

Sequences, Topology of Sets

Question 1:

Given a sequence $(a_n)_{n=1}^{\infty}$, we say that a is a *foo-point* of (a_n) if for every $\varepsilon > 0$ and for every $n_0 \in \mathbb{N}$ there exists an $N \in \mathbb{N}$ satisfying $N > n_0$ such that $|a_N - a| < \varepsilon$.

- (a) Give an example of a sequence with two distinct *foo-points*.
- (b) Give an example of a sequence which has a single *foo-point*, but does not converge.

Proof:

- a. Let $(a_n) = (-1)^n$. Pick $\epsilon > 0$, $n_0 \in \mathbb{N}$. Choose $N = n_0 + 1 > n_0$.

If N is even, then $a = 1$ is a *foo-point* since $|a_N - a| = |(-1)^N - 1| = 0 < \epsilon$.

If N is odd, then $a = -1$ is a *foo-point* since $|a_N - a| = |(-1)^N - (-1)| = 0 < \epsilon$.

Now suppose there exists another *foo-point* b where $b \neq -1$ and $b \neq 1$.

Pick $\epsilon = \min\left(\frac{|b-1|}{2}, \frac{|b+1|}{2}\right)$. It follows that there exists $N > n_0$ s.t. $|a_N - b| < \epsilon$.

Thus $|a_N - b| < \frac{|b+1|}{2}$ and $|a_N - b| < \frac{|b-1|}{2} \Rightarrow 2 < 1$ which is a contradiction.

Thus, $(a_n) = (-1)^n$ is a sequence with exactly two *foo-points*.

- b. Let $(a_n) = n \sin\left(\frac{n\pi}{2}\right)$.

To prove (a_n) is not convergent, we take the subsequence (a_{n_k}) where $n_k = 4k + 1$.

Suppose (a_{n_k}) converges to L . Pick $\epsilon = \frac{1}{2}$. $\exists M$ s.t. $\forall m > M$, $|a_m - a_{m+1}| < \epsilon$.

Thus:

$$\begin{aligned} |a_{4k+1} - a_{4k+5}| &= |a_{4k} - L - a_{4k+5} + L| \\ \left| (4k+1) \sin\left(\frac{(4k+1)\pi}{2}\right) - (4k+5) \sin\left(\frac{(4k+5)\pi}{2}\right) \right| &= |a_{4k} - L| + |L - a_{4k+5}| \\ \left| (4k+1) \sin\left(2k\pi + \frac{1}{2}\pi\right) - (4k+5) \sin\left(2k\pi + \frac{5}{2}\pi\right) \right| &\leq \frac{1}{2} + \frac{1}{2} \\ |(4k+1)(1) - (4k+5)(1)| &\leq 1 \\ |-4| &= 4 \leq 1 \quad (\text{contradiction}) \end{aligned}$$

Thus, since there exists a non-convergent subsequence of (a_n) , by the result in Q1, (a_n) is not convergent.

Now, pick $\epsilon > 0$, $n_0 \in \mathbb{N}$.

If n_0 is odd, choose $N = n_0 + 1$. If n_0 is even, choose $N = n_0 + 2$. Thus, $N > n_0$ and N can be written as $2k$ for some $k \in \mathbb{N}$ since N is even.

$a = 0$ is a *foo-point* since $|a_N - a| = \left| N \sin\left(\frac{2k\pi}{2}\right) - 0 \right| = |0 - 0| = 0 < \epsilon$.

Suppose for sake of contradiction that there is another *foo-point* $b \neq 0$.

Pick $\epsilon > 0$ and $n_0 \in \mathbb{N}$. We know $|a_N - b| < \epsilon$. Now pick $n'_0 > N$. There exists $N' > n'_0$ s.t. $|a'_{N'} - b| < \epsilon$.

If $a_N = 0 = a'_N$, pick $\epsilon = \frac{b}{2}$ and $|b - 0| < \frac{b}{2}$ will contradict our hypothesis $b \neq 0$.

Suppose either $a_N \neq 0$ or $a'_N \neq 0$ and pick $\epsilon = \frac{1}{4}$. Since $a_n = n, -n$ or 0 by definition, $a_N, a'_N \in \mathbb{N}$. Since the difference of two natural numbers is at least 1:

$$\begin{aligned} 1 &\leq |a_N - a'_N| && \text{(since } a_N \neq a'_N\text{)} \\ &= |a_N - b - a'_N + b| \\ &= |a_N - b| + |a'_N - b| \\ &< \frac{1}{4} + \frac{1}{4} = \frac{1}{2} && \text{(contradiction)} \end{aligned}$$

Therefore, $(a_n) = n \sin \frac{n\pi}{2}$ is a non-convergent sequence with a single *foo-point*.

Question 2:

Construct an open set of arbitrarily small size which contains \mathbb{Q} , but which is a proper subset of \mathbb{R} . In this context we define the size of (a, b) for $a < b$ to be $b - a$. We define the size of a union of open intervals to be the sum of their sizes

Proof:

We know that \mathbb{Q} is a countable set, and can be indexed entirely by set I . Let the desired size of the set be $\varepsilon > 0$. For any rational number r at index i , choose the interval $\left(r - \frac{\varepsilon}{2^{i+1}}, r + \frac{\varepsilon}{2^{i+1}}\right)$. The length of this interval is $\frac{2\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2^i}$. Its total length:

$$\sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon \sum_{i=1}^{\infty} \frac{1}{2^i} = \varepsilon$$

And this is an infinite union of open sets, which we know to be open.

Question 3:

Construct a sequence $\{x_k\} \subseteq \mathbb{R}^2$ with the property that for any $x \in \mathbb{R}^2$, there exists a subsequence $\{x_{k_n}\}$ which converges to x .

Proof:

Claim: Since $\mathbb{Q} \times \mathbb{Q}$ is countable, it can be indexed into indexing set I . Let our sequence $\{a_i\}$ be defined by $a_i = i^{th}$ element of I .

Pick limit point $x \in \mathbb{R}^2$. Consider subsequence $\{a_{i_k}\}$ where $a_{i_k} \in B_{1/k}(x)$. Since $\mathbb{Q} \times \mathbb{Q}$ is dense in \mathbb{R}^2 , there will be an infinite number of choices. Pick a_{i_k} such that $i_{k-1} < i_k$ which can be guaranteed since there are infinite choices.

Notice that $|a_{i_k} - x| < \frac{1}{k}$. So, as $k \rightarrow \infty$, $|a_{i_k} - x| \rightarrow 0$. Therefore, $a_{i_k} \rightarrow x$, completing the proof.

Question 4:

Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be continuous functions and suppose that $D \subseteq \mathbb{R}^n$ is a dense set. If $f(x) = g(x)$ for every $x \in D$, then $f(x) = g(x)$ for every $x \in \mathbb{R}^n$.

Proof:

Remember that if $D \subseteq \mathbb{R}^n$ is dense in \mathbb{R}^n , then $\bar{D} = \mathbb{R}^n$.

Define function $h = f - g$ which is continuous.

Note that for any $d \in D$, $h(d) = (f(d) - g(d)) = 0$.

Now, if $x \in \mathbb{R}^n = \bar{D}$, there exists a sequence $\{d_n\} \in D$ such that $d_n \xrightarrow{n \rightarrow \infty} x$.

Since h is continuous:

$$\lim_{n \rightarrow \infty} h(d_n) = h\left(\lim_{n \rightarrow \infty} d_n\right)$$

$$\lim_{n \rightarrow \infty} 0 = h(x)$$

$$0 = f(x) - g(x)$$

$$g(x) = f(x)$$

as required