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What is Linear Algebra?

- study of vectors and their transformations
- lots of stuff in EECS can be treated as vectors [ex. application tomography]

(System of) Linear Equations A linear equation is an equation where each variable has degree 1. A system of linear equations is multiple linear equations that summarizes the known relationships between the variables we want to solve for.

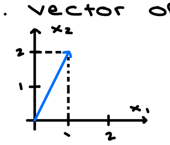
Linear Function A linear function f is a function of one scalar argument with the property that, for arbitrary scalars a and x, $f(ax) = af(x)$. ex. $f(x) = x^2$

- **Affine Functions** Set of functions that can be written as a sum of linear function and a scalar function. Affine functions are not always linear. ex. $g(x) = 2x+1$

Linear Equations A linear equation w/ the unknown scalars $x_1, x_2, \ldots, x_n$ is an equation where each side is a sum of scalar-valued linear functions of each of the unknowns plus a scalar constant. Algebraically, if f_i and g_i are linear equations with a single scalar input and output. b_f and b_g are two scalar constants. $f_1(x_1) + f_2(x_2) + f_3(x_3) + \ldots + f_n(x_n) + b_f = g_1(x_1) + g_2(x_2) + \ldots + g_n(x_n) + b_g$.
Recall $f_i(x) = a_i \cdot x$ and $g_i(x) = a'_i \cdot x$ where a_i and a'_i are scalar constants
So $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n + b_f = a'_1 x_1 + a'_2 x_2 + \ldots + a'_n x_n + b_g$. The LHS alone can be thought of as a "weighted sum" of the x_i where the weights are the a_i. We can call the weighted sum a **linear combination** of the x_i. So, a linear equation is one that equates 2 linear combinations of the unknowns plus a constant term.

Vectors A vector is an ordered list of numbers. Suppose we have a collection of n real numbers: $x_1, x_2, \ldots, x_n$. This collection can be written as single point in an n-dimensional space, denoted as: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. We call $\vec{x}$ a vector. Because $\vec{x}$ contains n real numbers, we can use $\in$ ("in" : is a member of). We can write $\vec{x} \in \mathbb{R}^n$. If the elements in x were complex #'s, we would write $\vec{x} \in \mathbb{C}^n$. Each x_i is called a **component/element** of the vector. The **size** of a vector is the # of component it contains.

ex. vector of size 2 $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



Matrix A matrix is a rectangular array of the numbers, written as:

$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$ Each A_{ij} (i is the row index, j is the column index) is a **component/element** of the matrix A.

Representing Linear Systems using matrices

Augmented Matrix used to represent the coefficients of a linear system of equations. Take the equations

$$x_{11} + x_{21} + x_{31} = 6$$
$$x_{12} + x_{22} + x_{32} = 6$$
$$x_{13} + x_{23} + x_{33} = 6$$
$$x_{23} + x_{22} + x_{21} = 5$$
$$x_{13} + x_{12} + x_{11} = 9$$
$$x_{33} + x_{32} + x_{31} = 4$$

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Rewrite them as

$1\times x_{11} + 0\times x_{12} + 0\times x_{13} + 1\times x_{21} + 0\times x_{22} + 0\times x_{23} + 1\times x_{31} + 0\times x_{32} + 0\times x_{33} = 6$

$0\times x_{11} + 1\times x_{12} + 0\times x_{13} + 0\times x_{21} + 1\times x_{22} + 0\times x_{23} + 0\times x_{31} + 1\times x_{32} + 0\times x_{33} = 6$

$0\times x_{11} + 0\times x_{12} + 1\times x_{13} + 0\times x_{21} + 0\times x_{22} + 1\times x_{23} + 0\times x_{31} + 0\times x_{32} + 1\times x_{33} = 6$

$1\times x_{11} + 1\times x_{12} + 1\times x_{13} + 0\times x_{21} + 0\times x_{22} + 0\times x_{23} + 0\times x_{31} + 0\times x_{32} + 0\times x_{33} = 9$

$0\times x_{11} + 0\times x_{12} + 0\times x_{13} + 1\times x_{21} + 1\times x_{22} + 1\times x_{23} + 0\times x_{31} + 0\times x_{32} + 0\times x_{33} = 5$

$0\times x_{11} + 0\times x_{12} + 0\times x_{13} + 0\times x_{21} + 0\times x_{22} + 0\times x_{23} + 1\times x_{31} + 1\times x_{32} + 1\times x_{33} = 4$

$$\left[\begin{array}{ccccccccc|c} 1&0&0&1&0&0&1&0&0&6\\ 0&1&0&0&1&0&0&1&0&6\\ 0&0&1&0&0&1&0&0&1&6\\ 1&1&1&0&0&0&0&0&0&9\\ 0&0&0&1&1&1&0&0&0&5\\ 0&0&0&0&0&0&1&1&1&4 \end{array}\right]$$

augmented matrix notation

$$\begin{bmatrix} 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1\\ 1&1&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&1 \end{bmatrix} \begin{bmatrix} x_{11}\\ x_{12}\\ x_{13}\\ x_{21}\\ x_{22}\\ x_{23}\\ x_{31}\\ x_{32}\\ x_{33} \end{bmatrix} = \begin{bmatrix} 6\\6\\6\\9\\5\\4 \end{bmatrix}$$

Gaussian Elimination An algorithim used to solve large systems of equations.

ex. $5x + 6y + z = 43$ (1)

$8x + 9y + 2z = 67$ (2)

$x + y + 4z = 19$ (3)

$x + 6/5\, y + 1/5\, z = 43/5$ (4) [1/5 × (1)]

$(8x + 9y + 2z) - 8(x - 6/5\, y + 1/5\, z) = 67 - 8 \cdot 43/5$ [(2) - 8 (4)]

$\longrightarrow -3/5\, y + 2/5\, z = -9/5$ (5)

$(x + y + 4z) - (x + 6/5\, y + 1/5\, z) = 19 - 43/5$ [(3) - (4)]

$\longrightarrow -1/5\, y + 19/5\, z = 52/5$ (6)

$y - 2/3\, z = 3$ (7) [(5) × -5/3]

$-1/5\, y + 19/5\, z + 1/5\,(y - 2/3\, z) = 52/5 + 1/5 \times 3$ [1/5 (7) + (6)]

$\longrightarrow 11/3\, z = 11$ (8) so $z = 3$, $y = 5$, $x = 2$.

Steps

1. select an equation involving x and scale it to make the x coefficient unity
2. add multiples of this equation from all the other equations to eliminate x, producing a system with one fewer unknown and one fewer equation.
3. Repeat steps 1 and 2 until arriving at an equation with exactly one unknown. solve
4. Substitute the obtained value until the remaining unknowns are calculated.

Steps 1-3 are known as row reduction. Step 4 is back substitution.

Operations

1. Multiplying an equation by a nonzero scalar constant.
2. Adding a scalar constant multiple of one equation to another
3. Swapping two equations

Gaussian Elimination with Matrices

ex. $5x + 3y = 5$
$-4x + y = 2$

$$\left[\begin{array}{cc|c} 5 & 3 & 5\\ -4 & 1 & 2 \end{array}\right]$$

multiply rows by scalars, swap rows, add scalar multiple of rows to others

1. Swap rows if needed so that an equation containing variable i is contained in row i (column i and row i should be nonzero in the

augmented matrix).

2. Divide row i by the coefficient of variable i in this row such that the i^{th} row and column of the augmented matrix is 1.

3. For rows $j = i+1$ to n, subtract row i times the entry in row j and column i to cancel variable i.

These steps will lead to a "triangular form" known as row echelon form. All nonzero rows are above all zero rows. The leading coefficient of a non-zero row is always to the right of the leading coefficient of the row above it. The leading coefficient of every non-zero row (which we call the pivot, and say it is in the pivot position) is 1. Each column with an element that is in the pivot position of some row has 0s everywhere else. This is known as reduced row echelon form.

ex. $2x + 4y + 2z = 8$ $\quad \left[\begin{array}{ccc|c} 2 & 4 & 2 & 8 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & 4 \end{array}\right]$

$x + y + z = 6$

$x - y - z = 4$

divide row 1 by 2. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & 4 \end{array}\right]$

Subtract row 1 from row 2 and 3 $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 0 & 2 \\ 0 & -3 & -2 & 0 \end{array}\right]$

multiply row 2 by -1 $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -2 & -6 \end{array}\right]$

To scale z by 1 in the final equation divide row 3 by -2 $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

we are at row echelon form

Subtract row 3 from row 1 to eliminate z from the first equation

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Subtract 2 times row 2 from row 1 $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$ $\quad x = 5$

$y = -2$

$z = 3$

ex. $\begin{array}{c} \begin{array}{cccc} x & y & z & = c \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 3 & 4 \end{array}\right] \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ infinite solutions

$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 1 & 2 & 8 & 0 \\ 1 & 3 & 5 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$ no solution

$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 2 & 2 & 7 & 6 \\ -1 & -1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $z = 2$, x and y have infinite solutions

Algorithim Stopping Point running the algorithim will tell us whether we have one, zero, or infinite solutions

m equations, n variables. If $m > n$, final rows of Gaussian reduction should all be 0s.

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Formal Algorithim [From F2019 EE16A Note 1B]

Data: Augmented matrix $A \in \mathbb{R}^{m \times (n+1)}$, for a system of m equations with n variables
Result: Reduced form of augmented matrix

```
# Forward elimination procedure:
for each variable index i from 1 to n do
    if entry in row i, column i of A is 0 then
        if all entries in column i and row > i of A are 0 then
            proceed to next variable index;
        else
            find j, the smallest row index > i of A for which entry in column i ≠ 0 ;
            # The following rows implement the "swap" operation:
            old_row_j ⟵ row j of A;
            row j of A ⟵ row i of A;
            row i of A ⟵ old_row_j;
        end
    end
    divide row i of A by entry in row i, column i of A;
    for each row index k from i+1 to m do
        scaled_row_i ⟵ row i of A times entry in row k, column i of A;
        row k of A ⟵ row k of A − scaled_row_i;
    end
end
# Back substitution procedure:
for each variable index u from n−1 to 1 do
    if entry in row u, column u of A ≠ 0 then
        for each row v from u−1 to 1 do
            scaled_row_u ⟵ row u of A times entry in row v, column u of A;
            row v of A ⟵ row v of A − scaled_row_u;
        end
    end
end
```

Algorithm 1: The Gaussian elimination algorithm.

Vectors (Review) & More

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

- Each x_i is a component / element. Do not have to be real.
- Size: # of x_i (n).
- Two vectors x, y are equal if they have the same size and if $x_i = y_i$ for all i.

Examples of Vectors:

- position vs. time $x \in \mathbb{R}^n$ can represent samples of a quantity at n time points

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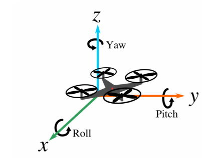
Imagine a particle moving along a line. Its position at $t_1, \ldots, t_n$ can be represented with a vector $\vec{x} = \begin{bmatrix} x_{t_1} \\ x_{t_2} \\ \vdots \\ x_{t_n} \end{bmatrix}$. x_{t_i} represents the position of the particle at time t_i.

• Quadrator state: vectors can be used to represent the state of a system, defined as follows: State — the minimum information you need to completely characterize a system at a given point in time, without need for info abt the past of the system.

The state of a quadator vector can be summarized by its 3D position, angular position, velocity, and angular velocity. $\vec{q} = \begin{bmatrix} x \\ y \\ z \\ \text{roll} \\ \text{pitch} \\ \text{yaw} \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\text{roll}} \\ \dot{\text{pitch}} \\ \dot{\text{yaw}} \end{bmatrix}$ (note $\dot{x} = dx/dt$).



• Electric circuit quantities (see current & voltage).

• vectors representing functions Consider $p(d) = \begin{cases} d & \text{if } d \leq 5 \\ -d & \text{if } d = 6 \end{cases}$

$\vec{p} = \begin{bmatrix} p(1) \\ p(2) \\ p(3) \\ p(4) \\ p(5) \\ p(6) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ -6 \end{bmatrix}$ is a vector that summarizes p if $1 \leq d \leq 6$ (ex. dice rolls).

• color $\vec{x} \in \underbrace{\mathbb{R}^3}_{\text{triples } (x,y,z) \text{ where } x,y, \text{ and } z \text{ are real numbers}}$ can represent a color with its components giving red, blue, and green intensity values.

• Image a grayscale image of $m \times n$ pixels is effectively a matrix w/ m rows and n columns. entries correspond to grayscale levels at pixel location.

Zero Vector A vector with all components equal to 0. Represented as $\vec{0}$.

Standard Unit Vector A vector with all components equal to 0 except for one element, which is equal to 1. A standard unit vector is a vector with all components equal to 0 except for one element, equal to 1. A SUV where the ith position is equal to 1 is written as $\vec{e}_i$. We denote the three SUVs in $\mathbb{R}^3$ as follows:

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ the word "pure" may be used to describe SUVs

Vector Addition If two vectors are the same size and in the same space, they can be added together by adding their components. Vector addition is:

— commutative $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
— associative $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
— has an additive identity $\vec{0}$ s.t. $\vec{x} + \vec{0} = \vec{x}$ $0\vec{x} = \vec{0}$ (see below)
— has an additive inverse $-\vec{x}$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$. $\vec{x} = -1 \times \vec{x}$ (see below)

$\underbrace{\text{Scalar}}_{\text{number}}$ Multiplication multiply the components of the vector by the scalar

— associative $(\alpha B)\vec{x} = \alpha(B\vec{x})$
— distributive $(\alpha + B)\vec{x} = \alpha\vec{x} + Bx$
— multiplicative identity $1\vec{x} = \vec{x}$.

Vector Transpose Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Then the transpose $\vec{x}^T = [x_1, \ldots, x_n]$. (i.e. column vector becomes row vector).

Vector Vector Multiplication Note a row vector can only be multiplied by a column vector. In addition, vector-vector multiplication is NOT commutative (i.e. $\vec{x}\vec{y}^T \neq \vec{y}^T\vec{x}$.). Also, $\vec{y}^T\vec{x}$ is only defined if both vectors

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have the same size. $\vec{x}\,\vec{y}^T$ is defined for vectors of any size.

Multiplying a column vector by a row vector on the left is one way of computing the inner product or dot product of two vectors.

$$\vec{y}^T \vec{x} = [y_1, y_2, \dots, y_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = y_1 x_1 + y_2 x_2 + \dots + y_n x_n$$

Multiplying a row vector by a column vector on the left generates a matrix

$$\vec{x}\,\vec{y}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} [y_1, y_2, \dots, y_m] = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_m \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_m \\ \vdots & \vdots & \vdots & \vdots \\ x_n y_1 & x_n y_2 & \cdots & x_n y_m \end{bmatrix}$$

The entry in row i and column j in $\vec{x}\,\vec{y}^T$ is the product of x_i and y_j.

Matrices A matrix is said to be $\mathbb{R}^{m\times n}$ if it has m rows, n columns.

ex. $A \in \mathbb{R}^{m\times n}$, $A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$. A matrix is said to be square if $m = n$.

Transpose of a Matrix $A^T = \begin{bmatrix} A_{11} & \cdots & A_{m1} \\ \vdots & \ddots & \vdots \\ A_{1n} & \cdots & A_{nm} \end{bmatrix}$. Mathematically, $(A^T)_{ij} = A_{ji}$

A square matrix is said to be symmetric if $A^T = A$ (i.e. $A_{ij} = A_{ji}\ \forall i, j$).

Examples of Matrices:

- Illumination pattern pixel illumination pattern can be represented as a square matrix of 1's and 0's. 1 if illuminated, 0 if not.
- Water Pumps assume a city has 3 water resoirvoirs, A B & C. 1/2 of water from A is transferred to B, 1/4 of water in B goes to C, and C is evenly split between the three.

$$P = \begin{bmatrix} P_{A\to A} & P_{B\to A} & P_{C\to A} \\ P_{A\to B} & P_{B\to B} & P_{C\to B} \\ P_{A\to C} & P_{B\to C} & P_{C\to C} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix}$$

Zero Matrix matrix with all components equal to 0. represented as 0.

Identity Matrix elements on diagonal equal to 1, rest equal to 0. represented as I. column vectors are unit vectors in $\mathbb{R}^n$. $I\vec{x} = \vec{x}$.

Matrix Addition Two matrices of the same size can be added together by adding their corresponding coordinates.

$$A + B = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mn} \end{bmatrix} = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & \cdots & A_{1n}+B_{1n} \\ A_{21}+B_{21} & A_{22}+B_{22} & \cdots & A_{2n}+B_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1}+B_{m1} & A_{m2}+B_{m2} & \cdots & A_{mn}+B_{mn} \end{bmatrix}$$

- commutative $A + B = B + A$
- associative $(A+B)+C = A+(B+C)$
- has an additive identity $A + 0 = A$ $\quad 0 \times A = 0$ (see below)
- has an additive inverse $A + (-A) = 0$. $-A = -1 \times A$ (see below).

Scalar Matrix Multiplication

$$\alpha A = \alpha \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} = \begin{bmatrix} \alpha A_{11} & \alpha A_{12} & \cdots & \alpha A_{1n} \\ \alpha A_{21} & \alpha A_{22} & \cdots & \alpha A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha A_{m1} & \alpha A_{m2} & \cdots & \alpha A_{mn} \end{bmatrix}$$

- associative $(\alpha \beta) \times A = (\alpha) \times (\beta A)$
- distributive $(\alpha + \beta) \times A = \alpha A + \beta A$ and $\alpha(A+B) = \alpha A + \alpha B$

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— multiplicative identity $I \times A = A$

Matrix Vector Multiplication Let $A \in \mathbb{R}^{m \times n}$, $\vec{x} \in \mathbb{R}^n$

$$\vec{b} = A\vec{x} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n \end{bmatrix}$$

$$b_i = \sum_{j=1}^{n} A_{ij}x_j$$

$$(I\vec{x})_i = \vec{x}_i$$

$$\longrightarrow I\vec{x} = \vec{x}$$

Vectors can be treated as $n \times 1$ matrices.

State Tranisition Matrices

refer to the water pumps example

$$x'_A = \tfrac{1}{2}x_A + \tfrac{1}{4}x_B + \tfrac{1}{3}x_C$$
$$x'_B = \tfrac{1}{2}x_A + 0x_B + \tfrac{1}{3}x_C$$
$$x'_C = 0x_A + \tfrac{3}{4}x_B + \tfrac{1}{3}x_C$$

$$\Rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix} = \begin{bmatrix} x'_A \\ x'_B \\ x'_C \end{bmatrix}$$

Matrix-Matrix Multiplication Matrix-Matrix Multiplication involves multiplying each row vector in A with each column vector in B. So the # of columns in A must be equal to the # of rows in B.

Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$; $A \times B \in \mathbb{R}^{m \times p}$

$$\begin{bmatrix} \cdots & \vec{r}_1^{\,T} & \cdots \\ \cdots & \vec{r}_2^{\,T} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & \vec{r}_m^{\,T} & \cdots \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_p \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vec{r}_1^{\,T}\vec{c}_1 & \vec{r}_1^{\,T}\vec{c}_2 & \cdots & \vec{r}_1^{\,T}\vec{c}_p \\ \vec{r}_2^{\,T}\vec{c}_1 & \vec{r}_2^{\,T}\vec{c}_2 & \cdots & \vec{r}_2^{\,T}\vec{c}_p \\ \vdots & \vdots & & \vdots \\ \vec{r}_m^{\,T}\vec{c}_1 & \vec{r}_m^{\,T}\vec{c}_2 & \cdots & \vec{r}_m^{\,T}\vec{c}_p \end{bmatrix}$$

ex. $$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (2)(1)+(4)(3) & (2)(2)+(4)(4) \\ (3)(1)+(1)(3) & (3)(2)+(1)(4) \end{bmatrix}$$

Note: matrix multiplication is associative but not commutative

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Linear Dependence 2 equivalent definitions

1. A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent if there exist scalars $\alpha_1, \ldots, \alpha_n$ s.t. $\alpha_1\vec{v}_1 + \ldots \alpha_n\vec{v}_n = \vec{0}$ and not all α_is are equal to zero
2. A set of vectors $\{\vec{v}_1, \ldots \vec{v}_n\}$ is linearly dependent if there exist scalars $\alpha_1, \ldots, \alpha_n$ and ao index i s.t. $\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j$. (i.e. one of the vectors can be written as a linear combination of the other vectors.

Linear Indepence A set of vectors is linearly independent if it is not linearly dependent (i.e. $\alpha_1\vec{v}_1 + \ldots + \alpha_n\vec{v}_n = 0$ implies $\alpha_1 = \ldots = \alpha_n = 0$).

ex. $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. Consider $\alpha_1\vec{a} + \alpha_2\vec{b} = 0$.

$$\alpha_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2\alpha_1 + \alpha_2 \\ \alpha_1 + 5\alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $\vec{a}$ and $\vec{b}$ are linearly independent

Theorems Recall a system of linear equations can be written in matrix vector form as $A\vec{x} = \vec{b}$.

1. If the system of linear equations $A\vec{x} = \vec{b}$ has infinite # of solutions then the columns of A are linearly dependent.
2. If the columns of A in the system of linear equations $A\vec{x} = \vec{b}$ are linearly dependent, then the system does not have a unique solution.

Span The span of a set of vectors $\{v_1, \ldots, v_n\}$ is the set of all linear combinations of $\{v_1, \ldots, v_n\}$. span $(v_1, \ldots, v_n) = \{\sum_{i=1}^{n} \alpha_i \vec{v}_i \mid \alpha_i \in \mathbb{R}\}$

- a set of vectors is linearly dependent if any one of the vectors is in the span of the remaining vectors.

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Proofs

1. Start with axioms
   statements we accept w/o proof

2. Logical Deductions: simple steps that apply the rules of logic to make new conclusions.

Water Resoirvoirs and Pumps

Three water resoirvoirs: A, B, and C. Let's say the initial amount of water they hold is A_0, B_0, and C_0 respectively. The vector $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ represents the amount of water currently in the resoirvoir.

Then, the system of pumps can be represented by the matrix
$$\begin{bmatrix} P_{A\to A} & P_{B\to A} & P_{C\to A} \\ P_{A\to B} & P_{B\to B} & P_{C\to B} \\ P_{A\to C} & P_{B\to C} & P_{C\to C} \end{bmatrix}.$$
Each element $P_{i\to j}$ represents the fraction of water in resoirvoir i that goes into resoirvoir j the next day. We call this matrix a state transition matrix. If the matrix is the identity matrix, no water is transferred. The 0 matrix can be thought of as a drain. (Generally however, we do not use drainage and water must thus be conserved). When will we use matrix multiplication? When we have multiple cities, multiple pumps that act sequentially, etc. Thus far, we have been talking about discrete pumps (act instantaneously). Continuos pumps require the usage of diffrentials [EE16B].

Matrix Inversion

A square matrix A is said to be invertible if there exists a matrix B s.t. $AB = BA = I$. where I is the identity matrix. In this case, we call B the inverse of matrix A, which we denote A^{-1}. The inverse is unique. If $QP = I$ and $RQ = I$, then $P = R$. P is the right inverse of Q and R is the left inverse of Q.

Finding Inverses with Gaussian Elimination

perform algorithm on $\left[\begin{array}{c|c} M & I_n \end{array}\right]$. Algorithm terminates when the left side is the identity matrix i.e. $\left[\begin{array}{c|c} I_n & M^{-1} \end{array}\right]$. If we cannot obtain I_n on the left, the matrix is not invertible.

Existence of an Inverse

- If a matrix A is invertible, there exists a unique solution to the equation $Ax = \vec{b}$ for all possible vectors $\vec{b}$.
- If a matrix A is invertible, its columns are linearly independent.
- A has a trivial null space if it is invertible
- the determinant of an invertible matrix cannot be 0.

Vector Space

A vector space V is a set of vectors and two operators that satisfy the following properties:

- vector addition for all $\vec{v}, \vec{u} \in V$
  - associative $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
  - commutative $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
  - additive identity $\exists \vec{0} \in V$ s.t. $\vec{v} + \vec{0} = \vec{v}$
  - additive inverse $\exists -\vec{v} \in V$ s.t. $\vec{v} + (-\vec{v}) = \vec{0}$
  - closed under addition $\vec{v} + \vec{u} \in V$
- Scalar multiplication. for all $\alpha, \beta \in \mathbb{R}$, $\vec{u}, \vec{v} \in V$.
  - associative: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$
  - multiplicative identity: $\exists 1 \in \mathbb{R}$ s.t. $1 \cdot \vec{v} = \vec{v}$
  - distributive in vector +: $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$
  - distributive in scalar +: $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$
  - closure under scalar multiplication $\alpha\vec{v} \in V$

Basis Given a vector space V, a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis of the vector space if it satisfies the following 2 properties:

- $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent
- For any vector $\vec{v} \in V$, there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ s.t. $\vec{v} = \alpha_1 \vec{v}_1 + \ldots + \alpha_n \vec{v}_n$

Intuitively, a basis is the minimum # of vectors needed to represent all vectors in the vector space. Bases are not unique, but every basis contains the same # of vectors

ex. a basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

dimension (of a vector space) is the # of basis vectors. dimension of $\mathbb{R}^3$ is 3.

Subspace A subspace U consists of a subset of the set V in the vector space (V, F) that satisfies:

1. $\vec{0} \in U$.
2. closed under vector addition
3. closed under scalar multiplication

Range We can think of matrices as linear functions that acts on vectors. Consider the matrix $A \in \mathbb{R}^{n \times m} \rightarrow$ takes vectors in $\mathbb{R}^m$ and outputs vectors in $\mathbb{R}^n$. The range of an operator is the space of all outputs that the operator can map to. Write our matrix as:

$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \\ | & | & & | \end{bmatrix}$ where the vectors $\vec{a} \in \mathbb{R}^n$. The matrix A operates on any vector $\vec{x}$ that lives in $\mathbb{R}^m$, where the operation on $\vec{x}$ is $A\vec{x}$.

Members of $\mathbb{R}^m$ can be written as $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$. From this we have

$$A\vec{x} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \sum_{k=1}^{m} x_k \vec{a}_k.$$

So we can conclude that the range of the operator A is the space of all possible linear combinations of its columns, or the span of (the columns of) A, which we can write as $\text{span}(A) = \{\vec{v} \mid \vec{v} = \sum_{i=1}^{m} x_i \vec{a}_i$ where x_is are scalars$\}$, also the **column space** of A. Note that range(A) is a subspace of $\mathbb{R}^n$.

The **rank** (dimension of range) of a matrix is the dimension of the span of its columns. **rank(A) = dim(span(A))**. # of linearly independent columns.

Nullspace The nullspace of A consists of all vectors $\vec{x}$ in $\mathbb{R}^m$ s.t. $A\vec{x} = \vec{0}$: $N(A) = \{\vec{x} \mid A\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^m\}$. The nullspace of A is the set of vectors that get mapped to $\vec{0}$ by A.

- **Rank-nullity theorem** $m - \dim(\text{range}(A)) = \dim(N(A))$

Aside: If a vector can be represented as a linear combination of linearly independent vectors, then this representation is unique.

ex. Let $A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

Consider $\left[\begin{array}{ccc|c} -1 & 3 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$

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$R_2 + R_1$, $R_3 - R_2$ $\left[\begin{array}{ccc|c} -1 & 3 & 2 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, $-R_1$, $R_2/4$ $\left[\begin{array}{ccc|c} -1 & 3 & 2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, $R_1 + 3R_2$ $\left[\begin{array}{ccc|c} -1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 - 0.5x_3 = 0$ $\quad x_2 + 0.5x_3 = 0.$

$x_1 = \frac{1}{2}x_3 \quad x_2 = -\frac{1}{2}x_3 \quad x_3 = x_3$

$\begin{bmatrix} x \\ -x \\ 2x \end{bmatrix}$ is the nullspace of A.

Eigenvectors and Eigenvalues Consider a square matrix $A \in \mathbb{R}^{m \times n}$. An **eigenvector** of A is a nonzero vector $\vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \lambda\vec{x}$ where λ is a scalar value, called the **eigenvalue** of $\vec{x}$.

Determinant $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$. If a matrix's columns are linearly dependent, then its determinant is zero. $(A - \lambda I_n)\vec{x} = 0 \rightarrow \det(A - \lambda I_n) = 0.$

ex. Find the eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

$$A - \lambda I_n = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_n) = (1-\lambda)(3-\lambda) - (4)(2) = 0$$
$$= \lambda^2 - 4\lambda - 5 = (\lambda+1)(\lambda-5) = 0$$

$\lambda = -1$, $\lambda = 5$ are the eigenvalues

$\lambda = 5$: $(A - 5I_2)\vec{x} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$ $\quad \left.\begin{array}{l} 4x_1 - 2x_2 = 0 \\ -4x_1 + 2x_2 = 0 \end{array}\right]$ equivalent to $x_2 = 2x_1$

So the eigenvectors associated with $\lambda = 5$ are of the form $\boxed{\alpha\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \alpha \in \mathbb{R}}$

$\lambda = -1$: $(A + 1I_2)\vec{x} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$ $\quad 2x_1 + 2x_2 = 0$
$x_1 = -x_2$

So the eigenvectors associated with $\lambda = -1$ are of the form $\boxed{\alpha\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \alpha \in \mathbb{R}}$

eigenspace: all eigenvalues

Every 2×2 matrix has 2 eigenvalues but they do not have to be real or unique.

- **Repeated** ex. $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$
- **Complex** ex. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Consider an arbitrary 2×2 matrix A. We know $\det(A - \lambda I) = 0$. Letting $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in \mathbb{R}$. So $\det(A - \lambda I_2) = \det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right)$

$= (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad - bc) = 0$. The polynomial is known as the **characteristic polynomial** for A. There are thus three possible cases:

1. There are 2, real distinct eigenvalues λ_1 and λ_2.
   - the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to λ_1 and λ_2 are linearly independent. More generally if $\vec{v}_1, \ldots, \vec{v}_n$ are the eigenvectors of an $n \times n$ matrix, all the $\vec{v}_i$ are linearly independent.
2. There is a single (repeated) eigenvalue λ.
   - **multiplicity**: # of times an eigenvalue is repeated (in this case 2)
   - when the dimension of the eigenspace is less than the multiplicity, we call the matrix **defective**
3. There are two, complex, distinct eigenvalues λ_1 and λ_2.

Steady States $A\vec{x}^* = \vec{x}^*$ (an element of a corresponding to $\lambda = 1$).

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Starting @ eigenvectors our initial state is an eigenvector $\vec{v}$ of A, but λ may not be equal to one. (i.e. $A\vec{v} = \lambda\vec{v}$, but possible that $\lambda\vec{v} \neq \vec{v}$). After t timesteps, our state is $x[t] = \lambda^t \vec{v}$. The states behaviour as $n \to \infty$ depends upon the value of λ.

- If $\lambda > 1$, then our state will keep growing along the direction of $\vec{v}$ towards ∞.
- If $\lambda = 1$, we are at steady state, so $\vec{x}[t] = \vec{v}$ as $t \to \infty$.
- If $0 < \lambda < 1$, our state will scale down and approach $\vec{0}$.
- If $\lambda = 0$, then our state drops to 0 from the first timestep onwards.
- If $\lambda < 0$, state will behave similar to $|\lambda|$ but sign will flip at each timestep.

General Initial States Let's consider an initial state that is not an eigenvector of A. For simplicity, only consider matrices whose eigenvectors form a basis (i.e. if A is $n \times m$, then any initial state $\vec{x}[0]$ can be written as a linear combination $\vec{x}[0] = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \ldots + \alpha_n\vec{v}_n$. where $\vec{v}_1 \ldots \vec{v}_n$ are linearly independent eigenvectors with eigenvalues λ_1 through λ_n and the α_i are real scalar coefficients).

$\vec{x}[t] = A^t \vec{x}[0] = \alpha_1(\lambda_1^t \vec{v}_1) + \ldots + \alpha_n(\lambda_n^t \vec{v}_n)$.

Criteria for Convergence Consider an arbitrary initial state $\vec{x}[0]$. We know that $\vec{x}[t] = \alpha_1(\lambda_1^t \vec{v}_1) + \ldots + \alpha_n(\lambda_n^t \vec{v}_n)$. We want to know if $\vec{x}[t]$ will converge to some fixed value, no matter what the $\{\alpha_i\}$ are.

- if $|\lambda_i| > 1$ for a single λ_i, then $\lambda_i^t \vec{v}_i \to \infty$ so $\vec{x}[t]$ could go to infinity
- if a single $\lambda_i = -1$, $\lambda_i^t \vec{v}_i$ will osillate, so $\vec{x}[t]$ could fail to converge to a fixed value
- if all λ_i are s.t. $-1 < \lambda_i \leq 1$, $x[t]$ will always converge to a fixed value

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Change of Basis for Vectors

Consider the vector $u = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. In this form, $\vec{u}$ is being represented in the standard basis for $\mathbb{R}^2$ ($e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$). Suppose we want to represent $\vec{u}$ as a linear combination of another set of basis vectors, say $\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Thus, we must solve $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_{a1} \\ u_{a2} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. $u_{a1} = 4$
$u_{a2} = 1$

In general, suppose we are given a vector $\vec{u} \in \mathbb{R}^n$ in the standard basis and want to change it to a different basis with linearly independent basis vectors $\vec{a}_1, \ldots, \vec{a}_n$. If we denote the vector in the new basis as $\vec{u}_a = \begin{bmatrix} u_{a1} \\ \vdots \\ u_{an} \end{bmatrix}$, we solve the following equation $A\vec{u}_a = \vec{u}$, where A is the matrix $[\vec{a}_1, \ldots, \vec{a}_n]$. Therefore the change of basis is given by: $\vec{u}_a = A^{-1}\vec{u}$. Suppose we have a vector $\vec{u}_a$ in the basis $\vec{a}_1, \ldots, \vec{a}_n$. Reverse the change of basis with $\vec{u} = A\vec{u}_a$ to have it back in the standard basis.

Change of Basis for Linear Transformations

Suppose we have a linear transformation T represented by a $n \times n$ matrix that transforms $\vec{u} \in \mathbb{R}^n$ to $\vec{v} \in \mathbb{R}^n$: $\vec{v} = T\vec{u}$. We can think of T as a geometric transformation on vectors. We assume the vectors lie in the coordinate system defined by the standard basis vectors. But what if $\vec{u}$ and $\vec{v}$ were represented in some basis with vectors $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^n$. Then $\vec{u} = u_{a1}\vec{a}_1 + \ldots + u_{an}\vec{a}_n$, $\vec{v} = v_{a1}\vec{a}_1 + \ldots + v_{an}\vec{a}_n$. Can we also represent T in this basis? Define $A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$. We can write $\vec{u} = A\vec{u}_a$ and $\vec{v} = A\vec{v}_a$.

So $T\vec{u} = \vec{v} \to A^{-1}TA\vec{u}_a = \vec{v}_a$. set $T_a = A^{-1}TA$ to get the desired relationship $T_a\vec{u}_a = \vec{v}_a$.

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A Diagnolizing Basis It can be useful to transform T into a diagonal matrix if we want to apply T repeatedly. Suppose that we chose our basis vectors $\vec{a}_1, \dots, \vec{a}_n$ to be the eigenvectors of the transformation matrix T, which have associated eigenvalues $\lambda_1, \dots, \lambda_n$. Recall $\vec{u}$ can be written in the new basis:

$$T\vec{u} = T(u_{a_1}\vec{a}_1 + \dots + u_{a_n}\vec{a}_n) = u_{a_1}T\vec{a}_1 + \dots + u_{a_n}T\vec{a}_n = u_{a_1}\lambda_1\vec{a}_1 + \dots + u_{a_n}\lambda_n\vec{a}_n$$

$$= \begin{bmatrix} | & \cdots & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} u_{a_1} \\ \vdots \\ u_{a_n} \end{bmatrix} = AD\vec{u}_a = ADA^{-1}\vec{u}$$

where D is the diagonal matrix of eigenvalues and A is the matrix with the corresponding eigenvectors as its columns. Thus we have shown that in an eigenvector basis, $T = ADA^{-1}$. In particular, T_a, the counterpart of T in the eigenvector basis, is a diagonal matrix.

Diagnolization A $n \times n$ matrix T is diagnizable if it has n linearly independent eigenvectors. If so, we can write $T = ADA^{-1}$, where $A = [\vec{a}_1, \dots, \vec{a}_n]$.

- If two $n \times n$ diagnizable matrices A and B have the same eigenvectors, then their matrix multiplication is commutative ($AB = BA$).

Electrical Circuit Analysis

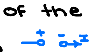
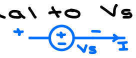
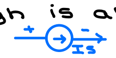
Current Flow of charges. Symbol I. Units Amperes (A).

Voltage potential energy per charge Symbol V. Units Volts (V).

Resistance material's tendency to resist the flow of current Symbol R. Units Ohms (Ω).

Circuit Diagram visual representation of how a collection of circuit elements are connected. Each voltage has voltage across it and some current through it. Voltage is measured relative to another point. In circuits, we define **ground** to be a reference point against which other voltages can be measured.

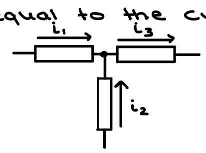
Circuit Elements:

- **wire**: drawn as a solid line. $V_{elem} = 0$. I_{elem} = any value, determined by the rest of the circuit. drawn as 
- **resistor**: IV relationship is given by **Ohm's Law — $V_{elem} = I_{elem}R$**. drawn as 
- **Open Circuit**: This element is the dual of the wire. $I_{elem} = 0$. V_{elem} = any value, determined by the rest of the circuit. drawn as 
- **Voltage Source**: component that forces a specific voltage across its terminals. The + and − sign indicate which way the voltage is pointing. The voltage difference is always equal to V_s. I_{elem} = any value, determined by the rest of the circuit. drawn as 
- **Current Source**: component that forces current in a specified direction. The current flowing through is always I_s. V_{elem} = any value, determined by the rest of the circuit. drawn as 

Rules for Circuit Analysis

Kirchoff's Current Law (KCL). A place in a circuit where two or more elements meet is called a **node**. KCL states that the net current flowing out of (or equivalently, into) any node of a circuit is zero (i.e. the current flowing into a node must be equal to the current flowing out of the node).

ex:



$$(-i_1) + (-i_2) + i_3 = 0.$$

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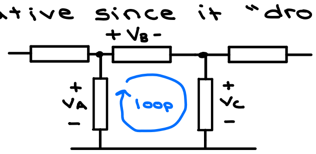
Kirchoff's Voltage Law (KVL) The sum of voltages across the elements connected in a loop must be zero. Convention: voltage that goes from + to − is negative since it "drops". Voltage that goes from − to + is positive. ex.



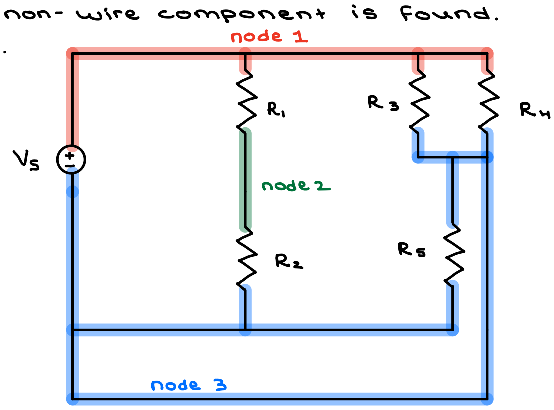
$$\sum_{Loop} V_k = 0$$

$$V_A - V_B - V_C = 0.$$

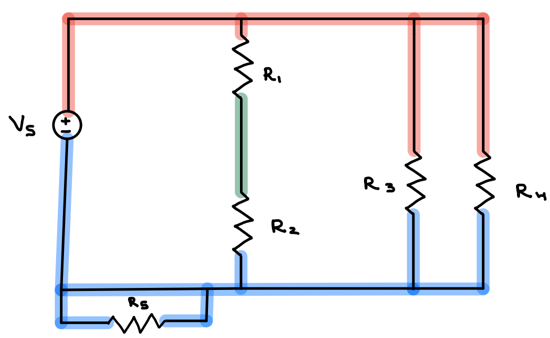
Ohm's Law $V_{elem} = I_{elem} R$

Guide to Finding Nodes Choose a starting point and trace ALL wires until a non-wire component is found.

ex.

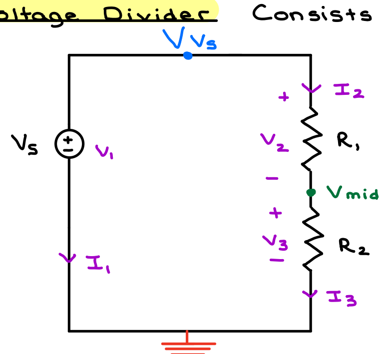


Once the nodes are identified, we can redraw the circuit as long as each non-wire component must be connected to the same nodes on either end.



is the same as the above circuit.

Voltage Divider Consists of a voltage source and two resistors



Step 1: pick a node and label it as ground

Step 2: label the node connected to the voltage supply as V_{Vs}, since the voltage supply goes between this node and the ground.

Step 3: label the remaining node as V_{mid}

Step 4: Label the voltages and currents through every element in the circuit with V_i and I_i

Step 5: write KCL equations for all nodes with unknown voltage. In this case, it is only V_{mid} since $V_{Vs} = V_s$. Since I_2 enters V_{mid} and I_3 exits V_{mid}, $I_2 = I_3$.

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Step 6: Find expressions for element currents for all elements (except the voltage source). $I_2 = V_2 / R_2$ and $I_3 = V_3 / R_3$.
We can express our element voltages as $V_2 = V_s - V_{mid}$ and $V_3 = V_{mid} - 0 = V_{mid}$. Substituting, $I_2 = [V_s - V_{mid}] / R_1$ and $I_3 = V_{mid} / R_2$.

Step 7: substitute into the KCL equation. $\frac{V_{mid}}{R_2} = \frac{V_s - V_{mid}}{R_1}$

Step 8: solve for $V_{mid} \longrightarrow V_{mid} = \frac{R_2}{R_1 + R_2} V_s = \frac{1}{1 + \frac{R_1}{R_2}} V_s$

This circuit is called a voltage divider because we can create any output voltage $V_{mid} = \alpha V_s$ for any $\alpha \in [0, 1]$ by varying the ratio of the resistance values.

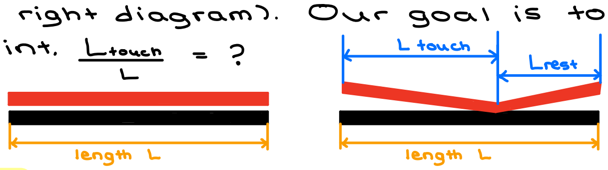
Resistive Touchscreen

For simplicity, we will begin by working one dimensional touch screens (i.e. we only detect the horizontal position of the touch.

Physical Structure of a Touch Screen

top layer is flexible
bottom layer is non-flexible | Both layers are conductive.

When we touch, the top layers contacts the bottom layer at the touchpoint (see right diagram). Our goal is to find the position of the touchpoint. $\frac{L_{touch}}{L} = ?$



Physics of Circuits

Charge is the basic underlying quantity associated with all electrical systems. can be positive or negative (although most systems carry negative electrons). We measure charges in coulombs (C) and use the symbol Q.

Current a measure of the movement of charge, specifically the net amount of charge, specifically, the net amount of charge crossing through a surface in a unit time. We usually use the symbol I to denote current and is defined by $I = dQ / dt$. The unit for current is an Ampere (A), which is equivalent to 1 C/s. We specify the direction of the net flow using an arrow.

Voltage the amount of energy needed to move a unit charge between two points. We usually denote voltage with the symbol V. The unit associated with a voltage is a volt (V), where 1 volt is defined s.t. it will require 1 Joule of energy to move 1 coulomb between the two points. Voltage is relative NOT absolute. Usually measured relative to the ground.

Resistance Energy must be "spent" to move current through a conductor. We call the energy spent resistance. It has unit ohms (Ω) and symbol R. Ohm's Law: $V = IR$.

- the value of resistance for a physical material is set by two things:
  1. Material properties Every conducting material has a property known as resistivity ρ.
  2. Physical Dimensions If we increase the length L, resistance will

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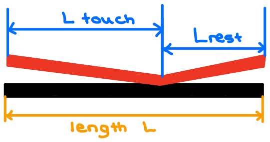
increase. If we increase the cross sectional area A, resistance will decrease.

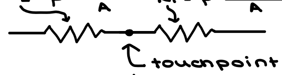
So, resistance $R = \rho \times \frac{L}{A}$

Resistive Touch Screen (Revisited)



Let the top layer have resistivity ρ and cross-sectional area

$R_2 = \rho \frac{L_{touch}}{A}$ $\quad R_1 = \rho \frac{L_{rest}}{A}$



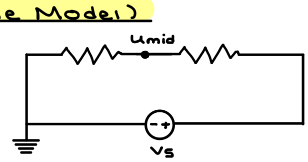
Notice that if we connect a voltage source to this circuit model, we have a voltage divider.

Thus $U_{mid} = \frac{R_2}{R_1 + R_2} V_s = \frac{\rho \frac{L_{touch}}{A}}{\rho \frac{L_{rest}}{A} + \rho \frac{L_{touch}}{A}} \times V_s = \frac{L_{touch}}{L_{touch} + L_{rest}} V_s = \frac{L_{touch}}{L} V_s$

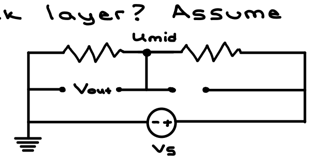
By connecting the screen to a power source, we can calculate U_{mid} and determine the relative position of L_{touch}. Also note that since U_{mid} is not dependent on any material properties (ρ and A), the top layer can be built with any material and the relationship is still valid.

Resistive Touchscreen (expanding the Model)

Recall our model of the red plate.

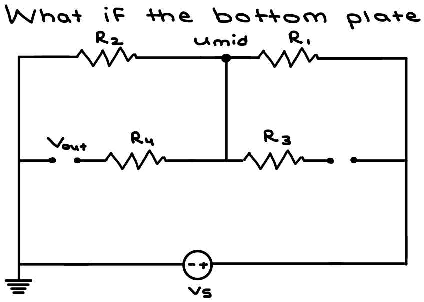


What is the purpose of the bottom black layer? Assume it is a perfect conductor ($\rho = 0$). Then the circuit becomes:

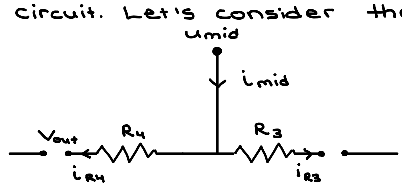


The bottom plate lets us take the measurement V_{out} using connection points on the edge of the plate.

What if the bottom plate is not a perfect conductor? Our circuit is:



Notice that there are more nodes in the circuit. Let's consider the abreviated schematic



Note that R4 and R3 are followed by open circuits, so $i_{mid} = i_3 + i_4 = 0 + 0 = 0$

So, even if the bottom plate is not perfectly conductive, $V_{out} = U_{mid}$.

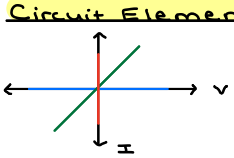
Power Rate of change of energy. $P = \frac{dE}{dt} = VI = I^2R = \frac{V^2}{R}$

- **voltmeter**: used to measure voltage
- **ammeter**: used to measure current

do not allow any power dissapated through the measurement devices

According to passive sign convention, positive current goes into the positive terminal of components. So if $P = IV > 0$, power is being dissipated. When $P = IV < 0$ power is being generated / delivered. So since we want $P = 0$, current through a voltmeter must be 0 (open circuit) and the voltage drop of an ammeter should be 0 (short circuit).

Circuit Element I-V Characteristics



open circuit

resistor $\frac{\partial I}{\partial V} = \frac{1}{R}$

wire

Notice that all 3 behave the same when $I = 0$, $V = 0$.

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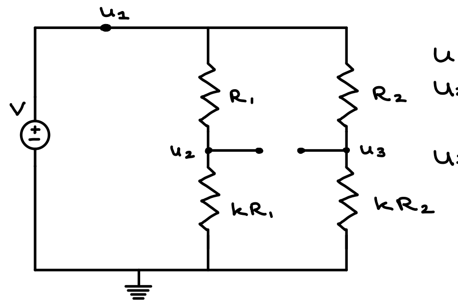
Important Circuit



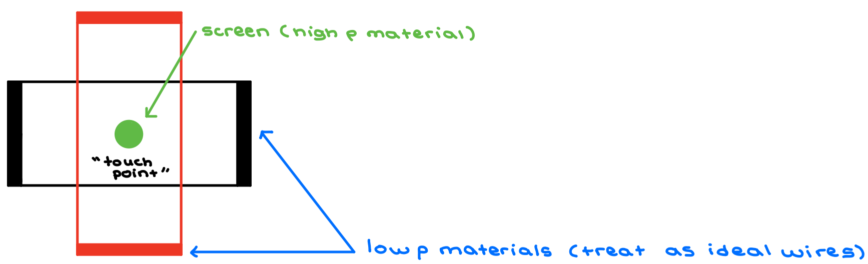
$$u_1 = V_S$$

$$u_2 = \frac{k}{1+k} V_S$$

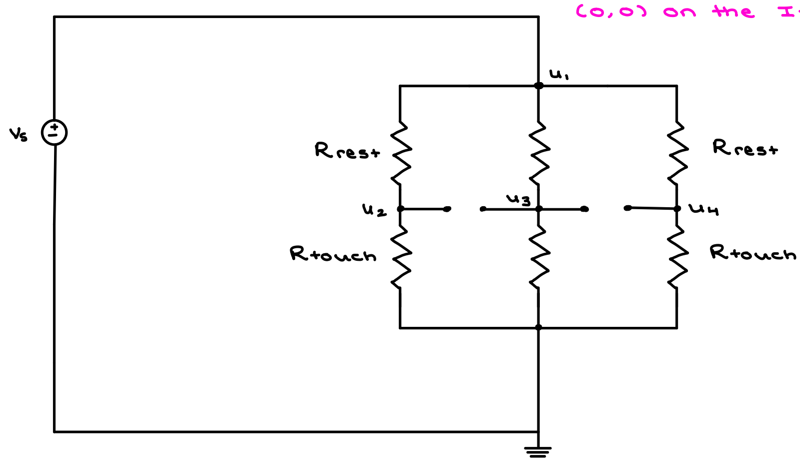
$$u_3 = \frac{k}{1+k} V_S$$

2D Resistive Touchscreen

Physical Structure Top red plate and a bottom black plate



In a 2D touchscreen, we want to figure out the vertical position and the horizontal position of the touch point: $L_{touch, vertical}$, $L_{touch, horizontal}$. We can treat the red plate as a bunch of vertical resistor strips, where each vertical strip is connected to the strips next to it by horizontal resistors as well. When we touch the plate, we split it into a top and bottom half R_{rest} and R_{touch}. For simplicitly let's divide it into just three vertical segments of equal width which are connected by horizontal resistors, which we can replace with open circuits: Note this can only be done since the resistors are at (0,0) on the I-V plot.



$$R_{touch} = \rho \frac{L_{touch}}{A}$$

$$R_{rest} = \rho \frac{L_{rest}}{A}$$

$$u_3 = \frac{R_{touch}}{R_{rest} + R_{touch}} V_S = \frac{L_{touch, vertical}}{L} V_S$$

We can follow a similar procedure for the black plate (the circuit will be rotated 90°. We will obtain a similar result, summarized below.

Vertical Position Measurement $V_{out} = \frac{L_{touch, vertical}}{L} \times V_S$

Horizontal Position Measurement $V_{out} = \frac{L_{touch, horizontal}}{L} \times V_S$

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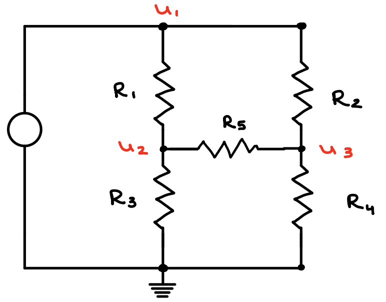
Faster Circuit Analysis Consider the following circuit. We would like to calculate the voltage at each node.



Step 1: Label the circuit (i.e. all unknown node voltage values).

Step 2: Write equations for nodes with voltage sources between them. $u_1 = V_s$.

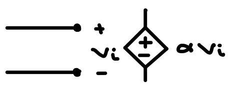
Step 3: Write KCL for any unknown nodes, using the $V = IR$ relationship and taking into account any current sources connected to the node.

$$-I_s + \frac{u_1 - u_2}{R_1} + \frac{u_1 - u_3}{R_2} = 0$$

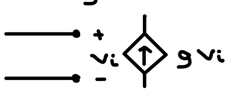
$$\frac{u_2 - V_s}{R_1} + \frac{u_2}{R_3} + \frac{u_2 - u_3}{R_5} = 0 \qquad \frac{u_3 - V_s}{R_2} + \frac{u_3}{R_4} + \frac{u_3 - u_2}{R_5} = 0.$$

Dependent Sources

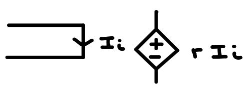
1. Voltage Controlled Voltage Source



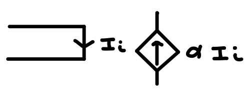
2. Voltage Controlled Current Source



3. Current-controlled voltage source



4. Current-controlled Current Source



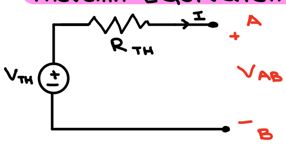
Superposition Recall that in our seven-step circuit analysis procedure we set up a matrix problem of the form $A\vec{x} = \vec{b}$ where $\vec{x}$ contained the unknown currents and node potentials, $\vec{b}$ contained the independent current and voltage sources, and A described the relationship between them. Observe that A is invertible: $\vec{x} = A^{-1}\vec{b}$. Thus, we can describe any current or node potential as a linear combination of the independent current and voltage sources. Consider a circuit with n independent sources, voltage sources $V_{s1}, \dots, V_{sn}$ and m independent current sources $I_{s1}, \dots, I_{sm}$. An arbitrary node potential $u_i = \alpha_1 V_{s1} + \dots + \alpha_n V_{sn} + \beta_1 I_{s1} + \dots + \beta_m I_{sm}$ where α and β are the coefficients from inverting A. We use the process of superposition to calculate each term:

- For each independent source k (either voltage source or current source)
  - Set all other independent sources to 0
    - Voltage Source: replace with a wire
    - Current source: replace with an open circuit
  - Compute the circuit voltages and currents due to this source k.
- Compute V_{out} by summing the $V_{out\,ks}$ for all k.

Zero voltage source and zero resistance are equivalent to wires, zero current source and infinite resistance are equivalent to open circuits.

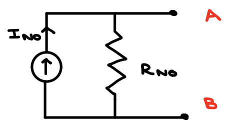
Equivalence Two circuits are equivalent if they have the same I-V relationship. Note this tells us nothing about power. If we pick two terminals within a circuit, we say that another circuit is equivalent to the original circuit if it exhibits the same I-V relationship at those two terminals. Two types of equivalent circuit:

- **Thevenin Equivalent Circuit**



To find V_{TH}, connect an open circuit between the two output terminals and measure the voltage across them. To find R_{TH}, zero out any independent sources then either (a) apply a test current and measure the resultant voltage or (b) apply a test voltage and measure current (then $R_{TH} = \frac{V}{I}$).

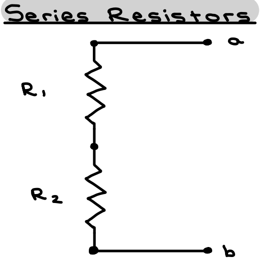
- **Norton Equivalent Circuit**



To find I_{NO}, connect a short circuit between the two output terminals and measure the current through it. To find R_{NO}, zero out any independent sources then either (a) apply a test current and measure the resultant voltage or (b) apply a test voltage and measure current (then $R_{NO} = \frac{V}{I}$).

Examples of TH Equivalence:

- **Series Resistors**

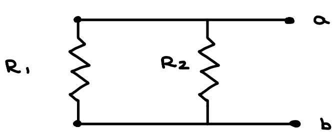


$V_{TH} = 0$

$R_{TH} = R_1 + R_2$

Note that the current through both resistors is the same

- **Parallel Resistors**

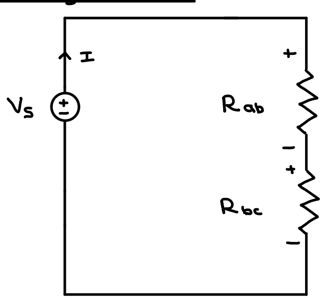


$V_{TH} = 0$

$R_{TH} = \frac{R_1 R_2}{R_1 + R_2}$

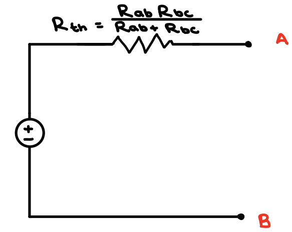
Note the voltage across R_1 and R_2 are the same

- **Voltage Divider**



$\longrightarrow$ $V_{th} = V_{AB,oc} = \frac{R_{bc}}{R_{ab} + R_{bc}} V_S$

$R_{th} = \frac{R_{ab} R_{bc}}{R_{ab} + R_{bc}}$



Ways to find TH/NO Equivalent Resistance

1. Zero out all independent sources and apply a V_{test} or I_{test} to calculate the resulting I_{test} or V_{test}, respectively. $R_{eq} = V_{test} / I_{test}$.
   - works for any circuit
2. Zero out all independent sources and reduce the entire remaining circuit into a

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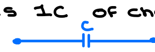
single resistor using the series and parallel resistor formulas.

- only works if there are no dependent sources

3. Calculate V_{TH} and I_{NO}, $R_{eq} = \frac{V_{TH}}{I_{NO}}$.

- only works if there is at least one independent source in the circuit.

Capacitor A circuit element that stores charge. has an associated quantity called **capacitance**, which represents how much charge can be stored for a given amount of voltage. Denote capacitance as C, voltage across the capacitor as V_c, and charge stored in the capacitor as Q. Then $Q = CV_c$. The unit of capacitance is the Farad (F). A capacitor with capacitance 1 F stores 1C of charge when it has 1V of potential energy across its electrodes. drawn as 

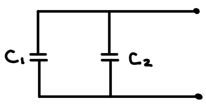
$I = C\frac{dV_c}{dt}$. Thus current is only flowing through a capacitor if the voltage across the capacitor is changing with time. $V_c(t) = \frac{I}{C}t + V_c(0)$.

Once voltage on a capacitor can no longer increase, current becomes 0.

Capacitor Equivalence Apply I_{test} (or $\frac{dV_{test}}{dt}$) and measure $\frac{dV_{test}}{dt}$ (or I_{test}).

$C_{eq} = I_{test} / (\frac{dV_{test}}{dt})$.

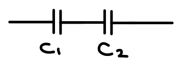
- **Capacitors in Parallel**



$C_{eq} = C_1 + C_2$

$C_{eq} = \varepsilon \frac{A_1 + A_2}{d}$

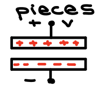
- **Capacitors in Series**



$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$

$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{\varepsilon}\frac{d_1 + d_2}{A}$

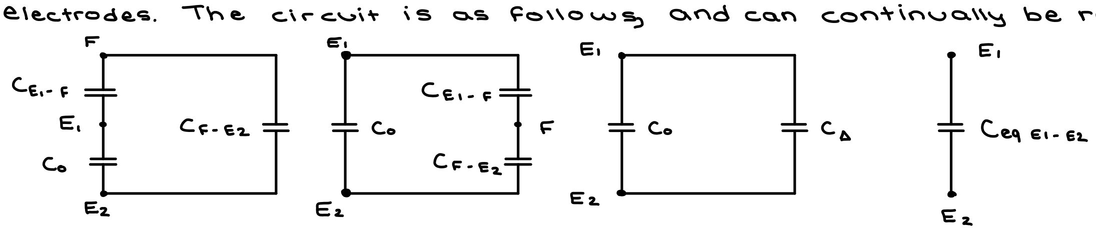
Capacitor Physics Two pieces of conductive material separated by an insulator. If there is voltage across the two pieces, charges will build up on the surface.



The capacitance of a capacitor is given by $C = \varepsilon\frac{A}{d}$, where A is the area of the surface of the plates, d is the separation between the two plates, and ε is the permitivity of the material between the plates (if air, $\varepsilon = 8.85 \times 10^{-12}$ F/m).

Energy $E = \frac{1}{2}CV^2$

Capacitive Touchscreen Each pixel of our touch screen consists of two pieces of metal with an insulator between them. This forms a capacitor with capacitance C_o. Another insulator (like glass) goes on top of the upper piece of metal. We want to detect if a finger touches it. When it does, two new capacitors are formed between the finger and each of the two electrodes. The circuit is as follows, and can continually be redrawn



We consider the parallel combination of C_o and C_Δ as some equivalent capacitance $C_{eq\,E_1-E_2} = C_o + C_\Delta$. When the finger is present, $C_\Delta > 0$ so $C_{eq\,E_1-E_2} > C_o$. Otherwise $C_{eq\,E_1-E_2} = C_o$. If we can measure this change in capacitance, we can detect the presence of a finger.

Touchscreen with multiple pixels Suppose we also want to determine WHERE the finger touches. Create a grid of the individual pixels as follows:

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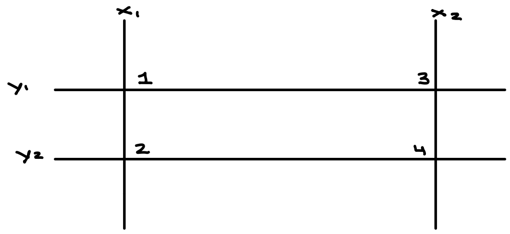


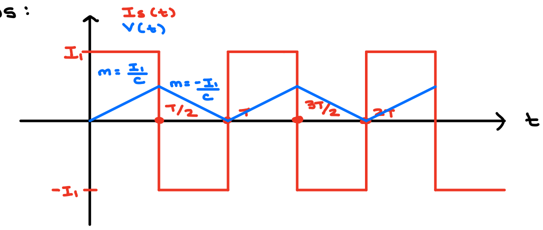
If we want to know if a finger is at pixel i, we can measure the capacitance between electrodes x_i and y_i — if the capacitance is increased, a finger is present

Capacitance Measurement

$V = (I_s t) / C$ assuming there was no voltage across the capacitor at $t = 0$.

Measuring Capacitance with a periodic current source

We apply a periodic current source, where the current function is as follows:



Assuming the capacitor is initially uncharged, voltage is given by

$$V_c(t) = \begin{cases} \frac{I_1}{C} t & \text{when } 0 \le t \le \frac{T}{2} \\ -\frac{I_1}{C}\left(t - \frac{T}{2}\right) + \frac{I_1 T}{2C} & \text{when } 0 \le t \le T \end{cases}$$

and continues.

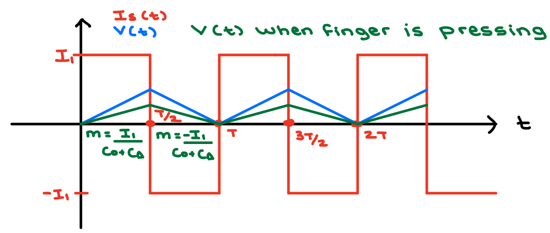
Switches

1. **on state**: perfectly conducting wire "closed"
2. **off-state**: open circuit  "open"

When a switch is closed, the voltage and charge of a capacitor connected in parallel will be zero.

Effect of a Finger Touch

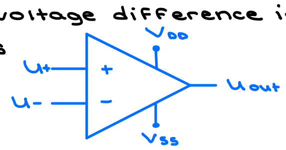
When a finger is pressing, the slope of the voltage function decreases.



Comparator and O-amp Basics

A o-amp (**operational amplifier**) is a device that transforms a small voltage difference into a very large voltage difference.

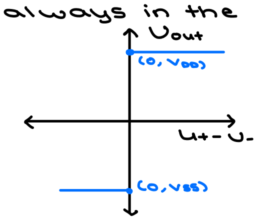
drawn as



. The bottom voltage source adds the voltage $\frac{V_{DD} - V_{SS}}{2}$ while the top source creates a voltage $A(U_+ - U_-)$ where A is a very large constant.

$$\text{So } U_{out} = \begin{cases} A(U_+ - U_-) + \frac{V_{DD} + V_{SS}}{2} & \text{if } V_{SS} \le A(U_+ - U_-) + \frac{V_{DD} + V_{SS}}{2} \le V_{DD} \\ = V_{SS} & \text{if } A(U_+ - U_-) + \frac{V_{DD} + V_{SS}}{2} < V_{SS} \\ = V_{DD} & \text{if } V_{DD} < A(U_+ - U_-) + \frac{V_{DD} + V_{SS}}{2} \end{cases}$$

For very large A, when $U_+ < U_-$, $A(U_+ - U_-) + \frac{V_{DD} + V_{SS}}{2} = -\infty$ and when $U_- < U_+$, $A(U_+ - U_-) + \frac{V_{DD} + V_{SS}}{2} = \infty$. So we are always in the 1st or 3rd case, giving us the following graph for U_{out}:



Thus, the opamp acts as a **comparator** because it indicates whether U_+ or U_- is larger, even if the difference is extremely small.

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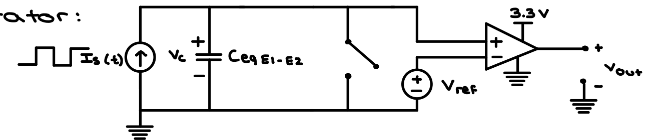
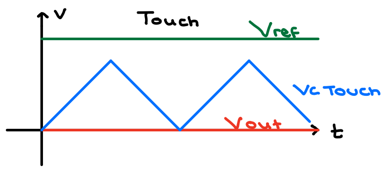
Capacitive Touchscreen with Comparator

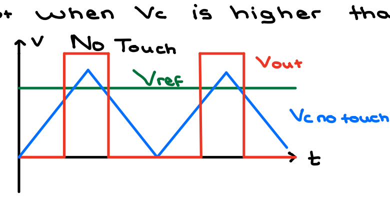
Complete our touchscreen by adding a comparator:



$$V_{out} = \begin{cases} 3.3\text{ V} & \text{if } V_c > V_{ref} \\ 0\text{ V} & \text{if } V_c < V_{ref} \end{cases}$$

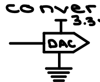
When a finger is present, $V_c < V_{ref}$ so $V_{out} = 0$. When there is no finger present, then there will be an increase in V_{out} when V_c is higher than V_{ref}.





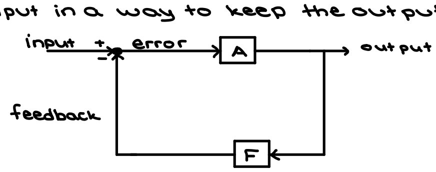
Design Example — DAC

A digital to analog converter is a component that translates digital signals into output analog voltages.

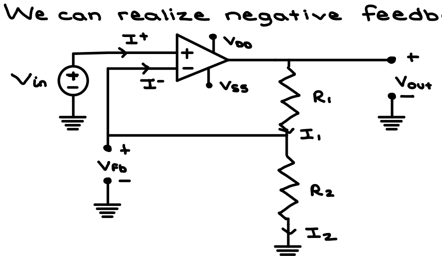


Suppose we have a song stored digitally and would like to convert it to a voltage then play it on the speaker. We will need to connect the DAC to an op-amp connected to the speaker. However, we cannot just scale the voltage linearly as we did when using op amps as comparators. Thus, another tool must be used.

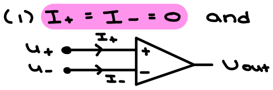
Negative Feedback when some function of the output of a system is fed back into the input in a way to keep the output at some finite value.



We can realize negative feedback in op-amps. Consider the circuit



To analyze this circuit, we need the following two rules: (1) $I_+ = I_- = 0$ and (2) $U_+ = U_-$ for

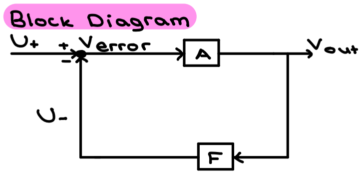


Apply KCL to the node between R_1 and R_2: $I_1 = I_2 + I_- = I_2$

By the second rule, $V_{in} = V_{Fb}$. Using Ohm's Law

$$I_2 = \frac{V_{Fb}}{R_2} = I_1 = \frac{V_{out} - V_{Fb}}{R_1} \longrightarrow V_{out} = V_{in}\left(1 + \frac{R_1}{R_2}\right)$$

Block Diagram



$V_{error} = U_+ - U_-$

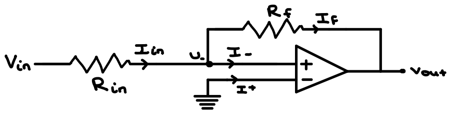
$V_{out} = A V_{error} = A(U_+ - U_-)$

$U_- = f V_{out}$

$\longrightarrow U_- = f\left(\frac{A}{1+Af} U_+\right)$. As $A \to \infty$, $U_+ = U_-$.

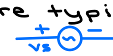
$$\boxed{V_{out} = \frac{A}{1 + A \cdot f} V_{in}}$$

Example: Inverting op amp

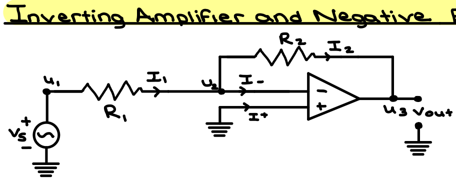


Circuit Analysis yields $V_{out} = -\frac{R_f}{R_{in}} V_{in}$.

When the output voltage is a multiple of the input voltage with a scaling factor of $\frac{-R_f}{R_{in}}$, the circuit is called an inverting amplifier. Note Vout and Vin have opposite sign.

Signals vs. Supply Voltages Op amps allow us to perform mathematical operations on the input voltages. In real systems, input voltages are typically small signals we have measured. Although they are voltages, they are typically much smaller and they change over time. We will thus use  to represent them. Such signals should not be used to power op-amps.

Inverting Amplifier and Negative Feedback

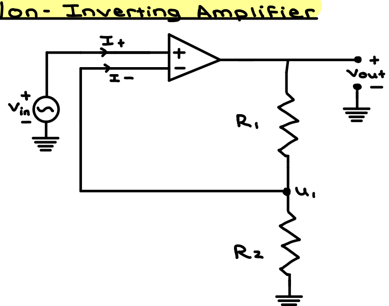


$$V_{out} = \frac{-R_2}{R_1} V_{in}$$

Checking for Negative Feedback

1. Zero out all independent sources
2. "Dink" the output. Check if the feedback coming from the circuit as a result of "dink-ing" the output has the opposite direction of "dinking". If so, it is negative feedback.

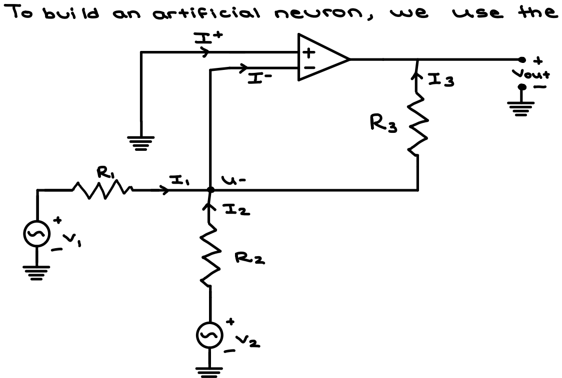
Non-Inverting Amplifier



$$V_{out} = \left(1 + \frac{R_1}{R_2}\right) \cdot V_{in}$$

Artificial Neuron Neurons in our brain are interconnected. The output of a single neuron is dependent on inputs from several other neurons. We can represent this idea with vector-vector multiplication, in which the output a linear combination of several inputs. $[a_1 a_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_1x_1 + a_2x_2$

To build an artificial neuron, we use the following circuit model:



$$\boxed{V_{out} = -\frac{R_3}{R_1} V_1 - \frac{R_3}{R_2} V_2}$$

"inverting summing amplifier"

However, suppose we want positive coefficients.

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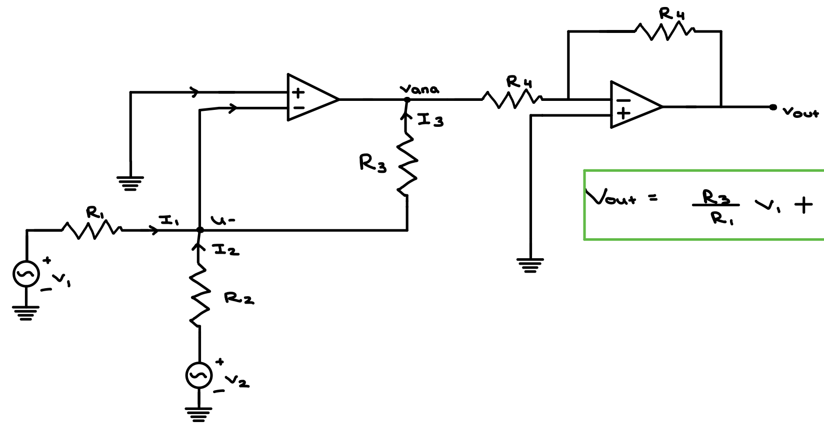
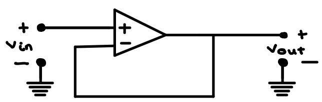


$$V_{out} = \frac{R_3}{R_1} v_1 + \frac{R_3}{R_2} v_2$$

Loading and Buffering



$V_{out} = V_{in}$

"unity gain buffer"

This answers our original question regarding connecting a DAC to a speaker; we connect them using such a buffer. Without a buffer, there will be a large voltage drop over R_{TH} since the speaker has small resistance. This effect is called loading.

Design Procedure We are ready to focus on designing interesting circuits circuits to perform specific tasks. The design procedure is as follows:

Step 1: Specification Concretely restate the goals for the design.

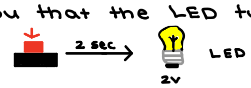
Step 2: Strategy Describe your strategy (in the form of a block diagram) to achieve your goal. Start by thinking about what you can measure vs. what you want to know. For example in our capacitive touchscreen: $\xrightarrow{\text{touch/no touch}}$ [convert touch to capacitance] $\rightarrow$ [convert capacitance to voltage] $\xrightarrow{\text{voltage}}$

Step 3: Implementation Implement the components described in your strategy.

Step 4: Verification Check your design from step 3 does what you specified in step 1. Check block-to-block connections, as these are the most common point for problems, check for contradictions.

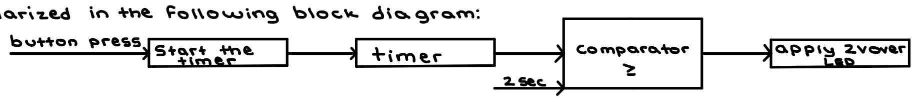
Design Examples:

- countdown timer build a countdown timer that will turn on LED 2 sec after a button is pressed. She tells you that the LED turns on when 2V is applied across it.



1. Specification: Build a circuit that, after a button is pressed, measures 2 seconds and then applies two volts across an LED.

2. Strategy: When the button is pressed we'd like to turn on a times. Then we'll need to know if the time elapsed has reached 2 sec which we can determine with a comparator. When our comparator tells up that time has exceeded 2 sec, we want to apply 2V across the LED. This is summarized in the following block diagram:



Note that "time" will need to be represented as either a current or voltage, since those are the only circuit quantities we can control. Therefore, we will also have to determine what current /

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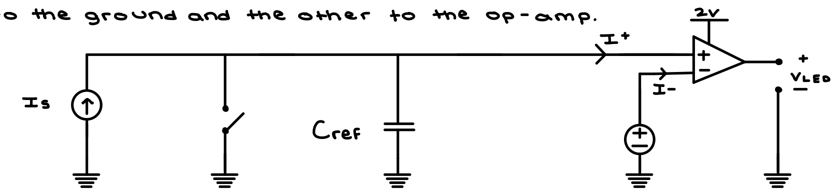
voltage is equivalent to 2 sec in our representation.

3. Implementation: Which components could be used to build each block. The first block we need to build is to "start the timer". What component can change based on a user input? a switch.

The next block is the timer itself. To build a timer, we need a component that changes either voltage or current as a function of time. A capacitor will work. The third block of our block diagram, the comparator, can be represented by a negative feedback op-amp.

We choose $V_{ref} = \frac{2I_s}{C_{ref}}$. If the voltage across C_{ref} becomes higher than V_{ref}, the comparator outputs $V_{DD} = 2V$. To apply 2V over the LED, we connect one terminal of the LED to the ground and the other to the op-amp.

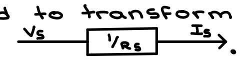


4. Verification Verify that when the button is pressed, the circuit acts as desired.

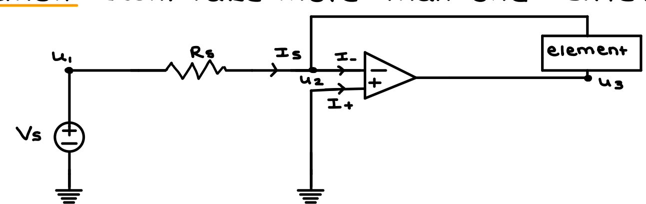
- "Almost" Current Source

1. Specification Build a current source that outputs constant current, I_s, regardless of the voltage across it.

2. Strategy We can use a voltage source, but we need to transform the voltage into a constant current. Consider using a resistor:

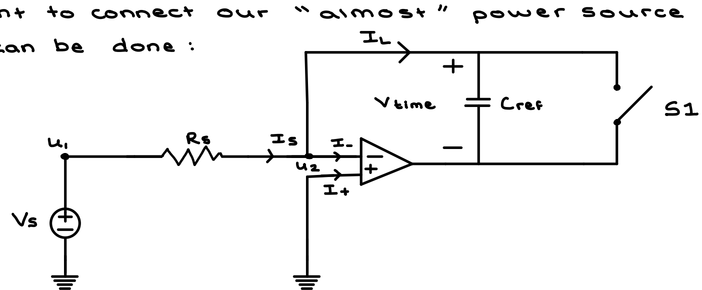


3. Implementation (will take more than one attempt).



4. Verification According to KCL $I_L = I_s = \frac{V_s}{R_s}$. $V_{RL} = \frac{V_s}{R_s} R_L$ ✓

Now we want to connect our "almost" power source to our countdown timer. This can be done:



Let's analyze this circuit.

$$I_s = C_{ref} \frac{dV_{time}}{dt} = C_{ref} \frac{d(u_2 - u_3)}{dt} = C_{ref} \frac{d(-u_3)}{dt}$$

$$\longrightarrow u_3(t) = -\frac{V_s}{R_s C_{ef}} t$$

units of RC are seconds

Although the designed "Almost Current Source" is very similar to an ideal current source, there are two important differences.

- do not connect the output of the current source to ground externally
- the circuit element we hook up to the current source must still keep the op-amp circuit in its negative feedback state.

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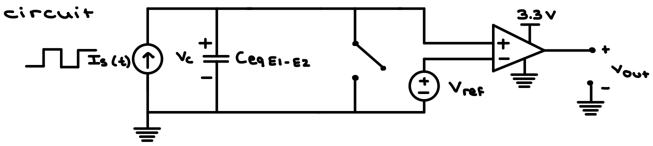
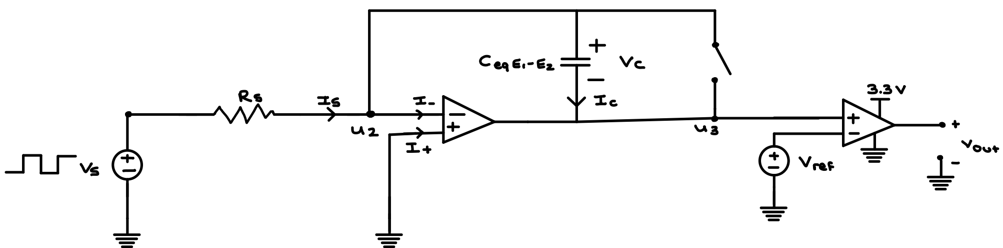
Capacitive Touchscreen Revisited

Recall our circuit



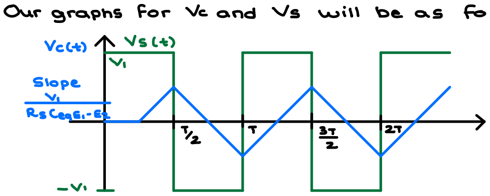
was powered by a periodic current source. Now suppose we instead attached an "almost current source to our touchscreen as follows:

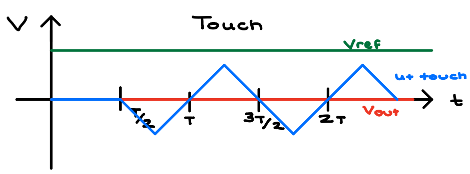


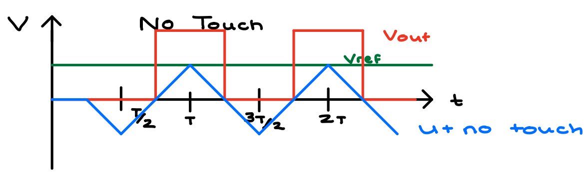
Analyzing this circuit, we get: $u_3 = -V_c(t)$ and

$$V_c(t) = \frac{I_c}{C_{eq\,E_1-E_2}}\, t = \frac{V_1}{R_S C_{eq\,E_1-E_2}}\, t$$

Our graphs for V_c and V_s will be as follows:







Introduction: Positioning Systems

Global Positioning System (GPS): a navigational system that we use all the time to tell us where we are. A reciever (ex. your phone) recieves messages from satelites orbiting to earth to find your distance from each one. The reciever knows the position of the satelites. The first satelite defines a set of all possible locations where the reciever could — a circle (or sphere in 3D). The others do the same, allowing the reciever to determine its precise position. Divide this into two parts:

1. Find the distances from the reciever to the satelites, based on the messages recieved.
2. Combine the measurements from each satelite to determine location.

Inner Products The (Euclidean) inner product between two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$

$$\langle \vec{x}, \vec{y} \rangle \equiv \vec{x} \cdot \vec{y} \equiv \vec{x}^T \vec{y} = [x_1 \;\; x_2 \dots \;\; x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

In physics, inner products are often called dot products. The inner product of $\vec{x}$ and $\vec{y}$ is their lengths multiplied by the angle between them. $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cos\theta$.

Properties

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- **Commutative** $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$
- **Scalar multiplication** $\langle \vec{x}, c\vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$ and $\langle c\vec{x}, \vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$
- distributive over vector addition $\langle \vec{x}+\vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

Orthogonal Vectors Two vectors $\vec{x}$ and $\vec{y}$ are orthogonal if $\langle \vec{x}, \vec{y} \rangle = 0$.

Special Vector Operations

Sum of Components

$$\langle [1\ 1 \dots 1]^T, [x_0, x_1, \dots, x_n]^T \rangle = x_0 + x_1 + \dots + x_n$$

Average

$$\langle [\tfrac{1}{n} \dots \tfrac{1}{n}]^T\ [x_0 \ \dots \ x_n]^T \rangle = \frac{x_0 + \dots + x_n}{n}$$

Sum of Squares

$$\langle [x_0 \dots x_n]^T, [x_0 \dots x_n]^T \rangle = x_0^2 + \dots + x_n^2$$

Selective Sum

$$\langle [0\ 0\ 1\ 0\ 1 \dots 0\ 1]^T, [x_0\ \ x_1 \dots x_n]^T \rangle = x_2 + x_4 + \dots + x_n$$

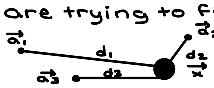
Norms The Euclidian norm of a vector is defined by $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\langle \vec{x}, \vec{x} \rangle}$

Properties

- **non-negativity** For $\vec{x} \in \mathbb{R}^n$, $\|\vec{x}\| \geq 0$
- **zero vector** $\|\vec{x}\| = 0$ iff. $\vec{x} = 0$.
- **Scalar multiplication** For vector $\vec{x} \in \mathbb{R}^n$ and scalar α, $\|\alpha\vec{x}\| = |\alpha|\|\vec{x}\|$
- **Triangle Inequality** For $\vec{x}, \vec{y} \in \mathbb{R}^n$, $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$
- **Cauchy-Schwarz Inequality** $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|\|\vec{y}\|$

P-Norm $\|\vec{x}\|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{1/p}$.

Trilateration Suppose we know the positions of the $\vec{a}_i$ and the d_i and are trying to find the position of $\vec{x}$.



$\|\vec{x} - a_1\|^2 = d_1^2$ $\quad \|\vec{x} - \vec{a}_2\| = d_2^2$ $\quad \|\vec{x} - \vec{a}_3\| = d_3^2$

We can create the matrix $\begin{bmatrix} 2(\vec{a}_1 - \vec{a}_2)^T \\ 2(\vec{a}_1 - \vec{a}_3)^T \end{bmatrix} = \begin{bmatrix} \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 - d_1^2 + d_2^2 \\ \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 - d_1^2 + d_3^2 \end{bmatrix}$

Signals message that contains info as a function of time. We can represent discrete-time signals (only at certain t) as vectors. Each element represents the signal value @ one time step.

Cross-correlation: measure of the similarity between two signals $\vec{x}$ and $\vec{y}$. $\text{corr}_{\vec{x}}(\vec{y})[k] = \sum_{i=-\infty}^{\infty} x[i]y[i-k]$ where $\text{corr}_{\vec{x}}(\vec{y})[k]$ is the k-th element of the cross correlation of $\vec{x}$ and $\vec{y}$. Make the assumption that $x[n]$ and $y[n]$ are 0 for n they are not defined for. When the inner product is large, $\vec{x}$ and $\vec{y}$ are more similar. Correlation is NOT commutative ($\text{corr}_{\vec{x}}(\vec{y}) \neq \text{corr}_{\vec{y}}(\vec{x})$).

- **autocorrelation**: correlation between a signal and itself
- **circular cross-correlation**: $\text{circcorr}(\vec{x}, \vec{y})[k] = \sum_{i=0}^{N-1} x[i]y[(i-k)_N]$
- **periodic correlation**: $\text{corr}_N(\vec{x}, \vec{y})[k] = \sum_{i=0}^{N-1} x[i]y[i-k]$

Least Squares Consider linear equations $A\vec{x} = \vec{b}$ corrupted by noise (i.e. there is some error $\vec{e} = \vec{b} - A\vec{x}$. We would like to choose an $\vec{x}$ which minimizes this error. To do so, we choose an $\vec{x}$ orthogonal to the span of all possible $A\vec{x}$ — in other words, an $\vec{x}$ that is orthogonal to the column space of A.

- A vector $\vec{e}$ is orthogonal to every vector in the column space of A iff. it is orthogonal to each of the columns $\vec{a}_i$ that form the basis of the column space.
- $\text{Null}(A^TA) = \text{Null}(A)$, even when A has a nontrivial nullspace.

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Application of Least Squares: planetary motion

Orthogonal Matching Pursuit (OMP)

Setup: Let the # of unique codes be m. Each code is length n. We represent each of the m codes with a vector $\vec{s_i}$ where i is an integer between 0 and m-1. Each code can potentially carry a message a_i along with it. Then the measurement at the reciever is $\vec{y} = a_0 \vec{s_0}^{(\tau_0)} + a_1 \vec{s_1}^{(\tau_1)} + \ldots + a_{m-1} \vec{s}_{m-1}^{(\tau_{m-1})}$. We will assume that the # of codes that are being brodcast at the same time is very small. If a code is not being broadcast, it's coefficient a_i will be zero. Thus, there are at most k non-zero a's.

Inputs:

- a set of m codes, each of length n. $S = \{\vec{s_0}, \vec{s_1}, \ldots, \vec{s}_{m-1}\}$
- An n-dimensional recieved signal vector: $\vec{y}$
- The sparsity level k of the signal. This is the # of codes w/ nonzero coefficients.
- Some threshold th. If norm of signal < th, it only contains noise

Outputs:

- A set of codes that were identified F, which will contain at most k elements
- A vector $\vec{x}$ containing the coefficients of the codes (a_1, etc.) which will be of length k or less
- An n-dimensional residual $\vec{e}$.

Procedure:

- Initialize $\vec{e} = \vec{y}$, $j = 1$, $A = [\,]$, $F = \{\varnothing\}$.
- while $((j < k) \,\&\, (\|\vec{e}\| \geq th))$:
  1. Cross correlate $\vec{e}$ with each of the codes. Find the code index, i, and the shifted version of the code $\vec{s_i}^{(\tau_i)}$ with which the recieved signal has the highest correlation value.
  2. Add i to the set of code indices, F.
  3. Column concatenate matrix A with the corrected shifted version of the code: $A = [A \mid \vec{s_i}^{(\tau_i)}]$
  4. Use the least squares to obtain the code coefficients: $\vec{x} = (A^T A)^{-1} A^T \vec{y}$
  5. Update the residual value $\vec{y}$ by subtracting: $\vec{e} = \vec{y} - A\vec{x}$.
  6. Update the counter $j = j + 1$.
- Stop after k iterations or when the norm of the residual drops below th.

$\longrightarrow$ used in machine learning

Computing distances from time delays

Imagine that many beacons begin broadcasting at t = 0 and our reciever begins recording at t = 0. There will be a time delay of $t_i = \frac{d_i}{v}$, meaning $d_i = t_i v$. Suppose we instead starting recording at some $t = \Delta t$. Then $d_{n-1} = v(\Delta t + t_{n-1})$. Define the time difference of arrival to be $\tau_i = t_i - t_0$.

Then $\sqrt{(\vec{x}-\vec{a}_{n-1})^T(\vec{x}-\vec{a}_{n-1})} - \sqrt{\vec{x}^T\vec{x}} = v\tau_{n-1}$. After many steps, we are left with:

$$\begin{bmatrix} 2\vec{a}_1^T/(v\tau_1) - 2\vec{a}_2^T/(v\tau_2) \\ 2\vec{a}_1^T/(v\tau_1) - 2\vec{a}_3^T/(v\tau_3) \\ \vdots \\ 2\vec{a}_1^T/(v\tau_1) - 2\vec{a}_{n-1}^T/(v\tau_{n-1}) \end{bmatrix} \vec{x} = \begin{bmatrix} v\tau_2 - v\tau_1 \\ v\tau_3 - v\tau_1 \\ \vdots \\ v\tau_{n-1} - v\tau_1 \end{bmatrix}.$$