LAWRGE 2023 NOTES

Contents

1. Lecture 1 (Monday)

Throughout the week we will use the language of TQFTs to relate physics and math. The goal of this talk is to introduce the relevant terminology and some definitions.

1.1. TQFTs.

1.2. Definition.

We begin by introducing the notion of a d-dimensional TQFT.

Definition 1.1. Let M, N be closed oriented $(d-1)$-manifolds. A ***d-dimensional cobordism W from M to N*** is an oriented d-dimensional manifold W together with an identification $\partial W \cong \overline{M} \coprod N$.

Remark 1.2. *There is also a notion of an unoriented cobordism between two unoriented manifolds, framed cobordism between two framed manifolds and, more generally, a cobordism equipped with a tangential structure.*

Cobordisms define a symmetric monoidal category $\mathrm{Cob}^{or}_{d,d-1}$ as follows:

- Its objects are closed oriented $(d-1)$-manifolds.
- Morphisms from M to N are diffeomorphism classes of oriented cobordisms from M to N.
- Composition of a cobordism W_1 from M to N and a cobordism W_2 from N to O is given by the cobordism $W_1 \coprod_N W_2$ from M to O.
- The symmetric monoidal structure is given by disjoint union of manifolds.

Definition 1.3. An ***oriented*** d***-dimensional TQFT*** is a symmetric monoidal functor $Z\colon \mathrm{Cob}_{d,d-1} \to \mathrm{Vect}$ to the category of ($\mathbb{C}$-)vector spaces with tensor product as the monoidal structure.

The physical idea of the definition is as follows:

- For a closed $(d-1)$-manifold M we have a vector space $Z(M)$. It is the vector space of states of the TQFT (often a Hilbert space in physical examples).
- For a closed d-manifold W we have a number $Z(W)$. It is the partition function of the TQFT on W.
- For a cobordism W from M to N we get a linear map $Z(W)\colon Z(M) \to Z(N)$. It is the *transition amplitude* (S-matrix) associated to the cobordism W.

1.3. **Extending down.** Given a decomposition $W = W_1 \coprod_M W_2$ of a closed oriented d-manifold into a union of two manifolds along their common boundary, one can compute the partition function as

$$Z(W) = Z(W_2)(Z(W_1)(1)),$$

where

$$Z(W_1)\colon \mathbb{C} \longrightarrow Z(M), \qquad Z(W_2)\colon Z(M) \longrightarrow \mathbb{C}.$$

This allows one to compute the partition function by decomposing a manifold into pieces. This is related to the principle of locality of a QFT. Full locality will also allow us to compute the partition function by decomposing the boundary M into pieces. This can be made precise by extending the category $\mathrm{Cob}^{or}_{d,d-1}$ to a 2-category or even a higher category as follows. Let $\mathrm{Cob}^{or}_{d,d-1,d-2}$ be the symmetric monoidal 2-category as follows:

- Its objects are closed oriented $(d-2)$-manifolds.
- 1-morphisms from M to N are oriented $(d-1)$-dimensional cobordisms W from M to N.
- 2-morphisms from $W_1\colon M \to N$ to $W_2\colon M \to N$ are diffeomorphism classes of d-dimensional cobordisms between W_1 and W_2.

One can also extend it all the way down and define the symmetric monoidal d-category Cob^{or}_d whose objects are closed oriented 0-manifolds (disjoint unions of oriented points), 1-morphisms are 1-dimensional cobordisms and so on.

To define TQFTs we also need to extend the target category Vect down. For instance, for once-extended TQFTs we are looking for a symmetric monoidal bicategory $\mathcal{C}$ (usually it is the bicategory of some class of categories) with the property that $\mathrm{Hom}_{\mathcal{C}}(1,1) \cong \mathrm{Vect}$. Similarly, for fully extended TQFTs we are looking for a symmetric monoidal d-category $\mathcal{C}$ with a similar property for top-level morphisms.

Definition 1.4. Let $\mathcal{C}$ be a (linear) symmetric monoidal d-category. A ***fully extended TQFT*** is a symmetric monoidal functor $Z\colon \mathrm{Cob}_d^{or} \to \mathcal{C}$.

Note that given any fully extended TQFT we obtain higher-categorical structures irrespectively of the target $\mathcal{C}$:

- If M is a closed oriented $(d-1)$-manifold, $Z(M)$ is a vector space. We can think of $Z(M)$ as an element of the vector space $\mathrm{Hom}_{\mathcal{C}}(Z(\varnothing^{d-1}), Z(M))$.
- If M is a closed oriented $(d-2)$-manifold, $\mathrm{Hom}_{\mathcal{C}}(Z(\varnothing^{d-2}), Z(M))$ is a category. In fact, the structure of an oriented TQFT will induce a Calabi–Yau structure on this.
- . . .

1.4. **Extending up.** Let M be a closed oriented d-manifold and $\mathrm{Diff}(M)$ the topological group of orientation-preserving diffeomorphisms of M. There is a natural map

$$\mathrm{MCG}(M) = \pi_0 \mathrm{Diff}(M) \longrightarrow \mathrm{Aut}_{\mathrm{Bord}_{d,d-1}^{or}}(M)$$

given by considering $W = M \times [0,1]$ with the identification $\partial W \cong \overline{M} \coprod M$ twisted by a diffeomorphism. The reason that isotopic diffeomorphisms give rise to the same morphisms is that in the definition of $\mathrm{Bord}_{d,d-1}^{or}$ we identify diffeomorphic cobordisms.

The full homotopy type of the diffeomorphism group can be encoded if we work in the framework of ∞-categories. Namely, there is a symmetric monoidal ∞-category $\mathrm{Bord}_{d,d-1}^{or}$ which has the following informal description:

- Its objects are closed oriented $(d-1)$-manifolds.
- 1-morphisms from M to N are oriented cobordisms from M to N.
- 2-morphisms are given by diffeomorphisms of cobordisms.
- 3-morphisms are given by isotopies of diffeomorphisms.
- ...

Similarly, there is a symmetric monoidal (∞, d)-category Bord_d^{or}.

Definition 1.5. Let $\mathcal{C}$ be a symmetric monoidal (∞, d)-category. A ***fully extended TQFT*** is a symmetric monoidal functor $Z\colon \mathrm{Bord}_d^{or} \to \mathcal{C}$.

In physics the state space on a closed oriented $(d-1)$-manifold M is often a chain complex $Z(M) \in \mathrm{Ch}$ (with the differential the BRST differential coming from gauge symmetries and/or supersymmetric twisting). So, while on the level of cohomology there is an action of the mapping class group $\mathrm{MCG}(M)$ on $\mathrm{H}^\bullet(Z(M))$, on the chain level it should extend to a homotopy-coherent action of $\mathrm{C}_\bullet(\mathrm{Diff}(M))$ (equipped with the Pontryagin product) on the chain complex $Z(M)$. We will encounter the following two versions of this action:

- Consider a d-dimensional TQFT Z (for $d \geq 2$) and the chain complex $Z(S^{d-1})$. The natural S^1-action on S^{d-1} induces a $\mathbb{C}_\bullet(S^1)$-action on the chain complex $Z(S^{d-1})$. This action boils down to a square-zero degree -1 operation $B\colon Z(S^{d-1}) \to Z(S^{d-1})$.
- Consider a d-dimensional TQFT Z (for $d \geq 3$) and the category $Z(S^{d-2})$. The natural S^1-action on S^{d-2} induces a natural automorphism of the identity functor on $Z(S^{d-2})$.

If M is a closed oriented d-manifold, the partition function $Z(M)$ is merely a number. So, the higher-categorical structure is irrelevant in this case and we simply have that $Z(M)$ is invariant under $\mathrm{Diff}(M)$. It turns out to be useful to phrase this condition by saying that

$$Z(M) \in \mathrm{H}^0(\mathrm{BDiff}(M); \mathbb{C}).$$

1.5. **Boundary conditions.** The notion of a relative TQFT was introduced by Freed–Teleman and Johnson-Freyd–Scheimbauer. We will not give a precise definition, but will just indicate the main idea.

Suppose $Z\colon \mathrm{Cob}^{or}_{d,d-1} \to \mathrm{Vect}$ be a d-dimensional TQFT. The partition function $Z(M)$ *as a number* makes sense only for a closed oriented d-manifold. Given a *boundary condition*, we can evaluate the theory on manifolds with boundary as follows:

- For any closed oriented $(d-1)$-manifold M we have the space of states $Z(M)$.
- The boundary condition defines a distinguished vector $Z^\partial(M) \in Z(M)$.
- Given a compact oriented d-manifold W with boundary M we may view it as a cobordism $M \to \varnothing$. In particular,
$$Z(W)\colon Z(M) \longrightarrow \mathbb{C}.$$
So, the partition function of the TQFT on W with the given boundary condition is
$$Z(W)(Z^\partial(M)).$$

We can also talk about boundary conditions to once-extended or fully extended TQFTs. Then:

- For any closed oriented $(d-k)$-manifold M we have a $(k-1)$-category $\mathrm{Hom}_{\mathcal{C}}(Z(\varnothing^{d-k}), Z(M))$.
- The boundary condition defines a distinguished object $Z^\partial(M) \in \mathrm{Hom}_{\mathcal{C}}(Z(\varnothing^{d-k}), Z(M))$.

For instance, on the level of the point we get a distinguished object $Z^\partial(\mathrm{pt}) \in \mathrm{Hom}_{\mathcal{C}}(1, Z(\mathrm{pt}))$ in the $(d-1)$-category $\mathrm{Hom}_{\mathcal{C}}(1, Z(\mathrm{pt}))$. So, we can think of this as the $(d-1)$-category of *boundary conditions* (more precisely, fully local boundary conditions correspond to suitably dualizable objects of this $(d-1)$-category).

1.6. **2d mirror symmetry.** I will end this lecture by explaining the TQFT ideas behind the usual two-dimensional mirror symmetry as a warm up for three-dimensional mirror symmetry. We have the following 2d TQFTs:

- Let M be a symplectic manifold. Then one can define the 2d A-model $Z_{2dA,M}$. The category of boundary conditions $Z_{2dA,M}(\mathrm{pt})$ is some version of the Fukaya category of M.
- M be a smooth complex algebraic variety. Then one can define the 2d B-model $Z_{2dB,M}$. The category of boundary conditions $Z_{2dB,M}(\mathrm{pt})$ is some version of the derived category of coherent sheaves on M.

There are also equivariant versions of these 2d TQFTs:

- Given a (real) Lie group G acting in a Hamiltonian way on a symplectic manifold M there is an equivariant 2d A-model.
- Given a complex algebraic group $G_{\mathbb{C}}$ acting on a smooth complex algebraic variety M there is an equivariant 2d B-model.

Remark 1.6. *If M is not compact, these TQFTs are not defined on all 2-dimensional cobordisms.*

Remark 1.7. *Even though the framed TQFTs are well-defined, there is an "orientation anomaly" which complicates the definition of the oriented TQFT. The partition function on a surface Σ_g of genus g defines an element of $\mathrm{H}^{2(g-1)\dim_{\mathbf{R}} M}(\mathrm{BDiff}(\Sigma_g);\mathbb{C})$ rather than an element of $\mathrm{H}^0(\mathrm{BDiff}(\Sigma_g);\mathbb{C})$. For instance, the underlying number is zero for $g \neq 1$.*

The statement of 2-dimensional homological mirror symmetry can be formulated as follows. We say a symplectic manifold M is 2d mirror to a complex algebraic variety $M^\vee$ if

$$Z_{2dA,M} \cong Z_{2dB,M^\vee}.$$

This contains the following statements:

- An equivalence of Calabi-Yau categories.

$$Z_{2dA,M}(\mathrm{pt}) \cong Z_{2dB,M^\vee}(\mathrm{pt}).$$

  In practice the left-hand side is a version of the Fukaya category of M and the right-hand side is a version of the derived category of coherent sheaves on $M^\vee$.
- An equivalence of commutative algebras

$$\mathrm{H}^\bullet(Z_{2dA,M}(S^1)) \cong \mathrm{H}^\bullet(Z_{2dB,M^\vee}(S^1)).$$

  In fact, there is a Gerstenhaber structure (explained in the next lecture) on both sides which is also preserved.

2. Lecture 2 (Monday)

In this lecture Z denotes some d-dimensional TQFT. For simplicity I will assume that it is fully extended, but many statements make sense with partially extended TQFTs. I will assume that the TQFT is valued in the (∞-)category of chain complexes.

2.1. **Local and line operators.** Besides computing partition functions, in physics one is often interested in computing correlation functions of some local operators. Let us introduce them using the following heuristic idea.

Suppose M is a closed oriented d-manifold and $x \in M$ is a point with an insertion of a "local operator" $\mathcal{O}$. By locality one should be able to compute the partition function as follows:

- Consider a ball $D \subset M$ around x and let $S^{d-1} \subset M$ be its boundary.
- $Z(M \setminus D)$ defines a map $Z(S^{d-1}) \to \mathbb{C}$. The local operator defines a map $Z(D_{\mathcal{O}}) \colon \mathbb{C} \to Z(S^{d-1})$ and the partition function on M is the composite of these two maps.

If we are being agnostic about local operators, we may observe that the only thing we have used about them is the vector of $Z(S^{d-1})$ that they define. This leads us to the following definition.

Definition 2.1. Let Z be a d-dimensional TQFT. The ***space of local operators*** is the chain complex $Z(S^{d-1})$.

One also considers defects given by extended objects: lines, surfaces, ... embedded in M. Besides local operators, we will only encounter line operators this week. We can think of them as follows:

- A line operator is specified by a defect supported on a knot $K \subset M$. The same analysis as before shows that we can compute the partition function if we know the corresponding vector in $Z(S^{d-2} \times K)$.
- One often only considers "local" line operators which themselves obey cutting and gluing axioms of a TQFT. These local line operators define an object of the category $Z(S^{d-2})$.

This motivates the following definition.

Definition 2.2. Let Z be a d-dimensional TQFT.

- The ***space of line operators*** is $Z(S^{d-2} \times S^1)$.
- The ***category of line operators*** is $Z(S^{d-2})$.

2.2. $\mathbb{E}_d$**-algebras.** Our next goal is to explain algebraic structures present on the space of local and line operators. Given any cobordism W from k copies of S^{d-1} to S^{d-1} we get an algebraic operation

$$Z(W) \colon Z(S^{d-1})^{\otimes k} \longrightarrow Z(S^{d-1})$$

on the space of local operators in any TQFT. We will now investigate operations coming from cobordisms "with no topology".

Definition 2.3. Fix a dimension d.

- $$\mathbb{E}_d(k) = \mathrm{Emb}^{fr}(D^{\amalg k}, D)$$
  is the space of (smooth) framed embeddings of k d-dimensional open disks D into a given disk D.
- $$\mathbb{E}_d^{fr}(k) = \mathrm{Emb}(D^{\amalg k}, D)$$
  is the space of (smooth) oriented embeddings of k d-dimensional open disks D into a given disk D.

There are natural composition maps which make $\mathbb{E}_d$ and $\mathbb{E}_d^{fr}$ into operads. In particular, we can talk about their algebras.

Example 2.4. *The operads $\mathbb{E}_1$ and $\mathbb{E}_1^{fr}$ are both equivalent to the associative operad.*

Example 2.5. *There is a natural action of $\mathrm{SO}(d)$ on $\mathbb{E}_d$, so that a $\mathbb{E}_d^{fr}$-algebra is an $\mathbb{E}_d$-algebra equipped with a compatible $\mathrm{SO}(d)$-action.*

Given an embedding $D^{\amalg k} \hookrightarrow D$ we obtain a cobordism from $(S^{d-1})^{\amalg k}$ to S^{d-1} by removing the interiors of the embedded disks. In particular, we obtain a natural map

$$\mathrm{C}_\bullet(\mathbb{E}_d^{fr}(k); \mathbb{C}) \otimes_{\mathbb{C}} Z(S^{d-1})^{\otimes k} \longrightarrow Z(S^{d-1})$$

for any oriented TQFT. Similarly, if Z is a framed TQFT we get a natural map

$$\mathrm{C}_\bullet(\mathbb{E}_d(k); \mathbb{C}) \otimes_{\mathbb{C}} Z(S^{d-1})^{\otimes k} \longrightarrow Z(S^{d-1}).$$

Both maps are compatible with compositions and we obtain the following result:

- If Z is a framed TQFT, the chain complex of local operators $Z(S^{d-1})$ is an $\mathbb{E}_d$-algebra.
- If Z is an oriented TQFT, the chain complex of local operators $Z(S^{d-1})$ is a framed $\mathbb{E}_d$-algebra.

Up to homotopy the spaces of embeddings may be identified as follows.

Proposition 2.6. *There are homotopy equivalences*

$$\mathbb{E}_d(k) \cong \mathrm{Conf}_k(\mathbf{R}^d), \qquad \mathbb{E}_d^{fr}(k) \cong \mathrm{SO}(d)^k \times \mathrm{Conf}_k(\mathbf{R}^d),$$

where $\mathrm{Conf}_k(\mathbf{R}^d)$ is the configuration space of k distinct ordered points in $\mathbf{R}^d$.

Using the above description one can show the following:

- An $\mathbb{E}_2$-algebra in categories is a braided monoidal category.
- An $\mathbb{E}_2^{fr}$-algebra in categories is a balanced monoidal category, i.e. there is an extra automorphism of the identity functor, the *balancing* θ, which satisfies
  $$\theta_{x\otimes y} = \sigma_{y,x} \circ \sigma_{x,y} \circ (\theta_x \otimes \theta_y).$$

So, the category of line operators in a 3-dimensional TQFT is a balanced monoidal category.

2.3. $\mathbb{P}_d$**-algebras.** To describe $\mathbb{E}_d$-algebras in chain complexes, let us first introduce a related notion.

Definition 2.7. A $\mathbb{P}_d$***-algebra*** is a commutative dg algebra A equipped with a bracket of cohomological degree $1-d$ (inducing a Lie structure on $A[d-1]$) satisfying the Leibniz rule

$$\{a, bc\} = \{a, b\}c + (-1)^{|b||c|}\{a, c\}b$$

for $a, b, c \in A$.

Remark 2.8. *A* $\mathbb{P}_2$*-algebra is known as a* Gerstenhaber algebra*.*

Let $\mathbb{P}_d(k)$ be the vector space of all operations $A^{\otimes k} \to A$ on a $\mathbb{P}_d$-algebra. We can formalize it as follows: define $\mathbb{P}_d(k)$ to be the subspace of the free $\mathbb{P}_d$-algebra on degree 0 variables $x_1, \dots, x_k$ consisting of expressions where each x_i appears exactly once. For instance:

- $\mathbb{P}_d(1) \cong \mathbb{C}$ spanned by the identity map $A \to A$.
- $\mathbb{P}_d(2) \cong \mathbb{C} \oplus \mathbb{C}[d-1]$ spanned by the commutative multiplication $m\colon A \otimes A \to A$ and $\{-,-\}\colon A \otimes A \to A[1-d]$ by the Poisson bracket.

We have the following claim.

Definition 2.9. Suppose $d \geq 2$. Then there is an isomorphism of graded vector spaces $\mathrm{H}_\bullet(\mathbb{E}_d(k); \mathbb{C}) \cong \mathbb{P}_d(k)$.

Remark 2.10. *In fact, both* $\mathbb{E}_d$ *and* $\mathbb{P}_d$ *are operads and there is an equivalence* $\mathrm{C}_\bullet(\mathbb{E}_d; \mathbb{C}) \cong \mathbb{P}_d$ *of graded linear operads.*

As a corollary, given an $\mathbb{E}_d$-algebra A, its homology $\mathrm{H}_\bullet(A)$ has a natural structure of a $\mathbb{P}_d$-algebra.

Example 2.11. *Let* Z *be a 3d TQFT. Then the cohomology of the space of local operators* $\mathrm{H}^\bullet(Z(S^2))$ *carries a graded commutative multiplication as well as Poisson bracket of degree* -2*.*

2.4. Ω**-deformation.** We will now explain an important construction with $\mathbb{E}_d$-operads which is known in physics as the procedure of Ω-deformation.

Consider the $\mathbb{E}_d$-operad equipped with its natural $\mathrm{SO}(d)$-action. There is a natural inclusion of operads $\mathbb{E}_{d-2} \hookrightarrow \mathbb{E}_d$ which is $\mathrm{SO}(d-2) \times \mathrm{SO}(2)$-equivariant, where the $\mathrm{SO}(2)$-action on the left is trivial.

Theorem 2.12. *The inclusion of operads* $\mathbb{E}_{d-2} \hookrightarrow \mathbb{E}_d$ *realizes* $\mathbb{E}_{d-2}$ *as the space of fixed points of the* $\mathrm{SO}(2)$*-action on* $\mathbb{E}_d$*.*

To state an important corollary, let us first recall a few basics of equivariant localization. Given a space X with an action of a topological group G we may consider equivariant homology $\mathrm{H}^G_\bullet(X)$ and cohomology $\mathrm{H}^\bullet_G(X)$ which are both modules over $\mathrm{H}^\bullet_G(\mathrm{pt}) = \mathrm{H}^\bullet(\mathrm{B}G)$.

Example 2.13. *We have* $\mathrm{BSO}(2) = \mathbf{CP}^\infty$*, so* $\mathrm{H}^\bullet(\mathrm{BSO}(2)) = \mathbb{C}[\epsilon]$*, where* $\deg(\epsilon) = 2$*.*

Theorem 2.14 (Equivariant localization)**.** *Let G be a topological group. Let Y be a space with a G-action. Let X be a topological space equipped with a trivial G-action and a G-equivariant map $X \to Y$ which realizes X as the space of fixed points of the G-action on Y. Then the induced map*

$$\mathrm{H}^G_\bullet(X) \otimes_{\mathbb{C}[\epsilon]} \mathbb{C}(\epsilon) \longrightarrow \mathrm{H}^G_\bullet(Y) \otimes_{\mathbb{C}[\epsilon]} \mathbb{C}(\epsilon)$$

is an isomorphism.

Combining the theory of equivariant localization and 2.12 we get the following.

Theorem 2.15. *Let A be a framed $\mathbb{E}_d$-algebra. Consider the induced* $\mathrm{SO}(2) \subset \mathrm{SO}(d)$*-action on A. Then* $A^{\mathrm{SO}(2)} \otimes_{\mathbb{C}[\epsilon]} \mathbb{C}(\epsilon)$ *is a framed* $\mathbb{E}_{d-2}$*-algebra.*

Example 2.16. *Let Z be a 3d TQFT and consider the framed $\mathbb{E}_3$-algebra structure on the space of local operators $Z(S^2)$. Recall that its cohomology* $\mathrm{H}^\bullet(Z(S^2))$ *carries a natural (graded) Poisson structure. The equivariant localization*

$$\mathrm{H}^\bullet_{\mathrm{SO}(2)}(Z(S^2)) \otimes_{\mathbb{C}[\epsilon]} \mathbb{C}(\epsilon)$$

is an associative algebra which provides a deformation quantization of $\mathrm{H}^\bullet(Z(S^2))$ *(with the quantization parameter being ϵ).*

2.5. **Swiss-cheese algebras.** I will end this lecture by describing algebraic structures appearing in TQFTs with boundary conditions.

Let Z be a d-dimensional TQFT with a chosen boundary condition Z^∂. We can extract the following kinds of algebras:

- The space of *bulk local operators* in Z, i.e.

$$A = Z(S^{d-1}),$$

  carries the structure of a framed $\mathbb{E}_d$-algebra.
- The space of *boundary local operators*, i.e.

$$B = Z(D^{d-1})(Z^\partial(S^{d-2})),$$

  carries the structure of a framed $\mathbb{E}_{d-1}$-algebra. Namely, let H be the d-dimensional half-ball. Consider the space

$$\mathrm{Emb}^\partial(H^{\amalg k}, H)$$

  of oriented embeddings of k d-dimensional half-balls into a single one so that the boundaries are embedded into the boundaries. Retracting the half-ball to its boundary (a $(d-1)$-dimensional ball) identifies this space with $\mathbb{E}^{fr}_{d-1}$.

- In addition, there is an action of A on B as follows. Let D be a d-dimensional ball and H a d-dimensional half-ball. Any embedding $D^{\amalg l} \coprod H^{\amalg k} \hookrightarrow H$ (so that the boundaries of the half-balls are embedded in the boundary of the bigger half-ball) gives rise to an operation

$$A^{\otimes l} \otimes B^{\otimes k} \longrightarrow B.$$

The pair (A, B) together with the operations described above is known as a d-dimensional ***Swiss-cheese algebra***. We will encounter the following manifestation of this structure.

Example 2.17. *Let Z be a 3d TQFT with a boundary condition Z^∂. Let A be the $\mathbb{E}_3$-algebra of bulk local operators and B the $\mathbb{E}_2$-algebra of boundary local operators. A carries a degree -2 Poisson structure. There is a map of graded commutative algebras $A \to B$ and the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is coisotropic.*

Remark 2.18. *There is a homotopy notion of a coisotropic submanifold which is precisely defined in terms of a Swiss-cheese algebra structure.*

Recall that $\mathbb{E}_n(k)$ is homotopy equivalent to $\mathrm{Conf}_k(\mathbf{R}^n)$, the configuration space of k distinct points in $\mathbf{R}^n$.

Recall that a $\mathbb{P}_n$-algebra is a dg commutative algebra A equipped with a bracket $\{-,-\}$ of cohomological degree $1-n$ (inducing a Lie structure on $A[n-1]$) satisfying the relation $\{a, bc\} = \{a, b\}c + (-1)^{|b||c|}\{a, c\}b$. Let $\mathbb{P}_n(k)$ be the subspace of the free $\mathbb{P}_n$-algebra on degree 0 variables $x_1, \dots, x_k$ consisting of expressions where each x_i appears exactly once. For instance, $\{\{x_1, x_2\}, \{x_3, x_4\}\}$ is an element of $\mathbb{P}_n(4)$ of cohomological degree $3(1-n)$.

2.6. Exercises.

Exercise 2.19. *Consider the map $\mathrm{Conf}_k(\mathbf{R}^n) \to \mathrm{Conf}_{k-1}(\mathbf{R}^n)$ given by forgetting the last point. Show that its fiber F_k is homotopy equivalent to a wedge of $(k-1)$ spheres S^{n-1}.*

Exercise 2.20. *The Leray–Serre spectral sequence for the fibration $F_k \hookrightarrow \mathrm{Conf}_k(\mathbf{R}^n) \to \mathrm{Conf}_{k-1}(\mathbf{R}^n)$ degenerates (using the Leray–Hirsch theorem), so that one may identify*

$$\mathrm{H}_\bullet(\mathrm{Conf}_k(\mathbf{R}^n); \mathbb{Q}) \cong \mathrm{H}_\bullet(\mathrm{Conf}_{k-1}(\mathbf{R}^n); \mathbb{Q}) \otimes \mathrm{H}_\bullet(F_k; \mathbb{Q}).$$

Find $\mathrm{H}_\bullet(\mathbb{E}_n(k); \mathbb{Q})$ for $k = 1, 2, 3$.

Exercise 2.21. *Describe the graded vector space $\mathbb{P}_n(k)$ for $k = 1, 2, 3$ and find an isomorphism*

$$\mathrm{H}_\bullet(\mathbb{E}_n(k); \mathbb{Q}) \cong \mathbb{P}_n(k)$$

for $n \geq 2$.

Exercise 2.22. *(*) Consider the S_2-action on $\mathbb{E}_2(2) \sim S^1$ given by reflection around the origin. Let $\mathcal{C}$ be a category. Show that an S_2-equivariant map*

$$S^1 \times \mathcal{C}^{\times 2} \longrightarrow \mathcal{C}$$

is the same as a pair $(\otimes, \sigma)$ *consisting of a functor* $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *as well as a natural isomorphism*

$$\sigma_{x,y}\colon x \otimes y \xrightarrow{\sim} y \otimes x.$$

3. Lecture 3 (Monday)

Hey this happened on Monday at 9

4. Lecture 4 (Monday)

Hey this happened on Monday at 9

5. Lecture 5 (Tuesday)

Hey this happened on Monday at 9

6. Lecture 6 (Tuesday)

Hey this happened on Monday at 9

7. Lecture 7 (Tuesday)

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8. Lecture 8 (Tuesday)

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9. Lecture 9 (Wednesday)

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10. Lecture 10 (Wednesday)

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11. Lecture 11 (Wednesday)

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12. Lecture 12 (Wednesday)

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13. Lecture 13 (Thursday)

Hey this happened on Monday at 9

14. Lecture 14 (Thursday)

Hey this happened on Monday at 9

15. Lecture 15 (Friday)

16. Lecture 16 (Friday)

17. Lecture 17 (Friday)

18. Lecture 18 (Friday)