Mixture Models & Latent Variable Models

Mark Andrews & Thom Baguley

September 15, 2016

Mixture models

- Given a set of observed variables $y_1, y_2 \ldots y_n$, it is common to model them as distributed according to a parameteric density function density function $P(y|\theta)$ whose parameter θ are unknown.
- For example, we could model $y_1, y_2 \ldots y_n$ as

$$y_i \sim N(\mu, \sigma^2), \quad \text{for } i \in 1 \ldots n.$$

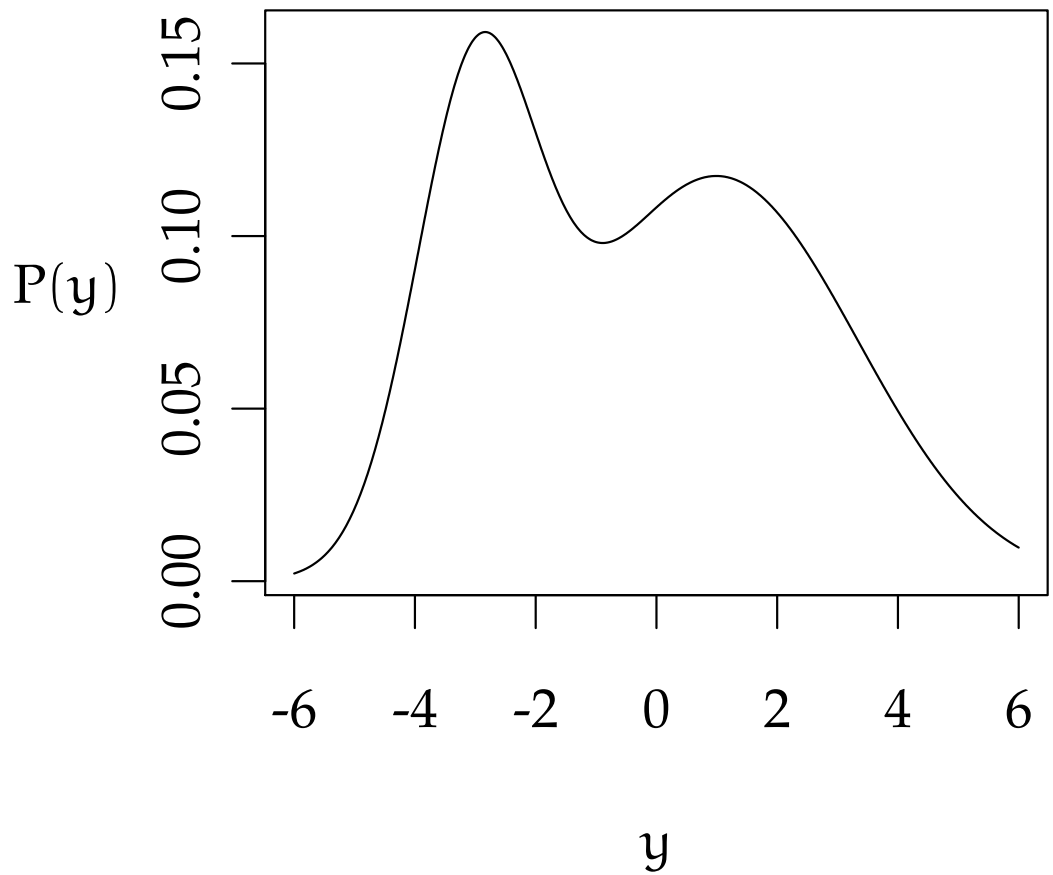
  where μ and σ are unknown.
- We could, however, models $y_1, y_2 \ldots y_n$ as distributed according to a finite *mixture* of parametric density functions. For example,

$$y_i \sim \sum_{k=1}^{K} N(\mu_k, \sigma_k)\pi_k, \quad \text{for } i \in 1 \ldots n.$$

  where π_k is the probability of component distribution k.

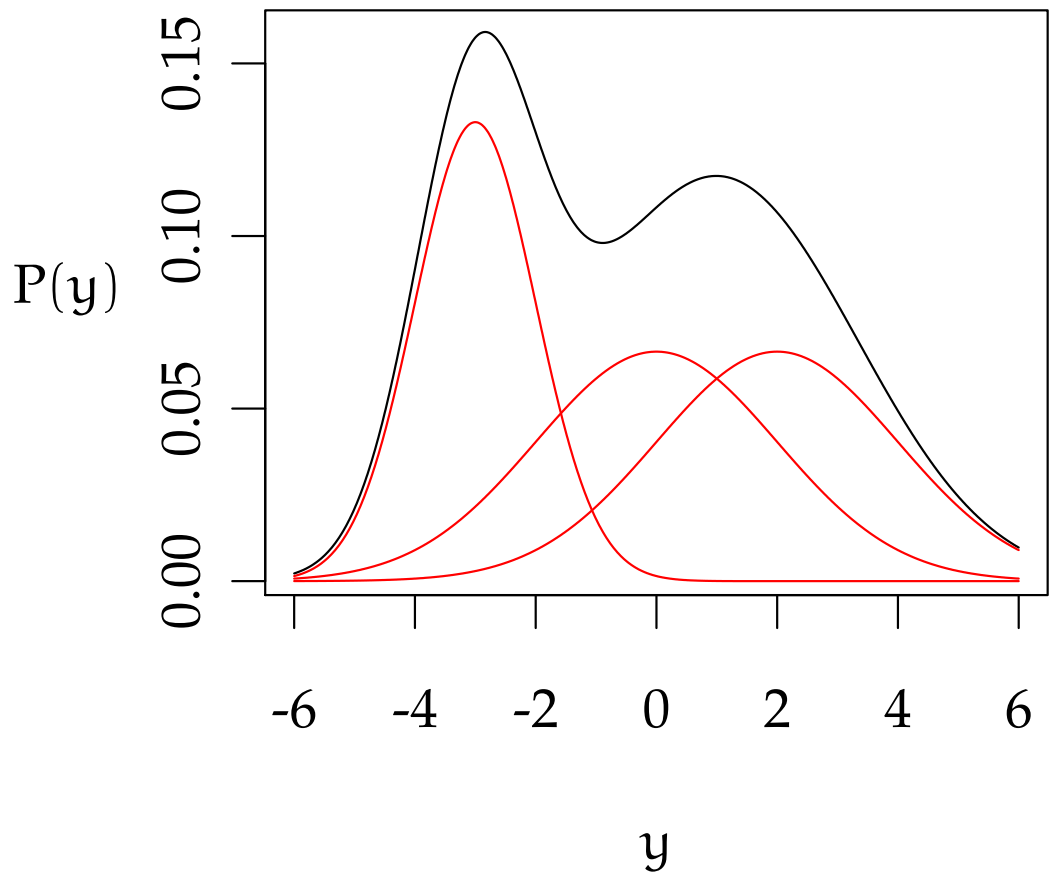
Mixture Models

A mixture of Gaussians density function.



Mixture Models

A mixture of Gaussians density function with component distributions shown in red.



Mixture models as latent variable models

- If $y_1, y_2 \dots y_n$ are modelled as

$$y_i \sim \sum_{k=1}^{K} N(\mu_k, \sigma_k)\pi_k, \quad \text{for } i \in 1 \dots n.$$

this is equivalent to each y_i corresponding to a discrete latent variable $x_i \in \{1, 2 \dots K\}$ and

$$y_i \sim N(\mu_{[x_i]}, \sigma_{[x_i]}), \quad \text{for } i \in 1 \dots n.$$

where

$$x_i \sim P(\pi).$$

Mixture models as latent variable models

- A general way of writing a mixture model is

$$P(y_i) \sim \sum_{k=1}^{K} P(y|\theta, x_i = k)P(x_i = k|\pi), \quad \text{for } i \in 1 \dots n.$$

- This is equivalent to the following generative model:
  For $i \in \{1, 2 \dots n\}$,

$$x_i \sim \text{Categorical}(\pi),$$
$$y_i \sim P(y|\theta_{[x_i]})$$

Bayesian inference in mixture models as latent variable models

- Given the general form

$$P(y_i) \sim \sum_{k=1}^{K} P(y|\theta, x_i = k)P(x_i = k|\pi), \quad \text{for } i \in 1 \dots n.$$

the observed data are

$$D = y_1, y_2 \dots y_n,$$

while the parameters

$$\theta = \theta_1, \theta_2 \dots \theta_K$$

and the values of the latent variables

$$x_1, x_2 \dots x_n$$

are unknown.

Bayesian inference in mixture models as latent variable models

- The full posterior is

$$P(\theta, \pi, x_{1:n}|y_{1:n}) \propto P(\pi)P(\theta)\prod_{i=1}^{n} P(y|\theta, x_i = k)P(x_i = k|\pi)$$

- We can often sample from this using a (blocked) Gibbs sampler. For example, the conditional distributions

$$P(x_i = k|\theta, \pi, y_{1:n}) \propto P(y_i|x_i = k, \theta)P(x_i = k|\pi)$$

and

$$P(\theta_k|x_{1:n}, y_{1:n}, \pi) \propto P(\theta_k) \prod_{\{i:\, x_i = k\}} P(y_i|\theta_k)$$

and

$$P(\pi|x_{1:n}) \propto P(x_{1:n}|\pi)P(\pi)$$

are often analytically tractable.

Example: Zero-Inflated Poisson distribution

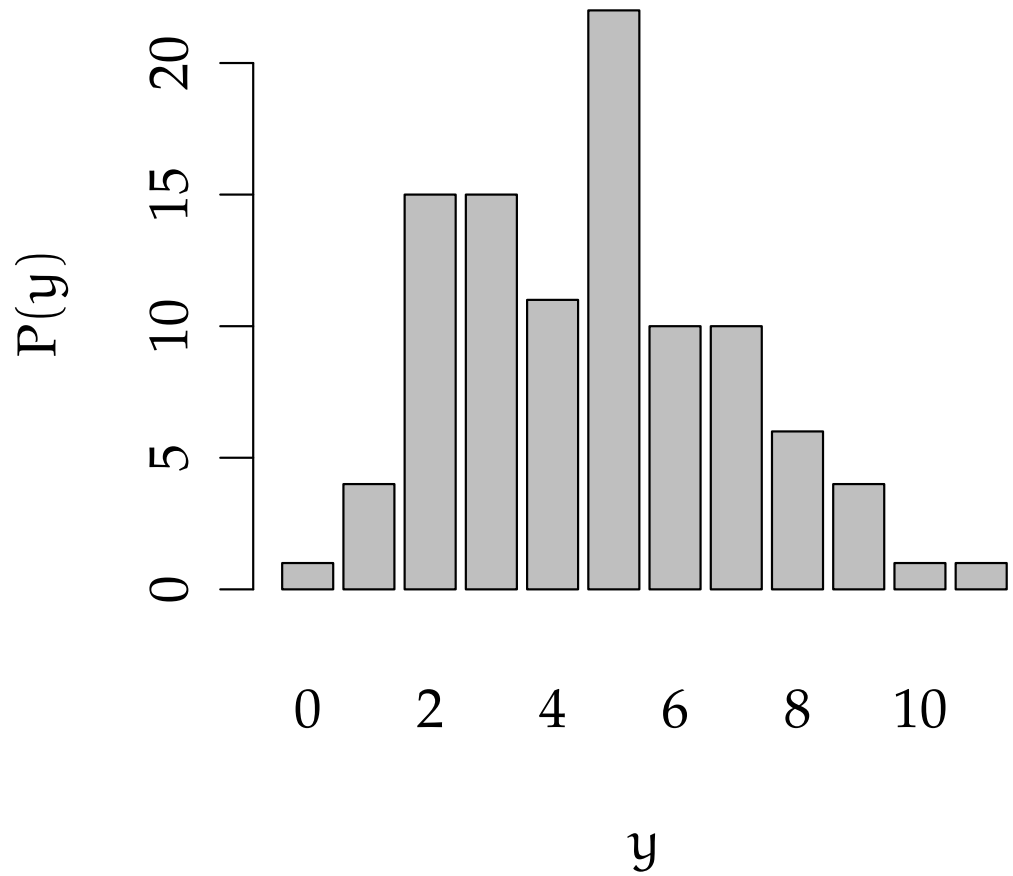
- A zero inflated Poisson model is a simple two component mixture model.
- According to this model, the observed $y_1, y_2 \dots y_n$ are assumed to generated by

$$y_i \sim \begin{cases} \text{Poisson}(\lambda) & \text{if } x_i = 0, \\ 0, & \text{if } x_i = 1 \end{cases},$$

$$x_i \sim \text{Bernoulli}(\pi).$$

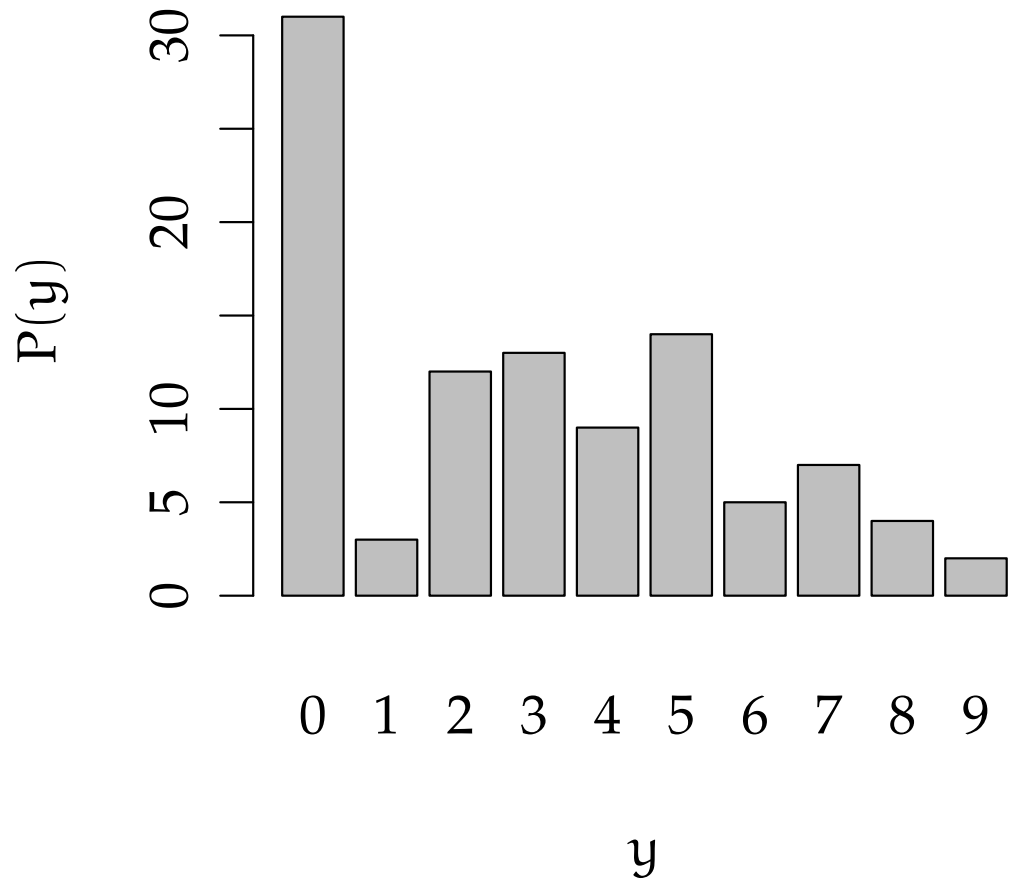
Poisson Distribution

A sample from a Poisson distribution with $\lambda = 5$.



Zero-Inflated Poisson Distribution

A sample from a Zero-inflated Poisson model, with Poisson distribution component with $\lambda = 5$, and probability of zero-model being .25.



Dirichlet Process mixture models

- Dirichlet Process mixture model can overcome the problem of choosing K, the number of component distributions.
- In a Dirichlet Process mixture model, our generative model is For $i \in \{1, 2 \dots n\}$,

$$\theta_i \sim DP(\alpha G_0),$$
$$y_i \sim P(y|\theta_i)$$

which is equivalent to

$$P(y_i) \sim \sum_{k=1}^{\infty} P(y|\theta, x_i = k)P(x_i = k|\pi), \quad \text{for } i \in 1 \dots n.$$

Dirichlet Process mixture models

- A Dirichlet Process with *base measure* G_0 and *concentration* parameter α can be defined by followng density:

$$f(\theta) = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k}(\theta)$$

where each π_k is drawn from a *stick-breaking* prior with parameter α, each θ_k is drawn G_0, and

$$\delta_{\theta_k}(\theta)$$

is an function over θ that takes the value of 1 if $\theta = \theta_k$ and 0 otherwise.

- In other words, a Dirchlet Process $f(\theta)$ is an infinite distribution of *spikes* on the θ parameter space.