Problem 1:
(a). It is clear that X and Y are dependent, that is $Y = X^2$ and $X = \pm\sqrt{Y}$, so given any $x \in X$ we know its corresponding $y \in Y$ and vice versa.
However, since X is uniformly distributed on $[-1, 1]$ and hence Y is uniformly distributed on $[0, 1]$, then:

$$\begin{aligned}
\mathbb{E}[X] = 0, \mathbb{E}[Y] &= \frac{1}{2} \\
Cov(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
Cov(X,Y) &= \mathbb{E}[X^3] \\
Cov(X,Y) &= 0
\end{aligned}$$

And since $Cov(X,Y) = 0$, then $Cor(X,Y) = 0$ which means X and Y are uncorrelated.

(b). The characteristic function of the distribution is of the form $\phi_X(t) = \mathbb{E}[e^{itX}], a \leq x \leq b$:

$$\begin{aligned}
\phi_X(t) &= \int_a^b \frac{1}{b-a} e^{itx} dx \\
&= \frac{1}{b-a} \int_a^b e^{itx} dx \\
&= \frac{1}{b-a} (\frac{e^{itx} - e^{itx}}{it}) \Big|_a^b \\
&= \frac{1}{it(b-a)} (e^{itb} - e^{ita})
\end{aligned}$$

Problem 2:

$$\begin{aligned}
Var[\mu_n] &= Var[\frac{1}{n} \sum_{k=1}^n x_k] \\
&= \frac{1}{n^2} Var[\sum_{k=1}^n x_k] \\
&= \frac{1}{n^2} \sum_{k=1}^n Var[x_k] \\
&= \frac{1}{n^2} \sum_{k=1}^n \mathbb{E}[(x_k - \mathbb{E}[x_k])(x_k - \mathbb{E}[x_k])^T]
\end{aligned}$$

This means that the summation describes n covariance matrices of X, thus:

$$Var[\mu_n] = \frac{1}{n^2} nC = \frac{1}{n} C$$

Problem 3: a. The scatterplot matrix is shown on the next page.

b.

	mpg	cylinders	displacement	horsepower	weight
mpg	1.000000	-0.777617	-0.805126	-0.778426	-0.832244
cylinders	-0.777617	1.000000	0.950823	0.842983	0.897527
displacement	-0.805126	0.950823	1.000000	0.897257	0.932994
horsepower	-0.778426	0.842983	0.897257	1.000000	0.864537
weight	-0.832244	0.897527	0.932994	0.864537	1.000000
acceleration	0.423328	-0.504683	-0.543800	-0.689195	-0.416839
year	0.580541	-0.345647	-0.369855	-0.416361	-0.309119
origin	0.565208	-0.568931	-0.614535	-0.455171	-0.585005

	acceleration	year	origin
mpg	0.423328	0.580541	0.565208
cylinders	-0.504683	-0.345647	-0.568931
displacement	-0.543800	-0.369855	-0.614535
horsepower	-0.689195	-0.416361	-0.455171
weight	-0.416839	-0.309119	-0.585005
acceleration	1.000000	0.290316	0.212745
year	0.290316	1.000000	0.181527
origin	0.212745	0.181527	1.000000

c.

```
Call:
lm(formula = mpg ~ . - name, data = Auto)

Residuals:
    Min      1Q  Median      3Q     Max
-9.5903 -2.1565 -0.1169 1.8690 13.0604

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -17.218435 4.644294 -3.707 0.00024 ***
cylinders    -0.493376 0.323282 -1.526 0.12780
displacement  0.019896 0.007515  2.647 0.00844 **
horsepower   -0.016951 0.013787 -1.230 0.21963
weight       -0.006474 0.000652 -9.929 < 2e-16 ***
acceleration  0.080576 0.098845  0.815 0.41548
year          0.750773 0.050973 14.729 < 2e-16 ***
origin        1.426141 0.278136  5.127 4.67e-07 ***
---
Signif. codes: 0   ***   0.001   **   0.01   *   0.05   .   0.1        1

Residual standard error: 3.328 on 384 degrees of freedom
Multiple R-squared: 0.8215,  Adjusted R-squared: 0.8182
F-statistic: 252.4 on 7 and 384 DF, p-value: < 2.2e-16
```

i. Displacement, weight, year, and origin all have p values lower than .05, so these predictors are likely to have a relationship with mpg.

ii. Weight and year seem to have the most significant relationship with very low p values and relatively low residual standard error.

iii. The coefficient for year suggests that the miles per gallon of a car increases by 0.75 every year.

d. The plots are attached on the next page. We can see that there is an overall correlation between fitted values and their residual errors, which may mean that the data would be better fit non-linearly. Our leverage plot indicates that there is indeed an observation(14) that has extremely high leverage, indicating that it had made a huge contribution towards our fitted values.

e. The following lists some of the interactions with significant statistical significance:

```
lm(formula = mpg ~ . - name + weight * year, data = Auto)
weight:year -4.879e-04 6.097e-05 -8.002 1.47e-14 ***

lm(formula = mpg ~ . - name + cylinders * displacement, data = Auto)
cylinders:displacement 0.0136081 0.0017209 7.907 2.84e-14 ***

lm(formula = mpg ~ . - name + horsepower * origin, data = Auto)
horsepower:origin -7.955e-02 1.074e-02 -7.405 8.44e-13 ***

lm(formula = mpg ~ . - name + acceleration * weight, data = Auto)
weight:acceleration -5.826e-04 8.408e-05 -6.928 1.81e-11 ***
```

There are likely many more, but the interactions above were found to be significant.

f. With the transformations we can note that the significant variables that were major in the determination of the fit remained significant, with only slight changes with residual errors p values. It can be noted that the R-squared value does increase, however.

Problem 4: We can prove this claim by deriving the B matrix using D. Consider that:

$$B = X^T X, B_{ij} = \sum_{k=1}^{p} x_{ik} x_{jk}$$

First break down the Euclidean distance D_{ij}:

$$D_{ij} = \sum_{k=1}^{p} (x_{ik} - x_{jk})^2 = \sum_{k=1}^{p} x_{ik}^2 - 2\sum_{k=1}^{p} x_{ik}x_{jk} + \sum_{k=1}^{p} x_{jk}^2$$

Now, using our breakdown, calculate the given s_i, s_j, and s.

$$s_i = \sum_{j=1}^{n} D_{ij} = \sum_{j=1}^{n} (\sum_{k=1}^{p} x_{ik}^2 - 2\sum_{k=1}^{p} x_{ik}x_{jk} + \sum_{k=1}^{p} x_{jk}^2)$$
$$s_j = \sum_{i=1}^{n} D_{ij} = \sum_{j=1}^{n} (\sum_{k=1}^{p} x_{ik}^2 - 2\sum_{k=1}^{p} x_{ik}x_{jk} + \sum_{k=1}^{p} x_{jk}^2)$$
$$s = \sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} (\sum_{k=1}^{p} x_{ik}^2 - 2\sum_{k=1}^{p} x_{ik}x_{jk} + \sum_{k=1}^{p} x_{jk}^2)$$

Using our centering condition, that is, $\sum_{i=1}^{n} x_i = 0$, then we can derive that:

$$\sum_{i=1}^{n}\sum_{k=1}^{p} x_{ik}x_{jk} = 0, \sum_{j=1}^{n}\sum_{k=1}^{p} x_{ik}x_{jk} = 0$$

Now using this condition for s_i, s_j, and s:

$$s_i = n\sum_{k=1}^{p} x_{ik}^2 + \sum_{j=1}^{n}\sum_{k=1}^{p} x_{jk}^2 \Leftrightarrow \sum_{k=1}^{p} x_{ik}^2 = \frac{1}{n}(s_i - \sum_{j=1}^{n}\sum_{k=1}^{p} x_{jk}^2)$$
$$s_j = n\sum_{k=1}^{p} x_{jk}^2 + \sum_{i=1}^{n}\sum_{k=1}^{p} x_{ik}^2 \Leftrightarrow \sum_{k=1}^{p} x_{jk}^2 = \frac{1}{n}(s_j - \sum_{i=1}^{n}\sum_{k=1}^{p} x_{ik}^2)$$
$$s = \frac{1}{2n}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{jk}^2$$

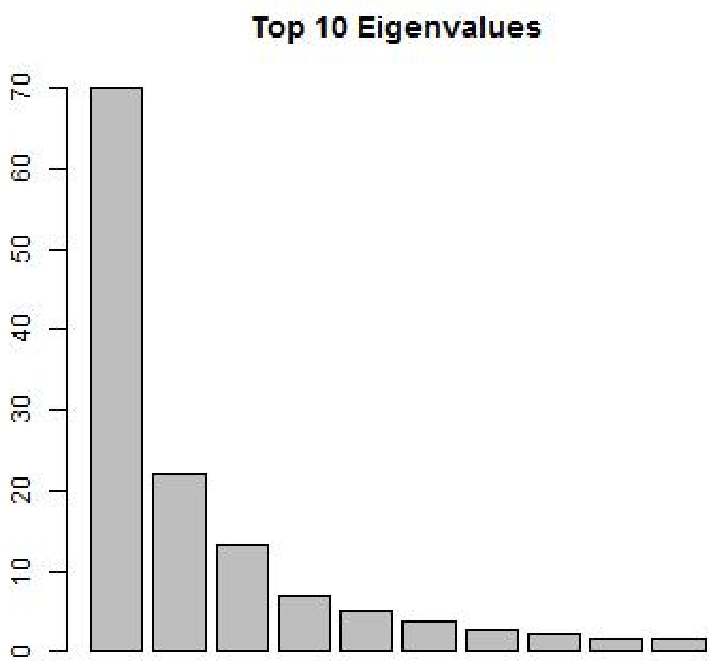
Now plugging that into our breakdown of D_{ij}:

$$D_{ij} = \frac{1}{n}(s_i - \sum_{j=1}^{n}\sum_{k=1}^{p} x_{jk}^2) - 2x_i^T x_j + \frac{1}{n}(s_j - \sum_{i=1}^{n}\sum_{k=1}^{p} x_{ik}^2)$$
$$D_{ij} - \frac{s_i}{n} - \frac{s_j}{n} + \frac{2}{n}\sum_{i=1}^{n}\sum_{k=1}^{p} x_{ik}^2 = D_{ij} - \frac{s_i}{n} - \frac{s_j}{n} + \frac{1}{n^2}s = -2x_i^T x_j$$

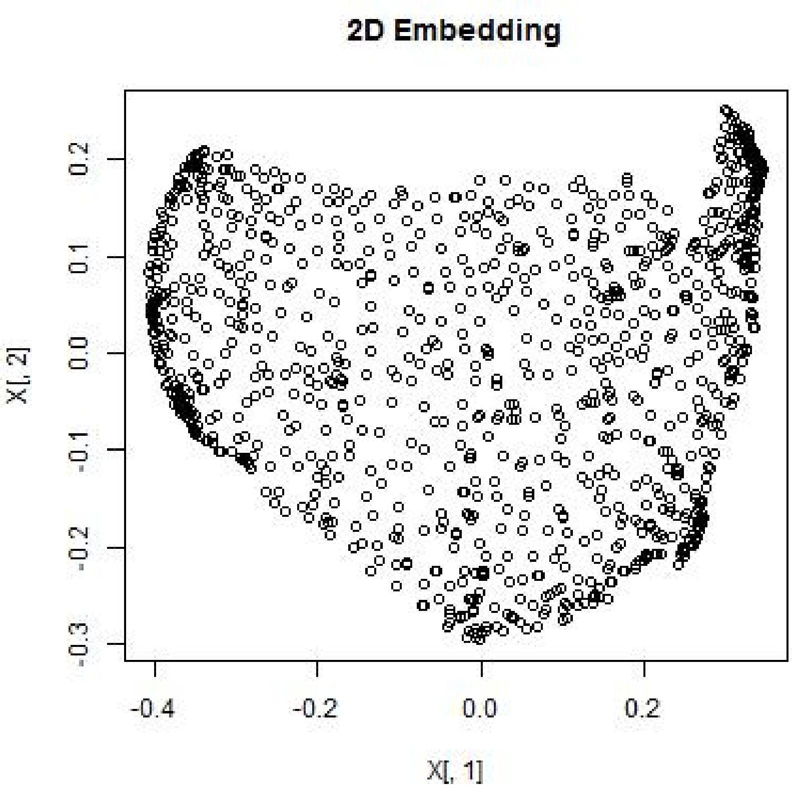
Problem 5:

```
constructB = function(distmatrix) {
#Part (i): Builds B matrix
  D = distmatrix;
  H = matrix( , nrow = dim(distmatrix)[1], ncol = dim(distmatrix)[1]);
  I = diag(dim(distmatrix)[1]);
  for (i in 1:dim(distmatrix)[1]) {
    for(j in 1:dim(distmatrix)[1]) {
    H[i,j] = 1/dim(distmatrix)[1];
    }
  }
  H = I - H;
  B = H %*% D;
  B = B %*% H;
  B = -1/2*B;
  return(B);
}

computeEmbedding = function(B) {
#Part (i), (ii): Creates eigenvalue barplot and then creates embedding
  E = eigen(B);
  Evals = E$values;
  barplot(Evals[1:10]);
  A = diag(dim(B)[1]);
  for (i in 1:dim(B)[1]) {
    A[i,i] = Evals[i];
  }
  A = sqrt(A);
  X = E$vectors %*% A;
  plot(X[,1], X[,2], main = "2D Embedding");
}
```



Top 10 Eigenvalues
0
10
20
30
40
50
60
70



2D Embedding
X[, 2]
X[, 1]
-0.4
-0.2
0.0
0.2
-0.3
-0.2
-0.1
0.0
0.1
0.2