Bayesian Inference in Single Parameter Models

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Inference in a Bernoulli Distribution

> *. . . Polish mathematicians Tomasz Gliszczynski and Waclaw Zawadowski. . . spun a Belgian one euro coin 250 times, and found it landed heads up 140 times . . . When tossed 250 times, the one euro coin came up heads 139 times and tails 111. . . .*
>
> *The Guardian*, January 4, 2002[1]

- A sample of $n = 250$ coin tosses can be modelled as n independent and identically dsibution Bernoulli random variables with parameter θ.
- In other words, our probabilitic model is

$$x_i \sim \text{Bernoulli}(\theta), \quad \text{for } i \in \{1, 2 \ldots n\}.$$

and we would like to inference the probable values of θ given an observation of $m = 139$ (or $m = 140$, etc.).

[1] See `http://bit.ly/1BOKu9b` for original story and `http://bit.ly/1BOKx4Q` for discussion.

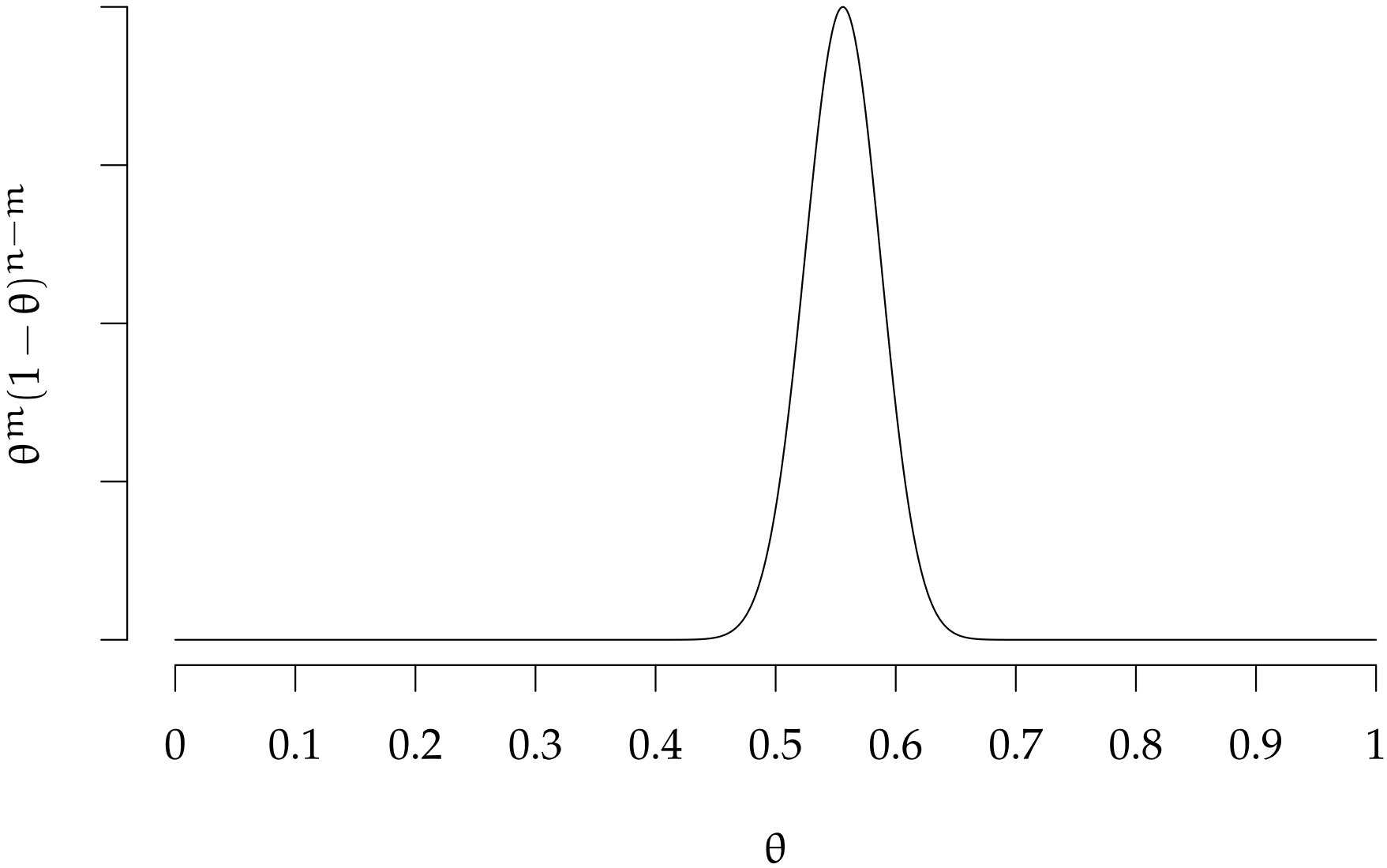
Inference in a Bernoulli Distribution

- The likelihood of the observed outcomes of the n coins is

$$\begin{aligned} P(x_1, x_2 \cdots x_n|\theta) &= \prod_{i=1}^{n} P(x_i|\theta), \\ &= \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}, \\ &= \theta^m(1-\theta)^{n-m}. \end{aligned}$$

- This is identical to a binomial distribution likelihood function.

Inference in a Bernoulli Distribution



The likelihood function for $n = 250$ and $m = 139$.

Inference in a Bernoulli Distribution

- The probabilistic generative model of the Euro coin toss data is as follows:
  - The coin's bias corresponds to the fixed but unknown value of the parameter θ of a Bernoulli random variable.
  - The observed outcomes $x_1, x_2 \cdots x_n$ are n iid samples from Bernoulli(θ).
- This generative model can be extended by assuming that θ is itself drawn from a prior distribution $P(\theta)$:

$$\begin{aligned} \theta &\sim P(\theta), \\ x_i &\sim \text{Bernoulli}(\theta), \quad \text{for } i \in \{1, 2 \ldots n\}. \end{aligned}$$

- In other words, we assume that a value for θ was randomly drawn from $P(\theta)$ and then n binary variables were sampled from Bernoulli(θ).

Conjugate Priors

- For a given likelihood function, a *conjugate prior* distribution is a prior probability distribution that leads to a posterior distribution of the same parameteric family.
- Using conjugate priors allows Bayesian inference and other probabilistic calculations to be performed analytically.
- Only a small subset of probabilistic models have conjugate priors.
- However, conjugate priors play a vital role in Monte Carlo methods like Gibbs sampling even in complex models.

The beta distribution

- For the binomial likelihood function

$$\theta^m(1-\theta)^{n-m}$$

a conjugate prior is the beta distribution

$$\text{Beta}(\theta|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}.$$

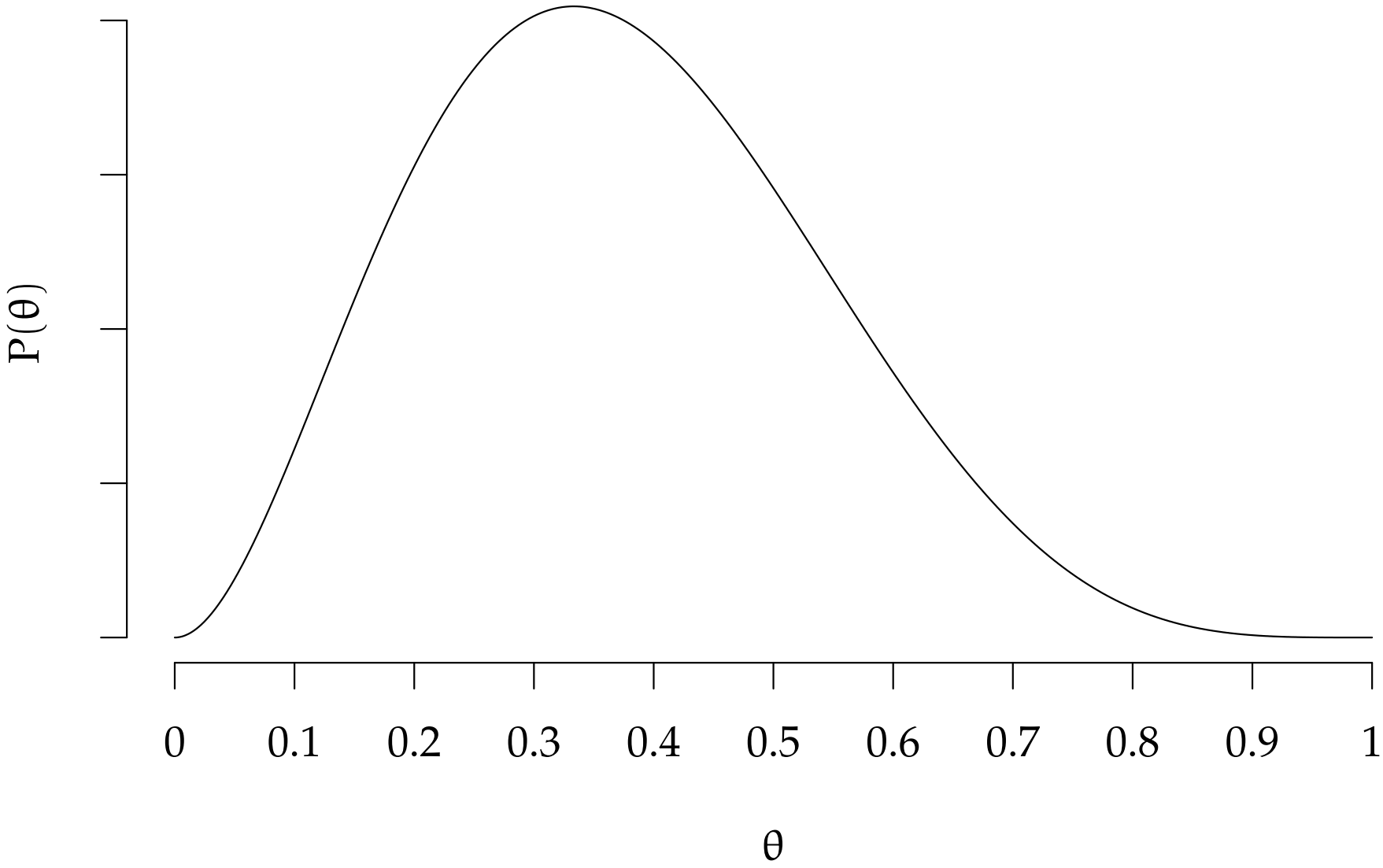
- The *normalizing constant* term is

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{1}{B(\alpha,\beta)},$$

where $B(\alpha,\beta)$ is the beta function:

$$B(\alpha,\beta) = \int \theta^{\alpha}(1-\theta)^{\beta-1}\,d\theta.$$

The beta distribution



The beta distribution with $\alpha = 3$ and $\beta = 5$.

Posterior distribution

- Denoting the observed data by $D = (n, m)$, with the beta prior, the posterior distribution is

$$
\begin{aligned}
P(\theta|D, \alpha, \beta) &= \frac{P(D|\theta)P(\theta|\alpha, \beta)}{\int P(D|\theta)P(\theta|\alpha, \beta)\, d\theta}, \\
&\propto \overbrace{\theta^m(1-\theta)^{n-m}}^{\text{likelihood}} \times \overbrace{\theta^{\alpha-1}(1-\theta)^{\beta-1}}^{\text{prior}}, \\
&\propto \theta^{m+\alpha-1}(1-\theta)^{n-m+\beta-1}, \\
&= \text{Beta}(m+\alpha, n-m+\beta).
\end{aligned}
$$

where the normalizing constant is the reciprocal of the beta function

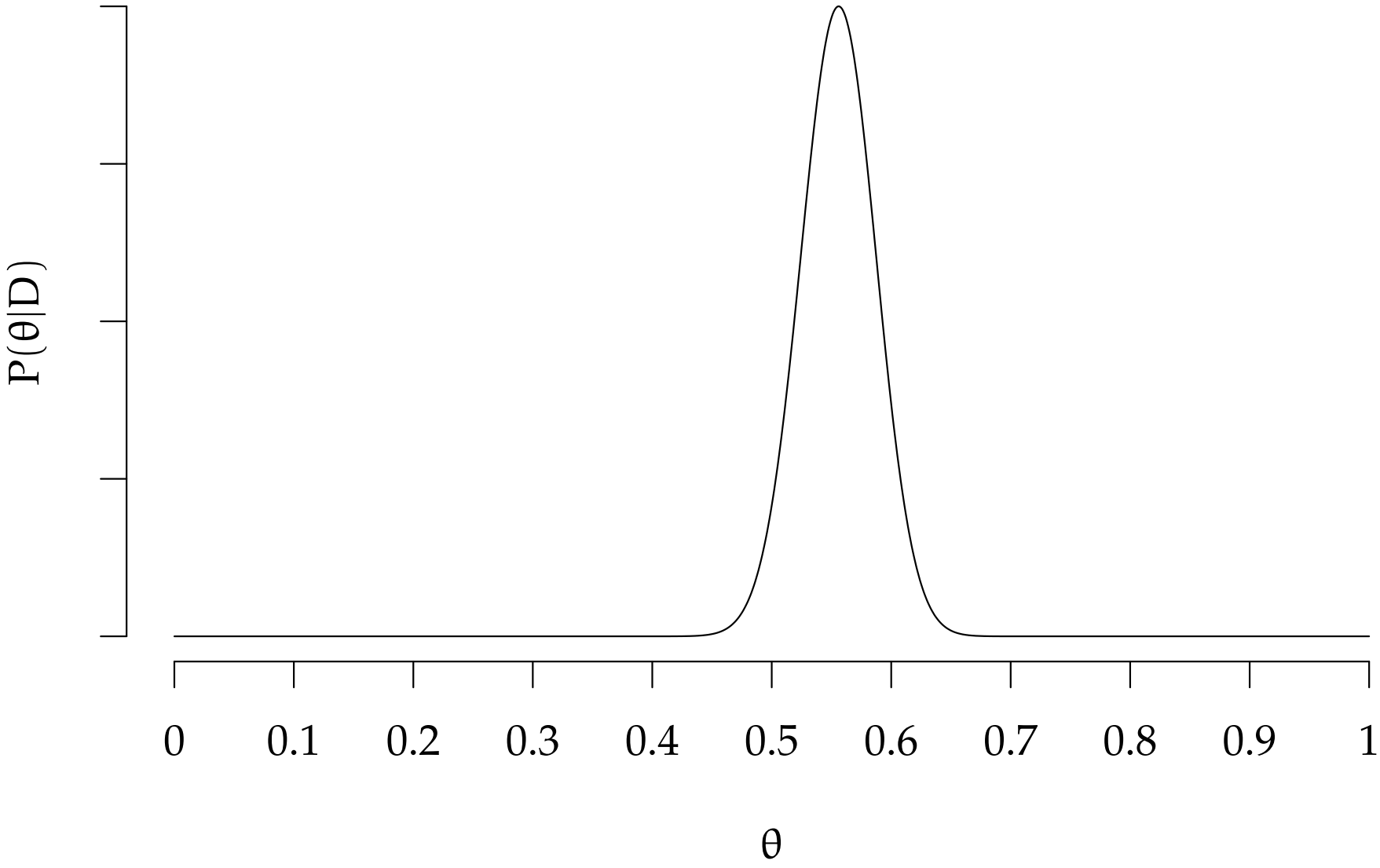
$$
\begin{aligned}
\frac{\Gamma(m+\alpha)\Gamma(n-m+\beta)}{\Gamma(n+\alpha+\beta)} &= \int \theta^{\alpha+m-1}(1-\theta)^{\beta+n-m-1}\, d\theta. \\
&= B(\alpha+m, \beta+n-m).
\end{aligned}
$$

Posterior distribution

- For our Euro coin example, our observed data are $n = 250$ and $m = 139$.
- A noninformative uniform prior on θ is $\text{Beta}(\alpha = 1, \beta = 1)$.
- With this prior, the posterior distribution is

$$\begin{aligned} \text{Beta}(m + \alpha, n - m + \beta) &= \text{Beta}(139 + 1, 250 - 139 + 1), \\ &= \text{Beta}(140, 112) \end{aligned}$$

Posterior distribution



The posterior distribution when $n = 250$, $m = 139$, $\alpha = 1$ and $\beta = 1$.

Summarizing the posterior distribution

- The mean, variance and modes of any beta distribution are as follows:

$$\langle \theta \rangle = \frac{\alpha}{\alpha + \beta},$$
$$V(\theta) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)},$$
$$\text{mode}(\theta) = \frac{\alpha - 1}{\alpha + \beta - 2}.$$

- Thus, in our case of Beta$(140, 112)$, we have

$$\langle \theta \rangle = 0.5556,$$
$$V(\theta) = 0.001, \quad \text{sd}(\theta) = 0.0312,$$
$$\text{mode}(\theta) = 0.556.$$

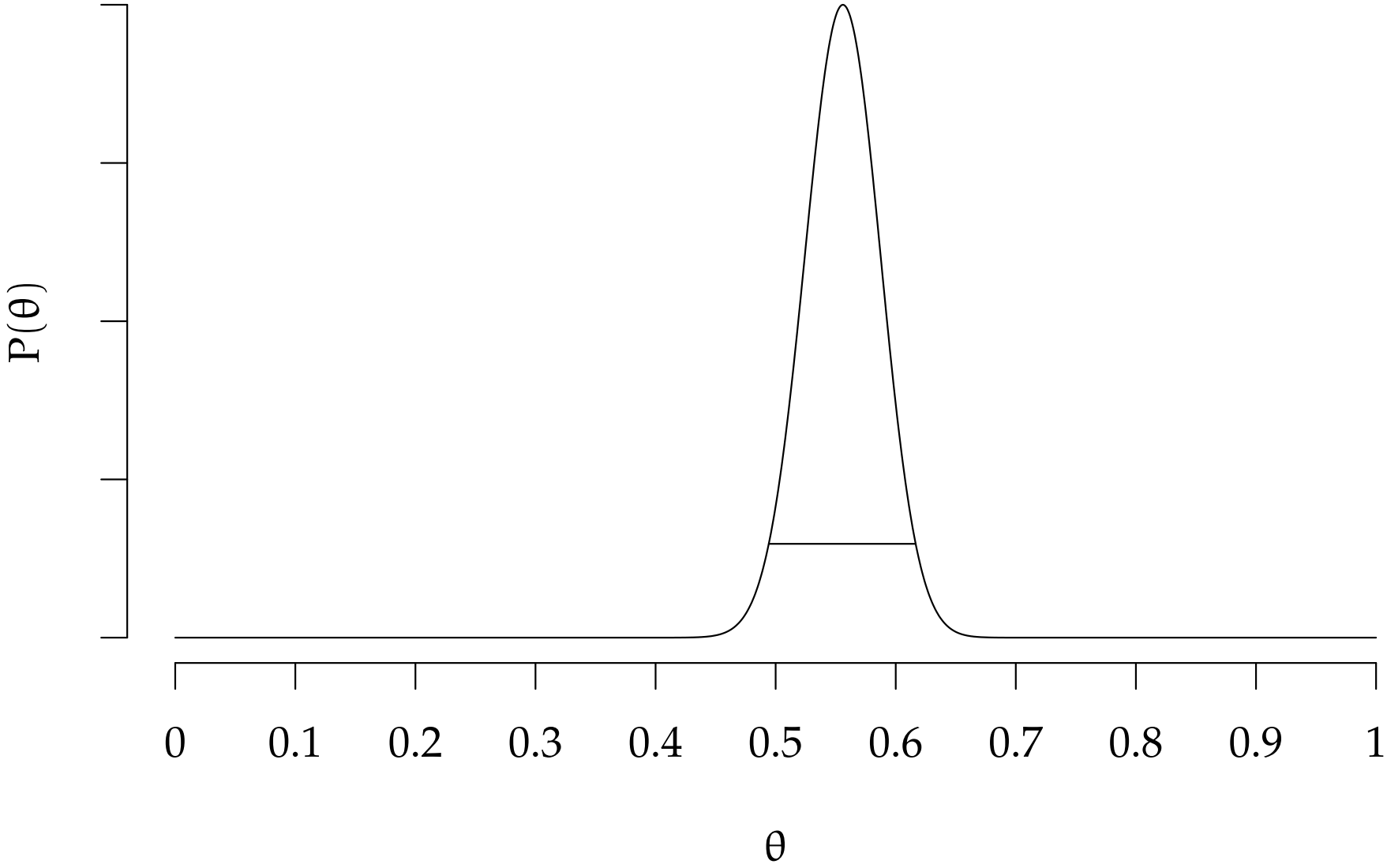
High posterior density (HPD) intervals

- HPD intervals provide ranges that contain specified probability mass. For example, the 0.95 HPD interval is the range of values that contain 0.95 of the probability mass of the distribution.
- The φ HPD interval for the probability density function $P(x)$ is computed by finding a probability density value p^* such that

$$P(\{x\colon P(x) \geqslant p^*\}) = \varphi.$$

- In other words, we find the value p^* such that the probability mass of the set of points whose density is greater than than p^* is exactly φ.
- In general, the HPD is not trivial to compute but in the case of symmetric distributions, it can be easily computed from the cumulative density function.

The 0.95 HPD *interval*



The posterior distribution, with its 0.95 HPD, when $n = 250$, $m = 139$, $\alpha = 1$ and $\beta = 1$. In this case, the HPD interval is $(0.494, 0.617)$.

Posterior predictive distribution

- Given that we have observed m heads in n coin tosses, what is the probability that the *next* coin toss is heads.
- This is given by the *posterior predictive* probability that $x = 1$:

$$\begin{aligned} P(x = 1|D, \alpha, \beta) &= \int P(x = 1|\theta) \overbrace{P(\theta|D, \alpha, \beta)}^{\text{Posterior}} \, d\theta, \\ &= \int \theta \times P(\theta|D, \alpha, \beta) \, d\theta, \\ &= \langle \theta \rangle, \\ &= \frac{\alpha + m}{\alpha + \beta + n}. \end{aligned}$$

- Thus, given 139 heads in 250 tosses, the predicted probability that the next coin will also be heads is ≈ 0.5556.

Marginal likelihood

- The posterior distribution is

$$P(\theta|D,\alpha,\beta) = \frac{\overbrace{P(D|\theta)}^{\text{Likelihood}}\ \overbrace{P(\theta|\alpha,\beta)}^{\text{Prior}}}{\underbrace{\int P(D|\theta)P(\theta|\alpha,\beta)\ d\theta}_{\text{Marginal likelihood}}}.$$

  where the *marginal likelihood* gives the likelihood of the model given the observed data:

$$\int P(D|\theta)P(\theta|\alpha,\beta)\ d\theta \stackrel{\text{def}}{=} P(D|\alpha,\beta).$$

- In this example, it has a simple analytical form:

$$P(D|\alpha,\beta) = B(\alpha+m,\beta+n-m) = \frac{\Gamma(m+\alpha)\Gamma(n-m+\beta)}{\Gamma(n+\alpha+\beta)}.$$

Model comparison

- Given D, we can compare the probability of model M_1 relative to model M_0 as follows:

$$\frac{P(M_1|D)}{P(M_0|D)} = \underbrace{\frac{P(D|M_1)}{P(D|M_0)}}_{\text{Bayes factor}} \times \underbrace{\frac{P(M_1)}{P(M_0)}}_{\text{Priors odds}} .$$

- When both models are equally probable a priori, then the relative posterior probabilities is determined by the Bayes factor.
- We can compare our model M_1, i.e. with $\alpha = \beta = 1$, with the M_0 model that $\theta = \frac{1}{2}$.

$$\frac{P(D|M_1)}{P(D|M_0)} = \frac{\int P(D|\theta)P(\theta|\alpha = 1, \beta = 1)\, d\theta}{\int P(D|\theta)\delta(\theta - \frac{1}{2})\, d\theta.}$$

- This effectively compares a model that assumes a completely random coin machine, i.e., coin biases are generated according to Beta$(\alpha = 1, \beta = 1)$, to a completely perfect coin machine, i.e., coin biases are generated according to $\delta(\theta - \frac{1}{2})$.

Model comparison

- We can compare our model M_1, i.e. with $\alpha = \beta = 1$, with the M_0 model that $\theta = \frac{1}{2}$.

$$\begin{aligned}\frac{P(D|M_1)}{P(D|M_0)} &= \frac{\int P(D|\theta)P(\theta|\alpha = 1, \beta = 1)\, d\theta}{\int P(D|\theta)\delta(\theta - \frac{1}{2})\, d\theta.}, \\ &= \frac{\Gamma(\alpha + m)\Gamma(\beta + n - m)}{\Gamma(\alpha + \beta + n)} \Big/ \tfrac{1}{2}^m (1 - \tfrac{1}{2})^{n-m}, \\ &= \frac{m!(n-m)!}{(n+1)!} \Big/ \tfrac{1}{2^n}.\end{aligned}$$

  If $n = 250$, $m = 139$, then

$$= \frac{139!111!}{251!} \Big/ \tfrac{1}{2^{250}} = 0.38.$$

- This is a factor of 2.65 in favour of the unbiased coin hypothesis.
- Note that the classical statistics null hypothesis test gives a p-value of $p = 0.0875$.

If n is a positive integer $\Gamma(n) = (n-1)!$.

Inference of the Upper Bound of a Uniform Distribution

- Given a sample of K values from a uniform distribution, bounded between 0 and λ, i.e.

$$x_1, x_2 \ldots x_K \sim \text{Uniform}(0, \lambda),$$

  what are the probable values of λ?
- This is related to the *German tank problem* whereby the total number of German tanks being produced during World War II was estimated from the serial numbers of captured tanks.
- In that problem, the tank serial numbers could be treated like random samples (without replacement) from a uniform distribution over the integers from 0 to T, where T is the total number of tanks.

Likelihood function of the Upper Bound

- The likelihood function over λ conditional on observations $x_1, x_2 \dots x_K$ is

$$P(x_1, x_2 \dots x_K|\lambda) = \prod_{k=1}^{K} P(x_k|\lambda)$$

where

$$P(x_k|\lambda) = \begin{cases} \frac{1}{\lambda}, & \text{for } \lambda \geqslant x_k, \\ 0, & \text{otherwise.} \end{cases}$$

- Therefore,

$$P(x_1, x_2 \dots x_K|\lambda) = \begin{cases} \frac{1}{\lambda}, & \text{for } \lambda \geqslant \max(x_{1:K}), \\ 0, & \text{otherwise.} \end{cases}$$

Posterior of the Upper Bound

- Assuming a uniform (improper) prior on λ, the posterior is obtained by normalizing the likelihood function.
- This leads to a posterior distribution

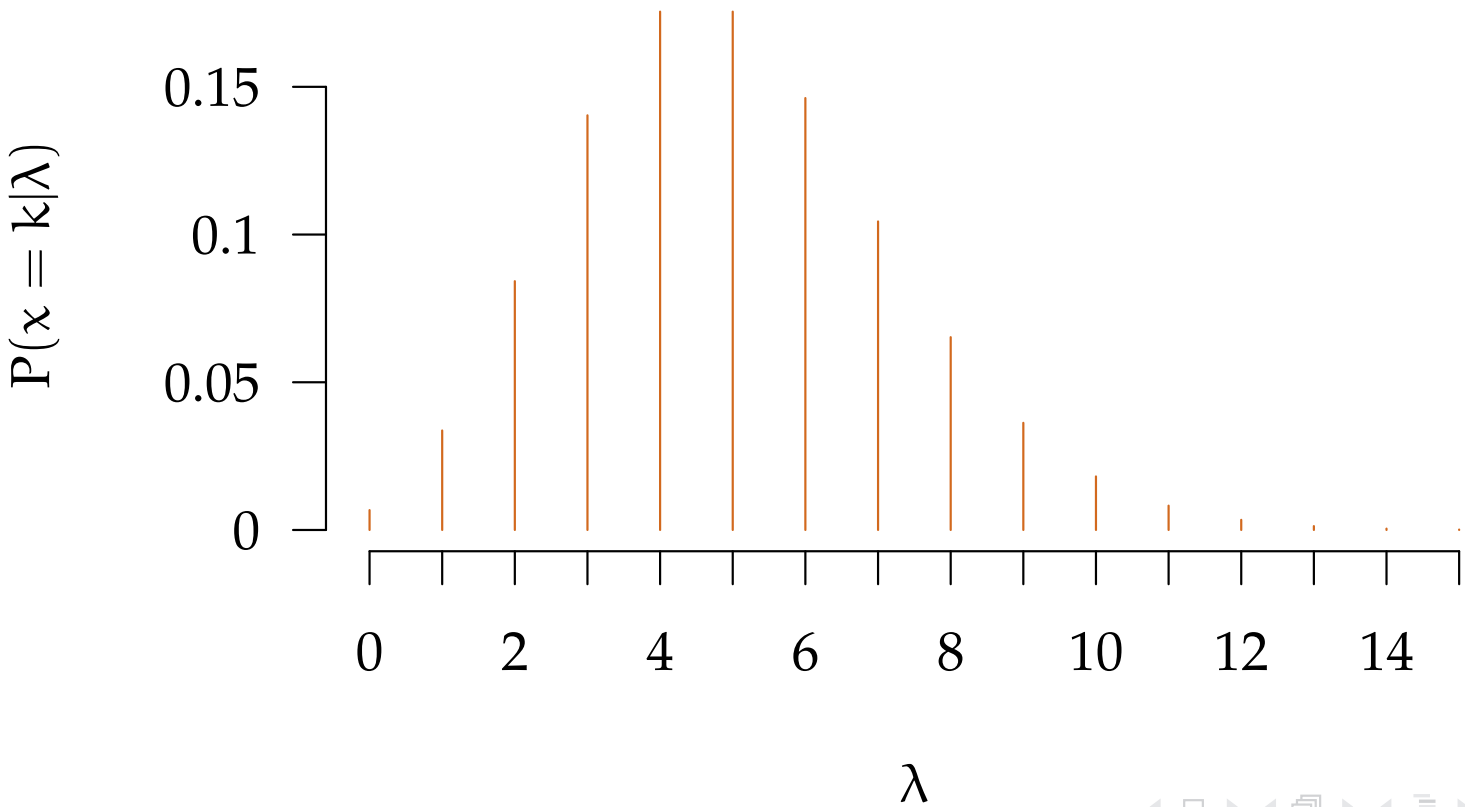
$$P(\lambda|x_1, x_2 \dots x_K) = \frac{\alpha x_0^\alpha}{\lambda^{\alpha+1}},$$

which is a *Pareto* distribution with shape parameter $\alpha = K - 1$ and scale parameter $x_0 = \max(x_{1:K})$.

Inference in Poisson models

- The Poisson distribution can be used to model the rare occurrences in fixed intervals.
- If x is a Poisson random variable with rate λ, then

$$P(x = k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}.$$



Inference in Poisson models

- Let's say that the number of emails I get per hour, in $n = 10$ hours, is
$$D = 6, 8, 12, 7, 11, 14, 10, 12, 11, 16.$$
- We can assume that these frequencies are generated according to a Poisson process with rate λ per hour.
- Given this assumption and this observed data, what is the probable value of λ?
- In other words, what is
$$P(\lambda|D)?$$

Likelihood of a Poisson model

- Given a known value for λ, the probability of $x_1, x_2 \cdots x_n$ is

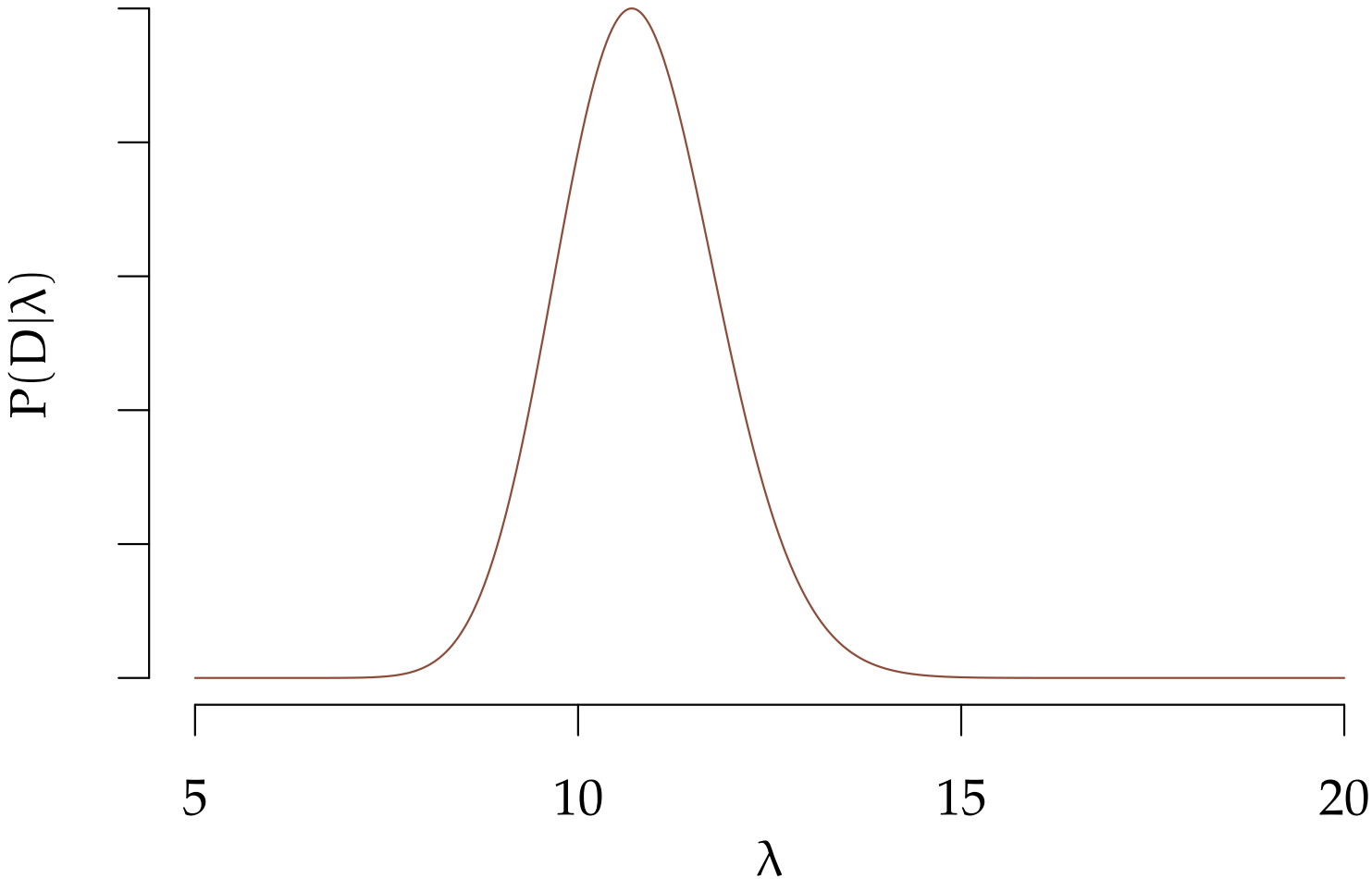
$$\begin{aligned} P(x_1, x_2 \cdots x_n|\lambda) &= \prod_{i=1}^{n} P(x_i|\lambda), \\ &= \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}, \\ &\propto e^{-n\lambda} \prod_{i=1}^{n} \lambda^{x_i} = e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}. \end{aligned}$$

When $D = x_1, x_2 \cdots x_n = 6, 8 \cdots 16$, the likelihood of λ is

$$P(D|\lambda) = e^{-n\lambda} \lambda^{S},$$

where $S = \sum_{i=1}^{n} x_i = 107$.

Likelihood of a Poisson model

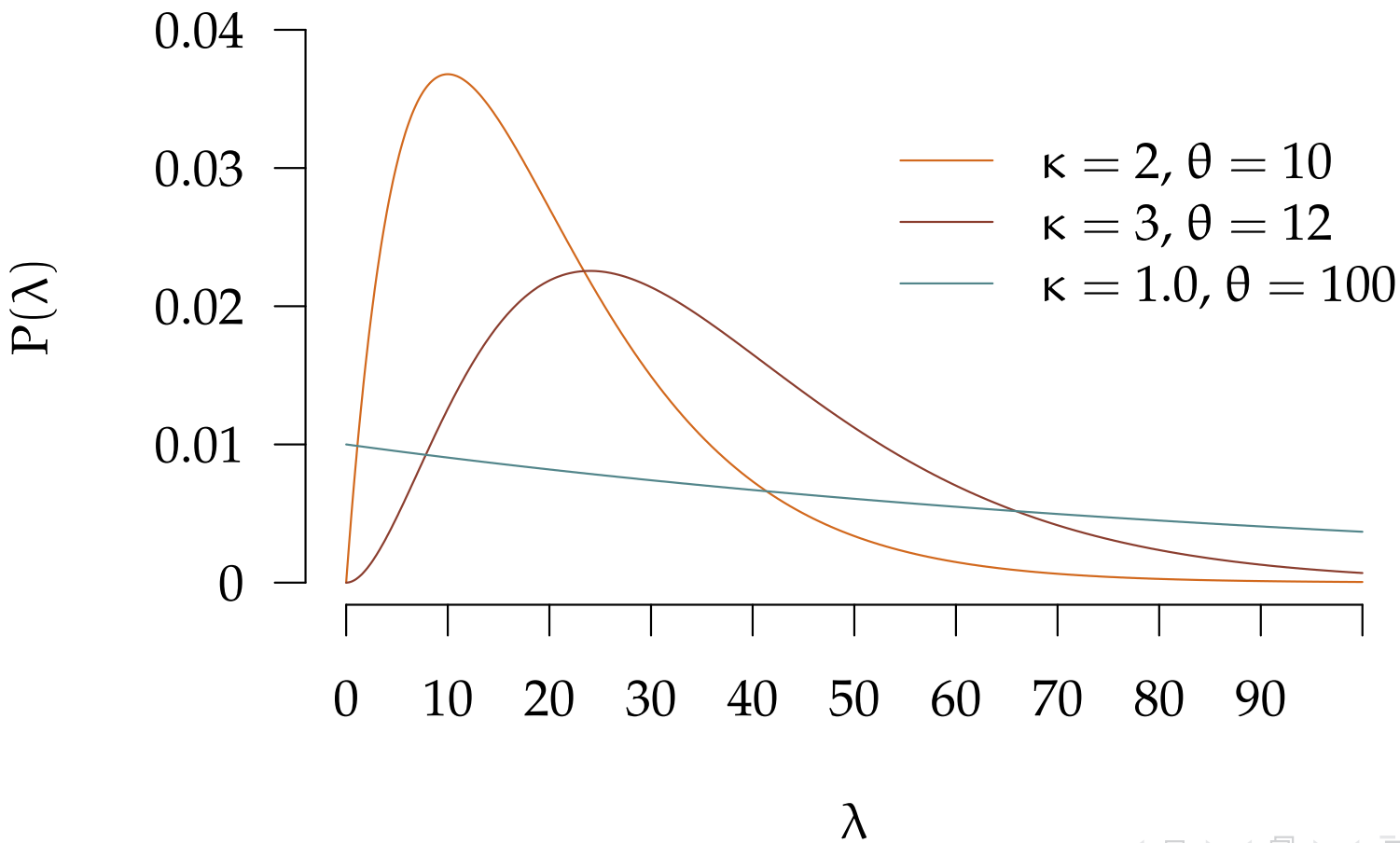


The likelihood of λ given the sufficient statistics $S = 107$ and $n = 10$.

Conjugate prior for the Poisson model

- The Gamma distribution with shape κ and scale θ is a conjugate prior for the Poisson model:

$$\text{Gamma}(\lambda|\kappa,\theta) = \frac{\lambda^{\kappa-1}e^{-\lambda/\theta}}{\theta^{\kappa}\Gamma(\kappa)}.$$



Posterior distribution

- With the Poisson likelihood, the Gamma prior leads to

$$P(\lambda|D,\kappa,\theta) \propto e^{-n\lambda}\lambda^{S} \times \frac{\lambda^{\kappa-1}e^{-\lambda/\theta}}{\theta^{\kappa}\Gamma(\kappa)},$$
$$\propto e^{-\lambda\left(n+\frac{1}{\theta}\right)}\lambda^{S+\kappa-1}.$$

Given that

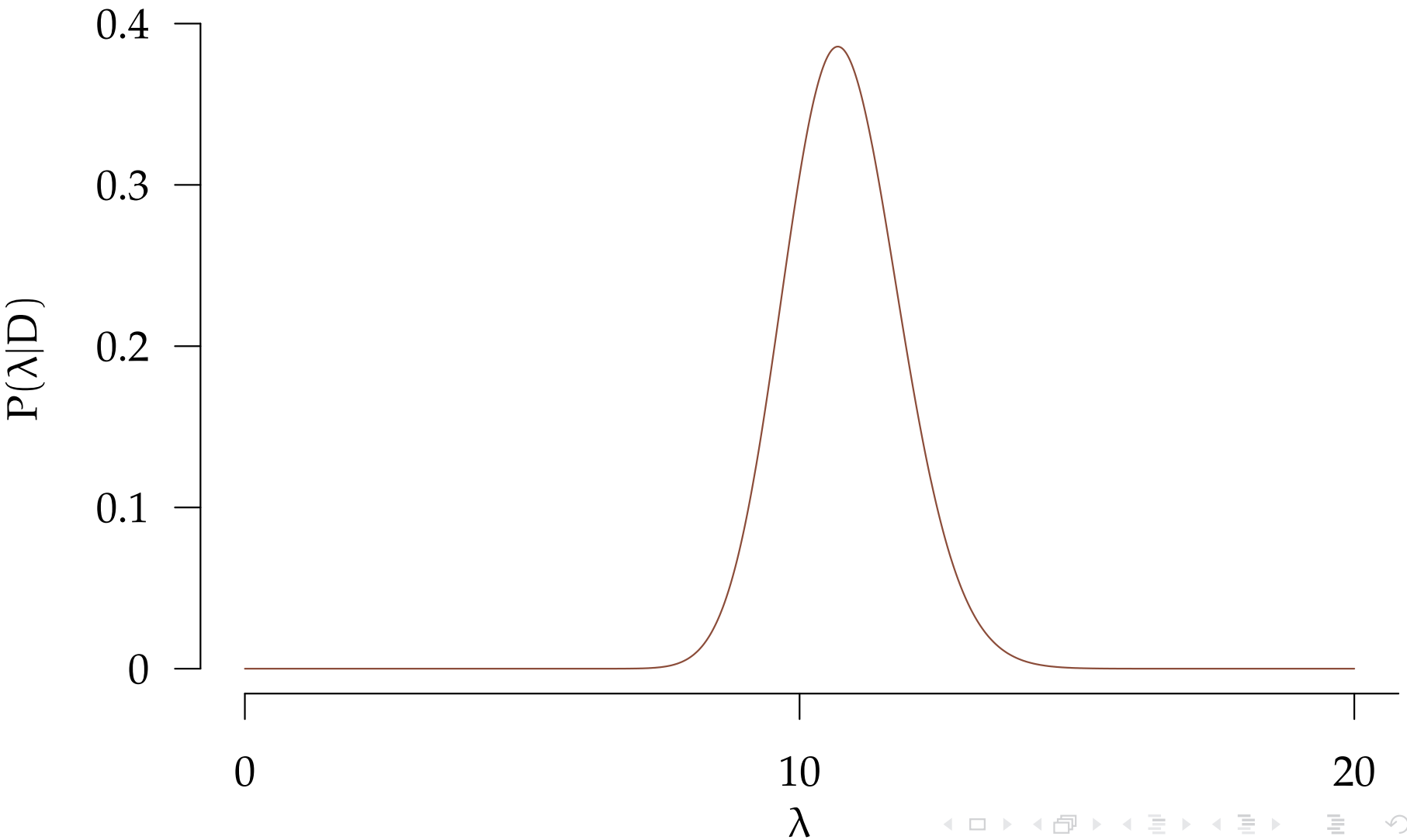
$$\int e^{-\lambda\left(n+\frac{1}{\theta}\right)}\lambda^{S+\kappa-1}\,d\lambda = \left(n+\tfrac{1}{\theta}\right)^{-(S+\kappa)}\Gamma(S+\kappa),$$

we have

$$P(\lambda|D,\kappa,\theta) = \text{Gamma}(\lambda|S+\kappa,(n+\tfrac{1}{\theta})^{-1}).$$

Posterior distribution

- With prior hyper-parameters of $\kappa = 1.0$, $\theta = 100.0$, and sufficient statistics of $S = 107$ and $n = 10$, we have Gamma distribution with shape 108 and scale ≈ 0.1



Summarizing the posterior

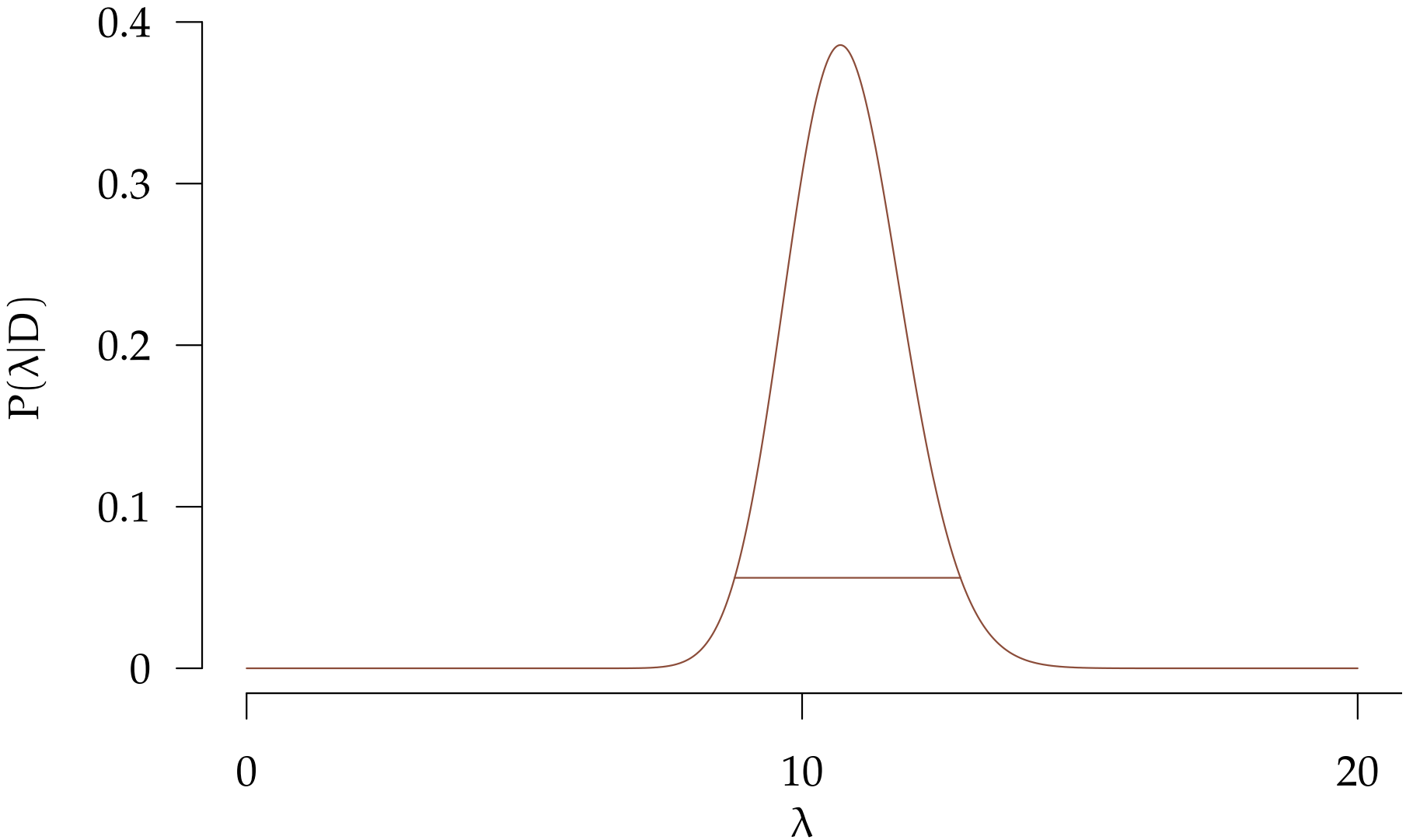
- The mean, variance and modes of any Gamma distribution with shape κ and scale θ are as follows:

$$\begin{aligned} \langle \lambda \rangle &= \kappa\theta, \\ V(\lambda) &= \kappa\theta^2, \\ \text{mode}(\lambda) &= (\kappa - 1)\theta. \end{aligned}$$

- Thus in our case, we have

$$\begin{aligned} \langle \lambda \rangle &= 108.0, \\ V(\lambda) &= 1.08, \quad \text{sd}(\theta) = 1.04, \\ \text{mode}(\lambda) &= 10.69. \end{aligned}$$

The 0.95 HPD *interval*



The 0.95 HPD interval is from 8.78 to 12.86.

Posterior predictive distribution

- Given that the number of emails every hour were

$$D = 6, 8, 12, 7, 11, 14, 10, 12, 11, 16,$$

what do we predict will be number of emails in the next hour?

- This is given by the posterior predictive distribution:

$$\begin{aligned} P(x_{\text{next}} = k|D, \kappa, \theta) &= \int P(x_{\text{next}} = k|\lambda) P(\lambda|D, \kappa, \theta) \, d\lambda, \\ &= \int \frac{e^{-k}\lambda^k}{k!} \times \frac{(n + \frac{1}{\theta})^{S+\kappa}}{\Gamma(S+\kappa)} e^{-\lambda(n+\frac{1}{\theta})} \lambda^{S+\kappa-1}, \\ &= \frac{\Gamma(S+\kappa+k)}{\Gamma(S+\kappa)\Gamma(k+1)} q^k (1-q)^{S+\kappa}, \\ &= \text{NegativeBinomial}(k|S+\kappa, q), \end{aligned}$$

with $q = \left(n + \frac{1}{\theta} + 1\right)^{-1}$.

Posterior predictive distribution

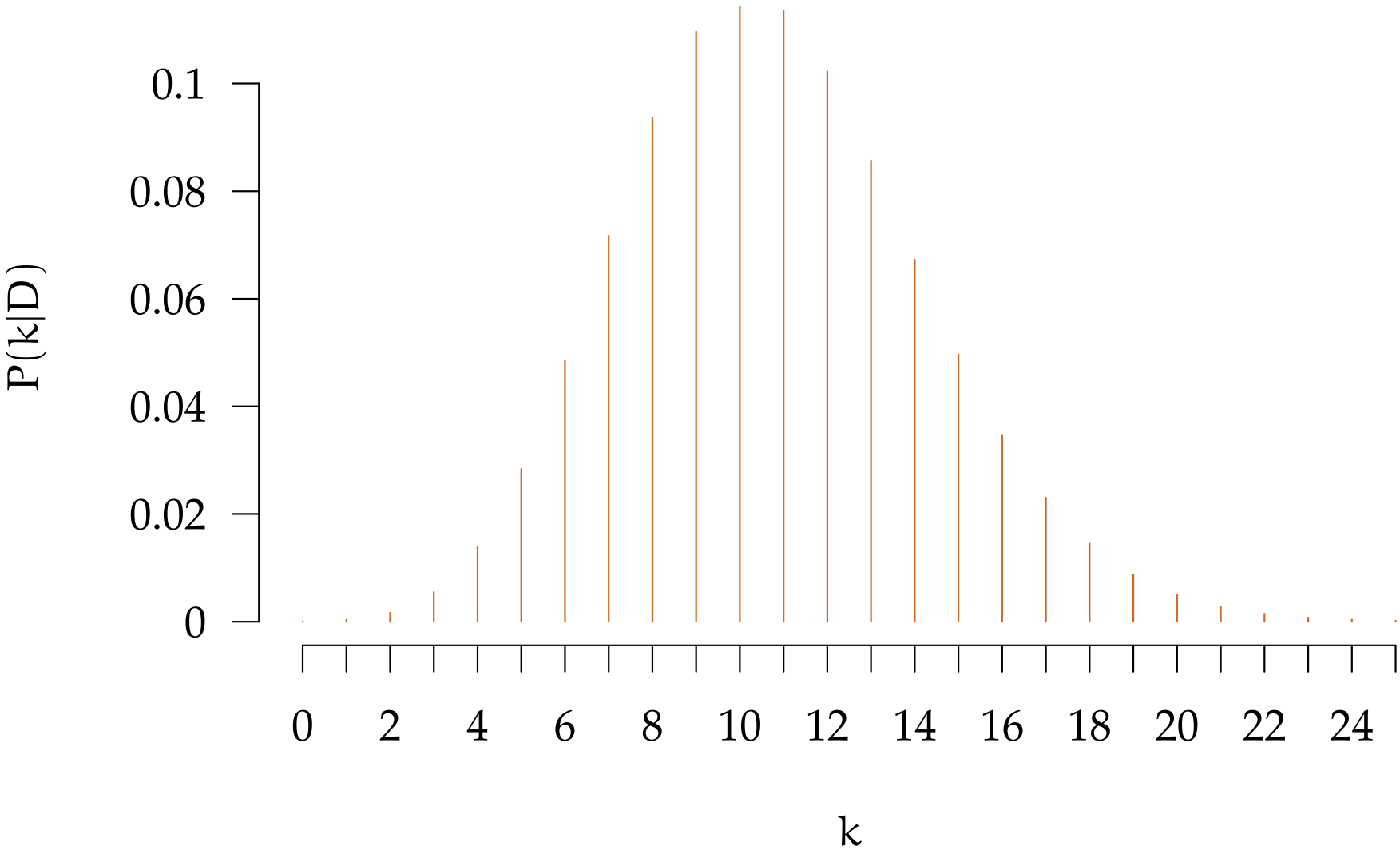
- The negative binomial distribution gives the number of "successes" until a predefined number r of "failures" have occurred, with the probability of a success being q.
- The mean and variance of the negative binomial are

$$\langle k \rangle = \frac{qr}{1-q}, \quad V(k) = \frac{qr}{(1-q)^2}.$$

- In our case, $r = S + \kappa = 108$ and
  $q = \left(n + \frac{1}{\theta} + 1\right)^{-1} = \left(10 + \frac{1}{100} + 1\right)^{-1} = 0.091.$
- The mean and variance of the negative binomial are

$$\langle k \rangle = \frac{S+\kappa}{n+\frac{1}{\theta}} = 10.79, \quad V(k) = \frac{S+\kappa}{n+\frac{1}{\theta}}(n + \frac{1}{\theta} + 1) = 11.87.$$

Posterior predictive distribution



The posterior predictive distribution $P(x_{next} = k|D, \kappa, \theta)$ is a negative binomial distribution with $r = 108$, $q = 0.091$.