NOTES ON SCHUBERT CALCULUS AND QUANTUM INTEGRABILITY

1. Lecture 16 (Allen Knutson 9)

1.1. Setup.

Consider the diagonal embedding $\mathrm{Gr}(k,n) \xrightarrow{\Delta} \mathrm{Gr}(k,n) \times \mathrm{Gr}(k,n)$. Recall from Knutson 7 that any continuous map between varieties/manifolds gives the following commutative diagram in the category of correspondences

$$\begin{tikzcd} \mathrm{Gr}(k,n) \times \mathrm{Gr}(k,n) \arrow[r, "graph(\Delta)^T"] \arrow[d, "graph(s)\times graph(s)"'] & \mathrm{Gr}(k,n) \arrow[d, "graph(s)"] \\ T^*\mathrm{Gr}(k,n) \times T^*\mathrm{Gr}(k,n) \arrow[r, "C_{graph(\Delta)^T}"] & T^*\mathrm{Gr}(k,n) \end{tikzcd}$$

where composition is given by fiber product, $s : \mathrm{Gr}(k,n) \hookrightarrow T^*\mathrm{Gr}(k,n)$ is the inclusion of the zero section and given A a correspondence in $M \times N$, C_A is the conormal bundle of A in $M \times N$. As it will turn out, we have a better understanding of $C_{graph(\Delta)^T}$ over $graph(\Delta)^T$.

Recall that given a correspondence $L \subset M \times N$ we have an induced map on (equivariant) cohomology[1] $\beta_L : H^*(M) \to H^*(N)$ given by the pull-push construction, i.e.

$$\beta_L(\alpha) = (\pi_N)_*([L] \cup \pi_M^*([\alpha]))$$

Moreover one has that $\beta_{graph(f)} = H^*(f)$ and in particular we see that $\beta_{graph(s)} : H^*(\mathrm{Gr}(k,n)) \to H^*(T^*\mathrm{Gr}(k,n))$ corresponds to multiplication by the Poincare dual of the zero section, so multiplication by 1 for ordinary cohomology. Also recall that the product in (equivariant) cohomology comes from the pullback map of Δ. As such, the transpose/dual map in (equivariant) Borel-Moore homology is given by $\beta_{graph(\Delta)^T}$. Therefore commutativity of the above diagram tells us that the product in $H^*(\mathrm{Gr}(k,n))$ can be computed from the map

$$\beta_{C_{graph(\Delta)^T}} : H^*(T^*\mathrm{Gr}(k,n)) \otimes H^*(T^*\mathrm{Gr}(k,n)) \to H^*(T^*\mathrm{Gr}(k,n))$$

Switching to $T \times \mathbb{C}^\times$ equivariant cohomology, the only thing that changes is that the class of the Poincare dual of the zero section is no longer trivial so that $\beta_{graph(s)}$ is no longer trivial. Thus if we know the structure constants for multiplication of the Maulik Okounkov classes in $H^*_{T\times\mathbb{C}^\times}(T^*\mathrm{Gr}(k,n))$, say

$$[MO_\lambda] \cdot [MO_\mu] = \sum_\nu \widetilde{c^\nu_{\lambda\mu}}[MO_\nu]$$

it follows from commutativty of the diagram above that in $H^*_{T\times\mathbb{C}^\times}(\mathrm{Gr}(k,n))$ we have the formula

$$\frac{[MO_\lambda]}{[s]} \cdot \frac{[MO_\mu]}{[s]} = \sum_\nu \widetilde{c^\nu_{\lambda\mu}} \frac{[MO_\nu]}{[s]}$$

[1]This should really be (equivariant) Borel-Moore homology but for smooth varieties Borel-Moore homology is isomorphic to cohomology via Poincare duality and $\mathrm{Gr}(k,n)$ is smooth.

Definition 1.1. The Serge-Schwartz-MacPherson class associated to a sequence λ is

$$SSM_\lambda := \frac{[MO_\lambda]}{[s]}$$

One might think that the classes SSM_λ correspond to Schubert classes, but notice that both $[MO_\lambda]$ and $[s]$ arise geometrically from classes of Langrangian subvarieties of $T^*\mathrm{Gr}(k,n)$ and so SSM_λ has degree 0. However it will turn out that

Proposition 1.2. *Let* $\mathrm{inv}(\lambda)$ *be the number of inversions in the sequence* λ*. Then*

$$\lim_{h\to 0} h^{\mathrm{inv}(\lambda)} SSM_\lambda = S_\lambda$$

where S_λ *is the corresponding Schubert class associated to* λ*.*

Heruistically, this is saying that the Schubert classes are the 5 vertex limit of the Maulik Okounkov classes which correspond to the 6 vertex model.

1.2. **Stable Envelope.**
The reason why we have a better understanding of $C_{graph(\Delta)^T}$ is that it factors

$$\begin{tikzcd} T^*\mathrm{Gr}(k,n) \times T^*\mathrm{Gr}(k,n) \arrow[rr, "C_{graph(\Delta)^T}"] \arrow[dr, "senv"'] & & T^*\mathrm{Gr}(k,n) \\ & T^*\mathcal{FL}(k, n+k; 2n) \arrow[ur, "hr"'] & \end{tikzcd}$$

Let us first describe the first correspondence/map *senv*, the stable envelope.

Definition 1.3. Let $\mathbb{C}^\times$ act on a variety M. Then define

$$\mathrm{attr}(M, \mathbb{C}^\times) = \left\{ m \in M \,\middle|\, \lim_{z\to 0} z\cdot m \text{ exists} \right\}$$

and similarly define

$$\mathrm{attrlang}(M, \mathbb{C}^\times) = \left\{ (m_\ell, m) \in M^{\mathbb{C}^\times} \times M \,\middle|\, m_\ell = \lim_{z\to 0} z\cdot m \text{ exists} \right\}$$

Remark 1.4. If M is compact, then $\mathrm{attr}(M) = M$. When M is not compact then $\mathrm{attr}(M)$ can be quite small. For example, let $\mathbb{C}^\times$ act on $\mathbb{C}$ with weight -1. Then only the origin $(0,0)$ is in $\mathrm{attr}(M)$, the limit for everything else goes to infinity.

Remark 1.5. $\mathrm{attrlang}(M, \mathbb{C}^\times)$ is a correspondence between $M^{\mathbb{C}^\times}$ and M, but isn't the one we are looking for because the map $\mathrm{attrlang}(M, \mathbb{C}^\times) \to M$ need not be proper[2]! For example, let $\mathbb{C}^\times$ act on $\mathbb{CP}^1$ by $z\cdot[a:b] = [za:b]$. $\mathbb{CP}^1$ is compact, so $\mathrm{attr}(M) = M$. Moreover, one can check that the only fixed points are $[1:0]$ (north pole) and $[0:1]$ (south pole) and that $m_\ell = [0:1]$ for all points $m \in \mathbb{CP}^1$ except the north pole. It follows that $\mathrm{attrlang}(M, \mathbb{C}^\times) = \mathbb{C} \sqcup pt$ topologically, and this isn't compact and so the map $\mathrm{attrlang}(M, \mathbb{C}^\times) \to M$ cannot be proper in this case.

Theorem 1.6. *If M is affine, then the map* $\mathrm{attrlang}(M, \mathbb{C}^\times) \to M$ *is proper.*

The following example will be very important.

[2]And so there will be no pushforward map in Borel-Moore homology

Example 1.7. Let

$$M' = \mathrm{GL}_n \cdot \begin{pmatrix} \epsilon_1 I_{n_1} & & & \\ & \epsilon_2 I_{n_2} & & \\ & & \ddots & \\ & & & \epsilon_d I_{n_d} \end{pmatrix}$$

where GL_n acts by conjugation and all the ϵ_i are fixed distinct elements of $\mathbb{C}$. This is what the general fiber of the partial Grothendieck Springer resolution $\mathrm{GS}_{n_1,\ldots,n_d}$ looks like. Recall that

$$M' \cong \frac{\mathrm{GL}_n}{\mathrm{GL}_{n_1} \times \ldots \mathrm{GL}_{n_d}}$$

And as a result, M will be an affine variety as it's the quotient of a reductive group acting on an affine variety.

Now, consider the case when $M = T^*\mathcal{FL}(k, n+k; 2n)$. This is not affine and so we can't use Theorem 1.6 to conclude that $\mathrm{attrlang}(M, \mathbb{C}^\times)$ gives rise to a map in cohomology. However, recall that M is the 0 fiber in the partial Grothendieck Springer resolution, i.e.

$$\mathrm{GS}_{k,n+k}|_{\vec{\epsilon}=\vec{0}} = T^*\mathcal{FL}(k, n+k; 2n)$$

So M in this case is the limit or special fiber of a family of varieties, most of which are affine by the example above (the set of points whose fibers are affine are dense). Therefore if we define

Definition 1.8.

$$env := \lim_{\vec{\epsilon}\to\vec{0}} \mathrm{attrlang}(T^*\mathcal{FL}(k, n+k; 2n), \mathbb{C}^\times)$$

this will give rise to a map

$$H^*(T^*\mathcal{FL}(k, n+k; 2n)^{\mathbb{C}^\times}) \xrightarrow{\beta_{env}} H^*(T^*\mathcal{FL}(k, n+k; 2n))$$

But what is $T^*\mathcal{FL}(k, n+k; 2n)^{\mathbb{C}^\times}$? Well, this will depend on the weights of the action of $\mathbb{C}^\times$ on $\mathbb{C}^{2n}$, but first consider an easier, related problem, namely $\mathrm{Gr}(k, V \oplus W)$ where $\mathbb{C}^\times$ acts on $V \oplus W$ with weights 0 on V and 1 on W. Explicitly this means that $z \cdot (\vec{v}, \vec{w}) = (\vec{v}, z\vec{w})$. Then because $\mathbb{C}^\times$ acts on V and W with different weights, we will have the following decomposition

$$\mathrm{Gr}(k, V \oplus W)^{\mathbb{C}^\times} = \bigsqcup_{i+j=k} \mathrm{Gr}(i, V) \times \mathrm{Gr}(j, W) \tag{1.9}$$

One way to get an element of the LHS is the following procedure. Consider any subspace $A \subseteq V \oplus W$ and note that

$$\lim_{z\to 0} z \cdot A = (\text{Projection of } A \text{ to } V) \oplus A \cap W \text{``} = \text{''} \mathrm{gr}(A)$$

The notation $\mathrm{gr}(A)$ is because we can think of the result as the associated graded subspace of A under the filtration $0 \subseteq W \subseteq V \oplus W$. By Eq. (1.9) we see that the LHS above is in $\mathrm{Gr}(k, n)^{\mathbb{C}^\times}$.

Now let $V = \mathbb{C}^n$ and $W = \mathbb{C}^n$ and again let $\mathbb{C}^\times$ act on V and W with weights 0 and 1. The same reasoning giving rise to the decomposition in Eq. (1.9) also applies to the two step partial flag variety, i.e. we will have the following decomposition

$$\mathcal{FL}(k,n+k;2n)^{\mathbb{C}^\times} = \bigsqcup_{\substack{a+c=k \\ b+d=n+k}} \mathcal{FL}(a,b;n) \times \mathcal{FL}(c,d;n)$$

And notice that for $a=0, b=k$, we have that the RHS above will be

$$\mathcal{FL}(0,k;n) \times \mathcal{FL}(k,n;n) = \mathrm{Gr}(k,n) \times \mathrm{Gr}(k,n)$$

$\mathrm{Gr}(k,n) \times \mathrm{Gr}(k,n)$ is one of the Now applying T^* to everything, and since $T^*(\mathrm{Gr}(k,n) \times \mathrm{Gr}(k,n)) \hookrightarrow T^*\mathcal{FL}(k,n+k;2n)^{\mathbb{C}^\times}$ is closed it's also proper and so we obtain our desired map

$$senv : H^*(T^*(\mathrm{Gr}(k,n)) \times T^*(\mathrm{Gr}(k,n))\,) \to H^*(T^*\mathcal{FL}(k,n+k;2n)^{\mathbb{C}^\times}) \xrightarrow{\beta_{env}} H^*(T^*\mathcal{FL}(k,n+k;2n))$$

1.3. **Hamiltonian Reduction and Nakajima quiver varieties.**

Definition 1.10. Suppose that G is a Lie group acting on symplectic manifold M such that the action is Hamiltonian. Let $\mu : M \to \mathfrak{g}^*$ be the associated moment map and let λ be a regular value of μ. Then the symplectic reduction is defined to be $\mu^{-1}(\lambda)/G$. If λ and μ are given, this will be abbreviated as $M//G$.

Theorem 1.11 (Marsden-Weinstein)**.** *$M//G$ is a symplectic manifold.*

Theorem 1.12. *If G is compact, the quotient map $\pi : \mu^{-1}(\lambda) \to M//G$ is proper.*

Proof. We need to show that π is closed with compact fibers. The fact that π is closed reduces to the fact that when G is compact, the $G-$orbit of a closed set is also a closed set. The fibers of π are all of the form $G \cdot m$ where $m \in M$ which is diffeomorphic to G/G_m where G_m is the stabilizer of m. Since G is compact, and $G \to G/G_m$ is surjective and continuous, it follows that G/G_m is also compact. □

Thus when G is compact, it follows from above that $\mu^{-1}(\lambda) \times M//G$ gives us a correspondence from M to the symplectic reduction $M//G$ such that we get an induced map on Borel-Moore homology. Moreover, since λ is a regular value, μ is a submersion at λ and it follows that $\mu^{-1}(\lambda)$ has codimension $\dim \mathfrak{g}^* = \dim G$ in M. It follows that

$$\dim M//G = \dim \mu^{-1}(\lambda) - \dim G = \dim M - 2\dim G \tag{1.13}$$

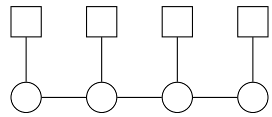
We want to apply the formalism above to $M = T^*\mathcal{FL}(k,n+k;2n)$ so we want to find a compact G such that $M//G = T^*\mathrm{Gr}(k,n)$. First note that

$$\mathcal{FL}(k,n+k;2n) \cong \frac{\mathrm{GL}_{2n}}{P_{k,n,n-k}} \qquad \mathrm{Gr}(k,n) \cong \frac{\mathrm{GL}_n}{P_{k,n-k}}$$

where $P_{k,n,n-k}$ and $P_{k,n-k}$ are the parabolic subgroups corresponding to the partition $k+n+n-k$ of $2n$ and $k+n-k=n$ of $2n$ and n respectively. As a result, one can then compute their dimensions by computing the dimensions of the tangent spaces resulting in

$$\dim T^*\mathcal{FL}(k,n+k;2n) = 2n^2 + 2k(n-k) \qquad \dim T^*\mathrm{Gr}(k,n) = 2k(n-k)$$

By Eq. (1.13), it follows that we should have $\dim G = n^2$. So where are we going to find a compact group of dimension n^2 acting on $T^*\mathcal{FL}(k,n+k;2n)$? It turns out that both $T^*\mathcal{FL}(k,n+k;2n)$ and $T^*\mathrm{Gr}(k,n)$ are Nakajima quiver varieties, and these come with natural actions of "$\mathrm{GL}(\vec{w})$" on the "framed" vertices. We won't define these varieties, but will say how these varieties are indexed. Given a Dynkin diagram I, construct the Nakajima diagram of I by attaching a framed vertex hanging off each vertex of I. For example the Nakajima diagram for A_4 looks like



Given a Nakajima diagram, by filling in the vertices and framed vertices of I with natural numbers with natural numbers $\vec{w} = (w^i)_{i\in I}$ and $\vec{v} = (v^i)_{i\in I}$, there is a procedure to turn this datum into an algebraic variety $\mathcal{M}(I, \vec{w}, \vec{v})$. For example,

$$\mathcal{M}(A_1, (n), (k)) = \begin{tikzcd} \boxed{n} \arrow[d, dash] \\ k \end{tikzcd} \cong T^*\mathrm{Gr}(k, n)$$

We should note that if $w^i = 0$ then we will not draw the framed vertex at i. In general if $n_1 < \ldots < n_{d-1} < n_d$ then we will have that

$$\begin{tikzcd} \boxed{n} \arrow[d, dash] & & & \\ n_d \arrow[r, dash] & n_{d-1} \arrow[r, dash] & \cdots \arrow[r, dash] & n_1 \end{tikzcd} \cong T^*\mathcal{FL}(n_1, \ldots, n_d; n)$$

In our case we will be working with the Nakajima quiver variety

$$\begin{tikzcd} \boxed{2n} \arrow[d, dash] & \\ n+k \arrow[r, dash] & k \end{tikzcd} \cong T^*\mathcal{FL}(k, n+k; 2n)$$

The framed vertex $\boxed{2n}$ has an action of GL_{2n}. If we write $T^*\mathcal{FL}(k, n+k; 2n)$ in Springer coordinates, e.g.

$$T^*\mathcal{FL}(k, n+k; 2n) = \left\{ (X, V^{n+k} \supset W^k) \,|\, X \in \mathrm{End}(\mathbb{C}^{2n}), \mathbb{C}^{2n} \xrightarrow{X} V^{n+k} \xrightarrow{X} W^k \xrightarrow{X} 0 \right\}$$

where V^{n+k} is a $n+k$ dimensional subspace of $\mathbb{C}^{2n}$ and W^k is a k dimensional subspace in V^{n+k}, then the action of $g \in \mathrm{GL}_{2n}$ is

$$g \cdot (X, V^{n+k} \supset W^k) = (gxg^{-1}, g(V^{n+k}) \supset g(W^k)) \tag{1.14}$$

Inside GL_{2n} we have the unitary subgroup

$$U_n := \left\{ \begin{bmatrix} I_n & 0 \\ E & I_n \end{bmatrix} \middle| E \in \mathrm{Mat}_{n\times n} \right\} \subseteq \mathrm{GL}_{2n}$$

which is a compact group of dimension n^2 so this fits our criteria from above. The moment map for the action of U_n turns out to send X to the northeast $n \times n$ quadrant of X when written in block matrix form. We will let $\lambda = I_n$ and as a result we have that

$$T^*\mathcal{FL}(k, n{+}k; 2n)//U_n = \left\{ (X, V^{n+k} \supset W^k) \,\middle|\, X = \begin{bmatrix} A & I_n \\ C & D \end{bmatrix}, \mathbb{C}^{2n} \xrightarrow{X} V^{n+k} \xrightarrow{X} W^k \xrightarrow{X} 0 \right\} /U_n$$

Proposition 1.15.

$$T^*\mathcal{FL}(k, n+k; 2n)//U_n \cong T^*\mathrm{Gr}(k, n)$$

Proof. The fiber of the moment map imposes more conditions on the set of tuples (X, V, W) than described above. In particular, let $\vec{v} = (\vec{v_1}\ \vec{v_2})^T \in \ker X$. Then it follows from

$$\begin{bmatrix} A & I_n \\ C & D \end{bmatrix} \begin{pmatrix} \vec{v_1} \\ \vec{v_2} \end{pmatrix} = \begin{pmatrix} A\vec{v_1} + \vec{v_2} \\ C\vec{v_1} + D\vec{v_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

that $\vec{v_2} = -A\vec{v_1}$ and thus $(C - DA)\vec{v_1} = 0$. We claim that $\dim \ker X \geq n$ from which it follows that $(C - DA)\vec{v_1} = 0\ \forall \vec{v_1} \in \mathbb{C}^n$ or in other words $C = DA$. Because X preserves the flag, X restricts to a map

$$X|_V : V \to W$$

Because $\dim V = n + k$ and $\dim W = k$, by rank-nullity it follows that

$$\dim \ker X|_V = (n + k) - \dim \mathrm{im} X|_V \geq (n + k) - k = n$$

and thus $C = DA$ as desired. In fact much more is true, as from before we know that

$$\ker X \subseteq \left\{ \begin{bmatrix} v \\ -Av \end{bmatrix}, v \in \mathbb{C}^n \right\} = V_0$$

and the subspace on the right has dimension n. Therefore

$$\dim \ker X = \dim \ker X|_V = n \implies \ker X = V_0 \subseteq V \text{ and } \mathrm{im} X|_V = W$$

Another consequence of the moment map condition is that since $X^3 = 0$ we have

$$X^3 = \begin{bmatrix} * & C + A^2 + AD + D^2 \\ * & * \end{bmatrix} = 0$$

As $C = DA$ the top right entry will be $(A + D)^2$ and so $(A + D)^2 = 0$.

We now consider the action of U_n. We claim that any element in $\mu^{-1}(I_n)$ is in the orbit of elements of the form

$$\left(X = \begin{bmatrix} 0 & I_n \\ 0 & F \end{bmatrix}, \mathbb{C}^n \oplus M \supset X(M) \right)$$

Given $(X, V^{n+k} \supset W^k) \in \mu^{-1}(I_n)$, let $V' := (0 \oplus \mathbb{C}^n) \cap V$. We claim that $V = \ker X \oplus V'$. The above description of $\ker X = V_0$ shows that $\ker X \cap V' = \{0\}$ and it is easy to see that these two subspaces span V. From this we see that $X|_{V'}$ gives an isomorphism $V' \xrightarrow{\simeq} W$ and thus W is extraneous data, as it can be recovered from X and V. Let $g = \begin{bmatrix} I_n & 0 \\ A & I_n \end{bmatrix}$. By the definition of the action Eq. (1.14) one can compute that

$$g \cdot (X, V) = g \cdot \left(\begin{bmatrix} A & I_n \\ DA & D \end{bmatrix}, \ker X \oplus V' \right) = \left(\begin{bmatrix} 0 & I_n \\ 0 & A + D \end{bmatrix}, \mathbb{C}^n \oplus V' \right)$$

which is exactly of the form above. Now, the RHS only depends on the datum of $A + D$ and V' and we claim that they in fact satisfy the conditions to be in $T^*\mathrm{Gr}(k, n)$. As $(A + D)^2 = 0$ from above, we only need to check that $(A + D)v \in V'\ \forall v \in \mathbb{C}^n$. This follows from $V_0 \subset V$ and so

$$\begin{pmatrix} 0 \\ (A + D)v \end{pmatrix} = \begin{pmatrix} v \\ Dv \end{pmatrix} - \begin{pmatrix} v \\ -Av \end{pmatrix} \in V$$

□

1.4. **Finale.** As explained in the paragraph before Eq. (1.13), the Hamiltonian reduction now gives us a correspondence

$$hr : T^*\mathcal{FL}(k, n+k; 2n) \to T^*\mathrm{Gr}(k,n)$$

and as alluded to at the beginning of Section 1.2 we have the following theorem

Theorem 1.16 (Knutson, Zinn-Justin, 2021)**.** *The two Lagrangian correspondences $senv, hr$*

$$\begin{tikzcd}[column sep=small] \boxed{n} \arrow[d, no head] & & & \boxed{n} \arrow[d, no head] & & & \boxed{2n} \arrow[d, no head] & & & & \boxed{n} \arrow[d, no head] \\ k \arrow[r, no head] & 0 & \times & n \arrow[r, no head] & k & \xrightarrow{senv} & n+k \arrow[r, no head] & k & \xrightarrow{hr} & k \arrow[r, no head] & k \end{tikzcd}$$

can be composed. Under the identification of first and third spaces with $T^*\mathrm{Gr}(k,n)^2$ and $T^*\mathrm{Gr}(k,n)$, the composite is the transpose $C_{graph(\Delta)^T}$ of the conormal bundle of the graph of the diagonal inclusion.

We should note that in order to make the identifications with $T^*\mathrm{Gr}(k,n)$ we are using the following theorem of Nakajima,

Theorem 1.17 (Nakajima, 2003)**.** *Let $i \in I$, define*

$$r_i(\vec{v})^j := \begin{cases} \vec{v}^j & \text{if } j \neq i \\ \text{sum of all adjacent labels } - \vec{v}^i & \text{if } j = i \end{cases}$$

Then

$$\mathcal{M}(I, \vec{w}, \vec{v}) \cong \mathcal{M}(I, \vec{w}, r_i(\vec{v}))$$

as complex varieties, equivariantly w.r.t. the framing group action $\prod_{i \in I} GL(w^i)$ on both sides.

Applying Nakajima's theorem above we find that

$$\begin{tikzcd}[column sep=small] \boxed{n} \arrow[d, no head] & & & \boxed{n} \arrow[d, no head] & & & \boxed{n} \arrow[d, no head] & & & & \boxed{n} \arrow[d, no head] & & & \boxed{n} \arrow[d, no head] \\ n \arrow[r, no head] & k & \cong & k \arrow[r, no head] & k & \cong & k \arrow[r, no head] & 0 & & k \arrow[r, no head] & k & \cong & 0 \arrow[r, no head] & k \end{tikzcd}$$

Remark 1.18. We can actually write out the two correspondences very explicitly in Springer coordinates

$$\begin{tikzcd}[column sep=tiny] & senv := \left\{ ((A, V', D, W'), (X, V, W)) \ : \ X = \begin{pmatrix} D & * \\ 0 & A \end{pmatrix}, \begin{array}{rcl} V & = & \mathbb{C}^n \oplus V' \\ W & = & W' \oplus 0 \end{array} \right\} \arrow[dl] \arrow[dr] & \\ \{(A \in \mathrm{End}(\mathbb{C}^n), V'^{\,j})\} \times \{(D \in \mathrm{End}(\mathbb{C}^n), W'^{\,j})\} & & \{(X \in \mathrm{End}(\mathbb{C}^{2n}), V^{n+j}, W^j)\} \end{tikzcd}$$

$$\begin{tikzcd}[column sep=tiny] & hr := \left\{ ((X, V, W), (Y, V'')) \ : \ \begin{array}{rcl} & X = \begin{pmatrix} A & Id \\ DA & D \end{pmatrix}, \ Y = A + D & \\ V \cap (0 \oplus \mathbb{C}^n) & = & 0 \oplus V'' \\ W/(0 \oplus \mathbb{C}^n) & = & (W' + \mathbb{C}^n)/(0 \oplus \mathbb{C}^n) \end{array} \right\} \arrow[dl] \arrow[dr] & \\ \{(X \in \mathrm{End}(\mathbb{C}^{2n}), V^{n+j}, W^j)\} & & \{(Y \in \mathrm{End}(\mathbb{C}^n), V''^{\,j})\} \end{tikzcd}$$