11

ROTATIONAL VECTORS AND ANGULAR MOMENTUM

EXERCISES

Section 11.1 Angular Velocity and Acceleration Vectors

11. **INTERPRET** This problem is an exercise in determining the direction and magnitude of the angular velocity vector. From the direction and speed at which the car is traveling, we are to deduce the angular velocity of its wheels.

DEVELOP From Chapter 10 (Equation 10.3), we know that the magnitude of the angular velocity (i.e., the angular speed) is given by $\omega = v_{\text{cm}} r.$ For this problem, we have $v_{\text{cm}} = (80\ \text{km/h})(10^3\ \text{m/km})(1\ \text{h}/3600\ \text{s}) = 22.22\ \text{m/s}$ and $r = d/2 = (0.60\ \text{m})/2 = 0.30\ \text{m}.$ The direction of the angular velocity vector can be determined using the right-hand rule (see Fig. 11.1).

EVALUATE Inserting the given quantities into Equation 10.3 gives an angular speed of

$$\omega = v_{\text{cm}}/r = (22.22\ \text{m/s})/(0.3\ \text{m}) = 74\ \text{s}^{-1}$$

to two significant figures. If the car is rolling north, the right-hand rule determines that the direction of the angular velocity vector is to the left, which is west. Therefore, $\vec{\omega} = 74\ \text{s}^{-1}$ west.

ASSESS Notice that the angular speed may be reported in units of $\text{rad/s},$ but since radians are a dimensionless quantity, they are often left out, leaving $\text{s}^{-1},$ which is a frequency (Hz).

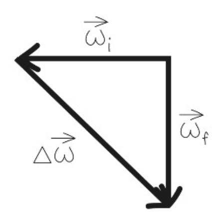
12. **INTERPRET** The problem asks us to determine the angular acceleration of the wheels of a car traveling north with a speed of 70 km/h and that makes a $90°$ left turn that lasts for 25 s.

DEVELOP The speed of the car is $v_{\text{cm}} = 70\ \text{km/h} = 19.44\ \text{m/s}.$ Assuming that the wheels are rolling without slipping, the magnitude of the initial angular velocity is

$$\omega = \frac{v_{\text{cm}}}{r} = \frac{19.44\ \text{m/s}}{0.31\ \text{m}} = 62.7\ \text{s}^{-1}$$

With the car going north, the axis of rotation of the wheels is east–west. Since the top of a wheel is going in the same direction as the car, the right-hand rule gives the direction of $\vec{\omega}_i$ as west. In unit vector notation, we write $\vec{\omega}_i = -\omega\hat{i}$.

After making a left turn, the angular speed remains unchanged, but the direction of $\vec{\omega}_f$ is now south (see sketch). In unit vector notation, we write $\vec{\omega}_f = -\omega\hat{j}$.



EVALUATE Using Equation 11.1, we find the angular acceleration to be

$$\vec{\alpha}_{\text{ave}} = \frac{\Delta\vec{\omega}}{\Delta t} = \frac{\vec{\omega}_f - \vec{\omega}_i}{\Delta t} = \frac{-\omega\hat{j} - (-\omega\hat{i})}{\Delta t} = \frac{\omega}{\Delta t}(\hat{i} - \hat{j})$$
$$= \frac{62.7\ \text{s}^{-1}}{25\ \text{s}}(\hat{i} - \hat{j}) = (2.5\ \text{s}^{-2})(\hat{i} - \hat{j})$$

The magnitude of $\vec{\alpha}_{\text{ave}}$ is

$$|\vec{\alpha}_{\text{ave}}| = \frac{\sqrt{2}\omega}{\Delta t} = \frac{\sqrt{2}(62.7\ \text{rad/s})}{25\ \text{s}} = 3.6\ \text{rad/s}^2$$

and $\vec{\alpha}_{\text{ave}}$ points in the southeast direction (in the direction of the vector $\hat{i} - \hat{j}$).

ASSESS Angular acceleration $\vec{\alpha}_{\text{ave}}$ points in the same direction as $\Delta\vec{\omega}$. The units can be reported as either rad/s^2 or s^{-2}.

13. **INTERPRET** This problem involves calculating the magnitude of the average acceleration given the initial and final angular velocities, and the time interval between the two. We are also asked to find the angle that the average angular acceleration vector makes with the horizontal.

DEVELOP Let the x-axis be the horizontal direction (positive to the right), and the upward direction be the y-axis. The average angular acceleration vector is simply the difference between the final and initial angular velocities divided by the time interval between these two speeds (i.e., Equation 11.1). The initial angular velocity is $\omega_i = (45\ \text{rpm})\,\hat{j}$, the final angular speed is $\omega_f = (60\ \text{rpm})\,\hat{i}$, and the time interval is $t = 15$ s. To find the angle θ the average angular acceleration vector makes with the horizontal, use the fact that that $\tan\theta = \bar{\alpha}_y / \bar{\alpha}_x$.

EVALUATE (a) Inserting the given quantities into Equation 11.1, we find

$$\bar{\alpha} = \frac{\omega_f - \omega_i}{\Delta t} = \frac{(60\ \text{rpm})\hat{i} - (45\ \text{rpm})\hat{j}}{0.25\ \text{min}^{-1}} = (240\ \text{rev}/\text{min}^{-2})\hat{i} - (180\ \text{rev}/\text{min}^{-2})\hat{j}$$
$$= (0.419\ \text{rad/s}^2)\hat{i} - (0.314\ \ \text{rad/s}^2)\hat{j}$$

The magnitude of the average acceleration is thus $\bar{\alpha} = \sqrt{(0.419\ \text{rad/s}^2)^2 + (-0.314\ \text{rad/s}^2)^2} = 0.524\ \text{rad/s}^2$ to two significant figures.

(b) The angle of the average angular acceleration vector with respect to the horizontal is

$$\theta = \tan^{-1}\left(\frac{\bar{\alpha}_y}{\bar{\alpha}_x}\right) = \tan^{-1}\left(\frac{-0.314\ \text{rad/s}^2}{0.419\ \text{rad/s}^2}\right) = -37°$$

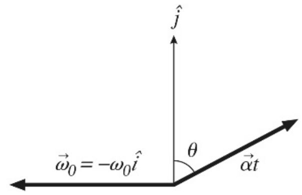
ASSESS Note that the quantities used to calculate part (b) were intermediate quantities, so more significant figures are retained. The final result, however, is reported to two significant figures, which reflects the precision of the data.

14. **INTERPRET** The problem involves angular velocity and angular acceleration. We are given the initial angular velocity of a wheel and asked to find its final angular velocity after an angular acceleration has been applied over a given time interval.

DEVELOP Draw a diagram of the situation with the initial vectors (see figure below). Take the x-axis east and the y-axis north, with positive angles measured CCW from the x-axis. In unit vector notation, the initial angular velocity ω_i and the angular acceleration $\bar{\alpha}$ can be expressed as

$$\omega_i = \omega_i\hat{i} = (140\ \text{rad/s})\hat{i}$$
$$\bar{\alpha} = \alpha(\cos\theta_\alpha\hat{i} + \sin\theta_\alpha\hat{j}) = (35\ \text{rad/s}^2)\left[\cos(90° + 68°)\hat{i} + \sin(90° + 68°)\,\hat{j}\right]$$
$$= (-32.45\ \text{rad/s}^2)\hat{i} + (13.11\ \text{rad/s}^2)\hat{j}$$

The final angular velocity can be found by using Equation 11.1.



EVALUATE Using Equation 11.1, the angular velocity at $t = 5.0$ s is

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t}$$

$$\omega_f = \omega_i + \bar{\alpha}t = (140\ \text{rad/s})\hat{i} + \left[\left(-32.45\ \text{rad/s}^2\right)\hat{i} + \left(13.11\ \text{rad/s}^2\right)\hat{j}\right](5.0\ \text{s})$$
$$= (-22\ \text{rad/s})\hat{i} + (66\ \text{rad/s})\hat{j}$$

to two significant figures. The magnitude and direction of $\vec{\omega}_f$ are

$$\omega_f = \sqrt{(-22.3\ \text{rad/s})^2 + (65.6\ \text{rad/s})^2} = 69\ \text{rad/s}$$

and

$$\theta_f = \tan^{-1}\left(\frac{\omega_{f,y}}{\omega_{f,x}}\right) = \tan^{-1}\left(\frac{-22.3\ \text{rad/s}}{65.6\ \text{rad/s}}\right) = -19°$$

or 19° west of north.

ASSESS Because the x-component of the angular acceleration is negative, $\Delta\omega_x$ is also negative. On the other hand, a positive α_y yields $\Delta\omega_y > 0$.

Section 11.2 Torque and the Vector Cross Product

15. **INTERPRET** This problem involves finding the torque about the origin given a force and the position vector that indicates where the force is applied.

DEVELOP Use Equation 11.2 to find the torque. The position vector r is $\vec{r} = (3\ \text{m})\hat{i} + (1\ \text{m})\hat{j}$.

EVALUATE (a) For a force $\vec{F} = (12\ \text{N})\hat{i}$, the torque is

$$\vec{\tau} = \vec{r} \times \vec{F} = \left[(3\ \text{m})\hat{i} + (1\ \text{m})\hat{j}\right] \times (12\ \ \text{N}\cdot\text{m})\hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\ \text{m} & 1\ \text{m} & 0\ \text{m} \\ 12\ \text{N} & 0\ \text{N} & 0\ \text{N} \end{vmatrix} = (-12\ \text{N}\cdot\text{m})\hat{k}$$

(b) For a force $\vec{F} = (12\ \text{N})\hat{j}$

$$\vec{\tau} = \vec{r} \times \vec{F} = \left[(3\ \text{m})\hat{i} + (1\ \text{m})\hat{j}\right] \times (12\ \ \text{N}\cdot\text{m})\hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\ \text{m} & 1\ \text{m} & 0\ \text{m} \\ 0\ \text{N} & 12\ \text{N} & 0\ \text{N} \end{vmatrix} = (36\ \text{N}\cdot\text{m})\hat{k}$$

(c) For a force $\vec{F} = (12\ \text{N})\hat{k}$

$$\vec{\tau} = \vec{r} \times \vec{F} = \left[(3\ \text{m})\hat{i} + (1\ \text{m})\hat{j}\right] \times (12\ \ \text{N}\cdot\text{m})\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\ \text{m} & 1\ \text{m} & 0\ \text{m} \\ 0\ \text{N} & 0\ \text{N} & 12\ \text{N} \end{vmatrix} = (12\ \text{N}\cdot\text{m})\hat{i} + (-36\ \text{N}\cdot\text{m})\hat{j}$$

ASSESS For part (c), the magnitude is $\tau = \sqrt{(12\ \text{N}\cdot\text{m})^2 + (-36\ \text{N}\cdot\text{m})^2} = 38\ \text{N}\cdot\text{m}$ and the direction is $\theta = \text{atan}(-36\ \text{N}\cdot\text{m}/12\ \text{N}\cdot\text{m}) = -72°$, or $72°$ clockwise from the x-axis and in the x–y plane.

16. **INTERPRET** We are asked to find the torque about two different points produced by an applied force. The problem is about taking a cross product.

DEVELOP The torque vector is defined as $\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{F}$ is the force vector and $\vec{r}$ is the position vector which points from the axis of rotation to the point where the force is acting. The direction of $\vec{\tau}$ is determined by the right-hand rule.

EVALUATE **(a)** For this part, $\vec{r} = (3\text{ m})\hat{\imath}$. Therefore, with $\vec{F} = (1.3\text{ N})\hat{\imath} + (2.7\text{ N})\hat{\jmath}$, the torque is

$$\vec{\tau} = \vec{r} \times \vec{F} = (3\text{ m})\hat{\imath} \times \left[(1.3\text{ N})\hat{\imath} + (2.7\text{ N})\hat{\jmath}\right] = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3\text{ m} & 0\text{ m} & 0\text{ m} \\ 1.3\text{ N} & 2.7\text{ N} & 0\text{ N} \end{vmatrix} = (8.1\text{ N}\cdot\text{m})\hat{k}$$

(b) Here we have $\vec{r} = (3\text{ m})\hat{\imath} - \left[(-1.3\text{ m})\hat{\imath} + (2.4\text{ m})\hat{\jmath}\right] = (4.3\text{ m})\hat{\imath} - (2.4\text{ m})\hat{\jmath}$. Therefore, the torque is

$$\vec{\tau} = \vec{r} \times \vec{F} = \left[(4.3\text{ m})\hat{\imath} - (2.4\text{ m})\hat{\jmath}\right] \times \left[(1.3\text{ N})\hat{\imath} + (2.7\text{ N})\hat{\jmath}\right] = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4.3\text{ m} & -2.4\text{ m} & 0\text{ m} \\ 1.3\text{ N} & 2.7\text{ N} & 0\text{ N} \end{vmatrix}$$

$$= (11.6\text{ N}\cdot\text{m})\hat{k} + (3.1\text{ N}\cdot\text{m})\hat{k} = (15\text{ N}\cdot\text{m})\hat{k}$$

ASSESS The torque vector $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$. It points in the direction normal to the plane formed by $\vec{r}$ and $\vec{F}$.

17. **INTERPRET** You want to know what torque is supplied by the deltoid muscle about the shoulder joint when your arm is outstretched.

DEVELOP From Equation 11.2, the torque is $\vec{\tau} = \vec{r} \times \vec{F}$, with the magnitude equaling $rF\sin\theta$.

EVALUATE The distance between the shoulder joint (i.e., where the arm pivots) and the point at which the deltoid force is applied is given as $r = 18\text{ cm}$. The angle between the corresponding radial vector and the muscle force is $\theta = 180^\circ - 15^\circ = 165^\circ$. The magnitude of the torque is then

$$\tau = rF\sin\theta = (0.18\text{ m})(62\text{ N})\sin 165^\circ = 2.9\text{ N}\cdot\text{m}$$

By the right-hand rule, we start with our fingers pointing to the right in the direction of $\vec{r}$, and then rotate them upward in the direction of $\vec{F}$. Our thumb points up, so the torque of $2.9\text{ N}\cdot\text{m}$ points out of the page.

ASSESS Is this enough torque to keep the arm outstretched? Let's assume the arm has a mass of about 3 kg (corresponding to a weight of about 30 N), and its center of mass is 30 cm from the shoulder joint. The gravitational force will pull the arm down at 90° to the horizontal arm direction, thus generating a torque in the opposite direction with a magnitude of $\tau \simeq (30\text{ N})(0.3\text{ m}) = 9\text{ N}\cdot\text{m}$. Therefore, the deltoid muscle would need help from other muscles to keep the arm horizontal.

Section 11.3 Angular Momentum

18. **INTERPRET** This problem involves a dimensional analysis of angular momentum. We are to express angular momentum in terms of the fundamental SI units, in terms of Newtons, and in terms of Joules.

DEVELOP Angular momentum is given by Equation 11.3, $\vec{L} = \vec{r} \times \vec{p}$. Given that r has units of distance (m) and p has units of mass times velocity, we can find the SI units of angular momentum. From Newton's second law (for constant mass) $F = ma$, we see that force is the product of mass and acceleration, so the units of a Newton are $\text{kg}\cdot\text{m}\cdot\text{s}^{-2}$. Similarly, energy (J) can be expressed as a force multiplied by distance (consider work, $W = \vec{F}\cdot\Delta\vec{r}$, Equation 6.5), so the SI units of a the joule are $\text{kg}\cdot\text{m}^2\cdot\text{s}^{-2}$.

EVALUATE (a). Using the dimensions of linear momentum (= $\text{kg}\cdot\text{m}\cdot\text{s}^{-1}$), the units of angular momentum are $(\text{kg}\cdot\text{m}\cdot\text{s}^{-1})(\text{m}) = \text{kg}\cdot\text{m}^2\cdot\text{s}^{-1}$.

(b) Because the units of a Newton are $\text{kg}\cdot\text{m}\cdot\text{s}^{-2}$, angular momentum can be expressed as $\text{kg}\cdot\text{m}^2\cdot\text{s}^{-1} = \text{N}\cdot\text{m}\cdot\text{s}$.

(c) Because energy (J) can be expressed as force times distance, we have $\text{J} = \text{N}\cdot\text{m}$, so the units of angular momentum are $\text{kg}\cdot\text{m}^2\cdot\text{s}^{-1} = (\text{N}\cdot\text{m})\cdot\text{s} = \text{J}\cdot\text{s}$.

ASSESS From Equation 11.2, we see that torque has units of N·m, so a torque multiplied by a time gives an angular momentum. This is just the definition of an angular impulse.

19. **INTERPRET** This problem asks us to estimate the angular momentum of our solar system about the galactic center using data from Appendix E.

DEVELOP The angular momentum of an object about a point is defined as (see Equation 11.3)

$$\vec{L} = \vec{r} \times \vec{p}$$

where $\vec{p}$ is the linear momentum and $\vec{r}$ is the position vector of the object relative to that point. We may also express $\vec{L}$ as

$$\vec{L} = \vec{r} \times \vec{p} = rp\sin\theta\hat{n}$$

where θ is the angle between $\vec{r}$ and $\vec{p}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{r}$ and $\vec{p}$. For this problem, we can assume that the Sun and the revolving bodies travel in a circle of radius r and speed v. Since the velocity of the system $\vec{v}$ is perpendicular to $\vec{r}$, the magnitude of the angular momentum about the galactic center is

$$L = |\vec{r} \times \vec{p}| = rp = rmv$$

Since the Sun is orders of magnitude more massive than all the other bodies combined, we can use its mass, mean distance from central body, and mean orbital speed for this calculation (all found in Appendix E).

EVALUATE Evaluating the angular momentum gives, approximately

$$L_S = r_S M_S v_S = \left(2.6\times10^{20}\text{ m}\right)\left(1.99\times10^{30}\text{ kg}\right)\left(2.5\times10^{5}\text{ m/s}\right) \approx 10^{56}\text{ kg}\cdot\text{m}^2\cdot\text{s}^{-1}$$

ASSESS The direction of $\vec{L}_S$ is parallel to the axis of rotation. It is perpendicular to both $\vec{v}_S$ and $\vec{r}_S$.

20. **INTERPRET** For this problem, we are to find the angular speed, given the angular momentum and the rotational inertia of an object.

DEVELOP Use Equation 11.4, $\vec{L} = I\vec{\omega}$, to find the angular speed of the gymnast.

EVALUATE Given that $L = 460\text{ kg}\cdot\text{m}^2\cdot\text{s}^{-1}$ and $I = 63\text{ kg}\cdot\text{m}^2$, the angular speed of the gymnast must be

$$\omega = \frac{L}{I} = \frac{460\text{ kg}\cdot\text{m}^2\cdot\text{s}^{-1}}{63\text{ kg}\cdot\text{m}^2} = 7.3\text{ s}^{-1}$$

ASSESS The angular speed has units of frequency, as expected. It may equivalently be expressed as 7.3 rad/s, because radians are dimensionless.

21. **INTERPRET** Given the elements of rotational inertia and the angular velocity of the hoop, we are to find the corresponding angular momentum. We will need to use Table 10.2 to find the rotational inertia.

DEVELOP For an object rotating about a fixed axis, its angular momentum can be expressed as (see Equation 11.4) $\vec{L} = I\vec{\omega}$, where I is the moment of inertia of the object, and $\vec{\omega}$ is its angular velocity about its axis. From Table 10.2, we find that the rotational inertia of a hoop rotating about its axis is $I = mr^2$.

EVALUATE With $\omega = 170\text{ rpm} = 17.8\text{ rad/s}$, the magnitude of $\vec{L}$ is

$$L = I\omega = mr^2\omega = (0.66\text{ kg})(0.475\text{ m})^2(17.8\text{ rad/s}) = 2.7\text{ J}\cdot\text{s}$$

The direction of $\vec{L}$ is along the axis of rotation according to the right-hand rule.

ASSESS The angular momentum vector $\vec{L}$ points in the same direction as $\vec{\omega}$.

22. **INTERPRET** The problem asks for the angular momentum of a spinning golf ball.

DEVELOP Equation 11.4 gives the angular momentum as $\vec{L} = I\vec{\omega}$. In this case, we are only concerned with the magnitude of the golf ball's angular momentum. We are told to treat the ball as a uniform solid sphere spinning about an axis through its center, in which case its rotational inertia is given by $I = \frac{2}{5}MR^2$ (from Table 10.1).

EVALUATE Taking care to convert the rotational speed to rad/s, the angular momentum is

$$L = \frac{2}{5}MR^2\omega = \frac{2}{5}(0.045\text{ kg})\left(\frac{1}{2}0.043\text{ m}\right)^2(3000\text{ rpm})\left[\frac{2\pi\text{ rad/s}}{60\text{ rpm}}\right] = 2.6\times10^{-3}\text{ J}\cdot\text{s}$$

ASSESS The value seems reasonable. The units are correct, since $\text{kg}\cdot\text{m}^2/\text{s} = (\text{kg}\cdot\text{m}^2/\text{s}^2)\cdot\text{s} = \text{J}\cdot\text{s}$.

Section 11.4 Conservation of Angular Momentum

23. **INTERPRET** This problem involves conservation of angular momentum, which we can use to find the angular speed of a spinning wheel after a piece of clay is dropped onto it and sticks to its surface.

DEVELOP If the clay is dropped vertically onto a horizontally spinning wheel, the angular momentum about the vertical spin axis is conserved. Conservation of angular momentum is expressed as

$$\vec{L}_{\mathrm{i}} = \vec{L}_{\mathrm{f}} \Rightarrow I_{\mathrm{i}}\vec{\omega}_{\mathrm{i}} = I_{\mathrm{f}}\vec{\omega}_{\mathrm{f}}$$

For this problem, the direction of the angular velocity does not change, so this expression for conservation of angular momentum reduces to its scalar form, $I_{\mathrm{i}}\omega_{\mathrm{i}} = I_{\mathrm{f}}\omega_{\mathrm{f}}$. The initial rotational inertia is $I_{\mathrm{i}} = I_{\mathrm{wheel}} = 6.20\ \mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}$, and the final rotational inertia is $I_{\mathrm{f}} = I_{\mathrm{wheel}} + m_{\mathrm{clay}}r^2$.

EVALUATE Inserting the given quantities into the expression for conservation of angular momentum, the final angular velocity is

$$\omega_{\mathrm{f}} = \frac{I_{\mathrm{i}}}{I_{\mathrm{f}}}\omega_{\mathrm{i}} = \left(\frac{I_{\mathrm{wheel}}}{I_{\mathrm{wheel}} + m_{\mathrm{clay}}r^2}\right)\omega_{\mathrm{i}} = \frac{6.20\ \mathrm{kg}\cdot\mathrm{m}^2}{6.20\ \mathrm{kg}\cdot\mathrm{m}^2 + (2.50\ \mathrm{kg})(0.48\ \mathrm{m})^2}(20.0\ \mathrm{rpm}) = 18.3\ \mathrm{rpm}$$

ASSESS The clay increases the total rotational inertia of the system, so the angular speed decreases, as required by conservation of angular momentum.

24. **INTERPRET** This problem involves conservation of angular momentum and the work-energy theorem. The former we can use to find the angular speed of the merry-go-round after the children sit on it, and the latter we can use to find the energy lost in the transaction.

DEVELOP Conservation of angular momentum demands that $I_{\mathrm{i}}\vec{\omega}_{\mathrm{i}} = I_{\mathrm{f}}\vec{\omega}_{\mathrm{f}}$. The initial rotational inertia is $I_{\mathrm{i}} = 120$ kg·m^2, and the initial angular velocity is $\omega_{\mathrm{i}} = 0.50$ rev/s. The final rotational inertia is $I_{\mathrm{f}} = I_{\mathrm{i}} + 4m_{\mathrm{c}}r^2$, where m_{c} is the mass of one child. To find the energy lost when the children jump onto the merry-go-round, consider the work-energy theorem Equation 6.14, $\Delta K = W_{\mathrm{net}} = \vec{f}_{\mathrm{k}}\cdot\Delta\vec{r}$, where f_{k} is the force due to friction, which acts parallel to $\Delta\vec{r}$. From the rotational version of the work-energy theorem, Equation 10.19, we see that we can find the change in kinetic energy using the result of part (a).

EVALUATE (a) From conservation of momentum, we have

$$\begin{aligned}
I_{\mathrm{i}}\omega_{\mathrm{i}} &= I_{\mathrm{f}}\omega_{\mathrm{f}}\\
\omega_{\mathrm{f}} &= \omega_{\mathrm{i}}\frac{I_{\mathrm{i}}}{I_{\mathrm{f}}} = \omega_{\mathrm{i}}\frac{I_{\mathrm{i}}}{I_{\mathrm{i}} + 4m_{\mathrm{c}}r^2} = (0.50\ \mathrm{rev/s})\frac{120\,\mathrm{kg\cdot m^2}}{120\,\mathrm{kg\cdot m^2} + 4(25\ \mathrm{kg})(3.0\ \mathrm{m})^2}\\
&= (0.174\ \mathrm{rev/s})\left(\frac{2\pi\ \mathrm{rad}}{\mathrm{rev}}\right) = 1.09\ \ \mathrm{rad/s}
\end{aligned}$$

(b) Using $\omega_{\mathrm{i}} = (0.50\ \mathrm{rev/s})(2\pi\,\mathrm{rad/rev}) = \pi\,\mathrm{rad/s}$, the change in kinetic energy is $\Delta K = K_{\mathrm{f}} - K_{\mathrm{i}}$, which gives

$$\begin{aligned}
\Delta K &= \frac{1}{2}I_{\mathrm{f}}\omega_{\mathrm{f}}^2 - \frac{1}{2}I_{\mathrm{i}}\omega_{\mathrm{i}}^2 = \frac{\left(I_{\mathrm{i}} + 4m_{\mathrm{c}}r^2\right)\omega_{\mathrm{f}}^2 - I_{\mathrm{i}}\omega_{\mathrm{i}}^2}{2}\\
&= \frac{\left[120\,\mathrm{kg\cdot m^2} + 4(25\ \mathrm{kg})(3.0\ \mathrm{m})^2\right](1.09\ \mathrm{rad/s})^2 - \left(120\,\mathrm{kg\cdot m^2}\right)(\pi\ \mathrm{rad/s})^2}{2}\\
&= 386\ \mathrm{J}
\end{aligned}$$

ASSESS We could also find the energy lost using the fact that $\Delta K = I\omega^2 = L^2/I$. This gives

$$\begin{aligned}
\Delta K &= \frac{1}{2}\left(\frac{1}{I_{\mathrm{f}}} - \frac{1}{I_{\mathrm{i}}}\right)L^2\\
&= \frac{1}{2}\left(\frac{1}{345\ \mathrm{kg\cdot m^2}} - \frac{1}{120\ \mathrm{kg\cdot m^2}}\right)\left(120\ \mathrm{kg\cdot m^2}\times\pi\ \mathrm{s^{-1}}\right)^2 = 386\ \mathrm{J}
\end{aligned}$$

where we have used the fact that angular momentum is conserved so $L_{\mathrm{i}} = L_{\mathrm{f}} = L$.

25. **INTERPRET** In this problem, we are asked about the period of a star formed by a collapsing cloud. We can use conservation of angular momentum to find the answer.

DEVELOP If we assume there are no external torques and no mass loss during the collapse of the star-forming cloud, its angular momentum is conserved, so $I_{\mathrm{i}}\vec{\omega}_{\mathrm{i}} = I_{\mathrm{f}}\vec{\omega}_{\mathrm{f}}$. The initial and final rotational inertias may be found from Table 10.2, which gives $I_\text{i} = 2mr_\text{i}^2/5$ and $I_\text{f} = 2mr_\text{f}^2/5$. From this, we can solve for the final period of the star using $T_\text{f} = 2\pi/\omega_\text{f}$.

EVALUATE Given that the mass involved does not change, conservation of angular momentum gives

$$I_\text{i}\vec{\omega}_\text{i} = I_\text{f}\vec{\omega}_\text{f}$$
$$\frac{2}{5}MR_\text{i}^2\omega_\text{i} = \frac{2}{5}MR_\text{f}^2\omega_\text{f}$$
$$\frac{\omega_\text{i}}{\omega_\text{f}} = \left(\frac{R_\text{f}}{R_\text{i}}\right)^2$$

Thus, the final period is

$$T_\text{f} = T_\text{i}\left(\frac{R_\text{f}}{R_\text{i}}\right)^2 = (1.5\times10^6\text{ y})\left(\frac{7.5\times10^8\text{ m}}{1.0\times10^{13}\text{ m}}\right)^2 = 7.88\times10^{-3}\text{ y} = 2.9\text{ days}$$

ASSESS In current models of star formation, the collapsing cloud does not maintain a spherical shape, forming a flattened disk instead, and the central star retains just a fraction of the original cloud's mass.

26. **INTERPRET** This problem involves a skater holding two weights in her hands. Her rotational speed will change when she brings the weights to her chest.

DEVELOP If the skater is twirling on frictionless horizontal ice, her angular momentum about the vertical rotation axis is conserved: $I_\text{i}\omega_\text{i} = I_\text{f}\omega_\text{f}$. When the arms are initially outstretched, the weights contribute to the rotational inertia: $I_\text{i} = I_\text{s,out} + 2MR^2$. Here, $I_\text{s,out}$ is the skater's rotational inertia when her arms are outstretched, and R is the distance of the weights from the rotational axis. When the arms are brought in to the chest, we assume they no longer contribute to the rotational inertia, since the distance from the axis goes to zero. The final rotational inertia is just that of the skater with arms to her chest: $I_\text{f} = I_\text{s,in}$.

EVALUATE Solving for the final rotational speed gives:

$$\omega_\text{f} = \frac{I_\text{i}}{I_\text{f}}\omega_\text{i} = \frac{5.3\text{ kg}\cdot\text{m}^2 + 2(1.8\text{ kg})(0.68\text{ m})^2}{3.8\text{ kg}\cdot\text{m}^2}(3.0\text{ rev/s}) = 5.5\text{ rev/s}$$

ASSESS The skater almost doubles her speed, which seems reasonable. If she didn't have the weights in her hand, her final rotational speed would be 4.2 rev/s, which is only a 40% increase over the initial speed.

EXAMPLE VARIATIONS

27. **INTERPRET** In this problem we are given the mass and velocity of an SUV as it makes a circular turn and are asked for the magnitude of its angular momentum.

DEVELOP Since we can easily calculate the linear momentum, $\vec{p} = m\vec{v}$, of the vehicle, we can use Equation 11.3, $\vec{L} = \vec{r}\times\vec{p}$, to determine its angular momentum.

EVALUATE Plugging in the given values for the SUV we get

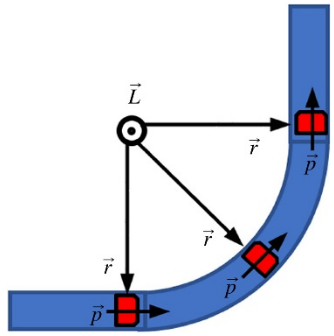
$$L = rmv = (175\text{m})(2150\text{kg})(18\text{m/s}) = 6.79\times10^6\text{ kg}\cdot\text{m}^2\cdot\text{s}^{-1}$$

Where we have converted from km/h to m/s.

ASSESS Here we are told the speed of the SUV as it makes a turn, so the velocity is perpendicular to the position vector that defines the circular turn for which the radius was given.

28. **INTERPRET** In this problem we are analyzing the angular momentum of a vehicle once it has completed a left turn. We want to know whether the direction and magnitude are unchanged once it has exited the turn.

DEVELOP Equation 11.3, $\vec{L} = \vec{r}\times\vec{p}$, describes the cross product that defines the direction and magnitude of an object's angular momentum. Looking at the figure below, we can apply the right-hand rule to determine the direction of the angular momentum at any point during the car's turn.



We can also use Equation 11.3 to determine the speed of the car since we have the magnitude of the angular momentum, the car's mass, and the radius of the turn.

EVALUATE Applying the right-hand rule at the point the car has exited the turn we find that the direction of the angular momentum is perpendicular to both $\vec{p}$ and $\vec{r}$, meaning it must be directed up (or out of the page). This is true right before, during, and right after the turn. Solving for the speed v from Equation 11.3 and plugging in our values we obtain

$$v = \frac{L}{rm} = \frac{\left(2.86\times10^{6}\,\mathrm{kg\cdot m^{2}\cdot s^{-1}}\right)}{(125\,\mathrm{m})(1150\,\mathrm{kg})} = 19.9\,\mathrm{m/s} = 71.6\,\mathrm{km/h}$$

This speed, along with the distance r from the center of the turn is maintained once the car has exited the turn, and thus the angular momentum remains unchanged at $2.86\times10^{6}\,\mathrm{kg\cdot m^{2}\cdot s^{-1}}$

ASSESS The direction of the angular momentum will remain pointing up relative to the car since the linear momentum and position vector of the vehicle will remain in the plane of its motion.

29. **INTERPRET** In this problem we are analyzing the angular momentum of a ball whirling in a horizontal circle. We want to determine the direction and magnitude of the angular momentum.

DEVELOP Equation 11.3, $\vec{L} = \vec{r}\times\vec{p}$, describes the cross product that defines the direction and magnitude of an object's angular momentum. Since we know the direction of its rotation, we can apply the right-hand rule to determine the direction along with the magnitude of the angular momentum.

EVALUATE Plugging in the given values we find the magnitude of the ball's angular momentum is equal to

$$L = rmv = (0.843\,\mathrm{m})(0.0582\,\mathrm{kg})(5.87\,\mathrm{m/s}) = 0.288\,\mathrm{kg\cdot m^{2}\cdot s^{-1}}$$

Looking at the ball from below we see it moves counterclockwise, meaning that from above the ball moves in a clockwise direction. Applying the right-hand rule, we see that an object rotating clockwise about a point will have its angular momentum pointing downward.

ASSESS By definition, counterclockwise rotation is defined as the positive angular direction of motion. Thus, any objects moving counterclockwise will have a positive angular momentum $\vec{L}$, which is parallel to the angular frequency $\vec{\omega}$, and perpendicular to both the linear momentum $\vec{p}$ and position vector $\vec{r}$.

30. **INTERPRET** In this problem we are analyzing the angular momentum of a ball whirling in a horizontal circle. We want to determine the angle the string makes with the horizontal, and the linear velocity of the ball.

DEVELOP Equation 11.3, $\vec{L} = \vec{r}\times\vec{p}$, describes the cross product that defines the direction and magnitude of an object's angular momentum. However, in this situation the vector $\vec{r}$ doesn't lie in the plane of the circular path made by the ball, but instead makes an angle φ with the vertical axis. The component of $\vec{r}$ which lies in the plane of the circle is what contributes to the magnitude of the angular momentum, while the component of $\vec{r}$ which is parallel to the vertical determines how the vector $\vec{L}$ is directed as the ball whirls around. Thus, we consider the magnitude of $\vec{L}$ to be $L = pl\sin\varphi = pl\cos\theta$, where the angle θ is between the string of length l and the horizontal direction. Looking at fig. 5.11 from Example 5.5 we see that the ball's velocity is given by

$$v = \sqrt{\frac{gl\cos^{2}\theta}{\sin\theta}}$$

This allows us to obtain an expression for θ in terms of the known angular momentum L, mass of the ball m, and string length l.

EVALUATE From our expression for the angular momentum and velocity we find

$$L = mvl\cos\theta = ml\cos\theta\sqrt{\frac{gl\cos^2\theta}{\sin\theta}}$$

$$\cos^3\theta\cot\theta = \frac{L^2}{gm^2l^3}$$

We can solve this numerically for the values given in the problem statement to obtain $\theta = 14.1°$. Plugging this value back into the expression for the velocity we obtain $v = 6.15\,\text{m/s}$.

ASSESS The direction of the angular momentum $\vec{L}$ is directed at angle θ from the vertical, about which it rotates as the ball whirls in a circle.

31. **INTERPRET** In this problem we are using the conservation of angular momentum to determine the change in rotational frequency for a star after its radius has changed by a given amount.

DEVELOP We are told no torques have acted on the core, meaning $L_\text{i} = L_\text{f}$, and thus: $I_\text{i}\omega_\text{i} = I_\text{f}\omega_\text{f}$. Modeling the rotational inertia as a solid uniform sphere using the expression found in Table 10.2: $I = \frac{2}{5}MR^2$, we can relate the change in angular frequency to the change in the core's radius.

EVALUATE Given I, the conservation of angular momentum becomes $\frac{2}{5}MR_\text{i}{}^2\omega_\text{i} = \frac{2}{5}MR_\text{f}{}^2\omega_\text{f}$, or

$$\omega_\text{f} = \omega_\text{i}\left(\frac{R_\text{i}}{R_\text{f}}\right)^2 \rightarrow T_\text{f} = T_\text{i}\left(\frac{R_\text{f}}{R_\text{i}}\right)^2 = (34.4\,\text{day})\frac{(60\cdot 24\,\text{min})}{(1\,\text{day})}\left(\frac{(4.21\times10^6\,\text{m})}{(4.96\times10^8\,\text{m})}\right)^2 = 3.57\,\text{min}$$

Where we have used the fact that the period of rotation is related to the angular frequency by $T = 2\pi/\omega$.

ASSESS As the rotational frequency increases, the rotational period decreases, due to their inverse relationship.

32. **INTERPRET** In this problem we are using the conservation of angular momentum to determine the change in radius for a star after its rotational frequency has changed by a given amount.

DEVELOP Assuming no torques have acted on the core, meaning $L_\text{i} = L_\text{f}$, then: $I_\text{i}\omega_\text{i} = I_\text{f}\omega_\text{f}$. Modeling the rotational inertia as a solid uniform sphere using the expression found in Table 10.2: $I = \frac{2}{5}MR^2$, we can relate the change in angular frequency to the change in the core's radius.

EVALUATE Given I, the conservation of angular momentum becomes $\frac{2}{5}MR_\text{i}{}^2\omega_\text{i} = \frac{2}{5}MR_\text{f}{}^2\omega_\text{f}$, or

$$\omega_\text{f} = \omega_\text{i}\left(\frac{R_\text{i}}{R_\text{f}}\right)^2 \rightarrow R_\text{i} = \sqrt{\frac{T_\text{i}}{T_\text{f}}}R_\text{f} = \sqrt{\frac{(49.3\,\text{day})}{(3.17\times10^{-5}\,\text{day})}}(7.10\,\text{km}) = 8.85\,\text{Mm}$$

Where we have used the fact that the period of rotation is related to the ordinary frequency by $T = 1/f$ to find the final rotational period $T_\text{f} = 1/\left[(21.9\,\text{rpm})\left(\frac{(1\,\text{min})}{(60\,\text{s})}\right)\right] = 2.74\,\text{s} = 3.17\times10^{-5}$ day.

ASSESS The ordinary frequency f is related to the angular frequency by $\omega = 2\pi f$, allowing us to easily relate the given revolutions per minute to a rate in Hertz from which we can then determine a final rotational period.

33. **INTERPRET** In this problem we are using the conservation of angular momentum to determine the change in a skater's spin rate after their rotational inertia has changed by a given amount.

DEVELOP Assuming no torques have acted on the core, meaning $L_\text{i} = L_\text{f}$, then: $I_\text{i}\omega_\text{i} = I_\text{f}\omega_\text{f}$. We can thus solve for the final rotational frequency and evaluate it using the given rotational inertias and initial rotational frequency of the skater.

EVALUATE From the angular momentum conservation we find that

$$\omega_f = \omega_i \frac{I_i}{I_f} = (1.66\ \text{rev/s})\frac{(3.56\,\text{kg}\cdot\text{m}^2)}{(1.21\,\text{kg}\cdot\text{m}^2)} = 4.88\ \text{rev/s}$$

ASSESS Since the angular momentum is conserved, so is the product of rotational inertia and rotational frequency. This means that if one decreases the other must increase, as is the case for the skater's rotational inertia and rotational frequency, respectively.

34. **INTERPRET** In this problem we are using the concept of angular momentum to determine the change in a skater's spin rate after they've caught a ball, which will impart a certain amount of torque on their body.

DEVELOP The ball will impart a torque on the skater's body as they catch it, which will cause their angular momentum to change by an amount equal to $\pm mvr$, depending on which hand they catch it with. This is because the ball arrives at the skater's hand, a distance r away from the axis of rotation, with a perpendicular velocity v. If the skater's angular momentum is pointing upward, they must be spinning counterclockwise, meaning that catching the ball with their right hand will increase their rotational frequency, whereas the opposite will be true if they catch it with their left hand. We can write, $L_f = L_i \pm mvr$, and thus: $I_f\omega_f = I_i\omega_i \pm mvr$. Solving for the final rotational frequency allows us to determine the skater's spin rate in both scenarios: catching with the left or right hand.

EVALUATE From our expression for the angular momentum we find that

$$\omega_f = \frac{I_i\omega_i \pm mvr}{I_f} = \frac{I_i\omega_i \pm mvr}{I_i + mr^2}$$

Where we note that final rotational inertia too has changed by an amount mr^2 due to the added mass m of the ball a distance r away from the rotational axis of the skater. Plugging in the given values from the problem statement we find that ω_f is equal to 1.04 rev/s if caught with the left hand, and 0.853 rev/s if caught with the right hand.

ASSESS The change in rotational inertia caused by catching the ball did not depend on which hand was used, whereas the change in rotational frequency did, due to its dependence on the direction of the torque which was applied.

PROBLEMS

35. **INTERPRET** This problem is an exercise in calculating torque, given the force and the position relative to an axis at which the force is applied.

DEVELOP Use Equation 11.2, $\vec{\tau} = \vec{r}\times\vec{F}$, to calculate the torque, given that $\vec{r} = (25\text{ cm})\hat{i} + (6.5\text{ cm})\hat{j}$ and $\vec{F} = (62\text{ N})\hat{i} - (16\text{ N})\hat{j}$.

EVALUATE Evaluating the cross product gives

$$\vec{\tau} = \left[(25\text{ cm})\hat{i} + (6.5\text{ cm})\hat{j}\right]\times\left[(62\text{ N})\hat{i} - (16\text{ N})\hat{j}\right] = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 25\text{ cm} & 6.5\text{ cm} & 0\text{ cm} \\ 62\text{ N} & -16\text{ N} & 0\text{ N}\end{pmatrix} = (-8.0\text{ N}\cdot\text{m})\hat{k}$$

ASSESS Thus, the torque is in the direction defined by the bolt.

36. **INTERPRET** The problem is an exercise in vector multiplication (cross product). It asks us to find the direction of a vector $\vec{B}$, given the directions of another vector $\vec{A}$ and their cross product $\vec{A}\times\vec{B}$.

DEVELOP The cross product, $\vec{A}\times\vec{B}$, is perpendicular to the plane defined by $\vec{A}$ and $\vec{B}$. We are given that $\vec{A}\times\vec{B} = -A^2\hat{k}$ and these vectors lie in the *x*–*y* plane. For simplicity, let's write the two vectors as

$$\vec{A} = A\left(\cos\theta_A\hat{i} + \sin\theta_A\hat{j}\right)$$
$$\vec{B} = B\left(\cos\theta_B\hat{i} + \sin\theta_B\hat{j}\right)$$

where $\theta_A = 30°$ and θ_B are measured counterclockwise from the *x*-axis. Using the above expressions, the cross product $\vec{A}\times\vec{B}$ is

$$\vec{A}\times\vec{B} = AB\left(\cos\theta_A\hat{i} + \sin\theta_A\hat{j}\right)\times\left(\cos\theta_B\hat{i} + \sin\theta_B\hat{j}\right) = AB\left(\cos\theta_A\sin\theta_B - \sin\theta_A\cos\theta_B\right)\hat{k}$$
$$= -AB\sin\left(\theta_A - \theta_B\right)\hat{k}$$

Using the information given in the problem statement, the angle θ_B can be calculated.

EVALUATE The problem states that $\vec{A}\times\vec{B} = -A^2\hat{k}$. The right-hand rule implies that the angle between $\vec{A}$ and $\vec{B}$, measured clockwise from $\vec{A}$, is less than 180°; namely, $\theta_A - \theta_B < 180°$ or $-150° < \theta_B < 30° = \theta_A$.
The magnitude of $\vec{A}\times\vec{B}$ is $AB\sin\left(\theta_A - \theta_B\right) = 2A^2\sin\left(\theta_A - \theta_B\right) = A^2$ (as given, with $B = 2A$), so

$$\sin\left(\theta_A - \theta_B\right) = \frac{1}{2}$$

or $\theta_A - \theta_B = 30°$ or 150°. When this is combined with the given value of θ_A and the range of θ_B, one finds that $\theta_B = 0°$ or $-120°$ (i.e., along the *x*-axis or 120° clockwise from the *x*-axis).

ASSESS The vector corresponding to $\theta_B = 0°$ can be written as $\vec{B}_1 = B\hat{i} = 2A\hat{i}$. Similarly, for $\theta_B = -120°$, we have

$$\vec{B}_2 = B\left[\cos(-120°)\hat{i} + \sin(-120°)\hat{j}\right] = 2A\left[(-1/2)\hat{i} - \left(\sqrt{3}/2\right)\hat{j}\right] = -A\hat{i} - \sqrt{3}A\hat{j}$$

With $\vec{A} = A\left[\cos(30°)\hat{i} + \sin(30°)\hat{j}\right] = A\left[\left(\sqrt{3}/2\right)\hat{i} + (1/2)\hat{j}\right]$ the cross products are

$$\vec{A}\times\vec{B}_1 = A\left[\left(\sqrt{3}/2\right)\hat{i} + (1/2)\hat{j}\right]\times(2A)\hat{i} = -A^2\hat{k}$$

and

$$\vec{A}\times\vec{B}_2 = A\left[\left(\sqrt{3}/2\right)\hat{i} + (1/2)\hat{j}\right]\times\left[-A\hat{i} - \sqrt{3}A\hat{j}\right] = \left(-\frac{3}{2}A^2 + \frac{1}{2}A^2\right)\hat{k} = -A^2\hat{k}$$

Both results indeed agree with the condition given in the problem statement.

37. **INTERPRET** We're asked to calculate the torque exerted by the ball player in order to bring the baseball to rest.
DEVELOP The player exerts a torque around his shoulder, which results in a stopping force on the ball. From Equation 11.2, the average torque is $\bar{\tau} = r\vec{F}_{\text{stop}}$, where we have taken into account that the vertically held arm and the horizontally directed force are at right angles, so $\sin\theta = 1$. We won't worry about the direction of the torque, just the magnitude. The average stopping force is equal to $\vec{F}_{\text{stop}} = m\vec{a}$, where the average acceleration can be found through Equation 2.11: $\bar{a} = v_0^2/2\Delta x$. Here, v_0 is the initial speed and Δx is the stopping distance. We have neglected the negative sign because we're only looking for magnitudes.
EVALUATE The average torque exerted by the player on the ball is:

$$\bar{\tau} = \frac{rmv_0^2}{2\Delta x} = \frac{(59\text{ cm})(0.145\text{ kg})(41\text{ m/s})^2}{2(3.00\text{ cm})} = 2400\text{ N}\cdot\text{m}$$

ASSESS One can arrive at the answer by using Equation 10.11: $\bar{\tau} = I\bar{\alpha}$. In this case, the rotational inertia is that of the ball rotating around the shoulder joint: $I = mr^2$. The average angular acceleration relates to the ball's average linear acceleration through Equation 10.5: $\bar{\alpha} = \bar{a}/r$, so the final expression is the same: $\bar{\tau} = rm\bar{a}$.

38. **INTERPRET** In this problem we are asked to verify the vector identity $\vec{A}\cdot(\vec{A}\times\vec{B}) = 0$.
DEVELOP The key to the proof is to realize that the cross product $\vec{A}\times\vec{B}$ is perpendicular to $\vec{A}$ and $\vec{B}$.
EVALUATE Let $\vec{C} = \vec{A}\times\vec{B}$. If $\vec{C}$ is perpendicular to $\vec{A}$ and $\vec{B}$, then their scalar products must vanish:

$$\vec{A}\cdot\vec{C} = \vec{A}\cdot\left(\vec{A}\times\vec{B}\right) = 0$$
$$\vec{B}\cdot\vec{C} = \vec{B}\cdot\left(\vec{A}\times\vec{B}\right) = 0$$

(Recall that $\vec{A}\cdot\vec{B} = AB\cos\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$.)
ASSESS An alternative approach is to use the component forms. Let's write the vectors as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}, \quad \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

The cross product $\vec{A}\times\vec{B}$ is

$$\begin{aligned}\vec{A}\times\vec{B} &= \left(A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\right)\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)\\ &= A_xB_x\hat{i}\times\hat{i}+A_xB_y\hat{i}\times\hat{j}+A_xB_z\hat{i}\times\hat{k}+A_yB_x\hat{j}\times\hat{i}+A_yB_y\hat{j}\times\hat{j}\\ &\quad +A_yB_z\hat{j}\times\hat{k}+A_zB_x\hat{k}\times\hat{i}+A_zB_y\hat{k}\times\hat{j}+A_zB_z\hat{k}\times\hat{k}\\ &= \left(A_yB_z-A_zB_y\right)\hat{i}+\left(A_zB_x-A_xB_z\right)\hat{j}+\left(A_xB_y-A_yB_x\right)\hat{k}\end{aligned}$$

The dot product $\vec{A}\cdot(\vec{A}\times\vec{B})$ then becomes

$$\begin{aligned}\vec{A}\cdot\left(\vec{A}\times\vec{B}\right) &= A_x\left(A_yB_z-A_zB_y\right)+A_y\left(A_zB_x-A_xB_z\right)+A_z\left(A_xB_y-A_yB_x\right)\\ &= \left(A_yA_z-A_zA_y\right)B_x+\left(A_zA_x-A_xA_z\right)B_y+\left(A_xA_y-A_yA_x\right)B_z\\ &= \left(\vec{A}\times\vec{A}\right)\cdot\vec{B}=0\end{aligned}$$

In general, $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)$ is called the triple scalar product and $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\left(\vec{A}\times\vec{B}\right)\cdot\vec{C},$ that is, the "dot" and the "cross" in the triple scalar product can be interchanged. This is equivalent to a cyclic permutation of the three vectors,

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\vec{C}\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\left(\vec{C}\times\vec{A}\right)$$

On the other hand, interchanging any two vectors introduces a minus sign,

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=-\vec{C}\cdot\left(\vec{B}\times\vec{A}\right)=-\vec{B}\cdot\left(\vec{A}\times\vec{C}\right)=-\vec{A}\cdot\left(\vec{C}\times\vec{B}\right)$$

39. **INTERPRET** This problem asks us to find an expression for the angular momentum of a rotating rod.

DEVELOP We are told that a rod of mass m and length a is rotating about a perpendicular axis through one of its ends, so its rotational inertia is given by the expression introduced in Chapter 10: $I=\frac{1}{3}ma^2.$ Since we are also given the speed at which the other end of the rod moves, we can apply Equation 11.4, $\vec{L}=I\vec{\omega},$ to find the angular momentum.

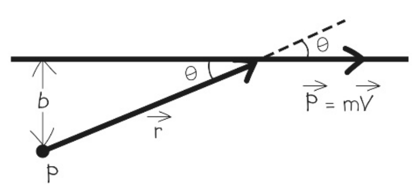
EVALUATE We are given the linear speed v at the other end of the rod, which means that the angular speed is $\omega=v/a,$ since a is the radius covered as the rod rotates. We find the angular momentum is given by

$$L=I\omega=\left(\frac{1}{3}ma^2\right)\left(\frac{v}{a}\right)=\frac{mva}{3}$$

ASSESS If the rod were instead rotating about an axis perpendicular to its center, the angular momentum would be smaller, since the rotational inertia would be smaller.

40. **INTERPRET** This problem involves calculating the angular momentum of an object of mass *m* traveling at speed *v* along a straight line. The point about which the angular momentum is to be calculated is a point a perpendicular distance *b* from the straight line. We are to show that the angular momentum is *mvb*, regardless of the position of the object on the line.

DEVELOP Apply Equation 11.3, $\vec{L}=\vec{r}\times\vec{p}$, using the geometry as drawn in the sketch below. Note that $\vec{p}=m\vec{v}$ and $b=r\sin\theta.$



EVALUATE Evaluating the cross product, we find $L=|\vec{r}\times\vec{p}|=pr\sin\theta=mvb$.

ASSESS The direction of $\vec{L}$ is also the same, for any position along the trajectory (in this case, into the page as sketched).

41. **INTERPRET** This problem involves calculating the angular momentum of two identical cars traveling in opposite directions at the same speed on a highway.

DEVELOP Let b be the perpendicular distance between the center of mass of the car and the center of the highway. Applying Equation 11.3, $\vec{L} = \vec{r} \times \vec{p}$, and noting that $\vec{p} = m\vec{v}$ and $b = r\sin\theta$, we find the angular momentum of the car moving to the right to be $L_R = |\vec{r} \times \vec{p}| = pr\sin\theta = mvb$, and the direction of $\vec{L}_R$ to be into the page. Similarly, for the car moving to the left, $L_L = mvb$, and $\vec{L}_L$ also points into the page.

EVALUATE With $m = 1900$ kg and $v = 75$ km / h $= 20.83$ m / s, the total angular momentum is

$$L_{\text{total}} = 2L = 2mvb = 2(1900 \text{ kg})(20.83 \text{ m / s})(2.3 \text{ m}) = 1.82\times10^5 \text{ J}\cdot\text{s}.$$

ASSESS The direction of $\vec{L}$ is the same for both cars. For the right-moving car, $\vec{L}_R = \vec{r} \times (m\vec{v})$, and for the left-moving car, $\vec{L}_L = (-\vec{r}) \times (-m\vec{v}) = \vec{r} \times (m\vec{v})$.

42. **INTERPRET** In this problem, we are asked to find the angle between two vectors, given that their dot product is one-third the magnitude of their cross product.

DEVELOP Consider two vectors, $\vec{A}$ and $\vec{B}$. The magnitude of their cross product $\vec{A}\times\vec{B}$ is $|\vec{A}\times\vec{B}| = AB\sin\theta$, where θ is the angle between the two vectors. On the other hand, the dot product of $\vec{A}$ and $\vec{B}$ is $\vec{A}\cdot\vec{B} = AB\cos\theta$.

EVALUATE The condition that $\vec{A}\cdot\vec{B} = \frac{1}{3}|\vec{A}\times\vec{B}|$ implies $\cos\theta = \frac{1}{3}\sin\theta$, or $\tan\theta = 3$. Thus, $\theta = \tan^{-1}3 = 71.6°$.

ASSESS The dot product $\vec{A}\cdot\vec{B}$ is a scalar quantity, while the cross product $\vec{A}\times\vec{B}$ is a vector quantity.

43. **INTERPRET** We need to find the angular momentum of a disk-shaped rotor that is part of a micromechanical device that measures blood flow.

DEVELOP The angular momentum of the rotor is $L = I\omega$, where the rotational inertia is that of a disk: $I = \dfrac{1}{2}MR^2$. We don't explicitly know the rotor's mass, but the material is silicon, which has a density of $\rho = 2.33 \text{ g / cm}^3$.

EVALUATE The mass of the rotor is the density times the volume: $M = \rho\left(d\cdot\pi R^2\right)$, where d is the rotor's thickness. The radius is half the diameter: $R = 145$ μm, and the 830-rpm rotational speed converted to SI units is $\omega = 86.9$ rad / s. So the angular momentum of the rotor during the tests is

$$\begin{aligned} L &= I\omega = \frac{\pi}{2}\rho dR^4\omega \\ &= \frac{\pi}{2}(2.33\times10^3 \text{ kg / m}^3)(2.3\times10^{-6} \text{ m})(145\times10^{-6} \text{ m})^4(86.9 \text{ rad / s}) = 3.2\times10^{-16} \text{ J}\cdot\text{s} \end{aligned}$$

ASSESS This is a very small angular momentum, but we expect it to be so. Otherwise, the device would significantly disturb the blood flow it is designed to measure.

44. **INTERPRET** This problem asks us to find the angular momentum of an object about a given point. To do so, we will need to calculate the rotational inertia of the object, given its rotational inertia about its center of mass (i.e., the rotational inertia if it were to rotate about an axis that goes through its center of mass). We are given the new axis about which the object rotates, so we can apply the parallel-axis theorem to find the rotational inertia about this new axis. The second part of the problem involves the rotational analog of Newton's second law, which we can use to find the torque required to achieve the given angular momentum in the given time.

DEVELOP Use Equation 11.4, $\vec{L} = I\vec{\omega}$, to find the angular momentum of the bat about point P. Applying the parallel-axis theorem (see Equation 10.17) to the bat gives us the rotational inertia about an axis through the point P as $I_p = Mh^2 + I_{\text{cm}}$, with $I_{\text{cm}} = 0.048 \text{ kg}\cdot\text{m}^2$ and $h = 43$ cm. The angular velocity can be found using Equation 10.3, $v = r\omega$, where r is the distance from P; $r = 43 \text{ cm} + 31 \text{ cm} = 74 \text{ cm}$ and $v = 47$ m / s. The direction of ω can be found using the right-hand rule; so if the bat is swung counterclockwise, the angular velocity vector is oriented out of the page, and if it is swung clockwise, the angular velocity vector is oriented into the page. To find the torque needed, apply Equation 11.5 in discrete form: $\Delta L / \Delta t = \tau$.

EVALUATE **(a)** Inserting the given quantities into Equation 11.4 gives

$$L = I\omega = \left(Mh^2 + I_{\text{cm}}\right)\frac{v}{r} = \left[(0.93 \text{ kg})(0.43 \text{ m})^2 + 0.048 \text{ kg}\cdot\text{m}^2\right]\left(\frac{47 \text{ m / s}}{0.74 \text{ m}}\right) = 14 \text{ J}\cdot\text{s}$$

The direction of the angular momentum is either out of or into the plane of the page, depending on whether the bat is rotated counterclockwise or clockwise, respectively.

(b) From the rotational analog of Newton's second law, the torque needed to achieve this angular momentum in 0.25 s is

$$\tau = \frac{\Delta L}{\Delta t} = \frac{L_\mathrm{f} - \overbrace{L_\mathrm{i}}^{=0}}{\Delta t} = \frac{L_\mathrm{f}}{\Delta t} = \frac{13.97\ \mathrm{J\cdot s}}{0.25\ \mathrm{s}} = 56\ \mathrm{N\cdot m}$$

The direction of the torque is the same as that of the angular momentum, because the initial momentum was zero.

ASSESS In ft-lbs, the torque is

$$\tau = (56\ \mathrm{N\cdot m})\left(\frac{1\ \mathrm{lb}}{4.448\ \mathrm{N}}\right)\left(\frac{1\ \mathrm{ft}}{0.3048\ \mathrm{m}}\right) = 41\ \mathrm{ft\cdot lb}$$

which is a reasonable result (i.e., possible for a human to achieve).

45. **INTERPRET** This problem asks us to calculate the rotational inertia of a wheel if the design reduces the angular momentum by a certain percentage while keeping the linear speed fixed.

DEVELOP The linear speed of the car is related to its angular speed as $v = \omega r$ (see Equation 10.3). Keeping v fixed implies

$$\omega_1 r_1 = \omega_2 r_2$$

From Equation 11.4, $L = I\omega$, the new rotational inertia can be computed.

EVALUATE The new specifications require that

$$\frac{L_2}{L_1} = \frac{I_2\omega_2}{I_1\omega_1} = 0.7 \;\Rightarrow\; \frac{I_2}{I_1} = 0.7\frac{\omega_1}{\omega_2}$$

Using $\omega_1 r_1 = \omega_2 r_2 / r_1$, we obtain

$$I_2 = (0.70)I_1\frac{\omega_1}{\omega_2} = (0.70)I_1\frac{R_2}{R_1} = (0.70)(0.29\ \mathrm{kg\cdot m^2})\left(\frac{35\ \mathrm{cm}}{38\ \mathrm{cm}}\right) = 0.19\ \mathrm{kg\cdot m^2}$$

ASSESS The general condition is

$$\frac{L_2}{L_1} = \frac{I_2\omega_2}{I_1\omega_1} = \frac{I_2R_1}{I_1R_2} \quad\Rightarrow\quad L_2 = \left(\frac{I_2}{I_1}\right)\left(\frac{R_1}{R_2}\right)L_1$$

A decrease in angular momentum $(L_2 < L_1)$ can be achieved by decreasing either r_1/r_2 or I_2/I_1. In our problem, the ratio $r_1/r_2 = (38\ \mathrm{cm})/(35\ \mathrm{cm}) = 1.09$ is actually increased. However, this change is accompanied by a greater decrease in rotational inertia, $I_2/I_1 = \left(0.19\ \mathrm{kg\cdot m^2}\right)/\left(0.29\ \mathrm{kg\cdot m^2}\right) = 0.66$.

46. **INTERPRET** This problem involves conservation of angular momentum, which we can use to find the angular speed when the mouse is at the center of the turntable. The second part involves the work-energy theorem, which we can use to find the work done by the mouse. The mouse does work when it exerts reaction forces to the friction between its feet and the turntable.

DEVELOP Apply conservation of angular momentum. Because the axis of rotation does not change, we can use the scalar form, so $L_\mathrm{f} = L_\mathrm{i}$. The final angular momentum is $L_\mathrm{f} = I_\mathrm{t}\omega_\mathrm{f}$, and the initial angular momentum is $L_\mathrm{i} = (I_\mathrm{t} + mr^2)\omega_\mathrm{i}$, where $m = 20.5\ \mathrm{g}$ is the mass of the mouse and $r = 15\ \mathrm{cm}$ is its distance from the axis of rotation. With the final angular velocity known, we can apply the work-energy theorem, Equation 10.19,

$$W = \Delta K_\mathrm{rot} = I_\mathrm{f}\omega_\mathrm{f}^2/2 - I_\mathrm{i}\omega_\mathrm{i}^2/2$$

to find the work done by the mouse.

EVALUATE (a) Inserting the given values into the expression for conservation of angular momentum gives

$$\omega_\mathrm{f} = \omega_\mathrm{i}I_\mathrm{i}/I_\mathrm{f} = (32.0\ \mathrm{rpm})\frac{\left[0.0115\ \mathrm{kg\cdot m^2} + (0.0205\ \mathrm{kg})(0.15\ \mathrm{m})^2\right]}{0.0115\ \mathrm{kg\cdot m^2}} = 33.3\ \mathrm{rpm}$$

to three significant figures.

(b) From the work-energy theorem, the work done by the mouse is

$$W = K_\mathrm{f} - K_\mathrm{i} = \frac{1}{2}I_\mathrm{f}\omega_\mathrm{f}^2 - \frac{1}{2}I_\mathrm{i}\omega_\mathrm{i}^2 = \frac{1}{2}I_\mathrm{f}\omega_\mathrm{f}^2\left(1-\frac{\omega_\mathrm{i}}{\omega_\mathrm{f}}\right)$$
$$= \frac{1}{2}(0.0115\ \mathrm{kg\cdot m^2})(33.28\ \mathrm{rev/min})^2\left(\frac{2\pi\ \mathrm{rad}}{1\ \mathrm{rev}}\right)^2\left(\frac{1\ \mathrm{min}}{60\ \mathrm{s}}\right)^2\left(1-\frac{32.0\ \mathrm{rev/min}}{33.3\ \mathrm{rev/min}}\right) = 2.69\ \mathrm{mJ}.$$

where we used conservation of angular momentum from part (a).

ASSESS The units of rev / min had to be changed to rad / s in part (b) because rev / min are not SI units.

47. **INTERPRET** This problem involves conservation of angular momentum, which we can use to calculate the motion of the dog relative to the ground.

DEVELOP Walking around once relative to the turntable, the dog describes an angular displacement of $\Delta\theta_\mathrm{D}$ relative to the ground, and the turntable one of $\Delta\theta_\mathrm{T}$ in the opposite direction, such that $\Delta\theta_\mathrm{D} - \Delta\theta_\mathrm{T} = 2\pi$. The vertical component of the angular momentum of the dog-and-turntable system is conserved (which was zero initially), so

$$L_\mathrm{i} = L_\mathrm{f} \Rightarrow 0 = I_\mathrm{D}\omega_\mathrm{D} + I_\mathrm{T}\omega_\mathrm{T} = I_\mathrm{D}\left(\frac{\Delta\theta_\mathrm{D}}{\Delta t}\right) - I_\mathrm{T}\frac{\Delta\theta_\mathrm{T}}{\Delta t}$$

where the angular velocities (which are in opposite directions) have been rewritten in terms of the angular displacements and the common time interval. The rotational inertias about the axis of rotation are

$$I_\mathrm{D} = mR^2 = (20\ \mathrm{kg})(1.95\ \mathrm{m})^2 = 76\ \mathrm{kg\cdot m^2}$$

and $I_\mathrm{T} = 85\ \mathrm{kg\cdot m^2}$. These results allow us to solve for $\Delta\theta_\mathrm{D}$.

EVALUATE Eliminating $\Delta\theta_\mathrm{T}$, we find

$$0 = I_\mathrm{D}\Delta\theta_\mathrm{D} - I_\mathrm{T}(2\pi - \Delta\theta_\mathrm{D}) = (I_\mathrm{D} + I_\mathrm{T})\Delta\theta_\mathrm{D} - 2\pi I_\mathrm{T}$$

or

$$\frac{\Delta\theta_\mathrm{D}}{2\pi} = \frac{I_\mathrm{T}}{I_\mathrm{D} + I_\mathrm{T}} = \frac{85\ \mathrm{kg\cdot m^2}}{76\ \mathrm{kg\cdot m^2} + 85\ \mathrm{kg\cdot m^2}} = 0.53$$

In other words, $\Delta\theta_\mathrm{D}$ is 53% of a full circle relative to the ground.

ASSESS We find that $\Delta\theta_\mathrm{D}$, the angular displacement relative to the ground, decreases with I_D. This is what we expect from conservation of angular momentum.

48. **INTERPRET** This problem involves conservation of momentum and the work-energy theorem. The former can be used to find the student's mass given the rotational parameters of the turntable on which she is standing and the wheel that she is holding, and the latter can be used to find the work she does in turning the wheel upside down.

DEVELOP Because the turntable is frictionless, there are no external torques about its axis, and the *z*-component of angular momentum is conserved. The initial angular momentum is just that due to the spinning wheel,

$$L_\mathrm{i} = I_\mathrm{W}\omega_\mathrm{W}$$

When the wheel is inverted, the student and turntable acquire an angular momentum

$$L_\mathrm{f} = L_\mathrm{T} + L_\mathrm{S} - L_\mathrm{W} = \left(I_\mathrm{T} + \overbrace{m_\mathrm{W}h^2}^{\text{neglect}}\right)\omega_\mathrm{T} + I_\mathrm{S}\omega_\mathrm{S} - I_\mathrm{W}\omega_\mathrm{W}$$
$$= I_\mathrm{T}\omega_\mathrm{T} + I_\mathrm{S}\omega_\mathrm{S} - I_\mathrm{W}\omega_\mathrm{W}$$

where we have neglected the rotational inertia due to the center of mass of the wheel, and we have subtracted the angular momentum of the wheel because it is not oriented in the opposite (i.e., downward) direction. We know that $\omega_\mathrm{T} = \omega_\mathrm{S} = 70$ rpm, and that $\omega_\mathrm{W} = 130$ rpm. The rotational inertias are $I_\mathrm{T} = 0.31\ \mathrm{kg\cdot m^2}$, $I_\mathrm{S} = m_\mathrm{S}r^2/2$ with $r = 0.30$ m, and $I_\mathrm{W} = 0.22\ \mathrm{kg\cdot m^2}$, so we can solve for the student's mass m_S. Apply the rotational version of the work-energy theorem (Equation 10.19) to find the work done reversing the wheel. This gives

$$W = \Delta K_{\text{rot}} = \frac{1}{2} I_{\text{tot}} \omega_{\text{f}}^2 - \frac{1}{2} I_{\text{tot}} \overbrace{\omega_{\text{i}}^2}^{=0} = I_{\text{tot}} \omega_{\text{f}}^2$$

where the initial angular velocity of the entire system is zero (note that the kinetic energy of the wheel does not change in this experiment) and the final angular velocity is $\omega_{\text{f}} = \omega_{\text{T}} = 70$ rpm. The total rotational inertia of the system is $I_{\text{tot}} = I_{\text{T}} + I_{\text{S}}$ (we are neglecting the rotational inertia due to the center of mass of the wheel).

EVALUATE (**a**) Equating the initial and final angular momenta, we find

$$I_W \omega_W = I_{\text{T}} \omega_{\text{T}} + I_{\text{S}} \omega_{\text{S}} - I_{\text{W}} \omega_{\text{W}}$$

$$I_{\text{S}} = \frac{2 I_W \omega_W - I_{\text{T}} \omega_{\text{T}}}{\omega_{\text{S}}} = \frac{m_{\text{S}} R^2}{2}$$

$$m_{\text{S}} = 2\left(\frac{2 I_W \omega_W - I_{\text{T}} \omega_{\text{T}}}{R^2 \omega_{\text{S}}}\right) = 2\left(\frac{2\left(0.22\ \text{kg}\cdot\text{m}^2\right)(130\ \text{rpm}) - \left(0.31\ \text{kg}\cdot\text{m}^2\right)(70\ \text{rpm})}{(0.15\ \text{m})^2 (70\ \text{rpm})}\right) = 45\ \text{kg}$$

(**b**) Evaluating the expression above for work gives

$$W = \frac{1}{2} I_{\text{tot}} \omega_{\text{f}}^2 = \frac{1}{2}\left(I_T + I_S\right)\omega_{\text{f}}^2 = \frac{1}{2}\left(I_T + m_S R^2/2\right)$$

$$= \frac{1}{2}\left(0.31\ \text{kg}\cdot\text{m}^2 + (45.1\ \text{kg})(0.15\ \text{m})^2/2\right)\left(70\ \frac{\text{rev}}{\text{min}}\right)^2 \left(\frac{2\pi\ \text{rad}}{\text{rev}}\right)^2 \left(\frac{1\ \text{min}}{60\ \text{s}}\right)^2 = 22\ \text{J}$$

ASSESS From the expression for work, we see that a student with a larger radius would have to do more work to flip the wheel, which is reasonable because the student's rotational inertia would be greater, so it would require more work to get him to rotate.

49. **INTERPRET** This problem is about the rotational motion of the skaters, given their initial linear speed and the radius of the circle they traverse. The aim is to keep the final linear speed and centripetal force below the stated maxima. The key concept here is conservation of angular momentum.

DEVELOP If the ice is frictionless, the only external force on the skaters is the force that brings the end skater to a sudden stop at a point that we'll call *P*. (Note: The forces they exert on each other through their hands are internal forces.) The stopping force exerts no torque about point *P*, so the total angular momentum about a vertical axis through *P* is conserved. Initially, each of the seven other skaters is moving with the same linear momentum $(p = mv_0)$ in a direction perpendicular to the line that connects them $(\sin\theta = 1)$. So, from Equation 11.3, the angular momentum of each skater about *P* is

$$L_{0n} = \left|\vec{r}_n \times \vec{p}_n\right| = r_n\left(mv_0\right)\sin\theta = mv_0 r_n$$

where r_n is the distance between the *n*th skater and the point $P: r_n = n(\ell/7)$ for *n* = 1, 2, …, 7 and $\ell = 11$ m. The total initial angular momentum is the sum $L_0 = \sum_{n=1}^{7} L_{0n}$, which will be conserved when the group starts rotating and has an angular momentum of $L_{\text{f}} = I\omega$. Here, the rotational inertia is $I = \sum_{n=1}^{7} mr_n^2$. From all this, we can determine the rotational speed, which will give us the linear speed and centripetal force on the outside skater $(n = 7)$.

EVALUATE The total initial angular momentum is

$$L_0 = \sum_{n=1}^{7} mv_0 r_n = \frac{mv_0 \ell}{7}\sum_{n=1}^{7} n = \frac{mv_0 \ell}{7}\left[\frac{7\times 8}{2}\right] = 4mv_0\ell$$

where we have used $\sum_{n=1}^{N} n = N(N+1)/2$. Similarly, the rotational inertia of the seven skaters around point *P* is

$$I = \sum_{n=1}^{7} mr_n^2 = \frac{m\ell^2}{49}\sum_{n=1}^{7} n^2 = \frac{m\ell^2}{49}\left[\frac{7\times 8\times 15}{6}\right] = \frac{20m\ell^2}{7}$$

where we have used $\sum_{n=1}^{N} n^2 = N(N+1)(2N+1)/6$. Since angular momentum is conserved $(L_0 = L_{\text{f}})$, we can solve for the angular speed:

$$\omega = \frac{L_0}{I} = \frac{4mv_0\ell}{\frac{20}{7}m\ell^2} = \frac{7v_0}{5\ell}$$

The outside skater will have a tangential speed of $v = \omega\ell;$ so in order to keep this below $8.0 \text{ m/s},$ the initial speed can't exceed:

$$v_0 = \frac{5}{7}v < \frac{5}{7}(9.8 \text{ m/s}) = 7 \text{ m/s}$$

The force on the outside skater's hand is the centripetal force: $F = ma_c = m\ell\omega^2.$ To keep this below 300 N, the initial speed can't exceed:

$$V_0 = \frac{5}{7}\sqrt{\frac{F\ell}{m}} < \frac{5}{7}\sqrt{\frac{(300 \text{ N})(11 \text{ m})}{(65 \text{ kg})}} = 5.1 \text{ m/s}$$

This limit is stricter than the one above. The greatest speed that the skaters can have before the rotational maneuver is $5.1 \text{ m/s}.$

ASSESS Notice that the outside skater will be going 1.4 times faster following the maneuver. By contrast, the skaters closer to the point *P* will slow down after the maneuver $(v_n = \omega r_n).$ This makes sense: to keep the total angular momentum constant, some skaters will gain angular momentum, while others will lose it.

50. **INTERPRET** In this problem we want to find the mass a projectile needs to have so it can impart the necessary change in angular momentum to the planet Mars.

DEVELOP The goal of the launch is to make the angular frequency of Mars's rotation ω_M equal to that of Earth's ω_E. We are told that initially a day on Mars lasts 1.03 times longer than on Earth ($\omega_E = 1.03\omega_{Mi}$), so the launch seeks to increase the angular rotation of Mars by an amount $\Delta\omega_M = \omega_{Mf} - \omega_{Mi} = \omega_E - \frac{1}{1.03}\omega_E$. We can express this change in rotational frequency with the use of Equation 11.4, by noting the change in angular momentum is given by: $\Delta L_M = I_M \Delta\omega_M$. The projectile that is causing the change in angular momentum experienced by Mars is doing so, because as it launches, it too gains angular momentum equal to ΔL_M. Considering the planet center as the point about which it moves with the given velocity v_p, the angular momentum of the projectile can be calculated by Equation 11.3 as: $L_p = \Delta L_M = m_p v_p R_M$, where R_M is the mean Martian radius found in Appendix E. Equating these expressions will allow us to determine the necessary mass to make this happen.

EVALUATE Equating our expressions for the change in angular momentum results in a projectile mass given by

$$\Delta L_M = I_M \Delta\omega_M = m_p v_p R_M$$

$$m_p = \frac{I_M \Delta\omega_M}{v_p R_M} = \left(\frac{2}{5}M_M R_M{}^2\right)\frac{\omega_E}{v_p R_M}\left(1 - \frac{1}{1.03}\right)$$

$$m_p = \frac{2M_M R_M v_E}{5v_p R_E}\left(\frac{0.03}{1.03}\right) = \frac{4\pi M_M R_M}{5v_p T}\left(\frac{0.03}{1.03}\right)$$

Where we have substituted the linear velocity from Earth's angular frequency, $\omega_E = v_E / R_E$, with $\frac{2\pi R_E}{T}$. We have also treated the planet Mars as a uniform solid sphere and utilized the rotational inertia introduced in Chapter 10. Plugging in the given projectile velocity and the astrophysical data values from Appendix E we obtain

$$m_p = \frac{(4\pi)(6.42\times10^{23} \text{ kg})(3.39\times10^6 \text{ m})}{(5)(2.44\times10^6 \text{ m/s})(24\cdot60\cdot60 \text{ s})}\left(\frac{0.03}{1.03}\right) = 7.56\times10^{17} \text{ kg}$$

ASSESS Such a projectile would need to be ~70 times more massive that Mars's larger moon, Phobos!

51. **INTERPRET** In this problem we derive a general expression for the cross product of two vectors in three dimensions.

DEVELOP Let's write the vectors as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}, \quad \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

We use the following properties of the unit vectors:

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0,$$
$$\hat{i}\times\hat{j} = \hat{k},\ \hat{k}\times\hat{i} = \hat{j},\ \hat{j}\times\hat{k} = \hat{i}$$
$$\hat{j}\times\hat{i} = -\hat{k},\ \hat{i}\times\hat{k} = -\hat{j},\ \hat{k}\times\hat{j} = -\hat{i}$$

EVALUATE The cross product $\vec{A}\times\vec{B}$ is

$$\begin{aligned}\vec{A}\times\vec{B} &= \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)\\ &= A_xB_x\hat{i}\times\hat{i} + A_xB_y\hat{i}\times\hat{j} + A_xB_z\hat{i}\times\hat{k} + A_yB_x\hat{j}\times\hat{i} + A_yB_y\hat{j}\times\hat{j}\\ &\quad + A_yB_z\hat{j}\times\hat{k} + A_zB_x\hat{k}\times\hat{i} + A_zB_y\hat{k}\times\hat{j} + A_zB_z\hat{k}\times\hat{k}\\ &= \left(A_yB_z - A_zB_y\right)\hat{i} + \left(A_zB_x - A_xB_z\right)\hat{j} + \left(A_xB_y - A_yB_x\right)\hat{k}\end{aligned}$$

ASSESS The cross product $\vec{A}\times\vec{B}$ is a vector that is perpendicular to both $\vec{A}$ and $\vec{B}$. Note that interchanging the two vectors introduces a minus sign: $\vec{B}\times\vec{A} = -\vec{A}\times\vec{B}$.

52. **INTERPRET** In this problem we show that the cross product of two vectors in three dimensions can be written as a determinant of a 3 ×3 matrix.

DEVELOP To calculate the determinant of

$$\vec{A}\times\vec{B} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix},$$

we expand the determinant along the first row:

$$\vec{A}\times\vec{B} = \hat{i}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix} - \hat{j}\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix} + \hat{k}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}$$

The determinant of a 2 ×2 matrix is simply $\begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc.$

EVALUATE Using the equation above, we find the desired expression to be

$$\begin{aligned}\vec{A}\times\vec{B} &= \hat{i}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix} - \hat{j}\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix} + \hat{k}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}\\ &= \hat{i}(A_yB_z - A_zB_y) + \hat{j}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)\end{aligned}$$

ASSESS The result is the same as that obtained in Problem 49.

53. **INTERPRET** The problem is about Jumbo, an elephant walking toward the center of a rotating turntable. The total angular momentum is conserved. The second part involves the work-energy theorem, which we can use to find the work done by the elephant.

DEVELOP As the elephant walks toward the center, there are no external forces that can generate a torque around the turntable's axis, so the angular momentum of the elephant-turntable system in the vertical direction is conserved. Because the axis of rotation does not change, we can use the scalar form, so $L_f = L_i$. The final angular momentum is $L_f = I_f\omega_f = (Mr^2/2)\omega_f$ and the initial angular momentum is $L_i = I_i\omega_i = (Mr^2/2 + mr^2)\omega_i$, where $M = 1.5\times10^4$ kg is the mass of the turntable, $m = 4.8\times10^3$ kg the mass of the elephant, and $r = 8.5$ m the radius and the elephant's distance from the axis of rotation. Once the final angular velocity is calculated, we can apply the work-energy theorem Equation 10.19,

$$W = \Delta K_{\text{rot}} = I_{\text{f}}\omega_{\text{f}}^2/2 - I_{\text{i}}\omega_{\text{i}}^2/2$$

to find the work done by the elephant.

EVALUATE (**a**) Inserting the given values into the expression for conservation of angular momentum gives

$$\omega_{\text{f}} = \omega_{\text{i}} I_{\text{i}}/I_{\text{f}} = (0.15 \text{ rad/s})\frac{\left[\frac{1}{2}(1.5\times10^4 \text{ kg}) + 4.8\times10^3 \text{ kg}\right](8.5 \text{ m})^2}{\frac{1}{2}(1.5\times10^4 \text{ kg})(8.5 \text{ m})^2} = 0.246 \text{ rad/s} \approx 0.25 \text{ rad/s}$$

(**b**) From the work-energy theorem, the work done by the elephant is

$$W = K_{\text{f}} - K_{\text{i}} = \frac{1}{2}I_{\text{f}}\omega_{\text{f}}^2 - \frac{1}{2}I_{\text{i}}\omega_{\text{i}}^2 = \frac{1}{2}I_{\text{f}}\omega_{\text{f}}^2(1-\omega_{\text{i}}/\omega_{\text{f}})$$
$$= \frac{1}{2}\left[\frac{1}{2}(1.5\times10^4 \text{ kg})(8.5 \text{ m})^2\right](0.246 \text{ rad/s})^2\left(1-\frac{0.15 \text{ rad/s}}{0.246 \text{ rad/s}}\right) = 6.4\times10^3 \text{ J.}$$

ASSESS As expected from angular momentum conservation, the angular velocity increases as the elephant walks toward the center of the turntable, lowering the rotational inertia of the elephant-turntable system.

54. **INTERPRET** The problem asks for the angular momentum of a spinning anemometer.

DEVELOP Equation 11.4 gives the angular momentum as $\vec{L} = I\vec{\omega}$. In this case, we are only concerned with the magnitude of the anemometer's angular momentum. The anemometer consists of two rods of length ℓ and four cups. We are told to treat the cups as point masses. The rotational inertia of the device is given by

$$I = 2I_{\text{rod}} + 4I_{\text{cups}} = 2\left(\frac{1}{12}m_{\text{rod}}\ell^2\right) + 4\left[m_{\text{cup}}(\ell/2)^2\right] = \left[\frac{1}{6}m_{\text{rod}} + m_{\text{cup}}\right]\ell^2$$
$$= \left[\frac{1}{6}(0.0675 \text{ kg}) + 0.120 \text{ kg}\right](0.295 \text{ m})^2 = 0.0114 \text{ kg}\cdot\text{m}^2$$

EVALUATE The angular velocity of the anemometer is $\omega = 11.7 \text{ rev/s} = 73.51 \text{ rad/s}$. Thus, we find the angular momentum to be

$$L = I\omega = (0.0114 \text{ kg}\cdot\text{m}^2)(73.51 \text{ rad/s}) = 0.84 \text{ kg}\cdot\text{m}^2/\text{s}.$$

ASSESS The value is reasonable. The units are correct, since $\text{kg}\cdot\text{m}^2/\text{s} = (\text{kg}\cdot\text{m}^2/\text{s}^2)\cdot\text{s} = \text{J}\cdot\text{s}$. Note that the angular velocity is related to the wind speed.

55. **INTERPRET** The problem is about the rotational motion of the turntable. Tossing a piece of clay onto its surface is like a totally inelastic collision from Section 9.5. In this case, the total angular momentum is conserved.

DEVELOP The forces that cause the clay to stick to the turntable are internal forces (i.e., between clay and turntable). There are no external forces that can generate a torque around the turntable's axis, so the angular momentum of the turntable/clay system in the vertical direction is conserved. If we take the sense of rotation of the turntable to define the positive direction of vertical angular momentum, then the system's initial angular momentum is

$$L_{\text{i}} = I\omega + mvd$$

where we assume here that the clay hits the turntable with the same direction that the table is turning. After the collision, the clay turns at the same speed as the table, so the final angular momentum is

$$L_{\text{f}} = I\omega_{\text{f}} + md^2\omega_{\text{f}} = \left(I + md^2\right)\omega_{\text{f}}$$

By conservation of angular momentum, the clay's initial velocity is equal to

$$v = d\omega_{\text{f}} + \frac{I(\omega_{\text{f}} - \omega)}{md}$$

EVALUATE (a) If $\omega_{\text{f}} = \frac{1}{2}\omega$, then the clay hits the table with speed:

$$v = \frac{d\omega}{2} + \frac{-I\omega}{2md} = d\,\omega\left(\tfrac{1}{2} - I/2md^2\right)$$

(b) If $\omega_{\text{f}} = \omega$, then the clay hits the table with speed: $v = d\omega$.

(c) If $\omega_f = 2\omega$, then the clay hits the table with speed:

$$v = 2d\omega + \frac{I\omega}{md} = d\omega\left(2 + I/md^2\right)$$

ASSESS We have written the clay velocity in terms of $d\omega$, which is the initial linear speed of the turntable at the radius where the clay hits. If the clay hits with $v < d\omega$, then the collision will slow down the turntable, but if $v > d\omega$, the turntable will speed up.

56. **INTERPRET** This problem asks us to calculate angular momenta given the mass distribution and rotational speeds (Appendix E) of the various planets of our solar system and the Sun. In particular, we are asked to estimate how much of the solar system's angular momentum about its center is associated with the Sun.

DEVELOP The planets orbit the Sun in planes approximately perpendicular to the Sun's rotation axis, so most of the angular momentum in the solar system is in this direction. We can estimate the orbital angular momentum of a planet by mvr, where m is its mass, v its average orbital speed, and r its mean distance from the Sun. Compared to the orbital angular momentum of the four giant planets, everything else is negligible, except for the rotational angular momentum of the Sun itself, which can be estimated by assuming the Sun to be a uniform sphere rotating with an average period of $\frac{1}{2}(27+36)$ days. (The Sun's period of rotation at the surface varies from approximately 27 days at the equator to 36 days at the poles.)

EVALUATE The numerical data in Appendix E results in the following estimates:

Orbital Angular Momentum (*mvr*)		**%**
Jupiter	$19.2\times10^{42}\ \text{J}\cdot\text{s}$	59.7
Saturn	$7.85\times10^{42}\ \text{J}\cdot\text{s}$	24.4
Uranus	$1.69\times10^{42}\ \text{J}\cdot\text{s}$	5.2
Neptune	$2.52\times10^{42}\ \text{J}\cdot\text{s}$	7.8
Rotational Angular Momentum ($\frac{2}{5}MR^2\omega$)		
Sun	$0.89\times10^{42}\ \text{J}\cdot\text{s}$	2.8
Total	$32.2\times10^{42}\ \text{J}\cdot\text{s}$	99.9

ASSESS With $L_{\text{orb}} \gg L_{\text{rot}}$, we find that more than 97% of the total angular momentum of the solar system comes from the orbital angular momentum. In particular, the orbital motion of Jupiter alone accounts for roughly 60% of the total angular momentum.

57. **INTERPRET** To increase the surface area of this alien planet, you plan to hollow out its center. This will increase the rotational inertia, so to conserve angular momentum, the planet's rotation will slow down.

DEVELOP The planet is originally a solid sphere of radius R_0. When the planet is hollowed out, the radius of its outer surface is R, and the radius of its inner surface is $\frac{4}{5}R$, such that the shell thickness is $\frac{1}{5}R$. No material is added or taken away during this alteration, so the total mass, $M = \int dm$, should remain constant. To calculate the mass, divide the planet up into concentric shells of infinitesimal thickness. For a given shell of radius R', the mass is $dm = \rho \cdot 4\pi R'^2 dR'$, where ρ is the planet's density and $4\pi R'^2 dR'$ is the volume of the given shell. For the original planet, R' varies from 0 to R_0, while for the hollowed out planet, R' varies from $\frac{4}{5}R$ to R. Equating the mass integrals for the two cases gives:

$$\int_0^{R_0} 4\pi\rho R'^2 dR' = \int_{4R/5}^{R} 4\pi\rho R'^2 dR' \quad\rightarrow\quad R = \frac{5R_0}{\sqrt[3]{5^3 - 4^3}} \approx 1.27R_0$$

This can be used to find the increase in surface area. But to find the change in the length of the day, you have to find the change in the rotational inertia of the planet. The original sphere has $I_0 = \frac{2}{5}MR_0^2$, from Table 10.1. However, the formula for a hollow sphere in Table 10.1, $I = \frac{2}{3}MR^2$, assumes the shell is thin, which is not the case here. What you can do is sum over the infinitesimal shells with mass dm defined above. Each of them has rotational inertia of:

$$dI = \frac{2}{3}(dm)\,R'^2 = \frac{8\pi}{3}\rho R'^4 dR' = \frac{2M}{R_0^3}R'^4 dR'$$

where the mass relation for the original sphere was used: $M = \rho\left(\frac{4\pi}{3}R_0^3\right)$. Integrating over all the shells in the hollowed sphere gives:

$$I = \int dI = \frac{2M}{R_0^3}\int_{4R/5}^{R} R'^4 dR' = \frac{2MR^5}{5R_0^3}\left[1-\left(\tfrac{4}{5}\right)^5\right]$$

Notice that if R' varies from 0 to R_0, as in the original case, the integration returns the familiar result of $I = \frac{2}{5}MR_0^2$.

EVALUATE By hollowing out the planet, the surface area increases by a factor of

$$\frac{A}{A_0} = \frac{4\pi R^2}{4\pi R_0^2} = \left(\frac{R}{R_0}\right)^2 = 1.27^2 = 1.61$$

The angular momentum is conserved, so $I_0\omega_0 = I\omega$. The period is inversely proportional to rotation speed $(T = 2\pi/\omega)$, so the length of the day will increase by a factor of

$$\frac{T}{T_0} = \frac{\omega_0}{\omega} = \frac{I}{I_0} = \frac{\frac{2}{5}MR^5/R_0^3}{\frac{2}{5}MR_0^2}\left[1-\left(\tfrac{4}{5}\right)^5\right] = \left(\frac{R}{R_0}\right)^5\left[1-\left(\tfrac{4}{5}\right)^5\right] = 2.22$$

ASSESS Suppose the hollowed sphere has thickness Δ, where $\Delta \ll R$. Then, the radii are related by $R \approx R_0/\sqrt[3]{3\Delta}$. Substituting this into the rotational inertia equation, and making a further approximation, gives

$$I = \frac{2MR^5}{5R_0^3}\left[1-(1-\Delta)^5\right] \approx \frac{2}{5}MR^2\left(\frac{R}{R_0}\right)^3[5\Delta] = \frac{2}{3}MR^2$$

This is the formula for a hollow sphere given in Table 10.1, which shows that this expression only becomes valid when the thickness of the shell is much less than the radius.

58. **INTERPRET** This problem looks just like an inelastic collision, but instead of using conservation of linear momentum, we will use conservation of *angular* momentum. The angular momentum of each disk is in a single direction, so we can treat this as a one-dimensional problem.

DEVELOP The masses of disks 1 and 2 are $m_1 = 480$ g and $m_2 = 250$ g, respectively. The radii are $r_1 = 3.2$ cm and $r_2 = 2.0$ cm. The initial angular speed of disk 1 is $\omega_1 = 180$ rpm. Use conservation of angular momentum, $L_\mathrm{i} = L_\mathrm{f}$, to find the final angular speed of both disks stuck together, and $\eta = 1 - K_\mathrm{f}/K_\mathrm{i}$, where $K = \dfrac{1}{2}I\omega^2$, to find the fraction of energy lost.

EVALUATE **(a)** The initial angular momentum is

$$L_1 = I_1\omega_{1\mathrm{i}} + I_2\overbrace{w_{2\mathrm{i}}}^{=0} = \frac{1}{2}m_1r_1^2\omega_{1\mathrm{i}}.$$

The final angular momentum is

$$L_2 - (I_1 + I_2)\omega_\mathrm{f} = \left(\frac{1}{2}m_1r_1^2 + \frac{1}{2}m_2r_2^2\right)\omega_\mathrm{f}.$$

Conservation of angular momentum tells us that

$$L_1 = L_2$$

$$\frac{1}{2}m_1r_1^2\omega_{1\mathrm{i}} = \left(\frac{1}{2}m_1r_1^2 + \frac{1}{2}m_2r_2^2\right)\omega_\mathrm{f}$$

which can be solved to give

$$\omega_\mathrm{f} = \omega_{1\mathrm{i}}\left(\frac{m_1r_1^2}{m_1r_1^2 + m_2r_2^2}\right) = (180\text{ rpm})\left(\frac{(480\text{ g})(3.2\text{ cm})^2}{(480\text{ g})(3.2\text{ cm})^2 + (250\text{ g})(2.0\text{ cm})^2}\right) = 150\text{ rpm}$$

(b) The initial kinetic energy is $K_\text{i} = \dfrac{L_\text{i}^2}{2I_\text{i}}$, while the final kinetic energy is $K_\text{f} = \dfrac{L_\text{f}^2}{2I_\text{f}}$. Since $L_\text{f} = L_\text{i}$, the fraction of the initial kinetic energy lost to friction is

$$\eta = 1 - \frac{K_\text{f}}{K_\text{i}} = 1 - \frac{I_\text{i}}{I_\text{f}} = 1 - \frac{\frac{1}{2}m_1r_1^2}{\frac{1}{2}m_1r_1^2 + \frac{1}{2}m_2r_2^2} = \frac{m_2r_2^2}{m_1r_1^2 + m_2r_2^2}$$

$$= \frac{(250\text{ g})(2\text{ cm})^2}{(480\text{ g})(3.2\text{ cm})^2 + (250\text{ g})(2.0\text{ cm})^2} = 0.169 \approx 17\%$$

to two significant figures.

ASSESS Note that the fractional energy loss doesn't depend on the initial energy. For this particular set of disks, 17% of the initial energy will be lost in the collision regardless of how fast the bottom disk is spinning!

59. **INTERPRET** A solid spinning ball drops onto a frictional surface. At first it slides, but due to friction its spin will slow down and its linear speed will increase until it is purely rolling without sliding. We want to find the ball's angular speed when it begins purely rolling, and how long it takes.

DEVELOP From the problem statement, we see that the ball's mass is *M*, its radius is *R*, and its initial angular velocity around the horizontal axis is ω_0. The coefficient of kinetic friction between the ball and the surface is μ_k, so the frictional force is $F_f = \mu_\text{s}F_\text{n} = \mu Mg$. A torque acts on the ball due to the frictional force, which acts on the edge of the ball. This torque $\tau = -\mu_\text{s}MgR$ serves to slow the ball's rotation. Use $\tau = I\alpha$ to find the angular acceleration α and then use $\omega = \omega_0 + \alpha t$ to find the resulting angular speed. The frictional force on the ball also accelerates the ball, so we can use $F = Ma$ and $v = v_0 + at$ to find the speed of the ball. Combining this with the fact that the ball is no longer sliding when $R\omega = v$ allows us to find the time it takes to achieve rolling motion.

EVALUATE (**a**) The angular acceleration is

$$\alpha = \frac{\tau}{I} = \frac{-\mu_\text{k}MgR}{2MR^2/5} = -\frac{5\mu_\text{k}g}{2R}$$

where the negative sign comes from the fact that the frictional force always acts to counter the motion. Inserting this into the kinematic equation $\omega = \omega_0 + \alpha t$ gives

$$\omega = \omega_0 - \frac{5\mu_\text{k}g}{2R}t$$

Using the result from part (b) that $t = R\omega/\mu_\text{k}g$, we find that

$$\omega = \omega_0 - \frac{5\mu_\text{k}g}{2R}\left(\frac{R\omega}{\mu_\text{k}g}\right) = \omega_0 - \frac{5\omega}{2}$$

$$= \frac{2}{7}\omega_0$$

(**b**) The time it takes to achieve rolling motion is found from

$$a = \frac{F}{M} = \frac{\mu_\text{k}Mg}{M} = \mu_\text{k}g$$

so

$$v = v_0 + \mu_\text{s}gt$$

Inserting the condition $R\omega = v$ for rolling motion gives

$$R\omega = \mu_\text{k}gt$$

$$t = \frac{R\omega}{\mu_\text{k}g}$$

Using the result from part (a) that $\omega = 2\omega_0/7$, we find that

$$t = \frac{2R\omega_0}{7\mu_\text{k}g}$$

ASSESS The answer to part **(a)** is surprising—it says that no matter what the size or speed of the ball, or the coefficient of friction, the angular speed of the ball when it stops sliding is 2/7 of its original value! However, the time it takes the ball to achieve rolling motion depends on the radius of the ball, its initial angular speed, and the coefficient of kinetic friction. Notice that the more slippery is the surface (i.e., for smaller μ_k), the longer it will take for the ball to achieve rolling motion, which is reasonable.

60. **INTERPRET** This problem looks at a time-varying torque.

DEVELOP The torque and angular momentum are related by the rotational analog of Newton's second law: $\vec{\tau} = d\vec{L}/dt$. If we integrate the torque with respect to time, we obtain the angular momentum as a function of time. The given torque points in one direction, and consequently so does the angular momentum, so the vector notation can be dropped.

EVALUATE The angular momentum is

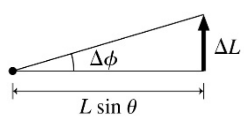
$$L = \int_0^t \tau dt' = \int_0^t (a + b\sin ct')dt' = \left[at' - \frac{b}{c}\cos ct'\right]_0^t = at + \frac{b}{c}(1 - \cos ct)$$

ASSESS We are told that the object is initially stationary, so we can verify that indeed $L(t=0) = 0$.

61. **INTERPRET** We're asked to derive the precession rate for a spinning gyroscope.

DEVELOP The torque is due to gravity. From Equation 11.2, it has a magnitude of $\tau = rF_g \sin\theta$, where θ is the angle between $\vec{L}$ and the vertical line extending up from the point where the gyroscope touches the bottom support. By the right-hand rule, the torque points in the direction perpendicular to the plane defined by $\vec{r}$ and the vertical.

EVALUATE Over a short time interval, Δt, the angular momentum changes in the direction given by the torque: $\Delta\vec{L} = \vec{\tau}\cdot\Delta t$, as shown in fig. 11.9. This change in $\vec{L}$ corresponds to a change in the rotational axis, since $\vec{L} = I\vec{\omega}$. We can characterize how the axis moves with a small angle $\Delta\phi = \Delta L / L\sin\theta$, as defined in the figure below. The view here is from above looking down at the gyroscope.



After the axis moves, the torque points in a new direction, but always in the direction perpendicular to the plane defined by $\vec{r}$ and the vertical. This leads to circular motion with a rotational speed of

$$\omega_p = \frac{\Delta\phi}{\Delta t} = \frac{1}{\Delta t}\left(\frac{\Delta L}{L\sin\theta}\right) = \frac{\tau}{L\sin\theta} = \frac{mgr\sin\theta}{L\sin\theta} = \frac{mgr}{L}$$

ASSESS This says the precession speed will be faster if the gyroscope has a larger mass and/or a longer radial length. It also says that the rate is inversely proportional to the angular momentum. Since $L = I\omega$, we have $\omega_p \propto 1/\omega$, which means that as the gyroscope gradually spins slower around its axis (due to friction forces), it will precess faster around the vertical. You may have observed this behavior in a gyroscope or a spinning top.

62. **INTERPRET** We use conservation of angular momentum to find the radius of a white dwarf star. We know the initial radius, mass, and rotational speed; this gives us the initial angular momentum. The final angular momentum will be the same, so we use it to find the radius, knowing the final mass and angular speed.

DEVELOP We're told that the star collapses with 55% of its original mass. That means 45% of the mass is "blown off." We'll assume these outer layers take their angular momentum with them. So we'll only deal with conservation of momentum in the star's core, with $M = 0.55\, M_{sun}$. Assuming the star is uniform, this core initially occupies a sphere with radius:

$$R_0 = \sqrt[3]{\frac{M}{\frac{4\pi}{3}\rho_{sun}}} = \sqrt[3]{\frac{0.55M_{sun}}{\frac{4\pi}{3}\left(M_{sun} / \frac{4\pi}{3}R_{sun}\right)}} = \sqrt[3]{0.55}R_{sun}$$

EVALUATE Before the collapse, the core's angular momentum is given by $L = I_0\omega_0,$ where $I_0 = \frac{2}{5}MR_0^2$ and $\omega_0 = 2\pi / 25$ d. After the collapse, the core still has the same angular momentum, but the expression is now $L = I\omega,$ where $I = \frac{2}{5}MR^2$ and $\omega = 2\pi / 131$ s. Solving for the unknown final radius, we get:

$$R = R_0\sqrt{\frac{\omega_0}{\omega}} = \left(\sqrt[3]{0.55}R_{\text{sun}}\right)\sqrt{\frac{131\text{ s}}{25\cdot 24\cdot 3600\text{ s}}} = 6.4\times10^{-3}R_{\text{sun}} = 4.45\times10^{6}\text{ m}$$

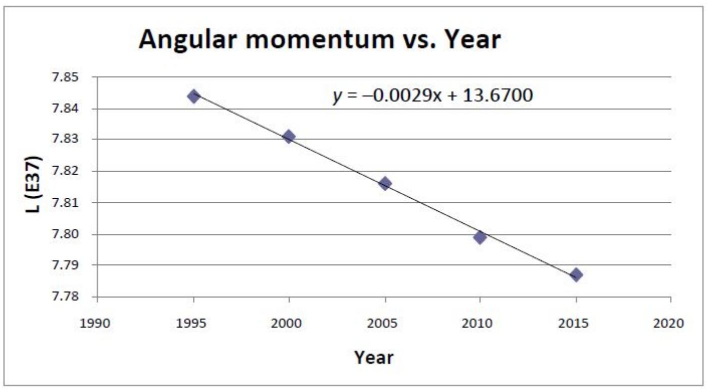
This radius is about 70% of the radius of Earth and 150 times smaller than the radius of the original star.

ASSESS One could assume that the outer layers blow off without taking away any of the angular momentum and that the core inherits *all* of the original angular momentum of the star before the collapse: $L = \frac{2}{5}M_{\text{sun}}R_{\text{sun}}^2\omega_0.$ (This is unlikely, but it can serve as an upper bound.) In such a case, the final radius would be 100 times smaller than that of the original star.

63. INTERPRET We are given the data of a pulsar's angular momentum as a function of time and asked to determine the torque due to its surrounding medium.

DEVELOP The torque that causes the pulsar's spinning rate (and thus the angular momentum) to change over a time interval Δt is given by $\tau = \Delta L / \Delta t.$ Thus, plotting L as a function of t will result in a straight line with slope τ.

EVALUATE The plot is shown below. We obtain a straight line with a slope of -0.0029×10^{37}/year.



The torque is

$$\tau = \frac{\Delta L}{\Delta t} = \frac{(-0.0029\times10^{37}\text{ kg}\cdot\text{m}^2/\text{s})}{\text{year}} = \frac{(-0.0029\times10^{37}\text{ kg}\cdot\text{m}^2/\text{s})}{(365\times86400\text{ s})} = -9.2\times10^{26}\text{ N}\cdot\text{m}.$$

ASSESS The negative sign in τ means that the torque results in a decrease of angular momentum. With $\tau = I\alpha$, the corresponding angular acceleration of the pulsar is

$$\alpha = \frac{\tau}{I} = \frac{-9.2\times10^{26}\text{ N}\cdot\text{m}}{1.12\times10^{38}\text{ kg}\cdot\text{m}^2} = -8.2\times10^{-12}\text{ rad/s}^2.$$

The small α implies that ω decreases very slowly.

64. INTERPRET We are asked to calculate a system's new total angular momentum if its axis of rotation makes a parallel shift.

DEVELOP The initial angular momentum of the system is given by $\vec{L} = \sum \vec{r}_i \times \vec{p}_i$. The vector between the two centers O and O' is $\vec{h} = \vec{r}_i - \vec{r}_i'$. The new angular momentum is $\vec{L}' = \sum \vec{r}_i' \times \vec{p}_i$.

EVALUATE Combining the expressions above, we obtain

$$\vec{L}' = \sum \vec{r}_i' \times \vec{p}_i = \sum\left(\vec{r} - \vec{h}\right)\times\vec{p}_i = \sum \vec{r}\times\vec{p}_i - \vec{h}\times\sum\vec{p}_i = \vec{L} - \vec{h}\times\vec{p}$$

ASSESS In the limit where $\vec{h} = 0,$ $\vec{r}_i = \vec{r}_i',$ and $\vec{L}' = \vec{L},$ as expected.

65. INTERPRET We are asked to determine what happens to a spinning gyroscope when different torques are applied to it.

DEVELOP Initially, the gyroscope has no torque on it, and the angular velocity and angular momentum both point to the right. By applying a force on the gyroscope between the arrowhead and disk, you exert a torque given by $\vec{\tau} = \vec{r} \times \vec{F}$ (Equation 11.2).

EVALUATE In this case, the force $\vec{F}$ points into the page and is applied at a radius $\vec{r}$ that points to the right. By the right-hand rule, the torque points upward. By Equation 11.5 $\left(d\vec{L} / dt = \vec{\tau} \right)$, the angular momentum will move in the torque's direction. Because the arrowhead points in the direction of the angular momentum, it too will move upward.

The answer is (d).

ASSESS It might seem odd that you push something in one direction, and it moves in a perpendicular direction. But this is just how the rotational analog of Newton's second law works.

66. **INTERPRET** We are asked to determine what happens to a spinning gyroscope when different torques are applied to it.

DEVELOP As described in the previous problem, the push results is a torque $\vec{\tau} = \vec{r} \times \vec{F}$.

EVALUATE In this case, the force $\vec{F}$ points upward and is applied at a radius $\vec{r}$ that points to the right. By the right-hand rule, the torque points out of the page. The angular momentum vector and the arrowhead will both move toward you, out of the page.

The answer is (b).

ASSESS Compared to the previous problem, the force has rotated by 90°, so we'd expect the torque would as well.

67. **INTERPRET** We are asked to determine what happens to a spinning gyroscope when different torques are applied to it.

DEVELOP The added weight means the gyroscope is no longer balanced on the stand. There will be more downward force on the left side than on the right side. By the right-hand rule, this generates a torque that points out of the page.

EVALUATE This torque will cause the angular momentum to move slightly in the direction of the torque, that is, out of the page. Recall fig. 11.9, where $\Delta\vec{L}$ points the same way as $\vec{\tau}$. This shift in the angular momentum will start the gyroscope turning in a clockwise direction as seen from above. As it moves, the torque will change so that the gyroscope continues to precess clockwise about the stand.

The answer is (d).

ASSESS One might have wrongly assumed that since the torque is out of the page it will "push" the left-hand side of the gyroscope (where the weight was added), thus resulting in a counterclockwise rotation. The torque does not act on a specific point, but instead acts on the whole system through its angular momentum.

68. **INTERPRET** We are asked to determine what happens to a spinning gyroscope when different torques are applied to it.

DEVELOP As the gyroscope precesses over a short time interval, Δt, the angular momentum changes by a small amount: $\Delta\vec{L} = \vec{\tau} \cdot \Delta t$. This shift corresponds to a change in the direction of $\vec{L}$ characterized by an angle: $\Delta\phi = \Delta L / L$.

EVALUATE The precession rate is equal to the rate at which $\Delta\phi$ changes with time:

$$\omega_{\text{p}} = \frac{\Delta\phi}{\Delta t} = \frac{\Delta L}{\Delta t \cdot L} = \frac{\tau}{L} = \frac{\tau}{I\omega}$$

This shows that the precession rate is inversely proportional to the rotation rate of the disk, ω So if the rotation rate increases, the precession rate will decrease.

The answer is (a).

ASSESS We can check the units on our expression for the precession rate. The ratio τ / I is equal to the angular acceleration, α (recall Equation 10.11). So the units are

$$\left[\omega_{\text{p}}\right] = \frac{[\alpha]}{[\omega]} = \frac{1/\text{s}^2}{1/\text{s}} = 1/\text{s}$$

This is what we would expect for the precession rate.