LAWRGE 2023 NOTES

Contents

1. Introduction

2. Lecture 1 (Monday)

Throughout the week we will use the language of TQFTs to relate physics and math. The goal of this talk is to introduce the relevant terminology and some definitions.

2.1. **TQFTs.**

2.2. **Definition.** We begin by introducing the notion of a d-dimensional TQFT.

Definition 2.1. Let M, N be closed oriented $(d-1)$-manifolds. A ***d-dimensional cobordism*** W ***from*** M ***to*** N is an oriented d-dimensional manifold W together with an identification $\partial W \cong \overline{M} \coprod N$.

Remark 2.2. *There is also a notion of an unoriented cobordism between two unoriented manifolds, framed cobordism between two framed manifolds and, more generally, a cobordism equipped with a tangential structure.*

Cobordisms define a symmetric monoidal category $\mathrm{Cob}^{or}_{d,d-1}$ as follows:

- Its objects are closed oriented $(d-1)$-manifolds.
- Morphisms from M to N are diffeomorphism classes of oriented cobordisms from M to N.
- Composition of a cobordism W_1 from M to N and a cobordism W_2 from N to O is given by the cobordism $W_1 \coprod_N W_2$ from M to O.
- The symmetric monoidal structure is given by disjoint union of manifolds.

Definition 2.3. An ***oriented d-dimensional TQFT*** is a symmetric monoidal functor $Z\colon \mathrm{Cob}_{d,d-1} \to \mathrm{Vect}$ to the category of ($\mathbb{C}$-)vector spaces with tensor product as the monoidal structure.

The physical idea of the definition is as follows:

- For a closed $(d-1)$-manifold M we have a vector space $Z(M)$. It is the vector space of states of the TQFT (often a Hilbert space in physical examples).
- For a closed d-manifold W we have a number $Z(W)$. It is the partition function of the TQFT on W.
- For a cobordism W from M to N we get a linear map $Z(W)\colon Z(M) \to Z(N)$. It is the *transition amplitude* (S-matrix) associated to the cobordism W.

2.3. **Extending down.** Given a decomposition $W = W_1 \coprod_M W_2$ of a closed oriented d-manifold into a union of two manifolds along their common boundary, one can compute the partition function as

$$Z(W) = Z(W_2)(Z(W_1)(1)),$$

where

$$Z(W_1)\colon \mathbb{C} \longrightarrow Z(M), \qquad Z(W_2)\colon Z(M) \longrightarrow \mathbb{C}.$$

This allows one to compute the partition function by decomposing a manifold into pieces. This is related to the principle of locality of a QFT. Full locality will also allow us to compute the partition function by decomposing the boundary M into pieces. This can be made precise by extending the category $\mathrm{Cob}^{or}_{d,d-1}$ to a 2-category or even a higher category as follows. Let $\mathrm{Cob}^{or}_{d,d-1,d-2}$ be the symmetric monoidal 2-category as follows:

- Its objects are closed oriented $(d-2)$-manifolds.
- 1-morphisms from M to N are oriented $(d-1)$-dimensional cobordisms W from M to N.
- 2-morphisms from $W_1\colon M \to N$ to $W_2\colon M \to N$ are diffeomorphism classes of d-dimensional cobordisms between W_1 and W_2.

One can also extend it all the way down and define the symmetric monoidal d-category Cob_d^{or} whose objects are closed oriented 0-manifolds (disjoint unions of oriented points), 1-morphisms are 1-dimensional cobordisms and so on.

To define TQFTs we also need to extend the target category Vect down. For instance, for once-extended TQFTs we are looking for a symmetric monoidal bicategory $\mathcal{C}$ (usually it is the bicategory of some class of categories) with the property that $\mathrm{Hom}_{\mathcal{C}}(1,1) \cong \mathrm{Vect}$. Similarly, for fully extended TQFTs we are looking for a symmetric monoidal d-category $\mathcal{C}$ with a similar property for top-level morphisms.

Definition 2.4. Let $\mathcal{C}$ be a (linear) symmetric monoidal d-category. A ***fully extended TQFT*** is a symmetric monoidal functor $Z\colon \mathrm{Cob}_d^{or} \to \mathcal{C}$.

Note that given any fully extended TQFT we obtain higher-categorical structures irrespectively of the target $\mathcal{C}$:

- If M is a closed oriented $(d-1)$-manifold, $Z(M)$ is a vector space. We can think of $Z(M)$ as an element of the vector space $\mathrm{Hom}_{\mathcal{C}}(Z(\varnothing^{d-1}), Z(M))$.
- If M is a closed oriented $(d-2)$-manifold, $\mathrm{Hom}_{\mathcal{C}}(Z(\varnothing^{d-2}), Z(M))$ is a category. In fact, the structure of an oriented TQFT will induce a Calabi–Yau structure on this.
- . . .

2.4. **Extending up.** Let M be a closed oriented d-manifold and $\mathrm{Diff}(M)$ the topological group of orientation-preserving diffeomorphisms of M. There is a natural map

$$\mathrm{MCG}(M) = \pi_0 \mathrm{Diff}(M) \longrightarrow \mathrm{Aut}_{\mathrm{Bord}_{d,d-1}^{or}}(M)$$

given by considering $W = M \times [0,1]$ with the identification $\partial W \cong \overline{M} \coprod M$ twisted by a diffeomorphism. The reason that isotopic diffeomorphisms give rise to the same morphisms is that in the definition of $\mathrm{Bord}_{d,d-1}^{or}$ we identify diffeomorphic cobordisms.

The full homotopy type of the diffeomorphism group can be encoded if we work in the framework of ∞-categories. Namely, there is a symmetric monoidal ∞-category $\mathrm{Bord}_{d,d-1}^{or}$ which has the following informal description:

- Its objects are closed oriented $(d-1)$-manifolds.
- 1-morphisms from M to N are oriented cobordisms from M to N.
- 2-morphisms are given by diffeomorphisms of cobordisms.
- 3-morphisms are given by isotopies of diffeomorphisms.
- ...

Similarly, there is a symmetric monoidal (∞, d)-category Bord_d^{or}.

Definition 2.5. Let $\mathcal{C}$ be a symmetric monoidal (∞, d)-category. A ***fully extended TQFT*** is a symmetric monoidal functor $Z\colon \mathrm{Bord}_d^{or} \to \mathcal{C}$.

In physics the state space on a closed oriented $(d-1)$-manifold M is often a chain complex $Z(M) \in \mathrm{Ch}$ (with the differential the BRST differential coming from gauge symmetries and/or supersymmetric twisting). So, while on the level of cohomology there is an action of the mapping class group $\mathrm{MCG}(M)$ on $\mathrm{H}^\bullet(Z(M))$, on the chain level it should extend to a homotopy-coherent action of $\mathrm{C}_\bullet(\mathrm{Diff}(M))$ (equipped with the Pontryagin product) on the chain complex $Z(M)$. We will encounter the following two versions of this action:

- Consider a d-dimensional TQFT Z (for $d \geq 2$) and the chain complex $Z(S^{d-1})$. The natural S^1-action on S^{d-1} induces a $\mathbb{C}_\bullet(S^1)$-action on the chain complex $Z(S^{d-1})$. This action boils down to a square-zero degree -1 operation $B\colon Z(S^{d-1}) \to Z(S^{d-1})$.
- Consider a d-dimensional TQFT Z (for $d \geq 3$) and the category $Z(S^{d-2})$. The natural S^1-action on S^{d-2} induces a natural automorphism of the identity functor on $Z(S^{d-2})$.

If M is a closed oriented d-manifold, the partition function $Z(M)$ is merely a number. So, the higher-categorical structure is irrelevant in this case and we simply have that $Z(M)$ is invariant under $\mathrm{Diff}(M)$. It turns out to be useful to phrase this condition by saying that

$$Z(M) \in \mathrm{H}^0(\mathrm{BDiff}(M); \mathbb{C}).$$

2.5. **Boundary conditions.** The notion of a relative TQFT was introduced by Freed–Teleman and Johnson-Freyd–Scheimbauer. We will not give a precise definition, but will just indicate the main idea.

Suppose $Z\colon \mathrm{Cob}^{or}_{d,d-1} \to \mathrm{Vect}$ be a d-dimensional TQFT. The partition function $Z(M)$ *as a number* makes sense only for a closed oriented d-manifold. Given a *boundary condition*, we can evaluate the theory on manifolds with boundary as follows:

- For any closed oriented $(d-1)$-manifold M we have the space of states $Z(M)$.
- The boundary condition defines a distinguished vector $Z^\partial(M) \in Z(M)$.
- Given a compact oriented d-manifold W with boundary M we may view it as a cobordism $M \to \varnothing$. In particular,

$$Z(W)\colon Z(M) \longrightarrow \mathbb{C}.$$

  So, the partition function of the TQFT on W with the given boundary condition is

$$Z(W)(Z^\partial(M)).$$

We can also talk about boundary conditions to once-extended or fully extended TQFTs. Then:

- For any closed oriented $(d-k)$-manifold M we have a $(k-1)$-category $\mathrm{Hom}_{\mathcal{C}}(Z(\varnothing^{d-k}), Z(M))$.
- The boundary condition defines a distinguished object $Z^\partial(M) \in \mathrm{Hom}_{\mathcal{C}}(Z(\varnothing^{d-k}), Z(M))$.

For instance, on the level of the point we get a distinguished object $Z^\partial(\mathrm{pt}) \in \mathrm{Hom}_{\mathcal{C}}(1, Z(\mathrm{pt}))$ in the $(d-1)$-category $\mathrm{Hom}_{\mathcal{C}}(1, Z(\mathrm{pt}))$. So, we can think of this as the $(d-1)$-category of *boundary conditions* (more precisely, fully local boundary conditions correspond to suitably dualizable objects of this $(d-1)$-category).

2.6. **2d mirror symmetry.** I will end this lecture by explaining the TQFT ideas behind the usual two-dimensional mirror symmetry as a warm up for three-dimensional mirror symmetry. We have the following 2d TQFTs:

- Let M be a symplectic manifold. Then one can define the 2d A-model $Z_{2dA,M}$. The category of boundary conditions $Z_{2dA,M}(\mathrm{pt})$ is some version of the Fukaya category of M.
- M be a smooth complex algebraic variety. Then one can define the 2d B-model $Z_{2dB,M}$. The category of boundary conditions $Z_{2dB,M}(\mathrm{pt})$ is some version of the derived category of coherent sheaves on M.

There are also equivariant versions of these 2d TQFTs:

- Given a (real) Lie group G acting in a Hamiltonian way on a symplectic manifold M there is an equivariant 2d A-model.
- Given a complex algebraic group $G_{\mathbb{C}}$ acting on a smooth complex algebraic variety M there is an equivariant 2d B-model.

Remark 2.6. *If M is not compact, these TQFTs are not defined on all 2-dimensional cobordisms.*

Remark 2.7. *Even though the framed TQFTs are well-defined, there is an "orientation anomaly" which complicates the definition of the oriented TQFT. The partition function on a surface Σ_g of genus g defines an element of $\mathrm{H}^{2(g-1)\dim_{\mathbf{R}} M}(\mathrm{BDiff}(\Sigma_g);\mathbb{C})$ rather than an element of $\mathrm{H}^0(\mathrm{BDiff}(\Sigma_g);\mathbb{C})$. For instance, the underlying number is zero for $g \neq 1$.*

The statement of 2-dimensional homological mirror symmetry can be formulated as follows. We say a symplectic manifold M is 2d mirror to a complex algebraic variety $M^\vee$ if

$$Z_{2dA,M} \cong Z_{2dB,M^\vee}.$$

This contains the following statements:

- An equivalence of Calabi-Yau categories.

  $$Z_{2dA,M}(\mathrm{pt}) \cong Z_{2dB,M^\vee}(\mathrm{pt}).$$

  In practice the left-hand side is a version of the Fukaya category of M and the right-hand side is a version of the derived category of coherent sheaves on $M^\vee$.
- An equivalence of commutative algebras

  $$\mathrm{H}^\bullet(Z_{2dA,M}(S^1)) \cong \mathrm{H}^\bullet(Z_{2dB,M^\vee}(S^1)).$$

  In fact, there is a Gerstenhaber structure (explained in the next lecture) on both sides which is also preserved.

3. Lecture 2 (Monday)

Hey this happened on Monday at 9

4. Lecture 3 (Monday)

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5. Lecture 4 (Monday)

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6. Lecture 5 (Tuesday)

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7. Lecture 6 (Tuesday)

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8. Lecture 7 (Tuesday)

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9. Lecture 8 (Tuesday)

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10. Lecture 9 (Wednesday)

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11. Lecture 10 (Wednesday)

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12. Lecture 11 (Wednesday)

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13. Lecture 12 (Wednesday)

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14. Lecture 13 (Thursday)

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15. Lecture 14 (Thursday)

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16. Lecture 15 (Friday)

17. Lecture 16 (Friday)

18. Lecture 17 (Friday)

19. Lecture 18 (Friday)