# MATH 355 Notes

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## 1 Topic 1 - Functions and Cardinality (cont)

**Example** Invertible functions

(i) $f := mapABRR, f(x) = x^3$
The inverse is $f^{-1}(x) = \sqrt[3]{(x)}$.

(ii) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 3x + 2$
Inverse is hard to find.
$f'(x) = 3x^2 + 3 > 0$
So $f$ is strictly increasing $\implies$ $f$ is injective.
$\lim_{x\to\infty} f(x) = \infty, \lim_{x\to-\infty} f(x) = -\infty$
So $f$ is surjective.

(iii) $g : \mathbb{Z} \to \mathbb{N}$

$$g(n) = \begin{cases} \frac{n}{2}, & n \text{ even} \\ \frac{n+1}{2}, & n \text{ n odd} \end{cases}$$

**Example** If $f : A \to B, g : B \to C$ are invertible, then so is $g \circ f : A \to C$ and $(g \circ f) = f^{-1} \circ g^{-1}$

**Example** (Composition of invertibles)

$$f : \mathbb{R} \to (0, \infty) f(x) = e^x$$
$$f^{-1} : (0, \infty) \to \mathbb{R} f^{-1}(x) = \log(x)$$
$$g : (0, \infty) \to (1, \infty) g(x) = x^3 + 1$$
$$g^{-1} : (1, \infty) \to (0, \infty) g^{-1}(x) = \sqrt[3]{x - 1}$$
$$\mathbb{R} \xrightarrow{f} (0, \infty) \xrightarrow{g} (1, \infty)$$
$$g \circ f : \mathbb{R} \to (1, \infty)$$
$$g \circ f(x) = g(f(x)) = (e^x)^3 + 1 = e^{3x} + 1 (g \circ f)^{-1} = f^{-1} \circ g^{-1} : (1, \infty) \to \mathbb{R}$$
$$f^{-1}(g^{-1}(x)) = f^{-1}(\sqrt[3]{x - 1}) = \log(\sqrt[3]{x - 1})$$

## 1.1 Cardinality

**Definition** Two sets $A, B$ are **equinumerous** if $\exists$ a bijection $f : A \to B$. We write $A \sim B$ in this case.

**Proposition** " " is an equivalence relation on sets. i.e.

(i) $A \sim A$

(ii) $A \sim B \implies B \sim A$

(iii) $A \sim B \wedge B \sim C \implies A \sim C$

**Definition** Suppose $A$ is a set and $I_n = \{1, 2, \ldots, n\}, n \in \mathbb{N}$.

(i) $A$ is **finite** if $A \sim I_n$ for some $n \in \mathbb{N}$.

(ii) $A$ is **infinite** if it is not finite.

(iii) $A$ is **denumerable** if $A \sim \mathbb{N}$.

(iv) $A$ is **countable** if $A$ is either finite xor denumerable.

(v) $A$ is **uncountable** if $A$ is not countable.

**Example** We know $\mathbb{N} \sim \mathbb{Z}$, so $\mathbb{Z}$ is denumerable.
The function $f : \mathbb{N} \cup \{\nvdash\} \to \mathbb{N}, f(n) = n + 1$ is invertible. So $\mathbb{N} \cup \{0\} \sim \mathbb{N}$.
This implies that if $A$ is denumerable then $A \cup \{a\}$ is denumerable.
This extends to: If $A$ is denumerable and B finite, then $A \cup B$ is denumerable.

**Proposition** If $B$ is countable and $A \subseteq$, then $A$ is countable.

**Proof** Check that if $B$ is finite, then so is $A$.
Assume $B$ is denumerable and $A \subseteq B$.
Write $B = \{b_1, b_2, \dots\}$ (here the map $f(n) = b_n$ is the bijection $\mathbb{N} \to B$).
Since $A \subseteq B$, we can find increasing natural numbers

$$n_1 < n_2 < n_3 \dots$$

so that $A = \{b_{n_1}, b_{n_2}, \dots\}$.
If there are only finitely many $n_i$, $A$ is finite.
If not, the map $g : A \to \mathbb{N}, g(b_{n_i}) = i$ is a bijection. (check). $\square$

**Corollary** If $A$ is uncountable and $A \subseteq B$, then $B$ is uncountable.

**Proof** This is the contrapositive of the proposition above.

**Theorem** The following are equivalent.

(i) $A$ is countable.

(ii) $\exists$ an injection $f : A \to \mathbb{N}$.

(iii) $\exists$ an surjection $g : \mathbb{N} \to A$.

**Proof** We will show that (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i).

(i) $\implies$ (ii) Either $A \sim I_n$ or $A \sim \mathbb{N}$. If $A \sim \mathbb{N}$, we are done, as a bijection is an injection.
If $A \sim I_n$, write $A = \{a_1, a_2, \dots, a_n\}$.
The function $\hat{f} : A \to I_n, \hat{(f)}(a_j) = j$ is a bijection.
Now define $f : A \to \mathbb{N}$ by $f(a_i) = \hat{f}(a_i) = i$. $f$ is an injection since $\hat{f}$ is.

(ii) $\implies$ (ii) Suppose $f : A \to \mathbb{N}$ is an injection.
Then $f : A \to f(A)$ is surjective and hence bijective. So $f^{-1} : f(A) \to A$

is also a bijection, and $f^{-1}(f(a)) = a$.
Define $g : \mathbb{N} \to A$ by

$$g(n) = \begin{cases} f^{-1}(n), & \text{if } n \in f(A) \\ a, & \text{if } n \notin f(A) \end{cases}$$

where $a$ is some fixed arbitrary element in $A$.
$g$ is a surjection, since if $b \in A$ we have $g(f(b)) = f^{-1}(f(b)) = b$.

(iii) $\implies$ (i) Let $g : \mathbb{N} \to A$ be a surjection.
Define $h : A \to \mathbb{N}$ by $h(a) = \min\{n \in \mathbb{N} | g(n) = a\}$, which always has a solution by the well-ordering principle, since $g$ is surjective.
$h$ is injective since if $h(a) = h(b)$, then $n$ is the minimal natural number with $g(n) = a$ and $g(n) = b$, so $a = b$.
Therefore $h : A \to h(A)$ is a bijection, so $A \sim h(A) \subseteq \mathbb{N}$, so $A$ and $h(A)$ are subsets of countable sets and so are also countable.

**Proposition** Suppose $A, B, A_1, A_2, A_3, \dots$ are countable sets. Then

(i) $A \times B$ is countable.

(ii) $\cup_{n=1}\infty A_n$ is countable.

**Proof** We assume the extreme case that all of these sets are denumerable.

(i) $A \sim B \sim \mathbb{N}$. So we show $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$. From (ii) in the previous theorem, it is enough to find an injection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. $f : \mathbb{N} \times \mathbb{N} \sim \mathbb{N}, f(m, n) = 2^m 3^n$, which is injective by the fundamental theorem of algebra.

(ii) Write the elements of each $A_i \sim \mathbb{N}$ as lists:

$$A_1 = (a_{11}, a_{12}, a_{13}, \dots)$$

$$A_2 = (a_{21}, a_{22}, a_{23}, \dots)$$

$$A_3 = (a_{31}, a_{32}, a_{33}, \dots)$$

$\vdots$

Without loss of generality, we can assume $A_i \cap A_j = \emptyset \forall i \neq j$. Now define $f : \cup_{n=1}\infty A_n \to \mathbb{N}$ by diagonalization, which is a bijection.
Or $g : \cup_{n=1}\infty A_n \to \mathbb{N}$ by $g(a_{ij}) = 2^i 3^j$, whih is an injection.