Control Principles of Complex Networks

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***Abstract*— The control of a network poses challenges that cannot always be answered by classical control theory. Controlling a complex network means steering its state variables using the available control inputs. To define controllability we have the ability to guide a system toward the desired state with a suitable choice of inputs.**

I. CONTROLLABILITY OF LINEAR SYSTEMS

A system is *controllable* if we can drive it from any initial state to any desired final state in finite time. Recent advances in the controllability of complex networks offer mathematical tools to identify the driver nodes, a subset of nodes whose direct control with appropriate signals can control the state of the full system.

A. Linear Time-Invariant Systems

For a large scale analysis, a usual starting point in the literature on control of complex networks is the linear time-invariant (LTI) control system,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1}$$

Consider the LTI dynamics, on a network G(A) of N nodes. The N nodes of the graph is associated a vector of finite size with state variables, $x_i(t) \in \mathbf{R}^n$. The state variable $x_i(t)$ can denote the amount of traffic that passes through a node i on a communication network.
The state matrix $A := (a_{ij})_{N\times N}$ represents the weighted wiring diagram of the underlying network, where a_{ij} is the strength or weight with which node j affects/influences node i: a positive (or negative) a_{ij} means the link $(j \to i)$ is excitatory (or inhibitory), and $a_{ij} = 0$ if node j has no direct influence on node i. Consider M independent control signals $\{u_1, ..., u_M\}$ applied to the network. The input matrix $B := (b_{im})_{N\times M}$ identifies the nodes that are directly controlled, where b_{im} represents the strength of an external control signal $u_m(t)$ injected into node i.
The input signal $u(t) = (u_1(t), ..., u_M(t))^T \in \mathbf{R}^M$ can be imposed on all nodes or only a preselected subset of the nodes. In general the same signal um(t) can drive multiple nodes. The nodes directly controlled by u(t) are called *actuator nodes* or simply *actuators*, like nodes x_1, x_2 and x_5 in Fig. 1. The number of actuators is given by the number of non-zero elements in B. The actuators that do not share input signals, e.g. nodes x_1 and x_2 in Fig. 1, are called *driver nodes* or simply *drivers*.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 \\ a_{31} & 0 & 0 & 0 & 0 \\ a_{41} & 0 & 0 & a_{34} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix} ; \ \mathbf{B} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & b_5 \end{bmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

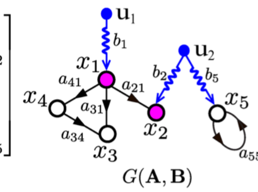

Fig. 1. Graphical representation of a linear time-invariant system.

B. Kalman's Criterion of Controllability

The best known controllability tests is Kalman's rank condition stating that the LTI system (A, B) is controllable if and only if the controllability matrix

$$C = [\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, ..., \mathbf{A}^{N-1}\mathbf{B}] \tag{2}$$

has full rank, i.e.

$$rank\, C = N. \tag{3}$$

If, however, $rank\, C = N$, then we can find an appropriate input vector $\mathbf{u}(t)$ to steer the system from $\mathbf{x}(0)$ to an arbitrary $\mathbf{x}(t)$. Hence, the system is controllable.

Example:

In the network control problem of Fig. 2a the controllability matrix

$$C = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & a_{21}b_1 & 0 \\ 0 & a_{31}b_1 & 0 \end{bmatrix} \tag{4}$$

is always rank deficient, as long as the parameters b_1, a_{21} and a_{31} are non-zero. Hence, the system is uncontrollable. In contrast, for Fig. 2c we have

$$C = \begin{bmatrix} b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & a_{21}b_1 & 0 & 0 & 0 \\ 0 & 0 & a_{31}b_1 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

which has full rank, as long as the parameters b_1, b_2, a_{21} and a_{31} are non-zero. Hence the system is controllable.

II. STRUCTURAL CONTROLLABILITY

For many complex networks a_{ij} (e.g. the elements in A) are not precisely known. Indeed, we are often unable to measure the weights of the links, knowing only whether there is a link or not. Hence, it is hard, if not conceptually impossible, to numerically verify Kalman's rank condition using fixed weights. Structural control, introduced by Lin

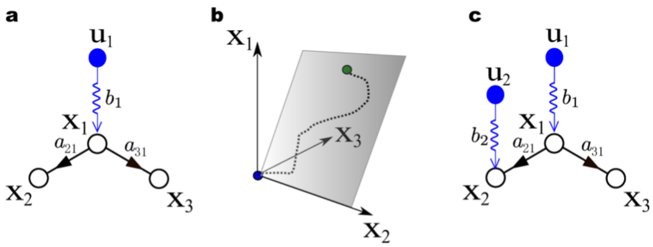

Fig. 2. Controlling star networks.

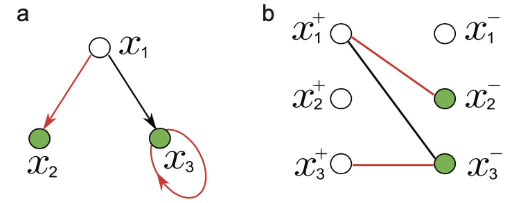

Fig. 3. Maximum Matching.

in 1970s, offers a framework to systematically avoid this limitation.

An LTI system (A,B) is a structured system if the elements in A and B are either fixed zeros or independent free parameters. A and B are called the structured matrices. The system (A,B) is structurally controllable if we can set the nonzero elements in A and B such that the resulting system is controllable in the usual sense (i.e., $rank C = N$).
This leads to the construction of the state matrix to show the absence of the link with zero and its existence with only one.

III. Minimum Inputs Theorem

In this section, we prove that in order to fully control a network G(A) the minimum number of input vertices (or equivalently the minimum number of driver nodes) we need is related to the size of maximum matching in the corresponding digraph G(A), which is one of our key results. To achieve this, we first generalize the concept of matching in undirected graph to digraph.

Definition: For an undirected graph, a matching M is an independent edge set, i.e. a set of edges without common vertices. A vertex is matched if it is incident to an edge in the matching. Otherwise the vertex is unmatched.

A matching of maximum cardinality/size is called a maximum matching. (Note that in general there could be many different maximum matchings for a given graph or digraph.) A maximum matching is called perfect if all vertices are matched. For example, in a directed elementary cycle, all vertices are matched. Note that a maximum matching of a digraph G(A) can be easily found in its bipartite representation, denoted as H(A). See Fig. 3. The bipartite graph is defined in the following way. $H(A) = (V^+ \cup V^-, \Gamma)$. Here, $V_A^+ = \{x_1^+, ..., x_N^+\}$ and $V^- = \{x_1^-, ..., x_N^-\}$ are the set of vertices corresponding to the N columns and rows of the state matrix A, respectively. Edge set $\Gamma = \{(x^+, x^-) | a \neq 0\}$. For a general bipartite graph, its maximum matching can be found efficiently using the well-known Hopcroft-Karp algorithm, which runs in $O(\sqrt{V}E)$ time .

IV. Control Energy

Indentifying the minimum number of driver or actuator nodes sufficient for control is only the first step of the control problem.

Consider the LTI system (1) driven from an arbitrary initial state x_i towards a desired final state x_f by the external signal $u(t)$ in the time interval $t \in [0,T]$. We define the associated control effort in the quadratic form

$$\mathscr{E} \equiv \int_0^{t_f} \|\mathbf{u}(t)\|^2 dt, \tag{6}$$

called the "control energy" in the literature.

For a fixed set of driver nodes the control input $u(t)$ that can drive the system from x_i to x_f can be chosen in many different ways, resulting in different trajectories followed by the system. Each of these trajectories has its own control energy. Of all the possible inputs, the one that yields the minimum control energy is

$$\mathbf{u}(t) = \mathbf{B}^T e^{\mathbf{A}^T (t_f - t)} W_B^{-1}(T) \mathbf{x}_f \tag{7}$$

where $\mathbf{W}_B(t)$ is the *gramian matrix* and $x_f = e^{At_f} x_0$.

$$\mathbf{W_B}(t) \equiv \int_0^{t_f} exp(\mathbf{A}\tau)\mathbf{B}\mathbf{B}^\mathbf{T} exp(\mathbf{A}^\mathbf{T}\tau) d\tau \tag{8}$$

Without loss of generality, we can set the final state at the origin, $x_f = 0$, and write the control energy as

$$\mathscr{E} = \mathbf{x}_0^T \mathbf{H}^{-1}(t_f) \mathbf{x}_0 \tag{9}$$

where $\mathbf{H}(t_f) = e^{-At_f} W_B e^{-A^T t_f}$ is the symmetric Gramian matrix.

Using the Rayleigh-Ritz theorem, the normalized control energy obeys the bounds

$$\eta_{max}^{-1} \equiv E_{min} \leq E(T) \leq E_{max} \equiv \eta_{min}^{-1} \tag{10}$$

where η_{max} and η_{min} the maximum and minimum eigenvalues of $\mathbf{H}$, respectively.

We consider a simple model: a unidirectional, one-dimensional string network, for which an analytic estimate of the control energy can be obtained as

$$\mathscr{E}_l \approx \lambda_H^{-1} \tag{11}$$

Where l is the chain length (the number of nodes on the link) and λ_H is the smallest eigenvalue of the symmetric Gramian matrix.

The matrix H is positive definite and symmetric with its inverse satisfying $H^{-1} = Q \Lambda Q^T$, where Λ and Q are the corresponding eigenvalue and eigenvector matrices, respectively.

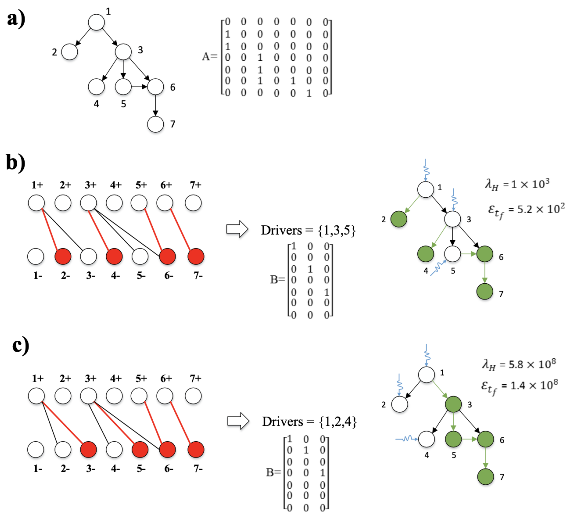

Fig. 4. Maximum Matching.

Thus, λ_H^{-1} is the largest eigenvalue of H^{-1} with λ_H denoting the smallest eigenvalue of H.

Where $\mathbf{q}_i$ is the ith column of Q. If the initial state x_0 is chosen to satisfy $\mathbf{q}_1^T.\mathbf{x}_0 = 1$, we obtain $\mathscr{E}(T) \approx \lambda_H^{-1}$.

A. Concepts of longest control chains

Identifying maximum matching so that the network is deemed structurally controllable does not guarantee convergent control energy.

To find the longest control chain (LCC) in a network, we first use the maximum matching algorithm to find all the driver nodes. We then identify the shortest paths from each of the driver nodes to each of the non-driver nodes. Finally, we pick out the longest such shortest paths as LCCs. The computational complexity of finding the LCCs are that associated with the maximum matching algorithm plus searching for the longest shortest paths, which is feasible for large networks. There can be multiple LCCs. The node at the end of an LCC is most difficult to be controlled in the sense that the largest amount of control energy is required. The number m of such end nodes dictates the degeneracy (multiplicity) of LCCs. An example is shown in Fig. 4. , where we see that, although there are multiple LCCs.

In Fig. 4b , Nodes 1, 3 and 5 are driver nodes. For this network $LCC = 2$ and in Fig. 4c driver nodes are 1,2 and 4. For the set of driver nodes we considered in part c, the longest control path is equal to three, $LCC = 3$.
As it turns out, any increase in LCC is accompanied by a huge increase in λ_H^{-1}. So we can say that LCC is an important parameter in determining the control energy.

V. Projected Gradient Method

By fixing the quadratic sum of all elements of matrix B and substituting (7) into (6), the minimum-cost control problem can be rewritten as

$$min \quad tr[W_B^{-1}X_f] \tag{12}$$

s.t.

$$tr(B^T B) - M - \varepsilon = 0 \tag{13}$$

Where X_f is a constant matrix given by $X_f = e^{At_f}X_0 e^{A^T t_f}$. $E(B) \triangleq tr[W_B^{-1}X_f]$ is the defined energy cost function to be minimized. We also define a norm function $N(B) \triangleq (tr(B^T B) - M)^2$ to associate with the equality constraint expressing the normalization condition on B, where ε is a positive constant to ensure that $N(B)$ is non-zero on the surface $tr(B^T B) = M + \varepsilon$.

We propose an efficient projected gradient method (PGM) to solve the optimization problem defined above. The main idea is to analytically derive the derivatives of the functions N(B) and E(B) with respect to B, and then project the negative of the gradient of E(B) onto the sphere surface $tr(B^T B) = M + \varepsilon$. By doing so, B can be searched via an iterative process until convergence. The gradients of functions N(B) and E(B) can be obtained as

$$\frac{\partial N(B)}{\partial B} = 4[tr(B^T B) - M].B \tag{14}$$

$$\frac{\partial E(B)}{\partial B} = -\int_0^{t_f} e^{A^T t} W_B^{-1} X_f W_B^{-1} e^{At} dt\, B \tag{15}$$

respectively. The two gradients point to the directions where $N(B)$ and $E(B)$ have the fastest increasing speed respectively. We also show that $\frac{\partial E(B)}{\partial B}$ is always non-zero as long as (A, B) is controllable, which explains the importance of imposing the normalization condition on B.

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