Linear Operators and Linear Functionals

Linear Operators

Definition. A linear operator T is an operator such that

(i) the domain $\mathscr{D}(T)$ of T is a vector space and the range $\mathscr{R}(T)$ lies in a vector space over the same field.

(ii) for all $x, y \in \mathscr{D}(T)$ and scalars α,

$$T(x+y) = Tx + Ty,$$

$$T(\alpha x) = \alpha Tx.$$

Observe the notation: we write Tx instead of $T(x)$; $\mathscr{D}(T)$ denotes the domain of T; $\mathscr{R}(T)$ denotes the range of T; $\mathscr{N}(T)$ denotes the **null space** of T (the set of all $x \in \mathscr{D}(T)$ such that $Tx = 0$).

We shall now consider some basic examples of linear operators:

1. **Identity operator.** The identity operator $I_X : X \longrightarrow X$ is defined by $I_X x = x$ for all $x \in X$. We also write simply I for I_X; thus, $Ix = x$.
2. **Zero operator.** The zero operator $\mathbf{0} : X \longrightarrow Y$ is defined by $\mathbf{0}x = 0$ for all $x \in X$.
3. **Differentiation.** Let X be the vector space of all polynomials on $[a, b]$. We may define a linear operator T on X by setting
$$Tx(t) = x'(t)$$
for every $x \in X$, where the prime denotes the differentiation with respect to t. This operator T maps X onto itself.
4. **Integration.** A linear operator T from $C[a, b]$ into itself can be defined by
$$Tx(t) = \int_a^t x(\tau) d\tau \qquad t \in [a, b].$$
5. **Multiplication by t.** A linear operator T from $C[a, b]$ into itself is defined by
$$Tx(t) = tx(t).$$
6. **Elementary vector algebra.** The cross product with one factor kept fixed defines a linear operator $T_1 : R^3 \longrightarrow R^3$. Similarly, the dot product with one fixed factor defines a linear operator $T_2 : R^3 \longrightarrow R$, say,
$$T_2 x = x \cdot a = \xi_1 \alpha_1 + \xi_2 \alpha_2 + \xi_3 \alpha_3$$
where $a = (\alpha_j) \in R^3$ is fixed.
7. **Matrices.** A real matrix $A = (\alpha_{jk})$ with r rows and n columns defines an operator $T : R^n \longrightarrow R^r$ by means of
$$y = Ax$$
where $x = (\xi_j)$ has n components and $y = (\eta_i)$ has r components and both vectors are written as column vectors beacuse of the usual convention of matrix multiplication.
If A were complex, it would define a linear operator from C^n into C^r.

Theorem. *Let T be a linear operator. Then $\mathscr{R}(T)$ is a vector space; if $\dim \mathscr{L}(T) = n < \infty$, then $\dim \mathscr{R}(T) \leq n$; the null space $\mathscr{N}(T)$ is a vector space.*

A important consequence of the second claim is that: *linear operators preserve linear dependence.*

Theorem (Inverse operator). *Let T be a linear operator. The inverse $T^{-1} : \mathscr{R}(T) \longrightarrow \mathscr{D}(T)$ exists if and only if*

$$Tx = 0 \qquad \Rightarrow \qquad x = 0.$$

If T^{-1} exists, it is a linear operator. Moreover, if T^{-1} exists and $\dim \mathscr{D}(T) = n < \infty$, then $\dim \mathscr{R}(T) = \dim \mathscr{D}(T)$.

Theorem. *Let $T : X \longrightarrow Y$ and $S : Y \longrightarrow Z$ be bijective linear operators, where X, Y, Z are vector spaces. Then the inverse $(ST)^{-1} : Z \longrightarrow X$ of the product ST exists, and*

$$(ST)^{-1} = T^{-1}S^{-1}.$$

Definition. Let X and Y be normed spaces and and $T : \mathscr{D}(T) \longrightarrow Y$ a linear operator, where $\mathscr{D}(T) \subset X$. The operator T is said to be **bounded** if there is a real number c such that for all $x \in \mathscr{L}(T)$,

$$\|Tx\| \leq c\|x\|.$$

Definition. Let T be a bounded linear operator. Then the **norm** of T is defined as

$$\|T\| = \sup_{\substack{x \in \mathscr{D}(T) \\ \|x\| \neq 0}} \frac{\|Tx\|}{\|x\|}.$$

It's clear that the norm defined by the above formula satisfies (N1) to (N4).

An alternative formula for the norm of T is

$$\|T\| = \sup_{\substack{x \in \mathscr{D}(T) \\ \|x\| = 1}} \|Tx\|.$$

Let's take a look at some typical examples:

1. **Identity operator.** The identity operator $I : X \longrightarrow X$ on a normed space $X \neq \{0\}$ is bounded and has norm $\|I\| = 1$.
2. **Zero operator.** The zero operator $\mathbf{0} : X \longrightarrow Y$ on a normed space X is bounded and has norm $\|\mathbf{0}\| = 0$.
3. **Differentiation operator.** Let X be the normed space of all polynomials on $J = [0, 1]$ with norm given $\|x\| = \max |x(t)|$, $t \in J$. A differentiation operator T is defined on X by

   $$Tx(t) = x'(t).$$

   This operator is linear but not bounded. Indeed, let $x_n(t) = t^n$, where $n \in N$. Then $\|x_n\| = 1$ and

   $$Tx_n(t) = x_n'(t) = nt^{n-1}$$

   so that $\|Tx_n\| = n$ and $\|Tx_n\|/\|x_n\| = n$. Since $n \in N$ is arbitrary, this shows that there is no fixed number c such that $\|Tx_n\|/\|x_n\| \leq c$.
4. **Integral operator.** W e can define an integral operator $T : C[0, 1] \longrightarrow C[0, 1]$ by

   $$y = Tx \quad \text{where} \quad y(t) = \int_0^1 k(t, \tau)x(\tau)d\tau.$$

   Here k is a given function, which is called the *kernel* of T and is assumed to be continuous on the closed square $G = J \times J$ in the $t\tau$-plane, where $J = [0, 1]$. This operator is linear and bounded.
5. **Matrix.** A real matrix $A = (\alpha_{jk})$ with r rows and n columns defines an operator $T : R^n \longrightarrow R^r$ by means of

   $$y = Ax.$$

   Then T is bounded.

Theorem. *If a normed space X is finite dimensional, then every linear operator on X is bounded.*

Sketch of Proof. Consider the basis of Y and make use of the lemma (bound for linear combinations). ■

Theorem. *Let $T : \mathscr{D}(T) \longrightarrow Y$ be a linear operator, where $\mathscr{D}(T) \subset X$ and X, Y are normed spaces. Then:*

(a) T is continuous if and only if T is bounded.

(b) If T is continuous at a single point, it is continuous.

Corollary. *Let T be a bounded linear operator. Then:*

(a) $x_n \longrightarrow x$ [where $x_n, x \in \mathscr{D}(T)$] implies $Tx_n \longrightarrow Tx$.

(b) The null space $\mathscr{N}(T)$ is closed.

The **restriction** of an operator $T : \mathscr{D}(T) \longrightarrow Y$ to a subset $B \subset \mathscr{D}(T)$ is denoted by $T|_B$ and is the operator defined by

$$T|_B : B \longrightarrow Y, \qquad T|_B x = Tx \text{ for all } x \in B.$$

An **extension** of T to a set $M \supset \mathscr{D}(T)$ is an operator

$$\tilde{T} : M \to Y \quad \text{such that} \quad \tilde{T}|_{\mathscr{D}(T)} = T,$$

that is, $\tilde{T}x = Tx$ for all $x \in \mathscr{D}(T)$.

Theorem. *Let $T : \mathscr{D}(T) \longrightarrow Y$ be a bounded linear operator, where $\mathscr{D}(T)$ lies in a normed space X and Y is a Banach space. Then T has an extension*

$$\tilde{T} : \overline{\mathscr{D}(T)} \to Y$$

where $\tilde{T}$ is a bounded linear operator of norm

$$\|\tilde{T}\| = \|T\|.$$

Linear Functionals

A **functional** is an operator whose range lies on the real line R or in the complex plane C.

Definition. A **linear functional** f is a linear operator with domain in a vector space X and range in the scalar field K of X; thus,

$$f : \mathscr{D}(f) \longrightarrow K,$$

where $K = R$ if X is real and $K = C$ if X is complex.

Definition. A **bounded linear functional** f is a bounded linear operator with range in the scalar field of the normed space X in which the domain $\mathscr{D}(f)$ lies. Thus there exists a real number c such that for all $x \in \mathscr{D}(f)$,

$$|f(x)| \leq c\|x\|.$$

Furthermore, the **norm** of f is

$$\|f\| = \sup_{\substack{x \in \mathscr{D}(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|}$$

or

$$\|f\| = \sup_{\substack{x \in \mathscr{D}(f) \\ \|x\|=1}} |f(x)|.$$

Theorem. *A linear functional f with domain $\mathscr{D}(f)$ in a normed space is continuous if and only if f is bounded.*

Here are some examples:

1. **Norm.** The norm $\|\cdot\| : X \longrightarrow R$ on a normed space $(X, \|\cdot\|)$ is a functional on X which is not linear.
2. **Dot product.** The dot product with one factor kept fixed defines a functional $f : R^3 \longrightarrow R$ by means of

   $$f(x) = x \cdot a = \xi_1\alpha_1 + \xi_2\alpha_2 + \xi_3\alpha_3,$$

   where $a = (\alpha_j) \in R^3$ is fixed. f is linear and bounded. In fact,

   $$|f(x)| = |x \cdot a| \leq \|x\|\|a\|,$$

   so that $\|f\| \leq \|a\|$. On the other hand, by taking $x = a$, we have

   $$\|f\| \geq \frac{|f(a)|}{\|a\|} = \frac{\|a\|^2}{\|a\|} = \|a\|.$$

   Hence, the norm of f is $\|f\| = \|a\|$.
3. **Definite integral.** Let the integral becomes a functional on the space $C[a, b]$. Then f is defined by

$$f(x)=\int_a^b x(t)dt \qquad x\in C[a,b].$$

f is linear. We prove that f is bounded and has norm $\|f\|=b-a$. (the norm of x is infinite norm)

In fact, writing $J=[a,b]$ and remembering the norm on $C[a,b]$, we obtain

$$|f(x)|=\left|\int_a^b x(t)dt\right|\le(b-a)\max_{t\in J}|x(t)|=(b-a)\|x\|.$$

Taking the supremum over all x of norm 1, we obtain $\|f\|\le b-a$. To get $\|f\|\ge b-a$, we choose the particular $x=x_0=1$, note that $\|x_0\|=1$ and use:

$$\|f\|\ge\frac{|f(x_0)|}{\|x_0\|}=|f(x_0)|=\int_a^b dt=b-a.$$

4. **Space $C[a,b]$.** Another important functional on $C[a,b]$ is obtained if we choose a fixed $t_0\in J=[a,b]$ and set

$$f_1(x)=x(t_0) \qquad x\in C[a,b].$$

f_1 is linear and bounded and has norm $\|f_1\|=1$. In fact, we have

$$|f_1(x)|=|x(t_0)|\le\|x\|,$$

and this implies $\|f_1\|\le 1$. On the other hand, for $x_0=1$ we have $\|x_0\|=1$ and obtain

$$\|f_1\|\ge|f_1(x_0)|=1.$$

5. **Space l^2.** We can obtain a linear functional f on the Hilbert space l^2 by choosing a fixed $a=(\alpha_j)\in l^2$ and setting

$$f(x)=\sum_{j=1}^{\infty}\xi_j\alpha_j$$

where $x=(\xi_j)\in l^2$. This series converges absolutely and f is bounded, since the Cauchy-Schwarz inequality gives

$$|f(x)|=|\sum\xi_j\alpha_j|\le\sum|\xi_j\alpha_j|\le\sqrt{\sum|\xi_j|^2}\sqrt{\sum|\alpha_j|^2}=\|x\|\|a\|.$$

It is of basic importance that the set of all linear functionals defined on a vector space X can itself be made into a vector space. This space is denoted by X^* and is called the **algebraic dual space** of X. Its algebra operations of vector space are defined in a natural way as follows. The sum f_1+f_2 of two functionals f_1 and f_2 is the functional s whose value at every $x\in X$ is

$$s(x)=(f_1+f_2)(x)=f_1(x)+f_2(x);$$

the product αf of a scalar α and a functional f is the functional p whose value at $x\in X$ is

$$p(x)=(\alpha f)(x)=\alpha f(x).$$

We may go a step further and consider the algebraic dual $(X^*)^*$ of X^*, whose elements are the linear functionals defined on X^*. We denote $(X^*)^*$ by X^{**} and call it the **second algebraic dual space** of X.

We choose the following notations:

Space	General element	Value at a point
X	x	$-$
X^*	f	$f(x)$
X^{**}	g	$g(f)$

We can obtain a $g\in X^{**}$, which is a linear functional defined on X^*, by choosing a fixed $x\in X$ and setting

$$g(f)=g_x(f)=f(x) \qquad (x\in X \text{ fixed}, f\in X^* \text{ variable}).$$

The subscript x is a little reminder that we got g by the use of a certain $x\in X$. The reader should observe carefully that here f is the variable whereas x is fixed. It's clear that g_x is linear:

$$g_x(\alpha f_1+\beta f_2)=(\alpha f_1+\beta f_2)(x)=\alpha f_1(x)+\beta f_2(x)=\alpha g_x(f_1)+\beta g_x(f_2).$$

So g_x is an element of X^{**}, by the definition of X^{**}.

To each $x \in X$ there corresponds a $g_x \in X^{**}$. This defines a mapping

$$\begin{aligned} C : X &\longrightarrow X^{**} \\ x &\longmapsto g_x. \end{aligned}$$

C is called the **canonical mapping** (or **canonical embedding**) of X into X^{**}. C is linear since its domain is a vector space and we have

$$\begin{aligned} (C(\alpha x + \beta y))(f) &= g_{\alpha x + \beta y}(f) \\ &= f(\alpha x + \beta y) \\ &= \alpha f(x) + \beta f(y) \\ &= \alpha g_x(f) + \beta g_y(f) \\ &= \alpha (Cx)(f) + \beta (Cy)(f). \end{aligned}$$

Isomorphism

In our work we are concerned with various spaces. Common to all of them is that they consist of a set, call it X, and a "structure" defined on X. For a metric space, this is the metric. For a vector space, the two algebraic operations form the structure. And for a normed space the structure consists of those two algebraic operations and the norm.

Given two spaces X and $\tilde{X}$ of the same kind (for instance, two vector spaces), it is of interest to know whether X and $\tilde{X}$ are "essentially identical," that is, whether they differ at most by the nature of their points. Then we can regard X and $\tilde{X}$ as identical-as two copies of the same "abstract" space-whenever the structure is the primary object of study, whereas the nature of the points does not matter. This situation occurs quite often. It suggests the concept of an **isomorphism**. By definition, this is a bijective mapping of X onto $\tilde{X}$ which preserves the structure.

Accordingly, an *isomorphism* T *of a metric space* $X = (X, d)$ onto a metric space $\tilde{X} = (\tilde{X}, \tilde{d})$ is a bijective mapping which preserves distance, that is, for all $x, y \in X$,

$$\tilde{d}(Tx, Ty) = d(x, y).$$

$\tilde{X}$ is then called *isomorphic* with X.

An *isomorphism* T *of a vector space* X onto a vector space $\tilde{X}$ over the same field is bijective mapping which preserves the two algebraic operations of vector space; thus, for all $x, y \in X$ and scalars α,

$$T(x + y) = Tx + Ty, \qquad T(\alpha x) = \alpha Tx,$$

that is, $T : X \longrightarrow \tilde{X}$ is a bijective linear operator. $\tilde{X}$ is then called isomorphic with X, and X and $\tilde{X}$ are called *isomorphic vector spaces.*

Isomorphisms for normed spaces are vector space isomorphisms which also preserve norms: a bijective linear operator $T : X \longrightarrow \tilde{X}$ preserves the norm, that is, for all $x \in X$,

$$\|Tx\| = \|x\|.$$

It can be shown that *the canonical mapping* C *is injective.* Since C is linear (see before), it is an isomorphism of X onto the range $\mathscr{R}(C) \subset X^{**}$.

If X is isomorphic with a subspace of a vector space Y, we say that X is **embeddable** in Y. Hence X is embeddable in X^{**}, and this is why C is also called the canonical embedding of X into X^{**}.

If C is surjective (hence bijective), so that $\mathscr{R}(C) = X^{**}$, then X is said to be **algebraically reflexive**.

Theorem. Let X be an n-dimensional vector space and $E = \{e_1, \cdots, e_n\}$ a basis for X. Then $F = \{f_1, \cdots, f_n\}$ given by

$$f_k(e_j) = \delta_{jk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

is a basis for the algebraic dual X^* of X, and $\dim X^* = \dim X = n$.

Lemma (Zero vector). Let X be a finite dimensional vector space. If $x_0 \in X$ has the property that $f(x_0) = 0$ for all $f \in X^*$, then $x_0 = 0$.

Theorem. *If X is finite dimensional, then X is algebraically reflexive.*

Sketch of proof. It follows almost directly from the theorem (inverse operator) and the lemma (zero vector). ■

Normed Dual Space

We take any two normed spaces X and Y (both real or both complex) and consider the set

$$B(X,Y)$$

consisting of all bounded linear operators from X to Y. We want to show that $B(X,Y)$ can itself be made into a normed space.

First of all, $B(X,Y)$ becomes a vector space if we define sum $T_1 + T_2$ of two operators $T_1, T_2 \in B(X,Y)$ in a natural way by

$$(T_1 + T_2)x = T_1 x + T_2 x$$

and the product αT of $T \in B(X,Y)$ and a scalar α by

$$(\alpha T)x = \alpha T x.$$

Now we introduce the norm:

Theorem. *The vector space $B(X,Y)$ of all bounded linear operators from a normed space X into a normed space Y is itself a normed space with norm defined by*

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\|=1}} \|Tx\|.$$

Then a central question is in what case will $B(X,Y)$ be a Banach space, which is answered in the following theorem. It's remarkable that the condition in the theorem does not involve X; that is, X may or may not be complete.

Theorem. *If Y is a Banach space, then $B(X,Y)$ is a Banach space.*

Proof. We consider an arbitrary Cauchy sequence (T_n) in $B(X,Y)$ and show that (T_n) converges to an operator $T \in B(X,Y)$. Since (T_n) is Cauchy, for every $\varepsilon > 0$ there is an N such that

$$\|T_n - T_m\| < \varepsilon \qquad (m, n > N).$$

For all $x \in X$ and $m, n > N$ we thus obtain

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\| \|x\| < \varepsilon \|x\|.$$

Now for any fixed x and given $\tilde{\varepsilon}$ we may choose $\varepsilon = \varepsilon_x$ so that $\varepsilon_x \|x\| < \tilde{\varepsilon}$. Then we have $\|T_n x - T_m x\| < \tilde{\varepsilon}$ and see that $(T_n x)$ is Cauchy in Y. Since Y is complete, $(T_n x)$ converges, say, $T_n x \longrightarrow y$. Clearly, the limit $y \in Y$ depends on the choice of $x \in X$. This defines an operator $T : X \longrightarrow Y$, where $y = Tx$. The operator T is linear since

$$\lim T_n(\alpha x + \beta z) = \lim(\alpha T_n x + \beta T_n z) = \alpha \lim T_n x + \beta \lim T_n z.$$

We prove that T is bounded and $T_n \longrightarrow T$, that is, $\|T_n - T\| \longrightarrow 0$. Using the continuity of the norm, we then obtain for every $n > N$ and all $x \in X$

$$\|T_n x - Tx\| = \|T_n x - \lim_{m \to \infty} T_m x\| = \lim_{m \to \infty} \|T_n x - T_m x\| \leq \varepsilon \|x\|.$$

This shows that $(T_n - T)$ with $n > N$ is a bounded linear operator. Since T_n is bounded, $T = T_n - (T_n - T)$ is bounded, that is, $T \in B(X,Y)$. Furthermore, if we take the supremum over all x of norm 1, we obtain

$$\|T_n - T\| \leq \varepsilon \qquad (n > N)$$

Hence $\|T_n - T\| \longrightarrow 0$. ■

This theorem has an important consequence with respect to the dual space X' of X, which is defined as follows.

Definition. Let X be a normed space. Then the set of all bounded linear functionals on X constitutes a normed space with norm defined by

$$\|f\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\|=1}} |f(x)|$$

which is called the **dual space** of X and is denoted by X'.

Since a linear functional on X maps X into a scalar field (R or C), and since R or C is complete, we see that X' is $B(X, Y)$ with the complete space Y. Hence, the above theorem is applicable and implies the basic.

Theorem. *The dual space X' of a normed space X is a Banach space (whether or not X is).*

It is a fundamental principle of functional analysis that investigations of spaces are often combined with those of the dual spaces. For this reason it is worthwhile to consider some of the more frequently occurring spaces and find out what their duals look like:

1. **Space R^n.** The dual space of R^n is R^n.
2. **Space l^1.** The dual space of l^1 is l^∞.
3. **Space l^p.** The dual space of l^p is l^q; here, $1 < p < \infty$ and q is the conjugate of p, that is, $1/p + 1/q = 1$.