}{g}} \quad \Rightarrow \quad g_{\text{eff}} = g\left(\frac{T}{T'}\right)^2$$

so

$$a = \sqrt{g_{\text{eff}}^2 - g^2} = \sqrt{g^2\left(\frac{T}{T'}\right)^4 - g^2} = g\sqrt{\left(\frac{T}{T'}\right)^4 - 1}$$

$$= \left(9.8 \text{ m/s}^2\right)\sqrt{\left(\frac{0.667 \text{ s}}{0.645 \text{ s}}\right)^4 - 1} = 3.7 \text{ m/s}^2$$

ASSESS It turns out that our use of ratios eliminates the need to know much about the physical setup. In fact, we get the same result even if we use the equation for the physical pendulum, rather than that for the simple pendulum.

83. **INTERPRET** We use energy methods to relate the period of a twisting swing to the rotational inertia I of the child on the swing.

DEVELOP For a torsional spring, the potential energy is $U(\theta) = \kappa\theta^2/2$. Differentiating this twice with respect to θ gives

$$\frac{d}{d\theta}U(\theta) = \kappa\theta$$

$$\frac{d^2}{d\theta^2}U(\theta) = \kappa = I\omega^2$$

where the last equality is from Equation 13.12. Solving this for ω gives

$$\omega = \sqrt{\frac{1}{I}\frac{d^2U(\theta)}{d\theta^2}}$$

The child rises a height $h = a\theta^2/(2\pi)^2$, where $a \equiv 0.01 \text{ m}$, so we can express her potential energy as

$$U(\theta) = mgh = \frac{mga\theta^2}{4\pi^2}$$

The period of her torsional oscillation is $T = 18.4 \text{s} = 2\pi/\omega$ and the mass of the child and swing is m = 32.6 kg.

EVALUATE Inserting the expression for potential energy into the expression for ω and performing the differentiation yields

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{1}{I}\frac{d^2U(\theta)}{d\theta^2}}$$

$$\frac{1}{I}\frac{d^2}{d\theta^2}\left[\frac{mg\theta^2 a}{4\pi^2}\right] = \frac{4\pi^2}{T^2}$$

$$\frac{1}{I}\frac{mga}{2\pi^2} = \frac{4\pi^2}{T^2}$$

$$I = \frac{mgaT^2}{8\pi^4} = \frac{(32.6\,\text{kg})(9.8\,\text{m/s}^2)(0.01\,\text{m})(18.4\,\text{s})^2}{8\pi^4} = 1.39\,\text{kg}\cdot\text{m}^2$$

ASSESS The problem assumes that the rise is proportional to θ^2, but an analysis of the geometry of the twisted ropes shows that this is not actually the case.

84. **INTERPRET** In this question we want to determine the change in period of oscillation of a physical pendulum when its distance from the pivot point changes by a small amount.

DEVELOP In the small amplitude approximation, the oscillatory motion of a pendulum can be treated as a differential equation in the angular displacement θ. Knowing this, one can arrive at Equation 13.13, $\omega = \sqrt{mgL/I}$, for the angular frequency of the pendulum. For a physical pendulum we want to express its rotational inertia I about a pivot point a distance L away from its center of mass. Since we know the object is a solid disk and the rod upon which it's mounted is massless, we can use the expressions from Table 10.2 to find the rotational inertia. Knowing the angular frequency, we can then obtain the rotational period using Equation 13.5, and determine the difference when the distance from the pivot point to the center of mass decreases by an amount h.

EVALUATE From the figure we can see that before the nut is moved upward the disk lies at the bottom of the rod of length L, which means the distance from the pivot to the center of mass is given by $L_{\text{cm}} = L - \frac{1}{2}D$, where $L = 0.3\,\text{m}$ and $D = 0.0635\,\text{m}$. After the nut has been moved upward, the distance from the pivot to the center of mass then changes to $L_{\text{cm}} = L - \frac{1}{2}D - h$, where $h = 0.001\,\text{m}$. Since the disk rotates about this pivot point a distance L_{cm} away, we use the expression from Table 10.2, for its rotational inertia about the center, and the parallel axis theorem to find its effective rotational inertia is given by

$$I = \frac{1}{2}m\left(\frac{D}{2}\right)^2 + mL_{\text{cm}}^{\ 2}$$

Plugging this into Equation 13.13, we calculate find the angular frequency is equal to

$$\omega = \sqrt{\frac{mgL_{\text{cm}}}{I}} = \sqrt{\frac{mgL_{\text{cm}}}{\frac{1}{2}m\left(\frac{D}{2}\right)^2 + mL_{\text{cm}}^{\ 2}}} = \sqrt{\frac{8gL_{\text{cm}}}{D^2 + 8L_{\text{cm}}^{\ 2}}}$$

Which means the period of oscillation is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{D^2 + 8L_{\text{cm}}^{\ 2}}{8gL_{\text{cm}}}}$$

We calculate these for the two lengths L_{cm} obtained before and after the nut has moved upward a distance h, and take their difference to obtain: $T_{\text{before}} - T_{\text{after}} = -0.002\,\text{s}$. Meaning after the nut has moved upward 1 mm the period has decreased by 20 ms.

ASSESS Although the expression found for this physical pendulum is not exactly the same as the one found in Example 13.4, we can see that in both cases decreasing the pendulum's length will make the period of oscillation shorter.

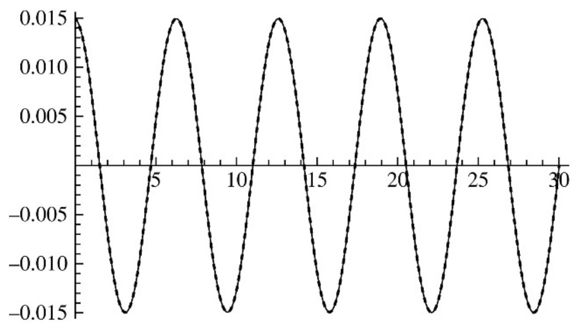
85. **INTERPRET** In this problem we explore the phenomena of nonlinear pendulum, where the amplitude of oscillation need not be small.

DEVELOP The general expression for Newton's law for a pendulum is $Id^2\theta/dt^2 = -mgL\sin\theta$. If the amplitude of the motion is small, then $\sin\theta \approx \theta$, and the equation simplifies to $d^2\theta/dt^2 = -\omega^2\theta$, where $\omega = \sqrt{mgL/I}$. For a simple pendulum, $I = mL^2$. The kinetic energy of the pendulum is $K = \frac{1}{2}I(d\theta/dt)^2 = \frac{1}{2}mL^2(d\theta/dt)^2$, and its potential energy is $U(\theta) = mgL(1-\cos\theta)$. When $\theta = \theta_{\max}$, $U_{\max} = mgL(1-\cos\theta_{\max})$. By energy conservation, at the bottom of the swing, we have $K_{\max} = \frac{1}{2}mv_{\max}^2 = mgL(1-\cos\theta_{\max})$, which implies $v_{\max} = \sqrt{2gL(1-\cos\theta_{\max})}$. In the extreme case where $\theta_{\max} = 180°$ (pendulum completely upside down), $U_{\max} = 2mgL$.

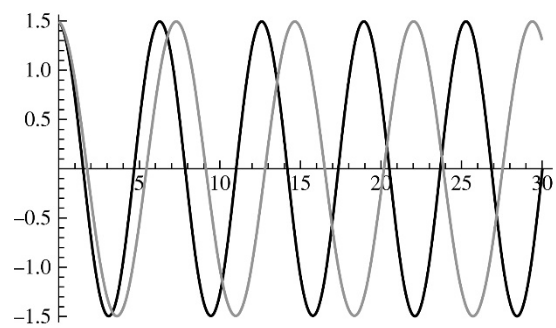
EVALUATE **(a)** Without small-amplitude approximation, the equation reads

$$d^2\theta/dt^2 = -(mgL/I)\sin\theta = -(g/L)\sin\theta$$

(b) (i) The case where $K_0 \ll U_{\max}$ implies that $v \ll v_{\max}$ or $\theta \ll \theta_{\max}$. In this case, the small-amplitude approximation applies. Plotted below are the numerical solutions with $\theta_{\max} = 0.015$ rad (in blue) to the function $(0.015\text{ rad})\cos\omega t$ (in red).



(ii) In the case where K_0 approaches $U_{\max}$, the small-amplitude approximation is no longer adequate. Plotted below are the numerical solutions with $\theta_{\max} = 1.5$ rad (in blue) to the function $(1.5\text{ rad})\cos\omega t$ (in red). The motion is still oscillatory, but with a longer period. The period lengthens indefinitely as K_0 approaches $U_{\max}$.



(iii) In the case where $K_0 > U_{\max}$, we have nonuniform circular motion, which is slowest at the top and fastest at the bottom of the circle.

ASSESS For large-amplitude swings, the restoring torque is less than it would be in a small-amplitude approximation, and the pendulum returns more slowly toward equilibrium, and hence the period increases.

86. **INTERPRET** We are asked to consider the body mass measuring (BMMD) device used to measure the mass of astronauts in space.

DEVELOP The oscillation period of the BMMD is related to the mass, m_a, of a given astronaut by

$$T = 2\pi\sqrt{\frac{m_c + m_a}{k}}$$

where m_c is the mass of the chair, and k is the spring constant.

EVALUATE The period for a 90-kg astronaut can be related to the period for a 60-kg astronaut:

$$\frac{T_{90}}{T_{60}}=\sqrt{\frac{20\text{ kg}+90\text{ kg}}{20\text{ kg}+60\text{ kg}}}=1.17$$

The time is therefore 17% longer with the more massive astronaut.

The answer is (c).

ASSESS The 90-kg astronaut is 50% more massive than the 60-kg astronaut, but the period does not scale linearly with the mass, as we have just shown.

87. **INTERPRET** We are asked to consider the body mass measuring device used to measure the mass of astronauts in space.

DEVELOP We don't know the exact orientation of the springs on the BMMD, but for simplicity, let's assume that there is just one spring and it is located just beneath the chair. On Earth, the force of gravity will pull the chair down, thus compressing the spring until $kx_0=(m_c+m_a)g$. The point, x_0, is the new equilibrium point of the system.

EVALUATE If the chair is displaced from the equilibrium point, the total force on the chair becomes: $F=(m_c+m_a)g-kx$, which can be written as:

$$F=-k(x-x_0)$$

Therefore, there will be the same restoring force around the new equilibrium point. As such, the system will still oscillate with the same period as it would up in space.

The answer is (a).

ASSESS Since gravity is a constant force, it does not disrupt the linear relation between force and displacement that is the hallmark of simple harmonic motion. But that said, springs are not ideal; there is only a certain range of stretching and compression that they obey $F=-kx$. It's possible that BMMD spring will not work exactly the same on Earth because the new equilibrium point is outside of its linear regime.

88. **INTERPRET** We are asked to consider the body mass measuring device used to measure the mass of astronauts in space.

DEVELOP Let's assume the astronaut's mass declines according to $m_a(t)=m_{a0}-bt$.

EVALUATE Taking the time derivative of the equation relating the period to the mass:

$$\frac{dT}{dt}=\frac{d}{dt}\left[2\pi\sqrt{\frac{m_c+m_{a0}-bt}{k}}\right]=\frac{T_0}{2}\frac{-b/M_0}{\sqrt{1-bt/M_0}}$$

Where T_0 and M_0 are the period and total mass at the beginning when $t=0$. Since the derivative is negative, the period is decreasing. We take the second derivative to see the rate at which the period is decreasing:

$$\frac{d^2T}{dt^2}=\frac{d}{dt}\left[\frac{T_0}{2}\frac{-b/M_0}{\sqrt{1-bt/M_0}}\right]=-\frac{T_0}{4}\left(\frac{b}{M_0}\right)^2(1-bt/M_0)^{-3/2}$$

The second derivative is negative, which means the rate is getting more negative with time. Or to say it another way it's decreasing at an ever-increasing rate.

The answer is (c).

ASSESS We can check that our result makes sense by putting in some representative values for the time. At $t=0$, the period is decreasing at $\dot{T}=-\frac{1}{2}(T_0b/M_0)$, whereas at $t=3M_0/4b$, it is decreasing at $\dot{T}=-(T_0b/M_0)$, which is twice as fast. So yes, the period is decreasing at an ever-increasing rate.

89. **INTERPRET** We are asked to consider the body mass measuring device used to measure the mass of astronauts in space.

DEVELOP The spring constant is given by $k=m(2\pi/T)^2$. Remark, however, that we are given the time it takes the chair with a 60-kg astronaut to complete three periods.

EVALUATE Taking the time derivative of the equation relating the period to the mass:

$$k = (m_\text{c} + m_\text{a})\left(\frac{2\pi}{\frac{1}{3}T_3}\right)^2 = (20\text{ kg} + 60\text{ kg})\left(\frac{6\pi}{6.0\text{ s}}\right)^2 = 80\pi^2\text{ N/m}$$

The answer is (d).

ASSESS Plugging this result back into Equation 13.7c with the mass of the 60-kg astronaut gives $T = 2.0\text{ s}$, which agrees with the 6.0 s given for three periods.