Homework 0: Basic Probability Review Problems **Name:**
ECE/CS 498 DS Spring 2019 **NetID:**
Issued: 01/16/2019
Due: 01/23/2019

Please submit Problem 1, 3, 4, 5, 8 for grading. The rest are for your practice.

① $P(A) \geq 0$ for all $A \subset S$
② $P(S) = 1$
③ 若 $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$

Problem 1 Basic Concepts

(a) **(5 points)** Write down the Probability Axioms.

概率公理：任何事件发生的概率的定义均满足概率公理

(b) **(5 points)** Explain the differences between a probability mass function (pmf) at a point and a probability density function (pdf) at a point.

give the probability that a discrete random variable is exactly equal to some value

associated with continuous rather than discrete random variables

(c) **(5 points)** If A and B are independent events with $P(A) = 0.6$, and $P(B) = 0.5$, find $P(A \bigcup B)$.

~~$P(A \cup B) = P(A) \cdot P(B) = 0.6 \times 0.5 = 0.3$~~

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= P(A) + P(B) - P(A)P(B)$
$= 0.6 + 0.5 - 0.6 \times 0.5 = 0.8$

(d) **(5 points)** Prove $P(A, B|C) = P(A|B, C) \times P(B|C)$.
Hint: Start from $P(A, B|C) = \frac{P(A,B,C)}{P(C)}$.

$P(A, B|C) = \frac{P(A,B,C)}{P(C)}$

$P(A|B, C) = \frac{P(A,B,C)}{P(B,C)}$

$P(B|C) = \frac{P(B,C)}{P(C)}$

$\Rightarrow \frac{P(A,B,C)}{P(B,C)} \cdot \frac{P(B,C)}{P(C)} = \frac{P(A,B,C)}{P(C)}$

Problem 2 Counting

(a) Find the number of solutions of $x + y + z = 12$ where x, y, z are all positive integers.

(b) Find the number of solutions of $x + y + z < 12$ where x, y, z are all positive integers.

(c) Find the number of solutions of $x + y + z = 12$ where x, y, z are all nonnegative integers.

Problem 3 Independence

There are 4 identical balls in an urn, marked with "1", "2", "3", and "1,2,3" (all), respectively. A ball is taken from the urn at random, event A_i = {"i" is on the ball}. For example, A_1 occurs when ball "1" is picked or when ball "1,2,3" is picked. Show your calculations for credit.

(1) 两两独立

① $P(X \cap Y) = P(X) \times P(Y)$
② $P(X \cap Z) = P(X) \times P(Z)$
③ $P(Y \cap Z) = P(Y) \times P(Z)$

①+②+③ + $P(X \cap Y \cap Z) = P(X) \cdot P(Y) P(Z)$
相互独立

(a) **(5 points)** What is the difference between pairwise independence and mutual independence? Illustrate your answer with respect to three random variables X, Y, and Z.

* 两两独立并不意味着相互独立，但相互独立则一定是两两独立

(b) **(5 points)** Are A_1, A_2, A_3 pairwise independent?

(c) **(5 point)** Are A_1, A_2, A_3 mutually independent?

(2) $P(A_1 \cap A_2) = [P(B_1) + P(B_{all})][P(B_2) + P(B_{all})]$
$= P(B_1)P(B_2) + P(B_1)P(B_{all}) + P(B_2)P(B_{all}) + P(B_{all})^2$
$= \frac{1}{4} \times \frac{1}{4} \times 4 = \frac{1}{4}$ $\Rightarrow$ pairwise ✓
$P(A_1) = P(A_2) = P(A_3) = \frac{1}{2}$
$P(A_1 \cap A_2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

(3) $P(A_1 \cap A_2 \cap A_3) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

Problem 4 Key Distributions

(a) An exponential random variable X (指数分布) can be parameterized by its *rate* λ ($\lambda > 0$) via the probability density function (pdf):

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0$$

(2) $P(A_1) = P(A_2) = P(A_3) = \frac{1}{2}$
$P(A_1 \cap A_2) = \frac{1}{4} = P(A_1)P(A_2)$ ✓

(2) mean $\int_0^{+\infty} f(x) x = \int_0^{\infty} \lambda e^{-\lambda x} x \, dx \Rightarrow E = \frac{1}{\lambda}$

(3) $P(A_1 A_2 A_3) = \frac{1}{4} \neq P(A_1) P(A_2) P(A_3)$ ✗

variance: $\frac{1}{\lambda^2}$

(a) $F(x) = \int_0^x f(x)dx = \int_0^x \lambda e^{-\lambda x}dx = -e^{-\lambda x}\Big|_0^x = -e^{-\lambda x}+1$

$P(X>t+t_0 \mid x>t_0) = \frac{P(X>t+t_0)}{P(X>t_0)} = \frac{1-F(t_0+t)}{1-F(t_0)} = \frac{1+e^{-\lambda(t+t_0)}-1}{1+e^{-\lambda t_0}-1} = \frac{e^{-\lambda(t+t_0)}}{e^{-\lambda t_0}} = e^{-\lambda t} = P(x>t)$

无记忆性

$\Rightarrow P(x>t+t_0) = P(x>t_0)P(x>t)$

(i) **(5 points)** Explain the memoryless property of the exponential distribution and provide the mathematical expression.

(ii) **(5 points)** Find the mean and variance of the exponential distribution.

(iii) **(5 points)** Derive the cumulative distribution function (cdf) of the exponential distribution.

$1-e^{-\lambda x}$ $x \geq 0$

(b) **(15 points)** The Poisson distribution can be seen as a limiting case to the binomial distribution as the number of trials goes to infinity and the expected number of successes remains fixed. Derive the Poisson distribution from the Binomial distribution.

二项: $P(X=k) = C_n^k p^k (1-p)^{n-k}$

Poison: $P\{x=k\} = \frac{e^{-\lambda}\lambda^k}{k!}$

$\int y\,dy$

Problem 5 Marginal/Joint Distribution

边缘分布 联合分布

$P(x) = \sum_y P(x,y) = \sum_y P(x|y)P(y)$

Find the marginal distribution of continuous random variables X and Y given their joint distributions:

(a) **(5 points)** $p(x, y) = x + y, \quad 0 < x, y < 1$

(b) **(10 points)** $q(x, y) = (x + \frac{1}{2})(y + \frac{1}{2}), \quad 0 < x, y < 1$

(a) $f_x(x) = \int_0^1 f(x,y)dy = \int_0^1 (x+y)dy = \int_0^1 x\,dy + y\,dy = xy\Big|_0^1 + \frac{1}{2}y^2\Big|_0^1 = x+\frac{1}{2}-0 = x+\frac{1}{2}$

$f_y(y) = \int_0^1 f(x,y)dx = \int_0^1 (x+y)dx = \int_0^1 (x\,dx + y\,dx) = \frac{1}{2}x^2\Big|_0^1 + yx\Big|_0^1 = \frac{1}{2}+y$

(b) $f_x(x) = \int_0^1 f(x,y)dy = \int_0^1 (xy+\frac{1}{2}x+\frac{1}{2}y+\frac{1}{4})dy = \frac{1}{2}xy^2\Big|_0^1 + \frac{1}{2}xy\Big|_0^1 + \frac{1}{4}y^2\Big|_0^1 + \frac{1}{4}y\Big|_0^1 = \frac{1}{2}x + \frac{1}{2}x + \frac{1}{4} + \frac{1}{4} = x+\frac{1}{2}$

$f_y(y) = \int_0^1 f(x,y)dx = \int_0^1 (xy+\frac{1}{2}x+\frac{1}{2}y+\frac{1}{4})dx = y+\frac{1}{2}$

Problem 6 Inequalities

Derive Chebyshev Inequality from Markov Inequality.

Problem 7 Central Limit Theorem

Assume the autonomous vehicle system consists of 1000 independent components. Say the probability that each component functions properly is 0.99.

1. Random variable X is the number of properly functioning components. Find the distribution of X.

2. The system requires at least 985 properly functioning components to work. Use the Central Limit Theorem to find the probability that the system works.

Problem 8 Bayes Theorem and Conditional Probabilities

$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$

(20 points) When autonomous vehicles have malfunctions, the probability of a disengagement is 0.8. When autonomous vehicles do not have malfunctions, the probability of a disengagement is 0.001. If the probability of a malfunction is 0.0001, evaluate the probability that a given disengagement is due to a malfunction.
FYI: A disengagement is a failure that causes the control of the vehicle to switch from the software to the human driver.

$P(U|B) =$

have malfunction: $P(M)$

disengagement: $P(D)$

$P(D|M) = 0.8 = \frac{P(D\cap M)}{P(M)} = \frac{P(D\cap M)P(M)}{P(M\cap D)+P(\bar{M}\cap D)} = \frac{0.8\times 0.0001}{P(D|M)P(M)+P(D|\bar{M})P(\bar{M})}$

$P(D|\neg M) = 0.001 = \frac{P(D\cap\bar{M})}{P(\bar{M})}$

$\cancel{P(D)} = 0.0001$

$P(M) = 0.0001$

$= \frac{0.8\times 0.0001}{0.8\times 0.0001 + 0.001\times 0.9999} = 0.074$

二项 $P\{x=k\} = C_n^k p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$

$$= \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{n^k} \cdot \frac{\lambda^k}{k!} \cdot \frac{(1-\lambda/n)^n}{(1-\lambda/n)^k}$$

$n\to+\infty$ $\left(1-\frac{\lambda}{n}\right)^n \approx e^{-\lambda}$

$\frac{n(n-1)\cdots(n-k+1)}{n^k} \approx 1$ $\Rightarrow P(x=k) \approx \frac{e^{-\lambda}\lambda^k}{k!}$

$\left(1-\frac{\lambda}{n}\right)^k \approx 1$