Matrices, Bases, Determinants

Question 1:

Suppose that $A \in M_{n\times n}(F)$ and that $AA^t = I$ (the identity map). Find all possible values of $\det(A)$.

Proof:

$$\begin{aligned}
AA^t &= I_n \\
\det(AA^t) &= \det(I_n) \\
\det(A)\det(A^t) &= 1 && \text{(by decomposition into elementary matrices)} \\
\det(A)\det(A) &= 1 && \text{(since for elementary matrix } E\text{, } \det(E) = \det(E^t)\text{)} \\
(\det(A))^2 &= 1 \\
\det(A) &= \pm 1
\end{aligned}$$

Question 2:

Prove that if $A, B \in M_{n\times n}(F)$ are similar, then $tr(A) = tr(B)$.

Proof:

First we prove $tr(AB) = tr(BA)$:
$$tr(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} \cdot B_{ji} \stackrel{(*)}{=} \sum_{j=1}^{n}\sum_{i=1}^{n} B_{ji} \cdot A_{ij} = \sum_{j=1}^{n}(BA)_{jj} = tr(BA)$$
*Since matrix entries are field elements, they are commutative and associative, and hence, the order of their summation is inconsequential to the final sum.

Using that fact, we can see that since $A = Q^{-1}BQ$ for some $Q \in M_{n\times n}(F)$:

$$tr(A) = tr(Q^-1BQ) = tr(BQQ^{-1}) = tr(BI) = tr(B)$$

Question 3:

Let V be a finite dimensional vector space over F and $\beta = \{v_1, ..., v_n\}$ is a basis for V. Suppose that Q is an invertible $n \times n$ matrix and define $x_j = \sum_{i=0}^{n} Q_{ij}v_i$. Set $\gamma = \{x_1, ..., x_n\}$. Prove that γ is a basis and hence any invertible matrix can be thought of as a change of coordinate basis for some basis γ.

Proof:

Consider the follwing equation:

$$\begin{aligned} c_1x_1 + c_2x_2 + ... + c_nx_n &= 0 \\ c_1\sum_{i=1}^{n} Q_{1i}v_1 + ... + c_n\sum_{i=1}^{n} Q_{ni}v_n &= 0 \\ \sum_{i=1}^{n} c_1Q_{1i}v_1 + ... + \sum_{i=1}^{n} c_nQ_{ni}v_n &= 0 \\ \Rightarrow \sum_{i=1}^{n} c_1Q_{1i}, ..., \sum_{i=1}^{n} c_nQ_{ni} &= 0 \qquad \text{(since } x_i\text{'s are linearly independent.)} \end{aligned}$$

The coordinates of x_i in β coordinates is $[x_i]_\beta =$ the i^{th} column of Q by definition. Since Q is invertible, all it's columns are linearly independent. Thus, you will always be able to write $u \in V$ as a unique sum of vectors in γ. Abd since $|\gamma| = \dim V$, γ is a basis.

In particular, Q is the change of coordinate basis from an arbitrary basis β to γ. That is $[v]_\gamma = Q[u]_\beta$