MATH 355 Notes

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September 10 2019

Definition Given $f : A \to B$ and $C \subseteq A, D \subseteq B$:

- The **image** of C under f is $f(C) = f(c)|c \in C$.

- The **preimage** of D under f is $f^{-1}(D) = a \in A|f(a) \in D$.

If f is invertible, the $f^{-1}(D)$ is the image of D under f^{-1}.

Example

$$\begin{aligned}
&f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 \\
&C = [-4, 1] \\
&f(C) = [0, 16] \\
\\
&D = [2, 16] \\
&f^{-1}(D) = [-4, -\sqrt{2}] \cup [\sqrt{2}, 4] \\
\\
&E = (-\infty, -2) \\
&f^{-1}(E) = x \in \mathbb{R}|x^2 \in E = \emptyset
\end{aligned}$$

Example

$$\begin{aligned}
&\chi_S : A \to \{0, 1\}, \chi_S = \begin{cases} 1, & x \in S \\ 0, & x \in A \setminus S \end{cases} \\
&\chi_S^{-1}(\{0\}) = A \setminus S \\
&\chi_S^{-1}(\{1\}) = S
\end{aligned}$$

Proposition Given $f : A \to B, C, C_1, C_2 \subseteq A, D, D_1, D_2 \subseteq B$,

a) $C \subseteq f^{-1}(f(C))$

b) $f(f^{-1}(D)) \subseteq D$

c) $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$

d) $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$

e) $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$

f) $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$

g) $f^{-1}(B \setminus D) = A \setminus f^{-1}(D)$

Proof

a) $f^{-1}(f(c)) \stackrel{\text{def}}{=} \{a \in A | f(a) \in f(C)\}$
This set includes C since, by definition, $f(c) \in f(C) \forall c \in C$.

b) $f(f^{-1}(D)) = \{f(a) | a \in f^{-1}(D)\} = \{f(a) | f(a) \in D\} \subseteq D$.

c) Suppose $b \in f(C_1 \cap C_2)$
$\implies \exists a \in C_1 \cap C_2 : b = f(a)$
$\implies b \in f(C_1) \wedge b \in f(C_2)$

d) Similar to c)

e) Tutorial this week

f) $a \in f^{-1}(D_1 \cap D_2) \iff f(a) \in D_1 \cap D_2 \iff a \in f^{-1}(D_1) \wedge a \in f^{-1}(D_2)$

g) $a \in A \setminus f^{-1}(D) \iff f(a) \notin D$
but $f : A \to B \implies f(a) \in B$ so $f(a) \in B \wedge f(a) \notin D \iff f(a) \in B \setminus D \iff a \in f^{-1}(B \setminus D)$.