35

QUANTUM MECHANICS

EXERCISES

Section 35.2 The Schrödinger Equation

11. **INTERPRET** We are given the wave function of a particle and are to deduce from this the most probable position of the particle and the position(s) where the probability of finding the particle is 50%.

DEVELOP The quantity $\psi^2(x)$ represents the probability of finding the particle at the position *x*. Therefore, the particle is most likely to be found at the position where the probability density $\psi^2(x)$ is a maximum.

EVALUATE **(a)** The maximum of $\psi^2(x) = A^2 e^{-2x^2/a^2}$ occurs where $d\left[\psi^2(x)\right]/dx = 0$ and $d^2\left[\psi^2(x)\right]/dx^2 < 0$. Evaluating the first derivative gives

$$\frac{d}{dx}\psi^2(x) = -\frac{4A^2 x}{a^2}e^{-2x^2/a^2} = 0$$
$$x = 0$$

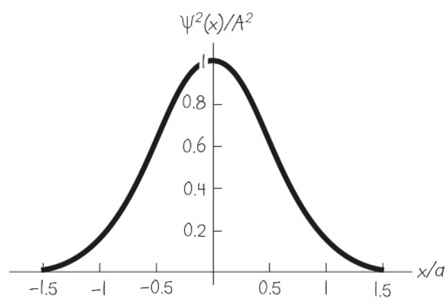
Evaluating the second derivative at $x = 0$ gives

$$\left.\frac{d^2}{dx^2}\psi^2(x)\right|_{x=0} = \left[-\frac{4A^2}{a^2}e^{-2x^2/a^2}\left(1-\frac{4x}{a^2}\right)\right]_{x=0} = -\frac{4A^2}{a^2} < 0$$

which shows that the extremum at $x = 0$ is indeed a maximum. Thus, most probable place to find the particle is at $x = 0$.

(b) The probability density $\psi^2(x)$ falls to half its maximum value when $\psi^2(x) = \psi^2(0)/2$. Solving this equation gives $x = \pm a\sqrt{\ln 2/2} = \pm 0.589a$.

ASSESS The probability distribution is shown below. Note that $\psi^2(x)/A^2 = e^{-2x^2/a^2}$ peaks at $x = 0$ and appears to be halved at $x/a \sim \pm 0.6$.



12. **INTERPRET** This problem is an exercise in normalizing the wave function $\psi(x)$ given the boundary conditions.

DEVELOP The wave function is like the first excited state of an infinite square well found in Section 35.3, except that $-a \le x \le a$ rather than $0 \le x' \le L$, as in Fig. 35.5. The correspondence is explicit if one takes $a = L/2$ and $x = x' - L/2$. Then

$$\sin\left(\frac{\pi x}{a}\right) = \sin\left(\frac{2\pi x'}{L} - \pi\right) = -\sin\left(\frac{2\pi x'}{L}\right)$$

which is the wave function in the equation $\psi(x) = A\sin(n\pi x/L)$ for $n = 2$, except for an overall phase.

EVALUATE The normalization constant for this wave function is therefore $A = \sqrt{2/L} = \sqrt{1/a}$.

ASSESS Of course, A can be determined by repeating the integration,

$$1 = A^2 \int_{-\infty}^{\infty} \sin^2\left(\frac{\pi x}{a}\right) dx = aA^2$$

which gives the same result.

Section 35.3 Particles and Potentials

13. **INTERPRET** We are to find the principal quantum number of a particle in an infinite square well, given the particle's energy relative to the ground state.

DEVELOP The energy levels for a particle in an infinite square well are (Equation 35.5) $E_n = E_0 n^2$, where $E_1 = h^2/(8mL^2)$ is the ground-state energy. Thus, we must find the quantum number n such that $E_n = 64E_0$.

EVALUATE Solving the equation gives

$$E_1 n^2 = 64E_0$$
$$n = 8$$

ASSESS The energy levels go as n squared.

14. **INTERPRET** The energy difference between two levels is 21 times the ground-state energy. We are to find the principal quantum numbers for each of the levels.

DEVELOP The energy levels in an infinite square well are $E_n = E_0 n^2$, (Equation 35.5) where $E_0 = h^2/(8mL^2)$ is the ground-state energy. We are given that

$$E_{n_a} - E_{n_b} = 21E_0$$
$$E_0 n_a^2 - E_0 n_b^2 = 21E_0$$
$$n_a^2 - n_b^2 = 21$$

EVALUATE The possible quantum numbers are 1, 4, 9, 16, 25, 36, 49…, so we see that 25 − 4 = 21. Thus, our quantum numbers are $n_a = 5$ and $n_b = 2$.

ASSESS Sometimes just looking at a problem and thinking about it like this is the best approach.

15. **INTERPRET** We have an electron confined in an infinite potential well and are to find its ground-state energy.

DEVELOP The energy levels for an infinite square potential well are given by Equation 35.5:

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2h^2}{8mL^2}$$

The ground-state energy corresponds to $n = 1$.

EVALUATE From the above equation, we find the ground-state energy to be

$$E_1 = \frac{h^2}{8mL^2} = \frac{(hc)^2}{8(mc^2)L^2} = \frac{(1240\ \text{eV}\cdot\text{nm})^2}{8(511\ \text{keV})(5\ \text{nm})^2} = 15\times10^{-3}\ \text{eV} = 2.4\times10^{-21}\ \text{J}$$

ASSESS A nonzero ground-state energy is a common feature of quantum systems. Note that the energy levels are quantized and proportional to n^2.

16. **INTERPRET** Given the energy of the first excited state of a proton in an infinite square well, we are to find the width of the well.

DEVELOP The energy levels for an infinite square well potential are given in Equation 35.5:

$$E_n = \frac{n^2h^2}{8mL^2}$$

The first excited state corresponds to $n = 2$, and we are told that the energy is $E_2 = 1.5$ keV = 1500 eV. The information allows us to find the well width L.

EVALUATE The well width is

$$E_2 = \frac{(2)^2 h^2}{8mL^2} = \frac{4h^2}{8mL^2} = \frac{h^2}{2mL^2}$$

$$L = hc\sqrt{\frac{1}{2mc^2E}} = (1240\text{ eV}\cdot\text{nm})\sqrt{\frac{1}{2(938\text{ MeV})(1500\text{ eV})}} = 7.39\times10^{-4}\text{ nm} \approx 0.74\text{ pm}$$

ASSESS This is much smaller than the Bohr radius a_0 = 52.9 pm, so this well is not realistic because it is smaller than the smallest atomic size.

17. **INTERPRET** The problem asks us to compute the lowest two energy states of a carbon nanotube.

DEVELOP The energy levels for an infinite square well potential are given in Equation 35.5:

$$E_n = \frac{n^2h^2}{8mL^2}$$

To simplify the calculation, we will multiply the top and bottom of the fraction by c^2 so that we can use the rest energy of the electron, $mc^2 = 511$ keV, and the shorthand $hc = 1240\text{ eV}\cdot\text{nm}$.

EVALUATE **(a)** Plugging in the tube diameter for the well width L = 0.48 nm, we get for the $n = 1$ ground state

$$E_1 = \frac{n^2(hc)^2}{8mc^2L^2} = \frac{(1)^2(1240\text{ eV}\cdot\text{nm})^2}{8(511\text{ keV})(0.48\text{ nm})^2} = 1.63\text{ eV} \approx 1.6\text{ eV}$$

(b) The first excited state has $n = 2$ and its energy is 4 times the ground state:

$$E_2 = n^2E_1 = (2)^2(1.63\text{ eV}) = 6.53\text{ eV} \approx 6.5\text{ eV}$$

ASSESS The answers seem reasonable. In the lowest energy states, electrons in the nanotube will have roughly the same energy as electrons accelerated by a 1 V potential difference.

18. **INTERPRET** In this problem, we want to demonstrate the smallness of the quantum effect when dealing with macroscopic objects. We are to imagine ourselves as a particle in a room-sized potential well and find the value of Planck's constant needed to give us the desired minimum speed.

DEVELOP In a one-dimensional infinite square well, the lowest energy of a particle is (see Equation 35.5) $E_1 = h^2/(8mL^2)$. Since we are in the nonrelativistic domain, we can set $E_1 = \frac{1}{2}mv^2$ to deduce the "would-be" Planck constant for the quantum effect to be noticeable.

EVALUATE Equating $E_1 = h^2/(8mL^2) = \frac{1}{2}mv^2$, we get $h^2 = 8mL^2\left(\frac{1}{2}mv^2\right) = 4m^2v^2L^2$, or

$$h = 2mvL = 2(60\text{ kg})(1.0\text{ m/s})(2.6\text{ m}) = 310\text{ J}\cdot\text{s}$$

to two significant figures. This is 4.7×10^{35} times larger than the actual value of Planck's constant.

ASSESS When dealing with motion of macroscopic objects, one may simply apply classical physics and ignore quantum effect.

19. **INTERPRET** This problem involves an infinite potential well in which an unknown particle is trapped. Given the difference in energy between the ground state and the first excited state, we are to determine if the particle is an electron or a proton.

DEVELOP Use the result of Schrödinger's equation applied to an infinite potential well (Equation 35.5). The energy difference ΔE between the first excited state (n = 2) and the ground state (n = 1) of a one-dimensional infinite square well is

$$\Delta E = \frac{(4-1)h^2}{8mL^2} = \frac{3}{8}\left(\frac{hc}{L}\right)^2\left(\frac{1}{mc^2}\right)$$

EVALUATE Using given values, we find that

$$mc^2 = \frac{3}{8}\left(\frac{hc}{L}\right)^2\frac{1}{\Delta E} = \frac{3}{8}\left(\frac{1240\text{ eV}\cdot\text{nm}}{1.0\text{nm}}\right)^2\frac{1}{1.13\text{ eV}} = 510\text{ keV}$$

which is very close to the electron's rest energy.

ASSESS The particle must be an electron.

20. **INTERPRET** We shall treat the snail as a particle confined in an infinite potential well. We want to show that a classical treatment of this problem is adequate for large quantum number n, in accordance with the correspondence principle.

DEVELOP The kinetic energy of the snail is

$$E = \frac{1}{2}mv^2 = \frac{1}{2}\left(3\times10^{-3}\ \text{kg}\right)(5\times10^{-4}\ \text{m/s})^2 = 3.75\times10^{-10}\ \text{J}$$

If this is regarded as the energy of a 3-g particle confined to a one-dimensional infinite square well of width L = 15 cm, the energy quantum number can be estimated from Equation 35.5:

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2h^2}{8mL^2}$$

EVALUATE Equating the kinetic energy with the energy derived from Schrödinger's equation,we find the quantum number to be

$$n = \frac{\sqrt{8mL^2E}}{h} = \frac{(0.15\ \text{m})\sqrt{8\left(3\times10^{-3}\ \text{kg}\right)\left(3.75\times10^{-10}\ \text{J}\right)}}{6.63\times10^{-34}\ \text{J}\cdot\text{s}} = 7\times10^{26}$$

The correspondence principle implies that classical theory ought to be quite adequate for quantum numbers this large.

ASSESS We expect classical physics to be adequate in characterizing the motion of the crawling snail. When we try to quantize the system, we find n to be very large, as expected from the correspondence principle.

21. **INTERPRET** We are to find the ground-state energy of an alpha particle (i.e., He^{2+}) trapped in the given infinite quantum well.

DEVELOP Use the result of Schrödinger's equation applied to an infinite potential well (Equation 35.5). For the ground state, n = 1. From Appendix C, we find that 1 u = 931.5 MeV/c^2.

EVALUATE With $n = 1$, the lowest alpha-particle energy in a one-dimensional infinite square well of width L = 15 fm is

$$E = \frac{h^2}{8mL^2} = \left(\frac{hc}{L}\right)^2 \frac{1}{8mc^2} = \left(\frac{hc}{L}\right)^2 \frac{1}{8(4u)c^2}$$
$$= \left(\frac{1240\ \text{MeV}\cdot\text{fm}}{15\ \text{fm}}\right)^2 \left(\frac{1}{8(4\times931.5\ \text{MeV})}\right) = 0.2\ \text{MeV}$$

ASSESS This is a rather high energy, which is not surprising given the small size of the potential well. Note that the Bohr radius a_0 = 52.9 pm gives the typical atomic size, which includes the electrons. Thus, our alpha particle is confined to a space much less than the size of an atom.

22. **INTERPRET** We are given the ground-state energy of a particle in a harmonic oscillator potential and asked to find the corresponding classical frequency of the oscillator.

DEVELOP The ground-state energy of a one-dimensional harmonic oscillator is given by Equation 35.7 with n = 0:

$$E_0 = \tfrac{1}{2}\hbar\omega = \tfrac{1}{2}hf$$

EVALUATE Thus, the classical frequency is

$$f = \frac{2E_0}{h} = \frac{2(0.14\ \text{eV})\left(1.6\times10^{-19}\ \text{J/eV}\right)}{6.63\times10^{-34}\ \text{J}\cdot\text{s}} = 6.8\times10^{13}\ \text{Hz}$$

ASSESS A nonzero ground-state energy is a common feature of quantum systems. Note that because the spacing between adjacent energy levels is $\Delta E = \hbar\omega = hf,\ f$ represents the frequency of the photon that must be absorbed or emitted for transitions to take place.

23. **INTERPRET** We are to find the energy of the lowest state for a particle in a harmonic oscillator potential.

DEVELOP Apply Equation 35.7. The lowest state is the ground state and has quantum number $n = 0$ for a harmonic oscillator.

EVALUATE With $n = 0$, Equation 35.7 gives the ground-state energy as

$$E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}\left(6.582\times10^{-16}\ \text{eV}\cdot\text{s}\right)\left(2.0\times10^{17}\ \text{s}^{-1}\right) = 66\ \text{eV}$$

ASSESS Note that the energy of a particle in a harmonic oscillator potential is independent of mass.

24. **INTERPRET** In this problem, we are asked to find the classical frequency of an oscillator, given the energy of photons emitted in a transition from an arbitrary state n to state $n - 1$.

DEVELOP If a harmonic oscillator emits a photon, conservation of energy demands that the final state of the harmonic oscillator be lower than the initial state. For a one-dimensional harmonic oscillator, the energy spacing between adjacent states is

$$\Delta E = E_{n-1} - E_n = -\hbar\omega = -hf$$

(see Equation 35.7). Conservation of energy demands that the initial energy be the same as the final energy, or

$$E_n = E_{n-1} + E_\gamma$$
$$E_\gamma = E_n - E_{n-1} = -\Delta E = hf$$

where $E_\gamma = 1.1\,\text{eV}$ is the energy of the emitted photon.

EVALUATE From the above equation, we find the classical oscillation frequency to be

$$f = \frac{E_\gamma}{h} = \frac{1.1\ \text{eV}}{4.136\times10^{-15}\ \text{eV}\cdot\text{s}} = 2.7\times10^{14}\ \text{Hz}$$

ASSESS For transition to occur, an oscillator must emit (or absorb) photons of this frequency.

Section 35.4 Quantum Mechanics in Three Dimensions

25. **INTERPRET** We are to consider a particle three-dimensionally confined in a cubic potential well. If the length of all sides of the box is doubled, how is the particle's ground-state energy affected?

DEVELOP From Equation 35.8, we see that the energy of a particle in a cubical box is inversely proportional to the square of the length of the sides of the box. In the ground state, $n_x = n_y = n_z = 1$, so the ground state energy is

$$E_0 = \frac{h^2}{8mL^2}$$

If we double the length, $L \to 2L$, we can recalculate the new ground-state energy.

EVALUATE Replacing L by $2L$, we find that the ground-state energy becomes

$$E_0' = \frac{h^2}{8m(2L)^2} = \frac{1}{4}\left(\frac{h^2}{8mL^2}\right) = \frac{1}{4}E_0$$

ASSESS The ground-state energy is inversely proportional to L^2, so doubling L reduces the ground-state energy by a factor of $L^2 = 4$.

26. **INTERPRET** This problem explores a crude model of an atomic nucleus in which the nucleus is considered to be a proton confined in a cubical potential well. We are to find the energy difference between the ground state and the first excited state.

DEVELOP The energy levels of a particle in a three-dimensional box are given by Equation 35.8:

$$E = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

The ground state corresponds to $n_x = n_y = n_z = 1$, whereas the first excited state has one of the quantum numbers equal to 2 (e.g., $n_x = 2,\ \ n_y = n_z = 1$).

EVALUATE The difference in energy between the first excited state and the ground state for a proton (mass 938 MeV/c^2) in a cubical box (side length 4 fm) is

$$\Delta E = \left(2^2 + 1^2 + 1^2 - 1^2 - 1^2 - 1^2\right)\frac{h^2}{8mL^2} = \frac{3(hc)^2}{8\left(mc^2\right)L^2} = \frac{3(1240\ \text{MeV}\cdot\text{fm})^2}{8(938\ \text{MeV})(4\ \text{fm})^2} = 38\ \text{MeV}$$

ASSESS Typical gamma-ray energies range from 100 keV to 10 MeV.

27. **INTERPRET** This problem is similar to the previous problem, except that we are now given energy of the photon emitted by an electron confined to a cubic quantum potential well and are asked for the size of the cube.

DEVELOP From the solution to the previous problem, we know that the energy difference between the energy of the ground state and of the first excited state is

$$\Delta E = \frac{3h^2}{8mL^2}$$

Set this equal to the photon energy (Equation 34.5) $E_\gamma = hf = hc/\lambda$ to find the cubic box size L.

EVALUATE Inserting the given quantities and solving for L gives

$$\frac{hc}{\lambda} = \frac{3h^2}{8mL^2} \quad \Rightarrow \quad L = \sqrt{\frac{3\lambda hc}{8mc^2}} = \sqrt{\frac{3(1240\ \text{eV}\cdot\text{nm})(950\ \text{nm})}{8(511\ \text{keV})}} = 930\ \text{pm}$$

where we have used me = 511 MeV/c^2.

ASSESS This cubic potential well is about the same size as in the previous problem, but the corresponding energy difference is much less because the electron's mass is much less than that of the proton.

EXAMPLE VARIATIONS

28. **INTERPRET** We're dealing with electrons in a one-dimensional infinite square well.

DEVELOP Equation 35.5 gives the energy of the nth state. We'll apply it for n = 1, 2, and 3.

$$E = \frac{h^2}{8mL^2}n^2$$

EVALUATE Using $L = 0.50\,\text{nm}$ and $m = m_e = 9.11\times10^{-31}\,\text{kg}$

$$E_n = \frac{n^2\left(6.63\times10^{-34}\,\text{J}\cdot\text{s}\right)^2}{8\left(9.11\times10^{-31}\,\text{kg}\right)\left(0.50\times10^{-9}\,\text{m}\right)^2} = 2.41\times10^{-19}n^2\ \text{J}$$

or $1.506n^2\ \text{eV}$. Evaluating with n = 1, 2, and 3 gives $E_1 = 1.51\,\text{eV}, E_2 = 6.03\,\text{eV}, \text{and}\ E_3 = 13.6\,\text{eV}$

ASSESS Constraining the electron to a small space gives energy values close to those of a small atom.

29. **INTERPRET** We're dealing with electrons in a one-dimensional infinite square well–like device, for which we know the energy in the n = 4 level and we are to find the layer thickness.

DEVELOP Equation 35.5 gives the energy of the nth state. We'll solve for L and evaluate using the given values for E_n and n

$$E = \frac{h^2}{8mL^2}n^2$$

EVALUATE Solving for L and evaluating we find

$$L = \frac{\left(6.63\times10^{-34}\,\text{J}\cdot\text{s}\right)4}{\sqrt{8\left(9.11\times10^{-31}\,\text{kg}\right)\left(3.98\times10^{-19}\,\text{J}\right)}} = 0.972\,\text{nm}$$

where we have expressed the particle's energy in terms of joules.

ASSESS Constraining the electron to a small space gives energy values close to those of a small atom.

30. **INTERPRET** This problem involves an electron transitioning between energy levels in an infinite potential well. We are to find the transition energies associated with the given transitions.

DEVELOP From Equation 35.5, we see that the transition energies are

$$\Delta E = \frac{\left(n_\text{i}^2 - n_\text{f}^2\right)h^2}{8mL^2}$$

and the wavelengths of the corresponding photons are

$$\lambda = \frac{hc}{\Delta E} = \frac{8mc^2L^2}{\left(n_\text{i}^2 - n_\text{f}^2\right)hc} = \frac{8(511\ \text{keV})(1.15\,\text{nm})^2}{\left(n_\text{i}^2 - n_\text{f}^2\right)(1240\ \text{eV}\cdot\text{nm}\ )} = \frac{4360\ \text{n m}}{\left(n_\text{i}^2 - n_\text{f}^2\right)}$$

EVALUATE Evaluating the wavelengths we find:

For $n_i = 3$ and $n_f = 1$, $\lambda = 545$ nm.

For $n_i = 2$ and $n_f = 1$, $\lambda = 1.45$ μm.

ASSESS These laser wavelengths range from the visible portion to the infrared portion of the electromagnetic spectrum.

31. **INTERPRET** This problem involves an electron transitioning between energy levels in an infinite potential well. We are to find the size of the well given the wavelength emitted when transitioning between two levels.

DEVELOP From Equation 35.5, we see that the transition energies are

$$\Delta E = \frac{\left(n_i^2 - n_f^2\right)h^2}{8mL^2} = \frac{hc}{\lambda}$$

Given the wavelength emitted, we can then express the size of the well as the electron transitions from the state $n = 3$ to $n = 2$.

EVALUATE Solving for L and evaluating we find

$$L = \sqrt{\frac{\left(n_i^2 - n_f^2\right)\lambda hc}{8mc^2}} = \sqrt{\frac{5(595\,\text{nm})(1240\ \text{eV}\cdot\text{nm})}{8(511\ \text{keV})}} = 0.950\,\text{nm}$$

ASSESS The width of such a well is comparable to the size of a small atom.

32. **INTERPRET** This is a question about probability, and we know that the probability density is the square of the wave function. So, our solution is going to involve ψ^2.

DEVELOP Following the same approach as the original example, we are to find the probability of finding the particle in a given region of the square well by integrating the square of its wave function over said region. In this case, the region is the left-hand third of a square well, considering the particle is in the ground state.

EVALUATE The probability becomes

$$P = \frac{2}{L}\int_0^{L/3} \sin^2\left(\frac{\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{x}{2} - \frac{\sin(2\pi x/L)}{4\pi/L}\bigg|_0^{L/3}\right) = \frac{1}{L}\left(\frac{L}{3} - \frac{\sqrt{3}}{4\pi}\right) = 0.20$$

ASSESS Although larger than the value found in the original example, it is still less than the probability $P = 0.33$ we would expect to classically for finding the particle in any third of the well.

33. **INTERPRET** This is a question about probability, and we know that the probability density is the square of the wave function. So, our solution is going to involve ψ^2. We are given the findings for a set of measurements and are asked to determine the region of the well which was searched to reach those results.

DEVELOP Following the same approach as the original example, we can express the probability of finding the particle in a given region of the square well by integrating the square of its wave function over said region. In this case, the region searched is starting at the left but does not reach the other edge. That is, $L/\alpha, \text{where} 1/\alpha < 1$. Considering the particle is in the ground state, we can express the probability for such a region and determine the value of α, and thus the size of the region searched, which resulted in 9090/10,000 findings.

EVALUATE The probability becomes

$$P = \frac{2}{L}\int_0^{L/\alpha} \sin^2\left(\frac{\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{x}{2} - \frac{\sin(2\pi x/L)}{4\pi/L}\bigg|_0^{L/\alpha}\right) = \frac{1}{L}\left(\frac{L}{\alpha} - \frac{1}{2\pi}\sin\left(\frac{2\pi}{\alpha}\right)\right) = 0.909$$

Considering this value is close to 1, it's likely a large portion of the well was searched. Looking at some possible values such as $L/2, 2L/3, \text{and } 3L/4$, we find probabilities of approximately 0.5, 0.804, and 0.909. Thus, the fraction of each square well which was searched was ~3/4.

ASSESS This value is larger than the probability $P = 0.75$ we would expect to classically for finding the particle when looking in three quarters of the way into the well.

34. **INTERPRET** This is a question about probability, and we know that the probability density is the square of the wave function. So, our solution is going to involve ψ^2.

DEVELOP Following the same approach as the original example, we are to find the probability of finding the particle in a given region of the square well by integrating the square of its wave function over said region. In this case, the region is the left-hand quarter of a square well, considering the particle is in the $n = 3$ state.

EVALUATE The probability becomes

$$P = \frac{2}{L}\int_0^{L/4} \sin^2\left(\frac{3\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{x}{2} - \frac{\sin(6\pi x/L)}{12\pi/L}\bigg|_0^{L/4}\right) = \frac{1}{L}\left(\frac{L}{4} + \frac{1}{6\pi}\right) = 0.30$$

ASSESS It's more than 3 times more likely to find the particle in the first quarter of the box while it's the $n = 3$ energy level than when in the ground state.

35. **INTERPRET** This is a question about probability, and we know that the probability density is the square of the wave function. So, our solution is going to involve ψ^2. We want to determine the region of the well which we need to search in order to have a 65% chance of finding it.

DEVELOP Following the same approach as the original example, we can express the probability of finding the particle in a given region of the square well by integrating the square of its wave function over said region. In this case, the region searched is starting at one end, which we consider the left of the well, but not reaching the other end. That is, L/α, where $1/\alpha < 1$. Considering the particle is in the $n = 3$ state, we can express the probability for such a region and determine the value of α, and thus the size of the region searched, which results in a 65% probability of findings.

EVALUATE The probability becomes

$$P = \frac{2}{L}\int_0^{L/\alpha} \sin^2\left(\frac{3\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{x}{2} - \frac{\sin(6\pi x/L)}{12\pi/L}\bigg|_0^{L/\alpha}\right) = \frac{1}{L}\left(\frac{L}{\alpha} - \frac{1}{6\pi}\sin\left(\frac{6\pi}{\alpha}\right)\right) = 0.65$$

Numerically solving for α we find $\alpha \sim 1.67$, meaning one would have to search 60% of the well.

ASSESS Due to the changes to shape of the wave function at different energy levels, and thus the probability distribution, searching different portions of the well for different energy states results in different findings.

PROBLEMS

36. **INTERPRET** We are to derive the normalization constant of the given wave function with the given boundary conditions.

DEVELOP Because the particle under consideration must be somewhere, the probability distribution y integrated over all space must be unity. This leads to the normalization condition of a wave function (Equation 35.3)

$$\int_{-\infty}^{\infty} \psi^2(x)\,dx = 1$$

This condition allows us to deduce the form of the normalization constant A.

EVALUATE Inserting the given wave function into the integral leads to

$$1 = \int_{-\infty}^{\infty} \psi^2(x)\,dx = \int_{-b}^{b} A^2\left(b^2 - x^2\right)^2 dx = 2A^2\int_0^b\left(b^4 - 2b^2x^2 + x^4\right)dx = 2A^2\left(b^4x - \frac{2b^2x^3}{3} + \frac{x^5}{5}\right)\bigg|_0^b$$
$$= 2A^2b^5\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16A^2b^5}{15}$$

Therefore, the normalization constant is $A = \frac{1}{4}\sqrt{15}\ b^{-5/2} = 0.968b^{-2.5}$.

ASSESS The normalization condition implies that $\psi(x)$ has dimension $(\text{length})^{-1/2}$. With $\psi(x) = A\left(b^2 - x^2\right)$, we expect A to have dimension $(\text{length})^{-5/2}$. Since b also has dimensions of length, we see that our result is consistent with dimensional analysis.

37. **INTERPRET** For this problem, we are to show that if two wave functions are solutions of the Schrödinger equation, then their linear combination must also be a solution.

DEVELOP The time-independent one-dimensional Schrödinger equation is given by Equation 35.1:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+U(x)\psi(x)=E\psi(x)$$

We want to show that for any constants *a* and *b*, $a\psi_1+b\psi_2$ is a solution if ψ_1 and ψ_2 are.

EVALUATE Substituting $\psi=a\psi_1+b\psi_2$ into the Schrödinger equation gives

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+U(x)\right](a\psi_1+b\psi_2)=a\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2}+U(x)\psi_1\right]+b\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2}+U(x)\psi_2\right]$$
$$=aE\psi_1+bE\psi_2=E(a\psi_1+b\psi_2)=E\psi$$

ASSESS The Schrödinger equation is a linear differential equation. Therefore, the result follows directly from the superposition principle.

38. **INTERPRET** This problem involves an electron trapped in an infinite potential well. We are to find the transition energies associated with the given transitions.

DEVELOP From Equation 35.5, we see that the transition energies are

$$\Delta E=\frac{\left(n_\text{i}^2-n_\text{f}^2\right)h^2}{8mL^2}$$

and the wavelengths of the corresponding photons are

$$\lambda=\frac{hc}{\Delta E}=\frac{8mc^2L^2}{\left(n_\text{i}^2-n_\text{f}^2\right)hc}=\frac{8(511\text{ keV})(15\text{ nm})^2}{\left(n_\text{i}^2-n_\text{f}^2\right)(1240\text{ eV}\cdot\text{nm})}=\frac{0.74\text{ mm}}{\left(n_\text{i}^2-n_\text{f}^2\right)}$$

EVALUATE To two significant figures, we find:

(a) For $n_\text{i}=2$ and $n_\text{f}=1$, $\lambda=0.25$ mm.

(b) For $n_\text{i}=20$ and $n_\text{f}=19$, $\lambda=19=\mu\text{m}$.

(c) For $n_\text{i}=100$ and $n_\text{f}=1$, $\lambda=74$ nm.

ASSESS These wavelengths range from the ultraviolet to the far infrared to the microwave portion of the electromagnetic spectrum.

39. **INTERPRET** We are to find the energy and wavelength of the photon emitted as an electron trapped in an infinite square well makes a transition to the adjacent energy level.

DEVELOP The energy levels for an infinite square potential well are given by Equation 35.5:

$$E_n=\frac{n^2h^2}{8mL^2}$$

Thus, the energy of the photon emitted when the electron drops from n_i to $n_\text{f}<n_\text{i}$ is

$$E_\gamma=\Delta E=\left(n_\text{i}^2-n_\text{f}^2\right)\frac{h^2}{8mL^2}$$

From Equation 34.6, the wavelength of the photon is $\lambda=hc/E_\gamma$.

EVALUATE **(a)** Substituting the values given, we find

$$E_\gamma=\left(n_\text{i}^2-n_\text{f}^2\right)\frac{h^2}{8mL^2}=\left(n_\text{i}^2-n_\text{f}^2\right)\frac{(hc)^2}{8\left(mc^2\right)L^2}=\left(8^2-6^2\right)\frac{(1240\text{ eV}\cdot\text{nm})^2}{8(511\text{ keV})(1.5\text{ nm})^2}=4.7\text{ eV}$$

(b) The wavelength of the photon is

$$\lambda=\frac{hc}{E_\gamma}=\frac{8mc^2L^2}{\left(n_\text{i}^2-n_\text{f}^2\right)hc}=264\,\text{nm}$$

to two significant figures.

ASSESS The wavelength is in the visible region of the electromagnetic spectrum.

40. **INTERPRET** This problem demonstrates an aspect of the correspondence principle, namely, that as quantum numbers become arbitrarily large, the quantum nature is lost and is replaced by the continuous nature of classical physics.

DEVELOP From Equation 35.5, we see that the difference between adjacent levels in an infinite square well is

$$\Delta E = E_0\left[(n+1)^2 - n^2\right] = E_0(2n+1)$$

so we can form the ratio between ΔE and E_{n+1} and take the limit as $n \to \infty$.

EVALUATE Comparing ΔE to the original energy level E_{n+1} and letting n go to infinity gives

$$\lim_{n\to\infty}\frac{\Delta E}{E_{n+1}} = \frac{2n+1}{(n+1)^2} = \frac{1}{n} = 0$$

so the energy different ΔE is zero.

ASSESS Thus, we have found that the possible energy levels form a continuum of states in the classical limit, in accordance with the correspondence principle.

41. **INTERPRET** An electron trapped in the given one-dimensional potential well must absorb a photon in order to make a transition from the ground state to an excited state. We want to know the maximum wavelength associated with this transition.

DEVELOP Equation 35.5 describes the allowed energy states for a particle trapped in a one-dimensional potential well. The allowed quantum numbers n are $n = 1, 2, 3, \ldots$, so the smallest transition energy is to the first excited state (i.e., $n=1$ to $n=2$). The energy difference may be found from inserting these quantum numbers into Equation 35.5, so

$$\Delta E = E_{n=2} - E_{n=1}$$

$$= \left(2^2 - 1^2\right)\frac{h^2}{8mL^2} = \frac{3(hc)^2}{8\left(mc^2\right)L^2} = \frac{3(1240 \text{ eV}\cdot\text{nm})^2}{8(511 \text{ keV})(3.6 \text{ nm})^2} = 0.087 \text{ eV}$$

The wavelength that corresponds to this energy is (see Equation 34.6) $E_\gamma = hf = hc/\lambda$. Because of the inverse relationship between wavelength and energy, the smallest energy corresponds to the largest wavelength. Therefore, the maximum wavelength that can cause a transition is

$$E_\gamma = \Delta E \Rightarrow \lambda_{\text{max}} = \frac{hc}{\Delta E}$$

EVALUATE Thus, the maximum wavelength that can be absorbed is

$$\lambda_{\text{max}} = \frac{hc}{\Delta E} = \frac{1240 \text{eV}\cdot\text{nm}}{0.087 \text{eV}} = 14.2 \text{ μm}$$

ASSESS The wavelength corresponds to the infrared region of the electromagnetic spectrum.

42. **INTERPRET** We are to compare the widths of two infinite square potential wells given the ratio of their ground-state energies.

DEVELOP Equation 35.5 shows that the energy of a one-dimensional infinite square well is given by

$$E = \frac{h^2}{8mL^2}n^2$$

The ground state corresponds to $n = 1$ and the first excited state to $n = 2$. We are given that $E^{\text{A}}_{n=1} = E^{\text{B}}_{n=2}$, so we can find the ratio of the well widths.

EVALUATE With $n_\text{A} = 1$ and $n_\text{B} = 2$,

$$\frac{h^2}{8mL_\text{A}^2} = 4\frac{h^2}{8mL_\text{B}^2} \quad\Rightarrow\quad L_\text{B} = 2L_\text{A}.$$

ASSESS The result of this depends on the dimensionality of the square well. For three dimensions, the result is $L_\text{A} = L_\text{B}/\sqrt{2}$.

43. **INTERPRET** There are various ways for the electron which is initially in the $n = 4$ state to make a transition to the ground state. We want to find the wavelengths associated with all possible spectral lines in this process.

DEVELOP The energy levels for a one-dimensional infinite square potential well are given by Equation 35.5:

$$E_n = \frac{n^2h^2}{8mL^2}$$

Thus, the energy of the photon emitted when the electron drops from n_i to $n_\mathrm{f} < n_\mathrm{i}$ is

$$E_\gamma = \Delta E = \left(n_\mathrm{i}^2 - n_\mathrm{f}^2\right)\frac{h^2}{8mL^2}$$

From Equation 34.6, we know that the corresponding photon wavelengths are

$$\lambda = \frac{hc}{E_\gamma} = \frac{8\left(mc^2\right)L^2}{\left(n_\mathrm{i}^2 - n_\mathrm{f}^2\right)hc} = \frac{8(511\text{ keV})(0.834\text{ nm})^2}{\left(n_\mathrm{i}^2 - n_\mathrm{f}^2\right)(1240\text{ eV}\cdot\text{nm})} = \frac{2293\text{ nm}}{n_\mathrm{i}^2 - n_\mathrm{f}^2}$$

EVALUATE **(a)** Starting from the $n = 4$ state, the possible transitions to the ground state are $4 \to 1, 4 \to 2, 4 \to 3,$ $3 \to 1, 3 \to 2,$ and $2 \to 1.$ So there are 6 ways.

(b) The corresponding wavelengths are

$$\lambda_{4\to1} = \frac{2293\text{ nm}}{4^2 - 1^2} = 153\text{ nm},\quad \lambda_{4\to2} = \frac{2293\text{ nm}}{4^2 - 2^2} = 191\text{ nm}$$
$$\lambda_{4\to3} = \frac{2293\text{ nm}}{4^2 - 3^2} = 328\text{ nm},\quad \lambda_{3\to1} = \frac{2293\text{ nm}}{3^2 - 1^2} = 287\text{ nm}$$
$$\lambda_{3\to2} = \frac{2293\text{ nm}}{3^2 - 2^2} = 459\text{ nm},\quad \lambda_{2\to1} = \frac{2293\text{ nm}}{2^2 - 1^2} = 764\text{ nm}$$

(c) Wavelengths in the range 400 – 10 nm correspond to the UV region of the electromagnetic spectrum; wavelengths in the range 400–750 nm correspond to the visible spectrum. The infrared region encompasses wavelengths between 1 mm and 750 nm.

ASSESS Only photons of these discrete wavelengths will be emitted during the transition from $n = 4$ to $n = 1$.

44. **INTERPRET** We are to find the mass of a particle in a well given the size of the well and the particle's energy.

DEVELOP Equation 35.5 shows that the energy of a one-dimensional infinite square well is given by

$$E_n = \frac{h^2}{8mL^2}n^2$$

We can solve for the mass m and use the given values for the well's size, particle's quantum state, and particle's energy to evaluate.

EVALUATE Solving for m we find

$$m = \frac{h^2n^2}{8EL^2} = \frac{\left(6.63\times10^{-34}\,\mathrm{J\cdot s}\right)^2 3^2}{8\left(6.23\times10^{-21}\,\mathrm{J}\right)(0.650\,\mathrm{nm})^2} = 1.88\times10^{-28}\text{ kg}$$

where we have expressed the particle's energy in terms of joules.

ASSESS We find the mass is that of a muon.

45. **INTERPRET** We are given a potential that is the same as in Fig. 35.5, except that the origin of coordinates is at the center of the well. We want to find expressions for the normalized wave function for even and odd quantum numbers and the corresponding energy levels.

DEVELOP The wave function for this well can be found by using Equation 35.6 (which is already normalized), but replacing x by $x' + \frac{1}{2}L,$ where $-\frac{1}{2}L \le x' \le \frac{1}{2}L$. The normalized wave function thus takes the form

$$\psi_n\left(x'\right) = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}\left(x' + \frac{L}{2}\right) = \sqrt{\frac{2}{L}}\left(\sin\frac{n\pi x'}{L}\cos\frac{n\pi}{2} + \cos\frac{n\pi x'}{L}\sin\frac{n\pi}{2}\right)$$

We need to distinguish between even and odd values of n.

EVALUATE **(a)** If n is odd, then $\cos\frac{1}{2}n\pi = 0$ and $\sin\frac{1}{2}n\pi = \pm1,$ so the wave function is

$$\psi_{n-\text{odd}}(x') = \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi x'}{L}\right)$$

The probability density $\psi_n^2(x)$ is unaffected by the overall sign of $\psi_n(x)$, so we choose the sign to be positive. If n is even, then $\cos\frac{1}{2}n\pi = \pm 1$ and $\sin\frac{1}{2}n\pi = 0$ and the wave function is

$$\psi_{n\text{-even}}(x') = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x'}{L}\right)$$

(b) The energy levels are the same, $E_n = n^2h^2/8mL^2$ regardless of how the potential is parameterized.

ASSESS These results can be confirmed by direct solution of the Schrödinger equation. For $-\frac{1}{2}L \le x \le \frac{1}{2}L$, Equation 35.4 has two solutions: $A\sin(kx)$ or $A\cos(kx)$. In order for ψ to vanish at $x = \pm\frac{1}{2}L$, one must use the cosine solution for odd quantum numbers [$\cos(\pm kL/2)$] which vanish for kL equal to odd multiples of π and the sine solution for even quantum numbers [$\sin(\pm kL/2)$] which vanish for kL equal to even multiples of π. Since the average of $\sin^2$ or $\cos^2$ over an integer number of half-cycles is ½, the normalization constant is $\sqrt{2/L}$ for either wave function (or use integrals in Appendix A). Note that these wave functions have even or odd parity about the center of the potential well (see Section 39.2).

46. **INTERPRET** For this problem, we are to modify Equation 35.8 for the case of an electron confined to a two-dimensional infinite square well and find the energy of the 10 lowest energy levels.

DEVELOP We can modify the time-independent one-dimensional Schrödinger equation for the infinite square well (Equation 35.4) by considering the wave function that depends on both the x and y coordinates as:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y) = -\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\psi(x,y) = E\psi(x,y)$$

where $\psi(x,y) = \psi(x)\psi(y) = A\sin\left(\frac{n_x\pi x}{L}\right)\sin\left(\frac{n_y\pi y}{L}\right)$ is the product of two sinusoidal waves along each dimension, each capable of oscillating independently and thus having their own quantum number n_i. We can substitute this function into the Schrödinger equation and solve for the energy levels of the two-dimensional well.

EVALUATE Substituting $\psi(x,y)$ into the Schrödinger equation gives

$$-\frac{\hbar^2}{2m}\left[\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right]A\left(\sin\left(\frac{n_x\pi x}{L}\right)\sin\left(\frac{n_y\pi y}{L}\right)\right) = (E_x+E_y)\psi(x,y)$$

$$-\frac{\hbar^2}{2m}\left[A\sin\left(\frac{n_y\pi y}{L}\right)\frac{d^2}{dx^2}\sin\left(\frac{n_x\pi x}{L}\right) + A\sin\left(\frac{n_x\pi x}{L}\right)\frac{d^2}{dy^2}\sin\left(\frac{n_y\pi y}{L}\right)\right] = E\psi(x,y)$$

$$\frac{\hbar^2\pi^2}{2mL^2}(n_x+n_y)^2\psi(x,y) = E\psi(x,y)$$

$$E_{n_x,n_y} = \frac{\hbar^2\pi^2}{2mL^2}(n_x+n_y)^2 = \frac{h^2\pi^2}{8mL^2}(n_x+n_y)^2$$

which is analogous to Equation 35.5. We can now find the 10 lowest energy levels of the well for the electron by plugging in its mass and the well size. Doing so we find:

$$E_{11} = 0.483 \text{ eV}$$
$$E_{12} = E_{21} = 1.21 \text{ eV}$$
$$E_{22} = 1.93 \text{ eV}$$
$$E_{13} = E_{31} = 2.41 \text{ eV}$$
$$E_{23} = E_{32} = 3.14 \text{ eV}$$
$$E_{41} = E_{14} = 4.10 \text{ eV}$$

ASSESS Similarly, we can solve the Schrödinger equation for a three-dimensional infinite square following the same approach to find an analogous expression with three quantum numbers.

47. **INTERPRET** We're given the energy of the photon emitted in a transition between adjacent quantum states and asked to find the width of the one-dimensional infinite potential well.

DEVELOP The energy levels for a one-dimensional infinite square potential well are given by Equation 35.5:

$$E_n = \frac{n^2h^2}{8mL^2}$$

Thus, the energy of the photon emitted when the electron drops from n_i to $n_\text{f} < n_\text{i}$ is

$$\Delta E = \left(n_i^2 - n_\text{f}^2\right)\frac{h^2}{8mL^2}$$

EVALUATE From the above equation, the energy difference between $n_\text{i} = 2$ and $n_\text{f} = 1$ is $\Delta E = 3h^2/\left(8mL^2\right)$, so the width of the potential well is

$$L = \frac{\sqrt{3}hc}{\sqrt{8mc^2\Delta E}} = \frac{\sqrt{3}\left(1240\text{ eV}\cdot\text{nm}\right)}{\sqrt{8\left(511\text{ keV}\right)\left(2.26\text{ eV}\right)}} = 0.707\text{ nm}$$

ASSESS The result is consistent with the typical quantum well width (about the size of an atom).

48. **INTERPRET** This problem involves a particle in a one-dimensional infinite square potential well. We are to find the probability of finding the particle in the middle 80% of the well.

DEVELOP From the derivation of Equation 35.5, we see that the ground-state wave function is

$$\psi(x) = A\sin\left(\pi x/L\right)$$

Because the particle must be somewhere, we know that

$$1 = \int_{-\infty}^{\infty}\psi(x)\,dx$$

which allows us to find the constant *A*. The result is

$$\psi(x) = \sqrt{2/L}\sin\left(\pi x/L\right)$$

so the probability of finding the particle in the central 80% of the well $(0.1L \le x \le 0.9L)$ is found, as in Example 35.2, by integrating the wave function squared from 0.1*L* to 0.9*L*.

EVALUATE Performing the integration gives

$$P = \frac{2}{L}\int_{0.10L}^{0.90L}\sin^2\left(\frac{\pi x}{L}\right)dx = \frac{2}{L}\left|\frac{x}{2} - \frac{\sin\left(2\pi x/L\right)}{4\pi/L}\right|_{0.10\,L}^{0.90\,L} = 0.80 - \frac{1}{2\pi}\left[\sin\left(2\pi - 36°\right) - \sin\left(36°\right)\right]$$
$$= 0.80 + \frac{\sin\left(36°\right)}{\pi} = 0.99$$

ASSESS The probability of finding the particle in either the left- or right-hand 5% of the well is 0.050.

49. **INTERPRET** The problem asks if quantum mechanics should be considered for macromolecules trapped inside a biological cell.

DEVELOP We will treat the biological cell like a one-dimensional square well so that the energy levels are given by Equation 35.5: $E = n^2h^2/8mL^2$.

EVALUATE The energy difference between the ground state and the first excited state is

$$\Delta E = \frac{\left(2^2 - 1^2\right)\left(6.63\times10^{-34}\text{ J}\cdot\text{s}\right)^2}{8\left(250{,}000\text{ u}\right)\left(10\ \mu\text{m}\right)^2}\left[\frac{1\text{ u}}{1.661\times10^{-27}\text{ kg}}\right] = 4.0\times10^{-36}\text{ J} = 2.5\times10^{-17}\text{ eV}$$

This is so much smaller than the energy of biochemical reactions (1 eV) that quantization is not relevant.

ASSESS Quantization is rarely considered in biology because the objects of interest are too large. One exception is photosynthesis, where the conversion of sunlight into chemical energy appears to exhibit some quantum effects.

50. **INTERPRET** You model the hydrogen in hydrogen chloride as a mass on a spring. Accordingly, the spring constant can be determined from the energy separation between the ground state and first excited state.

DEVELOP In terms of a simple harmonic oscillator, the energy difference between the ground state and first excited state is $\Delta E = \hbar\omega$. The angular frequency for the mass-spring system is $\omega = \sqrt{k/m}$. In this case, the mass is that of the hydrogen atom: $m_\text{H} = 1.008$ u.

EVALUATE Combining the above equations, the spring constant in the model for HCl is

$$k = m_{\text{H}}\left(\frac{\Delta E}{\hbar}\right)^2 = (1.008\ \text{u})\left[\frac{1.661\times10^{-27}\,\text{kg}}{1\ \text{u}}\right]\left(\frac{0.358\ \text{eV}}{\frac{1}{2\pi}4.14\times10^{-15}\text{eV}\cdot\text{s}}\right)^2 = 493\ \text{N/m}$$

ASSESS You can try to estimate the size of this molecule using the relation between the energy of a spring and the amplitude of its oscillations: $E = \frac{1}{2}kA^2$. Using the energy of the first excited state, the amplitude of the above system is roughly:

$$A = \sqrt{\frac{2E}{k}} = \sqrt{\frac{2(0.358\ \text{eV})}{(493\ \text{N/m})}\left[\frac{1.6\times10^{-19}\,\text{J}}{1\ \text{eV}}\right]} \sim 10^{-11}\text{m}$$

This is about an order of magnitude too small but it gives an idea of the size of the HCl molecule.

51. **INTERPRET** This problem concerns a particle in a one-dimensional infinite square potential well. We are asked to find the probability that the particle will be found within a given ranged centered at two different points in the square well.

DEVELOP As in Problem 48, the probability of finding the particle between X_1 and X_2 (where $0 \le x_1 < x_2 \le L$) is

$$P = \frac{2}{L}\int_{x_1}^{x_2}\sin^2\left(\frac{\pi x}{L}\right)dx = \frac{(x_2 - x_1)}{L} - \frac{1}{2\pi}\left[\sin\left(\frac{2\pi x_2}{L}\right) + \sin\left(\frac{2\pi x_1}{L}\right)\right]$$

We shall evaluate this probability function for the two ranges given.

EVALUATE **(a)** The probability P of finding the particle between $x_1 = 0.500L - 0.075L = 0.425L$ and $x_2 = 0.500L +0.075L = 0.575L$ is

$$P = \frac{0.575L - 0.425L}{L} - \frac{1}{2\pi}\left\{\sin\left[2\pi(0.575)\right] - \sin\left[2\pi(0.5425)\right]\right\} = 0.29$$

(b) The probability P of finding the particle between $x_1 = 0.250L - 0.075L = 0.425L$ and $x_2 = 0.250L + 0.075L = 0.575L$ is

$$P = \frac{0.175L - 0.325L}{L} - \frac{1}{2\pi}\left\{\sin\left[2\pi(0.325)\right] - \sin\left[2\pi(0.175)\right]\right\} = 0.15$$

Assess The probability is greater for the particle to be found in the center of the well than near the edges, which is reasonable in view of the probability distribution function (see Figure 35.7).

52. **INTERPRET** We are to calculate the probability of finding a particle in the central quarter of an infinite square well for various energy states and compare our answers with the classical probability.

DEVELOP From the discussion preceding Equation 35.5, we see that the wave function for the infinite square well is $\psi_n = \sqrt{\frac{2}{L}}\sin\,(n\pi x/L)$. The region in which we are interested is the central fourth: $\frac{3}{8}L \le x \le \frac{5}{8}L$. We will calculate the probability by integrating $\psi^2(x)dx$ from $x = 3L/8$ to $x = 5L/8$.

EVALUATE First, we find the general solution for any value of n:

$$P(n) = \frac{2}{L}\int_{3/8}^{5/8}\sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{1}{4} + \frac{1}{2n\pi}\left[\sin\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{5n\pi}{4}\right)\right]$$

(a) $n = 1 \quad\Rightarrow\quad P(1) = \frac{2\sqrt{2}+\pi}{4\pi} = 0.475$

(b) $n = 2 \quad\Rightarrow\quad P(2) = 0.0908$

(c) $n = 5 \quad\Leftrightarrow\quad P(5) = 0.205$

(d) $n = 20 \quad\Rightarrow\quad P(20) = 0.250$

(e) The classical model predicts that the particle would be anywhere in the box with equal probability, so the total probability of being in the central ¼ of the box is 0.25.

ASSESS Note that as n becomes higher, the quantum probability becomes closer to the classical probability.

53. **INTERPRET** We are to show that the Schrödinger equation has nonzero solutions in classically forbidden regions where $E < U$.

DEVELOP We are given solutions of the form $\psi(x) = Ae^{\pm\sqrt{2m(U-E)}x/\hbar}$, so all we need to do is substitute this wave function into the time-independent Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+U\right)\psi = E\psi$$

(see Section 35.3) and see if it fits.

EVALUATE Inserting the trial solution into the Schrödinger equation gives

$$\frac{d^2}{dx^2}\psi = A\frac{2m(U-E)}{\hbar^2}e^{\pm\sqrt{2m(U-E)}x/\hbar}$$

Substitute this into the time-independent Schrödinger equation:

$$-\frac{\cancel{\hbar^2}}{\cancel{2m}}\left[\cancel{A}\frac{\cancel{2m}(U-E)}{\cancel{\hbar^2}}\cancel{e^{\pm\sqrt{2m(U-E)}x/\hbar}}\right]+U\cancel{A}\cancel{e^{\pm\sqrt{2m(U-E)}x/\hbar}} = E\cancel{A}\cancel{e^{\pm\sqrt{2m(U-E)}x/\hbar}}$$

$$-(U-E)+U = E$$

$$E = E$$

Thus, we have shown that the proposed solution is a valid solution for the time-independent Schrödinger equation.

ASSESS This form of $\psi(x)$ is a solution to the time-independent Schrödinger equation. Note that this form of solution is not "wave like": it exponentially decays (or increases) instead of oscillating.

54. **INTERPRET** This problem involves quantum mechanics in a three-dimensional cubical box. We are to make an energy diagram and show the degeneracy of each level.

DEVELOP The energy levels of a particle in a three-dimensional box are given by Equation 35.8:

$$E_{n_{x,y,z}} = \frac{h^2}{8mL^2}\left(n_x^2+n_y^2+n_z^2\right)$$

The ground state corresponds to $n_x = n_y = n_z = 1,$ while the first excited state has one of the quantum numbers equal to 2 (e.g., $n_x = 2, n_y = n_z = 1$).

EVALUATE The quantum numbers, energy, and degeneracy of the first six levels in the three-dimensional infinite square well are summarized below:

Energy level	(n_x, n_y, n_z)	$E/(h^2/8mL^2)$	Degeneracy factor
1	(1,1,1)	3	1
2	(2,1,1), (1,2,1), (1,1,2)	6	3
3	(2,2,1), (1,2,2), (2,1,2)	9	3
4	(3,1,1), (1,3,1), (1,1,3)	11	3
5	(2,2,2)	12	1
6	(1,2,3), (1,3,2), (2,1,3) (2,3,1), (3,1,2), (3,2,1)	14	6

The energy-level diagram looks similar to that shown in Fig. 35.6.

ASSESS In general, degeneracy arises from symmetry in the system. In our case, the symmetry is due to the fact that we have a cube with $L_x = L_y = L_z = L.$

55. **INTERPRET** We are to verify that the given wave function is a solution to the three-dimensional Schrödinger equation and that the derived energy levels match those given by Equation 35.8.

DEVELOP For the given wave function $\psi(x,y,z),$ the second partial derivatives are

$$\frac{\partial^2\psi}{\partial x^2} = -\left(\frac{n_x\pi}{L}\right)^2\psi,\quad \frac{\partial^2\psi}{\partial y^2} = -\left(\frac{n_y\pi}{L}\right)^2\psi \quad\text{and}\quad \frac{\partial^2\psi}{\partial z^2} = -\left(\frac{n_z\pi}{L}\right)^2\psi$$

Substituting into the Schrödinger equation, with a potential for a cubical box (U = 0 inside and $U = \infty$ outside), we find

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)=-\frac{1}{2m}\left(\frac{h}{2\pi}\right)^2\left[-\left(\frac{n_x\pi}{L}\right)^2-\left(\frac{n_y\pi}{L}\right)^2-\left(\frac{n_z\pi}{L}\right)^2\right]\psi$$

$$=\frac{h^2}{8mL^2}\left(n_x^2+n_y^2+n_z^2\right)\psi=E\psi$$

EVALUATE The above derivation **(a)** demonstrates that $\psi(x,y,z)$ is a solution, and **(b)** shows that the energy levels match those given in Equation 35.8.

ASSESS This wave function also satisfies the boundary conditions appropriate for confinement ($\psi=0$ at $x=0$ or L, $y=0$ or L, and $z=0$ or L, which are points where $U=\infty$) since n_x, n_y and n_z are integers.

56. **INTERPRET** We are given the number of photons per unit time that impinge upon an ensemble of electrons in quantum wells of the given size. Each photon is of sufficient energy to raise an electron to the first excited state, and we are to find the total number of electrons thus excited in a given time.

DEVELOP The energy levels of a one-dimensional infinite quantum well are given by Equation 35.5, $E=n^2h^2/\left(8mL^2\right)$. The ground state has $n=1$, and the first excited state has $n=2$, so the energy difference between the two is

$$\Delta E=\left(2^2-1^2\right)\frac{h^2}{8mL^2}=\frac{3h^2}{8mL^2}$$

This is the energy required for the photon to excite an electron from the ground state to the first excited state. The average power in the light beam corresponds to a number of photons N per unit time of

$$N=\frac{P}{\Delta E}$$

The number of electrons N_e excited to the first excited state in a time Δt is $N_e=N\Delta t$.

EVALUATE Evaluating the expression for N_e gives

$$N_e=N\Delta t=\frac{P}{\Delta E}\Delta t=\frac{8mc^2L^2P}{3h^2c^2}\Delta t=\frac{8(511\text{ keV})(0.82\text{ nm})^2(6\text{ W})}{3(1240\text{ eV}\cdot\text{nm})^2}\left(\frac{1\text{ eV}}{1.6\times10^{-19}\text{ J}}\right)\left(5\times10^{-3}\text{ s}\right)=1.1\times10^{17}$$

ASSESS The ensemble contains many more electrons (i.e., $10^{24}\gg10^{17}$) than are excited. Since the photon energy is insufficient to excite transitions other than the first transition, the result we found is also the maximum number of electrons that can be excited.

57. **INTERPRET** When transitioning to lower states, electrons emit photons to release energy. We are interested in all the visible photon wavelengths associated with all possible electronic transitions in the given quantum well.

DEVELOP The energy levels for an infinite square potential well are given by Equation 35.5:

$$E_n=\frac{n^2h^2}{8mL^2}$$

Thus, the energy of the photon emitted when the electron drops from initial state n_i to final state $n_\text{f}<n_\text{i}$ is

$$E_\gamma=\Delta E=\left(n_\text{i}^2-n_\text{f}^2\right)\frac{h^2}{8mL^2}$$

The possible photon wavelengths are

$$\lambda=\frac{hc}{E_\gamma}=\frac{8mc^2L^2}{\left(n_\text{i}^2-n_\text{f}^2\right)hc}=\frac{8(511\text{ keV})(1.2\text{ nm})^2}{\left(n_\text{i}^2-n_\text{f}^2\right)(1240\text{ eV}\cdot\text{nm})}=\frac{4.75\ \mu\text{m}}{n_\text{i}^2-n_\text{f}^2}$$

EVALUATE **(a)** Visible photons fall within the wavelength range $0.4\ \mu\text{m}<\lambda<0.7\ \mu\text{m}$, or

$$4.75/0.7=6.78<n_\text{i}^2-n_\text{f}^2<11.9=4.75/0.4$$

There are four transitions satisfying this condition: $3\to1$, $4\to3$, $5\to4$, and $6\to5$.

ASSESS Since energy is quantized, only photons with wavelengths that satisfy the above condition will be emitted during the transitions.

58. **INTERPRET** Since this is a question about probability, we analyze ψ^2, which represents the probability density. The result here is a generalization of that in Example 35.1.

DEVELOP The wave function for the *n*th quantum state of a square potential is given in Equation 35.6:

$$\psi_n = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

To find the probability that a particle will be found in the left-hand quarter of the well, we square the wave function and integrate from $x = 0$ to $x = L/4$.

EVALUATE **(a)** We use the integral for the $\sin^2$ function from Appendix A to solve for the probability:

$$P = \int_0^{L/4} \psi_n^2(x)\, dx = \frac{2}{L}\left(\frac{x}{2} - \frac{\sin(2n\pi x/L)}{(4n\pi/L)}\right)\Bigg|_0^{L/4} = \frac{1}{4} - \frac{\sin(n\pi/2)}{2\pi n}$$

(b) If *n* is odd, then $\sin(n\pi/2) = \pm 1$ and the probability reduces to:

$$P = \frac{1}{4} \mp \frac{1}{2\pi n}$$

As $n \to \infty$, the second term disappears and the probability approaches ¼. Notice that the probability is exactly equal to ¼ when *n* is any even integer.

ASSESS The classical case can be thought of as a ping-pong ball in a box. If you shake the box randomly, the probability that the ball is in the left-hand quarter is 25%.

59. **INTERPRET** You're asked to develop the Schrödinger equation and the ground-state wave function for the simple harmonic oscillator.

DEVELOP You start with the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$$

EVALUATE **(a)** Plugging in the potential energy of a harmonic oscillator, you arrive at

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \tfrac{1}{2}m\omega^2 x^2\psi(x) = E\psi(x)$$

(b) For $n = 0$, the energy is $E = \tfrac{1}{2}\hbar\omega$ and the Schrödinger equation reduces to:

$$\frac{d^2\psi(x)}{dx^2} = \left(\frac{m^2\omega^2}{\hbar^2}x^2 - \frac{m\omega}{\hbar}\right)\psi(x) = \alpha^2\left(\alpha^2 - 1\right)\psi(x)$$

where $\alpha^2 = m\omega/\hbar$. If $\psi(x) = A_0 e^{-\alpha^2 x^2/2}$, then the first derivative is $\frac{d\psi(x)}{dx} = A_0\left(-\alpha^2 x\right)e^{-\alpha^2 x^2/2}$ and the second derivative is

$$\frac{d^2\psi(x)}{dx^2} = -\alpha^2 A_0\left[e^{-\alpha^2 x^2/2} + x\left(-\alpha^2 x\right)e^{-\alpha^2 x^2/2}\right] = \alpha^2\left(\alpha^2 - 1\right)\psi(x)$$

This proves that $\psi(x) = A_0 e^{-\alpha^2 x^2/2}$ is a solution to the ground state.

(c) To find the normalization constant, we use the normalization condition (Equation 35.3):

$$\int_{-\infty}^{+\infty} \psi^2 dx = \int_{-\infty}^{+\infty} A_0^2 e^{-\alpha^2 x^2} dx = 1$$

The integral is a Gaussian. One can find the value in an integral table, but we will give a short derivation here. We first define $I = \int_{-\infty}^{+\infty} e^{-\alpha^2 x^2} dx$ and then square both sides. We combine the right-hand side into a single exponent and then change to polar coordinates, which puts the integral into a more solvable form:

$$I^2 = \int_{-\infty}^{+\infty} e^{-\alpha^2 x^2} dx \cdot \int_{-\infty}^{+\infty} e^{-\alpha^2 y^2} dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\alpha^2\left(x^2 + y^2\right)} dxdy$$

$$= \int_0^{2\pi}\int_0^{+\infty} e^{-\alpha^2 r^2} rdrd\theta = 2\pi\left[\frac{-1}{2\alpha^2}e^{-\alpha^2 r^2}\right]_0^{+\infty} = \frac{\pi}{\alpha^2}$$

Therefore, $I = \sqrt{\pi}/\alpha$ and the normalization constant must be $A_0 = \left(\alpha^2/\pi\right)^{1/4}$. In summary, the normalized ground-state wave function of the simple harmonic oscillator is

$$\psi(x) = \left(\frac{\alpha^2}{\pi}\right)^{1/4} e^{-\alpha^2 x^2/2}$$

ASSESS The probability density for the ground state is a Gaussian, or "bell curve," as depicted in Figure 35.11a.

60. **INTERPRET** You want to show why fuel for a nuclear reactor is more concentrated in energy than traditional fuels that burn through chemical reactions.

DEVELOP To make a rough comparison of the potential energy in nuclear and chemical or atomic physics, you model both the nucleus and the atom as square potentials, confining a proton and an electron in their respective cases. The ground-state energy will be inversely proportional to the mass of the confined particle and the square of the potential's width: $E \propto 1/mL^2$.

EVALUATE Using the simplified model above, the ratio of nuclear energy to chemical energy is

$$\frac{E_\text{N}}{E_\text{C}} = \frac{m_\text{e}c^2 L_\text{atom}^2}{m_\text{p}c^2 L_\text{nuc}^2} = \frac{(511\text{ keV})(0.1\text{ nm})^2}{(938\text{ MeV})(1\text{ fm})^2} = 5\times10^6$$

ASSESS This says that there is a million times more energy in the nucleus of an atom than in its electrons. This sounds about right since a kilogram of pure uranium 235 contains $8.2\times10^{13}\text{ J}$, whereas a kilogram of gasoline has $43\text{ MJ} = 4.3\times10^7\text{ J}$ (see Appendix C).

61. **INTERPRET** When transitioning to the ground state, electrons emit photons to release energy. In this problem, we are given the photon wavelengths as a function of the initial states of the electron in a square well. We are interested in determining the width of the quantum well.

DEVELOP The energy levels for an infinite square potential well are given by Equation 35.5:

$$E_n = \frac{n^2h^2}{8mL^2}$$

Thus, the energy of the photon emitted when the electron drops from initial state n_i to final state $n_\text{f} = 1 < n_\text{i}$ is

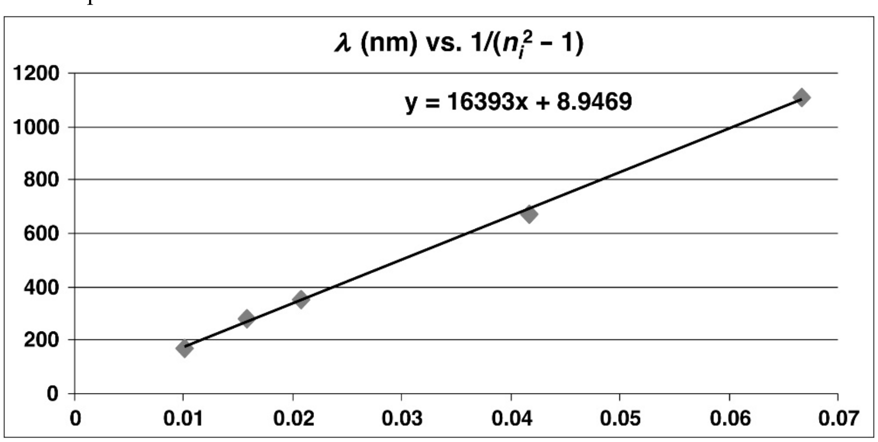
$$E_\gamma = \Delta E = \left(n_\text{i}^2 - 1\right)\frac{h^2}{8mL^2}$$

The possible photon wavelengths are

$$\lambda = \frac{hc}{E_\gamma} = \frac{8mc^2L^2}{\left(n_\text{i}^2 - 1\right)hc}$$

Thus, plotting λ vs. $1/(n_\text{i}^2 - 1)$ will yield a straight line with slope equal to $8mcL^2/h$, which allows us to deduce the value of L.

EVALUATE The plot is shown below.



The slope of the best-fit line is $8mcL^2/h = 16393$ nm, from which we calculate the width of the square well to be

$$L = \sqrt{\frac{(16393\text{ nm})hc}{8mc^2}} = \sqrt{\frac{(16393\text{ nm})(1240\text{ eV}\cdot\text{nm})}{8(0.511\times10^6\text{ eV})}} = 2.23\text{ nm}$$

ASSESS Since energy is quantized, only photons with wavelengths that satisfy the above condition will be emitted during the transitions.

62. **INTERPRET** We evaluate the energy levels of a quantum dot.

DEVELOP A quantum dot is basically a three-dimensional square well. If we assume the qdot is a cube, the energy levels are given by Equation 35.8:

$$E_{n_{x,y,z}} = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

EVALUATE The ground state has $n_x = n_y = n_z = 1$, whereas the first excited state has one of the n's equal to 2. The energy difference between the two states is

$$\Delta E = \frac{h^2}{8mL^2}\left[\left(2^2+1^2+1^2\right)-\left(1^2+1^2+1^2\right)\right] = \frac{3h^2}{8mL^2}$$

If the qdot decreases in size, the energy difference increases. The photon emitted when the qdot drops to its ground state will, therefore, have a smaller wavelength since

$$\lambda = \frac{hc}{\Delta E} \propto L^2$$

The answer is **(b)**.

ASSESS The advantage of qdots is that they are like tunable atoms. You can essentially choose the wavelength at which it absorbs or emits by simply adjusting its size.

63. **INTERPRET** We evaluate the energy levels of a quantum dot.

DEVELOP As was mentioned in the previous problem, a cubically symmetric qdot has a ground state given by

$$E = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2 + n_z^2\right),$$

where $n_x = n_y = n_z = 1.$ In other words,

$$E = \frac{3h^2}{8mL^2}.$$

EVALUATE There is only one ground state since there are no ways to rearrange the n's of the three dimensions. Another way to say this is that the state is nondegenerate.

The answer is **(a)**.

ASSESS The ground state could be degenerate when other quantum numbers are considered. As we'll learn in Chapter 36, the spin of an electron is specified by a quantum number, m_s, which can be either +1/2 or –1/2. Assuming the spin doesn't affect the energy, the state with n_x, n_y, n_z, m_s equal to 1,1,1,+1/2 is degenerate with the state with n_x, n_y, n_z, m_s equal to 1,1,1,–1/2.

64. **INTERPRET** We evaluate the energy levels of a quantum dot.

DEVELOP The first excited state of a cubically symmetric qdot has energy of

$$E = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2 + n_z^2\right),$$

where one of the three quantum numbers equals 2, while the other two equal 1.

EVALUATE There are three states that have the energy

$$E = \frac{6h^2}{8mL^2},$$

that is, n_x, n_y, n_z can equal 2,1,1 or 1,2,1 or 1,1,2. See Figure 35.17. We say this state is threefold degenerate.

The answer is **(c)**.

ASSESS Degeneracy often depends on there being some sort of symmetry. In this case, it is the symmetry of the cube. If the qdot's three sides were not equal, then the first excited state would nondegenerate.

65. **INTERPRET** We evaluate the energy levels of a quantum dot.

DEVELOP For the general case of a quantum with sides of different length, the ground-state energy is written as

$$E_{n_{x,y,z}} = \frac{h^2}{8m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right)$$

EVALUATE If all three sides are reduced in length by half, then the ground-state energy will increase by a factor of 4.

The answer is **(d)**.

ASSESS The special case of a cubical qdot has $E \propto 1/L^2$, which clearly shows that the energy quadruples when the size of the cube shrinks by half.