MATH 355 Notes

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September 10 2019

1 Topic 1 - Functions and Cardinality (cont)

Definition Given $f : A \to B$ and $C \subseteq A, D \subseteq B$:

- The **image** of C under f is $f(C) = f(c)|c \in C$.

- The **preimage** of D under f is $f^{-1}(D) = a \in A|f(a) \in D$.

If f is invertible, the $f^{-1}(D)$ is the image of D under f^{-1}.

Example

$$
\begin{aligned}
&f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 \\
&C = [-4, 1] \\
&f(C) = [0, 16] \\
\\
&D = [2, 16] \\
&f^{-1}(D) = [-4, -\sqrt{2}] \cup [\sqrt{2}, 4] \\
\\
&E = (-\infty, -2) \\
&f^{-1}(E) = x \in \mathbb{R}|x^2 \in E = \emptyset
\end{aligned}
$$

Example

$$
\begin{aligned}
&\chi_S : A \to \{0, 1\}, \chi_S = \begin{cases} 1, & x \in S \\ 0, & x \in A \setminus S \end{cases} \\
&\chi_S^{-1}(\{0\}) = A \setminus S \\
&\chi_S^{-1}(\{1\}) = S
\end{aligned}
$$

Proposition Given $f : A \to B, C, C_1, C_2 \subseteq A, D, D_1, D_2 \subseteq B$,

a) $C \subseteq f^{-1}(f(C))$

b) $f(f^{-1}(D)) \subseteq D$

c) $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$

d) $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$

e) $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$

f) $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$

g) $f^{-1}(B \setminus D) = A \setminus f^{-1}(D)$

Proof

a) $f^{-1}(f(c)) \stackrel{\text{def}}{=} \{a \in A | f(a) \in f(C)\}$
This set includes C since, by definition, $f(c) \in f(C) \forall c \in C$.

b) $f(f^{-1}(D)) = \{f(a) | a \in f^{-1}(D)\} = \{f(a) | f(a) \in D\} \subseteq D$.

c) Suppose $b \in f(C_1 \cap C_2)$
$\implies \exists a \in C_1 \cap C_2 : b = f(a)$
$\implies b \in f(C_1) \wedge b \in f(C_2)$

d) Similar to c)

e) Tutorial this week

f) $a \in f^{-1}(D_1 \cap D_2) \iff f(a) \in D_1 \cap D_2 \iff a \in f^{-1}(D_1) \wedge a \in f^{-1}(D_2)$

g) $a \in A \setminus f^{-1}(D) \iff f(a) \notin D$
but $f : A \to B \implies f(a) \in B$ so $f(a) \in B \wedge f(a) \notin D \iff f(a) \in B \setminus D \iff a \in f^{-1}(B \setminus D)$.

Example (failure of equality for a) - c))

$$f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$$

Let $C_1 = [-1, 0], C_2 = [0, 1]$. $C_1 \cap C_2 = 0$, so $f(C_1 \cap C_2) = 0$, but $f(C_1) = f(C_2) = [0, 1] \implies f(C_1) \cap f(C_2) = [0, 1] \supsetneq 0$.
Let $C = \{1\}$ so $f(C) = 1$, but $f^{-1}(\{1\}) = \{-1, 1\}, soC \subsetneq f^{-1}(f(C))$.
Let $D = [-1, 1]$. Then $f^{-1}(D) = \{x \in \mathbb{R} | -1 \leq x^2 \leq 1\} = [-1, 1]$ and $f([-1, 1]) \subsetneq D$.

Proposition

(i) If $f : A \to B$ is injective, statements a) and c) become equalities.

(ii) If $f : A \to B$ is surjective, then statement b) is an equality.

Proof

(i) We already know that $C \subseteq f^{-1}(f(C))$. Then let $f(a) \in f(C)$. This means $\exists c \in C : f(a) = f(c)$. Because f is injective, $f(a) = f(c) \implies a = c$, so $a \in C$, so $f^{-1}(f(C)) \subseteq C$, so $C = f^{-1}(f(C))$.
Statement c) is similar.

(ii) We already know that $f(f^{-1}(D)) \subseteq D$. Let $d \in D$. Then $\exists a \in A : d = f(a)$, by surjectivity of f. Then $a \in f^{-1}(D) \implies f(a) \in f(f^{-1}(D))$. Therefore $D \subseteq f(f^{-1}(D)) \implies D = f(f^{-1}(D))$.

Definition Suppose $f : A \to B, g : B \to C$.
The **composition** $g \circ f : A \to C$ is the function

$$g \circ f(a) = g(f(a))$$

Remarks:

- Even if $g \circ f$ is defined, $f \circ g$ may not be.

- Even if both $g \circ f$ and $f \circ g$ are defined with $A = B = C$, they are not generally the same functions.

Proposition Suppose $f : A \to B, g : B \to C$.

a) If f and g are injective, then $g \circ f$ is injective.

b) If f and g are surjective, then $g \circ f$ is surjective.

c) If f and g are bijective, then $g \circ f$ is bijective.

Definition Suppose $f : A \to B$. f is **invertible** if and only if $\exists g : B \to A : g \circ f = id_A \wedge f \circ g = id_B$.

Proposition The function g above is unique when it exists.

Proof Suppose $g, g_1 : B \to A$ so $g \circ f = g_1 \circ f = id_A$ and $f \circ g = f \circ g_1 = id_B$. Let $b \in B$. Then

$$\begin{aligned} g(b) &= g(f \circ g_1(b)), \text{since} f \circ g_1 = id_B \\ &= g(f(g_1(b))) = (g \circ f)(g_b(b)) = g_1(b), \text{since} g \circ f = id_A \end{aligned}$$

So $g(b) = g_1(b) \forall b \in B \implies g = g_1$.

Definition If f is invertible, the unique function g above is called the **inverse** of f, and is denoted f^{-1}.

Proposition $f : A \to B$ is invertible if and only if f is a bijection.

Proof

(i) $f : A \to B$ is invertible $\implies$ f is a bijection.
Suppose f is invertible, and $f(a_1) = f(a_2)$. Then $f^{-1}(f(a_1)) = f^{-1}(f(a_2)) \implies a_1 = a_2$, so f is injective.
Suppose $b \in B$. Then $f(f^{-1}(b)) = b$, so $\forall b \in B, \exists a \in A : f(a) = b$, with $a = f^{-1}(b)$.

(ii) f is a bijection $\implies$ $f : A \to B$ is invertible.
Suppose f is a bijection. Then for $\overbrace{\text{every } b \in B}^{surjectivity}$ there is a $\overbrace{\text{unique } a \in A}^{injectivity}$ so that $f(a) = b$. Then for each $b \in B$, define $g : B \to A$ by $g(b) = a$. Then $g = f^{-1}$.
Check: g is well defined