Prior Exposure 1: Bayes for beginners: **Bayes factors**

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Reprise: An approximate Bayes factor

A difference in BIC between models on the same data can be expressed as a likelihood ratio:

$$e.g., \Delta_{BIC} = BIC_{M0} - BIC_{M1}$$

$$LR_{BIC} = e^{1/2\Delta_{BIC}}$$

This quantity is a well-known approximation to a Bayes factor with a *unit-information prior*:

$$BF \approx LR_{BIC} = e^{1/2\Delta_{BIC}}$$

Here the prior is meant to carry as much information as a single observation (Kass & Wasserman, 1995)

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They are equal to likelihood ratios only for nested models with simple hypotheses and no nuisance parameters (Kass & Raftery, 1995)

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... thus the posterior probability is proportional to the likelihood times the prior

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... so the Bayes factor is the degree to which the data change the prior odds to obtain the posterior odds (It can also be interpreted as the strength of evidence provided by the data in the context of the prior)

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- promising 'default' objective priors have been developed

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Unlike AIC or BIC there is no need to add a penalty for complex models or hypotheses

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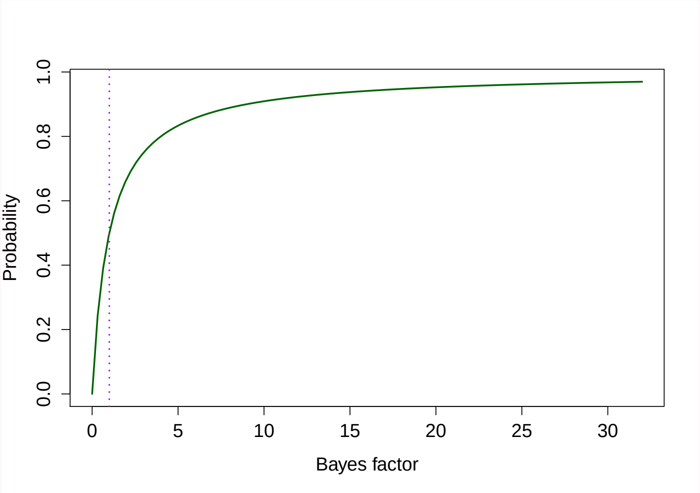
$$Pr\left(H_0\middle|\mathcal{D}\right) = \frac{BF_{01}}{1 + BF_{01}}$$

... assuming H_0 and H_1 are equally likely *a priori*

(It can also be useful to use the logarithm $ln(BF)$ when the Bayes factor is large)

Interpreting a Bayes factor (BF)

Transforming from an odds to a probability scale



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Jeffries (1961) proposed the following interpretation:

BF_{10}	Strength of evidence for H_1
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10 - 30	Strong
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... but remember that it is a continuous measure!

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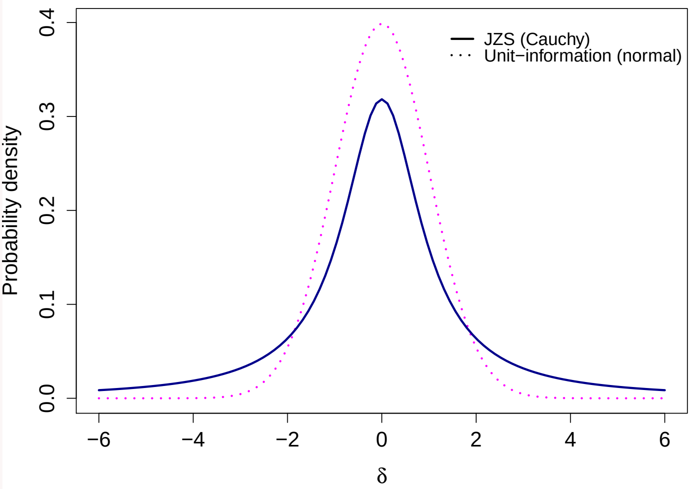
- unit-information prior
  (e.g., Kass & Raftery, 1995; Wagenmakers, 2007)
- Cauchy or JZS prior
  (e.g., Liang et al., 2008; Rouder et al., 2009)

Objective Bayes factors with default priors

Standard Cauchy versus standard normal priors



Bayesian t test with unit-information prior

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However, when H_0 is that $\delta = 0$ it can be calculated by hand (e.g., for one sample *t*):

$$BF_{01} = \frac{\left(1 + \frac{t^2}{n-1}\right)^{-n/2}}{(1 + nr^2)^{-1/2}\left(1 + \frac{t^2}{(1+nr^2)(n-1)}\right)^{-n/2}}$$

... here r is the scale factor for the variance (typically set at 1)

Bayesian t test with a Jeffreys-Zellner-Siow (JZS) prior

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The analytic solution for the one sample case is:

$$BF_{01} = \frac{\left(1 + \frac{t^2}{n-1}\right)^{-n/2}}{\int_0^\infty (1 + ngr^2)^{-1/2} \left(1 + \frac{t^2}{(1+ngr^2)(n-1)}\right)^{-n/2} (2\pi)^{-1/2} g^{-3/2} e^{-1/(2g)} \, dg}$$

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... and here r is the scale factor for the variance typically set at $1/2$ for a one sample or $\sqrt{2}/2$ for two samples

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This is around $n > 5{,}000$ for the unit-information prior or and $n > 50{,}000$ for the JZS prior

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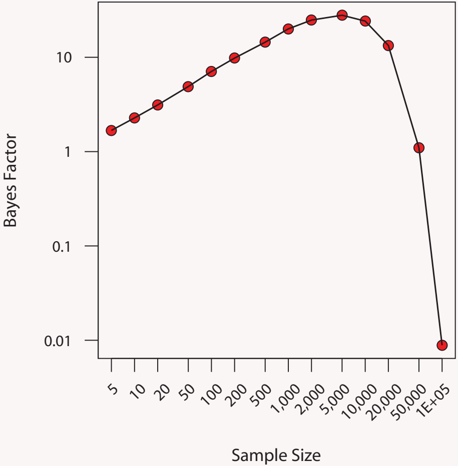
Tipping point for the unit-information prior when $\delta = 0.02$



Reprise: Voting intention example

Does a single exposure to an US flag influence voting intention?

Carter et al. (2011) report a priming study in which a single exposure to a US flag changes voting intentions

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How do our objective Bayes factors work in this case?

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```
> unit.prior.Bf.2s(t=2.02, n1=91.5, n2=91.5)
$`Bayes factor for H0`
[1] 0.9394962

$`Bayes factor for H1`
[1] 1.0644
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> exp(log.bf[['bf']])
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Baguley (2012) also provides R functions for these tests.

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