ON THE ECCENTRICITY OF SATELLITE ORBITS IN THE SOLAR SYSTEM

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Summary

In this paper the secular changes in the eccentricities of satellite orbits in the solar system are investigated. Two mechanisms which affect the eccentricities are considered. One of them is the tide raised on the planet by the satellite, which has been the subject of discussion in the past; the other is the tide raised on the satellite by the planet. It is seen that cases arise in the solar system in which each of these tide's effect on eccentricity is dominant.

1. Darwin (1), Groves (2) and Jeffreys (3) have given arguments to show that in most cases the tide raised by a satellite on a planet tends to increase the eccentricity of the satellite's orbit. If we look at the values of the eccentricities which arise in the orbits of the inner satellites of planets, we see that they range down as low as 10^{-4} for Tethys. Since it seems hard to imagine any process of formation of satellites which could produce initial values of eccentricity as low or lower than these, it would appear necessary to look for some mechanism that could produce a secular decrease of eccentricity which could rival in magnitude the secular change due to the tides raised on the planets. Such a mechanism was proposed by Urey, Elsasser and Rochester (4) in the form of tidal working in the satellite due to tides raised by the planet. If we only consider the case where the satellite always presents the same face to the planet (this is the only case which is observed in the solar system and includes the greater satellites of Jupiter and Saturn as well as the Moon (Jeffreys (5)) then it is easy to see why the tide raised on a satellite tends to decrease its eccentricity. The eccentricity

$$e = \sqrt{1 + \frac{2EL^2}{M_s^3 M_p^2 G}}, \tag{1}$$

where E is the energy of the orbit, L is the angular momentum and M_p and M_s are the planet and satellite masses. If the satellite is not spinning, then the tide raised on it can only produce a radial perturbation force. This means that L is not changed by the tide. Since any energy dissipation in the satellite decreases E and since we have $E < 0$, $0 < e < 1$ and L constant, we find that e is decreasing also. If $e \neq 0$ such dissipation must take place since the height of the tide will vary with the oscillation in distance between the satellite and planet.

In this paper it will be shown that insofar as their effects on eccentricity are concerned, the tides on the satellites are probably more important than tides on the planets in all cases where tidal effects might be significant except for Phobos, Deimos and possibly the Moon and Jupiter V. In the cases of Phobos, Deimos and Jupiter V, however, the tides raised in their planets tend secularly to decrease their eccentricities anyway so it is not surprising to find that the orbits of these satellites have low values of eccentricity.

2. In this section the following idealized case of a planet with a single satellite will be considered. The assumptions are as follows:

(i) The mass of the satellite is neglected in comparison with that of the planet.

(ii) The inclination of the satellite's orbit plane to the planet's equator is taken as zero.

(iii) The planet and satellite are both considered to be homogeneous incompressible spheres which can be characterized by two parameters, μ and Q. μ is the rigidity and Q the specific dissipation function

$$Q = \frac{2\pi E^*}{\oint (dE/dt)dt}, \tag{2}$$

where E^* is the peak energy stored in the system during a cycle and $\oint(dE/dt)dt$ is the energy dissipated over a complete cycle. Q will in general vary with the frequency and amplitude of the tide and the size of the sphere in addition to its composition.

(iv) Finally, I will work only to first order in eccentricity in the interest of simplicity.

The idealized problem that I have set up, but neglecting the tide raised on the satellite, has been solved by Darwin (**6**) and this solution was reviewed by Jeffreys (**7**). The following will be a summary of Jeffreys's paper with several corrections of misprints. Jeffreys's notation will not be followed; instead we shall use the following:

a = semi-major axis of the satellite's orbit.
e = eccentricity of the satellite's orbit.
G = gravitational constant.
n = satellite's mean motion $= 2\pi/T$ where
T = satellite's period.

Quantities pertaining to the planet will have a subscript p and those pertaining to the satellite will have a subscript s.

M = mass of sphere.
R = radius of sphere.
ρ = density of sphere.
g = surface gravity of sphere.
μ = rigidity of sphere.
$Q(\nu) = Q$ as a function of frequency ν.
ω = angular rotation velocity of sphere.
$2\epsilon_i$ $(i = 1 \to 3)$ are phase lags in the periodic tides raised by the tidal potential.

These phase lags arise as follows: the tidal potential acting is written as a sum of periodic terms with different frequencies. The response of the tide to any periodic component of the potential will be in phase with the potential only if no energy is dissipated by the tide. If the tide dissipates energy, then its phase will lag that of the potential (at least on the average). $2\epsilon_0$, $2\epsilon_1$, $2\epsilon_2$, and $2\epsilon_3$ are the lags of the tides with frequencies $2\omega - 2n$, $2\omega - 3n$, $2\omega - n$, and $3/2n$ respectively.

Finally, using the notation set up above, we can state Darwin's and Jeffreys's result.

$$\overline{\left(\frac{de}{dt}\right)}_p = -\frac{6}{5}(GM_p)^{1/2}R_p H_p \frac{e}{a^{7/2}}\left(\epsilon_{0p} - \frac{49}{4}\epsilon_{1p} + \frac{1}{4}\epsilon_{2p} + \frac{3}{2}\epsilon_{3p}\right), \qquad (3)$$

where

$$\overline{\frac{d}{dt}} = \overline{\left(\frac{d}{dt}\right)}_p + \overline{\left(\frac{d}{dt}\right)}_s.$$

$\overline{(d/dt)}_p$ denotes secular rate of change due to tides on the planet and $\overline{(d/dt)}_s$ denotes secular range of change due to tides on the satellite.

The formulae for λ and H in Jeffreys's paper, however, contain several mistakes, so they will be corrected here. The elastic tide raised by a potential

$$U = k_2 r^2 s_2 \qquad (4)$$

(where s_2 is a spherical harmonic of order two and r is the distance from the centre of the planet) is equal to

$$\lambda R_p s_2 \qquad (5)$$

at the surface of the sphere where

$$\lambda = \frac{5\rho_p R_p^2 k_2}{(19\mu_p + 2g_p\rho_p R_p)} \quad \text{and} \quad k_2 = \frac{3GM_s}{4a^4}. \qquad (6)$$

If we write $\lambda R_p = H_p$ then we find that the surface inequality of the planet produces an external potential

$$U_{1p}(r) = \tfrac{3}{5}\frac{GM_p R_p H_p s_2}{r^3}.$$

Using the correct expression for H we get:

$$\overline{\left(\frac{de}{dt}\right)}_p = \frac{9(GM_p)^{1/2}(GM_s)R_p^4\rho_p e}{2a^{13/2}(19\mu_p + 2g_p\rho_p R_p)}\left[\epsilon_{0p} - \frac{49}{4}\epsilon_{1p} + \frac{1}{4}\epsilon_{2p} + \frac{3}{2}\epsilon_{3p}\right]. \qquad (7)$$

If we next consider the tides raised on the satellite we can modify the above argument slightly to get $\overline{(de/dt)}_s$. We must first realize that $\omega_s = n$. Hence, $2\omega_s - 2n = 0$, $2\omega_s - 3n = -n$, $2\omega_s - n = n$ and as a consequence $\epsilon_{0s} = 0$, and $\epsilon_{1s} = -\epsilon_{2s}$. Furthermore, U_{1p} above is the potential energy per unit satellite mass due to tides on the planet.

$$U_{1s} = \frac{3GM_s R_s H_s s_2}{r^3}$$

is then the potential energy per unit planet mass due to tides on the satellite. To change this to potential energy per unit satellite mass, we must multiply U_{1s} by M_p/M_s. Using the preceding results we have from Jeffreys's formula that

$$\overline{\left(\frac{de}{dt}\right)}_s = -\frac{9M_p(GM_p)^{1/2}(GM_s)R_s^4\rho_s e}{2M_s a^{13/2}(19\mu_s + 2g_s\rho_s R_s)}\left[\frac{25}{2}\epsilon_{2s} + \frac{3}{2}\epsilon_{3s}\right]$$

or

$$\overline{\left(\frac{de}{dt}\right)}_s = -\frac{9(GM_p)^{3/2}R_s^4\rho_s e}{2a^{13/2}(19\mu_s + 2g_s\rho_s R_s)}\left[\frac{25}{2}\epsilon_{2s} + \tfrac{3}{2}\epsilon_{3s}\right]. \qquad (8)$$

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3. Before we can compare the magnitudes of $\overline{(de/dt)}_p$ and $\overline{(de/dt)}_s$, we must relate the phase lags, $2\epsilon_i$, to the specific dissipation function Q. This is accomplished in the following manner. Let W be the work done on one of our homogeneous, incompressible spheres by a body force $\mathbf{f}(\mathbf{r})$ which is derivable from a potential $U(\mathbf{r})$. Then

$$\frac{dw}{dt} = \int_v \rho \mathbf{v}(\mathbf{r}) . \mathbf{f}(\mathbf{r})\, d^3r, \tag{9}$$

where $\mathbf{v}(\mathbf{r})$ is the velocity of the material at the point $\mathbf{r}$. Using the equation of continuity we get $\rho\mathbf{v} . \nabla \mathbf{U} = \nabla . (\rho\mathbf{v}U) - U\nabla . (\rho\mathbf{v}) = \nabla . (\rho\mathbf{v}U) + (\partial\rho/\partial t)U$. Since $\partial\rho/\partial t = 0$, we have

$$\frac{dw}{dt} = \int_v \rho U\mathbf{v} . \hat{n}\, ds, \tag{10}$$

where $\hat{n}$ is the outward normal to the surface of the sphere. If $U \propto \cos \nu t$ and the surface inequality is $\propto \cos(\nu t - 2\epsilon)$ then $\mathbf{v} . \hat{n} \propto \sin(\nu t - 2\epsilon)$. Hence, we get

$$\frac{dw}{dt} = K \cos \nu t \sin(\nu t - 2\epsilon).$$

A simple integration gives to first order in ϵ:

$$\oint \frac{dw}{dt}\, dt = -\frac{K\pi}{\nu} \sin \epsilon = \oint \frac{dE}{dt}\, dt,$$

$$E^* = \int_0^{\pi/2\nu} \frac{dw}{dt}\, dt = \frac{K}{2\nu} \cos 2\epsilon.$$

Using (2) we get

$$Q = \frac{2\pi E^*}{\oint (dE/dt) dt} = \frac{1}{\tan 2\epsilon} \quad \text{or for large } Q \quad 2\epsilon = \frac{1}{Q}. \tag{11}$$

Incidentally, this calculation tells us that $\epsilon(\nu)$ has the same sign as ν since $\oint (dE/dt)\, dt < 0$.

4. Next I will show how Q compares for two spheres of the same material but of different size. This behaviour is important for explaining why the inner satellites of the large planets have circular orbits (e.g. Saturn I $\rightarrow$ V).

Qualitatively one would expect Q to increase with the size of the body for the following reason. The energy dissipated, per unit volume, in a cycle, will depend only on the square of the strains since this frictional energy dissipation is a local phenomenon. The peak energy stored per unit volume, on the other hand, increases for a fixed strain with the size of the body since the stresses are increased over the purely elastic ones by the self-gravitation of the sphere. Hence, Q increases as the body's self-gravity becomes more important than its elasticity. The relative importance of elasticity and self-gravity of the sphere enters into the formula for the tidal surface inequality in the denominator $(19\mu + 2g\rho R)$. To examine the quantitative dependence of Q on size, we proceed as follows. From formulae (4), (5), (6) and (10) we have the energy dissipated in a cycle, for each periodic component of the tide equals

$$E_\iota^+ = \oint \frac{dw_\iota}{dt}\, dt = \frac{C_\iota R^7 \epsilon_\iota}{(19\mu + 2g\rho R)},$$

where C_ι is independent of μ and R. However, the energy dissipated must be proportional to the square of the strain as mentioned previously. Therefore, the energy dissipated in a cycle equals

$$\frac{D_\iota R}{(19\mu+2g\rho R)}$$

where D_ι is independent of μ and R also. This last formula is derived as follows. Using (5) and (6) we have the surface strains proportional to

$$\frac{R^2}{(19\mu+2g\rho R)},$$

hence the square of the strain integrated over the volume of the sphere gives a result proportional to

$$\frac{R^7}{(19\mu+2g\rho R)}$$

where we assume the surface value of strain throughout the sphere as an approximation to get the proportional result. Comparing these expressions we get

$$\epsilon_\iota = \frac{\gamma}{(19\mu+2g\rho R)},$$

where γ is independent of μ and R. This gives

$$\frac{Q}{Q_0} = 1 + \frac{2g\rho R}{19\mu} \tag{12}$$

where Q_0 is the value of Q for a body where self-gravity is negligible. In the case of liquid or gaseous spheres $\mu = 0$ and Q_0 is not defined.

5. In this section the two rates of change of e will be compared both in magnitude and in sign.

First I shall deal with the question of sign. $(\overline{de/dt})_p$ has the sign of

$$-\left(\epsilon_{0p} - \frac{49}{4}\epsilon_{1p} + \frac{\epsilon_{2p}}{4} + \frac{3}{2}\epsilon_{3p}\right).$$

For the Earth, Q and hence ϵ, varies by less than a factor of four over a range of one cycle per second to one cycle per year (**8**). In this case ϵ_1 must be the dominant term and the sign of $(\overline{de/dt})_p$ is the same as the sign of $2\omega - 3n$. While this constant behaviour of Q with frequency may not be true for all planets (especially not the major ones) it is still likely that the ϵ_1 term is dominant because of its relatively large coefficient. If this ϵ_1 term is dominant, we have $(\overline{de/dt})_p > 0$ for all satellites except Phobos, Deimos, Jupiter V and the retrograde ones.

In the case of tides raised on satellites the sign of $(\overline{de/dt})_s$ is the sign of

$$-\left(\frac{25}{2}\epsilon_{2s} + \frac{3}{2}\epsilon_{3s}\right),$$

where ϵ_{2s} and ϵ_{3s} have the sign of n and therefore are positive. Hence, $(\overline{de/dt})_s < 0$ for all satellites which keep the same face toward their planets.

Next, we will compare the magnitudes of these two rates of change of e. From formulae (4) and (8) we get

$$\frac{\overline{(de/dt)}_s}{\overline{(de/dt)}_p} = \left(\frac{R_s}{R_p}\right)\frac{(19\mu_p+2g_p\rho_pR_p)}{(19\mu_s+2g_s\rho_sR_s)} + \frac{\left\{\frac{25}{2}\epsilon_{2s}+\frac{3}{2}\epsilon_{3s}\right\}}{\left\{\epsilon_{0p}-\frac{49}{4}\epsilon_{1p}+\frac{\epsilon_{2p}}{4}+\frac{3}{2}\epsilon_{3p}\right\}}. \tag{13}$$

I will examine this ratio in three limiting cases:

(i) $\mu_p \gg 2g_p\rho_pR_p \quad \mu_s \gg 2g_s\rho_sR_s$

This is the case of small satellite and small planet and yields the result

$$\frac{\overline{(de/dt)}_s}{\overline{(de/dt)}_p} \simeq \left(\frac{R_s}{R_p}\right)\left(\frac{\mu_p}{\mu_s}\right)\frac{\{25\epsilon_{2s}+3\epsilon_{3s}\}}{\left\{2\epsilon_{0p}-\frac{49}{2}\epsilon_{1p}+\frac{\epsilon_{2p}}{2}+3\epsilon_{3p}\right\}}. \tag{14}$$

We see that if the satellite is appreciably smaller than the planet and has approximately the same rigidity and specific dissipation function, we get the tides raised on the planet dominating. This case certainly applies to Phobos and Deimos.

(ii) $\mu_p \ll 2g_p\rho_pR_p \qquad \mu_s \gg 2g_s\rho_sR_s$

This is the case of large planet and small satellite and we get

$$\frac{\overline{(de/dt)}_s}{\overline{(de/dt)}_p} \simeq \frac{2g_p\rho_pR_s}{19\mu_s}\frac{\left\{\frac{25}{2}\epsilon_{2s}+\frac{3}{2}\epsilon_{3s}\right\}}{\left\{\epsilon_{0p}-\frac{49}{4}\epsilon_{1p}+\frac{\epsilon_{2p}}{4}+\frac{3}{2}\epsilon_{3p}\right\}}. \tag{15}$$

In this case no general conclusion can be drawn about this ratio.

(iii) $\mu_p \ll 2g_p\rho_pR_p \qquad \mu_s \ll 2g_s\rho_sR_s$

This is the case of large planet and large satellite and gives

$$\frac{\overline{(de/dt)}_s}{\overline{(de/dt)}_p} \simeq \left(\frac{\rho_p}{\rho_s}\right)^2\left(\frac{R_p}{R_s}\right)\times\frac{\left\{\frac{25}{2}\epsilon_{2s}+\frac{3}{2}\epsilon_{3s}\right\}}{\left\{\epsilon_{0p}-\frac{49}{4}\epsilon_{1p}+\frac{\epsilon_{2p}}{4}+\frac{3}{2}\epsilon_{3p}\right\}}. \tag{16}$$

In this case we see that the satellite tide wins. This is made even more striking when we remember that if the satellite has the same composition as the planet then

$$\frac{Q_p}{Q_s} = \left(\frac{R_p}{R_s}\right)^2, \text{ so that } \frac{\overline{(de/dt)}_s}{\overline{(de/dt)}_p} \simeq \left(\frac{R_p}{R_s}\right)^3. \tag{17}$$

6. A discussion of the results obtained in the preceding sections is presented. This discussion is intended to give an explanation of the results obtained so far and to indicate the range of their validity.

The first point dealt with will be the effect that tides raised on the planet have on the eccentricity. Let us consider the case where the period of the planet's rotation is much shorter than the period of the satellite's revolution and the satellite is a direct one (not retrograde). Since the tide raised on the planet is dissipative, we have a time lag between the applied tidal force and the tidal bulge it raises. Because the day is shorter than the month this time lag means that the tidal bulge precedes the satellite in longitude. The effect of this tidal bulge lead in longitude is to produce a couple between the satellite and planet which adds angular momentum to the satellite's orbit, at the expense of the rotational angular momentum of the planet. This is the well-known tidal couple which is responsible for the secular acceleration of the Moon.

So far our argument has been independent of the eccentricity of the satellite's orbit. Let us next consider what additional complications arise when we take the eccentricity into account and how they feed back to effect the eccentricity which produced them. In order to simplify the picture let us think of a very eccentric satellite orbit. The height of the tide raised depends inversely on the third power of the satellite's distance and the force it produces on the satellite involves four more reciprocal powers of distance. Hence, we have the torque on the satellite decreasing as the sixth power of the satellite's distance from the planet. This steep decrease with distance enables us to approximate the effect of the tidal bulge on the satellite's orbit by an impulse at pericentre. With this approximation the satellite must again pass through the same point at pericentre, since bound orbits in inverse square force fields are periodic. But, angular momentum has been added to the satellite's orbit; hence, the apocentre distance, and therefore the eccentricity, must have been increased. In fact the angular momentum of the satellite per unit mass is

$$L = \sqrt{GM_p a(1-e^2)} = \sqrt{GMr_p(1+e)}$$

where $r_p = a(1-e)$ is the distance to pericentre. If we have $\Delta L > 0$ and $\Delta r_p = 0$ as discussed above, then

$$\Delta L = \tfrac{1}{2}\sqrt{\frac{GMr_p \Delta e}{(1+e)}}$$

so Δe is positive. Also from $\Delta r_p = 0$ we get

$$\Delta[a(1-e)] = \Delta a(1-e) - a\Delta e = 0 \quad \text{or} \quad \Delta a = \frac{a\Delta e}{(1-e)}$$

so that Δa is positive also, as we would expect. The previous discussion, when modified to hold for smaller values of e, accounts for the tendency of the tide raised on the planet, to increase the satellite's eccentricity.

The considerations presented above are concerned solely with the tidal torque on the satellite. That is, they only make use of the component of the disturbing force which is perpendicular to the satellite's radius vector. In his paper on the Moon's eccentricity, Groves (**2**) also considered only the tidal torque. It is not surprising, then, that he found the Moon's eccentricity could only increase due to tides on the Earth. This neglect of the radial components of the disturbing force renders the above arguments, and those of Groves as well, incomplete. We shall consider how the picture presented up to now is altered by the inclusion of radial forces.

We shall again take an eccentric orbit about our planet. The relevant points are identical with those presented in Section 1. There it was shown that the tide raised on the satellite produces only radial perturbation forces and since these cannot change the satellite's angular momentum, but must decrease its energy, they must also decrease its eccentricity. The preceding argument, when applied to the planetary tide, shows us that this tide may decrease as well as increase eccentricity. The details of whether we have decreasing or increasing eccentricity depend on the satellite's revolution period, the planet's rotation period and the amplitude and frequency dependence of $Q(\nu)$.

The applicability of our results to the actual planet-satellite systems extant involves two questionable assumptions.

The first assumption is the neglect of all tides except the solid body ones. It has recently been demonstrated that, in the Earth–Moon systems, the ocean tides which in the past were thought to be of major importance are really much less important than the solid body tides. This conclusion would undoubtedly also pertain to Mars. For Jupiter, Saturn, Uranus and Neptune, however, turbulent tides in their atmospheres or possibly in any liquids which may be found on these planets, might be of greater importance than the solid body tides. We can still use the two parameters μ and Q to fit the tides on these planets, although we can no longer hope to make very good estimates of their frequency and amplitude dependence. The satellites of these planets are almost certainly solid since they are not big enough to have held the heat necessary to keep them liquid and are not receiving enough heat to do this either. Before leaving this question of the composition of the major planets, it should be mentioned that for Jupiter, measurements exist which have been used in the past to calculate a lower bound for Q. This question will be taken up in the following paper where it will be shown that although the measurements may be correct, their interpretation is not.

The second serious approximation we have made is the linear superposition of tides of different frequencies. In developing formula (7), Darwin and Jeffreys both wrote the tide raising potential as the sum of periodic potentials. They then proceeded to consider the response of the planet to each of the potentials separately. At first glance this might seem proper since the tidal strains are very small and should add linearly. The stumbling block in this procedure, however, is the amplitude dependence of the specific dissipation function. In the case of the Earth, it has been shown by direct measurement that Q varies by an order of magnitude if we compare the tide of frequency $2\omega-2n$ with the tides of frequencies $2\omega-n$, $2\omega-3n$ and $\frac{3}{2}n$. This is because these latter tides have amplitudes which are smaller than those of the principal tide (of frequency $2\omega-2n$) by a factor of eccentricity or about 0·05. It may still appear that we can allow for this amplitude dependence of Q merely by adopting an amplitude dependence for the phase lags of the different tides. Unfortunately, this is really not sufficient since a tide of small amplitude will have a phase lag which increases when its peak is reinforcing the peak of the tide of major amplitude. This non-linear behaviour cannot be treated in detail since very little is known about the response of the planets to tidal forces, except for the Earth. In our discussions we shall use the language of linear tidal theory, but we must keep in mind that our numbers are really only parametric fits to a non-linear problem.

There is one more assumption which is implied in this paper. It concerns the neglect of direct gravitational interactions between bodies in influencing the eccentricities of the satellites. Celestial mechanicians, in particular, would consider this omission to be a very serious one over periods greater than a few thousand years. This is because their calculations will not guarantee the stability of satellite eccentricities, perturbed by direct gravitational interactions, for periods greater than these. To this objection one can only offer the belief that, for well-spaced orbits, direct gravitational interactions alone will not endanger stability in eccentricity, even over ages comparable to those of the solar system.

7. The development of the orbit in time will be taken up in this section for use later on. From Jeffreys's paper (**9**) we have

$$(\overline{da/dt}) = \frac{18(GM_p)^{1/2}(GM_s)\rho_p R_p^4 \epsilon_{0p}}{a^{11/2}(19\mu_p + 2g_p\rho_p R_p)}. \tag{18}$$

In the absence of better information, Q is taken independent of frequency and amplitude in this section. This enables us to integrate the above equation which gives:

$$a = \left[\frac{117(GM_p)^{1/2}(GM)\rho_p R_p^4 \epsilon_p t}{(19\mu_p + 2g_p\rho_p R_p)} + a_0^{13/2}\right]^{2/13}$$

where $a = a_0$ at $t = 0$.

If we set ϵ_ι, $\iota = 1 \rightarrow 3$ equal to $\pm\epsilon$ then

$$(\overline{de/dt})_p = \frac{e\alpha(GM_p)(GM_s)R_p^4\rho_p\epsilon_p}{a^{13/2}(19\mu_p + 2g_p\rho_p R_p)},$$

$$(\overline{de/dt})_s = \frac{-e72(GM_p)R_s\rho_s\epsilon_s}{a^{13/2}(19\mu_s + 2g_s\rho_s R_s)}.$$

α is a numerical coefficient which depends on the sign of the various ϵ_ι.

In any case

$$\frac{\overline{de}}{da} = \frac{\gamma e}{a} \quad \text{so} \quad \left(\frac{e}{e_0}\right) = \left(\frac{a}{a_0}\right)^\gamma$$

where $e = e_0$ when $a = a_0$. γ will be evaluated separately in each case in the following sections.

8. Numerical estimates for $(\overline{de/dt})$ are made in this section for several different satellites.

(i) *The Earth–Moon System*

Since we have more information in this case than in any other, it will be investigated in greatest detail. Jeffreys (**10**) uses $19/2g\rho R = 3$ for the Earth, with a homogeneous sphere model. If we use the same value of μ for the Moon as for the Earth, then we get

$$\frac{(\overline{de/dt})_s}{(\overline{de/dt})_p} = -\frac{28}{19}\left(\frac{1738}{6378}\right)\left(\frac{4}{3}\right)^2 \frac{Q_{0p}}{Q_{0s}} = -0{\cdot}7\frac{Q_{0p}}{Q_{0s}}$$

where Q_0 is the value of Q for the material when self-gravity is negligible, as discussed in section 4. If $Q_{0s} = Q_{0p}$ then the eccentricity of the Moon's orbit would be increasing. However, there are two considerations which tend to increase the importance of the tides on the Moon. In the first place, the rigidity of the Moon is likely to be smaller than that for the Earth since the high rigidity of the Earth is due to high pressures in the interior. Furthermore, the strains on the Moon are larger than those on the Earth by a factor of 4·9, for the value of rigidity given above. It is known that for the Earth Q decreases with the amplitude of strain (**11**) and this would probably also be the case for the Moon.

The above would be considerably altered if ocean tides were significant contributors to Q on the Earth, but recent evidence (12) seems to rule out this possibility. If we take

$$\mu_s = \frac{\mu_p}{2} \quad \text{then} \quad \frac{\overline{(de/dt)}_p}{\overline{(de/dt)}_s} = -1 \cdot 4 \frac{Q_{0p}}{Q_{0s}}.$$

We thus see that small changes in the parameters μ and Q alter the sign of $\overline{(de/dt)}$ so that the results for the Moon must be considered inconclusive.

If we determine the value of

$$\frac{1}{(19\mu_p + 2g_p\rho_p R_p)Q_p},$$

for the Earth, from the present observed secular acceleration of the Moon, we can use this information to determine the evolution of the semi-major axis of the Moon. MacDonald has done this and claims that the data are consistent with having the Moon close to the Earth between one-half and one billion years ago (13). Ignoring the difficulties which arise from such a situation, we can make the following observation in regard to eccentricity: if the rate of eccentricity change is dominated by tides on the Earth, then we get

$$\left(\frac{e}{e_0}\right) = \left(\frac{a}{a_0}\right)^{\frac{48}{19}}. \tag{20}$$

For $a_0 = 10{,}000$ km, $a = 380{,}000$ km and $e = 0 \cdot 055$ we get $e_0 = 5 \times 10^{-6}$ (Groves gets 2×10^{-6}) which seems very small, especially when compared with the eccentricities of other satellites in the solar system, for which planet tides produce negligible secular accelerations. On the other hand, if e has been decreasing and we assume its initial value was less than $e_0 = 0 \cdot 5$, we get the following: from the result $(e/e_0) = (a/a_0)^\alpha$ we get $-0 \cdot 6 < \gamma < 0$, so we see that $|\overline{(de/dt)}_s| < 1 \cdot 25 |\overline{(de/dt)}_p|$ and the two rates of change must have almost cancelled each other. In any case, it seems likely from the preceding that whether the Moon's eccentricity is increasing or decreasing, the two tidal effects are close to being equal in magnitude. It is worth noting, in this context, that the Moon's eccentricity is higher than that of all other inner satellites in the solar system.

(ii) *Mars*

As explained in Section 5, Phobos and Deimos are covered by case (i), Section 5 and $\overline{(de/dt)}_s / \overline{(de/dt)}_p \ll 1$.

(*a*) For Phobos ϵ_{0p}, ϵ_{1p}, and ϵ_{2p} are all negative while ϵ_{3p} is positive. This tells us that $\overline{(de/dt)}_b$ is almost certainly negative. This agrees with the observed low eccentricity of 0·019 for Phobos. It is still necessary to show whether the tides on Mars could have appreciably altered the eccentricity of Phobos in the age of the solar system, which we take as four billion years. Using the same μ_p for Mars as is used by Jeffreys (14), we have:

$$\overline{(de/dt)}_p = -\frac{9 \cdot 75 \times 10^{-15} e}{Q_p} \text{ sec}^{-1}$$

at the present time. Next, using $Q = 100$, which is a typical value for low amplitude tides on the Earth, we get

$$(1/e)\, \overline{(de/dt)}_p = -9 \cdot 75 \times 10^{-17} \text{ sec}^{-1}.$$

Since 4×10^9 years $= 1{\cdot}2 \times 10^{17}$ sec we see that e could have been appreciably decreased by tides on Mars. It should be borne in mind that the semi-major axis of this satellite is decreasing, since the satellite's period is shorter than the Martian day. This means that $-(1/e)\,(de/dt)_p$ was smaller in the past than it is now. Using formulae (19) and (20), one could carry this analysis out to include the integration over the past four billion years.

(*b*) For Deimos, we have ϵ_{1p} negative while ϵ_{0p}, ϵ_{2p} and ϵ_{3p} are all positive. This assures us that $\overline{(de/dt)}_p$ is definitely negative for Deimos. Again using Jeffreys's numbers we have, taking $Q_p = 100$:

$$(1/e)\,(de/dt)_p = 4{\cdot}5 \times 10^{-20}\,\text{sec}^{-1},$$

so we see that in 4×10 years, or $1{\cdot}2 \times 10^{17}$ sec, the tidal forces will make a negligible change in e. For Deimos, the semi-major axis is increasing due to the tides but at such a rate that this effect may also be safely neglected.

Since the eccentricity of Deimos is 0·003, we are forced either to accept the conclusion that Deimos was formed with this initial eccentricity or to find some other process that might have decreased the eccentricity of this satellite. In the following paper one such mechanism is proposed. It involves the direct gravitational interaction of Phobos with Deimos coupled with the tidal forces on Phobos. It is shown there that the rate of change of Deimos' eccentricity, due to this process, could have been significant over a period of four billion years.

(iii) *Jupiter, Saturn, Uranus and Neptune*

As discussed in section 6, the satellites of these planets are likely to be solid, while the state or states of the planets are uncertain.

For Jupiter V the planetary, as well as the satellite, tides decrease the satellite's eccentricity. These tidal effects are very likely to be significant and probably account for the satellite's low eccentricity of 0·003.

For all the other satellites of these planets, except for the retrograde ones, the planetary tide increases the satellite's eccentricity. These satellite planet systems probably can be approximated by case (ii) of section 5, if we assume μ's of the order of those of ice. If we write equation (15) in terms of Q_{0s} and Q_{0p}, we get:

$$\frac{\overline{(de/dt)}_s}{\overline{(de/dt)}_p} = -\frac{28}{19}\,\frac{(2g_p\rho_pR_p)^2\,Q_{0p}}{19\mu_p \,.\, 19\mu_s\,Q_{0s}}\left(\frac{R_s}{R_p}\right).$$

If, for simplicity, we consider the case of a planet and satellite of the same material, this becomes

$$\frac{\overline{(de/dt)}_s}{\overline{(de/dt)}_p} = -\frac{28}{19}\left(\frac{2g_p\rho_pR_p}{19\mu}\right)^2\frac{R_s}{R_p}.$$

Since

$$\frac{2g_p\rho_pR_p}{19\mu_p} > 1,$$

we see that all satellites with

$$R_s \geqslant \frac{19R_p(19\mu)^2}{28(2g_p\rho_pR_p)^2}$$

will have decreasing eccentricities, while those with

$$R_s < \frac{19R_p(19\mu)^2}{28(2g_p\rho_p R_p)^2}$$

will have increasing eccentricities.

After this brief and very speculative discussion, we can only appeal to observation, which shows small eccentricities for the five inner satellites of Jupiter, the six inner satellites of Saturn, for four major satellites of Uranus and the inner satellite of Neptune. In all cases where the eccentricity is less than 0·01 we find:

$$-(1/e)\,\overline{(de/dt)}_s \geqslant 1\cdot2\times10^{17}\,\mathrm{sec}^{-1}$$

for reasonable values of Q's and μ's. This seems to indicate that tides raised on satellites are of great significance in the evolution of the eccentricities of these satellites.

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