Normed Spaces and Banach Spaces

Vector Space

Definition. A **vector space** (or **linear space**) over a field K is a nonempty set X of elements x, y, ... (called **vectors**) together with two algebraic operations. These operations are called vector addition and multiplication of vectors by scalars, that is, by elements of K.

Vector addition associates with every ordered pair (x, y) of vectors a vector $x + y$, called the sum of x and y, in such a way that the following properties hold! Vector addition is commutative and associative, that is, for all vectors we have

$$x + y = y + x$$
$$x + (y + z) = (x + y) + z;$$

furthermore, there exists a vector 0, called the **zero vector**, and for every vector x there exists a vector $-x$, such that for all vectors we have

$$x + \mathbf{0} = x$$
$$x + (-x) = \mathbf{0}.$$

Multiplication by scalars associates with every vector x and scalar α a vector αx, called the product of α and x, in such a way that for all vectors x, y and scalars a, b we have

$$\alpha(\beta x) = (\alpha\beta)x$$
$$1x = x$$

and the distributive laws

$$\alpha(x + y) = \alpha x + \alpha y$$
$$(\alpha + \beta)x = \alpha x + \beta x.$$

Normed Space, Banach Space

Definition. A **normed space** X is a vector space with a norm defined on it. A **Banach space** is a complete normed space (complete in the metric defined by the norm). Here a **norm** on a vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ and which has the properties:

(N1) $\|x\| \geq 0$.

(N2) $\|x\| = 0 \quad \Leftrightarrow \quad x = 0$.

(N3) $\|\alpha x\| = |\alpha|\|x\|$.

(N4) $\|x + y\| \leq \|x\| + \|y\|$ (*Triangle inequality*).

A norm on X naturally defines a metric d on X which is given by

$$d(x, y) = \|x - y\|$$

and is called the **metric induced by the norm**. The normed space just defined is denoted by $(X, \|\cdot\|)$ or simply by X.

Note that

$$|\|y\| - \|x\|| \leq \|y - x\|.$$

And this implies that: *The norm is continuous, i.e.* $x \mapsto \|x\|$ *is a continuous mapping of* $(X, \|\cdot\|)$ *into* R.

A metric d induced by a norm satisfies translation invariance:

$$d(x + a, y + a) = d(x, y)$$
$$d(\alpha x, \alpha y) = |\alpha| d(x, y).$$

Here are some useful spaces:

1. *Euclidean space R^n and unitary space C^n are Banach spaces with norm defined by*

$$\|x\| = (\sum_{j=1}^{n} |\xi_j|^2)^{1/2} = \sqrt{|\xi_1|^2 + \cdots + |\xi_n|^2}.$$

2. *Space l^p is a Banach space with norm defined by*

$$\|x\| = (\sum_{j=1}^{\infty} |\xi_j|^p)^{1/p}.$$

3. *Space l^∞ is a Banach space with norm defined by*

$$\|x\| = \sup_j |\xi_j|.$$

4. *Space $C[a, b]$ is a Banach space with norm defined by*

$$\|x\| = \max_{t \in [a,b]} |x(t)|,$$

but is incomplete with norm defined by

$$\|x\| = \int_0^1 |x(t)| dt.$$

5. The vector space of all continuous real-valued functions on $[a, b]$ forms a normed space X with norm defined by

$$\|x\| = \left(\int_a^b x(t)^2 dt \right)^{1/2}.$$

This space is not complete. The space X can be completed by the theorem introduced in previous chapter. The completion is denoted by $L^2[a, b]$. This is a Banach space.

More generally, for any fixed real number $p \geq 1$, the Banach space $L^p[a, b]$ is the completion of the normed space which consists of all continuous real-valued functions on $[a, b]$, as before, and the norm defined by

$$\|x\|_p = \left(\int_a^b |x(t)|^p dt \right)^{1/p}.$$

Definition. A **subspace** Y of a normed space X is a subspace of X considered as a vector space, with the norm obtained by restricting the norm on X to the subset Y. This norm on Y is said to be induced by the norm on X. If Y is closed in X, then Y is called a **closed subspace** of X.

Theorem. A subspace y of a Banach space X is complete if and only if the set Y is closed in X.

In a normed space we may go an important step further and use series as follows.

If (x_k) is a sequence in a normed space X, we can associate with (x_k) the sequence (s_n) of **partial sums**

$$s_n = x_1 + x_2 + \cdots + x_n$$

where $n = 1, 2, \cdots$. If (s_n) is convergent, say,

$$s_n \to s, \quad \text{that is,} \quad \|s_n - s\| \to 0,$$

then the **infinite series** or, briefly, **series**

$$\sum_{k=1}^{\infty} x_k = x_1 + x_2 + \cdots$$

is said to **converge** or to be **convergent**, s is called the sum of the series and we write

$$s = \sum_{k=1}^{\infty} x_k = x_1 + x_2 + \cdots$$

If $\|x_1\| + \|x_2\| + \cdots$ converges, the above series is said to be **absolutely convergent**.

In a normed space X, absolute convergence implies convergence if and only if X is complete.

Definition. If a normed space X contains a sequence (e_n) with the property that for every $x \in X$ there is a unique sequence of scalars (α_n) such that

$$\|x - (\alpha_1 e_1 + \cdots + \alpha_n e_n)\| \to 0 \quad (\text{as } n \to \infty)$$

then (e_n) is called a **Schauder basis** (or **basis**) for X. The series $\sum_{k=1}^{\infty} \alpha_k e_k$ which has the sum x is then called the **expansion** of x with respect to (e_n), and we write

$$x = \sum_{k=1}^{\infty} \alpha_k e_k.$$

For example, l^p has a basis $e_n = (\delta_{nj})$. Thus,

$$\begin{aligned} e_1 &= (1, 0, 0, 0, \cdots) \\ e_2 &= (0, 1, 0, 0, \cdots) \\ e_3 &= (0, 0, 1, 0, \cdots) \end{aligned}$$

etc.

If a normed space X has a Schauder basis, then X is separable. But conversely, a separable normed space may not have a Schauder basis according to the counterexample constructed by P. Enflo (1973).

Theorem. *Let $X = (X, \|\cdot\|)$ be a normed space. Then there is a Banach space $\hat{X}$ and an isometry A from X onto a subspace W of $\hat{X}$ which is dense in $\hat{X}$. The space $\hat{X}$ is unique, except for isometries.*

Finite Dimension

Lemma (Bound for linear combinations). *Let $\{x_1, \cdots, x_n\}$ be a linearly independent set of vectors in a normed space X. Then there is a number $c > 0$ such that for every choice of scalars $\alpha_1, \cdots, \alpha_n$, we have*

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|) \qquad (c > 0).$$

Theorem. *Every finite dimensional subspace Y of a normed space X is complete. In particular, every finite dimensional normed space is complete.*

Sketch of Proof. Consider the basis of Y and make use of the lemma (bound for linear combinations). ■

Theorem. *Every finite dimensional subspace Y of a normed space X is closed in X.*

Definition. A norm $\|\cdot\|$ on a vector space X is said to be **equivalent** to a norm $\|\cdot\|_0$ on X if there are positive numbers a and b such that for all $x \in X$ we have

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0.$$

In fact, *equivalent norms on X define the same topology for X.*

Theorem. *On a finite dimensional vector space X, any norm $\|\cdot\|$ is equivalent to any other norm $\|\cdot\|_0$.*

Sketch of Proof. Consider the basis of X and make use of the lemma (bound of linear combinations). ■

This theorem is of considerable practical importance. For instance, it implies that convergence or divergence of a sequence in a finite dimensional vector space does not depend on the particular choice of a norm on that space.

Compactness

Definition. A metric space X is said to be **compact** if every sequence in X has a convergent subsequence. A subset M of X is said to be **compact** if M is compact considered as a subspace of X, that is, if every sequence in M has a convergent subsequence whose limit is an element of M.

Lemma. *A compact subset M of a metric space is closed and bounded.*

The converse of this lemma is in general false.

However, for a finite dimensional normed space, we do have:

Theorem. *In a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded.*

Sketch of the Proof. Compactness implies closedness and boundedness by the above lemma. For the converse, consider the basis of X and make use of the lemma (bound of linear combinations). ∎

F. Riesz's Lemma. *Let Y and Z be subspaces of a normed space X (of any dimension), and suppose that Y is closed and is a proper subset of Z. Then for every real number θ in the interval $(0, 1)$ there is a $z \in Z$ such that*

$$\|z\| = 1, \quad \|z - y\| \geq \theta \text{ for all } y \in Y.$$

Proof. We consider any $v \in Z - Y$ and denote its distance from Y by a, i.e.

$$a = \inf_{y \in Y} \|v - y\|.$$

Clearly, $a > 0$ since Y is closed. We now take any $\theta \in (0, 1)$. By the definition of an infimum there is a $y_0 \in Y$ such that

$$a \leq \|v - y_0\| \leq \frac{a}{\theta}.$$

Let

$$z = c(v - y_0) \quad \text{where} \quad c = \frac{1}{\|v - y_0\|}.$$

Then $\|z\| = 1$, and we show that

$$\begin{aligned} \|z - y\| &= \|c(v - y_0) - y\| \\ &= c\|v - y_0 - c^{-1}y\| \\ &= c\|v - y_1\| \end{aligned}$$

where $y_1 = y_0 + c^{-1}y \in Y$.

Thus

$$\|z - y\| = c\|v - y_1\| \geq ca = \frac{a}{\|v - y_0\|} \geq \frac{a}{a/\theta} = \theta.$$

Since $y \in Y$ was arbitrary, this completes the proof. ∎

Theorem. *If a normed space X has the property that the closed unit ball $M = \{x | \|x\| \leq 1\}$ is compact, then X is finite dimensional.*

Proof. We assume that M is compact but $\dim X = \infty$, and show that this leads to a contradiction. We choose any x_1 of norm 1. This x_1 generates a one dimensional subspace X_1 of X, which is closed and is a proper subspace of X. By Riesz's lemma, there is a n $x_!$ $\in X$ of norm 1 such that

$$\|x_2 - x_1\| \geq \theta = \frac{1}{2}.$$

The elements x_1, x_2 generate a two dimensional proper closed subspace X_2 of X. By Riesz's lemma there is an x_3 of norm 1 such that for all $x \in X_2$ we have

$$\|x_3 - x\| \geq \frac{1}{2}.$$

In particular,

$$\|x_3 - x_1\| \geq \frac{1}{2},$$

$$\|x_3 - x_2\| \geq \frac{1}{2}.$$

Proceeding by induction, we obtain a sequence (x_n) of elements $x_n \in M$ such that

$$\|x_m - x_n\| \geq \frac{1}{2} \quad (m \neq n).$$

Obviously, (x_n) cannot have a convergent subsequence. This contradicts the compactness of M. Hence our assumption $\dim X = \infty$ is false, and $\dim X < \infty$. ■

Compact sets are important since they are "well-behaved": they have several basic properties similar to those of finite sets and not shared by noncompact sets. In connection with continuous mappings a fundamental property is that compact sets have compact images, as follows.

Theorem. *Let X and Y be metric spaces and $T : X \longrightarrow Y$ a continuous mapping. Then the image of a compact subset M of X under T is compact.*

Theorem. *A continuous mapping T of a compact subset M of a metric space X into R assumes a maximum and a minimum at some points of M.*

Linear Operators

Definition. A linear operator T is an operator such that

(i) the domain $\mathscr{D}(T)$ of T is a vector space and the range $\mathscr{R}(T)$ lies in a vector space over the same field.

(ii) for all $x, y \in \mathscr{D}(T)$ and scalars α,

$$T(x + y) = Tx + Ty,$$

$$T(\alpha x) = \alpha Tx.$$

Observe the notation: we write Tx instead of $T(x)$; $\mathscr{D}(T)$ denotes the domain of T; $\mathscr{R}(T)$ denotes the range of T; $\mathscr{N}(T)$ denotes the **null space** of T (the set of all $x \in \mathscr{D}(T)$ such that $Tx = 0$).

We shall now consider some basic examples of linear operators:

1. **Identity operator.** The identity operator $I_X : X \longrightarrow X$ is defined by $I_X x = x$ for all $x \in X$. We also write simply I for I_X; thus, $Ix = x$.
2. **Zero operator.** The zero operator $\mathbf{0} : X \longrightarrow Y$ is defined by $\mathbf{0}x = 0$ for all $x \in X$.
3. **Differentiation.** Let X be the vector space of all polynomials on $[a, b]$. We may define a linear operator T on X by setting

   $$Tx(t) = x'(t)$$

   for every $x \in X$, where the prime denotes the differentiation with respect to t. This operator T maps X onto itself.
4. **Integration.** A linear operator T from $C[a, b]$ into itself can be defined by

   $$Tx(t) = \int_a^t x(\tau) d\tau \qquad t \in [a, b].$$
5. **Multiplication by t.** A linear operator T from $C[a, b]$ into itself is defined by

   $$Tx(t) = tx(t).$$
6. **Elementary vector algebra.** The cross product with one factor kept fixed defines a linear operator $T_1 : R^3 \longrightarrow R^3$. Similarly, the dot product with one fixed factor defines a linear operator $T_2 : R^3 \longrightarrow R$, say,

   $$T_2 x = x \cdot a = \xi_1 \alpha_1 + \xi_2 \alpha_2 + \xi_3 \alpha_3$$

   where $a = (\alpha_j) \in R^3$ is fixed.
7. **Matrices.** A real matrix $A = (\alpha_{jk})$ with r rows and n columns defines an operator $T : R^n \longrightarrow R^r$ by means of

   $$y = Ax$$

   where $x = (\xi_j)$ has n components and $y = (\eta_i)$ has r components and both vectors are written as column vectors beacuse of the usual convention of matrix multiplication.
   If A were complex, it would define a linear operator from C^n into C^r.

Theorem. *Let T be a linear operator. Then $\mathscr{R}(T)$ is a vector space; if $\dim \mathscr{L}(T) = n < \infty$, then $\dim \mathscr{R}(T) \leq n$; the null space $\mathscr{N}(T)$ is a vector space.*

A important consequence of the second claim is that: *linear operators preserve linear dependence.*

Theorem (Inverse operator). *Let T be a linear operator. The inverse $T^{-1} : \mathscr{R}(T) \longrightarrow \mathscr{D}(T)$ exists if and only if*

$$Tx = 0 \qquad \Rightarrow \quad x = 0.$$

If T^{-1} exists, it is a linear operator. Moreover, if T^{-1} exists and $\dim \mathscr{D}(T) = n < \infty$, then $\dim \mathscr{R}(T) = \dim \mathscr{D}(T)$.

Theorem. *Let $T : X \longrightarrow Y$ and $S : Y \longrightarrow Z$ be bijective linear operators, where X, Y, Z are vector spaces. Then the inverse $(ST)^{-1} : Z \longrightarrow X$ of the product ST exists, and*

$$(ST)^{-1} = T^{-1}S^{-1}.$$

Definition. Let X and Y be normed spaces and and $T : \mathscr{D}(T) \longrightarrow Y$ a linear operator, where $\mathscr{D}(T) \subset X$. The operator T is said to be **bounded** if there is a real number c such that for all $x \in \mathscr{L}(T)$,

$$\|Tx\| \leq c\|x\|.$$

Definition. Let T be a bounded linear operator. Then the **norm** of T is defined as

$$\|T\| = \sup_{\substack{x \in \mathscr{D}(T) \\ \|x\| \neq 0}} \frac{\|Tx\|}{\|x\|}.$$

It's clear that the norm defined by the above formula satisfies (N1) to (N4).

An alternative formula for the norm of T is

$$\|T\| = \sup_{\substack{x \in \mathscr{D}(T) \\ \|x\| = 1}} \|Tx\|.$$

Let's take a look at some typical examples:

1. **Identity operator.** The identity operator $I : X \longrightarrow X$ on a normed space $X \neq \{0\}$ is bounded and has norm $\|I\| = 1$.
2. **Zero operator.** The zero operator $\mathbf{0} : X \longrightarrow Y$ on a normed space X is bounded and has norm $\|\mathbf{0}\| = 0$.
3. **Differentiation operator.** Let X be the normed space of all polynomials on $J = [0, 1]$ with norm given $\|x\| = \max |x(t)|$, $t \in J$. A differentiation operator T is defined on X by
$$Tx(t) = x'(t).$$
This operator is linear but not bounded. Indeed, let $x_n(t) = t^n$, where $n \in N$. Then $\|x_n\| = 1$ and
$$Tx_n(t) = x_n'(t) = nt^{n-1}$$
so that $\|Tx_n\| = n$ and $\|Tx_n\|/\|x_n\| = n$. Since $n \in N$ is arbitrary, this shows that there is no fixed number c such that $\|Tx_n\|/\|x_n\| \leq c$.
4. **Integral operator.** W e can define an integral operator $T : C[0, 1] \longrightarrow C[0, 1]$ by
$$y = Tx \quad \text{where} \quad y(t) = \int_0^1 k(t, \tau)x(\tau)d\tau.$$
Here k is a given function, which is called the *kernel* of T and is assumed to be continuous on the closed square $G = J \times J$ in the $t\tau$-plane, where $J = [0, 1]$. This operator is linear and bounded.
5. **Matrix.** A real matrix $A = (\alpha_{jk})$ with r rows and n columns defines an operator $T : R^n \longrightarrow R^r$ by means of
$$y = Ax.$$
Then T is bounded.

Theorem. *If a normed space X is finite dimensional, then every linear operator on X is bounded.*

Sketch of Proof. Consider the basis of Y and make use of the lemma (bound for linear combinations). ∎

Theorem. *Let* $T : \mathscr{D}(T) \longrightarrow Y$ *be a linear operator, where* $\mathscr{D}(T) \subset X$ *and* X, Y *are normed spaces. Then:*

(a) T *is continuous if and only if* T *is bounded.*

(b) If T *is continuous at a single point, it is continuous.*

Corollary. *Let* T *be a bounded linear operator. Then:*

(a) $x_n \longrightarrow x$ *[where* $x_n, x \in \mathscr{D}(T)$*] implies* $Tx_n \longrightarrow Tx$.

(b) The null space $\mathscr{N}(T)$ *is closed.*

The **restriction** of an operator $T : \mathscr{D}(T) \longrightarrow Y$ to a subset $B \subset \mathscr{D}(T)$ is denoted by $T|_B$ and is the operator defined by

$$T_{|B} : B \longrightarrow Y, \qquad T|_B x = Tx \text{ for all } x \in B.$$

An **extension** of T to a set $M \supset \mathscr{D}(T)$ is an operator

$$\tilde{T} : M \to Y \quad \text{such that} \quad \tilde{T}|_{\mathscr{D}(T)} = T,$$

that is, $\tilde{T}x = Tx$ for all $x \in \mathscr{D}(T)$.

Theorem. *Let* $T : \mathscr{D}(T) \longrightarrow Y$ *be a bounded linear operator, where* $\mathscr{D}(T)$ *lies in a normed space* X *and* Y *is a Banach space. Then* T *has an extension*

$$\tilde{T} : \overline{\mathscr{D}(T)} \to Y$$

where $\tilde{T}$ *is a bounded linear operator of norm*

$$\|\tilde{T}\| = \|T\|.$$