MATH 355 Notes

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September 5 2019

1 Topic 1 - Functions and Cardinality

Definition Given sets A and B, the **Cartesian Product of** A **and** B is the set of ordered pairs

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

Given any finite list of N sets $A_1, A_2, \ldots, A_N$, the **Cartesian Product of** A_i is

$$A_1 \times A_2 \times \cdots \times A_N = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i; i = 1, 2, \ldots, N\}$$

Example

$$\begin{aligned} \mathbb{R}^2 =& \mathbb{R} \times \mathbb{R} \\ \mathbb{R}^N =& \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{N} \end{aligned}$$

When $\mathbb{T}$ is the unit circle (i.e. $\mathbb{T} = \{(a, b) | a, b \in \mathbb{R}, a^2 + b^2 = 1\}$), $\mathbb{T}^2$ is the unit torus.

Definition A **function** f from A to B is a subset $f \subseteq A \times B$ with the additional property that for all $a \in A$, there is a single $b \in B$ so that $(a, b) \in f$. Then $b = f(a)$. Given a function $f : A \to B$, A is the **domain** of f (denoted $\mathrm{dom}(f)$), and B is the **codomain** of f (denoted $\mathrm{codom}(f)$). The **range** of f is the set $f(a) | a \in A$ (denoted $\mathrm{ran}(f)$).

Example

$$\begin{aligned}
&f:\mathbb{R}\to\mathbb{R}, f(x)=x^2\\
&\text{dom}(f)=\text{codom}(f)=\mathbb{R}\\
&\text{ran}(f)=[0,\infty)\\
\\
&g:\mathbb{R}\to\mathbb{R}, g(x)=x^3\\
&\text{dom}(g)=\text{codom}(g)=\text{ran}(g)=\mathbb{R}\\
\\
&h:[0,\pi]\to\mathbb{R}, h(x)=cos(x)\\
&\text{dom}(h)=[0,\pi]\\
&\text{codom}(h)=\mathbb{R}\\
&\text{ran}(h)=[-1,1]
\end{aligned}$$

Definition Given $f : A \to B$,

(i) f is injective $\iff f(a_1)=f(a_2) \implies a_1=a_2 \forall a_1, a_2 \in A$

(ii) f is surjective $\iff \forall b \in B, \exists a \in A : f(a) = b$

(iii) f is bijective $\iff f$ is injective and f is surjective

Example

$$k:\mathbb{Z}^+\to\mathbb{Z}, k(n)=\begin{cases}\frac{n}{2}, & n \text{ even}\\ -\left(\frac{n-1}{2}\right), & n \text{ odd}\end{cases}$$

is a bijection.

Proposition k is injective.
Let $a, b \in \mathbb{Z}^+$. Prove injectivity by contradiction.