Calculas of Variations

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The *calculus of variations* is a field of mathematical analysis that uses variations, which are small changes in functions and functionals, to find maxima and minima of *functionals*: mappings from a set of functions to the real numbers.

1 Background

Consider the famous **brachistochrone problem**:

> Find the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip (without friction) from one point to another in the least time.

The time to travel from a point P_1 to another point P_2 is given by the integral:

$$t_{12} = \int_{P_1}^{P_2} \frac{ds}{v},$$

where s is the arc length and v is the speed. The speed at any point is given by a simple application of conservation of energy equating kinetic energy to gravitational potential energy,

$$\frac{1}{2}mv^2 = mgy,$$

giving

$$v = \sqrt{2gy}.$$

Plugging this together with the identity

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}dx,$$

then gives

$$t_{12} = \int_{P_1}^{P_2} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx.$$

So the function to be varied is thus

$$L = \sqrt{\frac{1+y'^2}{2gy}}.$$

Therefore, we are to find a function $y = y(x)$ that minimize the functional t_{12}, or formally

$$\begin{aligned} &\min \ t_{12} \\ &\text{s.t. } \ y(x_1) = y_1, \ y(x_2) = y_2, \ y(x) \in C^2[a, b]. \end{aligned}$$

2 Euler–Lagrange Equation

Consider the *cost functional*

$$J[y] = \int_{x_1}^{x_2} L(x, y(x), y'(x)) dx,$$

where x_1, x_2 are constants, $y(x)$ is twice continuously differentiable and L is twice continuously differentiable w.r.t. its arguments x, y, y'.

If the functional $J[y]$ attains a local minimum at f, and $\eta(x)$ is an arbitrary function that has at least one derivative and vanishes at the endpoints x_1 and x_2, then for any number ε close to 0,

$$J[f] \leq J[f + \varepsilon\eta].$$

Let

$$\Phi(\varepsilon) = J[f + \varepsilon\eta].$$

Now, we've converted the functional extremum problem to a function extremum problem! Since the functional $J[y]$ has a minimum for $y = f$, the function $\Phi(\varepsilon)$ has a minimum at $\varepsilon = 0$ and thus,

$$\Phi'(0) \equiv \frac{d\Phi}{d\varepsilon}|_{\varepsilon=0} = \int_{x_1}^{x_2} \frac{dL}{d\varepsilon}|_{\varepsilon=0} dx = 0.$$

Using the chain rule and integration by parts, we have

$$\begin{aligned}\int_{x_1}^{x_2} \frac{dL}{d\varepsilon}|_{\varepsilon=0} dx &= \int_{x_1}^{x_2} (\frac{\partial L}{\partial f}\eta + \frac{\partial L}{\partial f'}\eta')dx \\ &= \int_{x_1}^{x_2} \frac{\partial L}{\partial f}\eta dx + \frac{\partial L}{\partial f'}\eta|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \frac{d}{dx}\frac{\partial L}{\partial f'}dx \\ &= \int_{x_1}^{x_2} (\frac{\partial L}{\partial f}\eta - \eta\frac{d}{dx}\frac{\partial L}{\partial f'})dx.\end{aligned}$$

Then, according to the fundamental lemma of calculus of variations, the part of the integrand in parentheses is zero, i.e.

$$\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} = 0,$$

which is called the **Euler-Lagrange equation**. The left hand side of this equation is called the *functional derivative* of $J[f]$ and is denoted $\delta J/\delta f(x)$.

In general this gives a second-order ordinary differential equation which can be solved to obtain the extremal function $f(x)$. In practice, if the ode is too difficult to solve, it's common for us to calculate the numerical solution instead (Euler's method, Runge-Kutta method ......).

3 A Sufficient Condition for Minimum

Note that the Euler–Lagrange equation is a necessary, but not sufficient, condition for an extremum $J[f]$. So we give following sufficient condition without proofs.

Sufficient condition for a minimum: The functional $J[y]$ has a minimum at $y = \hat{y}$ if its first variation $\delta J[h] = 0$ at $y = \hat{y}$ and its second variation $\delta^2 J[h]$ is strongly positive at $y = \hat{y}$.