MATH 355 Assignment 1

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Contents

Definitions

Injective

Given sets A and B, and a function $f : A \to B$,

$$f \text{ is injective} \iff \forall a_1, a_2 \in A, a_1 \neq a_2 \implies f(a_1) \neq (a_2)$$

Surjective

Given sets A and B, and a function $f : A \to B$,

$$f \text{ is surjective} \iff \forall b \in B, \exists a \in A : f(a) = b$$

Bijective

Given sets A and B, and a function $f : A \to B$,

$$f \text{ is bijective} \iff f \text{ is injective} \land f \text{ is surjective}$$

Continuous

Given intervals A, B on $\mathbb{R}$, and a function $f : A \to B$

Continuous at a point f is continuous at a point $c \in A$ if and only if

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x \in A, |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Continuous on an interval f is continuous on an interval $(a, b) \subseteq A$ if and only if

$$\forall c \in A, a < c < b \implies f \text{ is continuous at c.}$$

Continuous f is continuous if and only if

$$A = \mathbb{R} \wedge \forall c \in A, f \text{ is continuous at c.}$$

Question 1

Question Determine whether or not the following functions are injective, surjective, or bijective.
NOTE: The natural numbers $\mathbb{N}$ exclude 0.

Part a

Function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 + 1$

Injective? f is not injective.

Proposition Show that $\exists x_1, x_2 \in \mathbb{R} : x_1 \neq x_2 \wedge f(x_1) = f(x_2)$.

Proof Consider $x_1 = 1, x_2 = -1$. Then $f(x_1) = (1)^2 + 1 = 1 + 1 = 2$, and $f(x_2) = (-1)^2 + 1 = 1 + 1$. Therefore, $x_1 \neq x_2$, and $f(x_1) = f(x_2)$, as required. Therefore, f is not injective. $\square$

Surjective? f is not surjective.

Proposition Show that $\exists y \in \mathbb{R} : \forall x \in \mathbb{R}, f(x) \neq y$.

Proof Consider $y = 0$. Then suppose $x \in \mathbb{R}$ satisfies $f(x) = y$, so $x^2 + 1 = 0$. Solving this equation for x, we obtain the possible solutions $x = \pm\sqrt{-1} \notin \mathbb{R}$, which contradicts the supposition that $x \in \mathbb{R}$. Therefore, x does not exist, and so f is not surjective. $\square$

Bijective? f is not bijective.

Proof The above results show that f is neither injective nor surjective, and it must be both injective and surjective in order to be bijective. Therefore, f is not bijective. $\square$

Part b

Function $f : \mathbb{R} \to [0, \infty), f(x) = (x - 1)^2$

Injective? f is not injective.

Proposition Show that $\exists x_1, x_2 \in \mathbb{R} : x_1 \neq x_2 \wedge f(x_1) = f(x_2)$.

Proof Consider $x_1 = 0, x_2 = 2$. Then $f(x_1) = (0 - 1)^2 = (-1)^2 = 1$, and $f(x_2) = (2 - 1)^2 = (1)^2 = 1$. Therefore, $x_1 \neq x_2$, and $f(x_1) = f(x_2)$, as required. Therefore, f is not injective. $\square$

Surjective? f is surjective.

Proposition Show that $\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : f(x) = y$.

Proof Let $y \in [0, \infty)$. Then consider $x = \sqrt{y}+1$. Because $y \geq 0$, $\sqrt{y} \in \mathbb{R} \implies x \in \mathbb{R}$. Then

$$\begin{aligned} f(x) &= (x-1)^2 \\ &= (\sqrt{y}+1-1)^2 \\ &= (\sqrt{y})^2 \\ &= y \end{aligned}$$

which shows that f is surjective. $\square$

Bijective? f is not bijective.

Proof The above results show that f is not surjective, and it must be both injective and surjective in order to be bijective. Therefore, f is not bijective. $\square$

Part c

Function $f : \mathbb{R} \to [-\frac{1}{20}, \frac{1}{20}], f(x) = \sin 5x$

Injective? f is injective.

Proposition Show that $\forall x_1, x_2 \in [-\frac{1}{20}, \frac{1}{20}], x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

Proof Let $x_1, x_2 \in [-\frac{1}{20}, \frac{1}{20}]$. Then $5x_1, 5x_2 \in [-\frac{1}{4}, \frac{1}{4}]$, and sin is strictly increasing on $[-\frac{1}{4}, \frac{1}{4}]$, so $x_1 < x_2 \implies f(x_1) < f(x_2)$, and vice versa. Therefore, $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$, and therefore f is injective. $\square$

Surjective? f is not surjective.

Proposition Show that $\exists y \in [-1, 1] : \forall x \in [-\frac{1}{20}, \frac{1}{20}], f(x) \neq y$.

Proof Consider $y = 1$. The values of x for which $sin(5x) = 1$ are those values for which $5x = 2\pi n, n \in \mathbb{Z}$, or $x = \frac{2\pi n}{5}$. The solutions to this equation with the smallest magnitudes are $x = \pm\frac{2\pi}{5}$, both of which are outside of $[-\frac{1}{20}, \frac{1}{20}]$. Therefore, there does not exist $x \in [-\frac{1}{20}, \frac{1}{20}]$ so that $f(x) = 1$, and so f is not surjective. $\square$

Bijective? f is not bijective.

Proof f is not surjective, and so by definition it is not bijective.

Part d

Function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^5 + 3x^3 + 2x + 1$

Injective? f is injective.

Proposition Show that $\forall x_1, x_2 \in \mathbb{R}, x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

Proof Suppose $x_1, x_2 \in \mathbb{R}$. Then suppose that $x_1 \neq x_2$, but$f(x_1) = f(x_2)$. Then, as polynomials are continuous and differentiable everywhere, by Rolle's Theorem, there exists some $c \in (x_1, x_2)$ so that $\frac{df}{dx}(c) = 0$. However, taking the derivative of f, we obtain $\frac{df}{dx} = 5x^4 + 9x^2 + 2$, so $5c^4 + 9c^2 + 2 = 0$. This is impossible, because c^2 & c^4 are both non-negative (because c is real), and so the left side is always strictly positive. Therefore, f is injective.$\square$

Surjective? f is surjective.

Proposition Show that $\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : f(x) = y$.

Proof The $\lim_{x\to\infty} f(x) = \infty$, and the $\lim_{x\to-\infty} f(x) = -\infty$, as the fifth order term dominates. Therefore, because $f(x)$ is a polynomial, and thus continuous everywhere, by the Intermediate Value Theorem, for any $y \in \mathbb{R}, \exists x \in (-\infty, \infty)$ so that $f(x) = y$, and therefore f is surjective. $\square$

Bijective? f is bijective.

Proof The above results show that f is both injective and surjective, and is therefore bijective. $\square$

Part e

Function $f : \mathbb{Z} \to \mathbb{Z}, f(n) = n^2 - n$

Injective? f is not injective.

Proposition Show that $\exists n_1, n_2 \in \mathbb{Z} : n_1 \neq n_2 \wedge f(n_1) = f(n_2)$.

Proof Consider $n_1 = 1, n_2 = 0$. Then $n_1 \neq n_2$, but $f(n_1) = 1^2 - 1 = 0$ and $f(n_2) = 0^2 - 0 = 0$, so $f(n_1) = f(n_2)$. Therefore, f is not injective. $\square$

Surjective? f is not surjective.

Proposition Show that $\exists m \in \mathbb{Z} : \forall n \in \mathbb{Z}, f(n) \neq m$.

Proof Consider $m = 1$. Then suppose that for some integer n, $f(n) = n^2 - n = m$. This, however, implies that $n^2 - n = n(n-1) = 1$, but n and $n-1$ are consecutive integers. As such, either n or $n-1$ is divisible by 2, so their product must also be divisible by 2. As 1 is not divisible by 2, this produces a contradiction, so n cannot exist, and so f is not surjective. $\square$

Bijective? f is not bijective.

Proof The above results show that f is neither injective nor surjective, and is therefore not bijective. $\square$

Part f

Function $f : \mathbb{N} \to \mathbb{N} \cup \{0\}, f(n) = n^2 - n$

Injective? f is injective.

Proposition Show that $\forall n_1, n_2 \in \mathbb{N} : n_1 \neq n_2 \implies f(n_1) = f(n_2)$.

Proof Suppose $n_1, n_2 \in \mathbb{N}$ so that $n_1 \neq n_2$. Without loss of generality, suppose $n_1 < n_2$. Then let $k = n_2 - n_1 \implies k > 0$. Then $f(n_2) = f(n_1 + k) = (n_1 + k)^2 - (n_1 + k) = n_1^2 + 2kn_1 + k^2 - n_1 - k = f(n_1) + k(2n_1 - 1)$. In order for the equality $f(n_1) = f(n_2)$ to hold, $k(2n_1 - 1) = 0$, so either $n_1 = 1/2$, or $k = 0$. However, n_1 is an integer, so the first case cannot hold, and we have already shown that $k > 0$. Therefore, $f(n_1) \neq f(n_2)$, so f is injective. □

Surjective? f is not surjective.

Proposition Show that $\exists m \in \mathbb{N} \cup \{0\} : \forall n \in \mathbb{Z}, f(n) \neq m$.

Proof Consider $m = 1$. Then suppose that for some integer n, $f(n) = n^2 - n = m$. This, however, implies that $n^2 - n = n(n-1) = 1$, but n and $n - 1$ are consecutive integers. As such, either n or $n - 1$ is divisible by 2, so their product must also be divisible by 2. As 1 is not divisible by 2, this produces a contradiction, so n cannot exist, and so f is not surjective. □

Bijective? f is not bijective.

Proof The above results show that f is neither injective nor surjective, and is therefore not bijective. □

Part g

Function $f : \mathbb{Z} \to \mathbb{Z}, f(n) = n^3 - n$

Injective? f is not injective.

Proposition Show that $\exists n_1, n_2 \in \mathbb{Z} : n_1 \neq n_2 \wedge f(n_1) = f(n_2)$.

Proof Consider $n_1 = 1, n_2 = 0$. Then $n_1 \neq n_2$, but $f(n_1) = 1^3 - 1 = 0$ and $f(n_2) = 0^2 - 0 = 0$, so $f(n_1) = f(n_2)$. Therefore, f is not injective. □

Surjective? f is not surjective.

Proposition Show that $\exists m \in \mathbb{Z} : \forall n \in \mathbb{Z}, f(n) \neq m$.

Proof Consider $m = 1$. Then suppose that for some integer n, $f(n) = n^3 - n = m$. This, however, implies that $n^3 - n = n(n^2 - 1) = n(n-1)(n+1) = 1$, but n, $n - 1$, and $n + 1$ are consecutive integers. As such, at least one of them must be divisible by 2, so their product must also be divisible by 2. As 1 is not divisible by 2, this produces a contradiction, so n cannot exist, and so f is not surjective. □

Bijective? f is not bijective.

Proof The above results show that f is neither injective nor surjective, and is therefore not bijective. □

Part h

Function $f : \mathbb{N} \to \mathbb{N} \cup \{0\}, f(n) = n^3 - n$

Injective? f is injective.

Proposition Show that $\forall n_1, n_2 \in \mathbb{N} : n_1 \neq n_2 \implies f(n_1) = f(n_2)$.

Proof Suppose $n_1, n_2 \in \mathbb{N}$ so that $n_1 \neq n_2$. Without loss of generality, suppose $n_1 < n_2$. Then let $k = n_2 - n_1 \implies k > 0$. Then $f(n_2) = f(n_1 + k) = (n_1+k)^3-(n_1+k) = n_1^3+3kn_1^2+3k^2n_1+k^2-n_1-k = f(n_1)+(3kn_1^2+3k^2n_1-1)$. In order for the equality $f(n_1) = f(n_2)$ to hold, $k(2n_1 - 1) = 0$, so either $n_1 = 1/2$, or $k = 0$. However, n_1 is an integer, so the first case cannot hold, and we have already shown that $k > 0$. Therefore, $f(n_1) \neq f(n_2)$, so f is injective. $\square$

Surjective? f is not surjective.

Proposition Show that $\exists m \in \mathbb{N} \cup \{0\} : \forall n \in \mathbb{Z}, f(n) \neq m$.

Proof Consider $m = 1$. Then suppose that for some integer n, $f(n) = n^2-n = m$. This, however, implies that $n^2 - n = n(n - 1) = 1$, but n and $n - 1$ are consecutive integers. As such, either n or $n - 1$ is divisible by 2, so their product must also be divisible by 2. As 1 is not divisible by 2, this produces a contradiction, so n cannot exist, and so f is not surjective. $\square$

Bijective? f is not bijective.

Proof The above results show that f is neither injective nor surjective, and is therefore not bijective. $\square$

Question 2

Question Suppose that f, g, h are functions from $\mathbb{R}$ to $\mathbb{R}$.

Part a

Question Show that there does not exist f, g satisfying $f(x) + g(y) = xy$ for all $x, y \in \mathbb{R}$.

Proposition $\forall f, g : \mathbb{R} \to \mathbb{R}, \exists x, y \in \mathbb{R} : f(x) + g(y) \neq xy$.

Proof Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are functions that do satisfy $f(x)+g(y) = xy$ for all $x, y \in \mathbb{R}$. Then the following four equations must hold:

$$f(1) + g(1) = 1 * 1 = 1 \tag{1}$$
$$f(1) + g(2) = 1 * 2 = 2 \tag{2}$$
$$f(2) + g(1) = 2 * 1 = 2 \tag{3}$$
$$f(2) + g(2) = 2 * 2 = 4 \tag{4}$$

However, combining (1) and (2) gives that $g(2) = g(1) + 1$, and similarly combining (1) and (3) gives that $f(2) = f(1) + 1$. These two facts show that

$$\begin{aligned} f(2) + g(2) =& (f(1) + 1) + (g(1) + 1) \\ =& f(1) + g(1) + 2, \text{ which by equation (1)} \\ =& 1 + 2 = 3 \end{aligned}$$

which contradicts equation (4) above. Therefore f, g cannot exist, and the proposition is true. □

Part b

Question Show that there does not exist f, g satisfying $f(x)g(y) = x + y$ for all $x, y \in \mathbb{R}$.

Proposition $\forall f, g : \mathbb{R} \to \mathbb{R}, \exists x, y \in \mathbb{R} : f(x)g(y) \neq x + y$.

Proof Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are functions that do satisfy $f(x)g(y) = x+y$ for all $x, y \in \mathbb{R}$. Then the following four equations must hold:

$$f(1)g(1) = 1 + 1 = 2 \tag{5}$$
$$f(1)g(2) = 1 + 2 = 3 \tag{6}$$
$$f(2)g(1) = 2 + 1 = 3 \tag{7}$$
$$f(2)g(2) = 2 + 2 = 4 \tag{8}$$

However, combining (5) and (6) gives that $g(2) = \frac{3}{2}g(1)$, and similarly combining (5) and (7) gives that $f(2) = \frac{3}{2}f(1)$. These two facts show that

$$\begin{aligned} f(2)g(2) =& \left(\frac{3}{2}f(1)\right)\left(\frac{3}{2}g(1)\right) \\ =& \frac{9}{4}f(1)g(1), \text{ which by equation (5)} \\ =& \frac{9}{4}(2) = \frac{9}{2} \end{aligned}$$

which contradicts equation (8) above. Therefore f, g cannot exist, and the proposition is true. □

Part c

Question Show that there does not exist three functions f, g, h which satisfy

$$f(x) + g(y) + h(z) = xyz$$

for all $x, y, z \in \mathbb{R}$.

Proposition $\forall f, g, h : \mathbb{R} \to \mathbb{R}, \exists x, y, z \in \mathbb{R} : f(x) + g(y) + h(z) \neq xyz$

Proof Suppose that f, g, h are functions that satisfy the above equation for all real numbers x, y, z. Then consider $z = 0$. In this case

$$\begin{aligned} f(x) + g(y) + h(0) =& 0 \\ f(x) + g(y) =& -h(0), \forall x, y \in \mathbb{R} \end{aligned}$$

where -h(0) is some constant real number. Then consider $x = (-h(0) + h(1) + 1), y = 1, z = 1$. In this case, $x, y, z \in \mathbb{R}$, and in order for the equation to hold

$$\begin{aligned} f(-h(0) - h(1) + 1) + g(1) + h(1) =& (-h(0) + h(1) + 1)(1)(1) \\ =& -h(0) + h(1) + 1 f(-h(0) + h(1) + 1) + g(1) = \quad -h(0) + 1 \end{aligned}$$

which is a contradiction, as we have already seen that for any real numbers x, y, $f(x) + g(y) = -h(0)$. Therefore, f, g, h cannot exist, and the proposition is true. □

Question 3

Question Show that the infinite product

$$\prod_{i=1}^{\infty}\{0,1\}$$

is uncountable (you can think of this as the set of infinite strings of 0s and 1s).

Proposition There does not exist a surjection from $\mathbb{N}$ to $\displaystyle\prod_{i=1}^{\infty}\{0,1\}$.

Proof This proof will use a modified version of Cantor's Diagonalization argument. Let S represent the set in question. Suppose that the described set is countable, so there exists a surjection $f : \mathbb{N} \to S$. We will now construct an item $X \in S$, so that there does not exist $n \in \mathbb{N}$ such that $f(n) = X$. Regarding S as the set of infinite strings of 0s and 1s, for all $n \in \mathbb{N}$, let the nth character of X be the opposite of the nth character of $f(n)$ (i.e. if the nth character of $f(n)$ is 0, the nth character of X is 1, and vice versa). Then, for every $n \in \mathbb{N}, X \neq f(n)$, because it differs in at least one position. Therefore, the surjection f cannot exist, and so S is uncountable.

Question 4

Question Suppose x_1, x_2, y_1, y_2 are real numbers. Show that

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}.$$

Fully describe the set of points for which the above inequality is an equality.

Answer Consider the vectors $\mathbf{x} =< x_1, x_2 >, \mathbf{y} =< y_1, y_2 >$. The dot product of these two vectors is $x_1y_1 + x_2y_2$, which is the left hand side of the inequality above. Then consider $||\mathbf{x}||||\mathbf{y}|| = \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$, which is the right hand side of the inequality above. Therefore, the proposition above is equivalent to

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^2, \mathbf{x} \cdot \mathbf{y} \leq ||\mathbf{x}||||\mathbf{y}||,$$

which is easy to show using the fact that $\mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}||||\mathbf{y}|| \cos(\theta)$, where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$. As $\cos(\theta) \in [-1, 1]$ for $\theta \in [0, 2\pi)$, the inequality is always true.
The inequality holds as an equality when $\cos(\theta) = 1$, which has one solution in $[0, 2\pi)$, namely $\theta = 0$. Therefore, the inequality is an equality if and only if the vectors $\mathbf{x}, \mathbf{y}$ are collinear, so when $y_1 = kx_1$ and $x_2 = kx_2$ for some real number k.

Question 5

Part a

Question Suppose $x < y$ are real numbers. Show that there are infinitely many distinct rational numbers q such that $x < q < y$.

Answer Let n be the smallest integer so that $1 < 10^n(y - x) \leq 10$. Then

$$a = \frac{\operatorname{sgn}(x)\lfloor 10^n|x|\rfloor + 1}{10^n}$$

is a rational number so that $x < a < y$. Then let m be the smallest integer so that $\frac{1}{m} < y - a$. Then, for all integers $i \geq m, x < a + \frac{1}{i} < y$, so there are infinitely many rational numbers between x and y.$\square$

Part b

Question Suppose $x < y$ are real numbers. Show that there are infinitely many distinct irrational numbers w such that $x < w < y$. Are there uncountable many?

Answer Let n be the smallest integer so that $1 < 10^n(y - x) \leq 10$. Then

$$a = \frac{\operatorname{sgn}(x)\lfloor 10^n|x|\rfloor + 1}{10^n}$$

is a rational number so that $x < a < y$. Then let m be the smallest integer so that $\frac{1}{m} < y - a$, and let S the set of all irrational numbers between 0 and 1. Note that S is uncountable. Now, for each element $k \in S, x < a + \frac{k}{m} < y$, and because a is rational and m is an integer, $a + \frac{k}{m}$ is irrational. Therefore, there are uncountably many irrational numbers between x and y.

Question 6

Part a

Question Find an explicit injection $f : \mathbb{Q} \to \mathbb{Z}$.

Answer Given $\frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0$, let $f(\frac{p}{q}) = 2^p 3^q$.

Part b

Question Find an explicit injection $g : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Z}$.

Answer Given $(\frac{n}{m}, \frac{p}{q}), n, m, p, q \in \mathbb{Z}, m, q \neq 0$, let $g(\frac{n}{m}, \frac{p}{q}) = 2^n 3^m 5^p 7^q$.

Part c

Question Find an explicit injection $h : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$.

Answer Given $(a, b, c), a, b, c \in \mathbb{Z}$, let $h(a, b, c) = (2^a 3^b 5^c, 2^a 3^b 5^c)$.

Question 7

Question Suppose A and B are both bounded subsets of $\mathbb{R}$. Find a property "X" so that the statement $\sup A = \inf B$ if and only if "X" holds is true.

Answer $|\mathrm{Conv}(A \cup B) - A - B| = 1$.