1 Problem 1

The animation is about flipping a special designed commemorative coin.

2 Problem 3

Assume we have a point at $P = [0, 0, 1]$.

Let the pitch angle be $\frac{\pi}{4}$ and the yaw angle be $\frac{\pi}{4}$. We have the following rotation matrix.

$$R_{pitch} = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\frac{\pi}{4}) & 0 & -\sin(\frac{\pi}{4}) \\ 0 & 1 & 0 \\ \sin(\frac{\pi}{4}) & 0 & \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_{yaw} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) & 0 \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since we do not rotate on roll axis, R_{roll} can be omitted.

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = R_{pitch} R_{yaw} P^T$$

$$\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = R_{yaw} R_{pitch} P^T$$

By the above example, we know that $R_{yaw} R_{pitch} P \neq R_{pitch} R_{yaw} P$.

3 Problem 4

3.1 Problem 4a

Let $0 \leq \theta \leq \pi$, we have $\sin(-\theta) = -\sin(\theta)$ and $\cos(\theta) = \cos(-\theta)$.

$$R_{yaw} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, R_{pitch} = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}, R_{roll} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R^T_{yaw} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, R^T_{pitch} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}, R^T_{roll} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R^T_{yaw} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, R^T_{pitch} = \begin{bmatrix} \cos(-\theta) & 0 & -\sin(-\theta) \\ 0 & 1 & 0 \\ \sin(-\theta) & 0 & \cos(-\theta) \end{bmatrix}, R^T_{roll} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\theta) & -\sin(-\theta) \\ 0 & \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

We know that $RR^T = R^T R = I \iff A^T = A^{-1}$. That is, R^T can cancel the effects of R. Obviously, rotate $-\theta$ would cancel the effect of rotate θ. As we can see from the equations above, R^T is rotate $-\theta$ degrees while R is rotate θ degrees. So we know that R^T will cancel the effects of R. Thus, R^T is the inverse of R.

3.2 Problem 4b

By the above argument, we know $A = R_{pitch} R_{yaw} R_{roll} U \iff R^T_{pitch} R^T_{yaw} R^T_{roll} A = U$. Let $U = \begin{bmatrix} x_1 & x_2 & x_3 \\ 0 & x_4 & x_5 \\ 0 & 0 & x_6 \end{bmatrix}$ where $x_i \in R, 1 \leq i \leq 6$.

It suffices to show $\exists R^T_{pitch} R^T_{yaw} R^T_{roll}$ rotates p_1, p_2 to x-axis, xy-plane where p_1, p_2 are points in R^3.

First, rotate the outer ring and the middle ring to let the inner ring lies on the plane made by p_1, p_2 and center. Second, we attach p_1, p_2 to the inner ring. Next, we rotate the outer ring and the middle ring to make the inner ring lies on x-y plane. Last, rotate the inner ring to make p_1 lies on x axis.

Since each ring represent a matrix, we may construct $R^T_{pitch} R^T_{yaw} R^T_{roll}$ by the procedure above.

4 Problem 5

A Gimbal lock is when 2 (or more) dimensions are parallel. That is, any of (pitch, yaw), (yaw, roll), (pitch, roll) are parallel.

An object may not rotate in such direction which Gimbal lock is formed. For example, when pitch and yaw become parallel, we may only rotate in 2 directions, roll or pitch (pitch is equal to yaw). The following example shows an example of Gimbal lock.