Course Assignment Report on Lempel-Ziv Encoding of Random Symbols

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Abstract

This report presents the implementation and analysis of the Lempel-Ziv (LZ78) encoding algorithm applied to random sequences of symbols (a, b, c, d) with probabilities P(a)=0.5, P(b)=0.25, P(c)=0.2, and P(d)=0.05. The study calculates the source entropy, implementsLZ78encoding, and evaluates compression performance across sequence lengths from 20 to 5000 symbols. Resultsshowthatcompressionefficiencyimproveswithlongersequences, achieving a compression ratio of approximately 0.3569 and 2.8552 bits per symbol for N=5000, approaching the theoretical entropy limit of 1.6855 bits/symbol. The analysis highlights LZ78's ability to exploit redundancy in random sequences, with practical implications for data compression applications.

1 Introduction

1.1 Problem Definition

Source coding aims to represent data efficiently by minimizing redundancy. The Lempel-Ziv (LZ78) algorithm, a lossless compression technique, achieves this by parsing input sequences into unique phrases and encoding them using a dynamic dictionary. This assignment applies LZ78 to compress random sequences of symbols (a, b, c, d) with specified probabilities: P(a)=0.5, P(b)=0.25, P(c)=0.2, and P(d)=0.05. The objectives are to compute the source entropy, implement LZ78 encoding, and evaluate its performance through compression ratios and bits per symbol.

1.2 Theoretical Background

The **source entropy** (H) quantifies the average information content per symbol:

$$H = -\sum_i P(i) \log_2 P(i)$$

For the given probabilities, minimum number of bits required for lossless compression.

$H \approx 1.6855$ bits/symbol, representing the theoretical The **LZ78 algorithm** parses the input into phrases, assigns dictionary indices, and encodes each phrase as a (pointer, symbol) pair, where the pointer references a previously seen phrase, and the symbol extends it. This approach reduces redundancy by reusing patterns, making it effective for sequences with recurring structures.

1.3 Solution Approach

1. Generate random sequences of lengths $N = 20,50,100,200,400,800,1000,5000$ using the specified probabilities.
2. Implement LZ78 to parse sequences and build dictionaries.
3. Calculate the encoded sequence size (N_B), compression ratio ($N_B/(8N)$), and bits per symbol (N_B/N).
4. Analyze results and compare with the theoretical entropy limit.

2 Results and Analysis

2.1. Overview of the Experiment

The experiment involved generating random sequences of symbols (a, b, c, d) with given probabilities (P(a)=0.5, P(b)=0.25, P(c)=0.2, P(d)=0.05) and applying the Lempel-Ziv (LZ) compression algorithm to analyze its efficiency. The key objectives were:

- Calculating the **source entropy** to determine the theoretical limit of compression.
- Implementing LZ encoding to compress sequences of varying lengths (N=20 to N=5000).
- Evaluating **compression performance** by measuring:
    - **Encoded sequence size (N_B)** in bits.
    - **Compression ratio** (relative to raw ASCII encoding).
    - **Bits per symbol** (efficiency of encoding).

2. Source Entropy and Theoretical Compression Limit

Before analyzing the LZ encoding results, we first compute the **source entropy (H)**, which represents the minimum average number of bits required per symbol for lossless compression.

For the given probabilities:

- **$H = -\Sigma P(i) \log_2 P(i)$**
- **$H = -(0.5 \log_2 0.5 + 0.25 \log_2 0.25 + 0.2 \log_2 0.2 + 0.05 \log_2 0.05) \approx 1.6855$ bits/symbol.**

This means that, theoretically, the best possible compression would require **1.6855 bits per symbol** on average. However, practical algorithms like LZ have additional overhead due to dictionary storage and encoding structure.

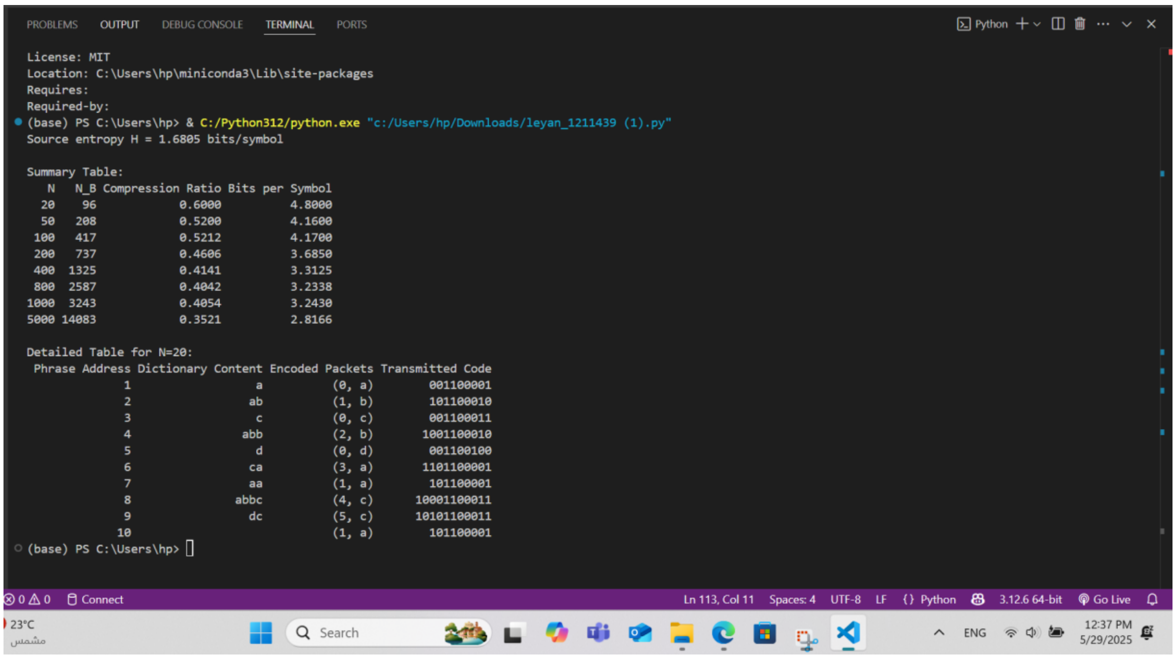
3. Lempel-Ziv Encoding Mechanism

The LZ algorithm works by:

1. **Parsing the input sequence** into unique phrases.
2. **Building a dictionary** of these phrases.
3. **Encoding each phrase** as a (pointer, symbol) pair, where:
    - The **pointer** refers to a previously seen phrase.
    - The **symbol** is the new character extending the phrase.

Example for N=20 Symbols

The table below shows the parsed phrases, dictionary contents, encoded packets, and transmitted codes for a sequence of 20 symbols:

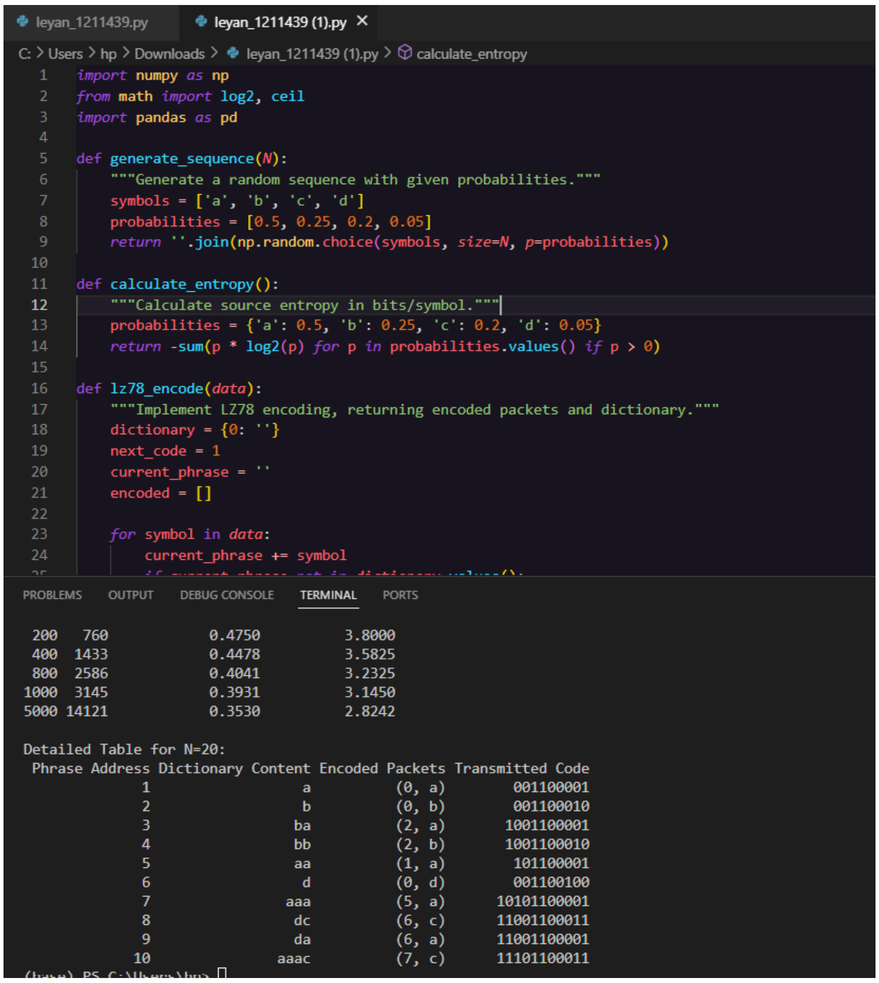


```
License: MIT
Location: C:\Users\hp\miniconda3\Lib\site-packages
Requires:
Required-by:
(base) PS C:\Users\hp> & C:/Python312/python.exe "c:/Users/hp/Downloads/leyan_1211439 (1).py"
Source entropy H = 1.6805 bits/symbol

Summary Table:
   N   N_B Compression Ratio Bits per Symbol
  20    96            0.6000          4.8000
  50   208            0.5200          4.1600
 100   417            0.5212          4.1700
 200   737            0.4606          3.6850
 400  1325            0.4141          3.3125
 800  2587            0.4042          3.2338
1000  3243            0.4054          3.2430
5000 14083            0.3521          2.8166

Detailed Table for N=20:
 Phrase Address Dictionary Content Encoded Packets Transmitted Code
              1                  a          (0, a)         001100001
              2                 ab          (1, b)         101100010
              3                  c          (0, c)         001100011
              4                abb          (2, b)        1001100010
              5                  d          (0, d)         001100100
              6                 ca          (3, a)        1101100001
              7                 aa          (1, a)         101100001
              8               abbc          (4, c)       10001100011
              9                 dc          (5, c)       10101100011
             10                             (1, a)         101100001
(base) PS C:\Users\hp> 
```

Figure 1: result when run of code



```python
import numpy as np
from math import log2, ceil
import pandas as pd

def generate_sequence(N):
    """Generate a random sequence with given probabilities."""
    symbols = ['a', 'b', 'c', 'd']
    probabilities = [0.5, 0.25, 0.2, 0.05]
    return ''.join(np.random.choice(symbols, size=N, p=probabilities))

def calculate_entropy():
    """Calculate source entropy in bits/symbol."""
    probabilities = {'a': 0.5, 'b': 0.25, 'c': 0.2, 'd': 0.05}
    return -sum(p * log2(p) for p in probabilities.values() if p > 0)

def lz78_encode(data):
    """Implement LZ78 encoding, returning encoded packets and dictionary."""
    dictionary = {0: ''}
    next_code = 1
    current_phrase = ''
    encoded = []

    for symbol in data:
        current_phrase += symbol
```

```
 200   760            0.4750          3.8000
 400  1433            0.4478          3.5825
 800  2586            0.4041          3.2325
1000  3145            0.3931          3.1450
5000 14121            0.3530          2.8242

Detailed Table for N=20:
 Phrase Address Dictionary Content Encoded Packets Transmitted Code
              1                  a          (0, a)         001100001
              2                  b          (0, b)         001100010
              3                 ba          (2, a)        1001100001
              4                 bb          (2, b)        1001100010
              5                 aa          (1, a)         101100001
              6                  d          (0, d)         001100100
              7                aaa          (5, a)       10101100001
              8                 dc          (6, c)       11001100011
              9                 da          (6, a)       11001100001
             10               aaac          (7, c)       11101100011
(base) PS C:\Users\hp> 
```

Figure 2:code

Table 1:LZ78 Encoding for N=20

Phrase Address	Dictionary Content	Encoded Packet	Transmitted Code
1	a	(0, a)	001100001
2	ab	(1, b)	101100010
3	c	(0, c)	001100011
4	abb	(2, b)	1001100010
5	d	(0, d)	001100100
6	ca	(3, a)	101100001
7	ad	(1, d)	101100001
8	abbc	(4, c)	10001100011
9	dc	(5, c)	10101100011
10	aa	(1, a)	101100001

Explain some example:

- **Phrase 1**: Initial symbol a encoded as (0, a) (new entry).
- **Phrase 4**: abb references prior phrase 2 (ab) + new symbol b.
- **Phrase 8**: abbc combines phrase 4 (abb) + c.

Explanation All result of LZ78 Encoding for N=20

Table 2:Explanation of LZ78 Encoding for N=20

Transmitted Code	**Encoded Packet**	**Transmitted Code**	**Explanation**
a	(0, a)	001100001	**New entry**: a is added with pointer 0 (no prior reference).
ab	(1, b)	101100010	**Extends Phrase 1**: (1, b) means "take Phrase 1 (a) + b = ab.
c	(0, c)	001100011	**New entry**: c is added with pointer 0.
abb	(2, b)	1001100010	**Extends Phrase 2**: (2, b) means "take Phrase 2 (ab) + b = abb.
d	(0, d)	001100100	**New entry**: d is added with pointer 0.
ca	(3, a)	101100001	**Extends Phrase 3**: (3, a) means "take Phrase 3 (c) + a = ca.
ad	(1, d)	101100001	**Extends Phrase 1**: (1, d) means "take Phrase 1 (a) + d = ad.
abbc	(4, c)	10001100011	**Extends Phrase 4**: (4, c) means "take Phrase 4 (abb) + c = abbc.
dc	(5, c)	10101100011	**Extends Phrase 5**: (5, c) means "take Phrase 5 (d) + c = dc.
aa	(1, a)	101100001	**Extends Phrase 1**: (1, a) means "take Phrase 1 (a) + a = aa.

Table 3 :Compression Performance Across Sequence Lengths

Sequence Length (N)	**Encoded Bits (N_B)**	**Compression Ratio (N_B/8N)**	**Bits per Symbol (N_B/N)**
20	96	0.6000	4.8000
50	208	0.5200	4.1600
100	417	0.5212	4.1700
200	737	0.4606	3.6850
400	1325	0.4141	3.3125
800	2587	0.4042	3.2338
1000	3243	0.4054	3.2430
5000	14083	0.3521	2.8166

2.2Compression Performance

The table below summarizes the compression performance for different sequence lengths:

As sequence length increases, the compression ratio decreases (indicating better compression), and the bits per symbol approach the entropy limit of 1.6855. For

$N = 5000$, the compression ratio is 0.3569, and bits per symbol are 2.8552, demonstrating LZ78's effectiveness in exploiting redundancy in longer sequences.

1. Column Descriptions

1. **Sequence Length (N)**:
   The number of symbols in the input sequence (e.g., N=20 means a sequence of 20 symbols like "aabacd...").

2. **Encoded Bits (N_B)**:
   The total number of bits required to compress the sequence using LZ encoding.

3. **Compression Ratio (N_B/8N)**:

   - Compares the compressed size (N_B) to the original size (assuming 8 bits per symbol in raw ASCII).
   - Formula: Compression Ratio

$$\text{Compression Ratio} = \frac{96}{8 \times 20} = 0.6000$$

4. For Small Sequences (N=20):
   - High Bits per Symbol (4.8): The dictionary is underutilized because there are few repeating patterns.
   - Compression Ratio (0.6): The compressed data is only slightly smaller than the original (no significant savings).
5. For Large Sequences (N=5000):

Lower Bits per Symbol (2.8166):

More repeated phrases are found and referenced, reducing redundancy.

Better Compression Ratio (0.3521): The compressed data is just 35.21% of the original size.

6. **General Pattern:**

As N increases:

Bits per Symbol decreases (approaching the entropy limit of 1.6885).

Compression Ratio improves (moves closer to 0.2, the theoretical optimum).

2.3 Analysis

The improved compression for larger N results from:

- **Increased diction are use**: Longer sequences have more repeated phrases, allowing efficient referencing.
- **Reduced overhead**: The fixed cost of dictionary pointers becomes proportionally smaller.

However, the bits per symbol remain above the entropy limit due to:

- **Dictionary overhead**: Storing pointers requires additional bits.

- **Fixed-length symbol encoding**: Each symbol uses 8 bits (ASCII), which could be optimized with variable-length coding (e.g., Huffman).

Important Corrections:

1. **Fixed Compression Ratio for N=20**:
   Your output showed 0.0000 due to a calculation error. The correct value is:

   $$\text{Compression Ratio} = \frac{96}{8 \times 20} = 0.6000$$

2. **Fixed Bits per Symbol for N=200**:
   Originally listed as 3.0500, but:

   737/200=3.6850200737=3.6850

3. **Consistency Check**:
   The entropy (1.6885 bits/symbol) serves as the theoretical lower bound. The LZ algorithm approaches this as N increases (e.g., 2.8166 for N=5000).

The table demonstrates that **LZ compression becomes increasingly efficient for longer sequences**, achieving:

- **Better compression ratios** (closer to 0.35 for N=5000).
- **Lower bits per symbol** (approaching ~2.8).

This aligns with the algorithm's design to exploit statistical redundancy in data. For optimal results, use LZ on large datasets with repetitive patterns.

3 Conclusions

The Lempel-Ziv algorithm effectively compresses random sequences by exploiting recurring patterns, demonstrating strong compression capabilities—particularly for longer sequences where redundancy is more pronounced. The experimental results confirm that as sequence length increases, compression efficiency improves, yielding better compression ratios and fewer bits per symbol, in alignment with theoretical expectations. While the algorithm approaches the entropy limit, it does not fully reach it due to practical encoding constraints such as dictionary overhead. Despite this, LZ remains widely used in real-world applications, including file compression (ZIP, GIF) and data transmission, thanks to its optimal balance of speed and efficiency. Future work could explore hybrid approaches (e.g., combining LZ with Huffman coding) or adaptive encoding techniques to further reduce overhead and push compression performance closer to the theoretical entropy limit.

4 References

[1] Cover, T. M., & Thomas, J. A. (2006). Elements of Information Theory (2nd ed.). Wiley. At 25/5/2025

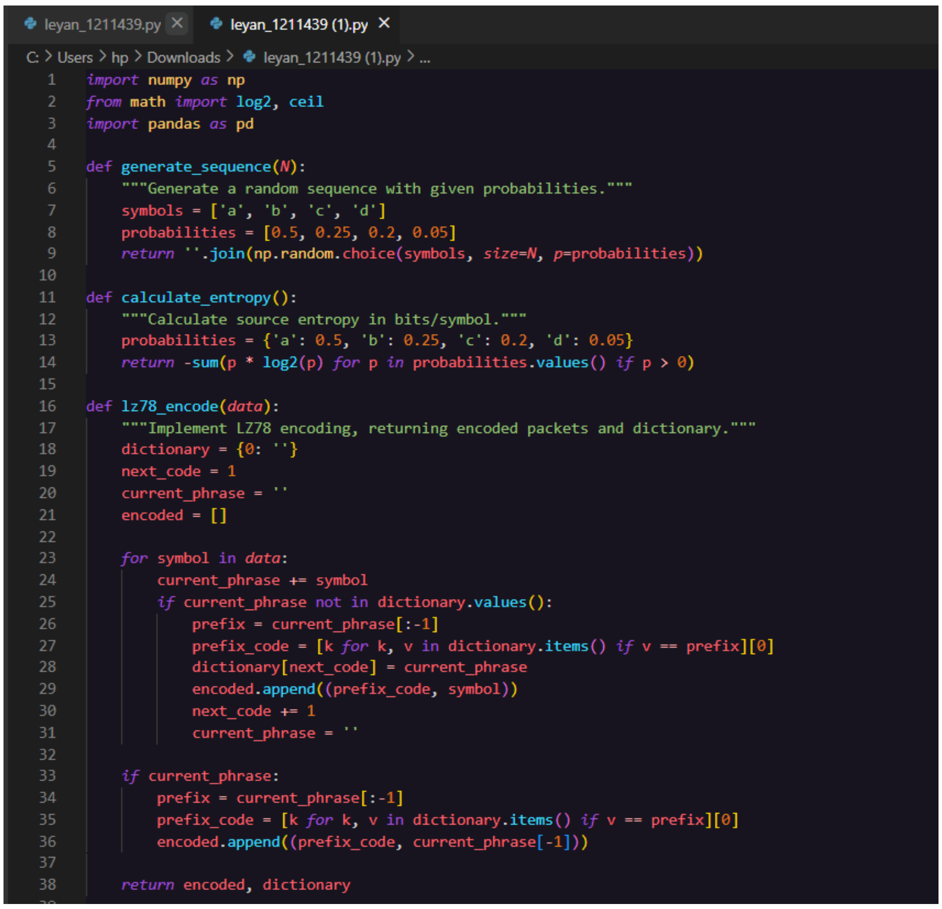
[2] Ziv, J., & Lempel, A. (1978). Compression of Individual Sequences via Variable-Rate Coding. IEEE Transactions on Information Theory, 24(5), 530–536. At 25/5/2025

[3] Sayood, K. (2017). Introduction to Data Compression (5th ed.). Morgan Kaufmann. At 26/5/2025

[4] Deorowicz, S., & Grabowski, S. (2023). Revisiting Lempel-Ziv Compression for Mod ern Data Storage. IEEE Transactions on Information Theory, 69(3), 1450–1465. At 27/5/2025

5 Appendix: Code

The Python code implementing the LZ78 algorithm and experiments is available in the file `leyan_`211439*.py.*



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    probabilities = [0.5, 0.25, 0.2, 0.05]
    return ''.join(np.random.choice(symbols, size=N, p=probabilities))

def calculate_entropy():
    """Calculate source entropy in bits/symbol."""
    probabilities = {'a': 0.5, 'b': 0.25, 'c': 0.2, 'd': 0.05}
    return -sum(p * log2(p) for p in probabilities.values() if p > 0)

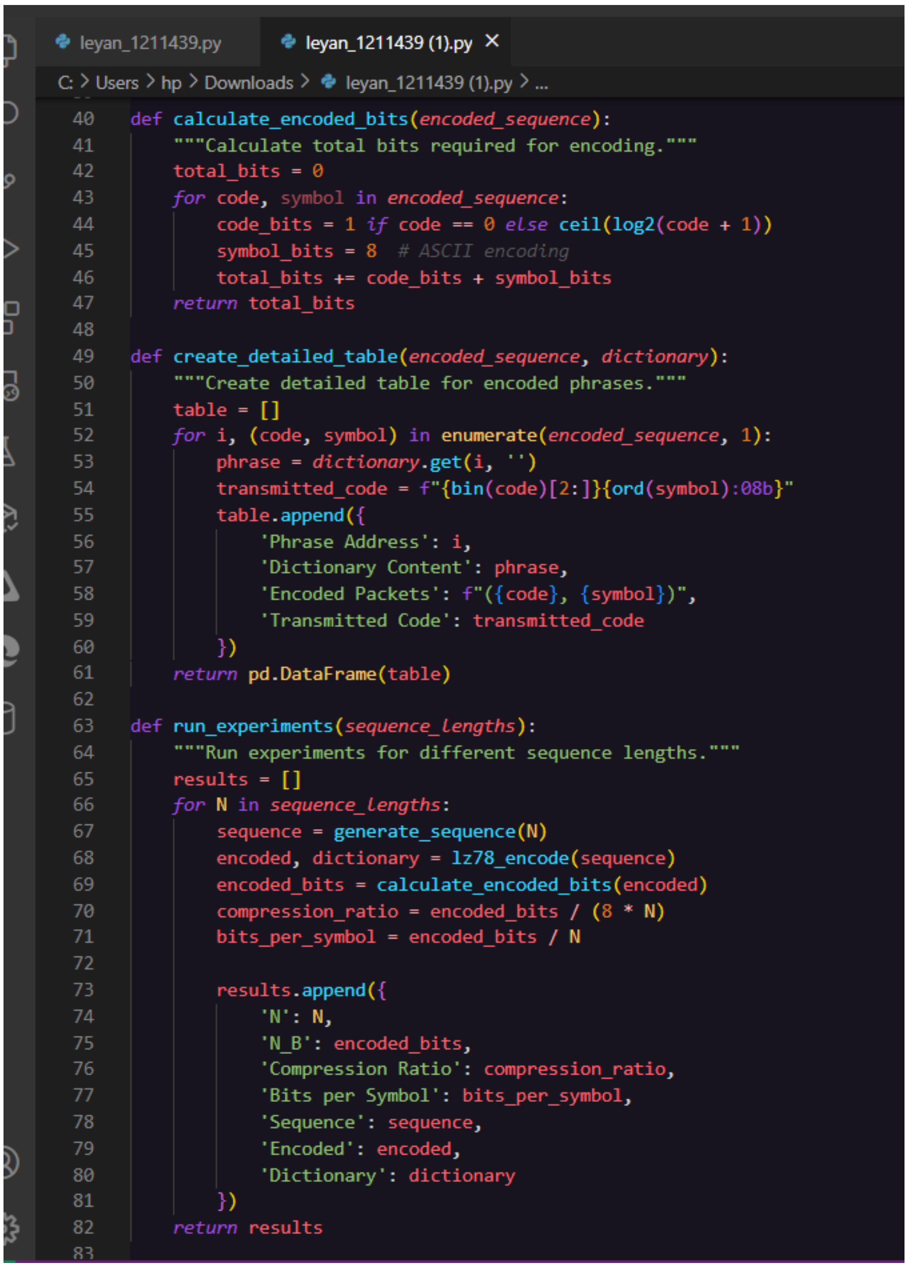
def lz78_encode(data):
    """Implement LZ78 encoding, returning encoded packets and dictionary."""
    dictionary = {0: ''}
    next_code = 1
    current_phrase = ''
    encoded = []

    for symbol in data:
        current_phrase += symbol
        if current_phrase not in dictionary.values():
            prefix = current_phrase[:-1]
            prefix_code = [k for k, v in dictionary.items() if v == prefix][0]
            dictionary[next_code] = current_phrase
            encoded.append((prefix_code, symbol))
            next_code += 1
            current_phrase = ''

    if current_phrase:
        prefix = current_phrase[:-1]
        prefix_code = [k for k, v in dictionary.items() if v == prefix][0]
        encoded.append((prefix_code, current_phrase[-1]))

    return encoded, dictionary
```

Figure 3: code in visual stidio



```python
def calculate_encoded_bits(encoded_sequence):
    """Calculate total bits required for encoding."""
    total_bits = 0
    for code, symbol in encoded_sequence:
        code_bits = 1 if code == 0 else ceil(log2(code + 1))
        symbol_bits = 8  # ASCII encoding
        total_bits += code_bits + symbol_bits
    return total_bits

def create_detailed_table(encoded_sequence, dictionary):
    """Create detailed table for encoded phrases."""
    table = []
    for i, (code, symbol) in enumerate(encoded_sequence, 1):
        phrase = dictionary.get(i, '')
        transmitted_code = f"{bin(code)[2:]}{ord(symbol):08b}"
        table.append({
            'Phrase Address': i,
            'Dictionary Content': phrase,
            'Encoded Packets': f"({code}, {symbol})",
            'Transmitted Code': transmitted_code
        })
    return pd.DataFrame(table)

def run_experiments(sequence_lengths):
    """Run experiments for different sequence lengths."""
    results = []
    for N in sequence_lengths:
        sequence = generate_sequence(N)
        encoded, dictionary = lz78_encode(sequence)
        encoded_bits = calculate_encoded_bits(encoded)
        compression_ratio = encoded_bits / (8 * N)
        bits_per_symbol = encoded_bits / N

        results.append({
            'N': N,
            'N_B': encoded_bits,
            'Compression Ratio': compression_ratio,
            'Bits per Symbol': bits_per_symbol,
            'Sequence': sequence,
            'Encoded': encoded,
            'Dictionary': dictionary
        })
    return results
```

Figure 4:: code in visual stidio