CS 441 Discrete Mathematics for CS
Lecture 4

Predicate logic

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Announcements

- **Homework assignment 1 due today**
- **Homework assignment 2:**
  - **posted on the course web page**
  - **Due on Thursday January 23, 2013**
- **Recitations today and tomorrow:**
  - **Practice problems related to assignment 2**

Propositional logic: limitations

Propositional logic: the world is described in terms of elementary propositions and their logical combinations

Elementary statements/propositions:

- **Typically refer to objects, their properties and relations. But these are not explicitly represented** in the propositional logic
  - **Example:**
    - "John is a UPitt student."

      John → **a Upitt student**

      object → a property
    - Objects and properties are hidden in the statement, it is not possible to reason about them

Propositional logic: limitations

(1) Statements that hold for many objects must be enumerated

- **Example:**
  - John is a CS UPitt graduate → John has passed cs441
  - Ann is a CS Upitt graduate → Ann has passed cs441
  - Ken is a CS Upitt graduate → Ken has passed cs441
  - …
- **Solution:** make statements with **variables**
  - x is a CS UPitt graduate → x has passed cs441

Propositional logic: limitations

(2) Statements that define the property of the group of objects

- **Example:**
  - All new cars must be registered.
  - Some of the CS graduates graduate with honor.
- **Solution:** make statements with **quantifiers**
  - **Universal quantifier** –the property is satisfied by all members of the group
  - **Existential quantifier** – at least one member of the group satisfy the property

Predicate logic

Remedies the limitations of the propositional logic

- Explicitly models objects and their properties
- Allows to make statements with variables and quantify them

Predicate logic:

- **Constant** –models a specific object

  Examples: "John", "France", "7"
- **Variable** – represents object of specific type **(defined by the *universe of discourse*)**

  Examples: x, y

  (universe of discourse can be people, students, numbers)
- **Predicate -** over one, two or many variables or constants.
  - Represents properties or relations among objects

  Examples: Red(car23), student(x), married(John,Ann)

Predicates

Predicates represent properties or relations among objects

- A predicate P(x) assigns a value **true or false** to each x depending on whether the property holds or not for x.
- The assignment is best viewed as a big table with the variable x substituted for objects from ***the universe of discourse***

Example:

- Assume **Student(x)** where the universe of discourse are people
- Student(John) …. T (if John is a student)
- Student(Ann) …. T (if Ann is a student)
- Student(Jane) ….. F (if Jane is not a student)
- …

Predicates

Assume a predicate P(x) that represents the statement:

- **x is a prime number**

Truth values for different x:

- P(2) T
- P(3) T
- P(4) F
- P(5) T
- P(6) F

All statements P(2), P(3), P(4), P(5), P(6) are propositions

…

But P(x) with variable x is not a proposition

Quantified statements

Predicate logic lets us to make statements about groups of objects

- To do this we use **special quantified expressions**

Two types of quantified statements:

- **universal**

  Example: ‘ all CS Upitt graduates have to pass cs441”

  – the statement is true for all graduates
- **existential**

  Example: ‘Some CS Upitt students graduate with honor.’

  – the statement is true for some people

Universal quantifier

Quantification converts a propositional function into **a proposition** by binding a variable to a set of values from the universe of discourse.

Example:

- Let P(x) denote x > x - 1. Assume x are real numbers.
- Is P(x) a proposition? **No.** Many possible substitutions.
- Is $\forall$x P(x) a proposition? **Yes.**
- What is the truth value for $\forall$x P(x) ?
  – **True**, since P(x) holds for all x.

Existential quantifier

Quantification converts a propositional function into **a proposition** by binding a variable to a set of values from the universe of discourse.

Example:

- Let T(x) denote x > 5 and x is from Real numbers.
- Is T(x) a proposition? **No.**
- Is $\exists$x T(x) a proposition? **Yes.**
- What is the truth value for $\exists$x T(x) ?
  - Since 10 > 5 is true. Therefore, $\exists$x T(x) is **true.**

Summary of quantified statements

- When $\forall$**x P(x)** and $\exists$**x P(x)** are true and false?

Statement	When true?	When false?
$\forall$x P(x)	P(x) true for all x	There is an x where P(x) is false.
$\exists$x P(x)	There is some x for which P(x) is true.	P(x) is false for all x.

Suppose the elements in the universe of discourse can be enumerated as x1, x2, ..., xN then:

- **$\forall$x P(x) is true whenever P(x1) $\wedge$ P(x2) $\wedge$ ... $\wedge$ P(xN) is true**
- **$\exists$x P(x) is true whenever P(x1) $\vee$ P(x2) $\vee$ ... $\vee$ P(xN) is true**.

Translation with quantifiers

Sentence:

- All Upitt students are smart.

- **Assume:** the domain of discourse of x are Upitt students
- **Translation:**
- $\forall x\ Smart(x)$

- **Assume:** the universe of discourse are students (all students):
- $\forall x\ at(x,Upitt) \rightarrow Smart(x)$

- **Assume:** the universe of discourse are people:
- $\forall x\ student(x) \wedge at(x,Upitt) \rightarrow Smart(x)$

Translation with quantifiers

Sentence:

- Someone at CMU is smart.

- **Assume:** the domain of discourse are all CMU affiliates
- **Translation:**
- $\exists\ x\ Smart(x)$

- **Assume:** the universe of discourse are people:
- $\exists\ x\ at(x,CMU) \wedge Smart(x)$

Translation with quantifiers

- **Assume two predicates S(x) and P(x)**

Universal statements typically tie with implications

- **All S(x) is P(x)**
  - $\forall x \, (S(x) \rightarrow P(x))$
- **No S(x) is P(x)**
  - $\forall x (S(x) \rightarrow \neg P(x))$

Existential statements typically tie with conjunctions

- **Some S(x) is P(x)**
  - $\exists x \, (S(x) \wedge P(x))$
- **Some S(x) is not P(x)**
  - $\exists x \, (S(x) \wedge \neg P(x))$

Nested quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:

- Every real number has its corresponding negative.
- **Translation:**
  - Assume:
    - a real number is denoted as x and its negative as y
    - A predicate P(x,y) denotes: "x + y =0"
- Then we can write:

  $\forall x \, \exists \, y \; P(x,y)$

Nested quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:

- There is a person who loves everybody.
- **Translation:**
  - Assume:
    - Variables x and y denote people
    - A predicate L(x,y) denotes: “x loves y”
- Then we can write in the predicate logic:

  $\exists x \forall y \; L(x,y)$

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Order of quantifiers

The order of nested quantifiers matters if quantifiers are of different type

- $\forall x \exists y \; L(x,y)$ is not the same as $\exists y \forall x \; L(x,y)$

Example:

- Assume L(x,y) denotes “x loves y”

- Then: $\forall x \exists y \; L(x,y)$
- Translates to: Everybody loves somebody.
- And: $\exists y \; \forall x \; L(x,y)$
- Translates to: There is someone who is loved by everyone.

The meaning of the two is different.

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Order of quantifiers

The order of nested quantifiers does not matter if quantifiers are of the same type

Example:

- For all x and y, if x is a parent of y then y is a child of x
- **Assume:**
  - Parent(x,y) denotes "x is a parent of y"
  - Child(x,y) denotes "x is a child of y"

- Two equivalent ways to represent the statement:
  - $\forall x\ \forall y\ Parent(x,y) \rightarrow Child(y,x)$
  - $\forall y\ \forall x\ Parent(x,y) \rightarrow Child(y,x)$

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Translation exercise

Suppose:

- Variables x,y denote people
- L(x,y) denotes "x loves y".

Translate:

- Everybody loves Raymond. $\forall x\ L(x,Raymond)$
- Everybody loves somebody. $\forall x \exists y\ L(x,y)$
- There is somebody whom everybody loves. $\exists y \forall x\ L(x,y)$
- There is somebody who Raymond doesn't love.
  $\exists y \neg L(Raymond,y)$
- There is somebody whom no one loves.
  $\exists y\ \forall x\ \neg L(x,y)$

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