Basic Concepts of Set Theory, Functions and Relations

Based on: Chapters 1 and 2 of Partee, Barbara H., Meulen, Alice ter, and Wall, Robert. 1990. *Mathematical Methods in Linguistics*. Dordrecht: Kluwer. Also "Preliminaries" from Partee 1979, *Fundamentals of Mathematics for Linguistics*.

1. Basic Concepts of Set Theory.

1.1. Sets and elements

Set theory is a basis of modern mathematics, and notions of set theory are used in all formal descriptions. The notion of set is taken as "undefined", "primitive", or "basic", so we don't try to *define* what a set is, but we can give an informal description, describe important properties of sets, and give examples. All other notions of mathematics can be built up based on the notion of set.

Similar (but informal) words: collection, group, aggregate.
Description: a *set* is a collection of objects which are called the *members* or *elements* of that set. If we have a set we say that some objects *belong* (or *do not belong*) to this set, *are* (or *are not*) *in* the set. We say also that sets *consist* of their elements.

Examples: the set of students in this room; the English alphabet may be viewed as the set of letters of the English language; the set of natural numbers[1]; etc.

So sets can consist of elements of various natures: people, physical objects, numbers, signs, other sets, etc. (We will use the words *object* or *entity* in a very broad way to include all these different kinds of things.)

A set is an ABSTRACT object; its members do not have to be physically collected together for them to constitute a set.

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[1] Natural numbers: 0,1,2,3,4,5,...   . No notion of positive or negative. The numbers used for "counting". Integers: positive, negative, and 0. See xeroxed section "Preliminaries" from Partee 1979.

The membership criteria for a set must in principle be well-defined, and not vague. If we have a set and an object, it is possible that we do not know whether this object belongs to the set or not, because of our lack of information or knowledge. (E.g. "The set of students in this room over the age of 21": a well-defined set but we may not know who is in it.) But the answer should exist, at any rate in principle. It could be unknown, but it should not be vague. If the answer is vague for some collection, we cannot consider that collection as a set. Another thing: If we have a set, then for any two elements of it, *x* and *y*, it should not be vague whether *x* = *y*, or they are different. (If they are identical, then they are not actually "two" elements of it; the issue really arises when we have two *descriptions* of elements, and we want to know whether those descriptions describe the same element, or two different elements.)

For example: is the letter q the same thing as the letter Q? Well, it depends on what set we are considering. If we take the set of the 26 letters of the English alphabet, then q and Q are the same element. If we take the set of 52 upper-case and lower-case letters of the English alphabet, then q and Q are two distinct elements. Either is possible, but we have to make it clear what set we are talking about, so that we know whether or not q = Q.

Sometimes we simply assume for the sake of examples that a description is not vague when perhaps for other purposes it would be vague – e.g., the set of all red objects.

Sets can be *finite* or *infinite*.

There is exactly one set, the *empty set*, or *null set,* which has no members at all.

A set with only one member is called a *singleton* or a *singleton set*. ("*Singleton of a*")

Notation: *A, B, C, ...* for sets; *a, b, c, ...* or *x, y, z, ...* for members.
$b \in A$ if *b* belongs to *A* ($B \in A$ if both *A* and *B* are sets and *B* is a member of *A*) and $c \notin A$, if *c* doesn't belong to *A*.

$\varnothing$ is used for the empty set.

1.2. Specification of sets

There are three main ways to specify a set:
(1) by listing all its members (*list notation*);
(2) by stating a property of its elements (*predicate notation*);
(3) by defining a set of rules which generates (defines) its members (*recursive rules*).

List notation. The first way is suitable only for finite sets. In this case we list names of elements of a set, separate them by commas and enclose them in braces:
Examples: {1, 12, 45}, {George Washington, Bill Clinton}, {a,b,d,m}.
"Three-dot abbreviation": {1,2, ..., 100}. (See xeroxed "preliminaries", pp xxii-xxiii)

{1,2,3,4,…} – this is not a real list notation, it is not a finite list, but it's common practice as long as the continuation is clear.
Note that we do not care about the order of elements of the list, and elements can be listed several times. {1, 12, 45}, {12, 1, 45,1} and {45,12, 45,1} are different representations of the same set (see below the notion of identity of sets).

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Predicate notation. Example:
$\{x \mid x$ is a natural number and $x < 8\}$
Reading: "the set of all x such that x is a natural number and is less than 8"
So the second part of this notation is a property the members of the set share (a condition or a predicate which holds for members of this set).

Other examples:
$\{x \mid x$ is a letter of Russian alphabet$\}$
$\{y \mid y$ is a student of UMass and y is older than 25$\}$

General form:
$\{x \mid P(x)\}$, where P is some predicate (condition, property).

The language to describe these predicates is not usually fixed in a strict way. But it is known that unrestricted language can result in *paradoxes*. Example: $\{x \mid x \notin x\}$. ("Russell's paradox") -- see the historical notes about it on pp 7-8. The moral: not everything that looks on the surface like a predicate can actually be considered to be a good defining condition for a set. Solutions – type theory, other solutions; we won't go into them. (If you're interested, see Chapter 8, Sec 2.)

Recursive rules. (Always safe.) Example – the set E of even numbers greater than 3:
a) $4 \in E$
b) if $x \in E$, then $x + 2 \in E$
c) nothing else belongs to E.

The first rule is the basis of recursion, the second one generates new elements from the elements defined before and the third rule restricts the defined set to the elements generated by rules a and b. (The third rule should always be there; sometimes in practice it is left implicit. It's best when you're a beginner to make it explicit.)

1.3. Identity and cardinality
Two sets are identical *if and only if* [2] they have exactly the same members. So $A = B$ iff for every $x, x \in A \Leftrightarrow x \in B$.
For example, $\{0,2,4\} = \{x \mid x$ is an even natural number less than 5$\}$

From the definition of identity follows that there exists only one empty set; its identity is fully determined by its absence of members. Note that empty list notation $\{\}$ is not usually used for the empty set, we have a special symbol $\varnothing$ for it.
The number of elements in a set A is called the *cardinality* of A, written $|A|$. The cardinality of a finite set is a natural number. Infinite sets also have cardinalities but they are not natural numbers. We will discuss cardinalities of infinite sets a little later (Chapter 4).

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[2] Be careful about "if and only if"; its abbreviation is *iff.* See Preliminaries, p. xxiii.

1.4. Subsets

A set *A* is a *subset of a set B* iff every element of *A* is also an element of *B*. Such a relation between sets is denoted by $A \subseteq B$. If $A \subseteq B$ and $A \neq B$ we call *A* a proper subset of *B* and write $A \subset B$. (Caution: sometimes $\subset$ is used the way we are using $\subseteq$.)

   Both signs can be negated using the slash / through the sign.
Examples:
$\{a,b\} \subseteq \{d,a,b,e\}$ and $\{a,b\} \subset \{d,a,b,e\}$, $\{a,b\} \subseteq \{a,b\}$, but $\{a,b\} \not\subset \{a,b\}$.

Note that the empty set is a subset of every set. $\varnothing \subseteq A$ for every set *A*. Why?

Be careful about the difference between "member of" and "subset of"!

1.5. Power sets

The set of all subsets of a set *A* is called the *power set* of A and denoted as $\wp(A)$ or sometimes as 2^A.
For example, if $A = \{a,b\}$, $\wp(A) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$.

From the example above: $a \in A$; $\{a\} \subseteq A$; $\{a\} \in \wp(A)$
   $\varnothing \subseteq A$; $\varnothing \notin A$; $\varnothing \in \wp(A)$; $\varnothing \subseteq \wp(A)$

1.6. Operations on sets: union, intersection.

We define several operations on sets. Let *A* and *B* be arbitrary sets.

   The *union* of *A* and *B*, written $A \cup B$, is the set whose elements are just the elements of *A* or *B* or of both. In the predicate notation the definition is
   $$A \cup B =_{\text{def}} \{ x \mid x \in A \text{ or } x \in B\}$$

Examples. Let $K = \{a,b\}$, $L = \{c,d\}$ and $M = \{b,d\}$, then

   $$
   \begin{array}{lll}
   K \cup L & = & \{a,b,c,d\} \\
   K \cup M & = & \{a,b,d\} \\
   L \cup M & = & \{b,c,d\} \\
   (K \cup L) \cup M = K \cup (L \cup M) & = & \{a,b,c,d\} \\
   K \cup K & = K \\
   K \cup \varnothing = \varnothing \cup K = K & = & \{a,b\}.
   \end{array}
   $$

   The *intersection* of *A* and *B*, written $A \cap B$, is the set whose elements are just the elements of both *A* and *B*. In the predicate notation the definition is
   $$A \cap B =_{\text{def}} \{ x \mid x \in A \text{ and } x \in B\}$$
Examples:

   $$
   \begin{array}{lll}
   K \cap L & = & \varnothing \\
   K \cap M & = & \{b\} \\
   L \cap M & = & \{d\} \\
   (K \cap L) \cap M = K \cap (L \cap M) & = & \varnothing \\
   K \cap K & = K \\
   K \cap \varnothing = \varnothing \cap K & = & \varnothing.
   \end{array}
   $$

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1.7 More operations on sets: difference, complement

Another binary operation on arbitrary sets is the *difference "A minus B"*, written $A - B$, which 'subtracts' from A all elements which are in B. [Also called *relative complement: the complement of B relative to A.*] The predicate notation defines this operation as follows:

$$A - B =_{\text{def}} \{\, x \mid x \in A \text{ and } x \notin B \}$$

Examples: (using the previous *K, L, M*)

$K - L$	=	$\{a,b\}$
$K - M$	=	$\{a\}$
$L - M$	=	$\{c\}$
$K - K$	=	$\varnothing$
$K - \varnothing$	=	K
$\varnothing - K$	=	$\varnothing$.

$A - B$ is also called the *relative complement* of B relative to A. This operation is to be distinguished from the *complement* of a set A, written A', which is the set consisting of everything not in A. In predicate notation

$$A' =_{\text{def}} \{\, x \mid x \notin A \}$$

It is natural to ask, where do these objects come from which do not belong to A? In this case it is presupposed that there exists a *universe of discourse* and all other sets are subsets of this set. The universe of discourse is conventionally denoted by the symbol U. Then we have

$$A' =_{\text{def}} U - A$$

1.8. Set-theoretic equalities

There are a number of general laws about sets which follow from the definitions of set-theoretic operations, subsets, etc. A useful selection of these is shown below. They are grouped under their traditional names. These equations below hold for any sets X, Y, Z:

1. *Idempotent Laws*
   (a) $X \cup X = X$ (b) $X \cap X = X$

2. *Commutative Laws*
   (a) $X \cup Y = Y \cup X$ (b) $X \cap Y = Y \cap X$

3. *Associative Laws*
   (a) $(X \cup Y) \cup Z = X \cup (Y \cup Z)$ (b) $(X \cap Y) \cap Z = X \cap (Y \cap Z)$

4. *Distributive Laws*
   (a) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ (b) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

5. *Identity Laws*
  (a) $X \cup \varnothing = X$ (c) $X \cap \varnothing = \varnothing$
  (b) $X \cup U = U$ (d) $X \cap U = X$

6. *Complement Laws*
  (a) $X \cup X' = U$ (c) $X \cap X' = \varnothing$
  (b) $(X')' = X$ (d) $X - Y = X \cap Y'$

7. *DeMorgan's Laws*
  (a) $(X \cup Y)' = X' \cap Y'$ (b) $(X \cap Y)' = X' \cup Y'$

8. Consistency Principle
  (a) $X \subseteq Y$ iff $X \cup Y = Y$ (b) $X \subseteq Y$ iff $X \cap Y = X$

Chapter 2. Relations and Functions

Much of mathematics can be built up from set theory – this was a project which was carried out by philosophers, logicians, and mathematicians largely in the first half of the 20th century. Whitehead and Russell were among the pioneers, with their great work *Principia Mathematica.* Defining mathematical notions on the basis of set theory does not add anything "mathematical", and is not of particular interest to the "working mathematician", but it is of great interest for the foundations of mathematics, showing how little needs to be assumed as "primitive".

We illustrate some bits of that project here, with some basic set-theoretic definitions of ordered pairs, relations, and functions, along with some standard notions concerning relations and functions.

2.1. Ordered pairs and Cartesian products

As we see, there is no order imposed on the elements of a set. To describe functions and relations we will need the notion of an ***ordered pair***, written $<a,b>$, for example, in which *a* is considered the ***first member (element)*** and *b* is the ***second member (element)*** of the pair. So, in general, $<a,b> \neq <b,a>$. (Whereas for a set, $\{a,b\} = \{b,a\}$.)

Is there a way to define ordered pairs in terms of sets? You might think not, since sets are themselves unordered. But there are in fact various ways it can be done. Here is one way to do it, usually considered the most conventional:

The ordered pair can be defined as follows:

Definition: $<a,b> =_{\text{def}} \{\{a\}, \{a,b\}\}$

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How can we be sure that that definition does the job it's supposed to do? What's crucial is that for every ordered pair, there is indeed exactly one corresponding set of the form {{a}, {a,b}}, and two different ordered pairs always have two different corresponding sets. We won't try to prove that that holds, but it does.

There would be nothing wrong with taking the notion of ordered pair as another primitive notion, alongside the notion of set. But mathematicians like seeing how far they can reduce the number of primitives, and it's an interesting discovery to see that the notion of order can be defined in terms of set theory.

Cartesian product. Suppose we have two sets *A* and *B* and we form ordered pairs by taking an element of *A* as the first member of the pair and an element of *B* as the second member. The *Cartesian product* of *A* and *B,* written $A \times B$, is the set consisting of all such pairs. The predicate notation defines it as:

$$A \times B =_{\text{def}} \{<x,y> \mid x \in A \text{ and } y \in B\}$$

What happens if either A or *B* is $\varnothing$? Suppose A = {a,b}. What is $A \times \varnothing$?

Here are some examples of Cartesian products:

Let *K* = {a,b,c} *and L* = {1,2}, then

$$
\begin{aligned}
K \times L &= \{<a,1>,<a,2>,<b,1>,<b,2>,<c,1>,<c,2>\} \\
L \times K &= \{<1,a>,<2,a>,<1,b>,<2,b>,<1,c>,<2,c>\} \\
L \times L &= \{<1,1>,<1,2>,<2,1>,<2,2>\}
\end{aligned}
$$

An aside on cardinality, and why Cartesian products are called **products** (the "Cartesian" part comes from the name of René Descartes, their inventor). Look at the cardinalities of the sets above, and see if you can figure out in general what the cardinality of the set $A \times B$ will be, given the cardinalities of sets *A* and *B*.

What about ***ordered triples***? The definition of ordered pairs can be extended to ordered triples and in general to ordered *n*-tuples for any natural *n*. For example, ordered triples are usually defined as:

$$<a,b,c> =_{\text{def}} <<a,b>,c>$$

And for three sets *A, B* and *C* the Cartesian product can be defined as
$$A \times B \times C =_{\text{def}} ((A \times B) \times C)$$

In the case when $A = B = C = \ldots$ a special notation is used: $A \times A = A^2,\ A \times A \times A = A^3$, etc. And we put $A^1 =_{\text{def}} A$. (This notation mimics the notation used for multiplication and exponents. It can, because the parallels hold quite uniformly.)

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2.2. Relations

In natural language *relations* are a kind of links existing between objects. Examples:
'mother of', 'neighbor of', "part of", 'is older than', 'is an ancestor of', 'is a subset of', etc.
These are binary relations. Formally we will define relations between elements of sets.

   We may write *Rab* or *aRb* for "*a* bears R to b". And when we formalize relations as
sets of ordered pairs of elements, we will officially write $<a,b> \in R$.

If *A* and *B* are any sets and $R \subseteq A \times B$, we call *R* a **binary relation from A to B** or a **binary
relation between A and B**. A relation $R \subseteq A \times A$ is called a relation *in* or *on A*.
The set **dom** $R = \{a \mid <a,b> \in R$ for some $b\}$ is called the *domain* of the relation *R* and the
set **range** $R = \{b \mid <a,b> \in R$ for some $a\}$ is called the *range* of the relation *R*.

   We may visually represent a relation *R* between two sets *A* and *B* by arrows in a
diagram displaying the members of both sets. In Figure 2-1 in PtMW [Partee, ter Meulen,
and Wall], $A = \{a.b\}$, $B = \{c,d,e\}$, and the arrows represent a set-theoretic relation $R = \{<a,d>,<a,e>, <b,c>\}$.
   [see Fig 2-1, p. 29.]

   Let us consider some **operations on relations**. The ***complement*** of a relation
$R \subseteq A \times B$ is defined as

$R' =_{def} (A \times B) - R$.

Note that what the complement of a relation is depends on what universe we are
considering. A given relation may certainly be a subset of more than one Cartesian
product, and its complement will differ according to what Cartesian product we are taking
to be the relevant universe.

What is the complement of the relation $R = \{<a,d>,<a,e>, <b,c>\}$ on the universe $\{a,b\} \times \{c,d,e\}$? (Answer: R' $= \{<a,c>, <b,d>, <b,e>\}$.)

   The ***inverse*** of a relation $R \subseteq A \times B$ is defined as the relation $R^{-1} \subseteq B \times A$,
$R^{-1} =_{def} \{<b,a> \mid <a,b> \in R\}$. Note that $(R^{-1})^{-1} = R$.

For example, for the relation *R* given above,
$R' = \{<a,c>,<b,d>,<b,e>\}$ and $R^{-1} = \{<d,a>,<e,a>,<c,b>\}$.

More examples: Let $\mathbb{N}$ be the set of natural numbers, $\{0, 1, 2, 3, 4, ... \}$
               Let R be "is less than" on $\mathbb{N}$ (i.e., on $\mathbb{N} \times \mathbb{N}$)
   Then what is R'?
   What is R^{-1}?

   We have focused so far on *binary* relations, i.e., sets of ordered pairs. In a similar way
we could define ***ternary, quaternary* or *just n-place* relations** consisting respectively of
ordered triples, quadruples or *n*-tuples. A unary relation *R* on a set *A* is just a subset of the
set *A*.
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2.3. Functions

Examples of functions: $f(x) = x^2 + 1$
                          $f(x) =$ the mother of x

Intuitively a function may be thought of as a "process" or as a **correspondence**.

A function is generally represented in set-theoretic terms as a special kind of relation.

Definition: A relation F from A to B is a **function from A to B** if and only if it meets both of the following conditions:

1. Each element in the domain of F is paired with just one element in the range, i.e., from $<a,b> \in F$ and $<a,c> \in F$ follows that $b = c$.

2. The domain of F is equal to A, **dom**$F = A$.

Equivalent definition: A function is a subset R of $A \times B$ such that each element of A occurs as the first member of exactly *one* ordered pair in R.

For example, consider the sets $A = \{a.b\}$ and $B = \{1,2,3\}$. The following relations from A to B are functions from A to B:

$P = \{<a,1>,<b,1>\}$
$Q = \{<a,2>,<b,3>\}$

The following relations from A to B are not functions from A to B:

$S = \{<a,1>\}$
$T = \{<a,2>,<b,1>,<b,3>\}$

S does not satisfy the condition 2, and T fails to meet condition 1. S is a function on the smaller domain $\{a\}$; T is not a function at all.

Much of the terminology used in talking about functions is the same as that for relations. We say that a function with domain A and range a subset of B is a function *from A to B*, while one in $A \times A$ is said to be a function *in* or *on A*. The notation '$F: A \rightarrow B$' is used for 'F is a function from A to B'. Elements of the domain of a function are called *arguments* and their correspondents in the range, *values*. If $<a,b> \in F$, the familiar notation $F(a) = b$ is used. 'Map', 'mapping' are commonly used synonyms for 'function'. A function **maps** each **argument** onto a corresponding **value**. A function $F: A^n \rightarrow A$ is also called an **n-ary operation** in **A.**

Functions as processes. Sometimes functions are considered in a different way, as processes, something like devices or boxes with inputs and outputs. We put the argument in the input and get the value of the function in output. In this case the set of ordered pairs in our definition is called the *graph* of the function.

Sometimes *partial* functions are considered. In this case the condition 2 in our definition can fail.

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Some terminology. Functions from *A* to *B* in the general case are said to be ***into B***. If the range of the function equals *B*, then the function is ***onto B*** (or ***surjection***). A function *F*: *A* → *B* is called ***one-to-one*** function (or ***injection***) just in case no member of *B* is assigned to more than one member of *A* (so if *a≠b*, then *F(a)≠F(b)*). A function which is both one-to-one and onto is called a ***one-to-one correspondence*** (or ***bijection***). It is easy to see that if a function *F* is one-to-one correspondence, then the relation F^{-1} is a function and one-to-one correspondence.

In Figure 2-2 three functions are indicated by the same sort of diagrams we introduced previously for relations. It is easy to see that functions *F* and *G* are <u>onto</u> but *H* is not.

[See PtMW, p. 32, Fig.2-2]

One useful class of functions are *characteristic functions* of sets. The *characteristic function of a set S,* considered as a subset of some larger domain *D,* is defined as follows:

$$F_S : D \rightarrow \{0,1\} : F_S(x) = 1 \text{ iff } x \in S$$
$$F_S(x) = 0 \text{ otherwise}$$

There is a one-to-one correspondence between sets and their characteristic functions. In semantics, where it is common to follow Frege in viewing much of semantic composition as carried out by function-argument application, it is often convenient to work with the characteristic functions of sets rather than with sets directly. Characteristic functions are used in many other applications as well.

2.4. Function composition

Given two functions *F*: *A* → *B* and *G*: *B* → *C*, we may form a new function from *A* to *C*, called the ***composition*** of *F* and *G,* written *G∘F*. Function composition is defined as

$$G \circ F =_{\text{def}} \{<x,z> | \text{ for some } y, <x,y> \in F \text{ and } <y,z> \in G \}$$

Figure 2-3 shows two functions *F* and *G* and their composition.

[See PtMW, p. 33, Fig.2-3]

The function *F*: *A* → *A* such that *F* = {<x,x> | x ∈ A} is called the ***identity function on A,*** written ***id***$_A$ (or 1$_A$). Given a function *F*: *A* → *B* that is a one-to-one correspondence, we have the following equations:

   $F^{-1} \circ F = id_A,$

   $F \circ F^{-1} = $ what? [if we don't get this far in class, the answer is in the book.]

The definition of composition need not be restricted to functions but can be applied to relations in general. Given relations $R \subseteq A \times B$ and $S \subseteq B \times C$ the composite of *R* and *S*, written $S \circ R =_{\text{def}} \{<x,z> | \text{ for some } y, <x,y> \in R \text{ and } <y,z> \in S \}$

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