MATH 355 Notes

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September 5 2019

1 Topic 1 - Functions and Cardinality

Definition Given sets A and B, the **Cartesian Product of** A **and** B is the set of ordered pairs

$$A \times B = \{(a, b)|a \in A, b \in B\}$$

Given any finite list of N sets $A_1, A_2, \dots, A_N$, the **Cartesian Product of** A_i is

$$A_1 \times A_2 \times \cdots \times A_N = \{(a_1, a_2, \dots, a_n)|a_i \in A_i; i = 1, 2, \dots, N\}$$

Example

$$\begin{aligned} \mathbb{R}^2 =& \mathbb{R} \times \mathbb{R} \\ \mathbb{R}^N =& \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{N} \end{aligned}$$

When $\mathbb{T}$ is the unit circle (i.e. $\mathbb{T} = \{(a, b)|a, b \in \mathbb{R}, a^2 + b^2 = 1\}$), $\mathbb{T}^2$ is the unit torus.

Definition A **function** f from A to B is a subset $f \subseteq A \times B$ with the additional property that for all $a \in A$, there is a single $b \in B$ so that $(a, b) \in f$. Then $b = f(a)$. Given a function $f : A \to B$, A is the **domain** of f (denoted $\mathrm{dom}(f)$), and B is the **codomain** of f (denoted $\mathrm{codom}(f)$). The **range** of f is the set $f(a)|a \in A$ (denoted $\mathrm{ran}(f)$).

Example

$$
\begin{aligned}
&f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 \\
&\mathrm{dom}(f) = \mathrm{codom}(f) = \mathbb{R} \\
&\mathrm{ran}(f) = [0, \infty) \\
\\
&g : \mathbb{R} \to \mathbb{R}, g(x) = x^3 \\
&\mathrm{dom}(g) = \mathrm{codom}(g) = \mathrm{ran}(g) = \mathbb{R} \\
\\
&h : [0, \pi] \to \mathbb{R}, h(x) = cos(x) \\
&\mathrm{dom}(h) = [0, \pi] \\
&\mathrm{codom}(h) = \mathbb{R} \\
&\mathrm{ran}(h) = [-1, 1]
\end{aligned}
$$

Definition Given $f : A \to B$,

(i) f is injective $\iff f(a_1) = f(a_2) \implies a_1 = a_2 \forall a_1, a_2 \in A$

(ii) f is surjective $\iff \forall b \in B, \exists a \in A : f(a) = b$

(iii) f is bijective $\iff f$ is injective and f is surjective

Example

$$k : \mathbb{Z}^+ \to \mathbb{Z}, k(n) = \begin{cases} \frac{n}{2}, & n \text{ even} \\ -\left(\frac{n-1}{2}\right), & n \text{ odd} \end{cases}$$

is a bijection.

Proposition k is injective.
Let $a, b \in \mathbb{Z}^+$. Prove injectivity by contradiction.