

# MATH 355 Notes

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## 1 Topic 1 - Functions and Cardinality (cont)

**Definition** Given  $f : A \rightarrow B$  and  $C \subseteq A, D \subseteq B$  :

- The **image** of  $C$  under  $f$  is  $f(C) = \{f(c) | c \in C\}$ .
- The **preimage** of  $D$  under  $f$  is  $f^{-1}(D) = \{a \in A | f(a) \in D\}$ .

If  $f$  is invertible, the  $f^{-1}(D)$  is the image of  $D$  under  $f^{-1}$ .

**Example**

$$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$$

$$C = [-4, 1]$$

$$f(C) = [0, 16]$$

$$D = [2, 16]$$

$$f^{-1}(D) = [-4, -\sqrt{2}] \cup [\sqrt{2}, 4]$$

$$E = (-\infty, -2)$$

$$f^{-1}(E) = \{x \in \mathbb{R} | x^2 \in E\} = \emptyset$$

**Example**

$$\chi_S : A \rightarrow \{0, 1\}, \chi_S = \begin{cases} 1, & x \in S \\ 0, & x \in A \setminus S \end{cases}$$

$$\chi_S^{-1}(\{0\}) = A \setminus S$$

$$\chi_S^{-1}(\{1\}) = S$$

**Proposition** Given  $f : A \rightarrow B, C, C_1, C_2 \subseteq A, D, D_1, D_2 \subseteq B$ ,

- a)  $C \subseteq f^{-1}(f(C))$
- b)  $f(f^{-1}(D)) \subseteq D$
- c)  $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$
- d)  $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$
- e)  $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$
- f)  $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$
- g)  $f^{-1}(B \setminus D) = A \setminus f^{-1}(D)$

**Proof**

- a)  $f^{-1}(f(C)) \stackrel{\text{def}}{=} \{a \in A \mid f(a) \in f(C)\}$   
This set includes  $C$  since, by definition,  $f(c) \in f(C) \forall c \in C$ .
- b)  $f(f^{-1}(D)) = \{f(a) \mid a \in f^{-1}(D)\} = \{f(a) \mid f(a) \in D\} \subseteq D$ .
- c) Suppose  $b \in f(C_1 \cap C_2)$   
 $\implies \exists a \in C_1 \cap C_2 : b = f(a)$   
 $\implies b \in f(C_1) \wedge b \in f(C_2)$
- d) Similar to c)
- e) Tutorial this week
- f)  $a \in f^{-1}(D_1 \cap D_2) \iff f(a) \in D_1 \cap D_2 \iff a \in f^{-1}(D_1) \wedge a \in f^{-1}(D_2)$
- g)  $a \in A \setminus f^{-1}(D) \iff f(a) \notin D$   
but  $f : A \rightarrow B \implies f(a) \in B$  so  $f(a) \in B \wedge f(a) \notin D \iff f(a) \in B \setminus D \iff a \in f^{-1}(B \setminus D)$ .

**Example** (failure of equality for a) - c))

$$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$$

Let  $C_1 = [-1, 0], C_2 = [0, 1]$ .  $C_1 \cap C_2 = \emptyset$ , so  $f(C_1 \cap C_2) = \emptyset$ , but  $f(C_1) = f(C_2) = [0, 1] \implies f(C_1) \cap f(C_2) = [0, 1] \supsetneq \emptyset$ .

Let  $C = \{1\}$  so  $f(C) = \{1\}$ , but  $f^{-1}(\{1\}) = \{-1, 1\}$ , so  $C \subsetneq f^{-1}(f(C))$ .

Let  $D = [-1, 1]$ . Then  $f^{-1}(D) = \{x \in \mathbb{R} \mid -1 \leq x^2 \leq 1\} = [-1, 1]$  and  $f([-1, 1]) \subsetneq D$ .

**Proposition**

- (i) If  $f : A \rightarrow B$  is injective, statements a) and c) become equalities.
- (ii) If  $f : A \rightarrow B$  is surjective, then statement b) is an equality.

**Proof**

- (i) We already know that  $C \subseteq f^{-1}(f(C))$ . Then let  $f(a) \in f(C)$ . This means  $\exists c \in C : f(a) = f(c)$ . Because  $f$  is injective,  $f(a) = f(c) \implies a = c$ , so  $a \in C$ , so  $f^{-1}(f(C)) \subseteq C$ , so  $C = f^{-1}(f(C))$ .

Statement c) is similar.

- (ii) We already know that  $f(f^{-1}(D)) \subseteq D$ . Let  $d \in D$ . Then  $\exists a \in A : d = f(a)$ , by surjectivity of  $f$ . Then  $a \in f^{-1}(D) \implies f(a) \in f(f^{-1}(D))$ . Therefore  $D \subseteq f(f^{-1}(D)) \implies D = f(f^{-1}(D))$ .

**Definition** Suppose  $f : A \rightarrow B, g : B \rightarrow C$ .

The **composition**  $g \circ f : A \rightarrow C$  is the function

$$g \circ f(a) = g(f(a))$$

Remarks:

- Even if  $g \circ f$  is defined,  $f \circ g$  may not be.
- Even if both  $g \circ f$  and  $f \circ g$  are defined with  $A = B = C$ , they are not generally the same functions.

**Proposition** Suppose  $f : A \rightarrow B, g : B \rightarrow C$ .

- a) If  $f$  and  $g$  are injective, then  $g \circ f$  is injective.
- b) If  $f$  and  $g$  are surjective, then  $g \circ f$  is surjective.
- c) If  $f$  and  $g$  are bijective, then  $g \circ f$  is bijective.

**Definition** Suppose  $f : A \rightarrow B$ .  $f$  is **invertible** if and only if  $\exists g : B \rightarrow A : g \circ f = id_A \wedge f \circ g = id_B$ .

**Proposition** The function  $g$  above is unique when it exists.

**Proof** Suppose  $g, g_1 : B \rightarrow A$  so  $g \circ f = g_1 \circ f = id_A$  and  $f \circ g = f \circ g_1 = id_B$ . Let  $b \in B$ . Then

$$\begin{aligned} g(b) &= g(f \circ g_1(b)), \text{ since } f \circ g_1 = id_B \\ &= g(f(g_1(b))) = (g \circ f)(g_1(b)) = g_1(b), \text{ since } g \circ f = id_A \end{aligned}$$

So  $g(b) = g_1(b) \forall b \in B \implies g = g_1$ .

**Definition** If  $f$  is invertible, the unique function  $g$  above is called the **inverse** of  $f$ , and is denoted  $f^{-1}$ .

**Proposition**  $f : A \rightarrow B$  is invertible if and only if  $f$  is a bijection.

**Proof**

(i)  $f : A \rightarrow B$  is invertible  $\implies f$  is a bijection.  
 Suppose  $f$  is invertible, and  $f(a_1) = f(a_2)$ . Then  $f^{-1}(f(a_1)) = f^{-1}(f(a_2)) \implies a_1 = a_2$ , so  $f$  is injective.  
 Suppose  $b \in B$ . Then  $f(f^{-1}(b)) = b$ , so  $\forall b \in B, \exists a \in A : f(a) = b$ , with  $a = f^{-1}(b)$ .

(ii)  $f$  is a bijection  $\implies f : A \rightarrow B$  is invertible.

Suppose  $f$  is a bijection. Then for  $\overbrace{\text{every } b \in B}^{\text{surjectivity}}$  there is a  $\overbrace{\text{unique } a \in A}^{\text{injectivity}}$  so that  $f(a) = b$ . Then for each  $b \in B$ , define  $g : B \rightarrow A$  by  $g(b) = a$ . Then  $g = f^{-1}$ .

Check:  $g$  is well defined