# Orthogonality and Inner Product Spaces

**Problem 1:** Let $F = \mathbb{R}$ or $\mathbb{C}$ and $V$ be an inner product space over $F$.

Show that the function $\alpha : V \to V^\vee$ defined by $\alpha(y) = \langle -, y\rangle$ is injective, and that it is linear when $F = \mathbb{R}$.

Let $V$ then be finite dimensional and $\{v_1, ..., v_n\}$ be an orthonormal basis of $V$. Let $f \in V^\vee$. Show that $f = \sum_{i=1}^{n} f(v_i)\langle -, v_i\rangle$

*Proof.* Let $y, z \in V$ where $\alpha(y) = \alpha(z)$. If $y = z$ then $\alpha$ is injective.

Indeed, since $\langle -, y\rangle = \langle -, z\rangle$ then $\langle -, y - z\rangle = 0$ by additivity in the second component. Consider $\langle y - z, y - z\rangle = 0$. That must mean $y - z = 0$. Thus $y = z$.

We can see $\alpha$ is linear over $\mathbb{R}$ since:

$$\alpha(y + z) = \langle -, y + z\rangle = \langle -, y\rangle + \langle -, z\rangle = \alpha(y) + \alpha(z)$$

$$\alpha(cy) = \langle -, cy\rangle = c\langle -, y\rangle = c\alpha(y)$$

for any $c \in \mathbb{R}$.

Since $f$ is linear, it can be fully described by where it takes the basis of its domain (for any vector in the domain can be written as a linear combination of basis vectors). So, it suffices to show $f(v_j) = \sum_{i=1}^{n} f(v_i)\langle v_j, v_i\rangle$ for any $v_j$ in the orthonormal basis. Indeed, since $\langle v_j, v_i\rangle = 0$ for $i \neq j$ and $\langle v_j, v_j\rangle = 1$:

$$\sum_{i=1}^{n} f(v_i)\langle v_j, v_i\rangle = f(v_j)$$

$\square$

**Problem 2:** Show that the differentiation map is not normal with respect to any inner product.

*Proof.* Suppose for sake of contradiction that $T$ is normal.

We know that differentiation is nilpotent. Thus, the characteristic polynomial of $T$ is $t^k$. In particular, this splits. And thus, by Schur's Theorem, there exist an orthonormal basis $\beta$ such that $[T]_\beta$ is upper triangular.

This combined with the fact that $[T]_\beta$ is normal implies that it is also diagonal based on Theorem 6.16 in *Linear Algebra* by Spence.

But the only eigenvalues of $T$ is 0 because it is nilpotent. Thus $[T]_\beta = 0$ which means $T = 0$ regardless of basis. And so, $0 = T(x) = 1$, which is a contradiction. $\square$

**Problem 3:** Let $T$ be a normal operator on $V$ Show the following:

(a) $||T(x)|| = ||T^*(x)||$ for all $x \in V$

(b) If $x$ is an eigenvector of $T$ with eigenvalue $\lambda$, then $x$ is an eigenvector of $T^*$ with eigenvalue $\bar{\lambda}$.

(c) If $\lambda$ and $\lambda'$ are distinct eigenvalues of $T$, $x \in E_\lambda$ and $x \in E_\lambda$, then $\langle x, x' \rangle = 0$

*Proof.*

(a) Since $T$ is normal,

$$||T(x)||^2 = \langle T(x), T(x) \rangle = \langle T^*T(x), x \rangle = \langle TT^*(x), x \rangle = \langle T^*(x), T^*(x) \rangle = ||T^*(x)||^2$$

which implies $||T(x)|| = ||T^*(x)||$.

(b) First we prove $T - \lambda I$ is normal if $T$ is normal:

$$\begin{aligned}
(T - \lambda I)^*(T - \lambda I) &= (T^* - \overline{\lambda} I)(T - \lambda I) \\
&= T^*T - T^*\lambda I - \overline{\lambda} T + |\lambda|^2 I \\
&= TT^* - T\overline{\lambda} I - \lambda T^* + |\lambda|^2 I \\
&= (T - \lambda I)(T^* - \overline{\lambda} I) \\
&= (T - \lambda I)(T - \lambda I)^*
\end{aligned}$$

Now, by part $(a)$, if $T(x) = \lambda(x)$ and $x \neq 0$,

$$||T^*(x) - \overline{\lambda} x|| = ||(T^* - \overline{\lambda} I)x|| \overset{(a)}{=} ||(T - \lambda)x|| = ||T(x) - \lambda x|| = 0$$

which implies $T^*(x) = \overline{\lambda} x$.

(c) By part $(b)$, if we let $x$ and $x'$ be non-zero,

$$\lambda \langle x, x' \rangle = \langle T(x), x' \rangle = \langle x, T^*(x') \rangle \overset{(b)}{=} \langle x, \overline{\lambda'} x \rangle = \lambda' \langle x, x' \rangle$$

And since $\lambda$ and $\lambda'$ are distinct, $\langle x, x' \rangle = 0$. $\square$

**Problem 4:** Let $F$ be $\mathbb{R}$ or $\mathbb{C}$, and $V$ a finite-dimensional inner product space over $F$. Let $T : V \to V$ be a linear operator. Show that $T$ is normal if and only if there exists a polynomial $f(t) \in F[t]$ such that $T^* = f(T)$.

*Proof.* If $T^* = f(T)$ then $T^*T = f(T)T = Tf(T) = TT^*$, proving the backward implication.

Now we prove the forward implication. Suppose $T$ is normal.

We know the characteristic polynomial of $T$ splits over $\mathbb{C}$ with complex roots.

And since $V$ is finite dimensional, we know there exist a orthonormal basis $\beta$ such that $[T]_\beta$ is upper triangular.

Since $T$ is normal, this means $[T]_\beta$ is diagonal. Furthermore, $[T^*]_\beta = [T]_\beta^*$.

So if $[T]_\beta = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ then $[T^*]_\beta = [T]_\beta^* = \begin{pmatrix} \overline{\lambda_1} & & \\ & \ddots & \\ & & \overline{\lambda_n} \end{pmatrix}$

We just need to show that $\exists f \in F[t]$ such that $f(T) = T^* \Rightarrow f([T]_\beta) = [T^*]_\beta$.

Since $[T]_\beta$ is diagonal, we need to find a polynomial $f$ where $f(\lambda_j) = \overline{\lambda_j}$ for all $j \in [1, n]$.

Let $\lambda_1, ..., \lambda_n$ be the distinct eigenvalues of $T$.

Define $f_j(t) = \displaystyle\prod_{i \neq j}^{n} \frac{t - \lambda_i}{\lambda_j - \lambda_i}$. Notice that $f_j(\lambda_j) = 1$ and $f_j(\lambda_i) = 0$ for all $j \neq i$.

Now define $f(t) = \displaystyle\sum_{i=1}^{n} \overline{\lambda_i} f_i(t)$. Observe that $f(\lambda_j) = \overline{\lambda_j}$ for any eigenvector of $T$.

If $F = \mathbb{C}$ we are done since $f \in \mathbb{C}[t]$. But we need to show that if $F = \mathbb{R}$ then $f \in \mathbb{R}[t]$.

By fundamental theorem of algebra, the roots of $P_T(t)$ are real, or come in conjugate pairs. Let $\lambda_r$ be a real root of $P_T(t)$ and consider $f_r$. By definition, the complex factors of $f_r$ come in conjugate pairs. Let $\mu$ and $\overline{\mu}$ be any arbitrary pair of complex roots.

Well,

$$\frac{(t-\mu)(t-\overline{\mu})}{(\lambda_r - \mu)(\lambda_r - \overline{\mu})} = \frac{t^2 - t\mu - t\overline{\mu} - |\mu|^2}{\lambda_r^2 - \lambda_r\mu - \lambda_r\overline{\mu} - |\mu|^2} = \frac{t^2 - 2tRe(\mu) - |\mu|^2}{\lambda_r^2 - 2\lambda_r Re(\mu) - |\mu|^2}$$

which is an entirely real polynomial. So $f_r$ only has real coefficients.

For $f_c$ where $\lambda_c$ is complex, only the factor $\dfrac{t - \overline{\lambda_c}}{\lambda_c - \overline{\lambda_c}}$ does not appear with a conjugate counterpart. Let $\lambda_d = \overline{\lambda_c}$. Well, $f$ is defined to have pairs $\overline{\lambda_c} f_c + \overline{\lambda_d} f_d$ as summands. We can factor out the real polynomial $g_c = \prod_{c \neq i \neq d}^{k} \frac{t - \lambda_i}{\lambda_c - \lambda_i}$ from $f_c$. Notice that this is the same real factor of $f_d$ since $|\lambda_d| = |\lambda_c|$ and $Re(\lambda_d) = Re(\lambda_c)$. And therefore:

$$\begin{aligned}
\overline{\lambda_c} f_c + \overline{\lambda_d} f_d &= \lambda_d f_c + \lambda_c f_d \\
&= g_c \left( \frac{\lambda_d (t - \lambda_d)}{(\lambda_c - \lambda_d)} - \frac{\lambda_c (t - \lambda_c)}{(\lambda_d - \lambda_c)} \right) \\
&= g_c \left( \frac{\lambda_d t - \lambda_d^2 + \lambda_c t - \lambda_c^2}{\lambda_c - \lambda_d} \right) \\
&= g_c \left( \frac{2Re(\lambda_c) t - 2Re(\lambda_c^2)}{2Re(\lambda_c)} \right) && \text{(since } \lambda_d^2 = (\overline{\lambda_c})^2 = \overline{\lambda_c^2}\text{)}
\end{aligned}$$

which is also an entirely real polynomial. Thus, $f \in \mathbb{R}[t]$, completing the proof. $\square$