Metric Space

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Definition. A **metric space** is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(M1) d is real-valued, finite and nonnegative.

(M2) $d(x, y) = 0$ if and only if $x = y$.

(M3) $d(x, y) = d(y, x)$ [*Symmetry*]

(M4) $d(x, y) \leq d(x, z) + d(z, y)$ [*Triangle inequality*]

In fact, the nonnegativity of a metric follows from (M2) to (M4):

$$2d(x, y) = d(x, y) + d(y, x) \geq d(x, x) = 0.$$

Definition. A **subspace** (Y, d) of (X, d) is obtained if we take a subset $Y \subset X$ and restrict d to $Y \times Y$; thus the metric on Y is the restriction

$$\tilde{d} = d|_{Y \times Y}.$$

$\tilde{d}$ is called the metric induced on Y by d.

We shall now list examples of metric spaces (for the rest of the notes, if there is no other explanation, we will adopt these metrics by default for these spaces):

1. **Real line** R with usual metric defined by $d(x, y) = |x - y|$.
2. **Euclidean plane** R^2 with *Euclidean metric* $d(x, y) = ||x - y||_2$.
3. **Euclidean space** R^n, **unitary space** C^n, **complex plane** C. The *Euclidean metric* defined on Euclidean space is
$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + \cdots + (\xi_n - \eta_n)^2},$$
where $x = (\xi_1, \cdots, \xi_n), \quad y = (\eta_1, \cdots, \eta_n)$. The space C^n can define the following metric
$$d(x, y) = \sqrt{|\xi_1 - \eta_1|^2 + \cdots + |\xi_n - \eta_n|^2}.$$
When $n = 1$, this is the complex plane C with usual metric
$$d(x, y) = |x - y|.$$
4. **Sequence space** l^∞. As a set X we take the set of all bounded sequences of complex numbers; that is, every element of X is a complex sequence
$$x = (\xi_1, \xi_2, \cdots) \quad \text{briefly} \quad x = (\xi_j)$$
such that for all $j = 1, 2, \cdots$ we have
$$|\xi_j| \leq c_x,$$
where c_x is a real number which may depend on x, but does not depend on j. We choose the metric defined by
$$d(x, y) = \sup_{j \in N} |\xi_j - \eta_j|.$$
5. **Continuous Functional space** $C[a, b]$. We can choose the metric (infinite norm) defined by
$$d(x, y) = \max_{t \in J} |x(t) - y(t)|.$$
We may also choose the following metric (1-norm) defined by integral
$$\tilde{d}(x, y) = \int_a^b |x(t) - y(t)| dt.$$
6. **Discrete metric space.** We take any set X and on it the so-called discrete metric for X, defined by

$$d(x,x)=0; \quad d(x,y)=1\ (x \neq y).$$

7. **Space** l^p. Let $p \geq 1$ be a fixed real number. By definition, each element in the space l^p is a sequence $x=(\xi_i)=(\xi_1,\xi_2,\cdots)$ of numbers such that $|\xi_1|^p+|\xi_2|^p+\cdots$ converges; thus

$$\sum_{j=1}^{\infty}|\xi_i|^p<\infty$$

and the metric is defined by

$$d(x,y)=(\sum_{j=1}^{\infty}|\xi_j-\eta_j|^p)^{1/p}$$

where $y=(\eta_j)$ and $\sum|\eta_j|^p<\infty$. It satisfies the triangle inequality according to the Minkowski inequality. If we take only real sequences, we get the real space l^p, and if we take complex sequences, we get the complex space l^p.

In the case $p=2$, we have the famous *Hilbert sequence space* l^2 with metric defined by

$$d(x,y)=\sqrt{\sum_{j=1}^{\infty}|\xi_j-\eta_j|^2}.$$

Open Set, Closed Set, Neighbourhood

Definition. Given a point $x_0 \in X$ and a real number $r>0$, we define three types of sets:

(a) $B(x_0;r)=\{x \in X | d(x,x_0)<r\}$ **[Open ball]**

(b) $\tilde{B}(x_0;r)=\{x \in X | d(x,x_0)\leq r\}$ **[Closed ball]**

(c) $S(x_0;r)=\{x \in X | d(x,x_0)=r\}$ **[Sphere]**

In all three cases, x_0 is called the *center* and r the *radius.*

Definition. A subset M of a metric space X is said to be **open** if it contains a ball about each of its points. A subset K of X is said to be **closed** if its complement (in X) is open, that is, $K^C=X-K$ is open.

Definition. An open ball $B(x_o;\varepsilon)$ of radius ε is often called an ε-neighbourhood of X_0. By a **neighbourhood** of x_0 we mean any subset of X which contains an ε-neighbourhood of x_0.

Definition. We call x_0 an **interior point** of a set $M \subset X$ if M is a neighbourhood of x_0. The **interior** of M is the set of all interior points of M and may be denoted by M^0 or $\mathrm{Int}(M)$. $\mathrm{Int}(M)$ is open and the largest open set contained in M.

Let's show that the collection of all open subsets of X, call it $\mathscr{T}$, has the following properties:

(T1) $\varnothing \in \mathscr{T}$, $X \in \mathscr{T}$.

(T2) *The union of any members of* $\mathscr{T}$ *is a member of* $\mathscr{T}$.

(T3) *The intersection of finitely many members of* $\mathscr{T}$ *is a member of* $\mathscr{T}$.

It follows from (T1) to (T3) that we've defined a *topological space* $(X,\mathscr{T})$ to be a set X and a collection $\mathscr{T}$ of subsets of X. In other words, *a metric space is a topological space.*

Definition. Let $X=(X,d)$ and $Y=(Y,\tilde{d})$ be metric spaces. A mapping $T: X \longrightarrow Y$ is said to be **continuous at a point** $x_0 \in X$ if for every $\varepsilon>0$ there is a $\delta>0$ such that

$$\tilde{d}(Tx,Tx_0)<\varepsilon \quad \text{for all } x \text{ satisfying} \quad d(x,x_0)<\delta.$$

T is said to be **continuous** if it is continuous at every point of X.

A mapping T *of a metric space* X *into a metric space* Y *is continuous if and only if the inverse image of any open subset of* Y *is an open subset of* X.

Definition. A point X_0 of X (which may or may not be a of M) is called an **accumulation point** of M if every neighbourhood of x_0 contains at least one point $y \in M$ distinct from x_0. The set consisting of the points of M and the accumulation points of M is called the **closure** of M and is denoted by $\bar{M}$. *It's the smallest closed set containing M.*

Definition. A subset M of a metric space X is said to be **dense** in X if $\bar{M} = X$. X is said to be **separable** if it has a countable subset which is dense in X.

Let's consider some important examples mentioned previously:

1. *Real line R is separable.*
2. *Complex plane C is separable.*
3. *A discrete metric space X is separable if and only if X is countable.*
4. *The space l^p with $1 \leq p < +\infty$ is separable.*

Convergence, Cauchy Sequence, Completeness

Definition. A sequence (x_n) in a metric space $X = (X, d)$ is said to **converge** if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

x is called the **limit** of (x_n) and we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \longrightarrow x$.

Definition. We call a nonempty subset $M \subset X$ a **bounded** set if its *diameter* $\delta(M) = \sup\limits_{x,y\in M} d(x, y)$ is finite. And we call a sequence (x_n) in X a **bounded sequence** if the corresponding point set is a bounded set of X.

Let $X = (X, d)$ be a metric space. *A convergent sequence in X is bounded and its limit is unique.* And *if $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in X, then $d(x_n, y_n) \longrightarrow d(x, y)$.*

Definition. A sequence (x_n) in a metric space $X = (X, d)$ is said to be **Cauchy** if for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that

$$d(x_m, x_n) < \varepsilon \quad \text{for every } m, n > N.$$

The space X is said to be **complete** if every Cauchy sequence in X converges.

Every convergent sequence in a metric space is a Cauchy sequence, but the convergence of a Cauchy sequence depend on the completeness of the corresponding space.

The real line and the complex plane are complete metric spaces. The rational line Q and the open interval (a, b) in R are incomplete metric space.

We can prove the following three important theorems:

Theorem. *Let M be a nonempty subset of a metric space (X, d) and $\bar{M}$ its closure. Then:*

(a) $x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \longrightarrow x$.

(b) M is closed if and only if the situation $x_n \in M$, $x_n \longrightarrow x$ implies that $x \in M$.

Theorem. *A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X*
.

Theorem. *A mapping $T\colon X \to Y$ of a metric space (X, d) into a metric space $(Y, \tilde{d})$ is continuous at a point $x_0 \in X$ if and only if*

$$\overrightarrow{x_n} \to x_0 \quad \text{implies} \quad Tx_n \to Tx_0.$$

To prove completeness, we take an arbitrary Cauchy sequence (x_n) in X and show that it converges in X. For different spaces, such proofs may vary in complexity, but they have approximately the same general pattern:

1. Construct an element x (to be used as a limit).
2. Prove that x is in the space considered.

3. Prove convergence $x_n \longrightarrow x$ (in the sense of the metric).

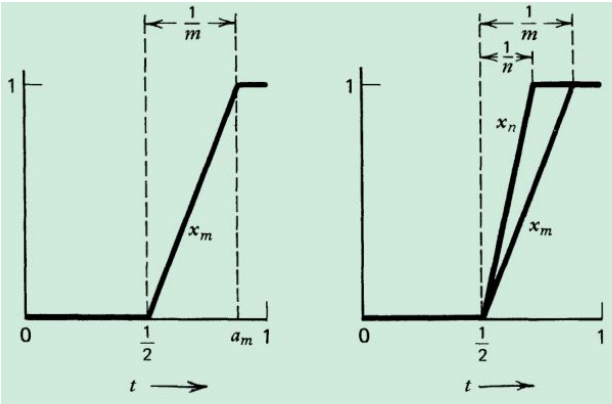
We shall present completeness conclusions for some metric spaces which occur quite frequently:

1. *Euclidean space R^n and unitary space C^n are complete.*
2. *The space l^∞ is complete.*
3. *The space c consisting of all convergent sequences $x = (\xi_j)$ of complex numbers, with the metric induced from the space l^∞ is complete.*
4. *The space l^p is complete, where p is fixed and $1 \le p < +\infty$.*
5. *The continuous function space $C[a, b]$ is complete under infinite-norm metric, but incomplete under 1-norm metric. Here, $[a, b]$ is any given closed interval on R.*

   We give a counterexample for 1-norm metric:

$$x_n(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{2} \\ n(x - \frac{1}{2}) & \frac{1}{2} < t < \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \le t \le 1 \end{cases}$$

   Here is an illustration:



6. *The space Q is incomplete.*

Completion of Metric Spaces

We know that the rational line Q is not complete but can be "enlarged" to the real line R which is complete. And this "completion" R of Q is such that Q is dense in R. It is quite important that an arbitrary incomplete metric space can be "completed" in a similar fashion, as we shall see.

Definition. Let $X = (X, d)$ and $\tilde{X} = (\tilde{X}, d)$ be metric spaces. Then: A mapping T of X into $\tilde{X}$ is said to be **isometric** or an **isometry** if T preserves distances, i.e. it for all $x, y \in X$,

$$\tilde{d}(Tx, Ty) = d(x, y),$$

where Tx and Ty are the images of x and y, respectively. The space X is said to be **isometric** with the space $\tilde{X}$ if there exists a bijective isometry of X onto $\tilde{X}$. The spaces X and $\tilde{X}$ are then called **isometric spaces**.

We can now state and prove the theorem that every metric space can be completed. The space X occurring in this theorem is called the **completion** of the given space X.

Theorem. *For a metric space $X = (X, d)$ there exists a complete metric space $\hat{X} = (\hat{X}, \hat{d})$ which has a subspace W that is isometric with X and is dense in $\hat{X}$. This space $\hat{X}$ is unique except for isometries, that is, if $\tilde{X}$ is any complete metric space having a dense subspace $\tilde{W}$ isometric with X, then $\tilde{X}$ and $\hat{X}$ are isometric.*

Sketch of Proof. First, we construct $\hat{X} = (\hat{X}, d)$ and an isometry T of X onto W, where $\bar{W} = \hat{X}$. Then we prove the completeness of $\hat{X}$ and the uniqueness of $\hat{X}$ (except for isometries).

Proof. **(a) (Construction of $\hat{X} = (\hat{X}, d)$)** Let (x_n) and (y_n) be Cauchy sequences in X. Define (x_n) to be equivalent to (y_n), written $(x_n) \sim (y_n)$, if

$$\lim_{n \to \infty} d(x_n, x_n') = 0.$$

Let $\hat{X}$ be the set of all equivalence classes $\{\hat{x}, \hat{y}, \cdots\}$ of Cauchy sequences in X. We write $(x_n) \in \hat{x}$ to mean that (x_n) is a member of $\hat{x}$ (a *representative* of the class $\hat{x}$). We are to define a metric on $\hat{X}$:

$$\hat{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n).$$

To show that the above metric definition is valid, we need to ensure that:

1. The limit $\lim_{n\to\infty} d(x_n, y_n)$ always exists and is independent of the particular choice of representatives.
2. $\hat{d}$ satisfies (M1) to (M4), thus being a metric.

These are not difficult to prove.

(b) (Construction of W, dense subspace, isometry) With each $b \in X$, we associate the class $\hat{b} \in \hat{X}$ which contains the constant Cauchy sequence $(b, b, \cdots)$. This defines a mapping $T : X \longrightarrow W$ onto the subspace $W = T(X) \subset \hat{X}$. The mapping T is given by $b \mapsto \hat{b} = Tb$, where $(b, b, \cdots) \in \hat{b}$.

It's not difficult to show that:

1. W is dense in $\hat{X}$.
2. W is isometric with X.

(c) (Completeness of $\hat{X}$) Let $(\hat{x}_n)$ be any Cauchy sequence in $\hat{X}$. Since W is dense in $\hat{X}$, for every $\hat{x}_n$ there is a $\hat{z}_n \in W$ such that

$$\hat{d}(\hat{x}_n, \hat{z}_n) < \frac{1}{n}.$$

Hence, by triangle inequality,

$$\begin{aligned} \hat{d}(\hat{z}_m, \hat{z}_n) &\leq \hat{d}(\hat{z}_m, \hat{x}_m) + \hat{d}(\hat{x}_m, \hat{x}_n) + \hat{d}(\hat{x}_n, \hat{z}_n) \\ &< \frac{1}{m} + \hat{d}(\hat{x}_m, \hat{x}_n) + \frac{1}{n}, \end{aligned}$$

which indicates that $(\hat{z}_n)$ is a Cauchy sequence. Let $\hat{x} \in \hat{X}$ be the class to which $(z_n = T^{-1}\hat{z}_n)$ belongs. Then $\hat{x}$ is the limit of $(\hat{x}_n)$.

Hence the arbitrary Cauchy sequence $(\hat{x}_n)$ in $\hat{X}$ has the limit $\hat{x} \in \hat{X}$, and $\hat{X}$ is complete.

(d) (Uniqueness of $\hat{X}$ except for isometries) If $(\tilde{X}, \tilde{d})$ is another complete metric space with a subspace $\tilde{W}$ dense in $\tilde{X}$ and isometric with X, for any $\tilde{x}, \tilde{y} \in \tilde{X}$ we have sequences $(\tilde{x}_n), (\tilde{y}_n)$ in $\tilde{W}$ such that $\tilde{x}_n \longrightarrow \tilde{x}$ and $\tilde{y}_n \longrightarrow \tilde{y}$; hence

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} \tilde{d}(\tilde{x}_n, \tilde{y}_n)$$

follows from

$$|\tilde{d}(\tilde{x}, \tilde{y}) - \tilde{d}(\tilde{x}_n, \tilde{y}_n)| \leq \tilde{d}(\tilde{x}, \tilde{x}_n) + \tilde{d}(\tilde{y}, \tilde{y}_n) \quad \to \quad 0.$$

Since $\tilde{W}$ is isometric with $W \subset \hat{X}$ and $\bar{W} = \hat{X}$, the distances on $\tilde{X}$ and $\hat{X}$ must be the same. Hence $\tilde{X}$ and $\hat{X}$ are isometric. ■