LAWRGE 2023 NOTES

Contents

1. Lecture 1 (Monday)

Throughout the week we will use the language of TQFTs to relate physics and math. The goal of this talk is to introduce the relevant terminology and some definitions.

1.1. TQFTs.

1.2. Definition.

We begin by introducing the notion of a d-dimensional TQFT.

Definition 1.1. Let M, N be closed oriented $(d-1)$-manifolds. A ***d-dimensional cobordism W from M to N*** is an oriented d-dimensional manifold W together with an identification $\partial W \cong \overline{M} \coprod N$.

Remark 1.2. *There is also a notion of an unoriented cobordism between two unoriented manifolds, framed cobordism between two framed manifolds and, more generally, a cobordism equipped with a tangential structure.*

Cobordisms define a symmetric monoidal category $\mathrm{Cob}^{or}_{d,d-1}$ as follows:

- Its objects are closed oriented $(d-1)$-manifolds.

- Morphisms from M to N are diffeomorphism classes of oriented cobordisms from M to N.
- Composition of a cobordism W_1 from M to N and a cobordism W_2 from N to O is given by the cobordism $W_1 \coprod_N W_2$ from M to O.
- The symmetric monoidal structure is given by disjoint union of manifolds.

Definition 1.3. An ***oriented d-dimensional TQFT*** is a symmetric monoidal functor $Z\colon \mathrm{Cob}_{d,d-1} \to \mathrm{Vect}$ to the category of ($\mathbb{C}$-)vector spaces with tensor product as the monoidal structure.

The physical idea of the definition is as follows:

- For a closed $(d-1)$-manifold M we have a vector space $Z(M)$. It is the vector space of states of the TQFT (often a Hilbert space in physical examples).
- For a closed d-manifold W we have a number $Z(W)$. It is the partition function of the TQFT on W.
- For a cobordism W from M to N we get a linear map $Z(W)\colon Z(M) \to Z(N)$. It is the *transition amplitude* (S-matrix) associated to the cobordism W.

1.3. **Extending down.** Given a decomposition $W = W_1 \coprod_M W_2$ of a closed oriented d-manifold into a union of two manifolds along their common boundary, one can compute the partition function as

$$Z(W) = Z(W_2)(Z(W_1)(1)),$$

where

$$Z(W_1)\colon \mathbb{C} \longrightarrow Z(M), \qquad Z(W_2)\colon Z(M) \longrightarrow \mathbb{C}.$$

This allows one to compute the partition function by decomposing a manifold into pieces. This is related to the principle of locality of a QFT. Full locality will also allow us to compute the partition function by decomposing the boundary M into pieces. This can be made precise by extending the category $\mathrm{Cob}^{or}_{d,d-1}$ to a 2-category or even a higher category as follows. Let $\mathrm{Cob}^{or}_{d,d-1,d-2}$ be the symmetric monoidal 2-category as follows:

- Its objects are closed oriented $(d-2)$-manifolds.
- 1-morphisms from M to N are oriented $(d-1)$-dimensional cobordisms W from M to N.
- 2-morphisms from $W_1\colon M \to N$ to $W_2\colon M \to N$ are diffeomorphism classes of d-dimensional cobordisms between W_1 and W_2.

One can also extend it all the way down and define the symmetric monoidal d-category Cob^{or}_d whose objects are closed oriented 0-manifolds (disjoint unions of oriented points), 1-morphisms are 1-dimensional cobordisms and so on.

To define TQFTs we also need to extend the target category Vect down. For instance, for once-extended TQFTs we are looking for a symmetric monoidal bicategory $\mathcal{C}$ (usually it is the bicategory of some class of categories) with the property that $\mathrm{Hom}_{\mathcal{C}}(1,1) \cong \mathrm{Vect}$.

Similarly, for fully extended TQFTs we are looking for a symmetric monoidal d-category $\mathcal{C}$ with a similar property for top-level morphisms.

Definition 1.4. Let $\mathcal{C}$ be a (linear) symmetric monoidal d-category. A ***fully extended TQFT*** is a symmetric monoidal functor $Z\colon \mathrm{Cob}_d^{or} \to \mathcal{C}$.

Note that given any fully extended TQFT we obtain higher-categorical structures irrespectively of the target $\mathcal{C}$:

- If M is a closed oriented $(d-1)$-manifold, $Z(M)$ is a vector space. We can think of $Z(M)$ as an element of the vector space $\mathrm{Hom}_{\mathcal{C}}(Z(\varnothing^{d-1}), Z(M))$.
- If M is a closed oriented $(d-2)$-manifold, $\mathrm{Hom}_{\mathcal{C}}(Z(\varnothing^{d-2}), Z(M))$ is a category. In fact, the structure of an oriented TQFT will induce a Calabi–Yau structure on this.
- ...

1.4. **Extending up.** Let M be a closed oriented d-manifold and $\mathrm{Diff}(M)$ the topological group of orientation-preserving diffeomorphisms of M. There is a natural map

$$\mathrm{MCG}(M) = \pi_0 \mathrm{Diff}(M) \longrightarrow \mathrm{Aut}_{\mathrm{Bord}_{d,d-1}^{or}}(M)$$

given by considering $W = M \times [0,1]$ with the identification $\partial W \cong \overline{M} \coprod M$ twisted by a diffeomorphism. The reason that isotopic diffeomorphisms give rise to the same morphisms is that in the definition of $\mathrm{Bord}_{d,d-1}^{or}$ we identify diffeomorphic cobordisms.

The full homotopy type of the diffeomorphism group can be encoded if we work in the framework of ∞-categories. Namely, there is a symmetric monoidal ∞-category $\mathrm{Bord}_{d,d-1}^{or}$ which has the following informal description:

- Its objects are closed oriented $(d-1)$-manifolds.
- 1-morphisms from M to N are oriented cobordisms from M to N.
- 2-morphisms are given by diffeomorphisms of cobordisms.
- 3-morphisms are given by isotopies of diffeomorphisms.
- ...

Similarly, there is a symmetric monoidal (∞, d)-category Bord_d^{or}.

Definition 1.5. Let $\mathcal{C}$ be a symmetric monoidal (∞, d)-category. A ***fully extended TQFT*** is a symmetric monoidal functor $Z\colon \mathrm{Bord}_d^{or} \to \mathcal{C}$.

In physics the state space on a closed oriented $(d-1)$-manifold M is often a chain complex $Z(M) \in \mathrm{Ch}$ (with the differential the BRST differential coming from gauge symmetries and/or supersymmetric twisting). So, while on the level of cohomology there is an action of the mapping class group $\mathrm{MCG}(M)$ on $\mathrm{H}^\bullet(Z(M))$, on the chain level it should extend to a homotopy-coherent action of $\mathrm{C}_\bullet(\mathrm{Diff}(M))$ (equipped with the Pontryagin product) on the chain complex $Z(M)$. We will encounter the following two versions of this action:

- Consider a d-dimensional TQFT Z (for $d \geq 2$) and the chain complex $Z(S^{d-1})$. The natural S^1-action on S^{d-1} induces a $\mathbb{C}_\bullet(S^1)$-action on the chain complex $Z(S^{d-1})$. This action boils down to a square-zero degree -1 operation $B\colon Z(S^{d-1}) \to Z(S^{d-1})$.
- Consider a d-dimensional TQFT Z (for $d \geq 3$) and the category $Z(S^{d-2})$. The natural S^1-action on S^{d-2} induces a natural automorphism of the identity functor on $Z(S^{d-2})$.

If M is a closed oriented d-manifold, the partition function $Z(M)$ is merely a number. So, the higher-categorical structure is irrelevant in this case and we simply have that $Z(M)$ is invariant under $\mathrm{Diff}(M)$. It turns out to be useful to phrase this condition by saying that

$$Z(M) \in \mathrm{H}^0(\mathrm{BDiff}(M); \mathbb{C}).$$

1.5. **Boundary conditions.** The notion of a relative TQFT was introduced by Freed–Teleman and Johnson-Freyd–Scheimbauer. We will not give a precise definition, but will just indicate the main idea.

Suppose $Z\colon \mathrm{Cob}^{or}_{d,d-1} \to \mathrm{Vect}$ be a d-dimensional TQFT. The partition function $Z(M)$ *as a number* makes sense only for a closed oriented d-manifold. Given a *boundary condition*, we can evaluate the theory on manifolds with boundary as follows:

- For any closed oriented $(d-1)$-manifold M we have the space of states $Z(M)$.
- The boundary condition defines a distinguished vector $Z^\partial(M) \in Z(M)$.
- Given a compact oriented d-manifold W with boundary M we may view it as a cobordism $M \to \varnothing$. In particular,

$$Z(W)\colon Z(M) \longrightarrow \mathbb{C}.$$

  So, the partition function of the TQFT on W with the given boundary condition is

$$Z(W)(Z^\partial(M)).$$

We can also talk about boundary conditions to once-extended or fully extended TQFTs. Then:

- For any closed oriented $(d-k)$-manifold M we have a $(k-1)$-category $\mathrm{Hom}_{\mathcal{C}}(Z(\varnothing^{d-k}), Z(M))$.
- The boundary condition defines a distinguished object $Z^\partial(M) \in \mathrm{Hom}_{\mathcal{C}}(Z(\varnothing^{d-k}), Z(M))$.

For instance, on the level of the point we get a distinguished object $Z^\partial(\mathrm{pt}) \in \mathrm{Hom}_{\mathcal{C}}(1, Z(\mathrm{pt}))$ in the $(d-1)$-category $\mathrm{Hom}_{\mathcal{C}}(1, Z(\mathrm{pt}))$. So, we can think of this as the $(d-1)$-category of *boundary conditions* (more precisely, fully local boundary conditions correspond to suitably dualizable objects of this $(d-1)$-category).

1.6. **2d mirror symmetry.** I will end this lecture by explaining the TQFT ideas behind the usual two-dimensional mirror symmetry as a warm up for three-dimensional mirror symmetry. We have the following 2d TQFTs:

- Let M be a symplectic manifold. Then one can define the 2d A-model $Z_{2dA,M}$. The category of boundary conditions $Z_{2dA,M}(\mathrm{pt})$ is some version of the Fukaya category of M.
- M be a smooth complex algebraic variety. Then one can define the 2d B-model $Z_{2dB,M}$. The category of boundary conditions $Z_{2dB,M}(\mathrm{pt})$ is some version of the derived category of coherent sheaves on M.

There are also equivariant versions of these 2d TQFTs:

- Given a (real) Lie group G acting in a Hamiltonian way on a symplectic manifold M there is an equivariant 2d A-model.
- Given a complex algebraic group $G_{\mathbb{C}}$ acting on a smooth complex algebraic variety M there is an equivariant 2d B-model.

Remark 1.6. *If M is not compact, these TQFTs are not defined on all 2-dimensional cobordisms.*

Remark 1.7. *Even though the framed TQFTs are well-defined, there is an "orientation anomaly" which complicates the definition of the oriented TQFT. The partition function on a surface Σ_g of genus g defines an element of $\mathrm{H}^{2(g-1)\dim_{\mathbf{R}} M}(\mathrm{BDiff}(\Sigma_g);\mathbb{C})$ rather than an element of $\mathrm{H}^0(\mathrm{BDiff}(\Sigma_g);\mathbb{C})$. For instance, the underlying number is zero for $g \neq 1$.*

The statement of 2-dimensional homological mirror symmetry can be formulated as follows. We say a symplectic manifold M is 2d mirror to a complex algebraic variety $M^\vee$ if

$$Z_{2dA,M} \cong Z_{2dB,M^\vee}.$$

This contains the following statements:

- An equivalence of Calabi-Yau categories.

$$Z_{2dA,M}(\mathrm{pt}) \cong Z_{2dB,M^\vee}(\mathrm{pt}).$$

  In practice the left-hand side is a version of the Fukaya category of M and the right-hand side is a version of the derived category of coherent sheaves on $M^\vee$.
- An equivalence of commutative algebras

$$\mathrm{H}^\bullet(Z_{2dA,M}(S^1)) \cong \mathrm{H}^\bullet(Z_{2dB,M^\vee}(S^1)).$$

  In fact, there is a Gerstenhaber structure (explained in the next lecture) on both sides which is also preserved.

2. Lecture 2 (Monday)

In this lecture Z denotes some d-dimensional TQFT. For simplicity I will assume that it is fully extended, but many statements make sense with partially extended TQFTs. I will assume that the TQFT is valued in the (∞-)category of chain complexes.

2.1. **Local and line operators.** Besides computing partition functions, in physics one is often interested in computing correlation functions of some local operators. Let us introduce them using the following heuristic idea.

Suppose M is a closed oriented d-manifold and $x \in M$ is a point with an insertion of a "local operator" $\mathcal{O}$. By locality one should be able to compute the partition function as follows:

- Consider a ball $D \subset M$ around x and let $S^{d-1} \subset M$ be its boundary.
- $Z(M \setminus D)$ defines a map $Z(S^{d-1}) \to \mathbb{C}$. The local operator defines a map $Z(D_{\mathcal{O}})\colon \mathbb{C} \to Z(S^{d-1})$ and the partition function on M is the composite of these two maps.

If we are being agnostic about local operators, we may observe that the only thing we have used about them is the vector of $Z(S^{d-1})$ that they define. This leads us to the following definition.

Definition 2.1. Let Z be a d-dimensional TQFT. The ***space of local operators*** is the chain complex $Z(S^{d-1})$.

One also considers defects given by extended objects: lines, surfaces, ... embedded in M. Besides local operators, we will only encounter line operators this week. We can think of them as follows:

- A line operator is specified by a defect supported on a knot $K \subset M$. The same analysis as before shows that we can compute the partition function if we know the corresponding vector in $Z(S^{d-2} \times K)$.
- One often only considers "local" line operators which themselves obey cutting and gluing axioms of a TQFT. These local line operators define an object of the category $Z(S^{d-2})$.

This motivates the following definition.

Definition 2.2. Let Z be a d-dimensional TQFT.

- The ***space of line operators*** is $Z(S^{d-2} \times S^1)$.
- The ***category of line operators*** is $Z(S^{d-2})$.

2.2. $\mathbb{E}_d$**-algebras.** Our next goal is to explain algebraic structures present on the space of local and line operators. Given any cobordism W from k copies of S^{d-1} to S^{d-1} we get an algebraic operation

$$Z(W)\colon Z(S^{d-1})^{\otimes k} \longrightarrow Z(S^{d-1})$$

on the space of local operators in any TQFT. We will now investigate operations coming from cobordisms "with no topology".

Definition 2.3. Fix a dimension d.

- $$\mathbb{E}_d(k) = \mathrm{Emb}^{fr}(D^{\amalg k}, D)$$
  is the space of (smooth) framed embeddings of k d-dimensional open disks D into a given disk D.
- $$\mathbb{E}_d^{fr}(k) = \mathrm{Emb}(D^{\amalg k}, D)$$
  is the space of (smooth) oriented embeddings of k d-dimensional open disks D into a given disk D.

There are natural composition maps which make $\mathbb{E}_d$ and $\mathbb{E}_d^{fr}$ into operads. In particular, we can talk about their algebras.

Example 2.4. *The operads $\mathbb{E}_1$ and $\mathbb{E}_1^{fr}$ are both equivalent to the associative operad.*

Example 2.5. *There is a natural action of $\mathrm{SO}(d)$ on $\mathbb{E}_d$, so that a $\mathbb{E}_d^{fr}$-algebra is an $\mathbb{E}_d$-algebra equipped with a compatible $\mathrm{SO}(d)$-action.*

Given an embedding $D^{\amalg k} \hookrightarrow D$ we obtain a cobordism from $(S^{d-1})^{\amalg k}$ to S^{d-1} by removing the interiors of the embedded disks. In particular, we obtain a natural map

$$\mathrm{C}_\bullet(\mathbb{E}_d^{fr}(k); \mathbb{C}) \otimes_{\mathbb{C}} Z(S^{d-1})^{\otimes k} \longrightarrow Z(S^{d-1})$$

for any oriented TQFT. Similarly, if Z is a framed TQFT we get a natural map

$$\mathrm{C}_\bullet(\mathbb{E}_d(k); \mathbb{C}) \otimes_{\mathbb{C}} Z(S^{d-1})^{\otimes k} \longrightarrow Z(S^{d-1}).$$

Both maps are compatible with compositions and we obtain the following result:

- If Z is a framed TQFT, the chain complex of local operators $Z(S^{d-1})$ is an $\mathbb{E}_d$-algebra.
- If Z is an oriented TQFT, the chain complex of local operators $Z(S^{d-1})$ is a framed $\mathbb{E}_d$-algebra.

Up to homotopy the spaces of embeddings may be identified as follows.

Proposition 2.6. *There are homotopy equivalences*

$$\mathbb{E}_d(k) \cong \mathrm{Conf}_k(\mathbf{R}^d), \qquad \mathbb{E}_d^{fr}(k) \cong \mathrm{SO}(d)^k \times \mathrm{Conf}_k(\mathbf{R}^d),$$

where $\mathrm{Conf}_k(\mathbf{R}^d)$ is the configuration space of k distinct ordered points in $\mathbf{R}^d$.

Using the above description one can show the following:

- An $\mathbb{E}_2$-algebra in categories is a braided monoidal category.
- An $\mathbb{E}_2^{fr}$-algebra in categories is a balanced monoidal category, i.e. there is an extra automorphism of the identity functor, the *balancing* θ, which satisfies
  $$\theta_{x\otimes y} = \sigma_{y,x} \circ \sigma_{x,y} \circ (\theta_x \otimes \theta_y).$$

So, the category of line operators in a 3-dimensional TQFT is a balanced monoidal category.

2.3. $\mathbb{P}_d$**-algebras.** To describe $\mathbb{E}_d$-algebras in chain complexes, let us first introduce a related notion.

Definition 2.7. A $\mathbb{P}_d$***-algebra*** is a commutative dg algebra A equipped with a bracket of cohomological degree $1-d$ (inducing a Lie structure on $A[d-1]$) satisfying the Leibniz rule

$$\{a, bc\} = \{a, b\}c + (-1)^{|b||c|}\{a, c\}b$$

for $a, b, c \in A$.

Remark 2.8. *A* $\mathbb{P}_2$*-algebra is known as a* Gerstenhaber algebra*.*

Let $\mathbb{P}_d(k)$ be the vector space of all operations $A^{\otimes k} \to A$ on a $\mathbb{P}_d$-algebra. We can formalize it as follows: define $\mathbb{P}_d(k)$ to be the subspace of the free $\mathbb{P}_d$-algebra on degree 0 variables $x_1, \dots, x_k$ consisting of expressions where each x_i appears exactly once. For instance:

- $\mathbb{P}_d(1) \cong \mathbb{C}$ spanned by the identity map $A \to A$.
- $\mathbb{P}_d(2) \cong \mathbb{C} \oplus \mathbb{C}[d-1]$ spanned by the commutative multiplication $m\colon A \otimes A \to A$ and $\{-,-\}\colon A \otimes A \to A[1-d]$ by the Poisson bracket.

We have the following claim.

Definition 2.9. Suppose $d \geq 2$. Then there is an isomorphism of graded vector spaces $\mathrm{H}_\bullet(\mathbb{E}_d(k); \mathbb{C}) \cong \mathbb{P}_d(k)$.

Remark 2.10. *In fact, both* $\mathbb{E}_d$ *and* $\mathbb{P}_d$ *are operads and there is an equivalence* $\mathrm{C}_\bullet(\mathbb{E}_d; \mathbb{C}) \cong \mathbb{P}_d$ *of graded linear operads.*

As a corollary, given an $\mathbb{E}_d$-algebra A, its homology $\mathrm{H}_\bullet(A)$ has a natural structure of a $\mathbb{P}_d$-algebra.

Example 2.11. *Let Z be a 3d TQFT. Then the cohomology of the space of local operators* $\mathrm{H}^\bullet(Z(S^2))$ *carries a graded commutative multiplication as well as Poisson bracket of degree* -2*.*

2.4. Ω**-deformation.** We will now explain an important construction with $\mathbb{E}_d$-operads which is known in physics as the procedure of Ω-deformation.

Consider the $\mathbb{E}_d$-operad equipped with its natural $\mathrm{SO}(d)$-action. There is a natural inclusion of operads $\mathbb{E}_{d-2} \hookrightarrow \mathbb{E}_d$ which is $\mathrm{SO}(d-2) \times \mathrm{SO}(2)$-equivariant, where the $\mathrm{SO}(2)$-action on the left is trivial.

Theorem 2.12. *The inclusion of operads* $\mathbb{E}_{d-2} \hookrightarrow \mathbb{E}_d$ *realizes* $\mathbb{E}_{d-2}$ *as the space of fixed points of the* $\mathrm{SO}(2)$*-action on* $\mathbb{E}_d$*.*

To state an important corollary, let us first recall a few basics of equivariant localization. Given a space X with an action of a topological group G we may consider equivariant homology $\mathrm{H}^G_\bullet(X)$ and cohomology $\mathrm{H}^\bullet_G(X)$ which are both modules over $\mathrm{H}^\bullet_G(\mathrm{pt}) = \mathrm{H}^\bullet(\mathrm{B}G)$.

Example 2.13. *We have* $\mathrm{BSO}(2) = \mathbf{CP}^\infty$*, so* $\mathrm{H}^\bullet(\mathrm{BSO}(2)) = \mathbb{C}[\epsilon]$*, where* $\deg(\epsilon) = 2$*.*

Theorem 2.14 (Equivariant localization)**.** *Let G be a topological group. Let Y be a space with a G-action. Let X be a topological space equipped with a trivial G-action and a G-equivariant map $X \to Y$ which realizes X as the space of fixed points of the G-action on Y. Then the induced map*

$$\mathrm{H}^G_\bullet(X) \otimes_{\mathbb{C}[\epsilon]} \mathbb{C}(\epsilon) \longrightarrow \mathrm{H}^G_\bullet(Y) \otimes_{\mathbb{C}[\epsilon]} \mathbb{C}(\epsilon)$$

is an isomorphism.

Combining the theory of equivariant localization and 2.12 we get the following.

Theorem 2.15. *Let A be a framed $\mathbb{E}_d$-algebra. Consider the induced* $\mathrm{SO}(2) \subset \mathrm{SO}(d)$*-action on A. Then* $A^{\mathrm{SO}(2)} \otimes_{\mathbb{C}[\epsilon]} \mathbb{C}(\epsilon)$ *is a framed $\mathbb{E}_{d-2}$-algebra.*

Example 2.16. *Let Z be a 3d TQFT and consider the framed $\mathbb{E}_3$-algebra structure on the space of local operators $Z(S^2)$. Recall that its cohomology* $\mathrm{H}^\bullet(Z(S^2))$ *carries a natural (graded) Poisson structure. The equivariant localization*

$$\mathrm{H}^\bullet_{\mathrm{SO}(2)}(Z(S^2)) \otimes_{\mathbb{C}[\epsilon]} \mathbb{C}(\epsilon)$$

is an associative algebra which provides a deformation quantization of $\mathrm{H}^\bullet(Z(S^2))$ *(with the quantization parameter being ϵ).*

2.5. **Swiss-cheese algebras.** I will end this lecture by describing algebraic structures appearing in TQFTs with boundary conditions.

Let Z be a d-dimensional TQFT with a chosen boundary condition Z^∂. We can extract the following kinds of algebras:

- The space of *bulk local operators* in Z, i.e.
$$A = Z(S^{d-1}),$$
carries the structure of a framed $\mathbb{E}_d$-algebra.
- The space of *boundary local operators*, i.e.
$$B = Z(D^{d-1})(Z^\partial(S^{d-2})),$$
carries the structure of a framed $\mathbb{E}_{d-1}$-algebra. Namely, let H be the d-dimensional half-ball. Consider the space
$$\mathrm{Emb}^\partial(H^{\amalg k}, H)$$
of oriented embeddings of k d-dimensional half-balls into a single one so that the boundaries are embedded into the boundaries. Retracting the half-ball to its boundary (a $(d-1)$-dimensional ball) identifies this space with $\mathbb{E}^{fr}_{d-1}$.

- In addition, there is an action of A on B as follows. Let D be a d-dimensional ball and H a d-dimensional half-ball. Any embedding $D^{\amalg l} \coprod H^{\amalg k} \hookrightarrow H$ (so that the boundaries of the half-balls are embedded in the boundary of the bigger half-ball) gives rise to an operation

$$A^{\otimes l} \otimes B^{\otimes k} \longrightarrow B.$$

The pair (A, B) together with the operations described above is known as a d-dimensional ***Swiss-cheese algebra***. We will encounter the following manifestation of this structure.

Example 2.17. *Let Z be a 3d TQFT with a boundary condition Z^∂. Let A be the $\mathbb{E}_3$-algebra of bulk local operators and B the $\mathbb{E}_2$-algebra of boundary local operators. A carries a degree -2 Poisson structure. There is a map of graded commutative algebras $A \to B$ and the induced map* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is coisotropic.*

Remark 2.18. *There is a homotopy notion of a coisotropic submanifold which is precisely defined in terms of a Swiss-cheese algebra structure.*

Recall that $\mathbb{E}_n(k)$ is homotopy equivalent to $\operatorname{Conf}_k(\mathbf{R}^n)$, the configuration space of k distinct points in $\mathbf{R}^n$.

Recall that a $\mathbb{P}_n$-algebra is a dg commutative algebra A equipped with a bracket $\{-,-\}$ of cohomological degree $1-n$ (inducing a Lie structure on $A[n-1]$) satisfying the relation $\{a, bc\} = \{a, b\}c + (-1)^{|b||c|}\{a, c\}b$. Let $\mathbb{P}_n(k)$ be the subspace of the free $\mathbb{P}_n$-algebra on degree 0 variables $x_1, \dots, x_k$ consisting of expressions where each x_i appears exactly once. For instance, $\{\{x_1, x_2\}, \{x_3, x_4\}\}$ is an element of $\mathbb{P}_n(4)$ of cohomological degree $3(1-n)$.

2.6. **Exercises.**

Exercise 2.19. *Consider the map* $\operatorname{Conf}_k(\mathbf{R}^n) \to \operatorname{Conf}_{k-1}(\mathbf{R}^n)$ *given by forgetting the last point. Show that its fiber F_k is homotopy equivalent to a wedge of $(k-1)$ spheres S^{n-1}.*

Exercise 2.20. *The Leray–Serre spectral sequence for the fibration* $F_k \hookrightarrow \operatorname{Conf}_k(\mathbf{R}^n) \to \operatorname{Conf}_{k-1}(\mathbf{R}^n)$ *degenerates (using the Leray–Hirsch theorem), so that one may identify*

$$\mathrm{H}_\bullet(\operatorname{Conf}_k(\mathbf{R}^n); \mathbb{Q}) \cong \mathrm{H}_\bullet(\operatorname{Conf}_{k-1}(\mathbf{R}^n); \mathbb{Q}) \otimes \mathrm{H}_\bullet(F_k; \mathbb{Q}).$$

Find $\mathrm{H}_\bullet(\mathbb{E}_n(k); \mathbb{Q})$ *for $k = 1, 2, 3$.*

Exercise 2.21. *Describe the graded vector space $\mathbb{P}_n(k)$ for $k = 1, 2, 3$ and find an isomorphism*

$$\mathrm{H}_\bullet(\mathbb{E}_n(k); \mathbb{Q}) \cong \mathbb{P}_n(k)$$

for $n \geq 2$.

Exercise 2.22. *(*) Consider the S_2-action on $\mathbb{E}_2(2) \sim S^1$ given by reflection around the origin. Let $\mathcal{C}$ be a category. Show that an S_2-equivariant map*

$$S^1 \times \mathcal{C}^{\times 2} \longrightarrow \mathcal{C}$$

is the same as a pair $(\otimes, \sigma)$ *consisting of a functor* $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *as well as a natural isomorphism*

$$\sigma_{x,y}\colon x \otimes y \xrightarrow{\sim} y \otimes x.$$

3. Lecture 3 (Monday)

3.1. **Supersymmetry.** My goals for this talk are to give an answer to the questions "what is supersymmetry?" and "what is a topological twist?".

We're going to be working in the wide world of models for classical and quantum field theory. I won't explain in detail what a quantum field theory is, and there are many different ways of modelling them, but part of the data will be a vector space (either of states, or of observables). In fact, usually a little more, a cochain complex $(\mathcal{E}, \mathrm{d})$. We can ask for $\mathcal{E}$ to be acted upon by some Lie algebra $\mathfrak{g}$ of symmetries.

Example 3.1. *If we're studying field theory on* $\mathbb{R}^4$*, we might ask for an action of the Lorentz algebra* $\mathfrak{so}(1,3)$*. Or of the associated Poincaré algebra* $\mathfrak{iso}(1,3) = \mathfrak{so}(1,3) \ltimes \mathbb{R}^4$*.*

The term *supersymmetry* is used for an enhancement of this sort of thing, where the Poincaré algebra is replaced by a $\mathbb{Z}/2\mathbb{Z}$-graded extension thereof.

Definition 3.2. A *super Lie algebra* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathfrak{g}$ equipped with a Lie bracket that is graded skew-symmetric:

$$[X, Y] = (-1)^{|X||Y|+1}[Y, X]$$

and satisfies the graded Jacobi identity

$$(-1)^{|X||Z|}[X, [Y, Z]] + (-1)^{|Y||X|}[Y, [Z, X]] + (-1)^{|Z||Y|}[Z, [X, Y]] = 0.$$

here $|X| \in \mathbb{Z}/2\mathbb{Z}$ denotes the degree of an element $X \in \mathfrak{g}$.

We will now define the super Lie algebras where supersymmetries live. For simplicity we will focus on the *complexified* Lie algebra of supersymmetries, which will be an extension of $\mathfrak{iso}(n, \mathbb{C}) = \mathfrak{iso}(p, q) \otimes_{\mathbb{R}} \mathbb{C}$ for any $p + q = n$.

Definition 3.3. A *super Poincaré algebra* in dimension n is a super Lie algebra with underlying super vector space

$$\mathfrak{siso}(n|\Sigma) = \mathfrak{iso}(n, \mathbb{C}) \oplus \Pi\Sigma$$

(where Π indicates an odd degree shift), where Σ is a *spinorial* representation of $\mathfrak{so}(n, \mathbb{C})$ (meaning all its irreducible summands are (semi)spin representations of $\mathfrak{so}(n, \mathbb{C})$), with an additional bracket given by a non-degenerate equivariant map $\Gamma\colon \mathrm{Sym}^2(\Sigma) \to \mathbb{C}^n$.

Remark 3.4. *We could of course also study the real forms of these complex super Lie algebras, which will depend on a choice of signature. We won't need these real forms this week.*

So we can now say what a supersymmetric field theory is (subject to the caveat that we haven't exactly defined a field theory, only stated that its structure should include a cochain complex!)

Definition 3.5. Suppose that $(\mathcal{E}, \mathrm{d})$ is a classical field theory on $\mathbb{R}^n$ with an action of the algebra $\mathfrak{iso}(n, \mathbb{C})$ of isometries. We say $(\mathcal{E}, \mathrm{d})$ is *supersymmetric* with supersymmetry Σ if $\mathcal{E}$ is equipped with an additional $\mathbb{Z}/2\mathbb{Z}$-grading (as well as its original $\mathbb{Z}$-grading) and the $\mathfrak{iso}(n, \mathbb{C})$ lifts to an action of the super Lie algebra $\mathfrak{siso}(n|\Sigma)$.

You might notice that we typically use slightly different terminology. We don't usually talk about "supersymmetry Σ", we usually say something like "$\mathcal{N} = 2$ supersymmetry" or "$\mathcal{N} = 4$ supersymmetry" or $\mathcal{N} = (2, 2)$ supersymmetry". We'll explain this now: it's because the possible odd terms Σ occuring in the supersymmetry algebra are generally highly constrained. They are given as sums of irreducible spinorial representations of which there are always either one or two. The bracket Γ of odd elements is also usually uniquely determined, maybe up to an overall scale. This is why it didn't appear in our notation.

Example 3.6 (3d supersymmetry)**.** *The key example this week is supersymmetry in 3d, so let's do this example first. As I mentioned, for simplicity I'm going to complexify my super Poincaré algebras, and the vector spaces $\mathcal{E}$ on which they act, that way I won't have to worry about a choice of signature.*

Recall that there is an exceptional isomorphism $\mathrm{Spin}(3, \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C})$ *(or perhaps more familiarly, in Euclidean signature* $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$*). The finite-dimensional irreducible representations of* $\mathrm{SL}(2, \mathbb{C})$ *or its Lie algebra* $(2, \mathbb{C})$ *are given as* $\mathrm{Sym}^k(V)$*, where V is the 2d defining representation. In particular, the 3d defining representation of* $\mathfrak{so}(3, \mathbb{C})$ *is isomorphic to* $\mathrm{Sym}^2(V)$*.*

The spin representation is, under this isomorphism, V itself, so there are potentially super Poincaré algebras with $\Sigma = V^k = V \otimes W$ where W is a k-dimensional vector space, for $k \geq 1$. To work out the possible brackets, we need an equivariant map

$$\mathrm{Sym}^2(\Sigma) = \mathrm{Sym}^2(V \otimes W) = \mathrm{Sym}^2(V) \otimes \mathrm{Sym}^2(W) \oplus \wedge^2(V) \otimes \wedge^2(W) \to \mathrm{Sym}^2(V).$$

So pretty clearly we get such a map for any linear map $g \colon$ $\mathrm{Sym}^2(W) \to \mathbb{C}$, and non-degeneracy of the bracket means g is precisely an inner product on W.

This example also illustrates an important concept: there are symmetries of the super Poincaré algebra in dimension 3 coming from elements of $\mathrm{O}(W)$. These are called *R-symmetries.*

Definition 3.7. The group G_R of *R-symmetries* of a super Poincaré algebra is the group of outer automorphisms of $\mathfrak{siso}(n|\Sigma)$ that act trivially on the even part. Write $\mathfrak{g}_R$ for its Lie algebra.

Remark 3.8. *If we like we can form the extension* $\mathfrak{g}_R \ltimes \mathfrak{siso}(n|\Sigma)$ *by the algebra of R-symmetries (or a subalgebra thereof). Such super Lie algebras are sometimes more generally referred to as "supersymmetry algebras".*

3.2. **Twisting.** To conclude, I'd like to talk about the concept of "twisting". Suppose $Q \in \mathfrak{siso}(n|\Sigma)$ is an odd element such that $[Q,Q]=0$. If $\mathcal{E}$ is a supersymmetric theory, we can use such a "square-zero" element Q to define a deformation to a new theory.

Definition 3.9. Let $(\mathcal{E},\mathrm{d})$ be a supersymmetric theory, and write α for the action of $\mathfrak{siso}(n|\Sigma)$ on $\mathcal{E}$. The *twist* of $(\mathcal{E},\mathrm{d})$ by a square-zero element Q is the theory $(\mathcal{E}_!, \mathrm{d}_Q) = (E, \mathrm{d}+\alpha(Q))$.

The square-zero condition is needed to ensure that d_Q is indeed a differential. Notice that d_Q is not of homogeneous degree anymore, so in general $(\mathcal{E}, \mathrm{d}+\alpha(Q))$ is only a $\mathbb{Z}/2\mathbb{Z}$-graded complex (though in many examples it is possible to cook up a $\mathbb{Z}$-grading after all!).

So, if we are studying twists, it is natural to ask exactly what sorts of elements Q we can twist by! Those elements satisfying the quadratic equation $[Q,Q]$ cut out a quadric subvariety of Σ.

Definition 3.10. The *nilpotence variety* associated to a super Poincaré algebra with odd summand Σ is the subvariety

$$\mathcal{N}\mathrm{ilp} = \{Q \colon [Q,Q]=0\} \subseteq \Sigma.$$

Notice that $\mathcal{N}\mathrm{ilp}$ is invariant under the rescaling action of $\mathbb{C}^\times$, by $Q \mapsto \lambda Q$. So we can instead study the projectivization of $\mathcal{N}\mathrm{ilp}$:

$$\mathbb{P}\mathcal{N}\mathrm{ilp} = (\mathcal{N}\mathrm{ilp}\backslash\{0\})/\mathbb{C}^\times \subseteq \mathbb{P}\Sigma.$$

You'll work through an example in detail during the exercise session shortly: the example of 3d $\mathcal{N}=4$ supersymmetry that is most relevant to this week's lectures.

3.3. **Twisting and Translation Invariance.** Let us conclude by talking about what twisting "buys us". In what sense are twists of supersymmetric theories more mathematically tractable? Well, in many cases, they are in fact topological! Here's the idea.

Suppose that our field theory $\mathcal{E}$ can be equipped with a Lie bracket for which the action of $\mathfrak{siso}(n|\Sigma)$ is *inner*. In other words, suppose that there is a Lie algebra map

$$H \colon \mathfrak{siso}(n|\Sigma) \to \mathcal{E}$$

so that the action of $X \in \mathfrak{siso}(n|\Sigma)$ coincides with the Lie bracket with $H(X) \in \mathcal{E}$ (these elements $H(X)$ are the *Hamiltonians* of the symmetries X). What happens in the twist $\mathcal{E}_Q$? Well, for any Q-exact symmetry $X=[Q,Q']$, the Hamiltonian $H(X)$ is *cohomologically trivial* in the twist by Q.

The image $\mathfrak{b}_Q$ of $[Q,-]$ is a subalgebra of the abelian Lie algebra $\mathbb{C}^n$ of translations. So this is saying that certain translations act cohomologically trivially in the twisted theory. If

$\mathfrak{b}_Q = \mathbb{C}^n$ contains *all* translations then we say Q is a *topological supercharge*. It turns out, that under some fairly mild assumptions, the twist by a topological supercharge is always a topological field theory!

In the exercise session you will show that there are two inequivalent families of topological supercharges in the 3d $\mathcal{N} = 4$ nilpotence variety.

3.4. **Exercises.** In this exercise we will learn about twists in a three-dimensional example. Recall that $\mathrm{Spin}(3, \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C})$. Let S be the two-dimensional spin representation of $\mathrm{Spin}(3, \mathbb{C})$ (i.e. the defining representation of $\mathrm{SL}(2, \mathbb{C})$) and V the three-dimensional vector representation (i.e. the adjoint representation of $\mathrm{SL}(2, \mathbb{C})$). There is an isomorphism $\mathrm{Sym}^2(S) \cong V$ of $\mathrm{Spin}(3, \mathbb{C})$-representations.

Spinorial representation take the form $\Sigma = S \otimes W$, where W is a complex vector space equipped with a nondegenerate symmetric bilinear pairing g. The super Poincaré Lie algebra is

$$\mathfrak{siso}(3|\Sigma) = \mathfrak{iso}(3, \mathbb{C}) \oplus \Pi\Sigma$$

with the only nonobvious bracket $\Gamma\colon \mathrm{Sym}^2(\Sigma) \to V$ defined using g and the isomorphism $\mathrm{Sym}^2(S) \cong V$. The R-symmetry group here is the group $\mathrm{O}(W)$ acting on W.

Consider the case $\dim W = 4$, i.e. we are working with 3d $\mathcal{N} = 4$ supersymmetry.

(1) For a basis $\{Q_1, Q_2\}$ of S let $Q = Q_1 \otimes u + Q_2 \otimes v \in \Sigma$. Give conditions under which Q lies in the nilpotence variety $\mathcal{N}$.
(2) What are the $\mathrm{Spin}(3, \mathbb{C}) \times \mathrm{SO}(W)$-orbits in the nilpotence variety $\mathcal{N}$ and its projectivization $\mathbb{P}\mathcal{N}$? (*Hint*: there are 3 orbits in the latter case.)
(3) Let $\mathfrak{b}_Q \subset V$ be the image of $\Gamma(Q, -)\colon \Sigma \to V$. Find the dimension of $\mathfrak{b}_Q$ for an element Q in each orbit.
(4) (*) Let $\mathfrak{z}_Q \subset \mathfrak{siso}(3|\Sigma)$ be the subalgebra of elements commuting with Q. This subalgebra has a nice interpretation: it consists of those symmetries that "survive" the twist, i.e. that continue to act on the twisted theory. Find this subalgebra for each orbit.

Remark 3.11. *The moduli space* $\mathrm{OGr}(1, 4)$ *of isotropic lines in* W *is a nondegenerate quadric surface in* $\mathbb{P}(W) \cong \mathbb{CP}^3$ *and therefore is isomorphic to* $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Remark 3.12. *The moduli space* $\mathrm{OGr}(2, 4)$ *of isotropic planes in* W *is isomorphic to the moduli space of lines in* $\mathrm{OGr}(1, 4)$ *by the map sending an isotropic plane* $\Pi \subseteq W$ *to the family of lines* $L \subseteq \Pi$*, which are automatically all isotropic. This moduli space of lines, and hence the moduli space* $\mathrm{OGr}(2, 4)$*, is isomorphic to* $\mathbb{CP}^1 \sqcup \mathbb{CP}^1$*. Given a point* $P \in \mathbb{CP}^1$ *there are two lines* $P \times \mathbb{CP}^1 \subset \mathrm{OGr}(1, 4)$ *and* $\mathbb{CP}^1 \times P \subset \mathrm{OGr}(1, 4)$.

4. Lecture 4 (Monday)

Hey this happened on Monday at 9

5. Lecture 5 (Tuesday)

The goal of this talk is to describe differential equations that appear in 2d and 3d A-models:

- Given a Kähler manifold M equipped with a Hamiltonian action of a Lie group G, the partition function $Z_{2dA,M//G}(\Sigma)$ on a Riemann surface Σ counts solutions of the symplectic vortex equations on Σ.
- Given a hyperKähler manifold M equipped with a triHamiltonian action of a Lie group G, the partition function $Z_{3dA,M///G}(N)$ on a 3-manifold N equipped with a Spin^c structure counts solutions of the nonlinear Seiberg–Witten equations on N.

For concreteness in this talk we restrict to the case of M being a vector space.

5.1. 2d equations.

In this section Σ denotes a Riemann surface.

5.1.1. *Cauchy–Riemann equations.* Let V be a complex vector space and $L \to \Sigma$ a Hermitian line bundle. It carries a compatible unitary connection.

Definition 5.1. The ***Cauchy–Riemann equation*** is the equation

$$\overline{\partial}\phi = 0 \in \Omega^{0,1}(\Sigma, L \otimes V)$$

for a smooth section ϕ of $L \otimes V$.

We will be interested in the moduli space of solutions of this equation. In the linear case we are considering here it is a chain complex.

Proposition 5.2. *The operator* $\overline{\partial}\colon \Omega^0(\Sigma, L\otimes V) \to \Omega^{0,1}(\Sigma, L \otimes V)$ *is elliptic. Assuming* Σ *is compact, its index is*

$$\mathrm{ind}_{\mathbb{C}}(\overline{\partial}) = ((1-g) + \deg(L)) \dim_{\mathbb{C}} V.$$

Remark 5.3. *In the above we are working with complex elliptic operators. Any complex elliptic operator has an orientation as a real elliptic operator and the index of the corresponding real elliptic operator is* $\mathrm{ind}_{\mathbf{R}} = 2\mathrm{ind}_{\mathbb{C}}$.

Remark 5.4. *If* L *is a square root of the canonical bundle (i.e. a theta characteristic), we have* $\deg(L) = g-1$. *So, in this case the index is zero, irrespectively of the dimension of* V *and the genus of* Σ.

5.1.2. *Flat connections.* Let G be a compact connected Lie group. Let $G_{\mathbb{C}}$ be its complexification.

Definition 5.5. The ***flatness equations*** are

$$F_\nabla = 0 \in \Omega^2(\Sigma; \mathrm{ad}\, P),$$

where $P \to \Sigma$ is a principal G-bundle equipped with a connection ∇ and F_∇ is its curvature. We consider the solutions modulo gauge transformations.

The flatness equation is nonlinear. Its linearization at a given flat connection ∇ is given by

$$\nabla\colon \Omega^1(\Sigma; \operatorname{ad} P) \longrightarrow \Omega^2(\Sigma; \operatorname{ad} P).$$

Incorporating linearized gauge transformations, we obtain a three-term chain complex concentrated in degrees $-1, 0, 1$.

Proposition 5.6. *The chain complex*

$$\Omega^0(\Sigma; \operatorname{ad} P) \xrightarrow{\nabla} \Omega^1(\Sigma; \operatorname{ad} P) \xrightarrow{\nabla} \Omega^2(\Sigma; \operatorname{ad} P)$$

is elliptic. If Σ is compact, its index is

$$\operatorname{ind} = (2g-2)\dim G.$$

Definition 5.7. We say a flat G-bundle is ***irreducible*** if the subspace of infinitesimal gauge transformations this flat G-bundle coincides with $Z(\mathfrak{g})$, the center of $\mathfrak{g}$.

Theorem 5.8 (Narasimhan–Seshadri–Ramanathan)**.** *The moduli space of irreducible flat G-bundles on Σ is a manifold of dimension $(2g-2)\dim G$. It is isomorphic to the moduli space of stable $G_{\mathbb{C}}$-bundles on Σ.*

5.1.3. *Symplectic vortex equations.* We will now combine the two equations. Suppose V is a unitary G-representation. Define a G-equivariant map

$$\mu\colon \operatorname{Sym}^2_{\mathbf{R}}(V) \longrightarrow \mathfrak{g}^*$$

by

$$\mu(v) = \frac{1}{2}(xv, v)$$

for $v \in V$ and $x \in \mathfrak{g}$ (note that the equation is quadratic in v).

Remark 5.9. *A unitary representation is, in particular, a Kähler manifold and μ defines a moment map for the G-action.*

Definition 5.10. The ***symplectic vortex equations*** are

$$\begin{aligned} \overline{\partial}\phi &= 0 \\ *F_\nabla + \mu(\phi) &= 0, \end{aligned}$$

where

- $P \to \Sigma$ is a principal G-bundle.
- ∇ is a connection on P.
- $\phi \in \Gamma(\Sigma, P \times^G V)$.

We again consider solutions modulo gauge transformations.

Proposition 5.11. *Consider a solution (ϕ, ∇) of the symplectic vortex equations. The linearization of these equations at (ϕ, ∇) is an elliptic PDE. If Σ is compact, its index is*

$$\mathrm{ind} = (2-2g)(\dim_{\mathbb{C}} V - \dim G) + 2\deg(P \times^G V).$$

Example 5.12. *The usual vortex equations correspond to the case $G = \mathrm{U}(1)$ and $V = \mathbb{C}$ the standard representation. In this case we have a Hermitian line bundle $L \to \Sigma$ with a unitary connection, ϕ is a section of L. The index of the linearized elliptic operator is $2\deg(L)$.*

5.2. **3d equations.** Let N be a 3-dimensional manifold.

5.2.1. *Dirac equation.* The Dirac (also known as Fueter) equation in 3 and 4 dimensions is a quaternionic analog of the Cauchy–Riemann equation. Let us recall a few facts about spin structures in 3 dimensions:

- The 3-dimensional spin group is $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$. It can be identified with the group of unit 1 quaternions. The spinor representation $\mathcal{S}$ may be identified with quaternions $\mathbf{H}$ with the left action of unit 1 quaternions.
- The vector representation V may be identified with the space $\Im\mathbf{H}$ of imaginary quaternions with the action of unit 1 quaternions by conjugation.
- There is an SU(2)-equivariant map $c\colon V \otimes_{\mathbf{R}} \mathcal{S} \to \mathcal{S}$ given by quaternion multiplication. It is the 3-dimensional version of the Clifford action of vectors on spinors.

Let N be a 3-dimensional manifold equipped with a spin structure; denote by $P \to N$ the corresponding Spin(3)-bundle. It carries a natural connection ∇ which induces the Levi-Civita connection on the oriented frame bundle. Let $\mathcal{S} \to N$ be the spinor bundle

$$\mathcal{S} = P \times^{\mathrm{SU}(2)} \mathbf{H}$$

which carries a right $\mathbf{H}$-module structure.

Let W be an $\mathbf{H}$-module. And let

$$\mathcal{S}_W = P \times^{\mathrm{SU}(2)} W = \mathcal{S} \otimes_{\mathbf{H}} W.$$

Consider a section

$$\phi \in \Gamma(N, \mathcal{S}_W) \cong \Gamma(N, \mathcal{S}_W).$$

We have

$$\nabla\phi \in \Omega^1(N, P \times^{\mathrm{SU}(2)} W) \cong \Gamma(N, P \times^{\mathrm{SU}(2)} (V \otimes_{\mathbf{R}} \mathcal{S} \otimes_{\mathbf{H}} W).$$

The Clifford multiplication $c\colon V \otimes \mathcal{S} \to \mathcal{S}$ therefore produces an element

$$\not{\nabla}\phi \in \Gamma(N, P \times^{\mathrm{SU}(2)} W).$$

We call $\not{\nabla}$ the ***Dirac operator***.

Definition 5.13. The ***Dirac equation*** is the equation

$$\not{\nabla}\phi = 0 \in \Gamma(N, \mathcal{S}_W)$$

for a smooth section ϕ of $\mathcal{S}_W$.

Proposition 5.14. *The Dirac operator $\not{\nabla}$ on $\Gamma(N, \mathcal{S}_W)$ is elliptic. If N is compact, its index is* 0*.*

5.2.2. *Bogomolny equation.* Let G be a compact connected Lie group.

Definition 5.15. The ***Bogomolny equation*** is

$$F_\nabla + *\nabla\sigma = 0,$$

where

- $P \to N$ is a principal G-bundle.
- ∇ is a connection on P.
- $\sigma \in \Gamma(N, \operatorname{ad} P)$.

We begin with the following observation. Using the Bianchi identity we get

$$\nabla * \nabla\sigma = 0.$$

Here

$$\nabla\colon \Omega^0(N, \operatorname{ad} P) \longrightarrow \Omega^1(N, \operatorname{ad} P).$$

Suppose N is closed and choose a nondegenerate pairing on $\mathfrak{g}$ which induces one on $\operatorname{ad} P$. Then ∇ has a formal adjoint

$$\nabla^*\colon \Omega^1(N, \operatorname{ad} P) \longrightarrow \Omega^0(N, \operatorname{ad} P)$$

given by $\nabla^* = *\nabla*$. In particular, the above equation implies that

$$\nabla^*\nabla\sigma = 0.$$

Pairing this equation with σ we get

$$|\nabla\sigma|^2 = 0.$$

So, $\nabla\sigma = 0$. In particular, the Bogomolny equation becomes $F_\nabla = 0$, i.e. for N closed solutions of the Bogomolny equation are the same as flat connections.

Example 5.16. *Suppose G is abelian. Then $\sigma \in C^\infty(N) \otimes \mathfrak{g}$ and $\nabla\sigma = 0$ implies that σ is constant.*

Remark 5.17. *A natural question is why we are adding the field σ at all. The linearization of the flatness equation in 3d is* not *elliptic while the linearization of the Bogomolny equation is.*

5.2.3. *Seiberg–Witten equations.* Let us now combine the Bogomolny and Dirac equations. Suppose W is a quaternionic G-representation. Let $\gamma\colon \Im\mathbf{H} \to \mathrm{End}(W)$ be the action of imaginary quaternions. Its adjoint defines a map $\gamma^*\colon \mathrm{End}(W) \to (\Im\mathbf{H})^*$.

Fix a central order 2 element $-1 \in G$ which acts on W by -1. Let

$$\mathrm{Spin}^G(3) = (\mathrm{Spin}(3) \times G)/(\mathbb{Z}/2\mathbb{Z}),$$

where $\mathbb{Z}/2\mathbb{Z} \subset \mathrm{Spin}(3) \cong \mathrm{SU}(2)$ is given by ± 1. Our assumption implies that W is a $\mathrm{Spin}^G(3)$-representation (where $\mathrm{SU}(2)$ acts by the multiplication by unit 1 quaternions). Moreover, $V = \mathbf{R}^3$ is also a representation of $\mathrm{Spin}^G(3)$ (via the homomorphism $\mathrm{Spin}^G(3) \to \mathrm{Spin}(3)/(\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{SO}(3)$). Let $\overline{G} = G/(\mathbb{Z}/2\mathbb{Z})$.

Define the map

$$\mu\colon \mathrm{Sym}^2_{\mathbf{R}}(W) \longrightarrow (\mathfrak{g} \otimes \Im\mathbf{H})^*$$

by

$$\mu(v) = \frac{1}{2}\gamma^*(vv^*).$$

By construction it will be $\mathrm{Spin}^G(3)$-equivariant, where $\mathrm{SU}(2)$ acts on $\Im\mathbf{H}$ by conjugation by unit quaternions.

Remark 5.18. *A quaternionic representation W is, in particular, a hyperKähler manifold and μ defines a moment map for the G-action.*

Choose a principal $\mathrm{Spin}^G(3)$-bundle $P \to N$ with an isomorphism $P \times^{\mathrm{Spin}^G(3)} V \cong \mathrm{T}M$ and let $\overline{P} = P \times^{\mathrm{Spin}^G(3)} \overline{G}$. Let

$$\mathcal{S}_W = P \times^{\mathrm{Spin}^G(3)} W.$$

For a section $\phi \in \Gamma(N, \mathcal{S}_W)$ we get $\Phi(\phi) \in \Omega^1(N, \mathrm{ad}\,\overline{P})$. A connection on P can be specified in terms of a sum of a connection ∇ on $\overline{P}$ and the Levi-Civita connection on $\mathrm{T}M$.

Definition 5.19. The (generalized) ***Seiberg–Witten equations*** are

$$\begin{aligned} \not{\nabla}\phi + [\sigma, \phi] &= 0 \\ *F_\nabla + \nabla\sigma + \mu(\phi) &= 0, \end{aligned}$$

where

- $P \to N$ is a principal $\mathrm{Spin}^G(3)$-bundle.
- ∇ is a connection on $\overline{P}$.
- $\sigma \in \Gamma(M, \mathrm{ad}\,\overline{P})$.
- $\phi \in \Gamma(M, \mathcal{S}_W)$.

In the case $W = 0$ we get the Bogomolny equation and, as in that case, for N closed we get $\nabla\sigma = 0$, so that we can remove it from consideration.

Let us consider a special case of these equations. The usual 3-dimensional Seiberg–Witten equations correspond to the case $G = \mathrm{U}(1)$. Then

$$\mathrm{Spin}^{\mathrm{U}(1)}(3) =: \mathrm{Spin}^c(3) = \mathrm{U}(2).$$

It has a natural complex 2-dimensional (real 4-dimensional) representation W, where $\mathrm{U}(2)$ acts by complex 2×2 matrices.

Suppose N is a closed oriented 3-manifold. There is a discrete set $\mathrm{Spin}^c(N)$ of Spin^c-structures on N. It carries a free transitive action of $\mathrm{H}^2(N; \mathbb{Z})$, so that for two Spin^c-structures σ_1, σ_2 we can define their difference $\sigma_2 - \sigma_1 \in \mathrm{H}^2(N; \mathbb{Z})$.

One can show that the linearization of the Seiberg–Witten equations defines an elliptic PDE whose index (if N is compact) is 0. So, assuming one applies an appropriate perturbation to make the moduli space of solutions a smooth manifold, we can "count" solutions of Seiberg–Witten equations. This count is independent of perturbations if $b_1(N) = \dim \mathrm{H}_1(N; \mathbb{C}) > 1$ (for $b_1(N) = 0, 1$ there is a wall-crossing behavior for the invariants). For a given element $\sigma \in \mathrm{Spin}^c(N)$ let $\mathrm{sw}_N(\sigma) \in \mathbb{Z}$ be the (signed) count. We can combine the different Seiberg–Witten invariants into a weighted count

$$\mathrm{SW}_N(\sigma) = \sum_{h \in H} \mathrm{sw}_N(\sigma - h) h \in \mathbb{Z}[H],$$

where $H = \mathrm{H}_1(N; \mathbb{Z}) \cong \mathrm{H}^2(N; \mathbb{Z})$.

Remark 5.20. *One can also consider the Seiberg–Witten equations on manifolds with boundaries. If we assume the metric of N near the boundary $\Sigma = \partial N$ looks like $\Sigma \times [0, \infty)$ (i.e. N has cylindrical ends), one can realize the moduli space of solutions of the Seiberg–Witten equations as providing a Lagrangian in the moduli space of solutions of the symplectic vortex equation on Σ.*

Exercise 5.21. *Let Σ be a Riemann surface and $N = \Sigma \times \mathbf{R}$. Show that a spin structure on Σ gives rise to a spin structure on N. In terms of Lie groups, this corresponds to describing a homomorphism $\mathrm{Spin}(2) \to \mathrm{Spin}(3)$ fitting into a commutative diagram*

$$\begin{tikzcd}
\mathrm{Spin}(2) \arrow[r, dashed] \arrow[d] & \mathrm{Spin}(3) \arrow[d] \\
\mathrm{SO}(2) \arrow[r] & \mathrm{SO}(3),
\end{tikzcd}$$

where $\mathrm{Spin}(2) \to \mathrm{SO}(2)$ is a connected $2:1$ cover.

Exercise 5.22. *Let V be the vector representation of $\mathrm{Spin}(3)$ and $\mathcal{S}$ the spin representation. Recall the Clifford multiplication map $V \otimes \mathcal{S} \to \mathcal{S}$ of $\mathrm{Spin}(3)$-representations. Using the homomorphism $\mathrm{Spin}(2) \to \mathrm{Spin}(3)$ defined in the previous exercise describe $\mathcal{S}$ and V as $\mathrm{Spin}(2)$-representations and describe the Clifford multiplication in these terms.*

Exercise 5.23. *Choose a spin structure on* Σ*. Let* $W = \mathbf{H}$*. Recall that for* $\phi \in \Gamma(N, \mathcal{S})$ *(where* $\mathcal{S}$ *is the spinor bundle on* N*) the Dirac equation is* $\not\nabla\phi = 0$*. Write explicitly the Dirac equation on* $\Sigma \times \mathbf{R}$*.*

6. Lecture 6 (Tuesday)

If M is a complex manifold, the partition function of the 2d B-model with target M as well as the partition function of the 3d B-model with target T^*M are both given by Reidemeister torsion. In this lecture we introduce what it is as well as state the first example of a mirror symmetry phenomenon.

6.1. **Torsion.** Let R be a commutative ring with a homomorphism $R \to K$ to a field K. Let N be a connected finite CW complex (let $x \in N$ be a chosen basepoint) and $\mathcal{L}$ a free R-module of finite rank equipped with a $\pi_1(N)$-action. It defines a homomorphism

$$\pi_1(N) \longrightarrow \mathrm{GL}_n(R)$$

and hence a homomorphism

$$\det\colon \mathrm{H}_1(N; \mathbb{Z}) \longrightarrow K^\times$$

by post-composing the first map with $\mathrm{GL}_n(R) \to \mathrm{GL}_n(K) \xrightarrow{\det} K^\times$.

Let $\tilde{N} \to N$ be the universal cover (with its natural $\pi_1(N)$-action) and lift the CW structure from N to $\tilde{N}$. Consider the complex of cellular chains

$$\mathrm{C}_0(\tilde{N}; \mathbb{Z}) \xleftarrow{d} \mathrm{C}_1(\tilde{N}; \mathbb{Z}) \xleftarrow{d} \dots$$

as a complex of $\pi_1(N)$-representations. Applying $\otimes_{\mathbb{Z}[\pi_1(N)]}\mathcal{L}$ we get the complex of cellular chains

$$\mathrm{C}_0(N; \mathcal{L}) \xleftarrow{d} \mathrm{C}_1(N; \mathcal{L}) \xleftarrow{d} \dots$$

Assume that this complex is acyclic when we apply $\otimes_R K$. Over a field an acyclic complex is contractible, so there is a contracting homotopy h (defined over K).

Definition 6.1. Let N be a connected finite CW complex with a chosen basepoint $x \in N$. A ***Turaev spider*** is a choice of a path from the center of each cell to x.

Note that given two spiders ${}_{1,2}$, we may measure their difference ${}_1 - {}_2 \in \mathrm{H}_1(N; \mathbb{Z})$ as follows. The difference of two paths from $\sigma \in A_d$ to x gives an element of $\mathrm{H}_1(N; \mathbb{Z})$; ${}_1 - {}_2$ is defined to be an alternating sum of these differences over each cell. We say two Turaev spiders are equivalent if the difference between them gives the zero element of $\mathrm{H}_1(N; \mathbb{Z})$.

Proposition 6.2 (Turaev)**.** *Let* N *be a closed oriented 3-manifold. There is a natural bijection between the set of Turaev spiders and the set of* Spin^c*-structures. This bijection is equivariant for the action of* $\mathrm{H}_1(N; \mathbb{Z}) \cong \mathrm{H}^2(N; \mathbb{Z})$*.*

Choose a basis of $\mathcal{L}$, an ordering of cells and a Turaev spider . The Turaev spider allows us to produce a canonical lift $\tilde{\sigma}$ to a cell of $\tilde{N}$ of each cell σ of N. Therefore, the complex of cellular chains $\mathrm{C}_\bullet(N;\mathcal{L})$ becomes a complex of free finite rank modules

$$R^{n_1} \xleftarrow{d} R^{n_2} \xleftarrow{d} \dots$$

Consider the map $d + h\colon \mathrm{C}_{even}(N;\mathcal{L}) \otimes_R K \to \mathrm{C}_{odd}(N;\mathcal{L}) \otimes_R K$. Since the complex of cellular chains is acyclic over K, this map is invertible. In particular, we can compute its determinant

$$\det(d+h) \in K^\times.$$

Let us analyze how this element changes if we make different choices:

- Reordering the cells introduces a sign ambiguity into $\det(d+h)$.
- Changing the basis of $\mathcal{L}_x$ changes $\det(d+h)$ by a unit of R.
- Changing the Turaev spider changes $\det(d+h)$ by the image of

$$\det\colon \mathrm{H}_1(N;\mathbb{Z}) \longrightarrow K^\times.$$

So, from this setup we get an element

$$\det(d+h) \in K^\times/R^\times.$$

6.2. **Milnor torsion.** Let us now specialize the discussion. Let $H = \mathrm{H}_1(N;\mathbb{Z})/\{torsion\}$, $R = \mathbb{Z}[H]$ and $K = \mathbb{Q}(H)$. We have $R^\times = \pm H$. Let $\mathcal{L} = \mathbb{Z}[H]$ equipped with the obvious homomorphism $\pi_1(M) \to \mathrm{H}_1(N;\mathbb{Z}) \to H \to \mathbb{Z}[H]$.

Definition 6.3. The ***Milnor torsion*** $\tau(N)$ of N is zero if the chain complex $\mathrm{C}_\bullet(N;\mathcal{L}) \otimes_R K$ is not acyclic and otherwise it is defined to be

$$\det(d+h) \in \mathbb{Q}(H)/(\pm H).$$

Example 6.4. *Consider the circle $N = S^1$. The Milnor torsion is*

$$\tau(S^1) = \frac{1}{1-t} \in \mathbb{Q}(t)/(\pm t^{\mathbb{Z}}).$$

Let us now indicate how to define torsion without the ambiguity of $\pm t^{\mathbb{Z}}$ (this is known as *Turaev (or refined) torsion*):

- The ambiguity of $t^{\mathbb{Z}}$ would be resolved if we choose a Turaev spider. A more homotopy-theoretic definition is as follows. For a finite CW complex N there is the homological Euler class $e(N)\mathrm{H}_0(N;\mathbb{Z})$. A bounding chain for this class is known as the ***Euler structure***.
- The sign ambiguity arose because we could reorder the cells. To fix this ambiguity we can fix an orientation of the determinant line of $\mathrm{H}_\bullet(N;\mathbf{R})$.

6.3. **Meng–Taubes–Turaev.** We are now ready to state one instance of 3-dimensional mirror symmetry. Let us recall that given the data of

- A compact Lie group G.
- A quaternionic G-representation W.

there are 3-dimensional TQFTs $Z_{3dA,W///G}$ and $Z_{3dB,W///G}$. Moreover, if the pairs (G, W) and $(G^\vee, W^\vee)$ are 3d mirror, we should have an equivalence of 3d TQFTs

$$Z_{3dA,W///G} \cong Z_{3dB,W^\vee///G^\vee}.$$

One basic example of 3d mirror symmetry we will often return to is the 3d mirror symmetry between $(\mathrm{U}(1),)$ and $(\mathrm{pt},)$.

In particular, we expect an equality

$$Z_{3dA,///\mathrm{U}(1)}(N) = Z_{3dB,}(N)$$

of the partition functions of these TQFTs. We will not give all details, but:

- The partition function of the 3d A-model for (G, W) counts solutions of the Seiberg–Witten equations.
- The partition function of the 3d B-model for $(G, V \otimes_{\mathbb{C}})$ computes the integral of torsion over the moduli space of pairs of a flat $G_{\mathbb{C}}$-bundle P on N and a flat section of the associated bundle $P \times^{G_{\mathbb{C}}} V$.

In the simplest case $(\mathrm{U}(1),)$ we are reduced to computing Seiberg–Witten invariants.

Theorem 6.5 (Meng–Taubes–Turaev)**.** *Let N be a 3-manifold with $b_1(N) > 1$. There is an equality*

$$\mathrm{SW}_N = \tau(N) \in \mathbb{Z}[H]/(\pm H).$$

Remark 6.6. *Choosing a* Spin^c *structure σ we can refine the statement to an equality*

$$\mathrm{SW}_N(\sigma) = \tau(N) \in \mathbb{Z}[H]/(\pm 1),$$

where $\tau(N)$ is the refined torsion which depends on σ.

Remark 6.7. *There are also more delicate statements when $b_1(N) = 1$ and $b_1(N) = 0$. The problem with these cases is that the Seiberg–Witten invariants depend on perturbation and exhibit a wall-crossing behavior.*

Exercise 6.8. *Consider the cell structure on $N = S^1$ with one 0-cell and one 1-cell. In this case $\pi_1(N) = \mathrm{H}_1(N; \mathbb{Z}) = H$ and $\mathbb{Z}[H] = \mathbb{Z}[t, t^{-1}]$.*

1. *Lift the cell structure on N to a cell structure on the universal cover $\tilde{N}$.*
2. *Show that the chain complex*

$$C_\bullet = \mathrm{C}_\bullet(\tilde{N}; \mathbb{Z}) \otimes_{\mathbb{Z}[H]} \mathbb{Q}(H)$$

is acyclic and find a contracting homotopy h (i.e. a map $h\colon C_0 \to C_1$ satisfying $dh + hd =$).

(3) Compute the Milnor torsion

$$\det(d+h)\colon C_{even} \longrightarrow C_{odd}$$

as an element $\tau(S^1) \in \mathbb{Q}(t)/(\pm t^{\mathbb{Z}})$.

Exercise 6.9. *Consider the cell structure on S^1 as before and the cell structure on S^2 with one 0-cell and one 2-cell. It induces the product cell structure on $N = S^1 \times S^2$. Compute the Milnor torsion $\tau(S^1 \times S^2) \in \mathbb{Q}(t)/(\pm t^{\mathbb{Z}})$. (*Hint*: the answer is the same as in the previous exercise.)*

Exercise 6.10. *Consider the cell structure on S^1 as before and the product cell structure on the two-torus $N = S^1 \times S^1$. Compute the Milnor torsion $\tau(S^1 \times S^1) \in \mathbb{Q}(t_1, t_2)/(\pm t_1^{\mathbb{Z}} t_2^{\mathbb{Z}})$.*

7. Lecture 7 (Tuesday)

Hey this happened on Monday at 9

8. Lecture 8 (Tuesday)

Hey this happened on Monday at 9

9. Lecture 9 (Wednesday)

Hey this happened on Monday at 9

10. Lecture 10 (Wednesday)

Hey this happened on Monday at 9

11. Lecture 11 (Wednesday)

Hey this happened on Monday at 9

12. Lecture 12 (Wednesday)

12.1. **Ω-Background and Quantization.** Recall that in an (fully extended) oriented 3d TQFT $Z : \mathrm{Bord}_3^{or} \to \mathrm{Ch}$, $Z(S^2)$ is an $\mathbb{E}_3$-algebra, i.e. there is an action for any $n \in \mathbb{N}$

$$C_*(\mathrm{Conf}_n(\mathbb{R}^3)) \otimes Z(S^2)^{\otimes n} \to Z(S^2).$$

In particular, it implies that the product structure on $Z(S^2)$ is homotopically commutative (we remark that this only requires the $\mathbb{E}_2$-structure). Consider the homotopy equivalence

$$\mathrm{Conf}_2(\mathbb{R}^3) \simeq S^2,\ (p_1, p_2) \mapsto \frac{p_1 - p_2}{|p_1 - p_2|}.$$

Then any closed 0-chain $p \in C_0(S^2)$ induces a product

$$\star_p : Z(S^2)^{\otimes 2} \to Z(S^2).$$

Denote the north and south poles by N, S, which correspond to the embedding $(e_1, -e_1) \subset \mathbb{R}^3$ and $(-e_1, e_1) \subset \mathbb{R}^3$. We have two products $\star_N, \star_S$, which correspond to the multiplication of $\mathcal{O}_1$ with $\mathcal{O}_2$ and respectively $\mathcal{O}_2$ with $\mathcal{O}_1$. Consider the 1-chain $\gamma \in C_1(S^2)$ such that $\partial\gamma = N - S$. γ defines a homotopy $\mathcal{O}_1 \star_N \mathcal{O}_2 \sim \mathcal{O}_1 \star_S \mathcal{O}_2$ for any $\mathcal{O}_1, \mathcal{O}_2 \in Z(S^2)$. More precisely, denote $\alpha_n : C_*(\mathrm{Conf}_n(\mathbb{R}^3)) \to \mathrm{Hom}(Z(S_1)^{\otimes n}, Z(S^1))$. Then

$$\partial\alpha_2(\gamma) = \alpha_2(\partial\gamma) = \alpha_2(N) - \alpha_2(S) = \star_N - \star_S.$$

(The reader may notice that the proof only requires an $\mathbb{E}_2$-structure.) Consider the fundamental class $[S^2] \in C_2(S^2)$. It induces a Poisson bracket $\{-,-\}$ by

$$\alpha_2([S^2]) : Z(S^2)^{\otimes 2} \to Z(S^2)[2].$$

$(Z(S^2), \star, \{-,-\})$ is a $\mathbb{P}_3$-algebra.

Recall that $SO(3)$ acts on $\mathrm{Conf}_2(\mathbb{R}^3) \simeq S^2$. Consider the subgroup $S^1 \subset SO(3)$ given by rotation along the $\overline{NS}$-axis. Then we get an action

$$C_*(S^1) \to C_*(SO(3)) \to \mathrm{End}(Z(S^2)).$$

By definition, the homotopy S^1-fixed points of $Z(S^2)$ are

$$Z(S^2)^{S^1} = \mathrm{Hom}_{C_*(S^1)-\mathrm{Mod}}(\mathbb{C}, Z(S^2)).$$

Question: What is $Z(S^2)^{S^1}$?

12.2. **S^1-Invariants and Equivariant Homology.** We recall some basic facts.

- Let V be an S^1-module. Then V^{S^1} is a $\mathbb{C}[\hbar]$-module, where $|\hbar| = 2$.
- Let X be an S^1-topological space. Then $C_*(S^1)$ acts on $C_*(X)$, and the homotopy fixed points are the equivariant Borel–Moore homology $C_*(X)^{S^1} \simeq C_*^{S^1}(X)$.

We only prove the first fact. In fact, it suffices to show that

$$\mathrm{Hom}_{C_*(S^1)}(\mathbb{C}, \mathbb{C}) \simeq C_*^{S^1}(\mathrm{pt}) \simeq \mathbb{C}[\hbar].$$

Note that $C_*(S^1) \simeq \mathbb{C}[\epsilon]/(\epsilon^2)$ where $|\epsilon| = -1$. We resolve the $C_*(S^1)$-module $\mathbb{C}$ by

$$\ldots \xrightarrow{\epsilon} \mathbb{C}[\epsilon]/(\epsilon^2) \xrightarrow{\epsilon} \mathbb{C}[\epsilon]/(\epsilon^2).$$

Take the total complex. Then the generator ϵ in each term of the resolution is supported in degree $0, -2, -4, \ldots$. Then by taking linear dual we can conclude that

$$\mathrm{Hom}_{C_*(S^1)}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C}[\hbar],\ |\hbar| = 2.$$

Given the S^1-action on S^2, by taking S^1-invariants we get an action

$$C_*^{S^1}(\mathrm{Conf}_2(\mathbb{R}^3)) \otimes (Z(S^2)^{S^1})^{\otimes 2} \to Z(S^2)^{S^1}.$$

Since N, S are invariant under the S^1-rotations, we still have two multiplications $\star_N, \star_S$, but they are no longer homotopic as the 1-chain γ whose boundary $\partial\gamma = N - S$ is no longer

S^1-invariant. To compute the difference between $\star_N$ and $\star_S$, we need to know the equivariant homology $C_*^{S^1}(S^2) \simeq C_*^{S^1}(\mathrm{Conf}_2(\mathbb{R}^3))$.

In order to compute $C_*^{S^1}(S^2)$, we need the following properties:

- Let T act on X, and $S^1 \subset T$ a closed subgroup that acts on X freely. Then
$$C_*^T(X) \simeq C_*^{T/S^1}(X).$$
- (Kirwan) Let $i : S^3 \hookrightarrow \mathbb{C}^2$ be the inclusion and $q : S^3 \to S^2$ the Hopf fibration. Let $T = T^2$ act on $\mathbb{C}^2$ be rotations on both components and $S^1 \subset T$ act on $\mathbb{C}^2$ diagonally. Then we have a surjective composition
$$C_*^T(\mathbb{C}^2) \xrightarrow{i^*} C_*^T(S^3) \xrightarrow{\sim} C_*^{S^1}(S^2).$$

Since there exists an S^1-equivariant contraction from $\mathbb{C}^2$ to pt, we know that
$$C_*^T(\mathbb{C}^2) \simeq C_*^T(\mathrm{pt}) \simeq \mathbb{C}[\mathfrak{t}] = \mathbb{C}[x_1, x_2],$$
where $\mathfrak{t}$ is the Lie algebra of T. Hence it suffices to compute the kernel of the composition.

- Let V be a T-representation and $W \subset V$ a T-subrepresentation. Then the (equivariant) fundamental class can be computed by
$$[W] = e^T(V/W) \cap [V] \in H_*^T(V),$$
where $e^T(V/W)$ is the (equivariant) Euler class of the normal bundle V/W of W. Moreover,
$$e^T(V/W) = \prod_{x\text{: weights that appear in } V/W} x \in \mathbb{C}[\mathfrak{t}].$$

Consider the maps $\mathbb{C}^2 \xhookleftarrow{i} S^3 \xrightarrow{q} S^2$. The normal bundle of the class $[\mathbb{C} \times 0]$ has weight $x_2 \in \mathfrak{t}$, and the normal bundle of the class $[0 \times \mathbb{C}]$ has weight $x_1 \in \mathfrak{t}$. Finally, the intersection $[0] = [\mathbb{C} \times 0] \cap [0 \times \mathbb{C}]$ is sent to $\varnothing$ in S^3. Therefore, under the map
$$C_*^T(\mathbb{C}^2) \to C_*^{S^1}(S^2),\ x_1 \cdot x_2 \mapsto 0.$$

Lemma 12.1. *Let S_1 act on S^2 by rotation. Then $C_*^{S^1}(S^2) \simeq \mathbb{C}[x_1, x_2]/(x_1 x_2)$.*

We can also write down the $\mathbb{C}[\hbar]$-module structure of $C_*^{S^1}(S^2) \simeq \mathbb{C}[x_1, x_2]/(x_1 x_2)$. Note that S^1 acts diagonally on $\mathbb{C}^2$, so the induced action of chain complexes is also diagonal:
$$\Delta : \mathbb{C}[\hbar] \to \mathbb{C}[x_1, x_2],\ \Delta(\hbar) = x_1 - x_2.$$

12.3. **Quantization of the $\mathbb{E}_3$-Algebra $Z_{3d}(S^2)$.** Under the diagram of maps $\mathbb{C}^2 \xhookleftarrow{i} S^3 \xrightarrow{q} S^2$, we know that $[\mathbb{C} \times 0]$ is mapped to the north pole $N \in S^2$, and $[0 \times \mathbb{C}]$ is mapped to the south pole $S \in S^2$. From the above computation, we then know that
$$x_1 = N \in C_*^{S^1}(S^2),\ x_1 = S \in C_*^{S^1}(S^2).$$

Moreover, under the map $C^T_*(\mathbb{C}^2) \to C^{S^1}_*(S^2)$, we know that the (equivariant) fundamental class $[\mathbb{C}^2]$ is sent to the (equivariant) fundamental class $[S^2]$. Since the normal bundle of $\mathbb{C}^2$ has no weights, we know that

$$1 = [S^2] \in C^{S^1}_*(S^2).$$

Therefore, writing $\alpha_2^{S^1} : C^{S^1}_*(\mathrm{Conf}_2(\mathbb{R}^3)) \to \mathrm{Hom}(Z(S^2)^{S^1})^{\otimes 2}, Z(S^2)^{S^1})$,

$$\alpha_2^{S^1}(N) - \alpha_2^{S^1}(S) = \alpha_2^{S^1}(x_1 - x_2) = \alpha_2^{S^1}(\hbar[S^2]).$$

We can conclude the following lemma (recall that $\star_N, \star_S$ represent the product $\mathcal{O}_1 \star' \mathcal{O}_2$ and respectively $\mathcal{O}_2 \star' \mathcal{O}_1$):

Lemma 12.2. *Let $\star_N, \star_S$ be the equivariant products on $Z^{S^1}(S^2)$. Then $\star_N - \star_S = \hbar\{-,-\}$.*

In general, given a Poisson algebra A, if there exists a $\star'$ product on $A[[\hbar]]$ such that

$$\mathcal{O}_1 \star' \mathcal{O}_2 - \mathcal{O}_2 \star' \mathcal{O}_1 = \hbar\{\mathcal{O}_1, \mathcal{O}_2\} + O(\hbar^2)$$

for any $\mathcal{O}_1, \mathcal{O}_2 \in A$, then $(A[[\hbar]], \star', \{-,-\})$ is called a ***deformation quantization*** of the Poisson algebra $(A, \star, \{-,-\})$. Hence here we call $Z(S^2)^{S^1}$ the quantization of $Z(S^2)$. One can show that $Z(S^2)^{S^1}$ is an $\mathbb{E}_1$-algebra.

12.4. **Quantization of the Coulomb Branch $Z_{3dA}(S^2)$.** Recall that for an algebraic group G acting on the cotangent matter T^*N, the $3d$ Coulomb branch is $Z_{3dA}(S^2) = C_*(\mathrm{Maps}(\mathbb{B}, [N/G]))$. Hence the quantization is

$$Z_{3dA}(S^2)^{S^1} = C^{S^1}_*(\mathrm{Maps}(\mathbb{B}, [N/G])),$$

where S^1, or equivalently $\mathbb{C}^\times$, acts on $\mathbb{B}$ by rotation. We consider two basic examples.

Example 12.3. Let $N = 0$. $\mathrm{Maps}(\mathbb{B}, [\mathrm{pt}/G]) = \mathrm{Maps}(\mathbb{B}, BG) = \mathrm{Bun}_G(\mathbb{B})$ is the moduli space of principal G-bundles over $\mathbb{B}$. Fix trivializations of the principal bundle on the two copies of $\mathbb{D}$ and denote them by P_0 and P_0'. Then the gluing map on $\mathbb{D}^\times$ determines an isomorphism

$$\alpha : P_0|_{\mathbb{D}^\times} \xrightarrow{\sim} P_0'|_{\mathbb{D}^\times},\ \alpha \in G((t)).$$

Since $G[[t]]$ acts on the space of trivializations P_0 and P_0', we get

$$\mathrm{Bun}_G(\mathbb{B}) = G[[t]]\backslash G((t))/G[[t]].$$

Meanwhile, $\mathbb{C}^\times$ also acts on the space $\mathrm{Bun}_G(\mathbb{B})$. Note that $\mathbb{C}^\times$ acts on $\mathrm{Spec}\,\mathbb{C}((t))$ by rotation so $s \cdot t^n = s^n t^n$. We can consider the semi-direct product

$$\mathbb{C}^\times \rtimes G((t)),\ (s, 1) \cdot (1, t^n) = (1, s^n t^n) \cdot (s, 1).$$

There is an isomorphism of moduli spaces

$$\mathbb{C}^\times \backslash (G[[t]]\backslash G((t))/G[[t]]) = (\mathbb{C}^\times \rtimes G[[t]])\backslash(\mathbb{C}^\times \rtimes G((t)))/(\mathbb{C}^\times \rtimes G[[t]]).$$

By projecting the left and right quotient space to a point, we get two projection maps

$$p_{L/R} : (\mathbb{C}^\times \rtimes G[[t]])\backslash(\mathbb{C}^\times \rtimes G((t)))/(\mathbb{C}^\times \rtimes G[[t]]) \to \mathrm{pt}/(\mathbb{C}^\times \rtimes G[[t]]).$$

Since taking equivariant cohomology with respect to G is equivalent to taking equivariant cohomology with respect to $G[[t]]$, the projection maps then induce maps on equivariant cohomologies

$$p_{L/R}^* : C^*_{S^1\times G}(\mathrm{pt}) \to C^*_{S^1}(\mathrm{Bun}_G(\mathbb{B})).$$

On the t^n-component of $G[[t]]\backslash G((t))/G[[t]]$, choosing $x \in C^*_G(\mathrm{pt})$, we have

$$p_L^*(x) = p_R^*(x) + p_R^*(n\hbar).$$

Here we have $x \cdot a = p_L^*(x)a$ and $a \cdot x = p_R^*(x)a$.

Now we can compute in $C_*^{S^1}(\mathrm{Bun}_G(\mathbb{B})) = \bigoplus_{n\in\mathbb{Z}} C_*^{S^1\times G}(\mathrm{pt})\left\langle t^n\right\rangle$,

$$t^n \cdot t^m = t^{n+m},\ x\cdot t^n - t^n \cdot x = p_L^*(x)t^n - p_R^*(x)t^n = n\hbar t^n.$$

This concludes the computation.

Proposition 12.4. *Let $G = \mathbb{C}^\times$. Then $C_*^{S^1}(\mathrm{Bun}_G(\mathbb{B})) = \mathbb{C}[\hbar]\langle x, t\rangle/([x,t] = \hbar t)$.*

Using the computation for $G = \mathbb{C}^\times$ and $N = 0$, we can compute the quantization of the Coulomb branch in more general abelian settings. Consider the vector bundle $T_{G,N} \to Gr_G$ and let $z : Gr_G \to T_{G,N}$ be the zero section. Consider the diagram

$$R_{G,N} \xhookrightarrow{i} T_{G,N} \xhookleftarrow{z} Gr_G.$$

Theorem 12.5. *Let G be an algebraic group and N be a G-representation. Then z^*i_* : $C_*^{S^1}(R_{G,N}/G[[t]]) \to C_*^{S^1}(\mathrm{Bun}_G(\mathbb{B}))$ is an algebra homomorphism, and when G is abelian, it is injective on homology.*

Example 12.6. Let $G = \mathbb{C}^\times$ and $N = \mathbb{C}$. Then the above diagram $R_{G,N} \xhookrightarrow{i} T_{G,N} \xhookleftarrow{z} Gr_G$ can be written as

$$\bigsqcup_{n\in\mathbb{Z}} t^n \times (t^n\mathbb{C}[[t]] \cap \mathbb{C}[[t]]) \xhookrightarrow{i} \bigsqcup_{n\in\mathbb{Z}} t^n \times t^n\mathbb{C}[[t]] \xhookleftarrow{z} \bigsqcup_{n\in\mathbb{Z}} t^n \times 0.$$

On each t^n-component, the homology class

$$[t^n \times (t^n\mathbb{C}[[t]] \cap \mathbb{C}[[t]])] = e^T(t^n\mathbb{C}[[t]]/(t^n\mathbb{C}[[t]] \cap \mathbb{C}[[t]]))t^n.$$

The computation will in the end show that

$$C_*^{S^1}(R_{G,N}/G[[t]]) \simeq \mathcal{D}_\hbar(\mathbb{C}) = \mathbb{C}[\hbar]\langle y, \partial_y\rangle/([\partial_y, y] = \hbar).$$

13. Lecture 13 (Thursday)

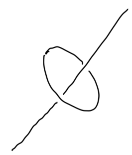
Hey this happened on Monday at 9

14. Lecture 14 (Thursday)

Hey this happened on Monday at 9

15. Lecture 15 (Friday)

In this lecture, we will study $Z(S^1)$, the ***category of line operators*** of our 3d TQFT. The name "line operator" comes from the fact that a line is the link of S^1 in $\mathbf{R}^3$:

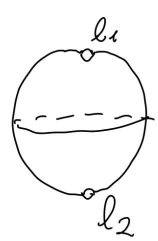


15.1. **General Structure of Line Operators.** Note that $Z(S^1)$ is a linear category, and objects $\ell \in Z(S^1)$ are used to "firm up" surfaces. More precisely, if Σ is a surface with holes, and to each hole we assign some $\ell_i \in Z(S^1)$, then we obtain a *vector space* $Z(\Sigma; \ell_1, \dots, \ell_n)$.

Morphisms in $Z(S^1)$ are defined as follows. Consider the cylinder $C = S^1 \times [0,1]$ as having two incoming boundary components and no outgoing boundary. If we place a line operator ℓ_i $(i = 1, 2)$ at each boundary component, we get

$$Z(C; \ell_1, \ell_2) = \mathrm{Hom}_{Z(S^1)}(\ell_1, \ell_2). \tag{15.1}$$

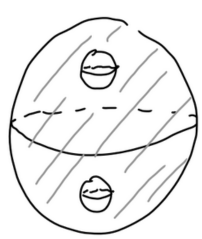
This corresponds to the following cobordism:



We can view $Z(C)$ as the functor

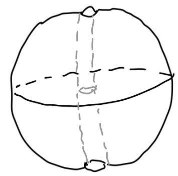
$$Z(C) = \mathrm{Hom}_{Z(S^1)}(-, -). \tag{15.2}$$

The product on $Z(S^2)$ comes from sticking small disks D^3_1, D^3_2 inside a larger disk D^3:



This defines a cobordism $D^3 \setminus (\mathring{D}^3_1 \sqcup \mathring{D}^3_1) : S^2_1 \sqcup S^2_2 \to S^2$ which gives an $\mathbb{E}_3$ multiplication on S^2.

We can adapt this picture to get a description of composition of morphisms in $Z(S^1)$. Namely, the surface (topologically a torus)



defines the composition map

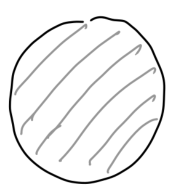
$$\circ : \mathrm{Hom}_{Z(S^1)}(\ell_2, \ell_3) \otimes \mathrm{Hom}_{Z(S^1)}(\ell_1, \ell_2) \to \mathrm{Hom}_{Z(S^1)}(\ell_1, \ell_3). \tag{15.3}$$

The category $Z(S^1)$ has an $\mathbb{E}_2$-structure. In other words, we can view $Z(S^1)$ as a braided tensor category. This structure comes from the following morphisms:

- A tensor operation $\otimes : Z(S^1) \boxtimes Z(S^1) \to Z(S^1)$ defined by

- A braiding defined by

- A tensor unit $\mathbb{1} \in Z(S^1)$, the ***trivial*** or ***identity line***, defined by



The category $Z(S^1)$ has an S^1-action coming from the group S^1 acting on the manifold S^1 by rotation. Passing to S^1-invariants (in physics language, "turning on an Omega background") takes us from an $\mathbb{E}_2$-category to an $\mathbb{E}_0$-category (i.e. a category with a distinguished object $\mathbb{1}$). One can compare this to how taking S^1-invariants sends the $\mathbb{E}_3$-algebra $Z(S^2)$ to an $\mathbb{E}_1$-algebra.

15.2. **State Spaces from $Z(S^1)$.** We can recover the value of Z on a surface (or a 3-manifold) from $Z(S^1)$.

Example 15.4. *To compute $Z(S^2)$, we note*

$$Z(S^2) = Z(D^2 \cup_{S^1} D^2) = \mathrm{Hom}_{Z(S^1)}(Z(D^2), Z(D^2)) = \mathrm{Hom}_{Z(S^1)}(\mathbb{1}, \mathbb{1}) = \mathrm{End}_{Z(S^1)}(\mathbb{1}). \tag{15.5}$$

From the physics of 3d $\mathcal{N} = 4$ gauge theories, we know that $Z(S^2) = \mathbb{C}[\mathcal{M}_{\mathrm{vac}}]$. If we know $\mathcal{M}_{\mathrm{vac}}$ (or its affinization $\mathcal{M}^{\mathrm{aff}}_{\mathrm{vac}}$), this gives us a good check for proposed definitions of $Z_A(S^1)$ and $Z_B(S^1)$.

Consider the functor[1]

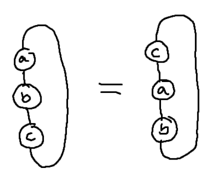
$$\mathrm{Hom}_{Z(S^1)}(-, \mathbb{1}) : Z(S^1)^{\mathrm{op}} \to \mathbb{C}[\mathcal{M}_{vac}] - \mathrm{mod} = \mathbf{D}^b\big(\mathrm{Coh}(\mathcal{M}^{\mathrm{aff}}_{\mathrm{vac}})\big). \tag{15.6}$$

If $\mathcal{M}_{\mathrm{vac}}$ is smooth, renormalization group flow arguments tell us that this functor should be an equivalence. In general, $Z(S^1)$ will contain more information than $\mathbf{D}^b\big(\mathrm{Coh}(\mathcal{M}^{\mathrm{aff}}_{\mathrm{vac}})\big)$, e.g. $Z(S^1)$ controls deformations and resolutions of $\mathcal{M}^{\mathrm{aff}}_{\mathrm{vac}}$.

Example 15.7. *To compute $Z(T^2)$, we note*

$$Z(T^2) = Z\big((S^1 \times [0,1])/((x,0) \sim (x,1)\big) = \text{“trace of Hom”} := \mathrm{HH}_\bullet(Z(S^1)), \tag{15.8}$$

the Hochschild homology of the category $Z(S^1)$. The “trace” here can be understood using the picture of beads on a string



where the “beads” (here a, b, c) are homomorphisms, and the picture defines the trace of the composite (here $\mathrm{tr}(abc) = \mathrm{tr}(cab)$).

15.3. **Line Operators in Topological Twists: Strategy.** To compute $Z(S^1)$ for the A- or B-twists of the 3d $\mathcal{N} = 4$ gauge theory corresponding to G and T^*V (where $\rho : G \to \mathrm{U}(V)$ is a unitary representation), we proceed by:

(1) Solving the equations of motion for the A- or B-twist.
(2) “Quantizing,” roughly by:
    (a) (A-twist) taking the Fukaya category of the moduli space of solutions.
    (b) (B-twist) taking $\mathbf{D}^b\mathrm{Coh}$ of the moduli space of solutions.

We will work this out in more detail as follows.

[1] Here we are making explicit the previously implicit assumption that the categories appearing in the study of 3d $\mathcal{N} = 4$ gauge theories are all derived / dg-categories.

15.4. **Line Operators in the 3d B-Model.** Fix coordinates $\vec{X}, \vec{Y}$ on $T^*V = V \oplus V^*$, and write a general $G_{\mathbb{C}}$-connection as $\mathcal{A}$. The equations of motion of the B-twist are given by setting $Q_B(-) = 0$ for all fields. This yields:

$$\begin{cases} \mathcal{F}_{\mathcal{A}} = 0 \\ \mathrm{d}_{\mathcal{A}}\vec{X} = \mathrm{d}_{\mathcal{A}}\vec{Y} = 0 \\ \mu_{\mathbb{C}}(\vec{X}, \vec{Y}) = \rho^*(\vec{X}, \vec{Y}) = 0 \\ \text{a reality equation} \end{cases} \tag{15.9}$$

To find the moduli space of solutions, we must quotient by G-gauge transformations. In fact, it is more convenient to quotient by $G_{\mathbb{C}}$-gauge transformations, which removes the reality equation.

In an infinitesimal neighborhood of S^1 in $\mathbf{R}^3$, the only information in the connection $\mathcal{A}$ is the holonomy $g = \mathrm{Hol}_{S^1_p}(\mathcal{A}) \in G_{\mathbb{C}}$ for some point $p \in S^1$. Let $\vec{X}_p$ and $\vec{Y}_p$ be the values of the fields $\vec{X}$ and $\vec{Y}$ at p. Our equations of motion become

$$\begin{cases} g\vec{X}_p = \vec{X}_p \\ \vec{Y}_p g^{-1} = \vec{Y}_p \\ \mu_{\mathbb{C}}(\vec{X}_p, \vec{Y}_p) = 0. \end{cases} \tag{15.10}$$

It is a good exercise to check that these equations are equivalent to the single equation $\mathrm{d}W = 0$ for $W : G_{\mathbb{C}} \times V \times V^* \to \mathbb{C}$ given by

$$W(g, \vec{X}, \vec{Y}) = \vec{Y} \cdot (\rho(g) - 1)\vec{X}. \tag{15.11}$$

Thus the moduli space of solutions is

$$\left\{ \left(g, \vec{X}, \vec{Y}\right) \in G_{\mathbb{C}} \times V \times V^* \,\middle|\, \mathrm{d}_{(g,\vec{X},\vec{Y})}W = 0 \right\} / G_{\mathbb{C}}. \tag{15.12}$$

Here $h \in G_{\mathbb{C}}$ acts by $h \cdot (g, \vec{X}, \vec{Y}) \mapsto (hgh^{-1}, h\vec{X}, \vec{Y}h^{-1})$.

The resulting moduli space is typically singular, so it is better to use the category of (equivatiant) matrix factorizations than the category of coherent sheaves. Thus

$$Z^B_{G,V}(S^1) = \mathrm{MF}^{G_{\mathbb{C}}}\left(\left\{ \left(g, \vec{X}, \vec{Y}\right) \in G_{\mathbb{C}} \times V \times V^* \,\middle|\, \mathrm{d}_{(g,\vec{X},\vec{Y})}W = 0 \right\} \right) \tag{15.13}$$

Here $\mathbb{1} = \mathcal{O}_{g=1}$, and $\mathrm{End}(\mathbb{1}) = \mathbb{C}[T^*V//G_{\mathbb{C}}] = \mathbb{C}[\mathcal{M}_H]$.

Example 15.14. *Consider the case where $G = 1$ and $V = \mathbb{C}$. The moduli space of solutions is smooth in this case, so we can use coherent sheaves in place of matrix factorizations, and*

$$Z^B_{1,\mathbb{C}}(S^1) = \mathbf{D}^b\mathrm{Coh}(T^*\mathbb{C}), \tag{15.15}$$

Here $\mathbb{1} = \mathcal{O}_{T^*\mathbb{C}}$ and $\mathrm{End}(\mathbb{1}) = \mathbb{C}[T^*\mathbb{C}]$.

It is more interesting to note that

$$Z^B_{1,\mathbb{C}}(T^2) = \mathrm{HH}_\bullet(\mathbf{D}^b\mathrm{Coh}(T^*\mathbb{C})) = \mathbb{C}[X, Y, \mathrm{d}\overline{X}, \mathrm{d}\overline{Y}] \tag{15.16}$$

by the Hochschild-Kostant-Rosenberg theorem. Here X and Y are in even degree, and $\mathrm{d}\overline{X}$ and $\mathrm{d}\overline{Y}$ are in odd degree.

15.5. **Line Operators in the 3d A-Model.** If we think algebraically and "squint hard enough," an infinitesimal neighborhood of S^1 in $\mathbf{R}^3$ looks like $D^\times = \mathrm{Spec}\mathbb{C}((z))$. The moduli space solutions to the 3d A-model equations of motion on $D^\times$ look like $T^*\big(V((z))/G_\mathbb{C}((z))\big)$, where the quotient by $G_\mathbb{C}((z))$ corresponds to modding out by gauge equivalence.

We expect $Z^A_{G,V}(S^1)$ to be something like the Fukaya category of this moduli space, but this is hard to define rigorously. Work of Justin Hilburn and Philsang Yoo proposes that we instead consider the category of D-modules (or constructible sheaves) on the base of the cotangent bundle. The modified proposal is then

$$Z^A_{G,V}(S^1) = D - \mathrm{mod}\big(V((z))/G_\mathbb{C}((z))\big) = D - \mathrm{mod}^{G_\mathbb{C}((z))}\big(V((z))\big), \tag{15.17}$$

the category of strongly equivariant D-modules on $V((z))$.

This definition may look abstract, but it is compatible with the BFN construction of Coulomb branches, providing evidence for its validity. Basic objects of $Z^A_{G,V}(S^1)$ are labeled by pairs

$$\{(L \subset V((z)) \text{ a subspace}, H \subset G_\mathbb{C}((z)) \text{ a subgroup}) \,|\, H \text{ stabilizes } L\}. \tag{15.18}$$

Here

(15.19)

$$\mathrm{Hom}_{Z^A_{G,V}(S^1)}\big((L,H),(L',H')\big) = \mathrm{H}_\bullet((L'/H') \times_{V((z))/G_\mathbb{C}((z))} L/H)$$

$$= \mathrm{H}_\bullet\left(H'\backslash\{(\vec{X},\vec{X}',g) \in L \times L' \times G_\mathbb{C}((z)) \,|\, \vec{X}' = g\vec{X}\}/H\right). \tag{15.20}$$

The unit object $\mathbb{1}$ is given by the pair $(V[[z]], G_\mathbb{C}[[z]])$ (this is an algebraic version of "filling in $D^\times$ to D"). One can check that $\mathrm{End}(\mathbb{1}) = \mathbb{C}[\mathcal{M}_C]$, as claimed above.

15.6. **3d Mirror Symmetry for Line Operators.** Mirror symmetry results have been proven for categories of line operators in the case when G is abelian. We discuss some of them here.

Theorem 15.21 (Hilburn, Raskin). *There exists an equivalence of categories between $Z^A_{1,\mathbb{C}}(S^1)$ and a de Rham version of $Z^B_{U(1),\mathbb{C}}(S^1)$.*

This theorem can be extended to other examples by looking at the behavior of the tensoring operation and the action of hyperkähler isometries.

Another approach proceeds by interpreting line operators in terms of vertex algebras.

Theorem 15.22 (Ballin, Creutzig, Dimofte, Niu). *If G is abelian and $G \curvearrowright V$ is faithful, then*

$$Z^B_{G,V}(S^1)^{\text{finitely supported on } G_\mathbb{C}} \simeq \mathbf{D}^b(\text{modules over some VOA})$$

as braided tensor categories. Furthermore, there is a 3d mirror symmetry statement relating this to an A-side category of VOA modules contained in $Z^A_{G^!,V^!}(S^1)$.

The slogan of this theorem is that 3d mirror symmetry for line operators looks like 2d mirror symmetry on loop spaces.

The nonabelian case is poorly understood, and work on understanding it is highly desirable. One viewpoint on the nonabelian case comes from Ben Webster's work interpreting 3d mirror symmetry combinatorially using KLRW algebras.

Theorem 15.23 (Webster)**.** *When* $\mathcal{M}_C$ *admits a full resolution* $\widetilde{\mathcal{M}_C} \to \mathcal{M}_C$*, there exists a deformation of* $Z^A_{G,V}(S^1)$ *to* $\mathbf{D}^b\big(\mathrm{Coh}(\widetilde{\mathcal{M}_C})\big)$.

This is a non-abelian mirror symmetry statement, but it misses information about the singularities. The deformation here is mirror to the deformation of $Z^B_{G^!,V^!}(S^1)$ obtained by imposing GIT stability parameters. It would be good to have an improved version of this theorem incorporating the full category of line operators in the A-model and singularities of the moduli space.

15.7. **Exercises.**

Exercise 15.24. *Suppose that* $G = U(1)$ *acts on* $V = \mathbb{C}^2$ *with weights* $(1,3)$*. (In other words* $e^{i\theta} : (X^1, X^2) \mapsto (e^{i\theta}X^1, e^{3i\theta}X^2)$*.)*

Write down explicitly the superpotential

$$W : G_{\mathbb{C}} \times V \times V^* \to \mathbb{C}$$

that appears in the solution to the B-twist equations of motion on S^1*. Differentiate with respect to* $g \in G_{\mathbb{C}}$ *to recover the complex moment map* $\mu_{\mathbb{C}}$ *for the action on* T^*V.

For the remaining problems, recall that in the A twist of (G, T^*V) gauge theory, the category $Z(S^1)$ has (some) objects labelled by (L, H) where L is a subspace of $V(\!(z)\!)$, and H is a subgroup of $G_{\mathbb{C}}(\!(z)\!)$ that stabilizes L. The morphism space between two such objects may be defined as Borel-Moore homology

$$\mathrm{Hom}\big((L,H),(L',H')\big) := H_\bullet\Big(H'\backslash\big\{(X',X,g) \in L' \times L \times G_{\mathbb{C}}(\!(z)\!) \,\big|\, X' = gX\big\}/H\Big)$$

Note that when a quotient is not free (as in the H or H' quotients above) it should be interpreted as taking equivariant cohomology.

Exercise 15.25. *Let* $\mathbb{1} := (V[\![z]\!], G_{\mathbb{C}}[\![z]\!])$*. Argue that* $End(\mathbb{1}) = \mathbb{C}[\mathcal{M}_C]$ *reproduces the BFN construction of the Coulomb branch.*

Exercise 15.26. *Take* $G = U(1)$ *and* $V = \mathbb{C}$ *(with weight* 1*). Recall that the Coulomb branch algebra may be presented as*

$$\mathbb{C}[\mathcal{M}_{\mathbb{C}}] \simeq \mathbb{C}[v_+, v_-, \varphi]/(v_+v_- = \varphi)\,.$$

Consider line operators

$$\mathcal{N}_+ = (\mathbb{C}(\!(z)\!), \mathbb{C}(\!(z)\!)^*)\,, \qquad \mathcal{N}_- = (0, \mathbb{C}(\!(z)\!)^*)\,.$$

Show that they give rise to modules

$$\mathrm{Hom}(\mathcal{N}_+, \mathbb{1}) = \mathbb{C}[\mathcal{M}_C]/(v_+ - 1)\,, \qquad \mathrm{Hom}(\mathcal{N}_i, \mathbb{1}) = \mathbb{C}[\mathcal{M}_C]/(v_- - 1)\,.$$