
Kinetic theory models for the distribution of wealth: power law from overlap of exponentials

Marco Patriarca^{1,3}, Anirban Chakraborti², Kimmo Kaski¹, and Guido Germano³

¹ Laboratory of Computational Engineering, Helsinki University of Technology,
POBoX 9203, 02015 HUT, Finland

² Brookhaven National Laboratory, Department of Physics, Upton, New York
11973, USA

³ Fachbereich Chemie, Philipps-Universität Marburg, 35032 Marburg, Germany

patriarc@staff.uni-marburg.de

anirban@bnl.gov

kimmo.kaski@hut.fi

germano@staff.uni-marburg.de

Summary. Various multi-agent models of wealth distributions defined by microscopic laws regulating the trades, with or without a saving criterion, are reviewed. We discuss and clarify the equilibrium properties of the model with constant global saving propensity, resulting in Gamma distributions, and their equivalence to the Maxwell-Boltzmann kinetic energy distribution for a system of molecules in an effective number of dimensions D_λ , related to the saving propensity λ [M. Patriarca, A. Chakraborti, and K. Kaski, Phys. Rev. E 70 (2004) 016104]. We use these results to analyze the model in which the individual saving propensities of the agents are quenched random variables, and the tail of the equilibrium wealth distribution exhibits a Pareto law $f(x) \propto x^{-\alpha-1}$ with an exponent $\alpha = 1$ [A. Chatterjee, B. K. Chakrabarti, and S. S. Manna, Physica Scripta T106 (2003) 367]. Here, we show that the observed Pareto power law can be explained as arising from the overlap of the Maxwell-Boltzmann distributions associated to the various agents, which reach an equilibrium state characterized by their individual Gamma distributions. We also consider the influence of different types of saving propensity distributions on the equilibrium state.

1 Introduction

'A rich man is nothing but a poor man with money' — W. C. Fields.

If money makes the difference in this world, then it is perhaps wise to dwell on what money, wealth and income are, to study models for predicting the respective distributions, how they are divided among the population of a

given country and among different countries. The most common definition of *money* suggests that money is the “Commodity accepted by general consent as medium of economics exchange” [1]. In fact money circulates from one economic agent (which can be an individual, firm, country, etc.) to another, thus facilitating trade. It is “something which all other goods or services are traded for” (for details see Ref. [2]). Throughout history various commodities have been used as money, termed usually as “commodity money” which include rare seashells or beads, and cattle (such as cows in India). Since the 17th century the most common forms have been metal coins, paper notes, and book-keeping entries. However, this is not the only important point about money. It is worth recalling the four functions of money according to standard economic theory:

- (i) to serve as a medium of exchange universally accepted in trade for goods and services
- (ii) to act as a measure of value, making possible the determination of the prices and the calculation of costs, or profit and loss
- (iii) to serve as a standard of deferred payments, i.e., a tool for the payment of debt or the unit in which loans are made and future transactions are fixed
- (iv) to serve as a means of storing wealth not immediately required for use.

A main feature that emerges from these properties and that is relevant from the point of view of the present investigation is that money is the medium in which prices or values of all commodities as well as costs, profits, and transactions can be determined or expressed. As for the *wealth*, it usually refers to things that have economic utility (monetary value or value of exchange), or material goods or property. It also represents the abundance of objects of value (or riches) and the state of having accumulated these objects. For our purpose, it is important to bear in mind that wealth can be measured in terms of money. Finally, *income* is defined as “The amount of money or its equivalent received during a period of time in exchange for labor or services, from the sale of goods or property, or as profit from financial investments” [3]. Therefore, it is also a quantity which can be measured in terms of money (per unit time). Thus, money has a two-fold fundamental role, as (i) an exchange medium in economic transactions, and (ii) a unit of measure which allows one to quantify (movements of) any type of goods which would otherwise be ambiguous to estimate. The similarity with e.g., thermal energy (and thermal energy units) in physics is to be noticed. In fact, the description of the mutual transformations of apparently different forms of energy, such as heat, potential and kinetic energy, is made possible by the recognition of their equivalence and the corresponding use of a same unit. And it so happens that this same unit is also the traditional unit used for one of the forms of energy. For example, one could measure energy in all its forms, as actually done in some fields of physics, in degree Kelvin. Without the possibility of expressing different goods in terms of the same unit of measure, there simply would not be any quantitative approach to economy models, just as there would be no

quantitative description of the transformation of the heat in motion and vice versa, without a common energy unit.

2 Multi-agent models for the distribution of wealth

In recent years several works have considered multi-agent models of a closed economy [4, 5, 6, 7, 8, 9, 10, 11, 12]. Despite their simplicity, these models predict a realistic shape of the wealth distribution, both in the low income part, usually described by a Boltzmann (exponential) distribution, as well in the tail, where a power law was observed a century ago by the Italian social economist Pareto [13]: the wealth of individuals in a stable economy follows the distribution, $F(x) \propto x^{-\alpha}$, where $F(x)$ is the upper cumulative distribution function, that is the number of people having wealth greater than or equal to x , and α is an exponent (known as the Pareto exponent) estimated to be between 1 and 2. In such models, N agents exchange a quantity x , that has sometimes been defined as wealth and other times as money. As explained in the introduction, here money must be interpreted all the goods that constitute the agents' wealth expressed in the same currency. To avoid confusion, in the following we will use only the term wealth. The states of agents are characterized by the wealths $\{x_n\}$, $n = 1, 2, \dots, N$. The evolution of the system is then carried out according to a prescription, which defines a "trading rule" between agents. The evolution can be interpreted both as an actual time evolution or a Monte Carlo optimization procedure, aimed at finding the equilibrium distribution. At every time step two agents i and j are extracted at random and an amount of wealth Δx is exchanged between them,

$$\begin{aligned} x'_i &= x_i - \Delta x, \\ x'_j &= x_j + \Delta x. \end{aligned} \quad (1)$$

It can be noticed that in this way the quantity x is conserved during the single transactions, $x'_i + x'_j = x_i + x_j$. Here x'_i and x'_j are the agent wealths after the transaction has taken place. Several rules have been studied for the model.

2.1 Basic model without saving: Boltzmann distribution

In the first version of the model, so far unnoticed in later literature, the money difference Δx is assumed to have a constant value [4, 5, 6],

$$\Delta x = \Delta x_0. \quad (2)$$

This rule, together with the constraint that transactions can take place only if $x'_i > 0$ and $x'_j > 0$, provides a Boltzmann distribution; see the curve for $\lambda = 0$ in Fig. 1. An equilibrium distribution with exponential tail is also obtained if

Δx is a random fraction ϵ of the wealth of one of the two agents, $\Delta x = \epsilon x_i$ or $\Delta x = \epsilon x_j$. A trading rule based on the random redistribution of the sum of the wealths of the two agents has been introduced by Dragulescu and Yakovenko [7],

$$\begin{aligned} x'_i &= \epsilon(x_i + x_j), \\ x'_j &= \bar{\epsilon}(x_i + x_j), \end{aligned} \quad (3)$$

where ϵ is a random number uniformly distributed between 0 and 1 and $\bar{\epsilon}$ is the complementary fraction, i.e. $\epsilon + \bar{\epsilon} = 1$. Equations (3) are easily shown to correspond to the trading rule (1), with

$$\Delta x = \bar{\epsilon}x_i - \epsilon x_j. \quad (4)$$

In the following we will concentrate on the latter version of the model and its generalizations, though both the versions of the basic model defined by Eqs. (2) or (4) lead to the Boltzmann distribution,

$$f(x) = \frac{1}{\langle x \rangle} \exp\left(-\frac{x}{\langle x \rangle}\right), \quad (5)$$

where the effective temperature of the system is just the average wealth $\langle x \rangle$ [4, 5, 6, 7]. The result (5) is found to be robust, in that it is largely independent of various factors. Namely, it is obtained for the various forms of Δx mentioned above, for pairwise as well as multi-agent interactions, for arbitrary initial conditions [8], and finally, for random or consecutive extraction of the interacting agents. The Boltzmann distribution thus obtained has been sometimes referred to as an “unfair distribution”, in that it is characterized by a majority of poor agents and very few rich agents, as signaled in particular by a zero mode and by the exponential tail. The form of distribution (5) will be referred to as the Boltzmann distribution and is also known as Gibbs distribution.

2.2 Minimum investment model without saving

Despite the Boltzmann distribution is robust respect to the variation of several parameters, the way it depends on the details of the trading rule is subtle. For instance, in the model studied in Ref. [9], the equilibrium distribution can have a very different shape. In that model it is assumed that both economic agents i and j invest the same amount x_{min} , which is taken as the minimum wealth of the two agents, $x_{min} = \min\{x_i, x_j\}$. The wealths after the trade are $x'_i = x_i + \Delta x$ and $x'_j = x_j - \Delta x$, where $\Delta x = (2\epsilon - 1)x_{min}$.

We note that once an agent has lost all his wealth, he is unable to trade because x_{min} has become zero. Thus, a trader is effectively driven out of the market once he loses all his wealth. In this way, after a sufficient number of transactions only one trader survives in the market with the entire amount of wealth, whereas the rest of the traders have zero wealth.

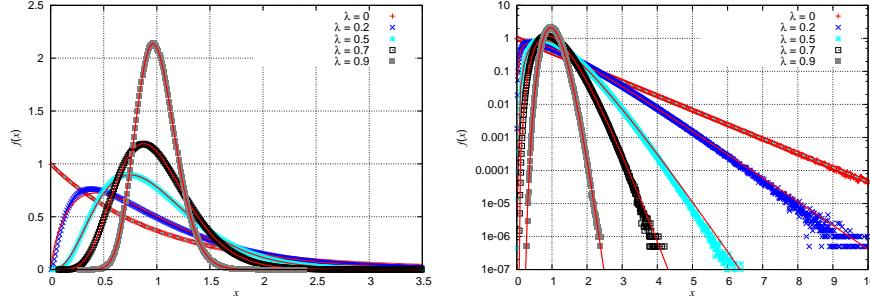


Fig. 1. Probability density for wealth x . The curve for $\lambda = 0$ is the Boltzmann function $f(x) = \langle x \rangle^{-1} \exp(-x/\langle x \rangle)$ for the basic model of Sec. 2.1. The other curves correspond to a global saving propensity $\lambda > 0$, see Sec. 2.3.

2.3 Model with constant global saving propensity: Gamma distribution

A step toward generalizing the basic model and making it more realistic is the introduction of a saving criterion regulating the trading dynamics. This can be achieved defining a saving propensity $0 < \lambda < 1$, that represents the fraction of wealth saved — and not reshuffled — during a transaction. The dynamics of the model is as follows [8, 9]:

$$\begin{aligned} x'_i &= \lambda x_i + \epsilon(1 - \lambda)(x_i + x_j), \\ x'_j &= \lambda x_j + \bar{\epsilon}(1 - \lambda)(x_i + x_j), \end{aligned} \quad (6)$$

with $\bar{\epsilon} = 1 - \epsilon$, corresponding to a Δx in Eq. (1) given by

$$\Delta x = (1 - \lambda)[\bar{\epsilon}x_i - \epsilon x_j]. \quad (7)$$

This model leads to a qualitatively different equilibrium distribution. In particular, it has a mode $x_m > 0$ and a zero limit for small x , i.e. $f(x \rightarrow 0) \rightarrow 0$, see Fig. 1. The functional form of such a distribution has been conjectured to be a Gamma distribution on the base of an analogy with the kinetic theory of gases, which is consistent with the excellent fitting provided to numerical data [14, 15]. Its form can be conveniently written by defining the effective dimension D_λ as [15]

$$\frac{D_\lambda}{2} = 1 + \frac{3\lambda}{1 - \lambda} = \frac{1 + 2\lambda}{1 - \lambda}. \quad (8)$$

According to the equipartition theorem, one can introduce a corresponding temperature defined by the relation $\langle x \rangle = D_\lambda T_\lambda / 2$, i.e.

$$T_\lambda = \frac{2 \langle x \rangle}{D_\lambda} = \frac{1 - \lambda}{1 + 2\lambda} \langle x \rangle. \quad (9)$$

Then the distribution for the reduced variable $\xi = x/T_\lambda$ reads

$$f(\xi) = \frac{1}{\Gamma(D_\lambda/2)} \xi^{D_\lambda/2-1} \exp(-\xi) \equiv \gamma_{D_\lambda/2}(\xi), \quad (10)$$

i.e. a Gamma distribution of order $D_\lambda/2$. For integer or half-integer values of $n = D_\lambda/2$, this function is identical to the equilibrium Maxwell-Boltzmann distribution of the kinetic energy for a system of molecules in thermal equilibrium at temperature T_λ in a D_λ -dimensional space (see Appendix A for a detailed derivation). For $D_\lambda = 2$, the Gamma distribution reduces to the Boltzmann distribution. This extension of the equivalence between kinetic theory and closed economy models to values $0 \leq \lambda < 1$ is summarized in Table 1. This equivalence between a multi-agent system with a saving propensity

Table 1. Analogy between kinetic and multi-agent model

	Kinetic model	Economic model
variable	K (kinetic energy)	x (wealth)
units	N particles	N agents
interaction	collisions	trades
dimension	integer D	real number D_λ [see Eq. (8)]
temperature	$k_B T = 2 \langle K \rangle / D$	$T_\lambda = 2 \langle x \rangle / D_\lambda$
reduced variable	$\xi = K/k_B T$	$\xi = x/T_\lambda$
equilibrium distribution	$f(\xi) = \gamma_{D/2}(\xi)$	$f(\xi) = \gamma_{D_\lambda/2}(\xi)$

$0 \leq \lambda < 1$ and an N -particle system in a space with effective dimension $D_\lambda \geq 2$ was originally suggested by simple considerations about the kinetics of a collision between two molecules. In fact, for kinematical reasons during such an event only a fraction of the total kinetic energy can be exchanged. Such a fraction is of the order of $1 - \lambda \approx 1/D$, to be compared with the expression $1 - \lambda = 3/(D/2 + 2)$ derived from Eq. (8) [15]. While λ varies between 0 and 1, the parameter D_λ monotonously increases from 2 to ∞ , and the effective temperature T_λ correspondingly decreases from $\langle x \rangle$ to zero; see Fig. 2. It is to be noticed that according to the equipartition theorem only in $D_\lambda = 2$ effective dimensions ($\lambda = 0$) the temperature coincides with the average value $\langle x \rangle$, $T_\lambda = 2 \langle x \rangle / 2 \equiv \langle x \rangle$, as originally found in the basic model [4, 5, 6, 7]. In its general meaning, temperature represents rather an estimate of the fluctuation of the quantity x around its average value. The equipartition theorem always gives a temperature smaller than the average value $\langle x \rangle$ for a number of dimensions larger than two. In the present case, Eqs. (8) or (9) show that this happens for any $\lambda > 0$.

The dependence of the fluctuations of the quantity x on the saving propensity λ was studied in Ref. [8]. In particular, the decrease in the amplitude of the fluctuations with increasing λ is shown in Fig. 3.

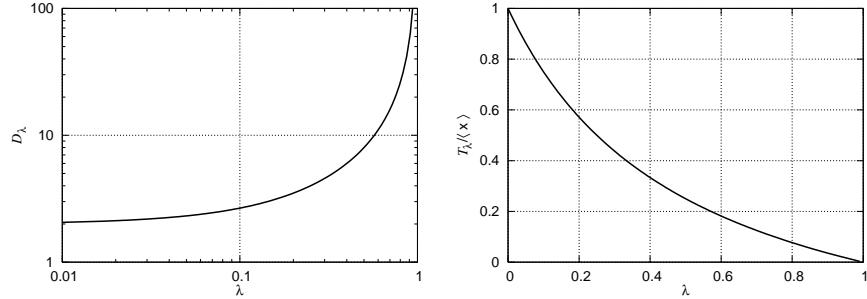


Fig. 2. Effective dimension D_λ , Eq. (8), and temperature, Eq. (9), as a function of saving propensity λ .

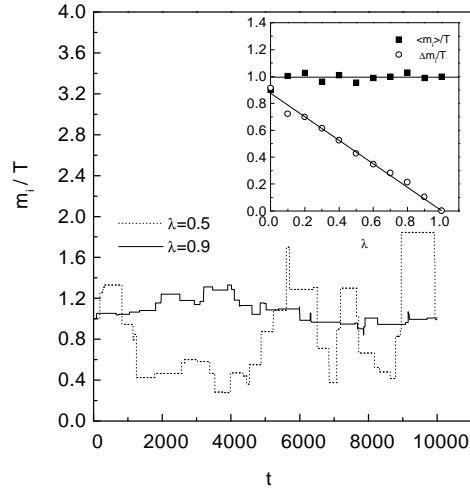


Fig. 3. Reproduced from Ref. [8] (only here $m \equiv x$ while $T \equiv 1$ is a constant). The continuous and the dotted curves are the wealths of two agents with $\lambda = 0.9$ and $\lambda = 0.5$ respectively: notice the larger fluctuations in correspondence of the smaller λ . The inset shows that $\Delta m \equiv \Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$ decreases with λ .

The fact that in general the market temperature T_λ decreases with λ means smaller fluctuations of x during trades, consistently with the saving criterion, i.e. with a $\lambda > 0$. One can notice that in fact $T_\lambda = (1 - \lambda) \langle x \rangle / (1 + 2\lambda) \approx (1 - \lambda) \langle x \rangle$ is of the order of the average amount of wealth exchanged during a single interaction between agents, see Eqs. (6).

2.4 Model with individual saving propensities: Pareto tail

In order to take into account the natural diversity between various agents, a model with individual propensities $\{\lambda_i\}$ as quenched random variables was studied in Refs. [10, 11]. The dynamics of this model is the following:

$$\begin{aligned} x'_i &= \lambda_i x_i + \epsilon[(1 - \lambda_i)x_i + (1 - \lambda_j)x_j], \\ x'_j &= \lambda_j x_j + \bar{\epsilon}[(1 - \lambda_i)x_i + (1 - \lambda_j)x_j], \end{aligned} \quad (11)$$

where, as above, $\bar{\epsilon} = 1 - \epsilon$. This corresponds to a Δx in Eq. (1) given by

$$\Delta x = \bar{\epsilon}(1 - \lambda_i)x_i - \epsilon(1 - \lambda_j)x_j. \quad (12)$$

Besides the use of this trading rule, a further prescription is given in the model, namely an average over the initial random assignment of the individual saving propensities: With a given configuration $\{\lambda_i\}$, the system is evolved until equilibrium is reached, then a new set of random saving propensities $\{\lambda'_i\}$ is extracted and reassigned to all agents, and the whole procedure is repeated many times. As a result of the average over the equilibrium distributions corresponding to the various $\{\lambda_i\}$ configurations, one obtains a distribution with a power law tail, $f(x) \propto x^{-\alpha-1}$, where the Pareto exponent has the value $\alpha = 1$. This value of the exponent has been predicted by various theoretical approaches to the modeling of multi-agent systems [16, 17, 12].

3 Further analysis of the model with individual saving propensities

On one hand, the model with individual saving propensities relaxes toward a power law distribution — with the prescription mentioned above to average the distribution over many equilibrium states corresponding to different configurations $\{\lambda_i\}$. On the other hand, the models with a global saving propensity $\lambda > 0$ and the basic model with $\lambda = 0$, despite being particular cases of the general model with individual saving propensities, relax toward very different distributions, namely a Gamma and a Boltzmann distribution, respectively. In this section we show that this difference can be reconciled by illustrating how the observed power law is due to the superposition of different distributions with exponential tails corresponding to subsystems of agents with the same value of λ .

3.1 The x - λ correlation

A key point which explains many of the features of the model and of the corresponding equilibrium state is a well-defined correlation between average wealth and saving propensity, which has been unnoticed so far in the literature

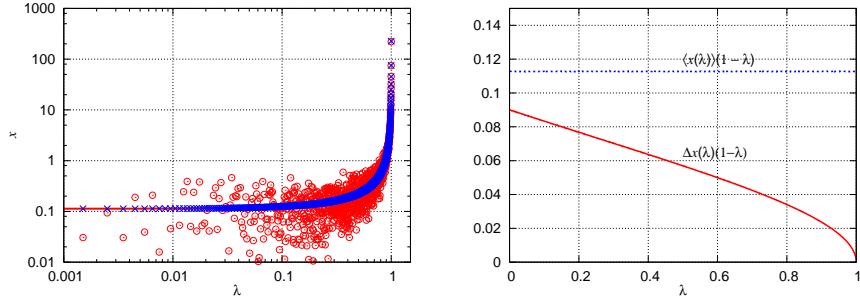


Fig. 4. Equilibrium state in the x - λ plane after $t = 10^9$ trades for a system of $N = 1000$ agents. Left: Circles (\circ) represent agents, crosses (\times) represent the average wealth $\langle x(\lambda) \rangle$, the continuous line is the function $\langle x(\lambda) \rangle = \kappa/(1 - \lambda)$, with $\kappa = 0.1128$. Right: The product $\langle x(\lambda) \rangle(1 - \lambda)$ (dotted line) is constant, in agreement with Eq. (14). The product $\Delta x(\lambda)(1 - \lambda)$ (continuous line), where $\Delta x(\lambda)$ is the standard deviation, shows that $\Delta x(\lambda)$ grows slower than $\langle x(\lambda) \rangle$.

[18]. The existence of such a correlation can be related to the origin of the power law and its cut-off at high values of x . It also explains the paradox according to which a very rich agent may lose all his wealth when interacting with poor agents, as a consequence of the stochastic character of the trading rule defined by Eq. (11). Figure 4 shows the equilibrium state for a system with $N = 1000$ agents after $t = 10^9$ trades. Each agent is represented by a circle (\circ) in the wealth-saving propensity x_i - λ_i plane. The correlation between wealth x and saving propensity λ becomes very high at large values of x and λ . Namely, one observes that the average wealth $\langle x(\lambda) \rangle$ [crosses (\times) in Fig. 4] diverges for $\lambda \rightarrow 1$. The average $\langle x(\lambda) \rangle$ was obtained by computing the probability density $f(x, \lambda)$ in the x - λ plane (normalized so that $\int dx d\lambda f(x, \lambda) = 1$) and averaging for a fixed value of λ ,

$$\langle x(\lambda) \rangle = \int dx x f(x, \lambda). \quad (13)$$

The observed correlation naturally follows from the structure of the trade dynamics (11). We remind that initially every agent has the same wealth $x_0 = \langle x \rangle$. During the initial phase of the evolution, when all agents have approximately the same wealth $\langle x \rangle$, an agent i with a large saving propensity λ_i can save more — on average — and therefore accumulate more. Afterwards, the agent i will continue to enter trades by investing only a small fraction $(1 - \lambda_i)x_i$ of his wealth x_i in the trade. Even when interacting with an agent j , with a smaller wealth $x_j < x_i$, agent i may still be successful in the trading, since agent j may have also a smaller saving propensity λ_j , so that the traded fraction of wealth $(1 - \lambda_j)x_j$ is comparable with or even larger than $(1 - \lambda_i)x_i$. Trading by agent i will very probably be successful on average with all agents j with a λ_j such that $(1 - \lambda_j)x_j$ is smaller than $(1 - \lambda_i)x_i$. These considerations

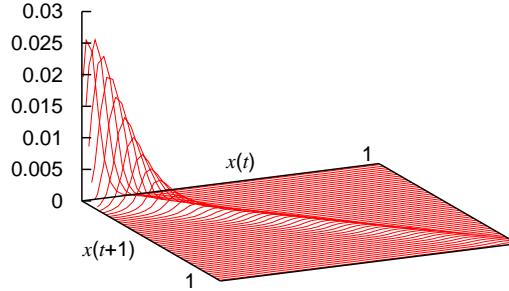


Fig. 5. Histogram of the wealths $x' = x(t + 1)$ after a trade versus $x = x(t)$ before the trade for all agents and trades, in a system with $N = 1000$ agents and 10^9 trades. The distribution is narrower for large x (rich agents), implying that it is unlikely that a rich agent becomes poor within a single trade.

suggest that agent i will reach equilibrium (and his maximum possible wealth) when $(1 - \lambda_i)x_i = \kappa \approx \langle(1 - \lambda)x\rangle$. The ratio between the constant κ and the average $\langle(1 - \lambda)x\rangle = \sum_j(1 - \lambda_j)x_j/N$ is actually found to be of the order of magnitude of 10. The formula

$$\langle x(\lambda_i) \rangle = \frac{\kappa}{1 - \lambda_i}, \quad (14)$$

however, shown as a continuous line in Fig. 4, provides an excellent interpolation of the average wealth $\langle x(\lambda) \rangle$ (also shown in the figure) computed numerically.

3.2 Variation of a single agent's wealth

The stability of the asymptotic state is also shown by the histogram in Fig. 5 of the wealths $x' \equiv x(t + 1)$ after a trade versus $x \equiv x(t)$ before it defined in Eqs. (1). The distribution is narrower at larger values of x than at smaller ones, implying that the probability that an agent i will undergo a large relative variation of his wealth x_i within a single trade is much higher for poor agents. The situation at small x (corresponding to agents with smaller saving propensities) is instead more similar to that of the trading rule without saving ($\lambda = 0$), Eqs. (6): the distribution is broader, indicating a higher probability of a large wealth reshuffling during a trade.

3.3 Power laws at small x and t scales

A peculiarity of the model with individual saving propensities is noteworthy. On one hand, in the procedure used to obtain a power law in Ref. [10] all agents

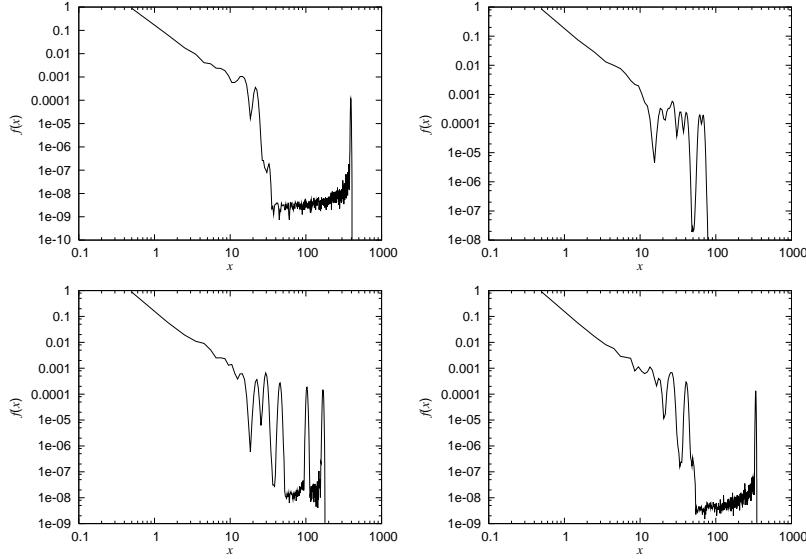


Fig. 6. The equilibrium configurations corresponding to four different random saving propensity sets $\{\lambda_i\}$, for a system with $N = 1000$ agents, differ especially at large x where the distribution deviates from a power law.

are equivalent to each other: they enter the dynamical evolution law on an equal footing, their saving propensities are reassigned randomly with the same uniform distribution between 0 and 1, and even their initial conditions can be set to be all equal to each other, $x_i = \langle x \rangle$, without loss of generality. Therefore the various equilibrium configurations, corresponding to different sets $\{\lambda_i\}$, are expected to be statistically equivalent to each other, in the sense that one should be able to obtain the power law distribution by a simple ensemble average for any fixed configuration of saving propensities $\{\lambda_i\}$, if the number of agents N is large enough. On the other hand, an averaging procedure over several $\{\lambda_i\}$ configurations is in practice necessary to obtain a power law distribution.

In order to understand this apparent paradox, we checked how the equilibrium distributions, corresponding to a given set of saving propensities $\{\lambda_i\}$, look like. One finds that every configuration $\{\lambda_i\}$ produces equilibrium distributions very different from each other; see Fig. 6 for some examples. The structures observed are very different from power laws, with well resolved peaks at large x . Only when an average over different $\{\lambda_i\}$ configurations is carried out, one obtains a smooth power law with Pareto exponent $\alpha = 1$. These same figures show, however, that for a given configuration $\{\lambda_i\}$ a power law is actually observed at small values of x . Another related interesting feature of simulations employing a single saving propensity configuration $\{\lambda_i\}$ is

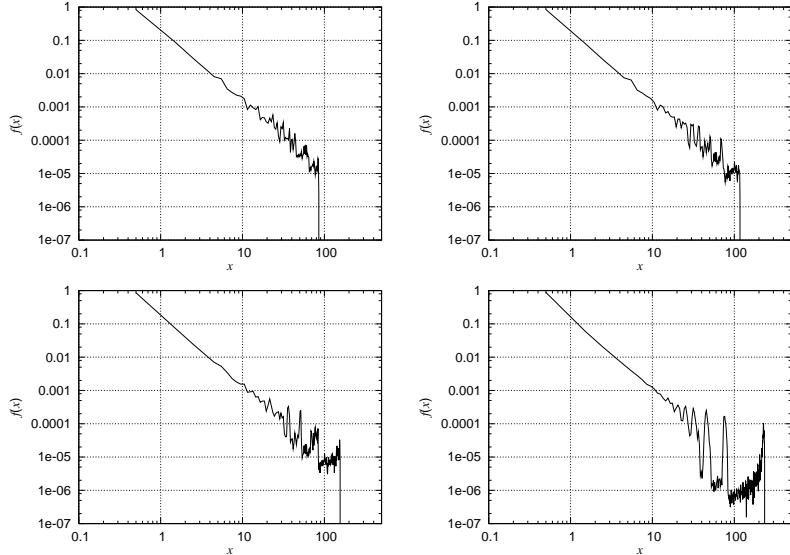


Fig. 7. Time evolution of the x distribution of a system with $N = 1000$ agents: $t = 2 \times 10^6$ (top left), 3×10^6 (top-right), 5×10^6 (bottom-left), and 2×10^7 trades (bottom-right). The distribution looks as a power law at small times, but develops into a structured distribution, maintaining a power law shape only at small x .

that a power law distribution is found only on a limited time scale, while it disappears partly or totally at equilibrium. Thus also in the time dimension one surprisingly finds a distribution much more similar to a power law at a smaller rather than larger scale. This is shown in the example in Fig. 7, where the distributions of a system of 1000 agents at four different times are compared to each other. These features suggest that the power law is intrinsically built into the dynamical laws of the model but that, for some reasons, it fades away at large x and t scales. The $x\text{-}\lambda$ correlation discussed above in Sec. 3.1 can provide an explanation of these features, both for those in the x and in the time dimension, as discussed below.

3.4 Origin of the power law

The peculiar features illustrated above, the necessity of averaging over different configurations $\{\lambda_i\}$ as done in Ref. [10] to obtain a power law distribution, as well as the power law distribution itself, are here explained in terms of equilibrium states of suitably defined subsystems and the $x\text{-}\lambda$ correlation illustrated above. This may seem odd since at first sight the averaging procedure of Ref. [10] defines a nonequilibrium process, the system being brought out of equilibrium from time to time by the reassignment of the saving propensities.

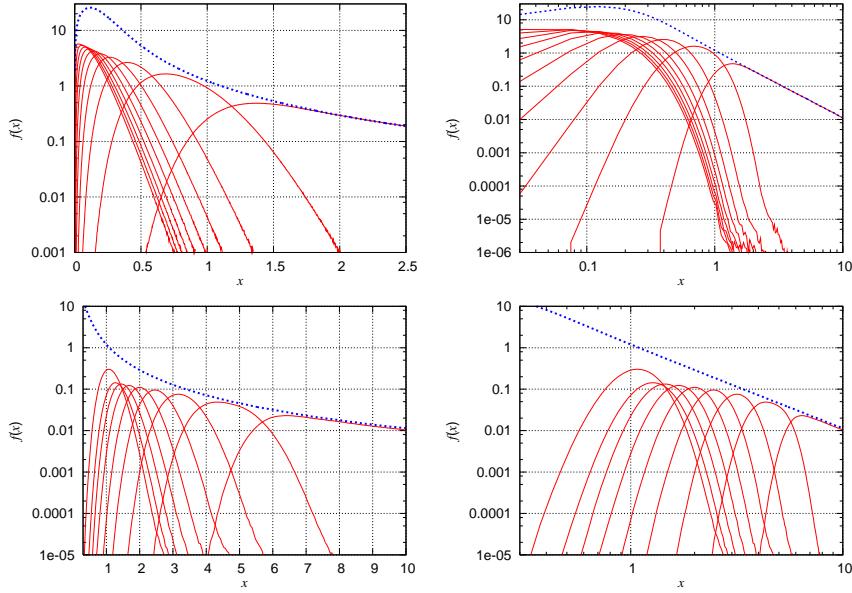


Fig. 8. Semi-log and log plots of partial distributions (continuous curves) and the resulting overlap (dotted line). Above: Partial distributions from the 10 intervals of width $\Delta\lambda = 0.1$ of the total λ range $(0,1)$. Below: The last partial distribution (with a power law tail) above, corresponding to the interval $\lambda = (0.9, 1.0)$, has been further resolved into ten partial distributions for the sub-intervals of width $\Delta\lambda = 0.01$.

Correspondingly, one may attribute the power law to the underlying dynamical process, as it is often the case in nonequilibrium models (e.g. models of markets on networks [19]).

However, if one considers the partial distributions of agents with a certain value of λ , one finds an unexpected result. For numerical reasons we consider the subsets made up of those agents with saving propensity λ within a window $\Delta\lambda$ around a given value λ . Figure 8 (upper row) shows the partial distributions (continuous lines) of the ten subsystems obtained by dividing the λ range $(0,1)$ into ten slices of width $\Delta\lambda = 0.1$ and average values $0.05, \dots, 0.95$ (curves from left to right respectively). Most of the partial distributions have an exponential tail, and only when summed up their overlap (dotted line) reproduces a power law. It can be noticed that the last partial distribution, corresponding to the interval $\lambda = (0.9, 1.0)$, is not of exponential form, but rather presents a power law tail, which overlaps with the total distribution at large x . However, its power law form is due only to the low resolution in λ employed. In fact even this partial distribution can in turn be shown to be given by the superposition of exponential tails. By increasing the resolution in λ , i.e. using a smaller interval $\Delta\lambda = 0.01$ to further resolve the interval

$\lambda = (0.9, 1.0)$ into subintervals with average values $\lambda = 0.905, \dots, 0.995$, one obtains the partial distributions shown in the lower row of Fig. 8. It is to be noticed that also in this case the last partial distribution corresponding to the interval $\lambda = (0.99, 1.00)$ has a power law tail. The procedure can then in principle be reiterated to resolve also this partial distribution by increasing the resolution in λ .

These facts also explain the origin of the peaks visible at large x in the plots in Figs. 6 and 7. Due to the high wealth-saving propensity correlation at large values of x , see Fig. 4, these peaks are due to agents with high λ . The reason why these agents give rise to resolved peaks instead of contributing to extending the power law tail is that the partial distributions (i.e. the average values) of single agents get farther and farther from each other for $\lambda \rightarrow 1$, while the corresponding widths do not grow enough to ensure the overlap of the distributions of neighbor agents in λ -space. Eventually, each agent (or cluster of agents) at high values of x will be resolved as an isolated peak against the background of the total distribution. In greater detail, one finds that the average value $\langle x(\lambda) \rangle$ diverges for $\lambda \rightarrow 1$ as $1/(1 - \lambda)$, as shown in Fig. 4. This implies that also the distance between two generic consecutive agents increases: if agents are labeled from $i = 1$ to $i = N$ in order of increasing λ ($\lambda_1 < \dots < \lambda_N$) and the λ distribution is uniform, then $\Delta\lambda = \lambda_{i+1} - \lambda_i = \text{constant}$. The distance between the average positions of the partial distributions of two consecutive agents is, from Eq. (14),

$$\delta\langle x(\lambda) \rangle = \langle x(\lambda + \Delta\lambda) \rangle - \langle x(\lambda) \rangle \approx \frac{\partial\langle x(\lambda) \rangle}{\partial\lambda} \Delta\lambda \approx \frac{\kappa\Delta\lambda}{(1 - \lambda)^2}, \quad (15)$$

where κ is a constant. Thus $\delta\langle x(\lambda) \rangle$ diverges even faster than $\langle x(\lambda) \rangle$. At the same time, the width of the partial distribution $\Delta x(\lambda)$, here estimated as $\Delta x(\lambda) = \sqrt{\langle x^2(\lambda) \rangle - \langle x(\lambda) \rangle^2}$, grows slower than $\langle x(\lambda) \rangle$, i.e. for $\lambda \rightarrow 1$ the ratio $\Delta x(\lambda)/\langle x(\lambda) \rangle \rightarrow 0$; see Fig. 4 (right). The breaking of the power law and the appearance of the isolated peaks takes place at a cutoff x_c where the distance $\delta\langle x(\lambda) \rangle$ between the peaks corresponding to consecutive agents i and $i + 1$ becomes comparable with the peak width $\Delta x(\lambda)$.

Also the origin of the peculiarities in the time evolution of the distribution function, mentioned in Sec. 3.3, can now be explained easily. In order to reach the asymptotic equilibrium state, agents can rely on an income flux which is on average proportional to $x_i(1 - \lambda_i)$. At the beginning, when all agents have the same wealth $x_i = x_0$, this quantity is smaller for agents with a larger λ_i ; and with this smaller flux agents with large λ_i have to reach their higher asymptotic value $\propto 1/(1 - \lambda_i)$. As a consequence, the relaxation time of an agent is larger for larger λ , a result already found in the numerical simulations of the multi-agent model with fixed global saving propensity (see Fig. 2 in Ref. [8]). Correspondingly, partial distributions of rich agents will reach their asymptotic form later (last frame in Fig. 7), while, at intermediate

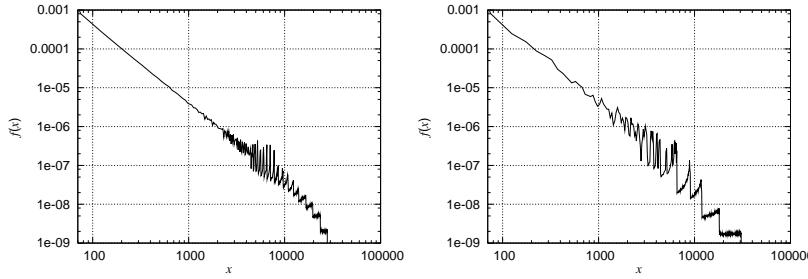


Fig. 9. Wealth distribution of a system of $N = 10^6$ agents after 10^{12} trades: the uniform λ distribution Eq. (16) produces a smoother distribution with a power law shape extending to higher x (left) than for a random λ distribution (right).

times, their distributions will be spread at smaller values of x , contributing to smoothen the total distribution (first frame in Fig. 7).

It is also possible to explain why the averaging procedure of Ref. [10] is successful in producing a power law distribution. Averaging over different configurations $\{\lambda_i\}$ is equivalent to simulate a very dense distribution in λ — which has large relaxation time and number of agents — with an affordable number of agents and computer time. However, the procedure is not needed in principle, since the power law can be obtained also when a single configuration with a proper density in λ -space is used.

3.5 Checking different λ distributions

A practical way to avoid the appearance of the peaks at large x and obtain a distribution closer to a power law is to increase the density of agents, especially at values of λ close to 1. In a random extraction of $\{\lambda_i\}$, it is natural that consecutive values of λ_i will not be equally spaced. Even small differences will be amplified at large x and will result in the appearance of peaks. A deterministic assignment of the λ , e.g. a uniform distribution achieved through the following assignment,

$$\lambda_i = \frac{i}{N}, \quad i = 0, N - 1, \quad (16)$$

is a uniform distribution of λ in the interval $[0,1]$ and will generate a smoother distribution of x . The comparison of the results for this distribution with those for a random distribution of λ is done in Fig. 9 (notice the high value of N). In the uniform case not only the power law extends to higher values of x but also that the distribution of peaks at large x is globally smoother, in the sense that on average the single peaks follow a power law better.

4 Conclusions

We have reviewed some multi-agent models for the distribution of wealth, in which wealth is exchanged at random in the presence of saving quantified by the saving propensity λ . We have shown how a distribution of λ generates a power law distribution of wealth through the superposition of Gamma distributions corresponding to particular subsets of agents. The physical picture for the model with individual saving propensities is thus more similar to that of the model with a constant global saving propensity than it may seem at first sight. In fact any subset of agents with the same value of the saving propensity λ equilibrates in a way similar to agents in the model with a global saving propensity, i.e. leading to a wealth distribution with an exponential tail. Correspondingly we have shown that both the noise in the power law tail and the cutoff in the power law depend on the coarseness of the λ distribution. This extends the analogy between economic and gas-like systems beyond the case of a global $\lambda \geq 0$, characterized by a Maxwell-Boltzmann distribution, to uniform continuous distributions in λ that span the whole interval $\lambda \in [0, 1]$.

Acknowledgment

This work was partially supported by the Academy of Finland, Research Centre for Computational Science and Engineering project nr. 44897 (Finnish Centre for Excellence Program 2000–2005). The work at Brookhaven National Laboratory was carried out under Contract nr. DE-AC02-98CH10886, Division of Material Science, U.S. Department of Energy.

A Maxwell-Boltzmann distribution in D dimensions

Here we show that for integer or half-integer values of the parameter n the Gamma distribution

$$\gamma_n(\xi) = \Gamma(n)^{-1} \xi^{n-1} \exp(-\xi), \quad (17)$$

where $\Gamma(n)$ is the Gamma function, represents the distribution of the rescaled kinetic energy $\xi = K/T$ for a classical mechanical system in $D = 2n$ dimensions. In this section, T represents the absolute temperature of the system multiplied by the Boltzmann constant k_B .

We start from a system Hamiltonian of the form

$$H(\mathbf{P}, \mathbf{Q}) = \frac{1}{2} \sum_{i=1}^N \frac{\mathbf{p}_i^2}{m_i} + V(\mathbf{Q}), \quad (18)$$

where $\mathbf{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_N\}$ and $\mathbf{Q} = \{\mathbf{q}_1, \dots, \mathbf{q}_N\}$ are the momentum and position vectors of the N particles, while $V(\mathbf{Q})$ is the potential energy contribution to the total energy. For systems of this type, in which the total energy factorizes as a sum of kinetic and potential contributions, the normalized probability distribution in momentum space is simply $f(\mathbf{P}) = \prod_i (2\pi m_i T)^{-D/2} \exp(-\mathbf{p}_i^2/2m_i T)$. Thus, since the kinetic energy distribution factorizes as a sum of single particle contributions, the probability density factorizes as a product of single particle densities, each one of the form

$$f(\mathbf{p}) = \frac{1}{(2\pi m T)^{D/2}} \exp\left(-\frac{\mathbf{p}^2}{2mT}\right), \quad (19)$$

where $\mathbf{p} = (p_1, \dots, p_D)$ is the momentum of a generic particle. It is convenient to introduce the momentum modulus p of a particle in D dimensions,

$$p^2 \equiv \mathbf{p}^2 = \sum_{k=1}^D p_k^2, \quad (20)$$

where the p_k 's are the Cartesian components, since the distribution (19) depends only on $p \equiv \sqrt{\mathbf{p}^2}$. One can then integrate the distribution over the $D - 1$ angular variables to obtain the momentum modulus distribution function, with the help of the formula for the surface of a hypersphere of radius p in D dimensions,

$$S_D(p) = \frac{2\pi^{D/2}}{\Gamma(D/2)} p^{D-1}. \quad (21)$$

One obtains

$$f(p) = S_D(p) f(\mathbf{p}) = \frac{2}{\Gamma(D/2)(2mT)^{D/2}} p^{D-1} \exp\left(-\frac{p^2}{2mT}\right). \quad (22)$$

The corresponding distribution for the kinetic energy $K = p^2/2m$ is therefore

$$f(K) = \left[\frac{dp}{dK} f(p) \right]_{p=\sqrt{2mK}} = \frac{1}{\Gamma(D/2)T} \left(\frac{K}{T} \right)^{D/2-1} \exp\left(-\frac{K}{T}\right). \quad (23)$$

Comparison with the Gamma distribution, Eq. (17), shows that the Maxwell-Boltzmann kinetic energy distribution in D dimensions can be expressed as

$$f(K) = T^{-1} \gamma_{D/2}(K/T). \quad (24)$$

The distribution for the rescaled kinetic energy,

$$\xi = K/T, \quad (25)$$

is just the Gamma distribution of order $D/2$,

$$f(\xi) = \left[\frac{dK}{d\xi} f(K) \right]_{K=\xi T} = \frac{1}{\Gamma(D/2)} \xi^{D/2-1} \exp(-\xi) \equiv \gamma_{D/2}(\xi). \quad (26)$$

References

1. Encyclopaedia Britannica, www.britannica.com/ebc/article?tocId=9372448.
2. F. Shostak, Quarterly J. Australian Econ. 3 (2000) 69.
3. www.answers.com/income&r=67.
4. E. Bennati, La simulazione statistica nell'analisi della distribuzione del reddito: modelli realistici e metodo di Montecarlo, ETS Editrice, Pisa, 1988.
5. E. Bennati, Un metodo di simulazione statistica nell'analisi della distribuzione del reddito, Rivista Internazionale di Scienze Economiche e commerciali (1988) 735–756.
6. E. Bennati, Il metodo di Montecarlo nell'analisi economica, Rassegna di lavori dell'ISCO 10 (1993) 31.
7. A. Dragulescu, V. M. Yakovenko, Statistical mechanics of money, Eur. Phys. J. B 17 (2000) 723.
8. A. Chakraborti, B. K. Chakrabarti, Statistical mechanics of money: how saving propensity affects its distribution, Eur. Phys. J. B 17 (2000) 167.
9. A. Chakraborti, Distributions of money in model markets of economy, Int. J. Mod. Phys. C 13 (2002) 1315.
10. A. Chatterjee, B. K. Chakrabarti, S. S. Manna, Money in gas-like markets: Gibbs and Pareto laws, Physica Scripta T106 (2003) 367.
11. A. Chatterjee, B. K. Chakrabarti, S. S. Manna, Pareto law in a kinetic model of market with random saving propensity, Physica A 335 (2004) 155.
12. A. Chatterjee, B. K. Chakrabarti, R. B. Stinchcombe, Master equation for a kinetic model of trading market and its analytic solution, <http://arxiv.org/abs/cond-mat/0501413>.
13. V. Pareto, Cours d'économie politique, Rouge, Lausanne, 1897.
14. M. Patriarca, A. Chakraborti, K. Kaski, Gibbs versus non-Gibbs distributions in money dynamics, Physica A 340 (2004) 334.
15. M. Patriarca, A. Chakraborti, K. Kaski, Statistical model with a standard Gamma distribution, Phys. Rev. E 70 (2004) 016104.
16. A. Das, S. Yarlagadda, A distribution function analysis of wealth distribution, <http://arxiv.org/abs/cond-mat/0310343>.
17. P. Repetowicz, S. Hutzler, P. Richmond, Dynamics of money and income distributions, <http://arxiv.org/abs/cond-mat/0407770>.
18. See however similar findings of S. S. Manna in this volume.
19. See the contribution by A. Chatterjee in this volume.