ALBERT & LAWRENCE
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6

1.

a) $\dfrac{11!}{4!\,3!\,2!\,2!}$ } ways of ordering 11 elements. } accounting for the groups of indistinguishable elements.

b) $\underbrace{\binom{5}{4}}_{\text{choose 4 spots out of first 5 for pink}} \cdot \underbrace{\binom{7}{3,2,2}}_{\text{choose remaining spots.}}$

90 + 60

$\binom{n}{k,\ n-k}$

c) $\underbrace{\_\ \_\ \_}_{\text{1 of the blue.}}\ \_\ \_\ \_\ \_\ \_\ \underbrace{\_\ \_\ \_}_{\text{remaining 2 blue}}$

$\dfrac{n!}{k!\,(}$

$\underbrace{3}_{\text{three possibilities for placing blue bottle on left}} \cdot \underbrace{\binom{3}{2}}_{\text{possible ways of arranging 2 remaining blue bottles}} \cdot \underbrace{\binom{8}{4,2,2}}_{\text{ways of arranging remaining spots:}}$

2. 8 professors 9 students, committee 5 prof 3 students

a) $\underbrace{1\cdot\binom{6}{4}\binom{9}{3}}_{\text{A in committee, B is not}} + \underbrace{1\cdot\binom{6}{4}\binom{9}{3}}_{\text{B in committee, A is not}} + \underbrace{\binom{6}{5}\binom{9}{3}}_{\text{neither A nor B in committee.}}$

Say prof A & prof B don't get along

b) Student A & B don't get along

$$\underbrace{\binom{8}{5}}_{\substack{\text{choose} \\ \text{5 profs} \\ \text{from a group} \\ \text{of 8}}}\left[\underbrace{1\cdot\binom{7}{2}}_{\substack{\text{A not} \\ \text{B}}} + \underbrace{1\cdot\binom{7}{2}}_{\substack{\text{B not} \\ \text{A}}} + \underbrace{\binom{7}{3}}_{\substack{\text{neither A} \\ \text{nor} \\ \text{B}}}\right]$$

c) Prof A and student B don't get along.

$$\underbrace{1\cdot\binom{7}{4}\binom{8}{3}}_{\substack{\text{Prof A} \\ \text{but not} \\ \text{student B}}} + \underbrace{1\cdot\binom{7}{5}\binom{8}{2}}_{\substack{\text{Student B} \\ \text{but not} \\ \text{Prof A}}} + \underbrace{\binom{7}{5}\binom{8}{3}}_{\substack{\text{neither} \\ \text{Student B or} \\ \text{prof A.}}}$$

3.

a) $12! \Rightarrow$ 12 choices for 1st spot, 11 for next and so on...

b) $\underbrace{2}_{\substack{\text{ways of} \\ \text{ordering A, B}}} \times \underbrace{11!}_{\substack{\text{ways in which} \\ \text{you can sit them}}}$

c) $\underset{m}{12} \cdot \underset{w}{6} \cdot 5 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 =$

$$= 2 \cdot 6 \cdot 6 \cdot 5 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = \underset{\substack{\uparrow \\ \text{start with} \\ \text{men or} \\ \text{with women}}}{2}\,\underbrace{(6!)^2}_{\substack{\text{ways of ordering} \\ \text{men or women}}}$$

d) _ _ _ _ _ _ _ _

8 spots to stick the men

$8 \cdot 7! \; 5! = 8!\,5!$

women order themselves

ways in which the men can order themselves

e) $2^6 \cdot 6!$ → ways to order 6 couples.

each couple can be ordered in 2 ways

4)

a) 7 kittens to 5 destinations.

total no. ways of kittens & destinations

$$\binom{\overbrace{7+5-1}}{7,\ 4}$$

choosing the 7 kittens.

choosing boundaries btn destinations.

– | – – | – | – – | –

boundaries of destinations.

b) 4 kittens · 5 destinations.

3 puppies.

$$\binom{11}{3,\ 4,\ 4}$$

total number of pets + destination − 1

choose puppies

choose kittens

choose destination −1.

5

a) Minimal investment

1+2+3+4 = $10 million dollars

$10 million left to invest with 4 companies. Use multinomial divider.

(k − 1)

— | — — — | — — — | — — —

$\binom{13}{10,\,3}$ total num of investments & dividers

+ $\binom{13}{10,\,3}$ all 4 from

c) $\underbrace{\binom{(20-6)+2}{14,\ 2}}_{\text{don't invest in 4}}$ + $\underbrace{\binom{(20-7)+2}{13,\ 2}}_{\text{don't invest in 3}}$ + $\underbrace{\binom{(20-8)+2}{12,\ 2}}_{\text{don't invest in 2}}$ + $\underbrace{\binom{(20-9)+2}{11,\ 2}}_{\text{don't invest in 1}}$

dividers − 1

minimal investment $6 million

minimal investment $7 million

minimal investment $8 million

minimal investment $9 million

6 $x[k] \in [1, n]$

$\underbrace{x[0]}_{\text{smallest}} < x[1] < \dots < \underbrace{x[k-1]}_{\text{largest int in the array}}$

k elements

k

— — — ... — —

n−(k−1) n−(k+2)

n−k+1

choices since we need k − 1 larger numbers

[1, n]

4

Simply pick groups of $\boxed{\binom{n}{k}}$ $\frac{n!}{k!\,(n-k)!}$

since order does not matter

— — — — — — — — — —

7. $\sum_{i=1}^{n} x_i \leq k.$

$$\overbrace{\underbrace{- \mid - \; - \mid \; \cdots \; - \mid - \mid -}_{k \text{ slots.}}}^{n-1 \text{ dividers.}}$$

$\sum_{i=1}^{n} x_i = k \quad \left\{ \binom{k+n-1}{k,\ n-1} \right\}$ ← Start with ways of pick vector such that $\sum_{i=1}^{n} x_i = k$.

$$\sum \left| \begin{array}{l} \text{No of ways} \\ \sum_{i=1}^{n} x_i \leq k \end{array} \right. = \begin{array}{c} \text{No of ways} \\ \sum_{i=1}^{n} x_i = k \end{array} + \begin{array}{c} \text{No of ways} \\ \sum_{i=1}^{n} x_i = k-1 \end{array} + \cdots + \begin{array}{c} \text{No of ways} \\ \sum_{i=1}^{n} x_i = 0 \end{array}$$

$$\boxed{\sum_{i=0}^{k} \binom{i+n-1}{i,\ n-1}} = \binom{n+k}{n}$$

We can say that the no of ways $\sum_{i=1}^{n} x_i \leq k = \sum_{i=1}^{n} x_i = k + \sum_{i=1}^{n} x_i = k-1$

$+ \cdots + \sum_{i=1}^{n} x_i = 1.$

8.

a) A B B C C C C

$\underset{1st}{\uparrow}$ $\underset{2nd}{\uparrow}$ $\underset{3rd}{\uparrow}$

$|S| = 6^7$

E be this event

$$|E| = \underbrace{\binom{7}{1,2,4}}_{\substack{\text{ways of arranging the}\\ \text{7 rolls}}} \cdot \underset{\substack{\text{choosing}\\ \text{the 1st}\\ \text{number}}}{6} \cdot \underset{\substack{\text{choosing}\\ \text{2nd}\\ \text{number}}}{5} \cdot 4 \leftarrow \text{Choosing 3rd number}$$

$$P(E) = \frac{|E|}{|S|} = \frac{6 \cdot 5 \cdot 4 \binom{7}{1,2,4}}{6^7}$$

b) 1 1 1 A A A B. A is distinct from B and 1.

$|S| = 6^7$

$$|E| = \underset{\substack{\text{choosing}\\ A}}{5} \cdot \underset{\substack{\text{choosing}\\ B}}{4} \cdot \underbrace{\binom{7}{3,3,1}}_{\text{ordering the 7 rolls}}$$

$$P(E) = \frac{|E|}{|S|} = \frac{5 \cdot 4 \cdot \binom{7}{3,3,1}}{6^7}$$

9.

a) $\underline{k \text{ attempts}}$. P(The candidate password selected logs into system) = $1/n$.

$n-k+1$

$n-k+1$

$\underline{k-1 \text{ failed attempts}}$.

$$\underbrace{\frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{n-3}{n-2} \cdots \frac{n-(k-1)}{n-(k-2)}}_{\substack{k-1 \text{ failed}\\ \text{attempts}}} \cdot \underbrace{\frac{1}{n-(k-1)}}_{\substack{k^{th} \text{ attempt}\\ \text{successful}}} = \boxed{\frac{1}{n}}$$

b) $\underbrace{\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{k-1} = \frac{k-1}{n}.$ } get it in less than k attempts

$\boxed{1 - \frac{k-1}{n}}$ which is the complement of ways in which we can get it in k attempts.

c) $\underbrace{\left(\frac{n-1}{n}\right)^{k-1}}_{\text{failed 1st } k-1 \text{ attempts}} \cdot \underbrace{\frac{1}{n}}_{\text{get it on kth attempt}}.$

$\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{2}$ $8!5!$

10.

a) flush.

$$|E| = \frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!} = \underbrace{4}_{\text{choice for suit}} \cdot \underbrace{\frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}}_{\text{pick 5 cards}} = 4\binom{13}{5}$$

$$|S| = \binom{52}{5}$$

$\binom{52}{2}\binom{50}{3}$

$$P(E) = \frac{\overset{\text{choose suit}}{4} \cdot \binom{13}{5} \}\text{ choosing 5 cards of same suit}}{\binom{52}{5} \}\text{ number of ways of choosing 5 cards.}}$$

52 - 2

b) $$|E| = \underbrace{\binom{13}{1}}_{\text{ways of choosing a number}} \underbrace{\binom{4}{2}}_{\text{out of suit pick 2 cards}} \underbrace{\binom{12}{3}}_{\text{pick 3 cards from remaining 12 numbers}} \underbrace{\binom{4}{1}^3}_{\text{pick suits for each of the cards independently}}$$

$$|S| = \binom{52}{5}$$

$$P(E) = \frac{|E|}{|S|} = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3}{\binom{52}{5}}$$

c) 4 of a kind

$$|E| = \underbrace{\binom{13}{1}}_{\text{choose a number}} \underbrace{\binom{4}{4}}_{\text{pick all 4 cards}} \underbrace{\binom{12}{1}}_{\text{pick second number}} \underbrace{\binom{4}{1}}_{\text{pick second suit}}$$

$$|S| = \binom{52}{5}$$

$$p(E) = \frac{|E|}{|S|} = \frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}}$$

11.

a) $|E| = 2^{n-1}$ $\qquad P(E) = \frac{2^{n-1}}{n!}$

$|S| = n!$

b) $\frac{2^{n-1}}{n!} = 0.01$

$\boxed{n = 8}$

n	Prob
$n=1$	1
$n=2$	1
$n=3$	2/3
$n=4$	1/3
$n=5$	0.1333
$n=6$	0.044
$n=7$	0.012
$n=8$	0.003

12. Since students are distinguishable

$$|E| = \underbrace{\binom{9}{4}}_{\substack{\text{pick 4 TAs that}\\ \text{will have groups of 4}}}\underbrace{\binom{5}{3}}_{\substack{\text{3 TAs with groups}\\ \text{of 5}}}\underbrace{\binom{40}{4}\binom{36}{4}\binom{32}{4}\binom{28}{4}\binom{24}{8}\binom{16}{8}\binom{8}{8}}_{\text{ways of picking groups of students}}$$

$|S| = \binom{40+9-1}{40,8}$ ways of distributing 40 students between 9 TAs

$$\therefore P(E) = \frac{|E|}{|S|} = \frac{\binom{9}{4}\binom{5}{3}\binom{40}{4}\binom{36}{4}\binom{32}{4}\binom{28}{4}\binom{24}{8}\binom{16}{8}\binom{8}{8}}{\binom{48}{40,8}}$$

13. 7 golden tickets in n chocolate bars.

$k \leq n$ chocolate bars.

$$|E| = \underbrace{\binom{7}{7}}_{\substack{\text{pick 7 bars}\\ \text{that have the}\\ \text{golden tickets}}}\underbrace{\binom{n-7}{k-7}}_{\substack{\text{pick the}\\ \text{remaining}\\ \text{bars to make up } k}} \qquad |S| = \underbrace{\binom{n}{k}}_{\substack{\text{ways of choosing}\\ k \text{ bars.}}}$$

$$P(E) = \frac{|E|}{|S|} \quad \frac{\binom{n-7}{k-7}}{\binom{n}{k}}.$$

14.

Say the largest number rolled is k and I rolled it

a) $\underbrace{\underset{\substack{\text{picked the}\\ \text{largest die}}}{\underline{1}}\ \underset{\substack{\text{can be less or equal}}}{\underline{k}}}_{\substack{\text{my die rolls}\\ \text{choices}}} \Bigg| \underbrace{\frac{k(k+1)}{2}}_{\text{roommates.}}$ > number of choices since order doesn't matter (subtract repeats off from k^2)

$$|E| = \sum_{k=1}^{6} k^2 \frac{(k+1)}{2}$$

is the sum that the largest no rolled is 1, 2, 3, 4, 5, 6 and I rolled it.

$$|S| = \underbrace{\left(\frac{6(6+1)}{2}\right)^2}_{\substack{\text{ways of}\\ \text{throwing 2}\\ \text{die unordered}}} \quad \therefore P(E) = \frac{|E|}{|S|} = \frac{266}{441}$$

C - event that I cleaned my room
b) P(T) T - event that there was a tie

$$P(T|C) = \frac{P(C|T)\,P(T)}{P(C)}$$

$P(C|T) = 1$ since if there is a tie, I always clean the room.

$P(C)$ is already calculated

$$P(T) = \frac{\sum_{i=1}^{6} k^2}{(36-15)^2}$$ ways of picking ties / Sample space

$$\therefore \quad P(T|C) = \frac{P(T)}{P(C)}$$

15.

a) 3

b) 4 hours

c) 13, 12, 14