Introduction to Predicate Logic

Ling324

Reading: *Meaning and Grammar*, pg. 113-141

Usefulness of Predicate Logic for Natural Language Semantics

- While in propositional logic, we can only talk about sentences as a whole, predicate logic allows us to decompose simple sentences into smaller parts: predicates and individuals.

(1) a. John is tall.
    b. T(j)

- Predicate logic provides a tool to handle expressions of generalization: i.e., quantificational expressions.

(2) a. Every cat is sleeping.
    b. Some girl likes David.
    c. No one is happy.

- Predicate logic allows us to talk about variables (pronouns).

  The value for the pronoun is some individual in the domain of universe that is contextually determined.

(3) a. It is sleeping.
    b. She likes David.
    c. He is happy.

Usefulness of Predicate Logic for Natural Language Semantics (cont.)

- Sentences with quantificational expressions can be divided into two interpretive components.

(4) Every cat is sleeping.

a. (every cat)(it is sleeping)

b. for all x, x is a cat, x is sleeping

c. = true iff 'it is sleeping' is true for all possible values for 'it' in the domain

  - A simple sentence containing a variable/pronoun (a place holder):

    It can be evaluated as true or false with respect to some individual contextually taken as a value for the pronoun.

  - The quantificational expression:

    It instructs us to limit the domain of individuals being considered to a relevant set (e.g., a set of cats), and tells us how many different values of the pronoun we have to consider from that domain to establish truth for the sentence.

Usefulness of Predicate Logic for Natural Language Semantics (cont.)

(5) Some girl likes David.

a. (some girl)(she likes David)

b. for some x, x is a girl, x likes David

c. = true iff ‘she likes David’ is true for at least one possible value for ‘she’

(6) No one is happy.

a. (no one)(s/he is happy)

b. for all x, x is a person, x is not happy

c. = true iff ‘s/he is happy’ is false for all possible values for ‘s/he’

Syntax of Predicate Logic

1. Primitive vocabulary

   (a) A set of terms:

   A set of individual constants: a, b, c, d, ...

   John, Mary, Pavarotti, Loren

   A set of individual variables: $x, y, z, x_0, x_1, x_2, \ldots$

   he, she, it

   (b) A set of predicates: P, Q, R, ...

   One-place predicates: *is happy, is boring*

   Two-place predicates: *like, hate, love, hit*

   Three-place predicates: *introduce, give*

   (c) A binary identity predicate: =

   (d) The connectives of propositional logic: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

   (e) Quantifiers: $\forall, \exists$

   (f) brackets: (,), [,]

2. Syntactic rules

(a) If P is an n-place predicate and $t_1, ..., t_n$ are all terms, then $P(t_1, ..., t_n)$ is an atomic formula.

John is happy: H(j)
John loves Mary: L(j,m)
John introduced Mary to Sue: I(j,m,s)

(b) If t_1 and t_2 are individual constants or variables, then $t_1 = t_2$ is a formula.

John is Bill: j=b

(c) If ϕ is a formula, then $\neg\phi$ is a formula.

(d) If ϕ and ψ are formulas, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ are formulas too.

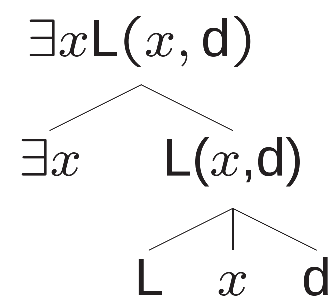
(e) If ϕ is a formula and x is a variable, then $\forall x\phi$, and $\exists x\phi$ are formulas too.

Everyone is happy: $\forall x\mathrm{H}(x)$
Someone loves John: $\exists x\mathrm{L}(x, \mathrm{j})$
Everyone loves someone: $\forall x\exists y\mathrm{L}(x, y)$, $\exists y\forall x\mathrm{L}(x, y)$

(f) Nothing else is a formula in predicate logic.

Syntactic Tree in Predicate Logic

- $\exists x \mathsf{L}(x, \mathsf{d})$



QUESTION: Draw the syntactic tree for the expression in (7) that are well-formed formulas of Predicate Logic.

(7) a. $\exists\forall$ (Qa $\longrightarrow$ PR(b)(c))

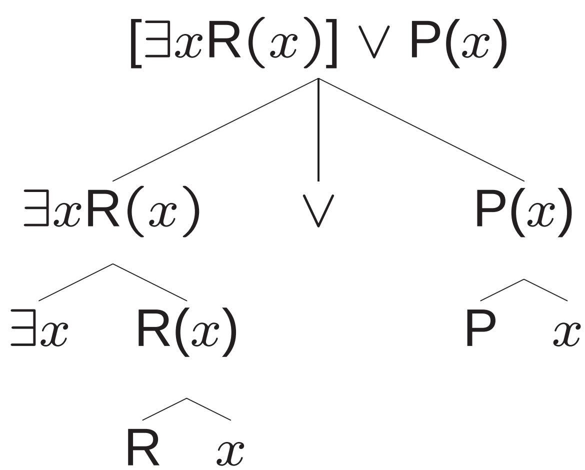
b. $\forall x$ (P(x) $\longrightarrow$ $\exists y$ Q(x, y))

c. $\exists x_1 \forall x_2 (\mathsf{P}(x_1, x_2) \longrightarrow (\mathsf{R}(x_1) \wedge \mathsf{Q}(x_2, \mathsf{a})))$

Some Syntactic Notions in Predicate Logic

- If x is a variable and ϕ is a formula to which a quantifier has been attached to produce $\forall x \phi$, or $\exists x \phi$, then we say that ϕ is the *scope* of the attached quantifier and that ϕ or any part of ϕ *lies in the scope* of that quantifier.

  Syntactically, the scope of a quantifier is what it c-commands.

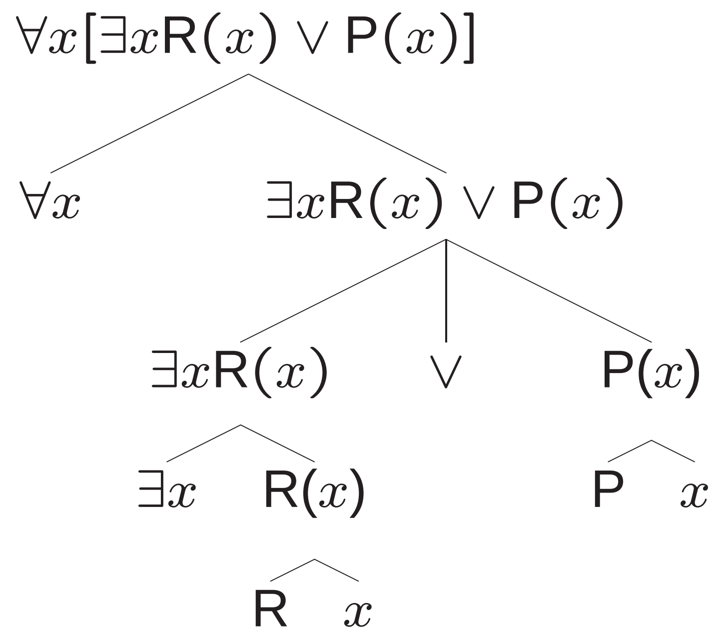


  What is the scope of $\exists x$?

Some Syntactic Notion in Predicate Logic (cont.)

- We say that an occurrence of a variable x is *bound* if it occurs in the scope of $\forall x$ or $\exists x$. A variable is *free* if it is not bound.

  Syntactically, an occurrence of x is bound by a lowest c-commanding quantifier Qx.



  x in R(x) is bound by $\exists x$;
  x in P(x) is bound by $\forall x$.

- $\forall y[\exists x \mathrm{R}(x) \vee \mathrm{P}(y)]$ and $\forall x[\exists x \mathrm{R}(x) \vee \mathrm{P}(x)]$ are *alphabetic variants* of each other and are semantically equivalent.

Some Syntactic Notions in Predicate Logic (cont.)

- Formulas with no free variables are called *closed formulas*, simply *formulas*, or *sentences*.

  H(j)
  $\forall x$ H(x)
  $\exists x$ L(j,x)
  $\exists x$ [H(x)$\vee$ L(x,j)]

  Those containing a free variable are called *open formulas*.

  H(x)
  [$\exists x$H(x)] $\vee$ L(x,j)
  $\forall x$ L(x, y)

Translations in Predicate Logic

(8) a. Every student is happy.

b. $\forall x$ [student(x) $\rightarrow$ happy(x)]

c. **wrong:** $\forall x$ [student(x) $\wedge$ happy(x)]

(9) a. Some students are happy.

b. $\exists x$ [student(x) $\wedge$ happy(x)]

c. **wrong:** $\exists x$ [student(x) $\rightarrow$ happy(x)]

(10) a. No student complained.

b. $\forall x$ [student(x) $\rightarrow$ $\neg$complained(x)]

c. $\neg\exists x$ [student(x) $\wedge$ complained(x)]

(11) a. Not every student complained.

b. $\neg\forall x$ [student(x) $\rightarrow$ complained(x)]

c. $\exists x$ [student(x) $\wedge$ $\neg$ complained(x)]

Translations in Predicate Logic (cont.)

QUESTION: Translate the following English sentences into predicate logic formula.

a. John likes Susan.

b. John has a cat.

c. A whale is a mammal.

d. Barking dogs don't bite.

e. Either every fruit is bitter or every fruit is sweet.

f. Every student heard some news. (possibly different news for each student)

g. There is some news that every student heard.

h. No student likes any exams.

Semantics of Predicate Logic: Trial 1

1. If α is a constant, then $[\![\alpha]\!]$ is specified by a function V (in the model M) that assigns an individual object to each constant.

   $[\![\alpha]\!]^M = V(\alpha)$

   If P is a predicate, then $[\![P]\!]$ is specified by a function V (in the model M) that assigns a set-theoretic objects to each predicate.

   $[\![P]\!]^M = V(P)$

2. If α is a variable, then ???

3. If P is an n-ary predicate and $t_1, ..., t_n$ are all terms (constants or variables), then for any model M, $[\![P(t_1, ..., t_n)]\!]^M = 1$ iff $< [\![t_1]\!]^M, ..., [\![t_n]\!]^M > \in [\![P]\!]^M$

4. If ϕ and ψ are formulas, then for any model M,

   $[\![\neg\phi]\!]^M$ = 1 iff $[\![\phi]\!]^M$ = 0

   $[\![\phi \wedge \psi]\!]^M$ = 1 iff $[\![\phi]\!]^M$ = 1 and $[\![\psi]\!]^M$ = 1

   $[\![\phi \vee \psi]\!]^M$ = 1 iff $[\![\phi]\!]^M$ = 1 or $[\![\psi]\!]^M$ = 1

   $[\![\phi \rightarrow \psi]\!]^M$ = 1 iff $[\![\phi]\!]^M$ = 0 or $[\![\psi]\!]^M$ = 1

   $[\![\phi \leftrightarrow \psi]\!]^M$ = 1 iff $[\![\phi]\!]^M$ = $[\![\psi]\!]^M$

5. If ϕ is a formula, and v is a variable, then, for any model M,

   $[\![\forall v\phi]\!]^M$ = 1 iff $[\![[c/v]\phi]\!]^M$ = 1 for all constants c.

   $[\![\exists v\phi]\!]^M$ = 1 iff $[\![[c/v]\phi]\!]^M$ = 1 for some constant c.

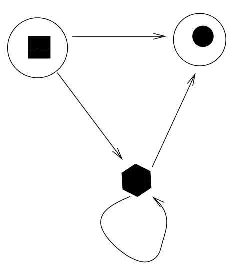
   $[c/v]\phi$: the formula resulting from having the constant c instead of the variable v in ϕ.

Semantics of Predicate Logic: Trial 1 (cont.)

- An example model

  Let us take the model M_1, depicted below. Let us take a language in Predicate Logic such that the constants a, b, and c denote the individuals dark box, dark circle and dark trapezoid, respectively, the unary predicate A denotes the set of individuals with a circle around, and the binary predicate R denotes the relation encoded by the arrows.

M1



U(niverse) = $\{$a, b, c$\}$

A = $\{$a, b$\}$

R = $\{<a,b>, <a,c>, <c,b>, <c,c>\}$

Determine the truth value of the following formulas in M_1.

(12) a. R(a,b) $\wedge$ R(b,b)

b. $\neg$A(c) $\rightarrow$ R(a,c)

c. $\forall x$ [R(x, x)]

d. $\forall x$ [R(x, x) $\leftrightarrow$ $\neg$A(x)]

e. $\exists x \exists y \exists z$ [R(x, y) $\wedge$ A(y) $\wedge$ R(x, z) $\wedge$ $\neg$A(z)]

Semantics of Predicate Logic: Trial 1 (cont.)

- Problem with the semantic rule having to do with quantifiers in Trial 1:

  In the semantic rule in Trial 1, quantifiers are ranging over constants, individuals that have names. But it could be that there is an individual that does not have a name. Then, this individual will be excluded from being considered when evaluation for truth.

  This suggests that when we are interpreting quantifiers, we need to range over individuals, not names.

Semantics of Predicate Logic: Trial 2

1. If α is a variable, then $[\![\alpha]\!]$ is specified by a variable assignment function g (in the model M) that assigns an individual object to each variable.

   $[\![\alpha]\!]^{M,g} = g(\alpha)$

   g: set of variables $\longrightarrow$ universe of individuals

   Let g_1 be an assignment function such that $g_1(x_1)$ = John, $g_1(x_2)$ = Mary, and for all n$\geq$3, $g_1(x_n)$ = Pete.

   $g_1 = \begin{bmatrix} x_1 \longrightarrow \text{John} \\ x_2 \longrightarrow \text{Mary} \\ x_n \longrightarrow \text{Pete} \end{bmatrix}$ where n$\geq$3

   Let g_2 be an assignment function such that $g_2(x_1)$ = Mary, $g_2(x_2)$ = John, and for all n$\geq$3, $g_2(x_n)$ = Pete.

   $g_2 = \begin{bmatrix} x_1 \longrightarrow \text{Mary} \\ x_2 \longrightarrow \text{John} \\ x_n \longrightarrow \text{Pete} \end{bmatrix}$ where n$\geq$3

   $[\![$He$_1$ is happy$]\!]^{M,g_1}$ = $[\![$happy$(x_1)]\!]^{M,g_1}$ = John is happy.
   $[\![$He$_3$ is happy$]\!]^{M,g_2}$ = $[\![$happy$(x_3)]\!]^{M,g_2}$ = Pete is happy.

2. If α is a constant, then $[\![\alpha]\!]$ is specified by a function V (in the model M) that assigns an individual object to each constant.

   $[\![\alpha]\!]^{M,g} = V(\alpha)$

   If P is a predicate, then $[\![P]\!]$ is specified by a function V (in the model M) that assigns a set-theoretic objects to each predicate.

   $[\![P]\!]^{M,g} = V(P)$

3. If P is an n-ary predicate and $t_1, ..., t_n$ are all terms (constants or variables), then for any model M and an assignment function g, $[\![P(t_1, ..., t_n)]\!]^{M,g} = 1$ iff $< [\![t_1]\!]^{M,g}, ..., [\![t_n]\!]^{M,g} > \in [\![P]\!]^{M,g}$

4. If ϕ and ψ are formulas, then for any model M and an assignment function g,

   $[\![\neg\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$
   $[\![\phi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$
   $[\![\phi \vee \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$
   $[\![\phi \rightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$
   $[\![\phi \leftrightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = [\![\psi]\!]^{M,g}$

5. If ϕ is a formula, and v is a variable, then, for any model M and an assignment function g,

$[\![\forall v\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g[d/v]} = 1$ for all individuals $d \in U$.
$[\![\exists v\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g[d/v]} = 1$ for some individual $d \in U$.

$g[d/v]$: the variable assignment g' that is exactly like g except (maybe) for $g(v)$, which equals the individual d.

$$g_1 = \begin{bmatrix} x_1 \longrightarrow \text{John} \\ x_2 \longrightarrow \text{Mary} \\ x_3 \longrightarrow \text{Pete} \\ x_n \longrightarrow \text{Pete} \end{bmatrix} \text{ where n}\geq 4$$

$$g_1[\text{John}/x_3] = \begin{bmatrix} x_1 \longrightarrow \text{John} \\ x_2 \longrightarrow \text{Mary} \\ x_3 \longrightarrow \text{John} \\ x_n \longrightarrow \text{Pete} \end{bmatrix} \text{ where n}\geq 4$$

$$g_1[[\text{John}/x_3]\text{Pete}/x_1] = \begin{bmatrix} x_1 \longrightarrow \text{Pete} \\ x_2 \longrightarrow \text{Mary} \\ x_3 \longrightarrow \text{John} \\ x_n \longrightarrow \text{Pete} \end{bmatrix} \text{ where n}\geq 4$$

Semantics of Predicate Logic: Trial 2 (cont.)

QUESTION: Complete the equivalences assuming: $g(x)$ = Mary, and $g(y)$ = Susan.

1. g[Paul/x](x) =
2. g[Paul/x](y) =
3. g[[Paul/x]Susan/x](x) =
4. g[[Paul/x]Susan/x](y) =
5. g[[Paul/x]Susan/y](x) =
6. g[[Paul/x]Susan/y](y) =

Compositional Interpretation

- Assume M_2, lexical meanings and assignment function g' specified below:

  U(niverse) = $\{$a, b, c$\}$
  $[\![$Allan$]\!]^{M_2,g'}$ = V(Allan) = a; $[\![$Betty$]\!]^{M_2,g'}$ = V(Betty) = b
  $[\![$H$]\!]^{M_2,g'}$ = V(H) = $\{$b, c$\}$
  $[\![$L$]\!]^{M_2,g'}$ = V(L) = $\{<$a,a$>$, $<$a,b$>$, $<$b,a$>$, $<$c,b$>\}$
  Assignment g':
  $[\![x]\!]^{M_2,g'}$ = $g'(x)$ = a; $[\![y]\!]^{M_2,g'}$ =$g'(y)$ = b.

- Syntax

  $\exists x$[H(x)]
  - $\exists x$
  - H(x)
    - H
    - x

  Semantic Interpretation

  $[\![\exists x$H$(x)]\!]^{M2,g'}$=1 iff for some $d \in U$, $[\![$H$(x)]\!]^{M2,g'[d/x]}$=1
  =1 iff for some $d \in U$, $d \in\{$b,c$\}$
  - $[\![$H$(x)]\!]^{M2,g'}$=1 iff $g'(x) \in \{$b,c$\}$
    - $[\![$H$]\!]^{M2,g'}$=V(H)=$\{$b,c$\}$
    - $[\![x]\!]^{M2,g'}$=$g'(x)$

If we do the same computation with respect to any other assignment g,
$[\![\exists x$H$(x)]\!]^{M_2,g}$ = 1. Hence, $[\![\exists x$H$(x)]\!]^{M_2}$ = 1.

Compositional Interpretation (cont.)

QUESTION: Draw the syntactic trees and spell out the compositional semantic interpretation of the following predicate logic formulas, against M2.

(13)
a. $\forall y[\mathrm{H}(y)]$
b. $\exists y[\mathrm{H}(\mathrm{Allan})]$
c. $\forall y[\mathrm{L}(y, x)]$
d. $\forall x \exists y[\mathrm{L}(x, y)]$
e. $\exists y \forall x[\mathrm{L}(x, y)]$

Compositional Interpretation: Examples

(13d) $\forall x \exists y[\mathsf{L}(x,y)]$

$[\![\forall x \exists y[\mathsf{L}(x,y)]]\!]^{M2,g'}$ = 1 iff for all $d' \in U$, $[\![\exists y[\mathsf{L}(x,y)]]\!]^{M2,g'[d'/x]}$ = 1.
= 1 iff for all $d' \in U$, there is a $d \in U$, such that $[\![\mathsf{L}(x,y)]\!]^{M2,g'[[d'/x]d/y]}$ = 1.
=1 iff for all $d' \in U$, there is a $d \in U$, such that $< d', d > \in \{$<a,a>,<a,b>,<b,a>,<c,b>$\}$.

$\forall x$

$[\![\exists y[\mathsf{L}(x,y)]]\!]^{M2,g'}$ = 1 iff there is a $d \in U$, such that $[\![\mathsf{L}(x,y)]\!]^{M2,g'[d/y]}$ = 1.
=1 iff there is a $d \in U$, such that $< [\![x]\!]^{M2,g'[d/y]}, [\![y]\!]^{M2,g'[d/y]} > \in [\![\mathsf{L}]\!]^{M2,g'[d/y]}$.
=1 iff there is a $d \in U$, such that $< g'(x), d > \in \{$<a,a>,<a,b>,<b,a>,<c,b>$\}$.

$\exists y$

$[\![L(x,y)]\!]^{M2,g'}$ = 1
iff $< g'(x), g'(y) > \in \{$<a,a>,<a,b>,<b,a>,<c,b>$\}$

$[\![\mathsf{L}]\!]^{M2,g'}$ = $V(\mathsf{L})$
= $\{$<a,a>,<a,b>,<b,a>,<c,b>$\}$

$[\![x]\!]^{M2,g'}$ = $g'(x)$

$[\![y]\!]^{M2,g'}$ = $g'(y)$

$\Longrightarrow [\![\forall x \exists y[\mathsf{L}(x,y)]]\!]^{M2,g'}$ = 1

Compositional Interpretation: Examples (cont.)

(13e) $\exists y \forall x[\mathsf{L}(x,y)]$

$[\![\exists y \forall x[\mathsf{L}(x,y)]]\!]^{M2,g'}$ = 1 iff there exists a $d' \in U$, such that $[\![\forall x[\mathsf{L}(x,y)]]\!]^{M2,g'[d'/y]}$ = 1.
=1 iff there is a $d' \in U$, such that for all $d \in U$, $[\![\mathsf{L}(x,y)]\!]^{M2,g'[[d'/y]d/x]}$ = 1.
=1 iff there is a $d' \in U$, such that for all $d \in U$, $< d, d' > \in \{$<a,a>,<a,b>,<b,a>,<c,b>$\}$.

$\exists y$

$[\![\forall x[\mathsf{L}(x,y)]]\!]^{M2,g'}$= 1 iff for all $d \in U$, $[\![\mathsf{L}(x,y)]\!]^{M2,g'[d/x]}$ = 1.
=1 iff for all $d \in U$, $< [\![x]\!]^{M2,g'[d/x]}, [\![y]\!]^{M2,g'[d/x]} > \in [\![\mathsf{L}]\!]^{M2,g'[d/x]}$.
=1 iff for all $d \in U$, $< d, g'(y) > \in \{$<a,a>,<a,b>,<b,a>,<c,b>$\}$.

$\forall x$

$[\![\mathsf{L}(x,y)]\!]^{M2,g'} = 1$
iff $< g'(x), g'(y) > \in \{$<a,a>,<a,b>,<b,a>,<c,b>$\}$

$[\![\mathsf{L}]\!]^{M2,g'} = V(\mathsf{L})$
= $\{$<a,a>,<a,b>,<b,a>,<c,b>$\}$

$[\![x]\!]^{M2,g'} = g'(x)$

$[\![y]\!]^{M2,g'} = g'(y)$

$\Longrightarrow [\![\exists y \forall x[\mathsf{L}(x,y)]]\!]^{M2,g'}$ = 0

Entailment, Logical Equivalence, Contradiction, Validity (cont.)

- A formula ϕ entails a formula ψ iff for every model M such that $[\![\phi]\!]$ = 1, $[\![\psi]\!]$ = 1.

  $\exists x \forall y [R(x, y)] \Rightarrow \forall y \exists x [R(x, y)]$

- A formula ϕ is valid iff for every model M, $[\![\phi]\!]$ = 1.

- A formula ϕ is contradictory iff for every model M, $[\![\phi]\!]$ = 0.

Entailment, Logical Equivalence, Contradiction, Validity

- A formula ϕ is logically equivalent to a formula ψ iff they entail each other (i.e., they are true in exactly the same models).

$$\neg\forall x[P(x)] \Leftrightarrow \exists x[\neg P(x)]$$

$$\forall x[P(x)] \Leftrightarrow \neg\exists x[\neg P(x)]$$

$$\neg\forall x[\neg P(x)] \Leftrightarrow \exists x[P(x)]$$

$$\forall x[\neg P(x)] \Leftrightarrow \neg\exists x[P(x)]$$

Entailment, Logical Equivalence, Contradiction, Validity (cont.)

Proof of $\forall x[\neg P(x)] \Leftrightarrow \neg\exists x[P(x)]$.

(i) Proof that $\forall x[\neg P(x)]$ entails $\neg\exists x[P(x)]$

– Assume that $[\![\forall x[\neg P(x)]]\!]^{M,g}$ = 1 for any model M and any assignment g.
– For all $d \in U$, $[\![\neg P(x)]\!]^{M,g[d/x]}$ = 1, by the semantics for $\forall$.
– For all $d \in U$, $[\![P(x)]\!]^{M,g[d/x]}$ = 0, by the semantics for $\neg$.
– There is no $d \in U$ such that $[\![P(x)]\!]^{M,g[d/x]}$ = 1.
– $[\![\exists x P(x)]\!]^{M,g}$ = 0, by the semantics of $\exists$.
– $[\![\neg\exists x P(x)]\!]^{M,g}$ = 1, by the semantics of $\neg$.

(ii) Proof that $\neg\exists x[P(x)]$ entails $\forall x[\neg P(x)]$

– Assume that $[\![\neg\exists x[P(x)]]\!]^{M,g}$ = 1 for any model M and any assignment g.
– $[\![\exists x[P(x)]]\!]^{M,g}$ = 0, by the semantics of $\neg$.
– There is no $d \in U$ such that $[\![P(x)]\!]^{M,g[d/x]}$ = 1, by the semantics of $\exists$.
– For all $d \in U$, $[\![P(x)]\!]^{M,g[d/x]}$ = 0.
– For all $d \in U$, $[\![\neg P(x)]\!]^{M,g[d/x]}$ = 1, by the semantics of $\neg$.
– $[\![\forall x \neg P(x)]\!]^{M,g}$ = 1, by the semantics of $\forall$.