Ambiguity, Disagreement, Price, and Volume

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Abstract

I construct a disagreement model where ambiguity-averse investors disagree about the interpretation of public information and face ambiguity when taking account of others' interpretations. I characterize how asset prices, trading volume, and investors' long, short, or non-participating decisions are linked to disagreement in equilibrium. The model provides a behavioral framework to study the relationship between returns, volume, and disagreement under ambiguity. In addition, the model can address several empirical evidence: positive correlation between volume and returns, high volume along with little price change, and intense trading activity around earnings announcements.

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1 Introduction

People disagree. In financial markets, investors usually have different interpretations of public information and are uncertain about others' interpretations. How investor disagreement is related to asset prices and trading volume is one of the most important questions in finance.[1] In this paper, I develop a behavioral multi-asset model where ambiguity-averse traders disagree on how to interpret public signals and face ambiguity about others' interpretations. The goal is to characterize the role of disagreement in asset pricing under ambiguity with a view to speaking directly to the joint behavior of stock prices and trading volume.[2]

Formally, for each risky asset there's a public signal, which equals to the sum of the true value of a asset and some noise. Two types of traders with equal mass, optimists and pessimists, know the true variance of the noise but have different prior beliefs on the mean of the noise. I use the word "interpretations" to describe traders' prior mean beliefs of the noise, and optimists' interpretation is lower than that of pessimists.[3] Investor disagreement is defined as the absolute value of the difference in two types of interpretations. Throughout the paper, when I refer to how investor disagreement affects trading volume or price, I am fixing the average interpretation of the public signal fixed as it directly affects the price.

Now, how do investors take account of the others' interpretation of the signal? Past studies typically assume that traders are absolutely convinced that their interpretations are correct.[4] As investors assign an excessively large weight on their own interpretations, they agree to disagree. However, this assumption seems too strong as in reality analysts and investors often incorporate others' interpretations to make revisions.

In the model, I assume that both types of investors form a convex combination of the two interpretations. That is, with a weight parameter $\alpha^i \in [0, 1]$, traders form a revised interpretation of the for risky asset i by assigning α^i weight to their own interpretation and $(1 - \alpha^i)$ weight to the other type's interpretation. When $\alpha^i > \frac{1}{2}$ ($\alpha^i < \frac{1}{2}$), traders believe

[1]See, for example, Harrison and Kreps (1978), Harris and Raviv (1993), Scheinkman and Xiong (2003), Cao and Ou-Yang (2008), and Banerjee and Kremer (2010).

[2]See Hong and Stein (2007) for discussion of the importance of investor disagreement model.

[3]By setup, lower interpretation means a more optimistic valuation on the asset compared to the signal.

[4]See, for example, Daniel et al. (2005), David (2008), Banerjee and Kremer (2010).

that they are superior (inferior) to others in terms of processing signals and hence assign disproportionately higher (lower) weight to their own interpretation.

In particular, α^i can take in a set of values, i.e., $\alpha^i \in [\underline{\alpha^i}, \overline{\alpha^i}] \subseteq [0, 1]$. The setup indicates that investors may experience ambiguity when incorporating the other type's interpretation and thus form a range of revised interpretations.[5] As a result, instead of forming a single revised interpretation, investors have in mind a set of revised interpretations. In reality, as investors typically have incomplete knowledge about other investors, it seems plausible that they experience ambiguity when thinking about interpretations belong to others. Furthermore, different values of $\underline{\alpha^i}$ and $\overline{\alpha^i}$ allow us to study investors' different ways of processing others' interpretations. In particular, investors can be unbiased, slightly overconfident, slightly underconfident, overconfident, and underconfident.

First, unbiased investors are impartial when picking the set of weight parameters, i.e., $\underline{\alpha^i} + \overline{\alpha^i} = 1$, $0 < \underline{\alpha^i} < 0.5 < \overline{\alpha^i} < 1$. Second, slightly overconfident investors ($\underline{\alpha^i} + \overline{\alpha^i} > 1$, $0 < \underline{\alpha^i} < 0.5 < \overline{\alpha^i} \leq 1$) believe that most of the time they are superior in processing the signal but others may have an upper hand from time to time. Slightly underconfident investors ($\underline{\alpha^i} + \overline{\alpha^i} < 1$, $0 \leq \underline{\alpha^i} < 0.5 < \overline{\alpha^i} < 1$) are the exact opposite of slightly overconfident investors. Third, overconfident investors ($\underline{\alpha^i} + \overline{\alpha^i} > 1$, $0.5 \leq \underline{\alpha^i} < \overline{\alpha^i} \leq 1$), always place themselves above others in terms of processing the signal. That is, they never assign a weight below $\frac{1}{2}$ to their own interpretation. Finally, underconfident investors ($\underline{\alpha^i} + \overline{\alpha^i} < 1$, $0 \leq \underline{\alpha^i} < \overline{\alpha^i} \leq 0.5$) are the exact opposite of overconfident investors.

The essential behavioral assumption is that investors are ambiguity-averse.[6] Several papers have provided evidence that many people exhibit ambiguity aversion. For example, in an experiment involving 104 individuals who are asked to choose between an ambiguous urn and a risky urn, Halevy (2007) finds that 61% are ambiguity-averse, 22% are ambiguity neutral, and 17% prefer the ambiguous urn. Using a Unicredit sample of 1,686 retail investors,

[5]Most ambiguity models such as Epstein and Schneider (2007), Epstein and Schneider (2008), Easley and O'Hara (2010), and Illeditsch (2011) assume that news or signals are of uncertainty quality. In this model, ambiguity lies in others' interpretations of the signal rather than the signal itself.

[6]The concept of ambiguity aversion in economics can be traced back to at least the Ellsberg Paradox (Ellsberg (1961)), which suggests that individuals are averse to vague probabilities and may not act as if they have a single prior.

Butler et al. (2014) find that 52% are ambiguity-averse and 25% are ambiguity-neutral. Dimmock et al. (2016) find that out of $3,258$ respondents in the American Life Panel (ALP), 52% are ambiguity averse, 10% are ambiguity-neutral, and 38% are ambiguity-seeking. In order to model ambiguity aversion, traders' preferences will be represented using the maxmin expected utility model of Gilboa and Schmeidler (1989).[7] Under the maxmin expected utility, agents have a set of probability measures and evaluate any action using the probability that minimizes the expected utility of that action.

There are many interesting properties in equilibrium. First, when investors are unbiased, both optimistic and pessimistic traders go long in the asset, price is decreasing in disagreement, and trading volume is zero. When investors are slightly overconfident, optimistic ones always go long in the asset while pessimistic ones go long in the asset when disagreement is low but step out of the market when disagreement is high. Price is always decreasing in disagreement, and trading volume is increasing in investor disagreement when both types of investors participate in the market but becomes fixed when only optimistic ones are present.[8]

Next, when investors are overconfident, optimistic ones always go long in the asset, while pessimistic ones go long in the asset at low disagreement, step out the market at medium disagreement, and go short in the asset at high disagreement. Price is decreasing in disagreement when both types of investors go long, and increasing in disagreement when only optimistic ones are present. When pessimistic ones go short in the asset, disagreement is unrelated to price. Volume, on the other hand, is increasing in investor disagreement when both types of investors participate in the market and is independent of disagreement when only optimistic ones are present.[9]

The disagreement model along with investors being ambiguity-averse provides a good framework to explain the joint behavior of stock returns and trading volume. First, past

[7] There are different forms of preferences in the literature that reflect ambiguity aversion, including the smooth ambiguity model by Klibanoff et al. (2005) and Klibanoff et al. (2009), "α^i-maxmin" model of Ghirardato et al. (2004), and "robust control" by Hansen and Sargent (2007) and Hansen and Sargent (2011). Though this paper adopts the maxmin expected utility formulation, a brief discussion of the extent to which alternative models would push on the results can be found in Section 3.3.

[8] When investors are slightly underconfident, everything is reversed. That is, pessimistic investors always go long in the asset while optimistic ones can either go long or step out of the market.

[9] When investors are underconfident, everything is reversed. That is, optimistic investors always go long in the asset while pessimistic ones can go long, go short, or step out of the market.

literature has documented a positive contemporaneous relation of volume and stock returns.[10] That is, stock prices rise on high volume but decline on low volume. In addition, Kandel and Pearson (1995) document that high trading volume can coexists with small price changes around earnings announcements. The above empirical evidence can be explained by my model when investors are either overconfident or underconfident.

Second, trading volume tends to rise around earning earnings announcements.[11] Intuitively, it is difficult to explain why trading volume would increase without some source of investor disagreement involved. Indeed, Beaver (1968), Bamber (1987), Ajinkya et al. (1991), and Garfinkel (2009) all argue that high trading volume is somehow related to high divergence of opinion around earnings announcements. Again, my model can explain the phenomenon under the assumption that disagreement is higher around announcement periods.

The paper contributes to investor disagreement, ambiguity, and earnings announcement literature. To the best of my knowledge, this is the first paper to incorporate ambiguity aversion into a disagreement model to study the role of disagreement in asset pricing. The framework of how investors incorporate others' interpretations is simple and tractable, which can be further studied in experimental studies. In addition, one prediction of the model is that trading volume is monotonically increasing in disagreement, which suggests that volume is good measure for disagreement.[12]

The paper is organized as follows. Section 2 provides a brief overview of the literature in disagreement, returns, and volume. Section 3 sets out a model with optimists and pessimists disagreeing on interpreting public signals and facing ambiguity when incorporating others' interpretations. Section 4 characterizes equilibrium asset prices and trading volume when there is with or without ambiguity. Section 5 speaks to several empirical facts. Section 6 concludes.

[10] See for example, Ying (1966), Clark (1973), Tauchen and Pitts (1983), Karpoff (1987), Gallant et al. (1992), and Andersen (1996).

[11] See Bamber et al. (2010) for detailed literature.

[12] See, e.g., Garfinkel and Sokobin (2006), Garfinkel (2009), and Berkman et al. (2009).

2 Literature Review

2.1 Disagreement and returns

The literature still disagrees on how investor disagreement should be related to stock returns. The two competing views are represented by Miller (1977) and Merton (1987). First, Miller (1977) posits that in the presence of short-sales constraints, stock prices are biased upward when disagreement among investors is high. This is because when pessimists can't freely trade on the negative information as a result of short-sales constraints, asset prices are mainly set by optimists. Morris (1996), Viswanathan (2001), and Chen et al. (2002) also suggest that prices typically reflect a more optimistic valuation due to high short-sale costs. Using analyst forecast dispersion as a proxy for investor disagreement, Diether et al. (2002) provides a negative relationship between investor disagreement and future stock returns in the cross-section.

In a traditional CAPM world, idiosyncratic risk is not priced since investors can hold efficiently diversified portfolios. Merton (1987), however, argues that investors tend to hold stocks they are familiar with and thus hold under-diversified portfolios. Naturally, they demand compensation to hold low visibility stocks with idiosyncratic risk. Since high disagreement indicates higher variation in earning streams, stocks with high divergence of opinion should earn higher future returns. David (2008) constructs a general equilibrium model in which two types of agents have heterogeneous beliefs about future fundamental growth. Agents face the risk that market prices move more in line with the trading models of competing agents than with their own, and thus speculate with each other. Gao et al. (2019) also argue that when investors agree to disagree, they both expect to profit at the expense of their trading counterparties. One assumption of David (2008) is that each trader is absolutely convinced that his own belief is correct.[13] Banerjee (2011) argues that when investors condition on prices, disagreement is related positively to expected returns but negatively to return autocorrelation.

[13]This assumption is similar to that of the overconfidence model of Daniel et al. (2005), in which each investor assigns an excessively large weight on his own model.

2.2 Disagreement and volume

Kim and Verrecchia (1991b) and Kim and Verrecchia (1991a) propose that trading volume is proportional to absolute price change when there is dispersion of risk tolerance coefficients and prior precision among investors. In addition, Shalen (1993) and Harris and Raviv (1993) indicate that dispersion in beliefs can be a factor contributing to the positive correlation between volume and absolute price changes. Kandel and Pearson (1995), on the other hand, provide evidence of high trading volume accompanied by close to zero price change around some earnings announcements. They set up a model where traders have differences of opinion about the meaning of the announcements. Scheinkman and Xiong (2003) and Hong et al. (2006) build dynamic models of investors continually updating their valuations based on their personal interpretations of incoming signals and argue that in the presence of short-sale constraints, a positive correlation exists between trading volume and the degree of overpricing. Banerjee and Kremer (2010) develop a dynamic model in which investors disagree about the interpretation of public information and show that when investors have infrequent but major disagreements, there is positive autocorrelation in volume and positive correlation between volume and volatility.

3 The Model

This section develops the model. In the market there are $I+1$ assets: a risk-free asset, money, which has a constant price of one, and I risky assets, denoted by $i = 1, ..., I$. There are N Bayesian investors and three dates.

At date 0, each investor begins with the same endowments of money and risky assets, $(\bar{m}, \bar{x}^1, ..., \bar{x}^I)$, and has identical prior beliefs for each risky asset.[14] All investors believe that the value of each risky asset, V^i, follows a normal distribution of mean v^i and variance σ^i, and that V^i's are independent. The equilibrium price for risky asset i at date 0 (p_0^i) equals to $v^i - \sigma^i \bar{x}^i$.

[14] In fact, under the CARA-normal structure, investors end up having the same holdings of money and risky assets even if they begin with different endowments but are allowed to trade at date 0.

Each investor has CARA utility of his wealth, w, with risk aversion parameter set equal to one:

$$u(w) = -exp(-w). \tag{1}$$

Each investor's budget constraint is

$$w = \bar{m} + \sum_i p^i \bar{x}^i, \tag{2}$$

where p^i is the price of risky asset i.

At date 1, there is a public signal for each risky asset and all signals are independent and observed by investors. Let S^i denote the public signal of risky asset i, where $S^i = V^i + \epsilon^i$, $\epsilon^i \perp V^i$, and $\epsilon^i \sim N(\mu^i, \phi^i)$. Investors know everything about S^i except for μ^i. In particular, half of the investors (type A) believe that $\mu^i = \mu^i_A$, while the other half (type B) believe that $\mu^i = \mu^i_B$. In addition, type A investors observe μ^i_B and type B investors observe μ^i_A.

I refer to μ^i_A and μ^i_B as type A's and type B's interpretation of S^i. Disagreement for risky asset i is defined as the absolute difference between two types of interpretations, $|\mu^i_A - \mu^i_B|$. If $\mu^i_A = \mu^i_B$, then there is no investor disagreement for risky asset i. If $\mu^i_B > \mu^i_A$, then type A investors are considered optimistic and type B investors are considered pessimistic for risky asset i, and vice versa.[15]

In the next section, I introduce how investors update their estimates of μ^i after observing the other type's interpretation. The goal is to set up a tractable framework that addresses how investors incorporate others' interpretations in different ways. At date 2, V^i's are realized and investors consume their wealth.

3.1 Updating under Ambiguity

At date 1, after observing the other type's interpretation, I assume that investors' updated estimates of μ^i are weighted arithmetic means of μ^i_A and μ^i_B with non-negative weights

[15] "Pessimistic" and "optimistic" traders are in a relative sense. For example, when $0 > \mu^i_B > \mu^i_A$, both types of investors think that V_i is higher than S_i. However, type A investors are relatively more optimistic than type B investors.

$\alpha^i \in [\underline{\alpha^i}, \overline{\alpha^i}] \subseteq [0, 1]$. That is, investors' updated estimates of μ^i are given by

$$\begin{cases} \{\widehat{\mu_A^i} \mid \widehat{\mu_A^i} = \alpha^i \mu_A^i + (1-\alpha^i)\mu_B^i, \alpha^i \in [\underline{\alpha^i}, \overline{\alpha^i}]\}, & \text{if type A investors} \\ \{\widehat{\mu_B^i} \mid \widehat{\mu_B^i} = \alpha^i \mu_B^i + (1-\alpha^i)\mu_A^i, \alpha^i \in [\underline{\alpha^i}, \overline{\alpha^i}]\}, & \text{if type B investors} \end{cases} \tag{3}$$

First, the set of weights $\alpha^i \in [\underline{\alpha^i}, \overline{\alpha^i}]$ indicates that investors may experience ambiguity when taking account of the other type's interpretation, which results in a set of updated estimates of μ^i. If, however, $\underline{\alpha^i} = \overline{\alpha^i}$, then investors experience no uncertainty. For example, when $\underline{\alpha^i} = \overline{\alpha^i} = 0.3$, type A investors' updated estimate of μ^i is $0.3\mu_A^i + 0.7\mu_B^i$ and type B investors' updated estimate of μ^i is $0.3\mu_B^i + 0.7\mu_A^i$.

Second, higher α^i indicates that investors value their own interpretation more when computing the updated estimates of μ^i for each risky asset i. As a result, the values of $\underline{\alpha^i}$ and $\overline{\alpha^i}$ enable us to characterize different investor behavior under ambiguity. In particular, investors are

$$\begin{cases} \text{unbiased,} & \text{if } (\underline{\alpha^i} + \overline{\alpha^i} = 1,\ 0 < \underline{\alpha^i} < 0.5 < \overline{\alpha^i} < 1) \\ \text{slightly overconfident,} & \text{if } (\underline{\alpha^i} + \overline{\alpha^i} > 1,\ 0 < \underline{\alpha^i} < 0.5 < \overline{\alpha^i} \leq 1) \\ \text{slightly underconfident,} & \text{if } (\underline{\alpha^i} + \overline{\alpha^i} < 1,\ 0 \leq \underline{\alpha^i} < 0.5 < \overline{\alpha^i} < 1) \\ \text{overconfident,} & \text{if } (\underline{\alpha^i} + \overline{\alpha^i} > 1,\ 0.5 \leq \underline{\alpha^i} < \overline{\alpha^i} \leq 1) \\ \text{underconfident,} & \text{if } (\underline{\alpha^i} + \overline{\alpha^i} < 1,\ 0 \leq \underline{\alpha^i} < \overline{\alpha^i} \leq 0.5). \end{cases} \tag{4}$$

This framework classifies investor behavior under ambiguity into five types. I exclude the case where α^i is below 0 or above 1 as it seems implausible in reality. In addition, whether α^i can take in discrete values rather than a continuum of values in $[\underline{\alpha^i}, \overline{\alpha^i}]$ doesn't change the results of the model. As I'll show in the next sections, only $\underline{\alpha^i}$ and $\overline{\alpha^i}$ matter under ambiguity aversion. Lastly, following Gilboa and Schmeidler (1989), investors choose a portfolio to maximize their minimum expected utility over the set of weights, $[\underline{\alpha^i}, \overline{\alpha^i}]$.

3.2 Asset Demands

We now solve for investors' asset demands. First, after taking account of S^i and the other type's interpretation, investors' posterior beliefs of V^i are given by

$$\begin{cases} V^i \sim N(\dfrac{\delta^i v^i + \gamma^i(S^i - \widehat{\mu^i_A})}{\delta^i + \gamma^i}, \dfrac{1}{\delta^i + \gamma^i}) := N(A^i, \dfrac{1}{\delta^i + \gamma^i}), & \text{if type } A \text{ investors} \\ \\ V^i \sim N(\dfrac{\delta^i v^i + \gamma^i(S^i - \widehat{\mu^i_B})}{\delta^i + \gamma^i}, \dfrac{1}{\delta^i + \gamma^i}) := N(B^i, \dfrac{1}{\delta^i + \gamma^i}), & \text{if type } B \text{ investors} \end{cases} \tag{5}$$

where $\delta^i = 1/\sigma^i$ and $\gamma^i = 1/\phi^i$. Let A^i and B^i denote type A and type B investors' possible posterior means of V^i, respectively. In addition, denote the minimum and maximum value of A^i and B^i by A^i_{min}, A^i_{max}, B^i_{min}, and B^i_{max}, respectively. Note that if $\mu^i_B > \mu^i_A$, then A^i_{min} and A^i_{max} are achieved at $\alpha^i = \underline{\alpha^i}$ and at $\alpha^i = \overline{\alpha^i}$, while B^i_{min} and B^i_{max} are achieved at $\alpha^i = \overline{\alpha^i}$ and at $\alpha^i = \underline{\alpha^i}$. If, on the other hand, $\mu^i_A > \mu^i_B$, then everything is reversed due to symmetry.

Let $(m_A, {x_A}^1, ..., {x_A}^I)$ and $(m_B, {x_B}^1, ..., {x_B}^I)$ denote type A and B investors' per capital asset demands, respectively. Demands for risky assets can be positive (go long), negative (go short), or zero (non-participating).[16] Each investor solves the following decision problem:

$$\begin{cases} \max\limits_{(m_A, {x_A}^1, ..., {x_A}^I)} \min\limits_{\alpha^i \in [\underline{\alpha^i}, \overline{\alpha^i}]} E\big[-e^{-(w + \sum_i (V^i - p^i) x^i_A)}\big], & \text{if type } A \text{ investors} \\ \max\limits_{(m_B, {x_B}^1, ..., {x_B}^I)} \min\limits_{\alpha^i \in [\underline{\alpha^i}, \overline{\alpha^i}]} E\big[-e^{-(w + \sum_i (V^i - p^i) x^i_B)}\big], & \text{if type } B \text{ investors,} \end{cases} \tag{6}$$

[16] Non-participating means that the final asset position is zero.

which can be rewritten as[17]

$$\begin{cases} \max\limits_{(m_A, x_A{}^1, \ldots, x_A{}^I)} \min\limits_{\alpha^i \in [\underline{\alpha^i}, \overline{\alpha^i}]} \quad w + \sum\limits_i (A^i - p^i) x_A^i - \frac{1}{2}(\frac{1}{\delta^i + \gamma^i})^2 (x_A^i)^2, & \text{if type } A \text{ investors} \\ \max\limits_{(m_B, x_B{}^1, \ldots, x_B{}^I)} \min\limits_{\alpha^i \in [\underline{\alpha^i}, \overline{\alpha^i}]} \quad w + \sum\limits_i (B^i - p^i) x_B^i - \frac{1}{2}(\frac{1}{\delta^i + \gamma^i})^2 (x_B^i)^2, & \text{if type } B \text{ investors,} \end{cases} \tag{7}$$

where the minimum is taken over the set $[\underline{\alpha^i}, \overline{\alpha^i}]$. By setup, it is equivalent to say that the minimum is taken over the possible posterior means of V_i, $[A^i_{min}, A^i_{max}]$ and $[B^i_{min}, B^i_{max}]$, for type A and type B investors, respectively.

3.3 Conservatism Under Ambiguity Aversion

Under the max-min expected utility, an investor contemplating either a long or a short position in risky asset i evaluates it using the posterior mean of V_i that yields the smallest expected utility before maximization. To figure out investor's demand function for each risky asset i, it is essential to consider whether an investor would prefer a long position, a short position, or zero position (non-participating).

For example, suppose $A^i_{min} > p^i$, a type A investor contemplating a long position ($x^i_A > 0$) would evaluate it using A^i_{min} before maximization as $(A^i - p^i)x^i_A$ is minimized at $A^i = A^i_{min}$. In contrast, a type A investor contemplating a short position ($x^i_A < 0$) would evaluate it using A^i_{max} before maximization as $(A^i - p^i)x^i_A$ is minimized at $A^i = A^i_{max}$. However, a long position of $(A^i_{min} - p^i)(\delta^i + \gamma^i)$ generates the highest expected utility, which is higher than the expected utility generated by any short positions or zero position. Following this logic, each type A investor's demand function for risky asset i is

$$x_A^{i*} = \begin{cases} (A^i_{min} - p^i)(\delta^i + \gamma^i), & \text{if } A^i_{min} > p^i \\ 0, & \text{if } A^i_{min} \le p^i \le A^i_{max} \\ (A^i_{max} - p^i)(\delta^i + \gamma^i), & \text{if } A^i_{max} < p^i. \end{cases} \tag{8}$$

[17] This is because $E[-e^{-(w+\sum_i (V^i - p^i) x^i_A)}]$ is a strictly increasing transformation of $w_A + \sum_i (A^i - p^i)x^i_A - 1/2(\frac{1}{\delta^i + \gamma^i})^2 (x^i_A)^2$ and $E[-e^{-(w+\sum_i (V^i - p^i) x^i_B)}]$ is a strictly increasing transformation of $w_B + \sum_i (B^i - p^i)x^i_B - 1/2(\frac{1}{\delta^i + \gamma^i})^2 (x^i_B)^2$.

Similarly, Each type B investor's demand function for risky asset i is given by

$$x_B^{i*} = \begin{cases} (B^i_{min} - p^i)(\delta^i + \gamma^i), & \text{if } B^i_{min} > p^i \\ 0, & \text{if } B^i_{min} \leq p^i \leq B^i_{max} \\ (B^i_{max} - p^i)(\delta^i + \gamma^i), & \text{if } B^i_{max} < p^i. \end{cases} \tag{9}$$

Equation (8) and (9) describes how ambiguity aversion affects an investor's demand. In particular, if the price of risky asset i is higher than his minimum posterior mean of V_i, than an investor goes long in risky asset i. If the price of risky asset i is higher than his maximum posterior mean of V_i, than an investor goes short in risky asset i. If the price of risky asset i is between than his minimum and posterior mean of V_i, then an investor will not participate in the market for risky asset i.

In other words, ambiguity-averse investors exhibit conservatism in the stock market. That is, investors go long in the asset only if the price is above their minimum possible estimate and go short in the asset only if the price is below their maximum possible estimate. In addition, if the price is above the minimum possible estimate and below the maximum possible estimate, investors will not participate in the market. These results are consistent with those findings in Easley and O'Hara (2009) and Easley and O'Hara (2010).

Other ambiguity aversion preferences such as the smooth ambiguity model by Klibanoff et al. (2005) and Klibanoff et al. (2009), "α-maxmin" model of Ghirardato et al. (2004), and "robust control" by Hansen and Sargent (2007) and Hansen and Sargent (2011) would also lead to investor conservatism and hence reach to the similar results in the following sections. The only difference is the extent of this conservatism. As the max-min expected utility is the most strict version of ambiguity aversion, it generates the highest level of conservatism.

4 Characterization of Equilibrium

In equilibrium the demand for each risky asset must equal its supply. That is, for each risky asset i:

$$\frac{N}{2}\, x_A^{i*} + \frac{N}{2}\, x_B^{i*} = N\bar{x}^i \tag{10}$$

Next, different values of $\underline{\alpha^i}$ and $\overline{\alpha^i}$ in equation (2) affect the relative magnitude of A^i_{min}, A^i_{max}, B^i_{min}, and B^i_{max}. I characterize the equilibrium under five types of investor behavior.

4.1 Benchmark Case: No Ambiguity

As a benchmark, I first solve the case where investors face no ambiguity, i.e., $\alpha^i = \underline{\alpha^i} = \overline{\alpha^i}$. Each type A and type B investor's demand function for risky asset i are:

$$\begin{cases} x_A^{i*} = \{\delta^i v^i + \gamma^i[S^i - (\alpha^i\mu_A^i + (1-\alpha^i)\mu_B^i)]\} - p^i(\delta^i + \gamma^i) \\ x_B^{i*} = \{\delta^i v^i + \gamma^i[S^i - (\alpha^i\mu_B^i + (1-\alpha^i)\mu_A^i)]\} - p^i(\delta^i + \gamma^i) \end{cases} \tag{11}$$

Using equation (10), the market-clearing price is

$$p^i_{benchmark} = \frac{\delta^i v^i + \gamma^i[S^i - \frac{(\mu_A^i + \mu_B^i)}{2}] - \bar{x}^i}{\delta^i + \gamma^i}, \tag{12}$$

In addition, since there are only two types of investors, trading volume for risky asset i, $Volume^i$, is the absolute change in equilibrium holdings for either type A or type B investors from before to after they observe the public signal S^i and the other type's interpretation. In particular, when investors face no ambiguity,

$$Volume^i = \frac{N}{2}|x_A^{i*}(p^i_{benchmark}) - \bar{x}^i| = \frac{N}{2}|x_B^{i*}(p^i_{benchmark}) - \bar{x}^i| = \frac{N}{2}|(0.5-\alpha^i)||\mu_A^i - \mu_B^i|. \tag{13}$$

Since the average interpretation of the public signal for risky asset i, $(\mu_A^i + \mu_B^i)/2$, clearly affects the market-clearing price, it is essential to hold the average interpretation fixed when we study the relationship between disagreement and price.[18]

[18]For example, higher average interpretation by setup implies a more negative view of the signal and

I define the abnormal price as the difference between the equilibrium price p^{i^*} under ambiguity and the benchmark price, i.e, $\widehat{p^i} = p^{i^*} - p^i_{benchmark}$. In this paper, I'll focus on

$$\frac{\partial \widehat{p^i}}{\partial |\mu^i_A - \mu^i_B|}, \tag{14}$$

which represents how price of risky asset i increases or decreases with respect to a unit increase in investor disagreement, holding the average interpretation of the public signal constant. The abnormal return for risky asset i from date 0 to date 1 is

$$\widehat{r^i} = \frac{\widehat{p^i} - p^i_0}{p^i_0} \tag{15}$$

Note that $\frac{\partial \widehat{p^i}}{\partial |\mu^i_A - \mu^i_B|}$ has the same sign as $\frac{\partial \widehat{r^i}}{\partial |\mu^i_A - \mu^i_B|}$. Hence, if disagreement is positively related to the abnormal price, it is also positively related to abnormal return, and vice versa.

Equation (12) indicates that when investors face no ambiguity, price of any risky asset is not related to disagreement. In addition, equation (13) indicates that $Volume^i$ is increasing in disagreement. For the following analysis, it is useful to denote the benchmark posterior mean of V_i by

$$V^i_{benchmark} = \frac{\delta^i v^i + \gamma^i [S^i - \frac{(\mu^i_A + \mu^i_B)}{2}]}{\delta^i + \gamma^i}. \tag{16}$$

It can be shown that $V^i_{benchmark} = (A^i_{min} + B^i_{max})/2 = (B^i_{min} + A^i_{max})/2$. As a result, on the real line A^i_{min} and B^i_{max} are symmetric with respect to $V^i_{benchmark}$, and so are B^i_{min} and A^i_{max}. In addition, A^i_{min}, A^i_{max}, B^i_{min}, and B^i_{max} can all be written as $V^i_{benchmark}$ plus some function of disagreement, $|\mu^i_A - \mu^i_B|$.

4.2 Unbiased investors

In this section I study the case where investors are unbiased ($\underline{\alpha^i} + \overline{\alpha^i} = 1$ and $0 < \underline{\alpha^i} < 0.5 < \overline{\alpha^i} < 1$) when thinking about the other type's interpretation. Figure 1 shows that $A^i_{max} = B^i_{max} > V^i_{benchmark} > A^i_{min} = B^i_{min}$.

naturally generates a lower price, and vice versa.

First, if at a price lower than $A^i_{min} = B^i_{min}$, both types of investors go long in risky asset i with their posterior means of V_i being A^i_{min} and B^i_{min}, respectively. Second, if $A^i_{min} = B^i_{min} \leq p^i \leq A^i_{max} = B^i_{max}$, then both types of investors won't participate in the market. Lastly, if at a price higher than $A^i_{max} = B^i_{max}$, then both types of investors go short in risky asset i, which can't be the equilibrium as risky asset i is in positive supply.

Hence, equilibrium only exists if $p^i < A^i_{min} = B^i_{min}$ and the average posterior mean of V_i in the market is equal to $(A^i_{min} + B^i_{min})/2$, which is smaller than $V^i_{benchmark}$ and thus a decreasing function of disagreement, $|\mu^i_A - \mu^i_B|$.

Proposition 1. *When investors are unbiased, the abnormal price for risky asset i is*

$$\widehat{p^i} = \frac{-\gamma^i}{\delta^i + \gamma^i} \frac{(\overline{\alpha^i} - \underline{\alpha^i})}{2} |\mu^i_A - \mu^i_B|. \tag{17}$$

In addition, there's no trading for risky asset i, i.e., $Volume^i = 0$.

Proof. See Appendix. □

Proposition 1 indicates that the equilibrium price for risky asset i is decreasing in investor disagreement, $|\mu^i_A - \mu^i_B|$, holding fixed the average interpretation of the public signal for risky asset i. That is, price reflects a more pessimistic valuation as disagreement increases. Figure 2 plots the relationship between the abnormal price of risky asset i and disagreement when investors are unbiased.

In particular, when disagreement increases by 1, price decreases by $\frac{\gamma^i}{\delta^i+\gamma^i}\frac{(\overline{\alpha^i}-\underline{\alpha^i})}{2}$, which is between 0 and 1/4. In addition, there's no trading volume for risky asset i as investors' prior and posterior means of V^i are the same.[19]

4.3 Slightly overconfident and slightly undeconfident investors

I then study the equilibrium where investors are slightly overconfident ($\underline{\alpha^i} + \overline{\alpha^i} > 1$ and $0 < \underline{\alpha^i} < 0.5 < \overline{\alpha^i} \leq 1$) when thinking about the other type's interpretation. Without loss

[19] $A^i_{min} = B^i_{min} = (A^i_{min} + B^i_{min})/2$

of generality, assume $\mu_B^i > \mu_A^i$, so type A investors are optimistic and type B investors are pessimistic. Figure 3 shows that $A_{max}^i > B_{max}^i > V_{benchmark}^i > A_{min}^i > B_{min}^i$ under $\mu_B^i > \mu_A^i$.

First, if at a price lower than B_{min}^i, both types of investors go long in risky asset i with their posterior means of V_i being A_{min}^i and B_{min}^i, respectively. Second, if $B_{min}^i \leq p^i < A_{min}^i$, only type A investors go long in risky asset i with their posterior mean of V_i being A_{min}^i. In the above two cases, the average posterior means of V_i in the market equal to $(A_{min}^i + B_{min}^i)/2$ and A_{min}^i, which are both smaller than $V_{benchmark}^i$ and thus are decreasing in disagreement. The corresponding market-clearing prices are $p^i = \frac{(A_{min}^i + B_{min}^i)}{2} - \frac{\bar{x}^i}{(\delta^i + \gamma^i)}$ and $p^i = A_{min}^i - \frac{2\bar{x}^i}{(\delta^i + \gamma^i)}$, respectively.[20]

Third, if $A_{min}^i \leq p^i \leq B_{max}^i$, both types of investors won't participate in the market so no equilibrium exists. Fourth, if $B_{max}^i < p^i \leq A_{max}^i$, only type B investors want to go short in risky asset i and thus no equilibrium exists. Lastly, if $p^i > A_{max}^i$, both types of investors want to go short in risky asset i and thus no equilibrium exists.

To summarize, when investors are slightly overconfident, optimistic traders always go long in the risky asset. Pessimistic traders, on the other hand, either go long in the risky asset or don't participate in the market.

Proposition 2. *When investors are slightly overconfident, the abnormal price for risky asset i is*

$$\widehat{p^i} = \begin{cases} \dfrac{-\gamma^i}{\delta^i + \gamma^i} \dfrac{(\overline{\alpha^i} - \underline{\alpha}^i)}{2} |\mu_A^i - \mu_B^i|, & \text{if } |\mu_A^i - \mu_B^i| < \dfrac{2\bar{x}^i}{\gamma^i(\overline{\alpha^i} + \underline{\alpha}^i - 1)} \\ \\ \dfrac{-\gamma^i}{\delta^i + \gamma^i} \dfrac{(1 - 2\underline{\alpha}^i)}{2} |\mu_A^i - \mu_B^i| - \dfrac{\bar{x}^i}{\delta^i + \gamma^i}, & \text{if } |\mu_A^i - \mu_B^i| \geq \dfrac{2\bar{x}^i}{\gamma^i(\overline{\alpha^i} + \underline{\alpha}^i - 1)}. \end{cases} \tag{18}$$

[20] The binding constraints are $\frac{(A_{min}^i + B_{min}^i)}{2} - \frac{\bar{x}^i}{(\delta^i + \gamma^i)} < B_{min}^i$ and $B_{min}^i \leq A_{min}^i - \frac{2\bar{x}^i}{(\delta^i + \gamma^i)} < A_{min}^i$.

In addition, trading volume for risky asset i, is given by

$$Volume^i = \begin{cases} \frac{N\gamma^i(\overline{\alpha^i} + \underline{\alpha^i} - 1)}{4}|\mu_A^i - \mu_B^i|, & \textit{if } |\mu_A^i - \mu_B^i| < \frac{2\bar{x}^i}{\gamma^i(\overline{\alpha^i} + \underline{\alpha^i} - 1)} \\ \\ \frac{N}{2}\,\bar{x}^i, & \textit{if } |\mu_A^i - \mu_B^i| \geq \frac{2\bar{x}^i}{\gamma^i(\overline{\alpha^i} + \underline{\alpha^i} - 1)}. \end{cases} \tag{19}$$

Proof. See Appendix. □

Figure 4 illustrates the idea of Proposition 2. When investors are slightly overconfident, the equilibrium price for risky asset i is decreasing in investor disagreement, $|\mu_A^i - \mu_B^i|$, holding fixed the average interpretation of the public signal for risky asset i. In addition, investor disagreement has a more negative effect on price when both types of investors participate in the market compared to the case where only optimistic traders are present, since $|\frac{-\gamma^i}{\delta^i+\gamma^i}\frac{(\overline{\alpha^i}-\underline{\alpha^i})}{2}| > |\frac{-\gamma^i}{\delta^i+\gamma^i}\frac{(1-2\underline{\alpha^i})}{2}|$.

On the other hand, trading volume is increasing in disagreement when both optimistic and pessimistic investors go long in risky asset i. However, when disagreement is high enough so that pessimistic traders step out of the market, volume is fixed at $\frac{N}{2}\bar{x}^i$ as optimistic traders buy out positions from pessimistic investors.

The case where investors are slightly underconfident ($\underline{\alpha^i} + \overline{\alpha^i} < 1$ and $0 \leq \underline{\alpha^i} < 0.5 < \overline{\alpha^i} < 1$) is similar. First, when $\mu_B^i > \mu_A^i$, Figure 5 shows that $B_{max}^i > A_{max}^i > V_{benchmark}^i > B_{min}^i > A_{min}^i$.

Proposition 3. *When investors are slightly underconfident, the abnormal price for risky asset i is*

$$\widehat{p^i} = \begin{cases} \frac{-\gamma^i}{\delta^i + \gamma^i}\frac{(\overline{\alpha^i} - \underline{\alpha^i})}{2}|\mu_A^i - \mu_B^i|, & \textit{if } |\mu_A^i - \mu_B^i| < \frac{2\bar{x}^i}{\gamma^i(1 - \underline{\alpha^i} - \overline{\alpha^i})} \\ \\ \frac{-\gamma^i}{\delta^i + \gamma^i}\frac{(2\overline{\alpha^i} - 1)}{2}|\mu_A^i - \mu_B^i| - \frac{\bar{x}^i}{\delta^i + \gamma^i}, & \textit{if } |\mu_A^i - \mu_B^i| \geq \frac{2\bar{x}^i}{\gamma^i(1 - \underline{\alpha^i} - \overline{\alpha^i})} \end{cases} \tag{20}$$

In addition, trading volume for risky asset i, is given by

$$Volume^i = \begin{cases} \frac{N\gamma^i(1-\underline{\alpha^i}-\overline{\alpha^i})}{4}|\mu_A^i - \mu_B^i|, & \textit{if } |\mu_A^i - \mu_B^i| < \frac{2\bar{x}^i}{\gamma^i(1-\underline{\alpha^i}-\overline{\alpha^i})} \\ \\ \frac{N}{2}\,\bar{x}^i, & \textit{if } |\mu_A^i - \mu_B^i| \geq \frac{2\bar{x}^i}{\gamma^i(1-\underline{\alpha^i}-\overline{\alpha^i})} \end{cases} \tag{21}$$

Figure 6 illustrates the idea of Proposition 3. The main results are very similar to the case where investors are slightly overconfident. The difference is that now pessimistic traders always go long in risky asset i, while optimistic traders go long under low disagreement and sit out of the market under high disagreement. Price is decreasing in disagreement and trading volume is increasing when both types of investors are in the market and remains fixed when only the pessimistic ones are present.

So far I've examined the cases where investors are slightly overconfident and slightly underconfident. In both cases, there always exists a negative relationship between disagreement and price. In addition, trading volume and disagreement are either positively related or unrelated. I proceed to examine the cases where investors are overconfident and underconfident.

4.4 Overconfident and underconfident investors

I then study the equilibrium where investors are overconfident ($\underline{\alpha^i} + \overline{\alpha^i} > 1$ and $0.5 \leq \underline{\alpha^i} < \overline{\alpha^i} \leq 1$) when thinking about the other type's interpretation. Without loss of generality, assume $\mu_B^i > \mu_A^i$, so type A investors are optimistic and type B investors are pessimistic. Figure 7 shows that $A_{max}^i > A_{min}^i > V_{benchmark}^i > B_{max}^i > B_{min}^i$ under $\mu_B^i > \mu_A^i$. Note that optimists' most conservative estimate, A_{min}^i, is higher than pessimist's boldest estimate, B_{max}^i.

First, if at a price lower than B_{min}^i, both types of investors go long in risky asset i with their posterior means of V_i being A_{min}^i and B_{min}^i, respectively. Second, if $B_{min}^i \leq p^i \leq B_{max}^i$, only type A investors participate in the market with their posterior mean of V_i being A_{min}^i. Third, if $B_{max}^i < p^i \leq A_{min}^i$, type A investors go long in risky asset i while type B investors go short in risky asset i, with their posterior means of V_i being A^i_{min} and B^i_{max}, respectively.

In the above three cases, the average posterior means of V_i in the market equal to $(A^i_{min} + B^i_{min})/2$, A^i_{min}, and $(A^i_{min} + B^i_{max})/2$, which are lower than, higher than, and equal to $V^i_{benchmark}$, respectively. The corresponding market-clearing prices are $p^i = \frac{(A^i_{min}+B^i_{min})}{2} - \frac{\bar{x}^i}{(\delta^i+\gamma^i)}$, $p^i = A^i_{min} - \frac{2\bar{x}^i}{(\delta^i+\gamma^i)}$, and $p^i = \frac{(A^i_{min}+B^i_{max})}{2} - \frac{\bar{x}^i}{(\delta^i+\gamma^i)}$, respectively.[21]

Fourth, if $A^i_{min} \leq p^i \leq A^i_{max}$, only type B investors want to go short in risky asset i and thus no equilibrium exists. Lastly, if $p^i > A^i_{max}$, both types of investors want to go short in risky asset i and thus no equilibrium exists.

To summarize, when investors are overconfident, optimistic traders always go long in the risky asset. Pessimistic traders, on the other hand, can go long in the risky asset, go short in the risky asset, or don't participate in the market at all. The following proposition presents the equilibrium abnormal price and trading volume.

Proposition 4. *When investors are overconfident, the abnormal price for risky asset i is*

$$\widehat{p^i} = \begin{cases} \dfrac{-\gamma^i}{\delta^i+\gamma^i}\dfrac{(\overline{\alpha^i} - \underline{\alpha^i})}{2}|\mu^i_A - \mu^i_B|, & \text{if } |\mu^i_A - \mu^i_B| < \dfrac{2\bar{x}^i}{\gamma^i(\underline{\alpha^i} + \overline{\alpha^i} - 1)} \\ \\ \dfrac{\gamma^i}{\delta^i+\gamma^i}\dfrac{(2\underline{\alpha^i} - 1)}{2}|\mu^i_A - \mu^i_B| - \dfrac{\bar{x}^i}{\delta^i+\gamma^i}, & \text{if } \dfrac{2\bar{x}^i}{\gamma^i(\underline{\alpha^i} + \overline{\alpha^i} - 1)} \leq |\mu^i_A - \mu^i_B| \leq \dfrac{2\bar{x}^i}{\gamma^i(2\underline{\alpha^i} - 1)} \\ \\ 0, & \text{if } |\mu^i_A - \mu^i_B| > \dfrac{2\bar{x}^i}{\gamma^i(2\underline{\alpha^i} - 1)} \end{cases} \tag{22}$$

[21]The binding constraints are $\frac{(A^i_{min}+B^i_{min})}{2} - \frac{\bar{x}^i}{(\delta^i+\gamma^i)} < B^i_{min}$, $B^i_{min} \leq A^i_{min} - \frac{2\bar{x}^i}{(\delta^i+\gamma^i)} \leq B^i_{max}$, and $B^i_{max} < \frac{(A^i_{min}+B^i_{max})}{2} - \frac{\bar{x}^i}{(\delta^i+\gamma^i)} \leq A^i_{min}$.

In addition, trading volume for risky asset i, is given by

$$Volume_i = \begin{cases} \dfrac{N\gamma^i(\overline{\alpha^i} + \underline{\alpha^i} - 1)}{4}|\mu_A^i - \mu_B^i|, & if \ \ |\mu_A^i - \mu_B^i| < \dfrac{2\bar{x}^i}{\gamma^i(\underline{\alpha^i} + \overline{\alpha^i} - 1)} \\ \\ \dfrac{N}{2}\ \bar{x}^i, & if \ \ \dfrac{2\bar{x}^i}{\gamma^i(\underline{\alpha^i} + \overline{\alpha^i} - 1)} \leq |\mu_A^i - \mu_B^i| \leq \dfrac{2\bar{x}^i}{\gamma^i(2\underline{\alpha^i} - 1)} \\ \\ \dfrac{N\gamma^i(2\underline{\alpha^i} - 1)}{4}|\mu_A^i - \mu_B^i|, & if \ \ |\mu_A^i - \mu_B^i| > \dfrac{2\bar{x}^i}{\gamma^i(2\underline{\alpha^i} - 1)} \end{cases} \tag{23}$$

Proof. See Appendix. □

Figure 8 illustrates the idea of Proposition 4. First, when investors are overconfident, the equilibrium price for risky asset i can be decreasing in, increasing in, or unrelated to investor disagreement, $|\mu_A^i - \mu_B^i|$, holding fixed the average interpretation of the public signal for risky asset i.

In particular, when disagreement is low, i.e, $|\mu_A^i - \mu_B^i| < \frac{2\bar{x}^i}{\gamma^i(\underline{\alpha^i}+\overline{\alpha^i}-1)}$, price for risky asset i is decreasing in disagreement. In this case, both optimistic and pessimistic traders go long in risky asset i and use their most conservative estimates of V^i. While A_{min}^i of optimistic traders exceeds $V_{benchmark}^i$, B_{min}^i of pessimistic traders is much lower than $V_{benchmark}^i$ and thus the average $(A_{min}^i + B_{min}^i)/2$ is lower than $V_{benchmark}^i$.

When disagreement is medium, i.e, $\frac{2\bar{x}^i}{\gamma^i(\underline{\alpha^i}+\overline{\alpha^i}-1)} \leq |\mu_A^i - \mu_B^i| \leq \frac{2\bar{x}^i}{\gamma^i(2\underline{\alpha^i}-1)}$, price for risky asset i is increasing in disagreement. In this case, only optimistic traders participate in the market and their most conservative estimate, A_{min}^i, is higher than $V_{benchmark}^i$.

When disagreement is high enough, i.e., $|\mu_A^i - \mu_B^i| > \frac{2\bar{x}^i}{\gamma^i(2\underline{\alpha^i}-1)}$, price for risky asset i is unrelated to disagreement. Optimistic traders go long in risky asset i while pessimistic go short in risky asset i. In particular, their average estimate of V^i equals to $V_{benchmark}^i$, so disagreement doesn't play a role in affecting the price.

On the other hand, trading volume is increasing in disagreement when both optimistic and pessimistic investors participate in the market. In addition, the model predicts that extremely high volume is caused by heavy selling. However, when disagreement is medium so that only optimistic traders participate in the market, volume is fixed at $\frac{N}{2}\bar{x}^i$ as optimistic traders buy out positions from pessimistic investors.

The case where investors are underconfident ($\underline{\alpha^i} + \overline{\alpha^i} < 1$ and $0 \leq \underline{\alpha^i} < \overline{\alpha^i} \leq 0.5$) is similar. Figure 9 shows that when $\mu_B^i > \mu_A^i$, $B^i_{max} > B^i_{min} > V^i_{benchmark} > A^i_{max} > A^i_{min}$.

Proposition 5. *When investors are underconfident, the abnormal price for risky asset i is*

$$\widehat{p^i} = \begin{cases} \dfrac{-\gamma^i}{\delta^i + \gamma^i}\dfrac{(\overline{\alpha^i} - \underline{\alpha^i})}{2}|\mu_A^i - \mu_B^i|, & \textit{if } \ |\mu_A^i - \mu_B^i| < \dfrac{2\bar{x}^i}{\gamma^i(1 - \underline{\alpha^i} - \overline{\alpha^i})} \\ \\ \dfrac{\gamma^i}{\delta^i + \gamma^i}\dfrac{(1 - 2\overline{\alpha^i})}{2}|\mu_A^i - \mu_B^i| - \dfrac{\bar{x}^i}{\delta^i + \gamma^i}, & \textit{if } \ \dfrac{2\bar{x}^i}{\gamma^i(1 - \underline{\alpha^i} - \overline{\alpha^i})} \leq |\mu_A^i - \mu_B^i| \leq \dfrac{2\bar{x}^i}{\gamma^i(1 - 2\overline{\alpha^i})} \\ \\ 0, & \textit{if } \ |\mu_A^i - \mu_B^i| > \dfrac{2\bar{x}^i}{\gamma^i(1 - 2\overline{\alpha^i})} \end{cases} \tag{24}$$

In addition, trading volume for risky asset i, is given by

$$Volume_i = \begin{cases} \dfrac{N\gamma^i(1 - \underline{\alpha^i} - \overline{\alpha^i})}{4}|\mu_A^i - \mu_B^i|, & \textit{if } \ |\mu_A^i - \mu_B^i| < \dfrac{2\bar{x}^i}{\gamma^i(1 - \underline{\alpha^i} - \overline{\alpha^i})} \\ \\ \dfrac{N}{2}\bar{x}^i, & \textit{if } \ \dfrac{2\bar{x}^i}{\gamma^i(1 - \underline{\alpha^i} - \overline{\alpha^i})} \leq |\mu_A^i - \mu_B^i| \leq \dfrac{2\bar{x}^i}{\gamma^i(1 - 2\overline{\alpha^i})} \\ \\ \dfrac{N\gamma^i(1 - 2\overline{\alpha^i})}{4}|\mu_A^i - \mu_B^i|, & \textit{if } \ |\mu_A^i - \mu_B^i| > \dfrac{2\bar{x}^i}{\gamma^i(1 - 2\overline{\alpha^i})} \end{cases} \tag{25}$$

Figure 10 illustrates the idea of Proposition 5. The results are similar to those of overconfident investors, with optimists and pessimists switching roles. Disagreement can be positively, negatively, or unrelated to abnormal price, while trading volume can be positively or unrelated to disagreement.

5 Empirical predictions

The equilibrium with overconfident or underconfident investors can explain many interesting empirical facts. However, as overconfidence is more likely to happen in reality, I use the case where investors are overconfident to discuss the predictions of the model.[22]

5.1 Volume-return relationship

Empirical evidence shows a positive correlation between volume and returns in the stock market. In particular, stock prices tend to rise on high volume but decline on low volume.[23]

I first define low and high trading volume as the volume that is below and above half of the outstanding shares. That is, for risky asset i, low volume is when $Volume^i < \frac{N\bar{x}^i}{2}$ and high volume is when $Volume^i \geq \frac{N\bar{x}^i}{2}$. From Figure 8, low volume occurs when disagreement is below $\frac{2\bar{x}^i}{\gamma^i(\underline{\alpha}^i+\overline{\alpha}^i-1)}$. Under this range of disagreement, abnormal price is decreasing in investor disagreement. Taken together, it means that stock price declines on low volume.

The intuition is that both optimists and pessimists go long in risky asset i and thus turnover ratio is never more than a half. In addition, optimists' most conservative estimate of the stock is not optimistic enough to balance that of pessimists, which eventually leads to a decrease in abnormal price.

Next, again from Figure 8, high volume occurs when disagreement is above $\frac{2\bar{x}^i}{\gamma^i(\underline{\alpha}^i+\overline{\alpha}^i-1)}$. Under this range of disagreement, price is either increasing in or unrelated to investor disagreement. Taken together, it means that stock price on average rises on high volume.

The intuition is that when optimistic investors go long in risky asset i while pessimistic ones sit out of the market, trading volume is fixed at one half of the total outstanding shares. In addition, market-clearing price is solely determined by optimists' most conservative estimate of the stock, which eventually generates an increase in abnormal price.

When investor disagreement is above $\frac{2\bar{x}^i}{\gamma^i(2\underline{\alpha}^i-1)}$ so that optimists go long and pessimists go short, there exists extremely large trading volume accompanied by zero price change.

[22]See, for example, Oskamp (1965), Scheinkman and Xiong (2003), Daniel et al. (2005), Van den Steen (2011), and Ortoleva and Snowberg (2015).

[23]See, for example,Ying (1966), Karpoff (1987), and Harris (1987).

Intuitively, optimist' most conservative estimate of the stock cancels out with pessimists' boldest estimate of the stock, and thus leaves the abnormal price unchanged. This situation corresponds to the findings in Kandel and Pearson (1995), where they document that stocks experience little or no price change at the time of their massive trading volume around earnings announcements. In addition, Bamber and Cheon (1995) document that trading is high relative to the magnitude of the price reaction when analysts forecasts are more diverse. Bailey et al. (2003) also find that after Regulation Fair Disclosure many earnings announcements have large trading reactions despite small price reactions.

5.2 Short selling and disagreement

Many papers use short sale constraints to discuss the relationship between disagreement and returns. For example, Mayshar (1983) and Miller (1977) argue that when there are constraints on short selling, pessimists cannot freely trade on their negative information. Hence, disagreement leads to overpricing in the presence of short-sale constraints. My model, on the other hand, doesn't require short sale constraints to generate overpricing. In fact, there's no shorting demand in the market when price is increasing in disagreement. If traders can't go short in the market, then pessimists can only go long or sit out of the market, with the former leading to negative abnormal return and the latter leading to positive abnormal return.

Short sale constraints seem to be more associated with the average interpretation of the signals when visibility is introduced into the context. Lee (1992), Gervais et al. (2001), Barber and Odean (2008), and Hirshleifer et al. (2008) propose the attention-grabbing hypothesis, which argues that individual investors rarely sell short and are net buyers of attention-grabbing stocks. Hence, the number of optimists increases while the number of pessimists remains virtually unchanged when a stock receives more attention, which generates high returns.

5.3 Volume and earnings announcements

Beaver (1968), Kiger (1972), and Morse (1981) document that trading volume tends to rise around earnings announcements. Graham et al. (2006) find that both anticipated and unanticipated announcements generate high trading volume. Several studies including Holthausen and Verrecchia (1990), Lee et al. (1993), Krinsky and Lee (1996), and Bamber et al. (1997) suggest that earnings announcements most likely generate dispersion in beliefs. While volume may proxy for more than disagreement, Garfinkel (2009) provides evidence that unexplained trading volume, which controls for both liquidity effect and informedness effect in volume, is high around earnings announcments.

Hence, I make a time-varying assumption on investor disagreement: disagreement is lower around non-announcement period but higher around announcements. Based on this assumption, as shown in Figure 8, volume is monotonically increasing in disagreement. In particular, when both optimists and pessimists participate in the market, volume is strictly increasing in disagreement.

6 Conclusion

This paper studies how disagreement is related to returns, volume, and the joint behavior between the two. Past studies typically assume that traders agree to disagree and ignore others' interpretations of signals, while in reality investors have many different ways to take account of others' interpretations.

I provide a behavioral framework that addresses this issue. In particular, investors can form a set of revised interpretations after observing others' interpretations. Based on this structure, I set up a multi-asset model in which optimistic and pessimistic investors interpret signals differently and characterize the corresponding equilibrium. The disagreement model provides insights on how disagreement shapes volume and returns when ambiguity-averse investors face ambiguity in a simple and tractable way.

For instance, when investors always value their own interpretations more, disagreement generates under-pricing at low disagreement while overpricing at medium disagreement.

When disagreement is extremely high, price is unrelated to disagreement. Trading volume, on the other hand, is strictly increasing in disagreement at low and high disagreement level, but remains fixed under medium disagreement level. Investors' long, short, or non-participating decisions are also specified.

The model can also speak to several empirical evidence. For example, stock prices rise on high volume but decline on volume, while extremely high volume can be accompanied by little price change. In addition, trading volume on average increases prior to or around earnings announcements.

The framework can be extended to a multi-period, multi-asset model that studies return and volume dynamics when additional assumptions on disagreement evolution are imposed. I leave this for future research.

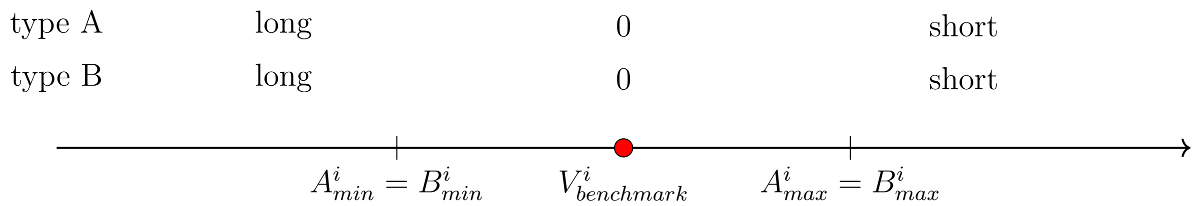

Figure 1. Unbiased investors. This figure plots the relative magnitude of A^i_{min}, A^i_{max}, B^i_{min}, B^i_{max}, and $V^i_{benchmark}$ when investors are unbiased. In addition, the figure shows the long, short, or non-participating positions investors take in each interval when the price p^i falls in that given interval.

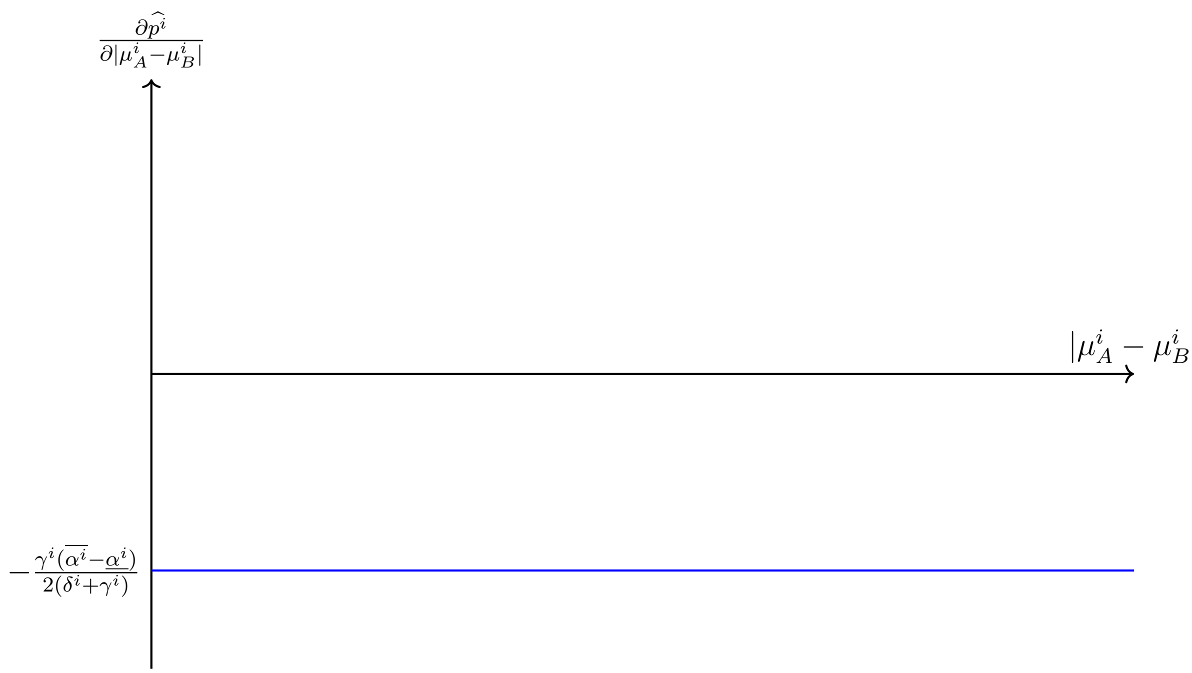

Figure 2. Unbiased investors: disagreement and price. This figure shows the relationship between risky asset i's price and disagreement when investors are unbiased. $\frac{\partial \widehat{p^i}}{\partial |\mu^i_A - \mu^i_B|}$ measures how price of risky asset i increases or decreases with respect to a unit increase in investor disagreement, holding the average interpretation of the public signal, $\frac{\mu^i_A + \mu^i_B}{2}$, constant. $|\mu^i_A - \mu^i_B|$ is disagreement for risky asset i.

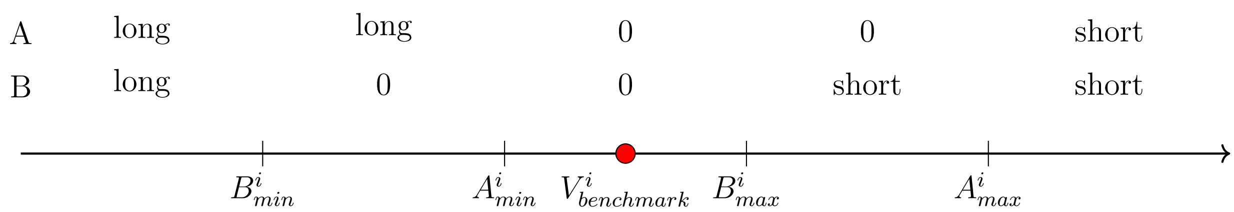

Figure 3. Slightly overconfident investors with $\mu^i_B > \mu^i_A$. This figure plots the relative magnitude of A^i_{min}, A^i_{max}, B^i_{min}, B^i_{max}, and $V^i_{benchmark}$ when investors are slightly overconfident under $\mu^i_B > \mu^i_A$ (type A investors are optimistic while type B investors are pessimistic). In addition, the figure shows the long, short, or non-participating positions investors take in each interval when the price p^i falls in that given interval.

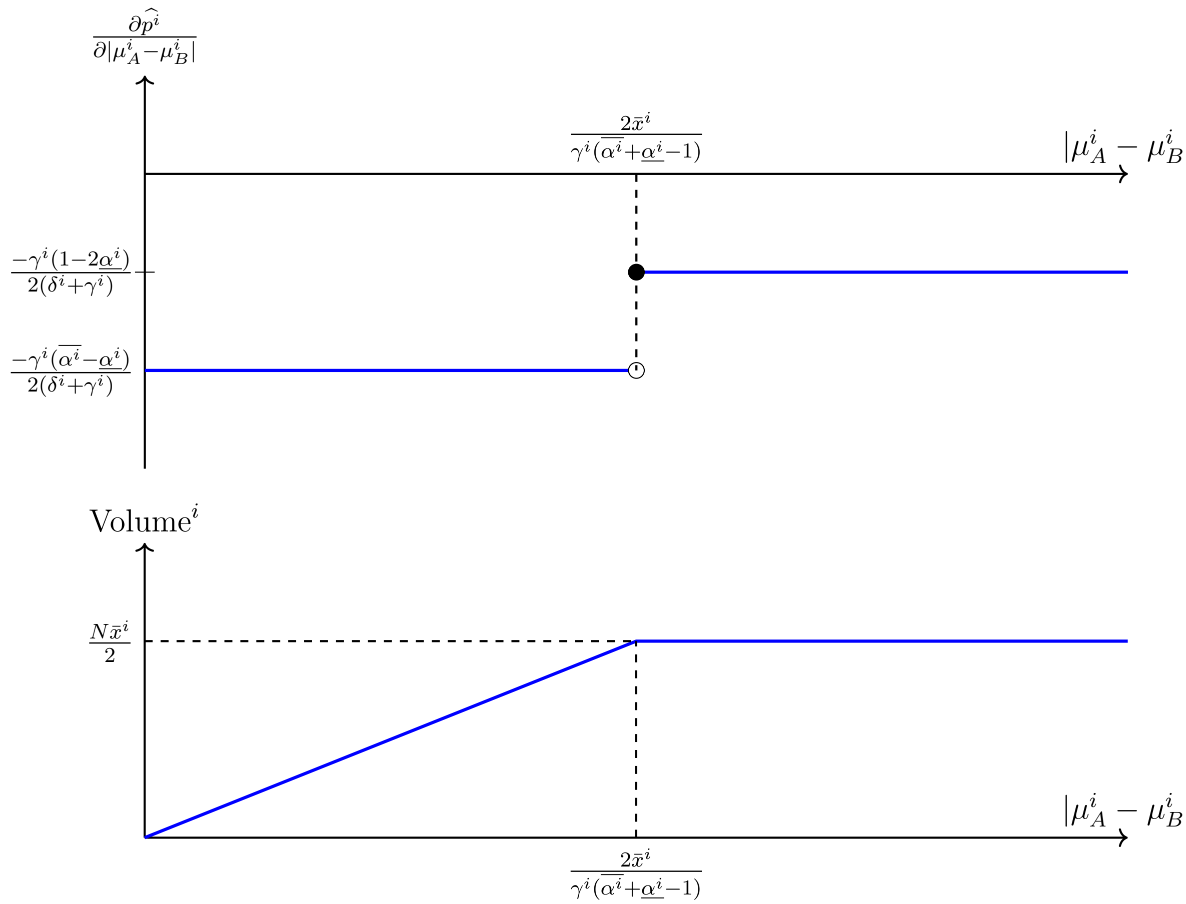

Figure 4. Slightly overconfident investors: disagreement, price, and volume. This figure shows the relationship between risky asset i's price and disagreement when investors are slightly overconfident. $\frac{\partial \widehat{p^i}}{\partial|\mu^i_A-\mu^i_B|}$ measures how price of risky asset i increases or decreases with respect to a unit increase in investor disagreement, holding the average interpretation of the public signal, $\frac{\mu^i_A+\mu^i_B}{2}$, constant. $|\mu^i_A-\mu^i_B|$ is disagreement for risky asset i.

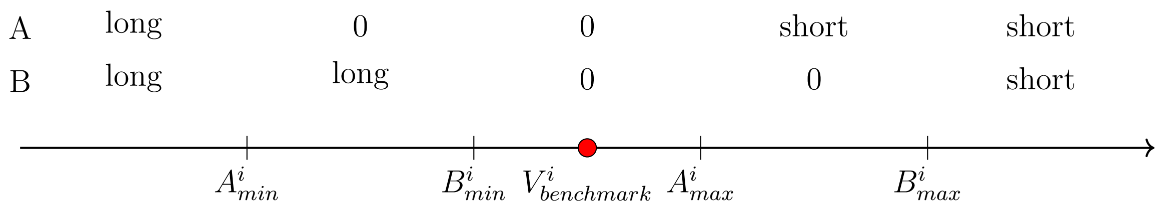

Figure 5. Slightly underconfident investors with $\mu^i_B > \mu^i_A$. This figure plots the relative magnitude of A^i_{min}, A^i_{max}, B^i_{min}, B^i_{max}, and $V^i_{benchmark}$ under $\mu^i_B > \mu^i_A$ (type A investors are optimistic while type B investors are pessimistic) when investors are slightly underconfident. In addition, the figure shows the long, short, or non-participating positions investors take in each interval when the price p^i falls in that given interval.

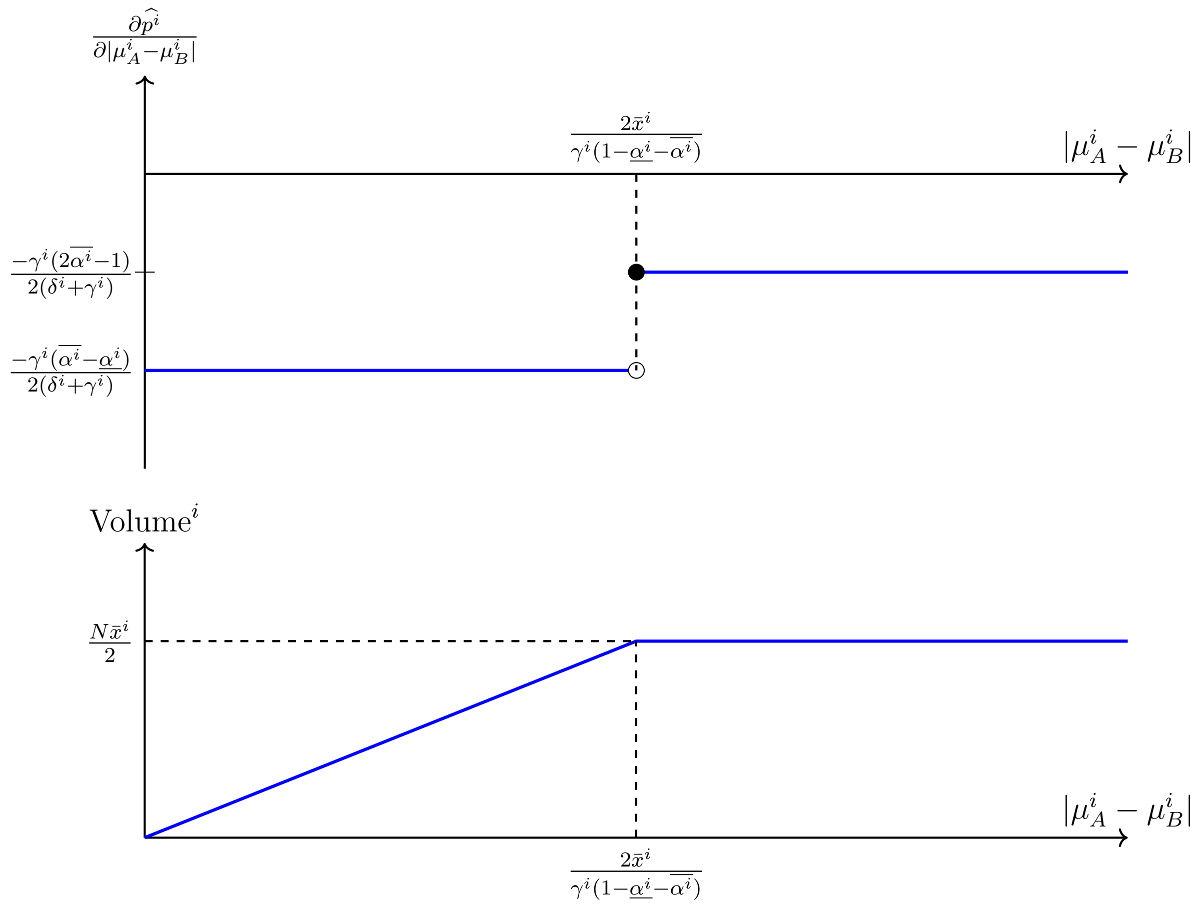

Figure 6. Slightly underconfident investors: disagreement, price, and volume. This figure shows the relationship between risky asset i's price and disagreement when investors are slightly underconfident. $\frac{\partial \widehat{p^i}}{\partial|\mu^i_A - \mu^i_B|}$ measures how price of risky asset i increases or decreases with respect to a unit increase in investor disagreement, holding the average interpretation of the public signal, $\frac{\mu^i_A+\mu^i_B}{2}$, constant. $|\mu^i_A - \mu^i_B|$ is disagreement for risky asset i.

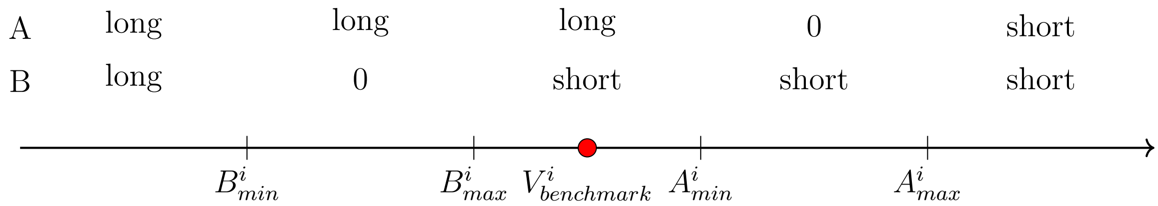

Figure 7. Overconfident investors with $\mu^i_B > \mu^i_A$. This figure plots the relative magnitude of A^i_{min}, A^i_{max}, B^i_{min}, B^i_{max}, and $V^i_{benchmark}$ under $\mu^i_B > \mu^i_A$ when investors are underconfident. In addition, the figure shows the long, short, or non-participating positions investors take in each interval when p^i falls in that given interval.

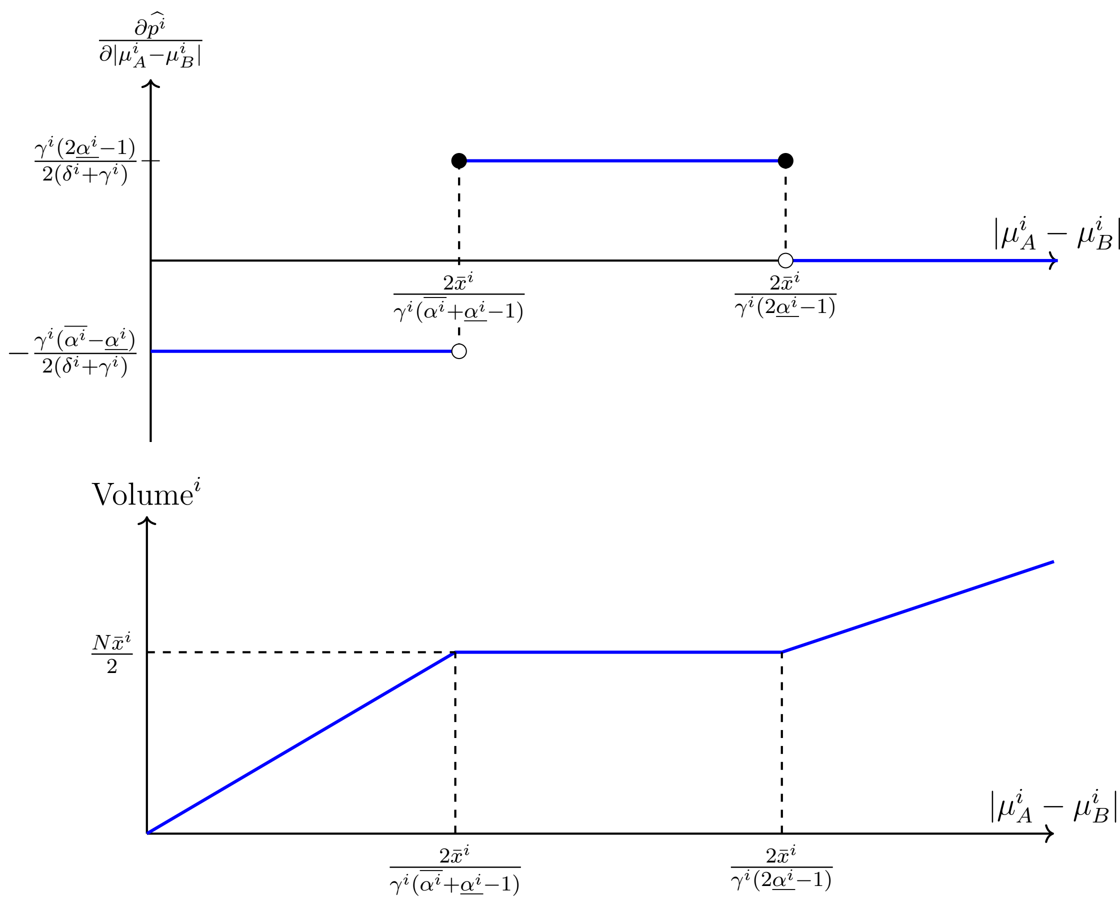

Figure 8. Overconfident investors: disagreement, price, and volume. This figure shows the relationship between risky asset i's price and disagreement when investors are overconfident. $\frac{\partial \widehat{p^i}}{\partial|\mu^i_A - \mu^i_B|}$ measures how price of risky asset i increases or decreases with respect to a unit increase in investor disagreement, holding the average interpretation of the public signal, $\frac{\mu^i_A + \mu^i_B}{2}$, constant. $|\mu^i_A - \mu^i_B|$ is disagreement for risky asset i.

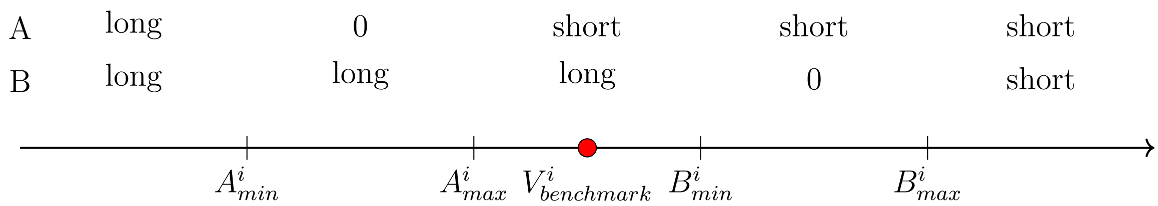

Figure 9. Underconfident investors with $\mu^i_B > \mu^i_A$. This figure plots the relative magnitude of A^i_{min}, A^i_{max}, B^i_{min}, B^i_{max}, and $V^i_{benchmark}$ under $\mu^i_B > \mu^i_A$ when investors are underconfident. In addition, the figure shows the long, short, or non-participating positions investors take in each interval when p^i falls in that given interval.

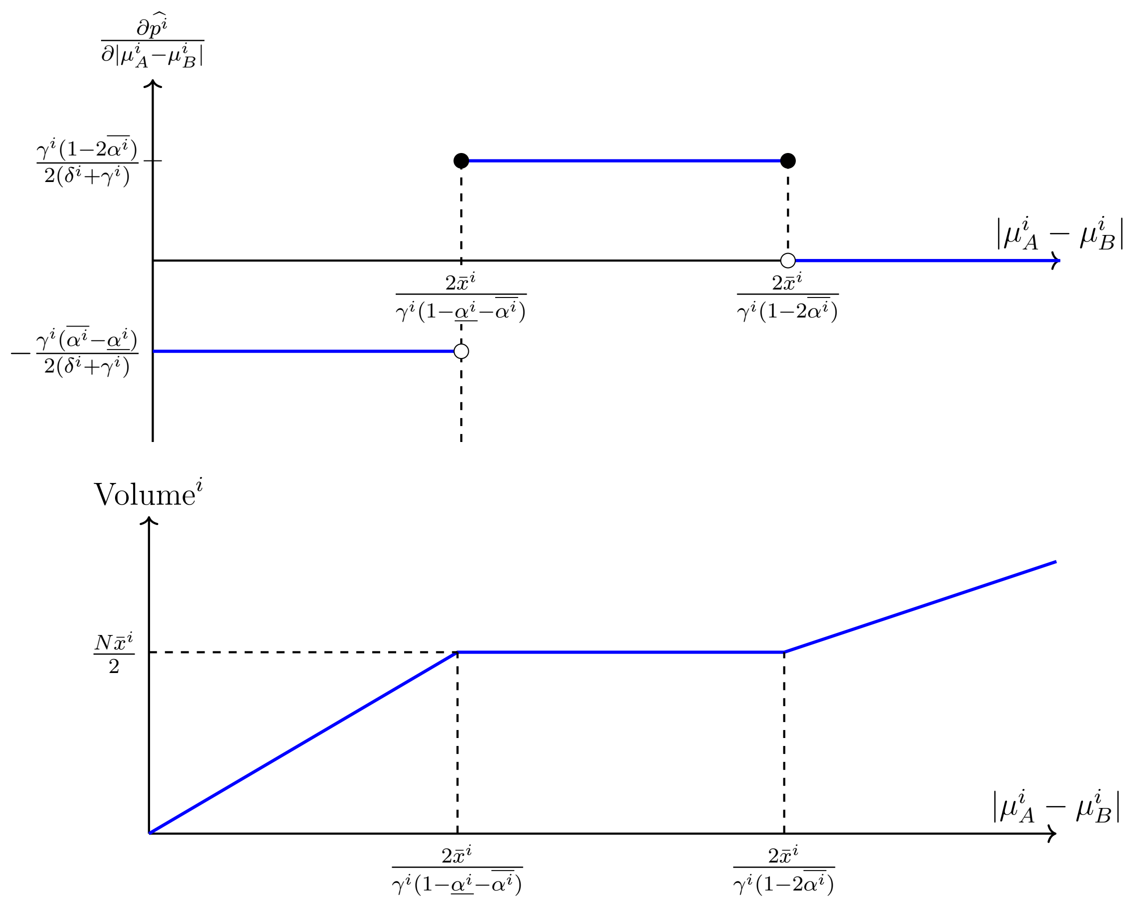

Figure 10. Underconfident investors: disagreement, price, and volume. This figure shows the relationship between risky asset i's price and disagreement when investors are underconfident. $\frac{\partial \widehat{p^i}}{\partial|\mu^i_A-\mu^i_B|}$ measures how price of risky asset i increases or decreases with respect to a unit increase in investor disagreement, holding the average interpretation of the public signal, $\frac{\mu^i_A+\mu^i_B}{2}$, constant. $|\mu^i_A - \mu^i_B|$ is disagreement for risky asset i.

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A Appendix

When $\mu_B^i > \mu_A^i$,

$$\begin{cases} A_{min}^i = \frac{\delta^i v^i + \gamma^i [S^i - (\underline{\alpha}^i \mu_A^i + (1-\underline{\alpha}^i)\mu_B^i)]}{\delta^i + \gamma^i} \\ B_{min}^i = \frac{\delta^i v^i + \gamma^i [S^i - (\overline{\alpha^i} \mu_B^i + (1-\overline{\alpha^i})\mu_A^i)]}{\delta^i + \gamma^i} \\ A_{max}^i = \frac{\delta^i v^i + \gamma^i [S^i - (\overline{\alpha^i} \mu_A^i + (1-\overline{\alpha^i})\mu_B^i)]}{\delta^i + \gamma^i} \\ B_{max}^i = \frac{\delta^i v^i + \gamma^i [S^i - (\underline{\alpha}^i \mu_B^i + (1-\underline{\alpha}^i)\mu_A^i)]}{\delta^i + \gamma^i}. \end{cases} \tag{26}$$

When $\mu_A^i > \mu_B^i$,

$$\begin{cases} A_{min}^i = \frac{\delta^i v^i + \gamma^i [S^i - (\overline{\alpha^i} \mu_A^i + (1-\overline{\alpha^i})\mu_B^i)]}{\delta^i + \gamma^i} \\ B_{min}^i = \frac{\delta^i v^i + \gamma^i [S^i - (\underline{\alpha}^i \mu_B^i + (1-\underline{\alpha}^i)\mu_A^i)]}{\delta^i + \gamma^i} \\ A_{max}^i = \frac{\delta^i v^i + \gamma^i [S^i - (\underline{\alpha}^i \mu_A^i + (1-\underline{\alpha}^i)\mu_B^i)]}{\delta^i + \gamma^i} \\ B_{max}^i = \frac{\delta^i v^i + \gamma^i [S^i - (\overline{\alpha^i} \mu_B^i + (1-\overline{\alpha^i})\mu_A^i)]}{\delta^i + \gamma^i}. \end{cases} \tag{27}$$

A.1 Proof of Proposition 1: unbiased investors

To begin with, assume $\mu_B^i > \mu_A^i$. First, if at a price lower than $A_{min}^i = B_{min}^i$, demand for risky asset i for type A and type B investors are

$$x_A^{i*} = (A_{min}^i - p^i)(\delta^i + \gamma^i) \text{ and } x_B^{i*} = (B_{min}^i - p^i)(\delta^i + \gamma^i), \tag{28}$$

respectively. Using equation (10), the market-clearing price is given by

$$p^{i^*} = \frac{1}{2}(A^i_{min} + B^i_{min} - 2\frac{\bar{x}^i}{\delta^i + \gamma^i}) \tag{29}$$

Thus, p^{i^*} will be the equilibrium market-clearing price for risky asset i if $p^{i^*} \leq A^i_{min} = B^i_{min}$, which automatically holds since $\bar{x}^i > 0$.

Second, if at a price between $A^i_{min} = B^i_{min}$ and $A^i_{max} = B^i_{max}$, both types of investors won't participate in the market for risky asset i. Hence, no equilibrium exists in this case. Lastly, if at a price higher than $A^i_{max} = B^i_{max}$, both types of traders want to go short so no equilibrium exists.

Plug in A^i_{min} and B^i_{min} into equation (28), the market-clearing price is

$$p^{i^*} = \frac{\delta^i v^i + \gamma^i [S^i - \frac{(\mu^i_A + \mu^i_B)}{2} - \frac{(\overline{\alpha^i} - \underline{\alpha^i})}{2}(\mu^i_B - \mu^i_A)]}{\delta^i + \gamma^i} - \frac{\bar{x}^i}{\delta^i + \gamma^i}. \tag{30}$$

Similarly, if $\mu^i_A > \mu^i_B$, then the market-clearing price is

$$p^{i^*} = \frac{\delta^i v^i + \gamma^i [S^i - \frac{(\mu^i_A + \mu^i_B)}{2} - \frac{(\overline{\alpha^i} - \underline{\alpha^i})}{2}(\mu^i_A - \mu^i_B)]}{\delta^i + \gamma^i} - \frac{\bar{x}^i}{\delta^i + \gamma^i}. \tag{31}$$

Combining the two cases, we have

$$p^{i^*} = \frac{\delta^i v^i + \gamma^i [S^i - \frac{(\mu^i_A + \mu^i_B)}{2} - \frac{(\overline{\alpha^i} - \underline{\alpha^i})}{2}|\mu^i_A - \mu^i_B|]}{\delta^i + \gamma^i} - \frac{\bar{x}^i}{\delta^i + \gamma^i}. \tag{32}$$

Since investors have same the same priors and posteriors on V_i, there's no trading activity.

A.2 Proof of Proposition 2: slightly overconfident investors

To begin with, assume $\mu^i_B > \mu^i_A$. From equation (25), we know that

$$B^i_{min} < A^i_{min} < B^i_{max} < A^i_{max}. \tag{33}$$

First, if $p^{i^*} < B^i_{min}$, type A and type B investors' demands for risky asset i are

$$x^{i^*}_A = (A^i_{min} - p^i)(\delta^i + \gamma^i) \text{ and } x^{i^*}_B = (B^i_{min} - p^i)(\delta^i + \gamma^i), \tag{34}$$

respectively. The market-clearing price is given by

$$p^{i^*} = \frac{1}{2}(A^i_{min} + B^i_{min} - 2\frac{\bar{x}^i}{\delta^i + \gamma^i}). \tag{35}$$

Note that p^{i^*} will be the market-clearing price for risky asset i if $p^{i^*} < A^i_{min}$, which is equivalent to

$$\gamma^i(\overline{\alpha^i} + \underline{\alpha}^i - 1)(\mu^i_B - \mu^i_A) < 2\bar{x}^i. \tag{36}$$

Second, if $B^i_{min} \leq p^i < A^i_{min}$, only type A investors participate in the market as type B investors prefer zero position:

$$x^{i^*}_A = (A^i_{min} - p^i)(\delta^i + \gamma^i),\ x^{i^*}_B = 0. \tag{37}$$

The market-clearing price is given by

$$p^{i^*} = A^i_{min} - 2\frac{\bar{x}^i}{\delta^i + \gamma^i} \tag{38}$$

Note that p^{i^*} will be the market-clearing price for risky asset i if $B^i_{min} \leq p^i < A^i_{min}$, which is equivalent to

$$\gamma^i(\overline{\alpha^i} + \underline{\alpha}^i - 1)(\mu^i_B - \mu^i_A) \geq 2\bar{x}^i. \tag{39}$$

Third, if $A^i_{min} \leq p^i \leq B^i_{max}$, both types of investors will not participate in the market, so no equilibrium exists. Fourth, if $B^i_{max} < p^i \leq A^i_{max}$, only type B investors go short in risky asset i as type A investors prefer zero position. Hence, no equilibrium exists in this case. Lastly, if $p^i > A^i_{max}$, both types of investors go short so there's no equilibrium. Based on the above analysis, when $\mu_B^i > \mu_A^i$, the market-clearing price is

$$p^{i^*} = \begin{cases} \dfrac{\delta^i v^i + \gamma^i[S^i - \frac{(\mu_A^i+\mu_B^i)}{2} - \frac{(\overline{\alpha^i}-\underline{\alpha}^i)}{2}(\mu_B^i - \mu_A^i)]}{\delta^i + \gamma^i} - \dfrac{\bar{x}^i}{\delta^i + \gamma^i}, & \text{if } \gamma^i(\overline{\alpha^i} + \underline{\alpha}^i - 1)(\mu_B^i - \mu_A^i) < 2\bar{x}^i \\ \\ \dfrac{\delta^i v^i + \gamma^i[S^i - \frac{(\mu_A^i+\mu_B^i)}{2} - (\frac{1}{2} - \underline{\alpha}^i)(\mu_B^i - \mu_A^i)]}{\delta^i + \gamma^i} - 2\dfrac{\bar{x}^i}{\delta^i + \gamma^i}, & \text{if } \gamma^i(\overline{\alpha^i} + \underline{\alpha}^i - 1)(\mu_B^i - \mu_A^i) \geq 2\bar{x}^i. \end{cases} \tag{40}$$

Similarly, if $\mu_A^i > \mu_B^i$, then the market-clearing price is

$$p^{i^*} = \begin{cases} \dfrac{\delta^i v^i + \gamma^i[S^i - \frac{(\mu_A^i+\mu_B^i)}{2} - \frac{(\overline{\alpha^i}-\underline{\alpha}^i)}{2}(\mu_A^i - \mu_B^i)]}{\delta^i + \gamma^i} - \dfrac{\bar{x}^i}{\delta^i + \gamma^i}, & \text{if } \gamma^i(\overline{\alpha^i} + \underline{\alpha}^i - 1)(\mu_A^i - \mu_B^i) < 2\bar{x}^i \\ \\ \dfrac{\delta^i v^i + \gamma^i[S^i - \frac{(\mu_A^i+\mu_B^i)}{2} - (\frac{1}{2} - \underline{\alpha}^i)(\mu_A^i - \mu_B^i)]}{\delta^i + \gamma^i} - 2\dfrac{\bar{x}^i}{\delta^i + \gamma^i}, & \text{if } \gamma^i(\overline{\alpha^i} + \underline{\alpha}^i - 1)(\mu_A^i - \mu_B^i) \geq 2\bar{x}^i. \end{cases} \tag{41}$$

Combining the two cases, we have

$$p^{i^*} = \begin{cases} \dfrac{\delta^i v^i + \gamma^i[S^i - \frac{(\mu_A^i+\mu_B^i)}{2} - \frac{(\overline{\alpha^i}-\underline{\alpha}^i)}{2}|\mu_A^i - \mu_B^i|]}{\delta^i + \gamma^i} - \dfrac{\bar{x}^i}{\delta^i + \gamma^i}, & \text{if } |\mu_A^i - \mu_B^i| < \frac{2\bar{x}^i}{\gamma^i(\overline{\alpha^i}+\underline{\alpha}^i-1)} \\ \\ \dfrac{\delta^i v^i + \gamma^i[S^i - \frac{(\mu_A^i+\mu_B^i)}{2} - (\frac{1}{2} - \underline{\alpha}^i)|\mu_A^i - \mu_B^i|]}{\delta^i + \gamma^i} - 2\dfrac{\bar{x}^i}{\delta^i + \gamma^i}, & \text{if } |\mu_A^i - \mu_B^i| \geq \frac{2\bar{x}^i}{\gamma^i(\overline{\alpha^i}+\underline{\alpha}^i-1)} \end{cases} \tag{42}$$

Trading volume can be computed accordingly using

$$\mathit{Volume}^i = \frac{N}{2}|x_A^{i^*}(p^{i^*}) - \bar{x}^i| = \frac{N}{2}|x_B^{i^*}(p^{i^*}) - \bar{x}^i|. \tag{43}$$

A.3 Proof of Proposition 4: overconfident investors

To begin with, assume $\mu_B^i > \mu_A^i$. From equation (26), we know that

$$B_{min}^i < B_{max}^i < A_{min}^i < A_{max}^i. \tag{44}$$

First, if $p^{i^*} < B^i_{min}$, type A and type B investors' demands for risky asset i are

$$x^{i^*}_A = (A^i_{min} - p^i)(\delta^i + \gamma^i) \text{ and } x^{i^*}_B = (B^i_{min} - p^i)(\delta^i + \gamma^i), \tag{45}$$

respectively. The market-clearing price is given by

$$p^{i^*} = \frac{1}{2}(A^i_{min} + B^i_{min} - 2\frac{\bar{x}^i}{\delta^i + \gamma^i}). \tag{46}$$

Note that p^{i^*} will be the market-clearing price for risky asset i if $p^{i^*} < B^i_{min}$, which is equivalent to

$$\gamma^i(\underline{\alpha^i} + \overline{\alpha^i} - 1)(\mu^i_B - \mu^i_A) < 2\bar{x}^i. \tag{47}$$

Second, if $B^i_{min} \leq p^i \leq B^i_{max}$, only type A investors participate in the market as type B investors prefer zero position:

$$x^{i^*}_A = (A^i_{min} - p^i)(\delta^i + \gamma^i),\ x^{i^*}_B = 0. \tag{48}$$

The market-clearing price is given by

$$p^{i^*} = A^i_{min} - 2\frac{\bar{x}^i}{\delta^i + \gamma^i}. \tag{49}$$

Note that p^{i^*} will be the market-clearing price for risky asset i if $B^i_{min} \leq p^i < B^i_{max}$, which is equivalent to

$$\gamma^i(2\underline{\alpha^i} - 1)(\mu^i_B - \mu^i_A) \leq 2\bar{x}^i \leq \gamma^i(\underline{\alpha^i} + \overline{\alpha^i} - 1)(\mu^i_B - \mu^i_A). \tag{50}$$

Third, if $B^i_{max} < p^i < A^i_{min}$, type A and type B investors' demands for risky asset i are

$$x^{i^*}_A = (A^i_{min} - p^i)(\delta^i + \gamma^i) \text{ and } x^{i^*}_B = (B^i_{max} - p^i)(\delta^i + \gamma^i), \tag{51}$$

respectively. The market-clearing price is given by

$$p^{i^*} = \frac{1}{2}(A^i_{min} + B^i_{max} - 2\frac{\bar{x}^i}{\delta^i + \gamma^i}). \tag{52}$$

Note that p^{i^*} will be the market-clearing price for risky asset i if $B^i_{max} < p^i < A^i_{min}$, which is equivalent to

$$\gamma^i(2\underline{\alpha}^i - 1)(\mu^i_B - \mu^i_A) > 2\bar{x}^i. \tag{53}$$

Fourth, if $A^i_{min} < p^i \le A^i_{max}$, only type B investors go short in risky asset i in the market as type A investors prefer zero position. Hence, no equilibrium exists in this case. Lastly, if $p^i > A^i_{max}$, both types of investors want to go short in risky asset i so no equilibrium exists. Based on the above analysis, when $\mu^i_B > \mu^i_A$, the market-clearing price is

$$p^{i^*} = \begin{cases} \dfrac{\delta^i v^i + \gamma^i[S^i - \frac{(\mu^i_A + \mu^i_B)}{2}]}{\delta^i + \gamma^i} - \dfrac{\bar{x}^i}{\delta^i + \gamma^i}, \text{ if } \ 2\bar{x}^i < \gamma^i(2\underline{\alpha}^i - 1)(\mu^i_B - \mu^i_A) \\ \\ \dfrac{\delta^i v^i + \gamma^i[S^i - \frac{(\mu^i_A + \mu^i_B)}{2} + (\underline{\alpha}^i - \frac{1}{2})(\mu^i_B - \mu^i_A)]}{\delta^i + \gamma^i} - 2\dfrac{\bar{x}^i}{\delta^i + \gamma^i}, \\ \text{if } \ \gamma^i(2\underline{\alpha}^i - 1)(\mu^i_B - \mu^i_A) \le 2\bar{x}^i \le \gamma^i(\underline{\alpha}^i + \overline{\alpha^i} - 1)(\mu^i_B - \mu^i_A) \\ \\ \dfrac{\delta^i v^i + \gamma^i[S^i - \frac{(\mu^i_A + \mu^i_B)}{2} - \frac{(\overline{\alpha^i} - \underline{\alpha}^i)}{2}(\mu^i_B - \mu^i_A)]}{\delta^i + \gamma^i} - \dfrac{\bar{x}^i}{\delta^i + \gamma^i}, \text{ if } \ 2\bar{x}^i > \gamma^i(\underline{\alpha}^i + \overline{\alpha^i} - 1)(\mu^i_B - \mu^i_A) \end{cases} \tag{54}$$

Similarly, if $\mu_A^i > \mu_B^i$, then the market-clearing price is

$$p^{i^*} = \begin{cases} \dfrac{\delta^i v^i + \gamma^i [S^i - \frac{(\mu_A^i + \mu_B^i)}{2}]}{\delta^i + \gamma^i} - \dfrac{\bar{x}^i}{\delta^i + \gamma^i}, \text{ if } \; 2\bar{x}^i < \gamma^i (2\underline{\alpha}^i - 1)(\mu_A^i - \mu_B^i) \\ \\ \dfrac{\delta^i v^i + \gamma^i [S^i - \frac{(\mu_A^i + \mu_B^i)}{2} + (\underline{\alpha}^i - \frac{1}{2})(\mu_A^i - \mu_B^i)]}{\delta^i + \gamma^i} - 2\dfrac{\bar{x}^i}{\delta^i + \gamma^i}, \\ \text{if } \; \gamma^i (2\underline{\alpha}^i - 1)(\mu_A^i - \mu_B^i) \leq 2\bar{x}^i \leq \gamma^i (\underline{\alpha}^i + \overline{\alpha^i} - 1)(\mu_A^i - \mu_B^i) \\ \\ \dfrac{\delta^i v^i + \gamma^i [S^i - \frac{(\mu_A^i + \mu_B^i)}{2} - \frac{(\overline{\alpha^i} - \underline{\alpha}^i)}{2}(\mu_A^i - \mu_B^i)]}{\delta^i + \gamma^i} - \dfrac{\bar{x}^i}{\delta^i + \gamma^i}, \text{ if } \; 2\bar{x}^i > \gamma^i (\underline{\alpha}^i + \overline{\alpha^i} - 1)(\mu_A^i - \mu_B^i) \end{cases} \tag{55}$$

Combining the two cases, we have

$$p^{i^*} = \begin{cases} \dfrac{\delta^i v^i + \gamma^i [S^i - \frac{(\mu_A^i + \mu_B^i)}{2}]}{\delta^i + \gamma^i} - \dfrac{\bar{x}^i}{\delta^i + \gamma^i}, \text{ if } \; |\mu_A^i - \mu_B^i| > \dfrac{2\bar{x}^i}{\gamma^i (2\underline{\alpha}^i - 1)} \\ \\ \dfrac{\delta^i v^i + \gamma^i [S^i - \frac{(\mu_A^i + \mu_B^i)}{2} + (\underline{\alpha}^i - \frac{1}{2})|\mu_A^i - \mu_B^i|]}{\delta^i + \gamma^i} - 2\dfrac{\bar{x}^i}{\delta^i + \gamma^i}, \\ \text{if } \; \frac{2\bar{x}^i}{\gamma^i (\underline{\alpha}^i + \overline{\alpha^i} - 1)} \leq |\mu_A^i - \mu_B^i| \leq \frac{2\bar{x}^i}{\gamma^i (2\underline{\alpha}^i - 1)} \\ \\ \dfrac{\delta^i v^i + \gamma^i [S^i - \frac{(\mu_A^i + \mu_B^i)}{2} - \frac{(\overline{\alpha^i} - \underline{\alpha}^i)}{2}|\mu_A^i - \mu_B^i|]}{\delta^i + \gamma^i} - \dfrac{\bar{x}^i}{\delta^i + \gamma^i}, \text{ if } \; |\mu_A^i - \mu_B^i| < \dfrac{2\bar{x}^i}{\gamma^i (\underline{\alpha}^i + \overline{\alpha^i} - 1)} \end{cases} \tag{56}$$

Trading volume can be computed accordingly using

$$Volume^i = \frac{N}{2} |x_A^{i^*}(p^{i^*}) - \bar{x}^i| = \frac{N}{2} |x_B^{i^*}(p^{i^*}) - \bar{x}^i|. \tag{57}$$