Prior Exposure 1: Bayes for beginners: **Introducing Bayesian data analysis**

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Reprise: Bayesian data analysis in three simple steps

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... this last step uses Bayes' theorem

Bayes' theorem

simple probability

$$\overbrace{Pr\left(A|B\right)}^{\text{Posterior}} = \frac{\overbrace{Pr\left(B|A\right)}^{\text{Evidence}} \quad \overbrace{Pr\left(A\right)}^{\text{Base rate (Prior)}}}{\underbrace{Pr\left(B\right)}_{\text{Probability of evidential event}}}$$

Bayes' theorem

Murder or SIDS: the Sally Clark case

What is the more likely cause of a cot death: Murder or SIDS?

$$\overbrace{Pr(A|B)}^{\text{Is it murder?}} = \frac{\overbrace{Pr(B|A)}^{\text{Prob. of death given murder}} \quad \overbrace{Pr(A)}^{\text{Prob. of murder}}}{\underbrace{Pr(B)}_{\text{Prob. of cot death}}}$$

Bayes' theorem

Murder or SIDS: the Sally Clark case

Trivially the probability of cot death given a murder is 1

$$\overbrace{Pr\left(A\,|\,B\right)}^{\text{Is it murder?}} = \frac{\overbrace{1}^{\text{Prob. of death given murder}}\;\overbrace{Pr\left(A\right)}^{\text{Prob. of murder}}}{\underbrace{Pr\left(B\right)}_{\text{Prob. of cot death}}}$$

Bayes' theorem

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Homicide rates for infants are about 1 in 21,700 (Hill, 2004)

$$\overbrace{Pr(A|B)}^{\text{Is it murder?}} = \frac{\overbrace{1}^{\text{Prob. of death given murder}} \times \overbrace{1/21700}^{\text{Prob. of murder}}}{\underbrace{Pr(B)}_{\text{Prob. of cot death}}}$$

Bayes' theorem

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Cot death rates are about 1 in 1,300 (Hill, 2004)

$$\overbrace{Pr(A|B)}^{\text{Is it murder?}} = \frac{\overbrace{1}^{\text{Prob. of death given murder}} \times \overbrace{1/21700}^{\text{Prob. of murder}}}{\underbrace{1/1300}_{\text{Prob. of cot death}}}$$

Bayes' theorem

Murder or SIDS: the Sally Clark case

So the probability that a cot death is murder is around 6%

$$\overbrace{1/16.7}^{\text{Is it murder?}} = \frac{\overbrace{1}^{\text{Prob. of death given murder}} \times \overbrace{1/21700}^{\text{Prob. of murder}}}{\underbrace{1/1300}_{\text{Prob. of cot death}}}$$

Bayes' theorem

probability distributions

... the same equation can be generalised to probability distributions (here Ω is the full set of parameters in the model including nuisance parameters)

$$\overbrace{\mathrm{P}(\theta|\mathcal{D},\Omega)}^{\text{Posterior}} = \frac{\overbrace{\mathrm{P}(\mathcal{D}|\theta)}^{\text{Likelihood}}\ \overbrace{\mathrm{P}(\theta|\Omega)}^{\text{Prior}}}{\underbrace{\int \mathrm{P}(\mathcal{D}|\theta)\mathrm{P}(\theta|\Omega)\,d\theta}_{\text{Marginal probability of data}}}$$

Marginal probability

... getting a posterior distribution for θ requires integrating out other parameters in the model

$$\int \mathrm{P}(\mathcal{D}|\theta)\mathrm{P}(\theta|\Omega)\,d\theta$$

Getting an analytic solution may be difficult or impossible

... but can also be very simple for some well-known problems

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... together imply:

$$\overbrace{\mathrm{P}(\mu|\mathcal{D}, \sigma^2)}^{\text{Posterior}} \sim N\left(\mu_{post}, \sigma^2_{post}\right)$$

Normal distributions with known variance

Posterior variance

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Posterior mean

The posterior mean is, in effect, a combination of the observed mean and the prior weighted by their relative precision

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... this weighting is sometimes termed a *fully automatic Occam's razor* (Spegelhalter & Smith, 1980)

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... but I prefer the more neutral *posterior probability interval* (or *probability interval* for short)

e.g., a 95% probability interval takes the form:

$$\mu_{post} \pm z_{.975}\sigma_{post} \approx \mu_{post} \pm 1.96\sigma_{post}$$

Normal distribution example

Standardized mean difference (Cohen's d)

A standardized mean difference is an example of a a normally distributed statistic with a known variance of 1

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This bias arises from multiple sources - notably from $p < .05$ acting as a filter that removes small effect sizes

Normal distribution example

Does a single exposure to an US flag influence voting intention?

Carter et al. (2011) report a priming study in which a single exposure to a US flag changes voting intentions (making them more Republican)

They report an 'effect size' of over a quarter of a standard deviation:

$$\hat{\mu}_E - \hat{\mu}_C = 0.142, \quad t_{181} = 2.02, \quad p = .04 \quad d = 0.298$$

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N.B. Cohen's *d* is really a family of statistics. The statistic here is probably Hedges' g

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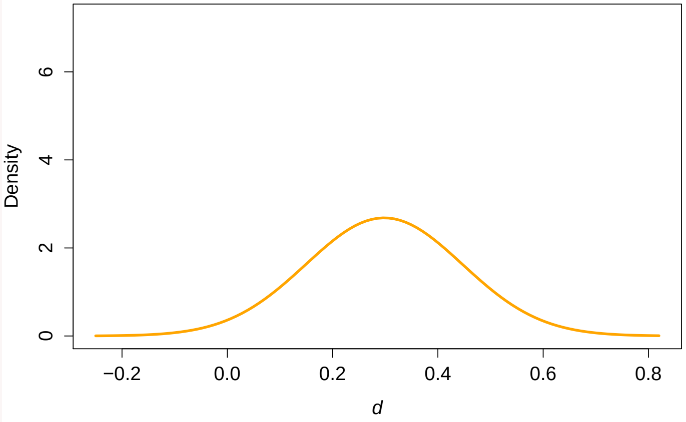
The likelihood (our probability model for the data) is therefore:

$$\mathrm{P}(\mathcal{D}|\mu) \sim N\left(0.298, 0.149^2\right)$$

Normal distribution example

Graphing the likelihood



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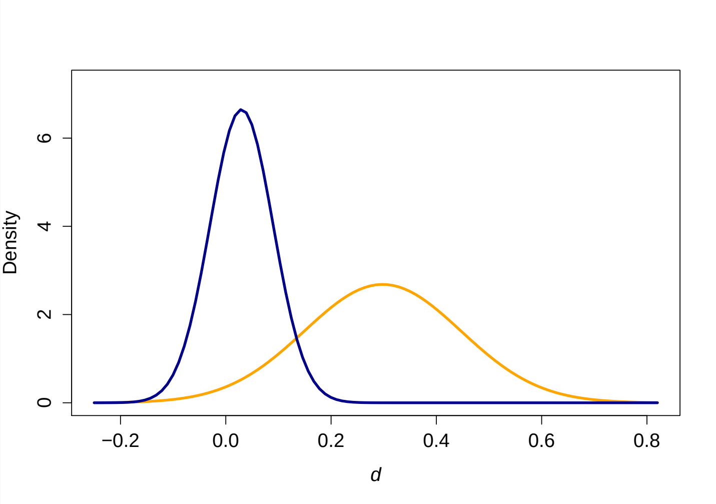
How might we determine our prior here?

A generous prior for the true effect size might be a 1% swing with a likely range of say $\pm 2\%$
If $g = 0.3$ corresponds to a 10% swing then this suggests the following normal prior:

$$\mathrm{P}(\mu|\sigma^2) \sim N\left(0.03, 0.06^2\right)$$

Normal distribution example

Adding the prior



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Posterior mean and standard deviation

$$\sigma_{post} = \sqrt{\left(\frac{1}{0.149^2} + \frac{1}{0.06^2}\right)^{-1}} \approx 0.056$$

$$\mu_{post} = \left(\frac{0.056^2}{0.149^2}\right) 0.298 + \left(\frac{0.056^2}{0.06^2}\right) 0.03$$

$$= 0.14 \times 0.298 + .86 \times 0.03 \approx 0.067$$

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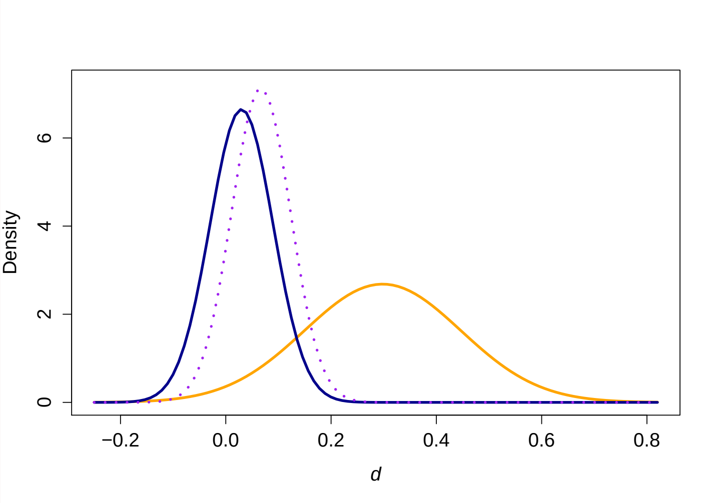
Normal distribution example

Calculating the posterior



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Evaluating the model

A 95% probability interval for the posterior distribution of the flag effect is:

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... but is my prior a good one?

Selecting a prior

Subjective versus objective priors

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>
> *A complete alternative is the fully subjectivist position, which compels one to elicit priors on all parameters based on the personal judgement of appropriate individuals.*

Spiegelhalter & Rice (2009)

Subjective Bayes

Eliciting a prior

A key task for a subjective Bayesian is to elicit a prior distribution:

> *... successful elicitation faithfully represents the opinion of the person being elicited. It is not necessarily "true" in some objectivistic sense, and cannot be judged that way. We see elicitation as simply part of the process of statistical modeling.*

Garthwaite et al. (2005)

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Objective Bayes

What is objectivity in Bayesian data analysis

... I take objectivity to mean that given the same data and the same assumptions regarding the model, different researchers will arrive at the same conclusions.

Wagenmakers (2007, Appendix)

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A viewpoint from subjective Bayes

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A viewpoint from objective Bayes

... it [objective Bayes] simultaneously achieves what should be a major goal of Bayesianism - ensuring that answers are conditional on the data actually obtained - while at the same time respecting the frequentist notion that the methodology must ensure success in repeated usage by scientists.

Berger (2006)

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- greater emphasis on transparency in modelling
- more flexibility in modelling complex, messy (real world) data sets