Confidence Intervals for Mean and Variance for Normal Distribution

1. Confidence intervals for the mean

   (a) Mean unknown and variance known

   $$P(-c < Z < c) = \gamma \qquad c = \Phi^{-1}\left(\tfrac{\gamma+1}{2}\right)$$

   $$P\left(\bar{X}_n - \tfrac{c}{\sqrt{n}}\sigma < \mu < \bar{X}_n + \tfrac{c}{\sqrt{n}}\sigma\right) = \gamma$$

   (b) Mean and variance unknown

   $$P(-c < t_{n-1} < c) = \gamma \qquad c = t_{n-1}^{-1}\left(\tfrac{\gamma+1}{2}\right)$$

   $$P\left(\bar{X}_n - \tfrac{c}{\sqrt{n}}\left[\tfrac{\sum(X_i-\bar{X}_n)^2}{n-1}\right]^{\frac{1}{2}} < \mu < \bar{X}_n + \tfrac{c}{\sqrt{n}}\left[\tfrac{\sum(X_i-\bar{X}_n)^2}{n-1}\right]^{\frac{1}{2}}\right) = \gamma$$

2. Confidence intervals for the variance

   (a) Variance unknown and mean known

   $$P(c_1 < \chi_n^2 < c_2) = \gamma \qquad c_2 = \left(\chi_n^2\right)^{-1}\left[\tfrac{\gamma+1}{2}\right] \qquad c_1 = \left(\chi_n^2\right)^{-1}\left[\tfrac{1-\gamma}{2}\right]$$

   $$P\left(\sum\left(X_i-\mu\right)^2/c_2 < \sigma^2 < \sum\left(X_i-\mu\right)^2/c_1\right) = \gamma$$

   (b) Mean and variance unknown

   $$P(c_1 < \chi_{n-1}^2 < c_2) = \gamma \qquad c_2 = \left(\chi_{n-1}^2\right)^{-1}\left[\tfrac{\gamma+1}{2}\right] \qquad c_1 = \left(\chi_{n-1}^2\right)^{-1}\left[\tfrac{1-\gamma}{2}\right]$$

   $$P\left(\sum\left(X_i-\bar{X}_n\right)^2/c_2 < \sigma^2 < \sum\left(X_i-\bar{X}_n\right)^2/c_1\right) = \gamma$$

   (c) Attempt to optimize c_1 and c_2 in both cases above using simple grid search. Start with c_1 and c_2 as defined above, then decrease both probabilities by 0.0001 until $\left[\tfrac{1-\gamma}{2}\right]$ is 0, then pick c_1 and c_2 with smallest confidence interval. For example for $\gamma = 0.95$ this will give 250 (0.025/0.0001) pairs of c_1 and c_2 to search. Among those 250 pairs the one that has the smallest $c_2 - c_1$ is selected.