Sequences, Topology of Sets

Problem 1: Given a sequence $(a_n)_{n=1}^{\infty}$, we say that a is a foo-point of (a_n) if for every $\varepsilon > 0$ and for every $n_0 \in N$ there exists an $N \in \mathbb{N}$ satisfying $N > n_0$ such that $|a_N - a| < \varepsilon$.

(a) Give an example of a sequence with two distinct foo-points.

(b) Give an example of a sequence which has a single foo-point, but does not converge.

Proof.

(a) Let $(a_n) = (-1)^n$. Pick $\epsilon > 0,\ n_0 \in \mathbb{N}$. Choose $N = n_0 + 1 > n_0$.

If N is even, then $a = 1$ is a *foo-point* since $|a_N - a| = |(-1)^N - 1| = 0 < \epsilon$.

If N is odd, then $a = -1$ is a *foo-point* since $|a_N - a| = |(-1)^N - (-1)| = 0 < \epsilon$.

Now suppose there exists another *foo-point* b where $b \neq -1$ and $b \neq 1$.

Pick $\epsilon = \min\left(\dfrac{|b-1|}{2}, \dfrac{|b+1|}{2}\right)$. It follows that there exists $N > n_0$ s.t. $|a_N - b| < \epsilon$.

Thus $|a_N - b| < \dfrac{|b+1|}{2}$ and $|a_N - b| < \dfrac{|b-1|}{2} \Rightarrow 2 < 1$ which is a contradiction.

Thus, $(a_n) = (-1)^n$ is a sequence with exactly two *foo-points.*

(b) Let $(a_n) = n \sin\left(\dfrac{n\pi}{2}\right)$.

To prove (a_n) is not convergent, we take the subsequence (a_{n_k}) where $n_k = 4k + 1$. Suppose (a_{n_k}) converges to L. Pick $\epsilon = \dfrac{1}{2}$. $\exists M$ s.t. $\forall m > M$, $|a_M - a_{M+1}| < \epsilon$. Thus:

$$
\begin{aligned}
|a_{4k+1} - a_{4k+5)}| &= |a_{4k} - L - a_{4k+5} + L| \\
\left|(4k+1)\sin\left(\frac{(4k+1)\pi}{2}\right) - (4k+5)\sin\left(\frac{(4k+5)\pi}{2}\right)\right| &= |a_{4k} - L| + |L - a_{4k+5}| \\
\left|(4k+1)\sin\left(2k\pi + \frac{1}{2}\pi\right) - (4k+5)\sin\left(2k\pi + \frac{5\pi}{2}\right)\right| &\leq \frac{1}{2} + \frac{1}{2} \\
|(4k+1)(1) - (4k+5)(1)| &\leq 1 \\
|-4| = 4 &\leq 1 \qquad \text{(contradiction)}
\end{aligned}
$$

Thus, since there exists a non-convergent subsequence of (a_n), by the result in Q1, (a_n) is not convergent.

Now, pick $\epsilon > 0,\ n_0 \in \mathbb{N}$.

If n_0 is odd, choose $N = n_0 + 1$. If n_0 is even, choose $N = n_0 + 2$. Thus, $N > n_0$ and N can be written as $2k$ for some $k \in \mathbb{N}$ since N is even.

$a = 0$ is a *foo-point* since $|a_N - a| = \left| N \sin\left(\frac{2k\pi}{2}\right) - 0 \right| = |0 - 0| = 0 > \epsilon$.

Suppose for sake of contradiction that there is another *foo-point* $b \neq 0$.

Pick $\epsilon > 0$ and $n_0 \in \mathbb{N}$. We know $|a_N - b| < \epsilon$. Now pick $n_0' > N$. There exists $N' > n_0'$ s.t. $|a_N' - b| < \epsilon$.

If $a_N = 0 = a_N'$, pick $\epsilon = \frac{b}{2}$ and $|b - 0| < \frac{b}{2}$ will contradict our hypothesis $b \neq 0$.

Suppose either $a_N \neq 0$ or $a_N' \neq 0$ and pick $\epsilon = \frac{1}{4}$. Since $a_n = n, -n$ or 0 by definition, $a_N, a_N' \in \mathbb{N}$. Since the difference of two natural numbers is at least 1:

$$\begin{aligned}
1 &\leq |a_N - a_N'| && \text{(since } a_N \neq a_N'\text{)} \\
&= |a_N - b - a_N' + b| \\
&= |a_N - b| + |a_N' - b| \\
&< \frac{1}{4} + \frac{1}{4} = \frac{1}{2} && \text{(contradiction)}
\end{aligned}$$

Therefore, $(a_n) = n \sin \frac{n\pi}{2}$ is a non-convergent sequence with a single *foo-point*. $\square$

Problem 2: Construct an open set of arbitrarily small size which contains $\mathbb{Q}$, but which is a proper subset of $\mathbb{R}$. In this context we define the size of (a, b) for $a < b$ to be ba. We define the size of a union of open intervals to be the sum of their sizes

Proof. We know that $\mathbb{Q}$ is a countable set, and can be indexed entirely by set I. Let the desired size of the set be $\varepsilon > 0$. For any rational number r at index i, choose the interval $\left(r + \frac{\varepsilon}{2^{i+1}}, r - \frac{\varepsilon}{2^{i+1}}\right)$. The length of this interval is $\frac{2\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2^i}$. Its total length:

$$\sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} \quad = \quad \varepsilon \sum_{i=1}^{\infty} \frac{1}{2^i} \quad = \quad \epsilon$$

And this is an infinite union of open sets, which we know to be open. $\square$

Problem 3: Construct a sequence $\{x_k\} \subseteq \mathbb{R}^2$ with the property that for any $x \in R^2$, there exists a subsequence $\{x_{k_n}\}$ which converges to x.

Proof. Claim: Since $\mathbb{Q} \times \mathbb{Q}$ is countable, it can be indexed into indexing set I. Let our sequence $\{a_i\}$ be defined by $a_i = i^{th}$ element of I.

Pick limit point $x \in \mathbb{R}^2$. Consider subsequence $\{a_{ik}\}$ where $a_{ik} \in B_{1/k}(x)$. Since $\mathbb{Q} \times \mathbb{Q}$ is dense in $\mathbb{R}^2$, there will be an infinite number of choices. Pick a_{ik} such that $i_{k-1} < i_k$ which can be guaranteed since there are infinite choices.

Notice that $|a_{ik} - x| < \dfrac{1}{k}$. So, as $k \to \infty$, $|a_{ik} - x| \to 0$. Therefore, $a_{ik} \to x$, completing the proof. $\square$

Problem 4: Let $f, g : \mathbb{R}^n \to \mathbb{R}^k$ be continuous functions and suppose that $D \subseteq \mathbb{R}^n$ is a dense set. If $f(x) = g(x)$ for every $x \in D$, then $f(x) = g(x)$ for every $x \in \mathbb{R}^n$.

Proof. Remember that if $D \in \mathbb{R}^n$ is dense in $\mathbb{R}^n$, then $\bar{D} = \mathbb{R}^n$.

Define function $h = f - g$ which is continuous.

Note that for any $d \in D$, $h(d) = (f(d) - g(d)) = 0$.

Now, if $x \in \mathbb{R}^n = \bar{D}$, there exists a sequence $\{d_n\} \in D$ such that $d_n \xrightarrow{n\to\infty} x$.

Since h is continuous:

$$\begin{aligned}
\lim_{n\to\infty} h(d_n) &= h\left(\lim_{n\to\infty} d_n\right) \\
\lim_{n\to\infty} 0 &= h(x) \\
0 &= f(x) - g(x) \\
g(x) &= f(x)
\end{aligned}$$

as required. $\square$