Homework 0: Basic Probability Review Problems
ECE/CS 498 DS Spring 2019
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Please submit Problem 1, 3, 4, 5, 8 for grading. The rest are for your practice.

Problem 1 Basic Concepts

(a) **(5 points)** Write down the Probability Axioms.

(a) ① $P(A) \geq 0$ for all $A \subset S$
② $P(S) = 1$
③ If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$

(b) **(5 points)** Explain the differences between a probability mass function (pmf) at a point and a probability density function (pdf) at a point.

(b) pmf: give the probability that a discrete random variable is exactly equal to some value
pdf: associated with conditions rather than discrete random variables
a point of pmf ≥ 0
a point of pdf $= 0$

(c) **(5 points)** If A and B are independent events with $P(A) = 0.6$, and $P(B) = 0.5$, find $P(A \bigcup B)$.

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = 0.6 + 0.5 - 0.6 \times 0.5 = 0.8$

(d) **(5 points)** Prove $P(A, B|C) = P(A|B, C) \times P(B|C)$.
Hint: Start from $P(A, B|C) = \frac{P(A,B,C)}{P(C)}$.

(d) $P(A,B|C) = \dfrac{P(A,B,C)}{P(C)}$ (1)

$P(A|B,C) = \dfrac{P(A,B,C)}{P(B,C)}$ (2)

$P(B|C) = \dfrac{P(B,C)}{P(C)}$ (3)

(1) = (2) × (3)

Problem 2 Counting

(a) Find the number of solutions of $x + y + z = 12$ where x, y, z are all positive integers.

(b) Find the number of solutions of $x + y + z < 12$ where x, y, z are all positive integers.

(c) Find the number of solutions of $x + y + z = 12$ where x, y, z are all nonnegative integers.

Problem 3 Independence

There are 4 identical balls in an urn, marked with "1", "2", "3", and "1,2,3", respectively. A ball is taken from the urn at random, event $A_i = \{$"i" is on the ball$\}$. For example, A_1 occurs when ball "1" is picked or when ball "1,2,3" is picked. Show your calculations for credit.

(a) ① pairwise independence
$P(X \cap Y) = P(X) \times P(Y)$
$P(Y \cap Z) = P(Y) \times P(Z)$
$P(Z \cap X) = P(Z) \times P(X)$
② mutual independence
$P(X \cap Y) = P(X) \times P(Y)$
$P(Y \cap Z) = P(Y) \times P(Z)$
$P(Z \cap X) = P(Z) \times P(X)$
$P(X \cap Y \cap Z) = P(X)P(Y)P(Z)$

(a) **(5 points)** What is the difference between pairwise independence and mutual independence? Illustrate your answer with respect to three random variables X, Y, and Z.

(b) **(5 points)** Are A_1, A_2, A_3 pairwise independent?

(b) $P(A_1) = P(A_2) = P(A_3) = \frac{1}{2}$
$P(A_1 A_2) = P(A_1)P(A_2) = \frac{1}{4}$
$P(A_2 A_3) = P(A_2)P(A_3) = \frac{1}{4}$
$P(A_3 A_1) = P(A_3)P(A_1) = \frac{1}{4}$
pairwise ✓

(c) **(5 point)** Are A_1, A_2, A_3 mutually independent?

(c) $P(A_1 A_2 A_3) = \frac{1}{4} \neq P(A_1)P(A_2)P(A_3) = \frac{1}{8}$ mutually ✗

Problem 4 Key Distributions

(a) An exponential random variable X can be parameterized by its *rate* $\lambda\,(\lambda > 0)$ via the probability density function (pdf):

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0$$

(i) memoryless property.
$F(x) = \int_0^x f(x)dx = \int_0^x \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^x = 1 - e^{-\lambda x}$

$P(x > t + t_0 \mid x > t_0) = \dfrac{P(x > t + t_0)}{P(x > t_0)} = \dfrac{1 - F(t_0 + t)}{1 - F(t_0)} = \dfrac{1 + e^{-\lambda(t + t_0)} - 1}{1 + e^{-\lambda t_0} - 1} = \dfrac{e^{-\lambda(t + t_0)}}{e^{-\lambda t_0}} = e^{-\lambda t} = P(x > t)$

$\Rightarrow P(x > t + t_0) = P(x > t_0)P(x > t) \Rightarrow$ memoryless property

(ii) $E(x)=\int_{-\infty}^{\infty} xf(x)dx=\int_0^{\infty} x\lambda e^{-\lambda x}dx = -\int_0^{\infty} x\,de^{-\lambda x} = (-xe^{-\lambda x})|_0^{\infty}+\int_0^{\infty} e^{-\lambda x}dx = 0-\frac{1}{\lambda}\int_0^{\infty} e^{-\lambda x}d(\lambda x) = -\frac{1}{\lambda}e^{-\lambda x}|_0^{\infty}=\frac{1}{\lambda}$

$E(x^2)=\int_0^{\infty} x^2\lambda e^{-\lambda x}dx = \frac{2}{\lambda^2}$ $Var(x)=E(x^2)-E^2(x)=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2}$

(iii) As showed in (i) $F(x)=1-e^{-\lambda x}$ $x>0$

(i) **(5 points)** Explain the memoryless property of the exponential distribution and provide the mathematical expression.

(ii) **(5 points)** Find the mean and variance of the exponential distribution.

(iii) **(5 points)** Derive the cumulative distribution function (cdf) of the exponential distribution.

(b) **(15 points)** The Poisson distribution can be seen as a limiting case to the binomial distribution as the number of trials goes to infinity and the expected number of successes remains fixed. Derive the Poisson distribution from the Binomial distribution.

(b) Binomial $P\{x=k\}=C_n^k p^k(1-p)^{n-k}=\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}=\frac{n!}{k!(n-k)!}(\frac{\lambda}{n})^k(1-\frac{\lambda}{n})^{n-k}$

$=\frac{n(n-1)\cdots(n-k+1)}{n^k}\cdot\frac{\lambda^k}{k!}\cdot\frac{(1-\lambda/n)^n}{(1-\lambda/n)^k}$

when $n\to\infty$ $(1-\frac{\lambda}{n})^n\approx e^{-\lambda}$

$\frac{n(n-1)\cdots(n-k+1)}{n^k}\approx 1$ $\Rightarrow P(x=k)\approx\frac{e^{-\lambda}\lambda^k}{k!}$ Poisson

$(1-\frac{\lambda}{n})^k\approx 1$

Problem 5 Marginal/Joint Distribution

Find the marginal distribution of continuous random variables X and Y given their joint distributions:

(a) **(5 points)** $p(x,y)=x+y, \quad 0<x,y<1$

(b) **(10 points)** $q(x,y)=(x+\frac{1}{2})(y+\frac{1}{2}), \quad 0<x,y<1$

(a) $f_X(x)=\int_0^1 f(x,y)dy=\int_0^1(x+y)dy=\int_0^1 xdy+ydy=xy|_0^1+\frac{1}{2}y^2|_0^1$
$=x+\frac{1}{2}-0=x+\frac{1}{2}$

$f_Y(y)=y+\frac{1}{2}$

(b) $f_X(x)=\int_0^1(x,y)dy=\int_0^1(xy+\frac{1}{2}x+\frac{1}{2}y+\frac{1}{4})dy=\frac{1}{2}xy^2|_0^1+\frac{1}{2}xy|_0^1$
$+\frac{1}{4}y^2|_0^1+\frac{1}{4}y|_0^1=\frac{1}{2}x+\frac{1}{2}x+\frac{1}{4}+\frac{1}{4}=x+\frac{1}{2}$

$f_Y(y)=y+\frac{1}{2}$

Problem 6 Inequalities

Derive Chebyshev Inequality from Markov Inequality.

Problem 7 Central Limit Theorem

Assume the autonomous vehicle system consists of 1000 independent components. Say the probability that each component functions properly is 0.99.

1. Random variable X is the number of properly functioning components. Find the distribution of X.

2. The system requires at least 985 properly functioning components to work. Use the Central Limit Theorem to find the probability that the system works.

Problem 8 Bayes Theorem and Conditional Probabilities

(20 points) When autonomous vehicles have malfunctions, the probability of a disengagement is 0.8. When autonomous vehicles do not have malfunctions, the probability of a disengagement is 0.001. If the probability of a malfunction is 0.0001, evaluate the probability that a given disengagement is due to a malfunction.
FYI: A disengagement is a failure that causes the control of the vehicle to switch from the software to the human driver.

have malfunction: $P(M)$
disengagement: $P(D)$
$P(D|M)=0.8$
$P(D|\bar{M})=0.001$
$P(M)=0.0001$

$P(D|M)=\frac{P(D,M)}{P(M)}=\frac{P(D|M)P(M)}{P(M,D)+P(\bar{M}D)}=\frac{P(D|M)P(M)}{P(D|M)P(M)+P(D|\bar{M})P(\bar{M})}$

$=\frac{0.8\times0.0001}{0.8\times0.0001+0.001\times0.9999}=0.074$