10/1/2020 INFORMATION THEORY | ee18btech11049

HOMEWORK - I

1.2 Prob of head = p out of n coin tossen

1.) $^nC_m \, p^m (1-p)^{n-m}$

2.) $p = 0.3 ; n = 10 ; m \leq 5$

$P(X \leq 5) = (1-p)^{10} + {^nC_1}\, p(1-p)^9 + {^nC_2} \cdot p^2 (1-p)^8 \ldots$

$\ldots$

$= 0.9527$

Markov's inequality:

3) $\mu = np$

$= 10(0.3)$

$= 3$

$a = 6$

$Pr\{X \geq 6\} \leq \frac{3}{6}$

$1 - Pr\{X \geq 6\} \geq 1 - \frac{1}{2}$

$$\boxed{Pr[X < 6] \geq 1/2}$$

Chebyshew's inequality

$Pr(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$

$\sigma^2 = 10(0.3)(0.7) = 2.1$

$\mu = 3$

$Pr\{|X-3| \geq 3\} \leq \frac{2.1}{a^2}$

$a = 3$

$Pr\{X \geq 6\} + Pr[X = 0] \leq \frac{7}{30}$

$Pr[X \geq 6] \leq \frac{7}{30} - (0.7)^{10}$

$1 - Pr[X \geq 6] \geq 1 - 0.205$

$$\boxed{Pr[X < 6] \geq 0.795}$$

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Chernoff bound:-

$$\Pr[X \geq \mu + a] \leq \min \frac{\mathbb{E}\, e^{t(X-\mu)}}{e^{ta}} \qquad \mu = 3, \quad \mu + a = 3$$

$$\Pr[X \geq 3+3] \leq \min_{t>0} \frac{\mathbb{E}\, e^{t(X-3)}}{e^{3t}}$$

$$\leq \min_{t>0} \frac{e^{-3t}\, \mathbb{E}(e^{tX})}{e^{3t}}$$

$$\leq \min_{t>0} e^{-6t} (p e^t + 1 - p)^n$$

$$\leq \min_{t>0} e^{-6t} (0.3 e^t + 1 - 0.3)^{10}$$

$$\leq \min_{t>0} \underbrace{e^{-6t} (0.3 e^t + 0.7)^{10}}_{H(t)}$$

$$\frac{d}{dt} H(t) = e^{-6t} \cdot (-6) (0.3 e^t + 0.7)^{10} + 10 e^{-6t} (0.3 e^t + 0.7)^9 (0.3 e^t)$$

$$= 0$$

$$6 \cancel{e^{-6t}} (0.3 e^t + 0.7)^{10} = 10 \cancel{e^{-6t}} \cancel{(0.3 e^t + 0.7)^9} (0.3 e^t)$$

$$1.8 e^t + 4.2 = 3 e^t$$

$$4.2 = 1.2 e^t \qquad e^t = 3.5$$

$\therefore\ M(t)$ is max at $e^t = 3.5$

i.e $M(t) = \cancel{0.88848}\ 0.1465$

$\therefore \Pr\{x \geq 6\} \leq \cancel{0.858}\ 0.1465$

$1 - \Pr\{x \geq 6\} \geq 1 - 0.14656$

$$\boxed{\Pr\{x < 6\} \geq 0.85345}$$

4) Markov's Ineq: $P\{x \geq a\} \leq \frac{\mu}{a}$ $\quad m = 350$

$\mu = (1009)(0.3) = 300$

$P[x \geq 3\cancel{50}] \leq \frac{\overset{6}{\cancel{300}}}{\underset{7}{\cancel{350}}}$

$P\{x < 3\cancel{50}\} \geq 1 - \cancel{\frac{6}{7}}\ \frac{51}{351} = 1 - \frac{300}{351}$

$P\{x < 351\} \geq 0.145$

Chebyshev's Ineq

$\Pr\{|x - 300| \geq 5\cancel{0}\} \leq \frac{21\cancel{0}}{5\cancel{0} \times 5\cancel{0}} \leq 0.08074$

$\Pr\{x \geq 351\} + \Pr\{x \leq 251\} \leq 0.08074$

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$$-\Pr[x \geq 350] - \Pr[x \leq 250] \geq -0.08074$$

$$+1 \qquad\qquad +1$$

$$\Pr[x < 350] - \Pr[x < 251] \geq 0.91926$$

$$\boxed{\therefore \Pr[x < 350] \geq 0.916}$$

Chernoff bound:-

$$\Pr\left[\sum_{i=1}^{n} x_i \geq np(1+\delta)\right] \leq e^{-\frac{\delta^2 np}{3}}$$

$$\delta = \frac{1}{6}$$

$$np = 300$$

$$\Pr\left[\sum_{i=1}^{n} x_i \geq 350\right] \leq e^{-\left(\frac{1}{6}\right)^2 \times \frac{\overset{100}{\cancel{300}}}{3}}$$

$$\leq 0.06217$$

$$1 - \Pr\left[\sum_{i=1}^{n} x_i \geq 350\right] \geq 1 - 0.06217$$

$$\boxed{\Pr\left[\sum_{i=1}^{n} x_i < 350\right] \geq 0.9378}$$

6. $\sum_{i=0}^{k} {}^nC_i \, p^i (1-p)^{n-i}$ ← Atmost k heads

1.5:

1.) $n > k > l$.

$$n > k$$

$$\frac{1}{n} < \frac{1}{k}$$

$$\frac{l}{n} < \frac{l}{k}$$

$$1 - \frac{l}{n} > 1 - \frac{l}{k}$$

$$\frac{n-l}{n} > \frac{k-l}{k}$$

$$\boxed{\frac{n-l}{k-l} > \frac{n}{k}}$$

2) T.P $\left(\frac{n}{k}\right)^k \leq {}^nC_k \leq \left(\frac{n^k}{k!}\right)$

case I: Proof $\left(\frac{n}{k}\right)^k \leq {}^nC_k$

$$\frac{n}{k} \times \frac{n-1}{(k-1)} \cdot \frac{n-2}{k-2} \cdots \frac{n-k+1}{1} \geq \frac{n}{k} \times \frac{n}{k} \cdots (k \text{ times})$$

$\therefore$ from $\frac{n-l}{k-l} \geq \frac{n}{k}$ in all terms

$$\therefore {}^nC_k \geq \left(\frac{n}{k}\right)^k$$

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case II :-

$\frac{n^k}{k!}$ and ^nC_k

$\frac{n^k}{k!} \geq {}^nC_k$

$\frac{n}{k} \cdot \frac{n}{(k-1)} \cdot \frac{n}{(k-2)} \cdots \frac{n}{1} \geq \frac{n}{k} \frac{(n-1)}{k-1} \cdots \frac{(n-k+1)}{1}$

$\downarrow$

comparing $\frac{n-l}{k-l} \ngtr \frac{n}{k}$

each consiqutive term
numerator > denominator

$\therefore$ Hence proved

3.) $n > k > 0$

Sterling's approx given $\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \leq n! \leq e\, n^{n+1/2} e^{-n}$ —(1)

put $n = k$ and reciprocal $\frac{1}{e\, k^{k+1/2} e^{-k}} \leq \frac{1}{k!} \leq \frac{1}{\sqrt{2\pi}\, k^{k+1/2} e^{-k}}$ —(2)

& $n = (n-k)$

$\therefore \sqrt{2\pi}\, (n-k)^{n+1/2-k} e^{-(n-k)} \leq (n-k)! \leq e\, (n-k)^{n-k+\frac{1}{2}} e^{-(n-k)}$

$\frac{1}{e\,(n-k)^{n-k-1/2} e^{-(n-k)}} \leq \frac{1}{(n-k)!} \leq \frac{1}{\sqrt{2\pi}\,(n-k)^{n-k+1/2} e^{-(n-k)}}$ —(3)

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①×②×③

$$\frac{\sqrt{2\pi}\, n^{n+1/2}\, \cancel{e^{-n}}}{e^2\, k^{k+1/2}\, (n-k)^{n-k-1/2}\, \cancel{e^{-n}}} \le \frac{n!}{(n-k)!\,k!} \le \frac{e\, n^{n+1/2}\, \cancel{e^{-n}}}{(2\pi)\, \cancel{e^{-n}}\, k^{k+1/2}\, (n-k)^{n-k+1/2}}$$

$$\frac{\sqrt{2\pi}}{e2}\; \frac{n^{n+1/2}}{k^{k+1/2}\, n^{n-k+1/2} \left(1-\frac{k}{n}\right)^{n-k+1/2}} \le {}^nC_k \le \frac{e}{2\pi}\; \frac{n^{n+1/2}}{k^{k+1/2}\, n^{n-k+1/2} \left(1-\frac{k}{n}\right)^{n-k-\frac{1}{2}}}$$

$$\frac{\sqrt{2\pi}}{e2}\; \frac{\overset{p^k}{\overbrace{\frac{n^k}{k^k}}}\,(1-p)^{k-n}}{\sqrt{np(1-p)}} \le {}^nC_k \le \frac{e}{2\pi}\; \frac{\overset{p^k}{\overbrace{\frac{n^k}{k^k}}}\,(1-p)^{k-n}}{\sqrt{np(1-p)}}$$

$$\frac{\sqrt{2\pi}}{e2} \cdot \frac{2^{\log_2\left(p^{-k}(1-p)^{k-n}\right)}}{\sqrt{np(1-p)}} \le {}^nC_k \le \frac{e}{2\pi}\; \frac{2^{\log_2\left(p^{-k}(1-p)^{k-n}\right)}}{\sqrt{np(1-p)}}$$

$k = np$

$$\frac{\sqrt{2\pi}}{e2}\; \frac{2^{-k\log_2 p + (k-n)\log_2(1-p)}}{\sqrt{np(1-p)}} \le {}^nC_k < \frac{e}{2\pi}\; \frac{2^{-k\log_2 k + (k-n)\log^{(1-p)}}}{\sqrt{np(1-p)}}$$

$$\frac{\sqrt{2\pi}}{e2}\; \frac{2^{-np\log_2 p - n(1-p)\log_2(1-p)}}{\sqrt{np(1-p)}} \le {}^nC_k \le \frac{e}{2\pi}\; \frac{2^{-np\log P - n(1-p)\log(1-p)}}{\sqrt{np(1-p)}}$$

$$H_2(p) = -p\log_2 p - (1-p)\log_2(1-p)$$

$$\boxed{\frac{\sqrt{2\pi}}{e^2}\frac{2^{nH_2(p)}}{\sqrt{np(1-p)}} \le {}^nC_k \le \frac{e}{2\pi}\frac{2^{nH_2(p)}}{np(1-p)}}$$

ii) To prove:-

$$H_2(p) = \lim_{n\to\infty}\frac{1}{n}\log_2({}^nC_k)$$

Previous result log(on) all

$$\log_2\left(\frac{\sqrt{2\pi}}{e^2}\frac{\sqrt{2\pi}}{e^2\sqrt{np(1-p)}}\right) \le \log_2({}^nC_k) \le \log_2\left(\frac{e}{2\pi}\frac{2^{nH_2(p)}}{\sqrt{np(1-p)}}\right)$$

$\frac{1}{n}$ on all sides

$$\underbrace{\frac{1}{n}\log_2\left(\frac{\sqrt{2\pi}}{e^2\sqrt{np(1-p)}}\right)+\frac{1}{n}\log_2\left(2^{nH_2(p)}\right)}_{1} \le \frac{1}{n}\log {}^nC_k \le \underbrace{\frac{1}{n}\log}_{2}$$

Apply $\lim_{n\to\infty}$ on all sides

case A:

$$1 \Rightarrow \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{\sqrt{2\pi}}{e^2\sqrt{np(1-p)}}\right)+\frac{1}{\cancel{n}}(\cancel{n}\,H_2(p))$$

$= H_2(P)$

case B:

② $\Rightarrow \lim_{n \to \infty} \left(\frac{1}{n} \log_2 \frac{e}{2\pi\sqrt{np(1-p)}} \right) + \cancel{\log}\ H_2(P)$

$= H_2(P)$

∴ from case A & B
we get By Sandwich theorem

$$\boxed{\lim_{n \to \infty} \frac{1}{n} \log_2 \binom{n}{k} = H_2(P)}$$

(1.4.) to show :- CDF is uniformly distributed

$F_x(x) = Pr(X < P)$
↓
∴ Prob has range [0,1]
∴ distributed ~~uniformly~~ in [0,1]

& let $F_Y(y)$ be CDF of $Y = F(X)$ then for any

$y \in [0,1]$ ∴ $F_Y(y) = Pr[Y \leq y] = Pr[F(X) \leq y]$

$= Pr[X \leq F^{-1}(y)]$

$= F(F^{-1}(y)) = y$

$\therefore$ It is continus uniform in $[0,1]$