Nonlinear regression

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May 12, 2016

Basis function regression

- Given a set of n observed predictor-outcome pairs $(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$, a linear model with normally distributed errors is, for $i \in 1 \ldots n$,

$$y_i = \alpha + \beta x_i + \epsilon_i, \quad \text{where } \epsilon_i \sim N(0, \sigma^2).$$

- How do we model a *nonlinear* relationship between x and y?
- One general solution is to use a weighted sum of K *basis functions* $\phi_1(x_i), \phi_2(x_i) \ldots \phi_3(x_i)$ and

$$y_i = \sum_{k=1}^{K} w_k \phi_k(x_i) + \epsilon_i$$

- Common choices of basis functions are *radial-basis* functions and *splines*.

Radial basis functions

- A radial, or squared exponential, basis function can have the following form:

$$\phi_k(x) = \exp\left(-\frac{|x - \mu_k|^2}{l^2}\right)$$

- Given a set of centers $\mu_1, \mu_2 \ldots \mu_K$, we can then model a smooth function between x and y with additive noise as

$$y \sim \sum_{k=1}^{K} w_k \phi_k(x) + \epsilon.$$

- If $\mu_1, \mu_2 \ldots \mu_K$ are known, then Bayesian inference in radial basis function regression involves inference of $w_1, w_2 \ldots w_K$, the width parameter l^2 and the observation noise term σ^2.

Gaussian Processes

- A Gaussian process can be seen as a generalization of a multivarate Gaussian to an infinite dimensional space.
- It can be seen as a distribution over smooth functions, i.e. a random function.
- For example, in a Gussian process over a space X, the covariance in the distribution between between any to points in that space is given by a positive definite *kernel* function.
- An example of a kernel could be

$$\kappa(x_i, x_j) = \exp\left(-\frac{|x_i - x_j|^2}{l^2}\right)$$

- The joint probability distribution over any set of points $x = x_1, x_2 \ldots x_n$ is then by a multivariate Gaussian with covariance matrix $K(x_1, x_2 \ldots x_n)$ where

$$K_{ij} = \kappa(x_i, x_j).$$

Gaussian Processes regression

- If $f \sim GP(m, \kappa)$ is a Gussian process, i.e. a random distribution over smooth functions, we can model a smooth function between x and y with additive noise as

$$y \sim f + \epsilon,$$

  with the prior on the mean function m as 0.
- Given a set of n observed predictor-outcome pairs $(\tilde{x}_1, y_1), (\tilde{x}_2, y_2) \ldots (\tilde{x}_n, y_n)$, the posterior distribution over f is itself a Gaussian process

$$P(f|\tilde{x}, y) \sim GP(\tilde{m}, \tilde{\kappa}),$$

- where

$$\tilde{m} = K(\tilde{x}, x)(K(\tilde{x}, x) + \sigma^2 I)^{-1} y,$$

  and

$$\tilde{\kappa} = K(\tilde{x}, \tilde{x}) - K(\tilde{x}, x)(K(\tilde{x}, x) + \sigma^2 I)^{-1} K(x, \tilde{x}).$$