Modern Math for Elementary Schoolers
Volume 1

Oleg Gleizer
oleg1140@gmail.com

October 2, 2011

Contents

1 Introduction

– What is an American university
math department?
– A place where the Russians
teach math to the Chinese.

A university joke.

Modern science, from economics and finance to cosmology and astronomy, is based on mathematics. Math models form the bedrock of technological advances, from nuclear power to computer animation. Here is just one example. It was predicted at the end of the 1970s that humanity will run out of oil in the next 20 years. Since then, oil consumption has grown by far exceeding the 1970s expectations, primarily due to the unforeseen industrial development in China. However, the modern day oil reserves cap those available at the end of the 1970s. The miracle is largely based on a mathematical procedure called the Radon transform[1]. Named after the Austrian mathematician Johann Radon, it enables geologists to literally see through the Earth. A branch of tomography, a way to non-invasively obtain three-dimensional (3D) pictures from the inside of a studied object, the novel method was quick to show that many oil fields considered exhausted in the 1970s had over 80% of their deposits untouched. In addition, the Radon transform-based imaging gave geologists new means to discover many more commercially viable oil reserves.

Tomography found an even more vital application in medicine, including the X-ray tomography (CT, CATScan), magnetic resonance imaging (MRI), and ultrasound. An advance in mathematics is not only filling our gas tanks, it is also saving lives on a daily basis. And the Radon transform is only one example out of thousands!

Yet as science and technology have greatly advanced in the last half of the 20th century, the level math is taught at in schools has degraded as greatly, especially in the US. According to the 2009 PISA[2] assessment of 15-year-olds, the United States was ranked 25th in mathematics out of the 34 OECD[3] countries, trailing far behind South Korea, Finland, Switzerland, Japan, and Canada.

Clifford F. Mass, a professor of Atmospheric Sciences at University of Washington, writes in his blog, "Last quarter I taught Atmospheric Sciences 101, a large lecture class with a mix of students, and gave them a math diagnostic test as I have done in the past.

The results were stunning, in a very depressing way. This was an easy test, including elementary and middle school math problems. And these are students attending a science class at the State's flagship university – these should be the creme of the crop of our high school graduates with high GPAs. And yet most of them can't do essential basic math – operations needed for even the most essential problem solving. ...

... If many of our state's best students are mathematically illiterate, as

[1] In this book, the blue-colored words and phrases are links to the corresponding web pages.

[2] The Program for International Student Assessment.

[3] The Organization for Economic Cooperation and Development.

shown by this exam, can you imagine what is happening to the others – those going to community college or no college at all?

Quite simply, we are failing our children and crippling their ability to participate in an increasingly mathematical world.”

1.1 What is this book?

This book has originated from the state of desperation parents of a 5-year-old boy, themselves mathematicians, have got into while searching for a proper school for their kid. We have looked both into the public and private sector, including some rather expensive private schools, and couldn't find any where they teach math, physics, and/or computer science at the 21st century level. The following are the two questions the author has asked the math teachers he had interviewed.

1. Given a straight line and a point away from it, how would you draw another straight line passing through the point and parallel to the original line, using a compass and straightedge as tools?

2. How would you draw a four-dimensional (4D) cube?

Instrumental in understanding geometry of the Euclidean plane, the first problem was solved by the ancient Greeks about two thousand years ago. The feature that distinguishes the Euclidean plane from all other 2D surfaces is the fact that the aforementioned straight line, the one passing through the given point parallel to the original line, exists and there is only one such. Unlike some modern mathematicians, the ancient Greeks were constructivists. They believed only the things they could construct. They definitely could draw parallel lines. The great majority of modern students, from elementary to high school, can't. Neither unfortunately can many of their teachers.

The second question is a good indicator of the knowledge of some elementary parts of modern math or of the absence thereof. Drawing a 4D cube is not hard at all. Please take a look at the model on page 120, another one on page 121, and at the corresponding explanations. All the children we have taught were able to grasp it in no more than four classes. Among the private schools we have called, there were two advertising themselves as the schools with emphasis in math and science. However, the teachers we have spoken to were not able to understand the second question, let alone answer it.

Eventually, we decided to send our kid to a local public school and to teach him math ourselves. This book is based on the lectures the author has given to his son and his friends. It is made to enhance the American kindergarten and elementary school math curriculum. That's how we do it. A common kindergarten and first grade problem is to break a positive integral number into a sum of two. In the first chapter of this book, we teach kids to break a positive integer n into a sum of positive integers in all possible ways, from one part, $n = n$, to n parts, $n = \underbrace{1 + \ldots + 1}_{n \text{ times}}$.

The second chapter of the book explores the notion of the straight line. The latter is as fundamental for understanding modern math and physics as it is, contrary to the intuitive feeling, exceptionally complicated. For example, to understand what is straight, one also has to understand what's not. So, we study the most important of the non-straight lines, the circumference. The straight line and the circumference are the most fundamental lines in mathematics and physics because they have a rich group of symmetries. A translation of the straight line along itself moves its points, but preserves the line as a whole, the fact responsible for our ability to add and subtract (real) numbers. Similarly, rotational symmetries of the circumference allow us to add and subtract angles. We emphasize this similarity between the lines, teaching the little ones to use a protractor for the angles' arithmetic on the way.

A straight line goes to infinity at both ends, so it makes sense to take a look at the notion of infinity. Segments of straight lines are the shortest paths between any two different points in a Euclidean space, such as the 3D world of our everyday physical reality. We discuss what happens to straight lines when the space gets warped, as it does in *Star Wars* and other sci-fi movies. And, yes, we teach kids to draw the straight line parallel to the given one and passing through the given point away from the original line in the Euclidean 2D, using a compass and a ruler as their tools.

To understand the difference between a warped space and a flat, or Euclidean one, we study geometries of three different 2D surfaces, the Euclidean plane, the cylinder, and the sphere. Our study is hands-on. We learn to draw straight lines, circles, angles, and polygons on the surfaces. Similar to the Euclidean plane, we define a straight, or geodesic, line on a surface as the shortest path connecting two different points. For example, straight lines on a sphere are arcs of great circles, the circles having the same radius as the sphere they "live" on. The latter fact is crucial for navigating the Earth, so we use geography to help us learn geometry of the sphere.

Enhancing the school studies of solid figures in the Euclidean 3D, we venture further to dimensions four and more. In particular, we discuss some magic properties of the 4D triangular pyramid and its 2D projection widely used in Hollywood suspense movies under the name of a pentagram.

It is the Earth's rotation that all our clocks and watches model with various degree of precision, so learning time in Chapter 3 takes us back to the coordinates on the globe. Understanding what longitude and latitude lines are in its turn gives the child a better grasp of spherical geometry as well as geography of our planet. Studying time enables us to invent a way to travel in the past and in the future, using nothing more than a long range commercial airliner, or even a sailing ship. If you think that the latter technology is not advanced enough for the task, please take a look at page 162.

Having introduced the number line as a tool to demonstrate geometric nature of addition and subtraction, we then study the *mod* n arithmetic, the one on a circle divided into n equal parts. When $n = 12$ (or 24), for example, this is nothing else, but the arithmetic of the face of a clock!

As you can see from the above brief outline of the book's content, there exists a huge gap between the modern school math curriculum, the good old 19th century curriculum watered down to almost nothingness by the recent "education reforms", and the 21st century demands. It is the goal of the author to fill the void.

The book is aimed at the parents ready to invest a lot of their precious time into giving a better education to their children. The book is also aimed at the school teachers, kindergarten through high school, aspiring to bring elements of modern math to their classrooms. Please note that the purpose of this book is not to teach the school curriculum better than it is done at school.

There exist lots of textbooks and after-school programs doing just that. The purpose of this book is to teach some important concepts of modern math absent from the school curriculum, although closely related to it, the concepts typically not known by professional educators. To the best of the author's knowledge, there is no after-school program or an enrichment program for homeschoolers where a child can learn what this book offers. Some university science camps use the approach similar to that of this book, but they can't be systematic in their studies due to the obvious time constrains. In fact, this book is nothing else but a manual for a two-year-long university level math camp.

1.2 How to use this book?

The book is written as a series of lectures for a parent/teacher to read, think through, and to present in a way that fits the needs and educational level of her/his particular little one(s). The theory is aimed at the adult rather than directly at the child, but the problems the book contains, and it contains over two hundred of them, are there for the children to solve. If the kid has no clue after a few minutes of pondering, please give her/him a helping hand. It is all right to solve a problem for the child. Then ask her/him to do it again, with less help. Begin the next class giving the little one the problems she/he was stuck with the previous time. Discuss the corresponding theory again, if needed. Reducing your help from 100% to zero, make sure that the child can independently solve most of the problems in the book. Kids are quick learners, but they forget things as quickly. Occasionally give the child problems from the lessons covered long ago. The book is written in an (almost) linear manner, but should be used as a two- or three-fold spiral.

The following is the difference between a scientist and an engineer. An engineer wants to know how a gadget functions or how to make one that functions according to the required specs. A scientist is mostly interested in why something works the way it works. Since at the age of five through seven the concept of a proof is completely alien to the child's mind, this book uses the engineering approach to teaching math. We want kids to learn solving problems without being able to articulate the logical foundations of what they have done. Still, all the mathematical statements appearing in the book are proven with various degree of rigor. The first reason is to make the book suitable for teaching older children. For example, the author has taught courses based on some parts of the book to high school students attending the UCLA Math Circle.

The second, and main, reason is ... to go through the proofs with the little ones! The proofs will give a child the feeling that the geometric constructions and algebraic manipulations she/he is asked to perform in homework problems do not come out of nowhere, but rather reveal small parts of a single picture of great beauty and enormous intricacy. Please discuss the proofs with your kids without requiring the latter to talk them back. Our brain is a machine functioning largely independent from our consciousness. As every scientist knows, if her/his mind finds something worth of giving a thought, it can analyze the matter subconsciously for a while (ten years in one particularly hard case the author of this book has lived through), informing the owner of its findings in the form of sudden revelations. Discussing the proofs with the child will plant the seeds for the crops that will be harvested a few years later.

Our son has learned everything this book has to offer, excluding the proofs, in two years, his kindergarten and first grade. We have discussed every lesson at least twice, often taking some considerable time between the discussions. Some harder topics required even more comebacks. If this sounds discouraging, please recall how much practice it takes to teach the little one lots of activities far less intellectual than mathematics.

Teaching your child yourself, using this book or by any other means, is a hard undertaking requiring a lot of time and effort from both you and the child. (It has made us greatly appreciate the work schoolteachers do!) You need to have regular classes. We have started with two 15-minutes-long sessions a week, gradually increasing the duration of each session to an hour. In addition to formal studies, we kept engaging our son in occasional conversations about the covered material. For example, we discussed recently studied topics driving the boy to and from his gymnastics club.

You can start using the book as soon as your child can count to twenty and is familiar with addition of single digit numbers, that is after her/his first half a year at the kindergarten. Since our son was taught to do such things earlier, we began teaching him by the book from his first kindergarten day. Giving, on average, two one-hour classes a week, including school breaks and holidays, we finished the book in two years. It will go faster, if you start out a year or two later. However, the second volume of this manuscript is already in the works!

The author has taught classes based on the book to children of various ages and educational needs, the former ranging from kindergarten to high school, the latter – from UCLA Math Circle to Creative Learning Place, an enhancement program for homeschooling families at Westside Jewish Community Center in Los Angels, CA. Older children tended to learn faster, needed fewer repetitions, and understood proofs. Since they were already familiar with addition and subtraction, the corresponding parts, primarily Lesson 5.1, were safely skipped. Other than that, very little adjustments were needed.

Among the words and notions used in the book, but not directly related to mathematics, there can be some unfamiliar to the child. Sometimes these words and notions are explained explicitly, some other times explanations are left to the parent/teacher. For example, exploiting popularity of the sci-fi movies, we boldly go to remote corners of the universe, leaving it to the adults to explain the meaning of the words such as a "star", "galaxy", "galactic cluster", "black hole", and so on. Explanations of this sort shouldn't be a problem in the age of Wikipedia and Internet.

The theoretical part of a lesson can serve as a starting point for a tour to the neighboring area of human knowledge, from history to architecture to navigation to cosmology. For example, introduction of cylindrical shapes has led the author from cylinders in geometry to castle towers, such as below. Looking at the castle of William the Conqueror in its turn has led us through some chapters of English history and medieval warfare. One of the boys in the class was an expert on the latter. He said, "I know, I know! When enemy soldiers tried to climb up the walls, castle defenders poured hot tea on them." The remark turned the next five minutes of the lesson into discussing throwing cakes and candies at the enemy as the ultimate weapon. Cosmology has led us to recalling the *Star Wars* movies as well as watching documentaries about manned space flight in an attempt to figure out the meaning of the words "up" and "down", and so on. Treating math as an organic part of human culture (which it is), not like a dry collection of formal manipulations (as it is mistakenly perceived by many), makes math classes fun to both teach and learn. However, in order to prevent this book from turning into “a book about everything”, most of the discussions not directly related to mathematics are left outside of it.

The castle of William the Conqueror, Château Guillaume-Le-Conquérant, in Falaie, Calvados, France. [4]

Closing this introduction, we’d like to mention that solutions to some of the book’s less obvious problems are provided in Section 7.

[4]Downloaded form http://en.wikipedia.org/wiki/William_I_of_England.

2 Copyright

3 Addition, subtraction and Young diagrams

3.1 Counting the stories

Not all the 5-year-olds are familiar with the meaning of the word "story" as in the "three-story house". If your child knows this meaning of the word, please skip this mini-section, except for Homework Problem 3.2.

Homework Problem 3.1 *How many stories are there in your house? In your neighbors' houses?*

The following problem was plagiarized from Prof. Zvonkin's book [2]. As bad as the author feels about plagiarizing, the fact that the book is unavailable in English gives him a bit of consolation.

Homework Problem 3.2 *Once upon a time on the tenth floor of a ten-story apartment building there lived a little boy who very much liked riding in the elevator. It was a very safe elevator, so the boy's parents didn't mind him* *taking rides unaccompanied by adults. Going down, the boy enjoyed it all the way to the first floor. On his way up however, he only took the elevator to the third floor and then walked seven stories up. A few months later, the boy started going in the elevator up to the fourth level instead of the third. Some more time has passed, and he began taking the elevator up to the fifth floor. Can you explain the boy's behavior?*

3.2 Parts of a number

It is a typical kindergarten problem to ask the child to represent a positive whole number as a sum of smaller positive integers. Many parents would be surprised to find out that the same problem appears in every college-level course of (enumerative) combinatorics, the art of counting objects satisfying certain criteria. Inspired by the general idea that children, when possible, should be taught math the same way it is used by grown-up professionals, let us see what we can do in this case.

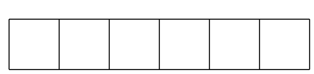
Here is a problem: represent 6 as a sum of smaller positive numbers. We shall approach the solution graphically. Let us call a square an "apartment". Let us suggest a child to draw a house built of 6 apartments in such a way that

1. an upper floor cannot have more apartments than the lower floor and there shouldn't be gaps between the apartments (otherwise the house will collapse);

2. the house wall facing the street must be built vertical (according to the city regulations); and

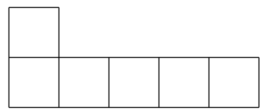
3. the street is always on the left.

First, here comes the one-story house. This is the partition of the number 6 into 1 piece.

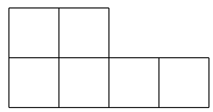


$$6=6$$

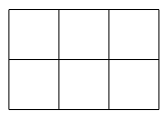
Then the two-story houses and the corresponding two-parts partitions.



$$6=5+1$$

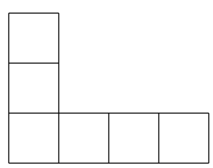


$$6=4+2$$

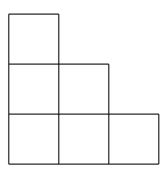


$$6=3+3$$

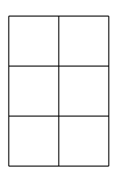
Ask the child if any other two-story house of 6 apartments is possible to build. Then the time comes for the three-story houses and a similar discussion on why the below list is exhaustive.



$$6=4+1+1$$

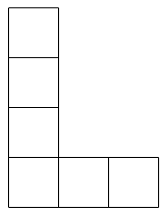


$$6=3+2+1$$

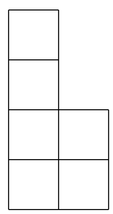


$$6=2+2+2$$

Here come the four-story houses, the corresponding partitions and a similar discussion:

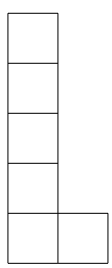


$$6=3+1+1+1$$



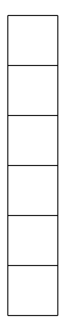
$$6=2+2+1+1$$

There is only one way to build a five-story building (why?).



$$6=2+1+1+1+1$$

And finally here comes the six-story tower:



$$6=1+1+1+1+1+1$$

Homework Problem 3.3 *Ask the child to draw all the possible houses with 4 apartments. Ask her/him to sum up the number of the apartments on each floor next to the corresponding picture as above.*

Homework Problem 3.4 *Ask the child to find the 2+3 sum by drawing a two-story house, 3 apartments on the first floor, 2 – on the second, and counting the total number of the apartments. Ask her/him to write 3+2=[the answer] next to the picture.*

Homework Problem 3.5 *Ask the child to find 3+4.*

Homework Problem 3.6 *Ask the child to find 7-3 by drawing a two-story house with seven apartments, 3 of them on the second floor, and by counting the number of the apartments on the first floor. In fact, having two stories is not necessary. The child can draw any house with seven apartments, precisely three of them on one floor. Then the floor gets marked out, and the remaining apartments – counted.*

Homework Problem 3.7 *Ask the child to find 7-4.*

Homework Problem 3.8 *Ask the child to draw all the possible houses with 7 apartments. Ask her/him to sum up the number of the apartments on each floor next to the corresponding picture. This is quite a laborious project. It's all right, if the child does only a part of it now.*

3.3 Odd and even numbers

As the reader can see from the first lesson, the above pictures are a great tool to visualize not only partitions of positive integers, but also addition and subtraction. We shall later introduce multiplication my means of counting the number of apartments in an $m \times n$ house. Grown-up scientists call these pictures either Young diagrams, after the British mathematician and clergyman Alfred Young[5], or Ferrers diagrams, after another British mathematician, Norman Macleod Ferrers.[6] The "Young diagram" notation is more

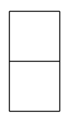
[5] 1873-1940

[6] 1829-1903. Mr. Ferrers seems to be the inventor. Mr. Young is honored this way because he is the inventor of Young tableaus, the objets of great importance in mathematics, based on Young diagrams. See `http://en.wikipedia.org/wiki/Partition_(number_theory)` and `http://en.wikipedia.org/wiki/Young_diagram` for more.

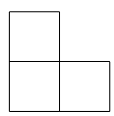
widespread than the "Ferrers diagram". As noticed by Vladimir Arnold[7], "A mathematical object most often bears the name of the last person to discover it." Upholding this law, we shall call the above pictures Young diagrams further in the book. As the author has observed in his numerous classes, children have no problem learning this name, or, in fact, any other. However, we shall smooth the transition by using both the "child" and "grown-up" notations for a while.

Definition 3.1 *A (positive) integer is called even, if it can be represented by a two-story house with an equal number of apartments on each floor. An integer that is not even is called odd.*

In other words, an even number can be represented by a Young diagram with two rows and an equal number of boxes in each row. For an odd number, the lower row will have an extra box. 1 is obviously not even. It cannot be represented by a two-story house, because we only have one apartment. So, 1 is an odd number. 2 is even.

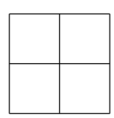


3 is odd.

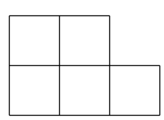


Discuss here why there is no way to represent 3 as a two story-house with an equal number of apartments on each floor.

4 is even.



5 is odd.



[7]1937–2010, a famous Russian mathematician.

It's time to notice the pattern: an odd number is obtained from the previous even number by adding an extra apartment to the first floor.

6 is even.

Observe that an even number is obtained from the previous odd number by adding an extra apartment on the top of the only first floor apartment which has no second floor pair. So on it goes: 1,3,5,7,... – odd; 2,4,6,8,... – even.

Homework Problem 3.9 *Is 9 odd or even?*

Homework Problem 3.10 *Using Young diagrams, find all the partitions of the number 5.*

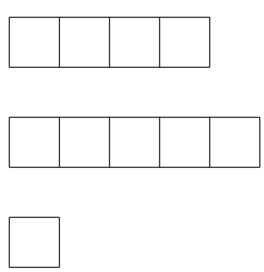
Homework Problem 3.11 *Find all the partitions of the number 6 into odd parts. In other words, draw all the possible houses with 6 apartments and with an odd number of apartments on every floor. Hint: first, list all the positive odd integers less than 6.*

Homework Problem 3.12 *Is 12 odd or even?*

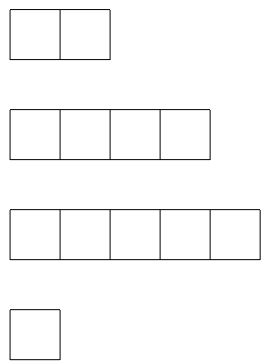
Example 3.1 *Using a Young diagram, compute*

$$12\text{-}4\text{-}5\text{-}1=$$

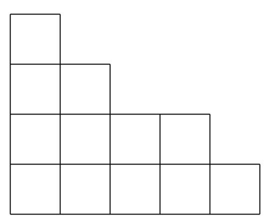
Let us first draw the numbers we need to subtract as floors "floating in the air".



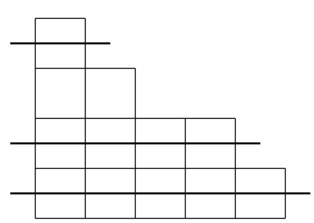
Let us add one more floor, also "floating in the air", so that the total number of the apartments becomes equal to 12.



We can solve the subtraction problem right now, but since our goal here is not only teaching the child subtraction, but also getting her/him used to Young diagrams, let's do two more steps. Let us assemble the "floating" floors into a proper house (Young diagram).



Finally, let us mark out the floors corresponding to the numbers we subtract. The last thing to be done is to count the apartments of the remaining floor.



$$12-4-5-1=2$$

Homework Problem 3.13 *Using Young diagrams, compute the following:*

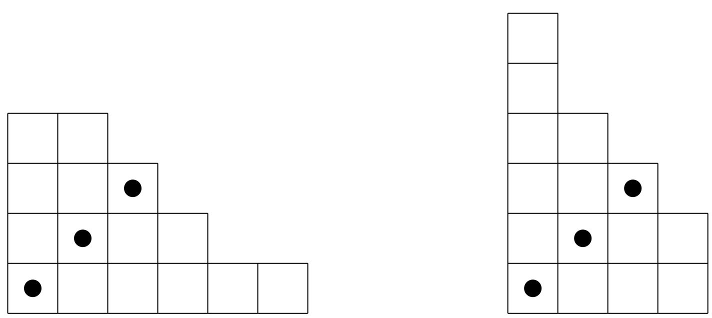
$3+3=$ $\quad$ $2+5=$ $\quad$ $5+3=$

$6-1=$ $\quad$ $8-5=$ $\quad$ $2+3+1=$

$7-6=$ $\quad$ $8-5-1=$ $\quad$ $3+3+2=$

3.4 Conjugation of Young diagrams

There exists an important operation on Young diagrams. A diagram can be flipped over the diagonal as shown on the following picture.



You can also see this process in motion by clicking here.[8]

The wall facing the street and the ground line change places. In the language of professional mathematicians, this operation is called conjugation. In the above picture, the partitions (6,4,3,2) and (4,4,3,2,1,1) are conjugate to each other. It is traditional in combinatorics to write down a partition not as a sum, but as a list of parts in parentheses, first the number of boxes

[8]This animation was created by a father of two of our students, an artist and web designer by profession, Cliff Kensinger.

in the first raw, then second, and so on. The partitioned number can be recovered by summing up the parts. It is called the volume of the partition. For example, the volumes of the conjugate partitions above are equal to 15.

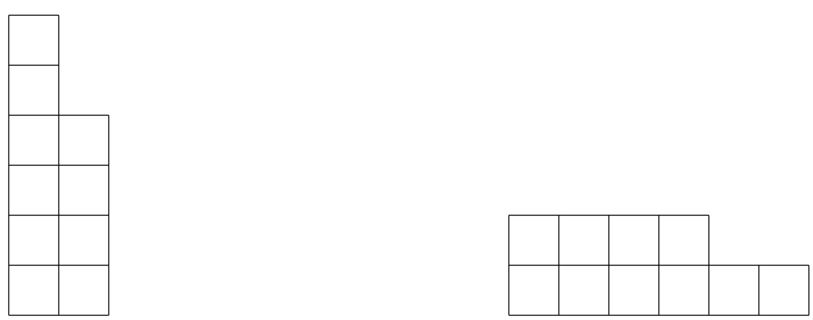
Conjugation is a difficult concept to understand for a five-year-old, so here come six similar problems. Please decrease your help from 100% with the first of them to 0% with the last one.

Homework Problem 3.14 *In the following problems a–f, draw the Young diagram of a given partition, the Young diagram of the conjugate partition, write down the conjugate partition, and find its volume.*

a. (3,1); b. (4,2,1); c. (3,3,1); d. (2,2,2,1,1); e. (4,3,1,1); f. (6,4)

Example 3.2 *Find all the partitions of 10 into parts not exceeding 2.*

In other words, count all the houses with 10 apartments and with no more than 2 apartments on every floor. The idea crucial for finding an "easy" and elegant solution is that a partition with parts not exceeding 2 is conjugate to a partition with no more than 2 parts.



Rather than counting the "tricky" partitions with small parts, we can count all the one- and two-story houses with 10 apartments instead: (10), (9,1), (8,2), (7,3), (6,4), and (5,5); 6 altogether. Your 5-year-old most likely would find this argument very complicated. However, the author believes that going through it with her/him at this point is worthwhile, even if the child doesn't understand all the details of the solution. The discussion of Young diagrams at the next level of the learning spiral will begin with getting back to this problem one more time.

Homework Problem 3.15 *Using Young diagrams, compute the following:*

$4+5+1=$ $\qquad$ $2+5+7=$

$5-3-1=$ $\qquad$ $10-2-1=$

$7-2-1-1=$ $\qquad$ $4+3+2+1=$

Homework Problem 3.16 *Is 15 odd or even? Why?*

Homework Problem 3.17 *Count all the partitions of the number 9 with odd distinct parts.*

It is time to get back to Homework Problem 3.8. Most likely, it was not finished the first time due to the big volume of the work needed. Now we have a tool to cut the work (almost) in half. Note that to list all the partitions of the number 7, it is enough to draw all the houses with seven apartments and with at most four stories. Then conjugate every picture and drop the partitions that appear twice.

Homework Problem 3.18 *Complete Homework Problem 3.8 using conjugation.*

4 Straight lines in geometry and physics

The purpose of the following classes is to study the straight line. As the number line, it provides a view on addition and subtraction different from the approach that uses counters (Young diagrams). The notion of a straight line seems very intuitive. However, as one of the most fundamental notions in mathematics, it is in fact rather complicated. To better understand the word "straight", for example, we have to get a feeling of what's not straight. A straight line goes to infinity at both ends, so it helps to figure out the meaning of the word "infinity". A line is made of points, so we need to learn what a point is, and so on.

4.1 A point

An axiom is a statement taken for granted as self-evident and not needed to be proved. All other math statements are derived from axioms using formal logic. Here is an example.

Axiom 1 *A point is that which has no part.*

A point is the most fundamental geometric object. All other geometric objects are made of points. The notion of a point is so basic that there is no meaningful way to define a point in terms of of simpler objects. Using Axiom 1, let us find out some properties of a point.

Proposition 4.1 *A point has no length, widths, or hight.*

Proof — Suppose that a point has some length. Then we can cut it in the middle into two smaller objects. This way, we can consider a point as an object made of two parts. But a point has no parts, so its length must be zero. Similarly, a point has zero width and height. □ (The little box means the end of the proof.)

The above reasoning is probably the first example of a mathematical proof in the child's life. A proof is nothing more than a way to convince skeptics in the correctness of your statement. The following anecdote illustrates the drastic change the concept of mathematical proof has undergone with time. One of the first great European mathematicians, René Descartes[9], tried to explain a new-found proof of an extremely beautiful theorem to his friend, a musketeer. The musketeer agreed that the theorem was beautiful indeed, but his friend's intension to prove it offended him. He said, "Monsieur Descartes, we are both noblemen. I trust your word".

[9] 1596-1650, a French philosopher, mathematician, physicist, and writer.

René Descartes[10]

In this book, most of the proofs will be geometric, rigorous enough to convince Monsieur Descartes, but not completely formal by modern standards.

Homework Problem 4.1 *Draw a point on a sheet of paper. Explore the drawing using significant magnification, a powerful lens or better a microscope. Is it really a point?*

Homework Problem 4.2 *Imagine that you look at an ideal point, an object with no length, height, or width, trough a microscope. Would the picture change if you increase magnification?*

Since a point has no size in any direction, it is sometimes called a *zero-dimensional space.*

Homework Problem 4.3 *Count all the partitions of the number 6 with more than 3 parts.*

Homework Problem 4.4 *Go through Example 3.2 one more time.*

[10] A portrait by Frans Hals, 1648.

Homework Problem 4.5 *Using Young diagrams, compute the following:*

$$3+5+1+2= \qquad 2+4+9+4=$$

$$10-3-3-1= \qquad 12-4-2-2-1=$$

4.2 A line

The Greek mathematician Euclid, who lived in the city of Alexandria[11] around 300 BC, is often referred to as the "Father of Geometry". By the way, "geometry" means nothing more than "measuring earth" in ancient Greek, the language spoken by Euclid and his Greek contemporaries. Indeed, this important part of mathematics was born when people began measuring their land lots. What a humble beginning for the science that explains, among other things, gravity as curvature of the warped space-time!

Euclid[12]

[11]Founded by Alexander the Great in 331 BC, currently the second-largest city in Egypt. Since ancient Greece has laid foundation of Western civilization, it's worthwhile here to take a short journey to the times of antiquity.

[12]This is an artist's rendering. No real portrait has reached our times.

Euclid's textbook, called "Elements", is the first geometry textbook on this planet. The most successful textbook of all times, it was in use from the moment of its creation until the end of the 19th century, about 2,200 years altogether! Many of the theorems presented in the "Elements" were discovered by mathematicians preceding Euclid. It is his achievement to organize them in a logically coherent manuscript, including a system of rigorous proofs that remains the basis of mathematics 23 centuries later.

"Elements" begin with a string of axioms, Axiom 1 being the first of them. Here is one more.

Axiom 2 *A line is breadthless length.*

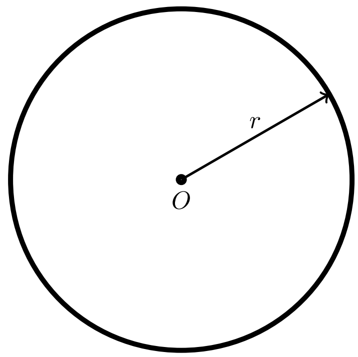
A line (not necessarily straight) has no height or width, only length. As any other geometric object, a line is made of points. Since, out of all dimensions, a line has only length, we call it a one-dimensional (1D) object.

Homework Problem 4.6 *Draw a line on a sheet of paper. Use a lens or a microscope to see if the drawing is truly 1D.*

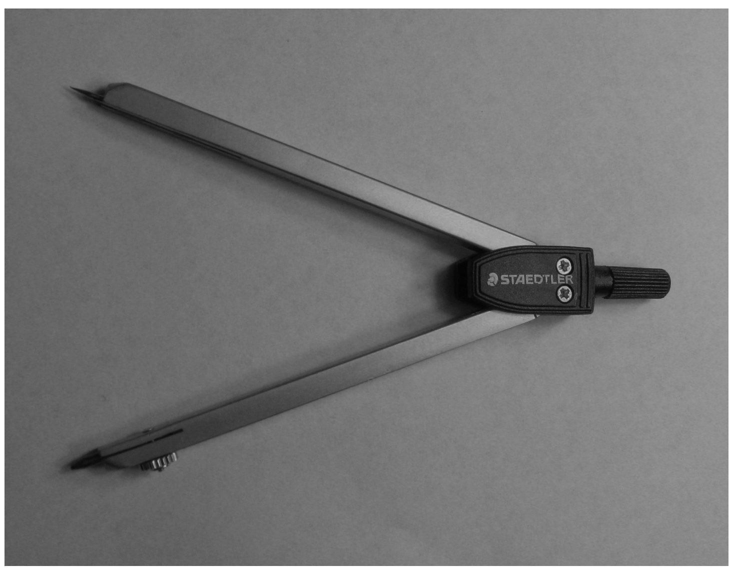
Below we shall familiarize the child with one of the most important lines in geometry, a circumference.

Definition 4.1 *A circumference is the set of all the points in the plane having an equal distance, called the radius, from a special point, called the center.*

The radius is often denoted by the lower case r, the center – by the upper case O. We haven't yet defined either the plane or the distance, but these notions are intuitive enough to start using them prior to giving a proper definition.



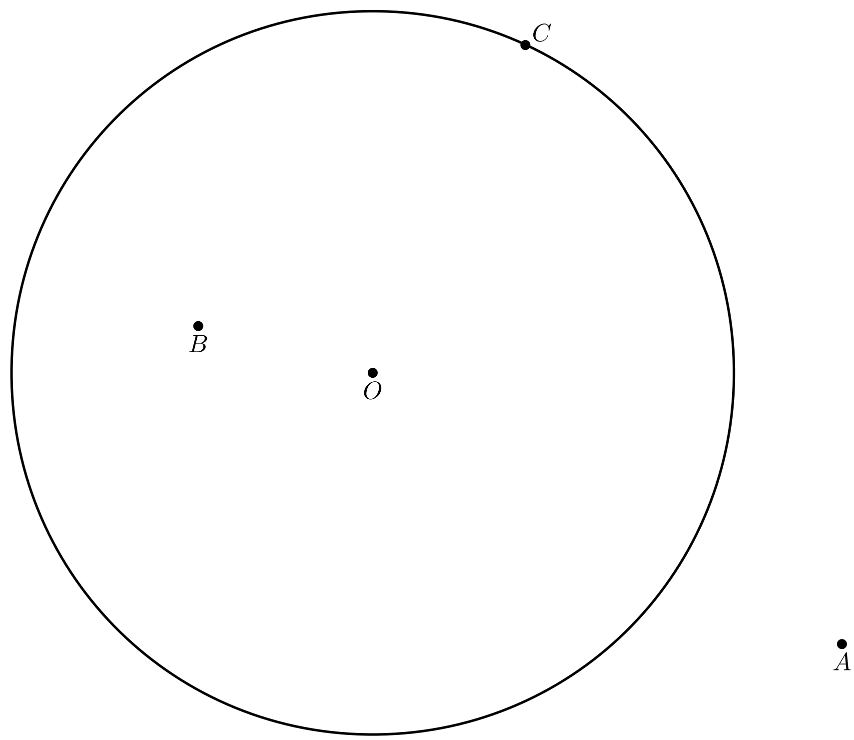
Although five-year-olds have no problem memorizing the circular shape, they have to do quite a bit of work to fully grasp Definition 4.1. We shall need a compass and a ruler marked with inches for the following.



a compass

Homework Problem 4.7 *Give the child a compass and a ruler. Show her/him how to use the ruler for setting the compass 2 inch (2") wide. Show the child how to draw a circumference with r=2". Ask the child to draw a circumference with r=1" by her/himself.*

Homework Problem 4.8 *Measure in inches the radius of the following circumference.*



Measure the distances $|OA|$, $|OB|$, and $|OC|$ with a ruler.

- *Which distance equals the radius? What point lies on the circumference?*

- *Which distance is less than the radius? What point lies inside the circumference?*

- *Which distance is greater than the radius? What point lies outside of the circumference?*

- *Pick a point on the circumference different from the point C. Tell the distance from the circumference center to the point without measuring it. Recall Definition 4.1 if necessary.*

- *Pick a point different from the point A outside of the circumference. Is the distance from the circumference center to the point greater or less than the radius? Check the answer by a direct measurement, comparing the distance to the 2" mark on the ruler.*

- *Pick a point different from the point B inside the circumference. Is the distance from the circumference center to the point greater or less than the radius? Check the answer by a direct measurement.*

- *Discuss Definition 4.1 with your parent/teacher one more time.*

Definition 4.2 *A circle of radius r is the set of all the points in the plane such that their distance to a special point called the center is less than or equals to r.*

Homework Problem 4.9 *Compare Definition 4.2 to Definition 4.1.*

Note that the circumference is the border line of the circle. In other words, a circle is its boundary circumference and all the points inside it.

Homework Problem 4.10 *Take another look at the picture in Homework Problem 4.8. Which of the points A, B, C, O belong to the circumference? To the circle?*

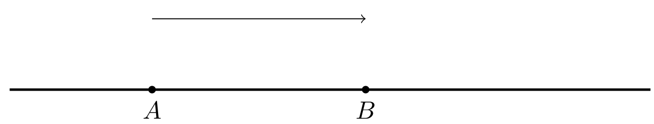
Homework Problem 4.11 *Suppose that you have to look after a goat. You hammer a stake in the center of a flat meadow, tie up the goat to the stake with a rope of length $r = 5$ feet (5') and let it graze. What figure do you get when the animal eats up all the grass it can reach?*

4.3 A straight line

That's how Euclid defines a straight line in the "Elements".

Axiom 3 *A straight line is a line which lies evenly with the points on itself.*

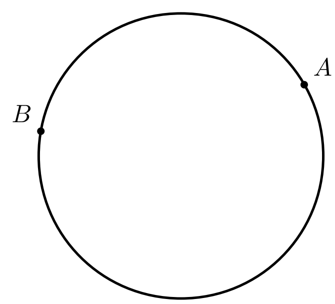
What Euclid means here is that for any two points on a straight line, one can be moved to the position of the other by sliding the straight line along itself.



Definition 4.3 *Transformations of a geometric object that preserve the object as a whole are called its symmetries.*

Axiom 3 states that, unlike a generic line, a straight line is a highly symmetric object. Unfortunately, Euclid does not specify that the symmetries that preserve the straight line are the linear translations along itself. (Dilations preserve a straight line as well, but we shall postpone studying them until the child learns multiplication.)

Homework Problem 4.12 *Can we slide point A to the position of point B by a move that preserves the following line?*

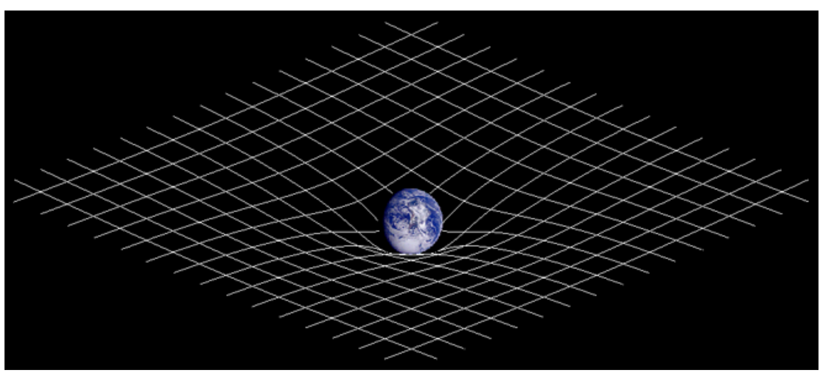


The points A and B in Homework Problem 4.12 can obviously be taken at random. For any two points on a circumference, there always exists a rotation sliding one in the position of the other. Similarly to a straight line, a circumference is highly symmetric (hence its importance in mathematics). The difference is that the symmetries of a circumference are rotations, not translations. However, being highly symmetric, a circumference is another "line which lies evenly with the points on itself", and there exist some more.

Axiom 3 doesn't work well enough. Using the notion of symmetry, it is possible to improve the axiom to the point where it actually defines nothing else but a straight line. Unfortunately, the child does not possess the necessary algebraic skills (multiplication/division) at the moment. Being stuck with the axiomatic approach, let us put the "Elements" aside for now and try to define a straight line by its properties that seem intuitively clear. First of all, like any line, a straight line has no width or height, only length. Second, it's infinite in both directions, and last, it's ... straight. The intuitive approach gets us stuck again. Let's try the experimental one.

Let us take a sheet of paper, a ruler, a pencil, and draw. We have no problem recognizing the straight line on the picture, haven't we? It looks straight, so all we need now is a minor mental effort. We have to imagine that the line we are looking at is one-dimensional and that it remains straight going to infinity at both ends. In doing so, it leaves the sheet of paper it's drawn upon and extends beyond the limits of the room, the house, the planet of Earth, the Solar System, reaches the boundaries of the galaxy we live in, the Milky Way, and bids them farewell.[13]

General relativity theory claims that mass is nothing but the space-time curvature. The heavier an object, the more it warps the four-dimensional space-time around itself. To remain straight, our line must avoid the curvature of the space-time fabric. A daunting task, indeed!



The Earth bending the space-time around it.[14]

[13]Raised on the *Star Wars* movies, our kids usually have no trouble venturing out of the confines of this planet, dealing with space-time warps and wormholes bringing us to remote corners of the universe. If your child hasn't seen any of the *Star Wars*, *Star Trek*, *Babylon 5*, *Battlestar Galactica* and the like yet, maybe it's time to get started.

[14]Downloaded from **here**.

Homework Problem 4.13 *Ask the child to take a sheet of paper, a ruler, a pencil, to put the sheet on the surface of a flat table, and to draw a straight line. Now try to do the same using, instead of the table, a ball big enough for the purpose. Can we draw a straight line on the surface of the ball? Why or why not?*

Homework Problem 4.14 *Do we see straight lines in nature? How about railroad tracks? Have they only one dimension? Are they always straight? Are they infinite?*

Homework Problem 4.15 *Using significant magnification, look at the straight line from Homework Problem 4.13. Is it really 1D?*

Homework Problems 4.1, 4.6, and 4.15 raise the following question: "If we can't draw geometric points and lines, are these mathematical abstractions any good?" The answer a few thousand years of human experience give us is, "Absolutely yes!" As history shows time and again, "There is nothing more practical than a good theory".[15] Most of the real life objects and

[15]The phrase is usually attributed to Kurt Lewin, 1890-1947, a German-American scientist recognized by many as the founder of social psychology.

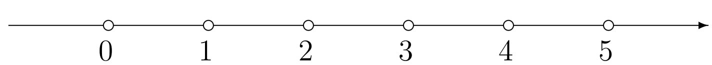
events are too complicated to understand them right away. So people make models, stripping the objects of their curiosity of all the features except for the essential ones. The resulting models exist in our minds only. They are simpler than the real life objects they approximate and sometimes allow us to understand some important properties of the latter. For example, it is much easier to calculate a spaceship orbit, if we consider the craft as a point.

In their turn, some mathematical abstractions are hard to grasp, too. They are so abstract that it prevents our intuition from giving our logic a helping hand. In this case, we model mathematical abstractions by means of the objects our intuition is comfortable with. A point and a line we draw on parer are not the abstract geometric figures defined by Euclid, but their real world models. To think about the former, we often need to look at the latter.

Homework Problem 4.16 *Take another look at the straight line on page 29. Using a ruler marked with centimeters (cm), find the distance we need to slide the line along itself in order to move point A to the position of point B. Will the line remain in place, if we slide it along itself by 7 cm to the right? By 2" to the left?*

4.4 The number line

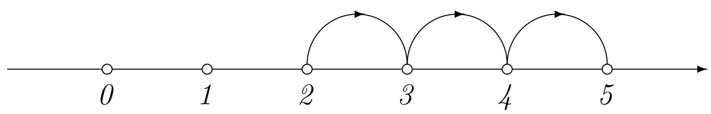
Let us mark a point on a straight line and call it zero. Let us take a point to the right of zero and call it 1. This point marks one step to the right of zero. Let us take one more. Since 1+1=2, we have to mark the point we end up at as 2. If we take another step to the right, we end up at 3, and so on.



As we shall see in no time, the number line is as convenient a tool for helping us with addition and subtraction as Young diagrams are. Consider the following example.

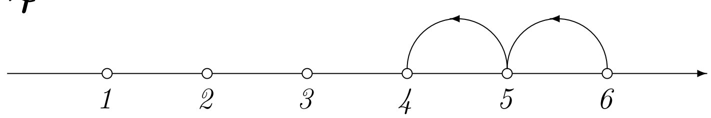
Example 4.1

$2+3=5$



We see that to add 3 to 2, we need to find point 2 on the number line and to slide it 3 one-unit steps to the right.

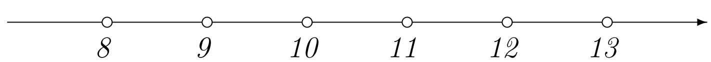
$6-2=4$



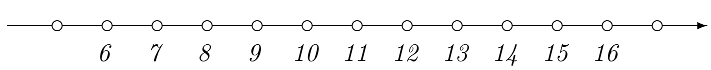
Similarly, to subtract 2 from 6, we find point 6 on the number line and slide it 2 one-unit steps to the left.

Homework Problem 4.17 *Marking steps on the number line as above, compute*

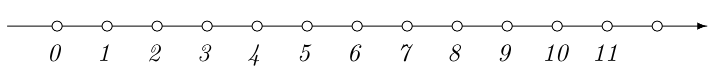
a. $12-3=$



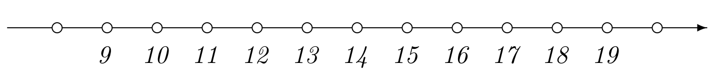
b. $7+8=$



c. $10-5-4=$



d. $11+2+2+1=$



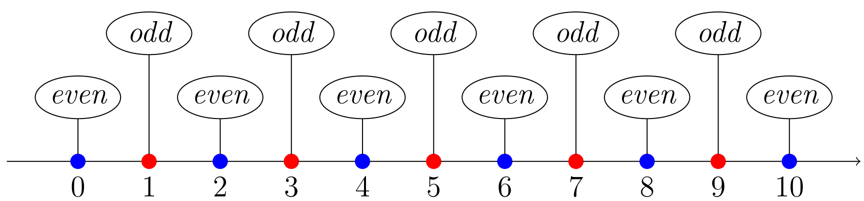
To check your answers, redo the computations using Young diagrams.

0 differs from all other numbers on the number line. It is the only number such that when we add or subtract it from any number, the latter doesn't change. We add nothing in the first case, we take nothing away in the second, so why should it? Despite the obvious simplicity of this observation, the idea to reserve a special number for *nothing* didn't occur to humanity for quite a while. Zero as we know it was invented in India around AD 500, thousands of years after people had learned to use numbers.

Homework Problem 4.18 *Compute the following:*

a. 0+2; b. 5-0; c. 3+0-0; d. 0-0; e. 0+0; f. 0+4-0; g. 6+0-2-0.

The number line is also a great place to see how odd and even numbers are located with respect to each other. Please get back to Lesson 3.3, recall the definition and then take a look at the following picture.

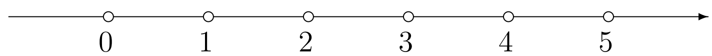


As we can see, odd and even numbers interlace on the number line. This is the reason we call 0 an even number. If we decide to go by the definition, we have to build a two-story house with equal number of apartments on each floor and with the total number of the apartments equal to 0.

Homework Problem 4.19 *Can we? Hint: do we need to do anything at all?*

Homework Problem 4.20 *What is a half of 0?*

An important feature of the number line is that it is *ordered*. This fact is expressed graphically by the arrow pointing to the right on the number line picture. If we move in the direction of the arrow, the numbers increase. If we move in the opposite direction, they decrease. For any two different numbers on the number line, we can always tell which one is greater. The greater number is located further in the direction of the arrow (further to the right in the standard notations). Looking at the following picture,



we can see that 0 is less than 2, 5 is greater than 4, 4 is greater than 1, and so on. We write this down as

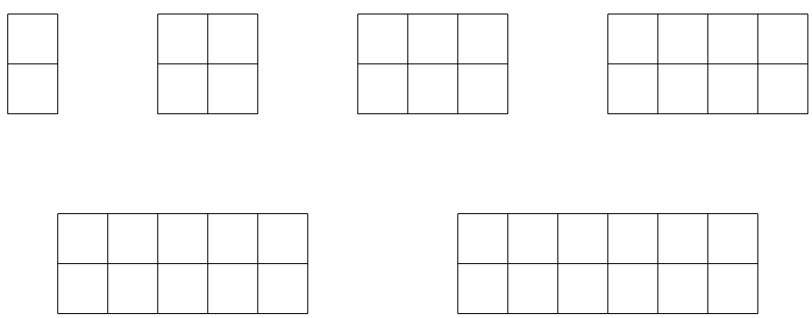
$$0 < 2, \qquad 5 > 4, \qquad 4 > 1.$$

The $>$ sign always opens up toward the greater number.

Homework Problem 4.21 *Put the correct sign, $>$ or $<$, in the box between the numbers. Use the number line if necessary.*

$$5 \square 10 \qquad 3 \square 1 \qquad 7 \square 8 \qquad 5 \square 0$$

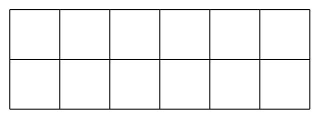
Homework Problem 4.22 *Draw Young diagrams conjugate to the diagrams below. What are their volumes?*



Homework Problem 4.23 *In Lesson 3.3, we have defined even numbers as integers represented by two-row Young diagrams with an equal number of boxes in each row. Using conjugation of Young diagrams, prove an equivalent definition: a (positive integral) number is even, if it can be represented as the volume of a Young diagram with two boxes in each row. Hint: take a good look at the previous problem before attempting to solve this one.*

Example 4.2 *Thomas and Daniela were given 12 candies. Can they divide them so that each gets an equal number of sweets?*

Let us represent the sweets by boxes of a Young diagram. 12 is an even number, so we can use this one.



The diagram corresponds to the partition of 12 into two equal parts.

$$12 = 6 + 6$$

Thus, each of the children should take 6 candies.

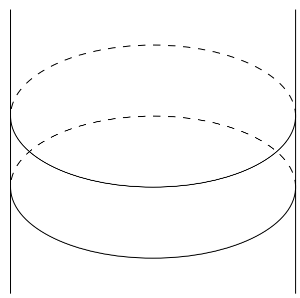
Homework Problem 4.24 *Thomas and Gregory have 7 slices of pizza to share. Can each of them have an equal number of slices? What is the best they can do? Hint: draw the appropriate Young diagram to see if 7 is odd or even.*

4.5 A 2D surface

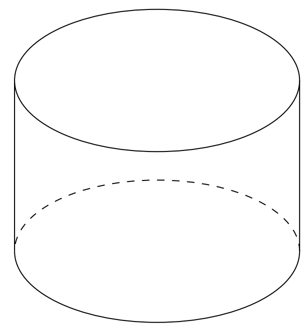
Let us get back to Euclid's "Elements".

Axiom 4 *A surface is that which has length and breadth only.*

In modern words, a surface is a two-dimensional (2D) geometric object. Like any other geometric object, it is made of points. A surface has two dimensions, the aforementioned length and width, but no height. Consider the following example. Let us take a circumference and a straight line and rotate the line around the circumference as on the left below.



a cylinder



a finite cylinder

What we get looks like a straight water pipe going to infinity at both ends. The surface, called a *cylinder*, can also be constructed by sliding the circumference along the straight line. If, instead of an infinite straight line, we rotate its final segment, we get a *finite cylinder* as on the right above. Note that a finite cylinder has only one wall and neither top nor bottom. The same, of course, is true for a cylinder.

Homework Problem 4.25 *Make a finite cylinder using paper, either glue or sticky tape, and, possibly, scissors.*

To better understand the meaning of dimensions, let us consider the above surfaces, the cylinder and the finite cylinder, as the worlds inhabited by some tiny bugs. The creatures can only crawl forward/backward or to the left/right. They cannot fly up, that is away from the surface, or dig down through it. The third, up/down, direction is unknown to them. Their worlds have only two.

Homework Problem 4.26 *Draw a bug on the finite cylinder from Homework Problem 4.25. Then draw the arrows pointing*

- *forward, backward, and*
- *to the left, to the right of the bug.*

Can you draw the arrows pointing up or down?

The cylinder looks very similar to the finite cylinder, but their geometries are very different. Suppose that the bugs inhabiting the cylinder live on its outer side. Since the surface extends to infinity, there is no way for the insects to get to the opposite, inner, side. To the contrary, the bugs inhabiting the finite cylinder can always crawl from the outer to the inner side and the other way around. All they need to do is to cross one of the two boundaries of the surface, the circumferences at the top and bottom of the finite cylinder. A finite cylinder is a 2D surface with a boundary, a cylinder – a 2D surface without a boundary.

Homework Problem 4.27 *Suppose that the cylinders are huge compared to the bugs, just like the planet of Earth is huge compared to us. Suppose, in addition, that the insects inhabiting the finite cylinder world live far away from the boundaries. Would the creatures believe that their home worlds are flat?*

Homework Problem 4.28 *If we slide a cylinder along itself, that is along its generating straight line, would it remain the same? How about a finite cylinder?*

Homework Problem 4.29 *If we rotate a cylinder along its generating circumference, would it stay in place? How about a finite cylinder?*

Homework Problem 4.30 *Which of the two has more symmetries, a cylinder or a finite cylinder? Recall Definition 4.3 if needed.*

Homework Problem 4.31 *What cylindrical structures were widely used in medieval times? (If you find it hard to answer this question, take a look at the photograph on page 10.)*

Homework Problem 4.32 *Is it possible to build a castle with towers having the shape of a cylinder instead of a finite cylinder? Has the Earth enough rocks to complete this project? How about the entire Solar system?*

Homework Problem 4.33 *Put the correct sign, > or <, in the box between the numbers. Use the number line if necessary. Remember, the sign opens up towards the greater number.*

$$7 \square 11 \qquad 0 \square 2 \qquad 5 \square 3 \qquad 4 \square 1$$

Homework Problem 4.34 *Gregory has 14 toy monsters. He wants to split them into two armies so that the forces are equal. Can he? If positive, how many monsters should there be in an army? What is the best he can do, if he has 15 monsters instead of 14?*

4.6 A 2D plane

Having obtained some initial experience with 2D surfaces, let us see how Euclid defines the most important of them, the 2D plane.

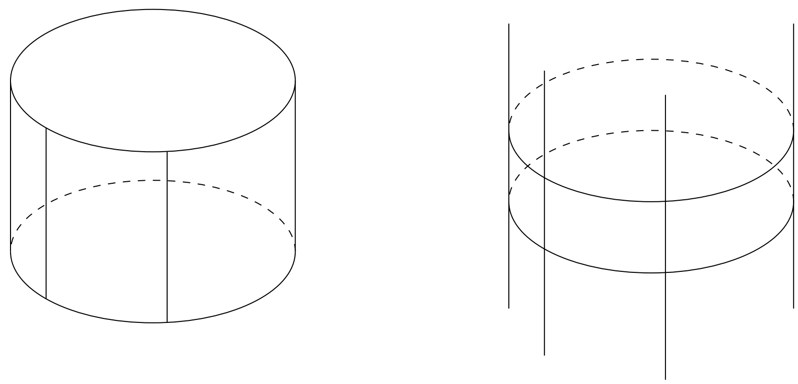
Axiom 5 *A plane is a surface which lies evenly with the straight lines on itself.*

Similarly to Axiom 3, the Greek geometer claims that a plane is a highly symmetric 2D surface. It is so symmetric that any straight line in a plane can be moved to the position of any other straight line in the plane by a transformation that preserves the plane itself.

Homework Problem 4.35 *Draw a straight line on a sheet of paper. Put a ruler on the sheet and consider the surface of the ruler as a model of the same plane as the one spanned by the paper sheet. Consider an edge of the ruler as another straight line in the plane. Can you move the edge to the position of the other straight line by means of rotating the ruler and sliding it along the plane while keeping the paper still?*

Homework Problem 4.35 shows that a 2D plane indeed has enough symmetries, rotations and translations, to move any straight line in the plane to the position of any other. Does this uniquely define the plane we intuitively expect? Obviously, not!

Let us take a stripe of paper similar to the one in Homework Problem 4.25. Using a ruler, let us draw two segments of straight lines parallel to the sides we are about to glue together and going all the way from the bottom to the top of the stripe. Then let us make a finite cylinder as in Homework Problem 4.25. Imagining our model going to infinity at both ends, let us consider the infinite cylinder instead of the finite one. Then the finite segments of the straight lines we have drawn will expand to infinity at both ends as well.



Homework Problem 4.36 *Can we rotate one of the two straight lines on the cylinder to the position of the other one? Can we rotate any straight line "living" on a cylinder to the position of any other straight line on the surface?*

Homework Problem 4.36 demonstrates that Axiom 5 is far from perfect. It defines not only a plane, but also a cylinder, and, as we shall soon see, many more symmetric 2D surfaces. Let us try to improve upon the Classics.

Homework Problem 4.37 *Take any two distinct points on a 2D plane modeled by a sheet of paper. Can we always connect them with a straight line? Is there only one straight line passing through the points? (Please make sure that the child takes the points far enough, otherwise, due to their non-zero size, you risk running into unnecessary complications.)*

Homework Problem 4.38 *Take two arbitrary points on a cylinder modeled by the finite cylinder from Homework Problems 4.25 and 4.36. Can we connect the points with a straight line?*

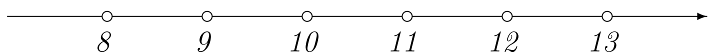
The above two Homework Problems suggest that the following definition is closer to our intuitive notion of the plane.

Definition 4.4 *A 2D plane is a 2D surface such that for any two distinct points on it, there exists one and only one straight line passing through them.*

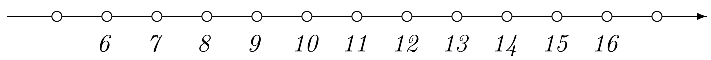
As we shall see, Definition 4.4 is not perfect either, but at least this time the cylinder doesn't get through the sieve.

Homework Problem 4.39 *Marking steps on the number line, compute*

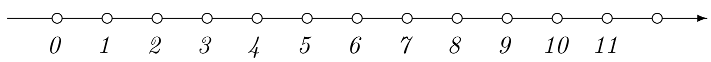
a. $9+2+1=$



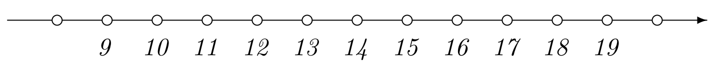
b. $6+4+3+2+1=$



c. $9-2-2-2=$

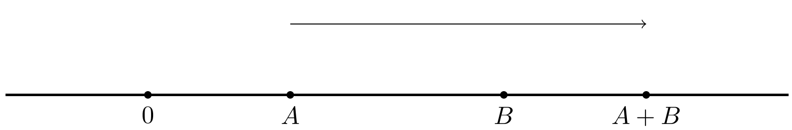


d. $19-4-2-2-1=$



To check your answers, redo the computations using Young diagrams.

Recall that addition/subtraction is nothing else but sliding the number line along itself to the right/left. According to Axiom 3, we can move any point on the line to the position of any other point this way. For a straight line equipped with zero, this allows us to introduce addition and subtraction for any two points on the line.



To add A to B, we first need to take a compass and measure the distance between 0 and A. Then let us place the compass's needle at point B and mark the point $A+B$ to the right of B (provided that A is positive, for a negative A the desired point would have been to the left of B).

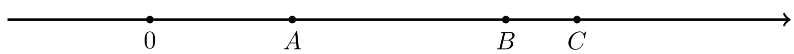
Homework Problem 4.40 *Using a compass and the above picture, find $B - A$.*

Homework Problem 4.41 *Put the correct sign, $>$ or $<$, in the box between the numbers. Use the number line if necessary.*

$$12 \square 14 \qquad 21 \square 17 \qquad 17 \square 15 \qquad 12 \square 11$$

Remember, the $\lesseqgtr$ relation depends only on the relative position of the numbers on the line.

Homework Problem 4.42 *Looking at the following picture,*



put the correct sign, $>$ or $<$, in the box between the numbers.

$$A \square 0 \qquad A \square B \qquad B \square C \qquad C \square 0$$

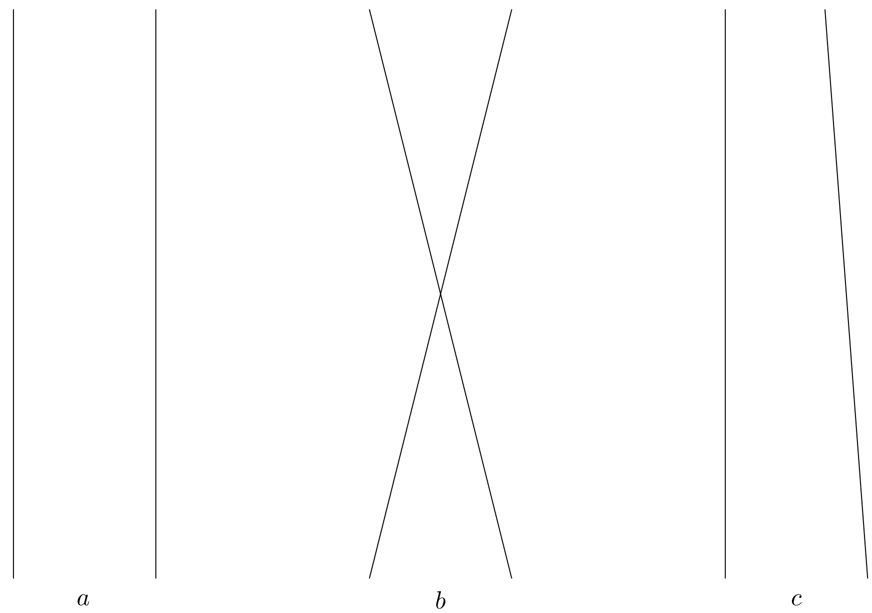
Homework Problem 4.43 *Using a compass and the previous Homework Problem picture, find the point $C - B$ on the number line. What number is greater, A or $C - B$?*

4.7 The Euclidean plane

As we have mentioned, Definition 4.4 does not uniquely define the 2D plane our intuition is comfortable with, the Euclidean plane. We shall define and study it here.

Definition 4.5 *Two straight lines in a 2D plane are called parallel, if they are distinct and do not intersect each other.*

Homework Problem 4.44 *What pair of the straight lines below, a, b, or c, is a pair of parallels?*

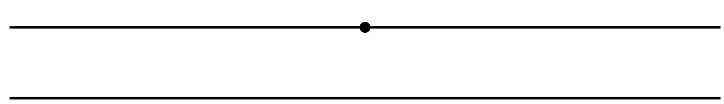


Homework Problem 4.45 *Using a ruler, try to draw a pair of parallel straight lines.*

Hard, isn't it? In Lesson 4.8, we shall learn how to construct a straight line parallel to the given one precisely, without guessing, using a ruler and compass as tools.

Homework Problem 4.46 *Have you seen parallel lines in nature? Take another look at the railroad tracks on page 31. Do they look parallel? Are they?*

Definition 4.6 *A 2D plane is called Euclidean if for any straight line in the plane and for any point in the plane not lying on the line, there exists one and only one straight line passing through the point parallel to the original line.*

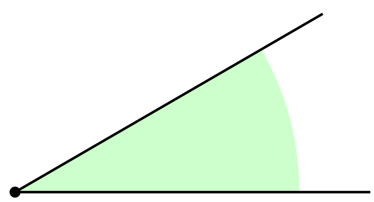


Homework Problem 4.47 *Draw a straight line on a sheet of paper and pick a point away from the line. Does there exist a straight line passing through the point parallel to the initial line? Is there only one?*

Homework Problem 4.47 shows that a sheet of paper on a flat table is a good model of the Euclidean plane. Let us meet the most elementary inhabitants of the latter and study their properties.

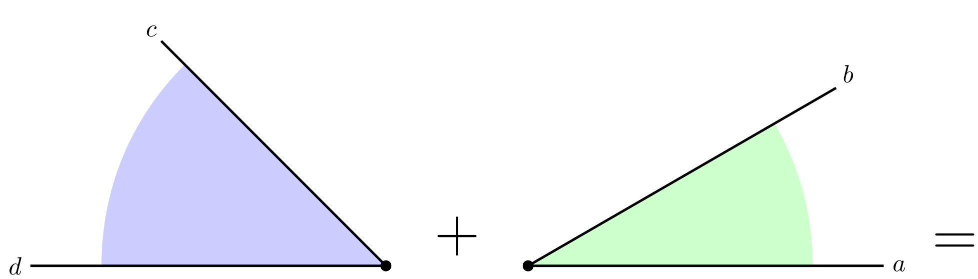
Definition 4.7 *A ray is a half of a straight line, finite in one direction, but infinite in the other. A ray contains its boundary point.*

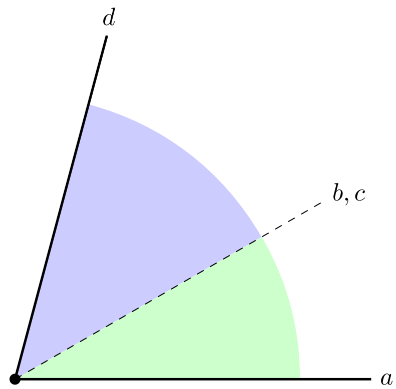
Definition 4.8 *A (plane) angle is a plane figure formed by two rays with the common vertex and by all the points in between.*



Looking at the above picture, we have to imagine that each of the two rays goes to infinity and so does the green coloring. This way, the rays split the plane into two parts, the green and white. The green-colored part, together with the boundary rays, is an angle. The white-colored region, bounded by the same rays, is also an angle. You have to specify the one you consider.

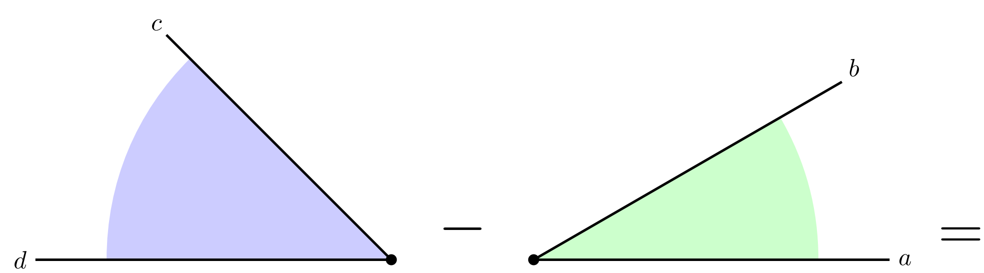
Just like translational symmetries of the straight line enable us to add and subtract (real) numbers, rotational symmetries of the circumference allow us to add and subtract angles. To add two angles, we need to put their vertices together in the center of a common circle. Keeping one of the angles still, let us rotate the other one until the angles' boundary rays coincide as on the picture below.

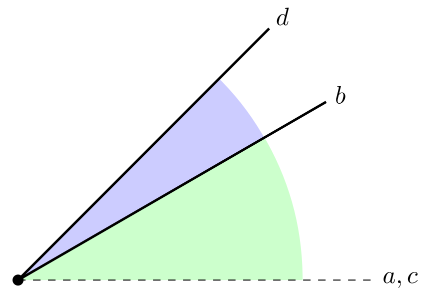




The resulting blue-and-green angle is the sum of the blue angle and the green one. The rays a and d are the sides of the sum, the rays b and c become its inner points. Alternatively, we can add the arcs the angles cut out of any circumference centered at their common vertex. The rays a and d originating at the center and passing through the endpoints of the resulting arc will form the boundaries of the new angle.

To subtract a smaller angle from a larger one, we similarly need to put the vertices together. Then an appropriate rotation, as on the picture below, does the job.





This time the result of subtraction is the blue part left from the larger angle after positioning the smaller one on top of it in such a way that the rays a and c coincide.

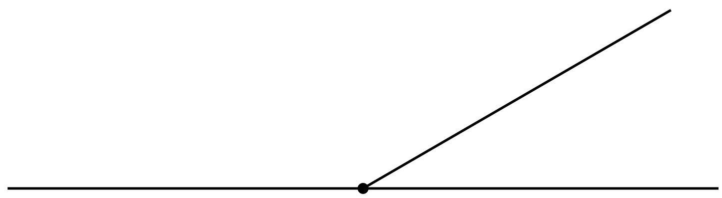
Homework Problem 4.48 *Draw two angles of different sizes on a sheet of paper, color them using contrasting colors, and cut out with scissors. Find their sum and difference.*

Definition 4.9 *If the rays forming an angle lie on one and the same straight line, the angle is called a straight angle.*

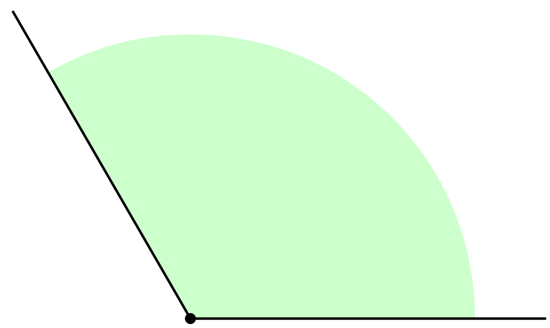
You have to imagine that the entire upper half-plane at the above picture is colored green. Note that the picture consists of two straight angles, the upper half-plane colored green and the lower half-plane colored white.

Homework Problem 4.49 *Draw a straight line on a sheet of paper. How can you make it into a straight angle?*

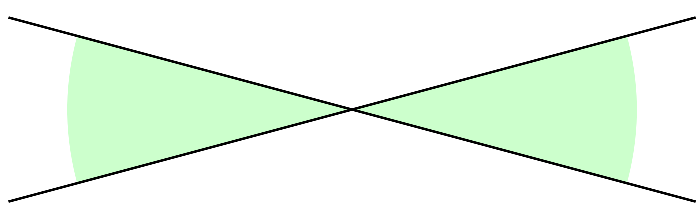
Definition 4.10 *Two angles are called supplementary, if they add up to a straight angle.*



Homework Problem 4.50 *Using a ruler, draw an angle supplementary to the green-colored angle below. How many ways are there to solve this problem?*



Definition 4.11 *Two angles are called opposite, if their sides form two straight lines as on the picture below.*



The green-colored angles on the above picture are opposite. The white-colored angles, supplementary to the green-colored angles, are opposite, too.

Proposition 4.2 *Opposite angles are equal to each other.*

Before giving a proof to this Proposition, let us consider the general idea of proving that something in the plane, or on another surface, is equal to something else. Recall that a symmetry of the surface is a move that changes positions of the points on the surface, but leaves the surface unchanged as a whole.

Definition 4.12 *A symmetry is called an isometry if it doesn't change the distance between any two points of the surface.*

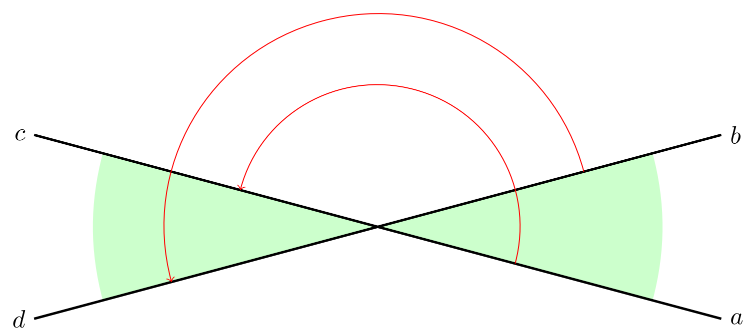
Translations and rotations of the plane are good examples of isometries.

Homework Problem 4.51 *Give an example of an isometry*
a. of a straight line,
b. of a circumference,
c. of a cylinder,
d. of a finite cylinder.

As Homework Problem 4.51 shows, all the symmetries we have considered so far were isometries. Translations, rotations, and combinations of the two do not change distances between the points. We shall study the symmetries that do change the distances later.

We shall call two objects equal, if there exists an isometry that moves one object to the position of the other. To prove Proposition 4.2, we need to find such a symmetry that does the trick for the opposite angles in consideration.

Proof — Consider rotating the plane by the straight angle around the angles' common vertex as on the picture below.



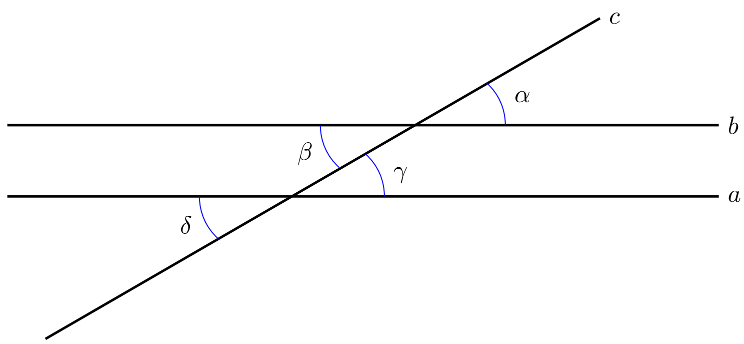
Under this transformation, ray a will become the opposite side of the straight angle it forms with ray c. In other words, a will coincide with c. Similarly, ray b will coincide with ray d. All the green-colored points on the right-hand side of the picture will rotate to the corresponding green-colored points on the left-hand side. The rotation will make the entire angles coincide. Thus, the opposite angles are equal to each other. □

Homework Problem 4.52 *Draw a pair of opposite angles on a sheet of paper and color them green or whatever other color you like. Cut them out with scissors. Then glue one of the angles to a blank paper sheet placing the angle's vertex not too far from the sheet's center. Attach the other angle's vertex to the vertex of the glued angle with a pin. Can you make the second angle opposite to the first one? Can you make it coincide with the first angle?*

Since the times of Euclid, it is customary to name angles with Greek letters. In the proof of the proposition below, we shall use the first four of them, α, called *alpha*; β, called *beta*; γ, called *gamma*, and δ, or *delta*. The first two and the last of these letters, changing the way they are written and pronounced, reached the English ABCs as a, b, and d. The third turned into g, the seventh letter of the English alphabet. It still holds the third position in modern Greek and in the Slavic alphabets such as the Russian, Ukrainian, and some more.

Homework Problem 4.53 *What do you think the word "alphabet" originates from?*

Proposition 4.3 *Let the straight lines a and b be parallel. Let the straight line c intersect them as on the picture below. Then the angles α, β, γ, and δ are all equal to one another.*



Proof — The angles α and β are opposite and so are the angles γ and δ. Thus, according to Proposition 4.2, $\alpha = \beta$ and $\gamma = \delta$. Once this observation is made, we only need to prove that $\alpha = \gamma$ in order to prove the proposition.

Consider sliding the plane down along the line c while keeping the line b parallel to a until the vertex of the angle α coincides with that of γ. This translation, an isometry, makes the angles α and γ coincide. Thus, $\alpha = \gamma$.[16] Finally, $\gamma = \delta$ implies $\alpha = \delta$. □

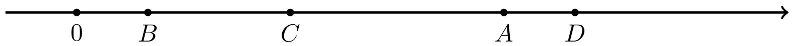
Note that Proposition 4.3 also holds for the four angles supplementary to α, β, γ, and δ.

Homework Problem 4.54 *Use your parent's/teacher's help to make two identical pictures similar to the one in Proposition 4.3. Cut the angle α out of one of the pictures. Slide it down along the c line to see whether it really coincides with the angle γ. Using rotations and translations, check out that $\alpha = \delta$ as well. Do a similar experiment for the angle supplementry to α.*

Homework Problem 4.55

• *Using a compass, mark the points $B+C$ and $D-A$ on the following number line. Remember, finding $B+C$ means sliding the number line by the distance from C to 0 to the right (in the case of positive C) and tracing where B ends up. We imitate the translation by marking the distance from C to 0 to the right of B with a compass. Similarly, to subtract A from D we need to slide the line to the left by the distance from A to 0 and trace where D goes.*

[16] This visually compelling argument would have satisfied Euclid and Descartes, but is not completely rigorous by modern standards. Unable to go too formal with kids of our target age, we shall leave it here for now.



• *Put the correct sign, >, <, or =, in the box between the numbers.*

$$B + C \square A \qquad B \square 0 \qquad B \square D - A$$

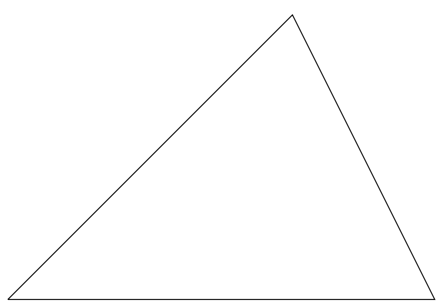
• *Using a ruler marked with centimeters, find the numerical values of the numbers A, B, C, D, B + C, and D − A. Having obtained the numerical data, check correctness of the above geometric equalities and inequalities.*

Homework Problem 4.56 *Use the number line to find all the numbers greater than 43 and less than 50. Which of these numbers are odd?*

4.8 Triangles and parallel lines in the Euclidean plane

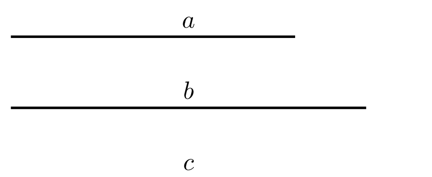
One of the goals of this lesson is to provide solutions to Homework Problems 4.45 and 4.47, that is to learn constructing a straight line parallel to the given one and passing through the given point away from it in the Euclidean plane. In order to achieve this goal, we'll need to develop some tools that turn out to be rather interesting objects themselves, the situation quite common in science in general and math in particular.

The word *polygon* means *multi-angled* in the language of Euclid, the ancient Greek. The simplest possible polygon in the Euclidean plane is a triangle, a polygon with three vertices, sides, and angles.



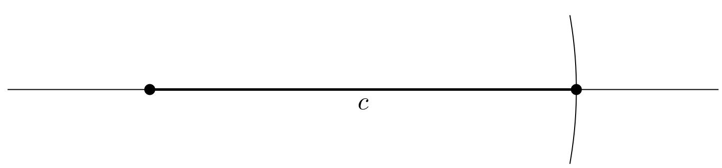
According to Definition 4.4, there always exists a straight line connecting any two distinct points of a plane and there is only one such. In the Euclidean plane, the finite segment of the line connecting the points is unique.[17] A part of the straight line, the segment is one-dimensional. It only has length, but no width or height. Thus, a 2D polygon with two vertices, sides, and angles does not exist in the Euclidean plane. The simplest possible 2D polygon in the Euclidean plane must have one vertex more than two. Let us see how we can make one.

Example 4.3 *Using a compass and a ruler, draw a triangle with the given sides a, b, and c.*



Step 1. Draw a straight line and mark a point on it.

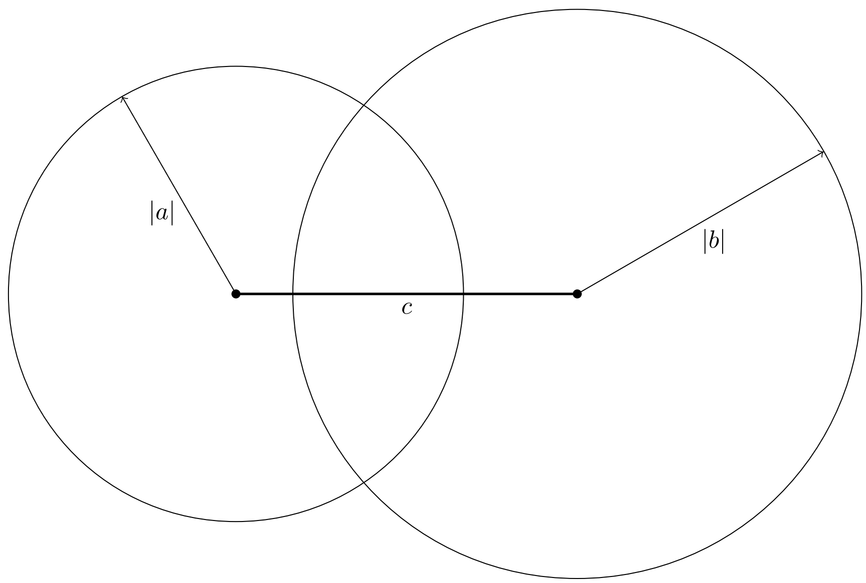
Step 2. Measure side c with a compass, place the needle at the marked point on the line, and mark side c on the line.



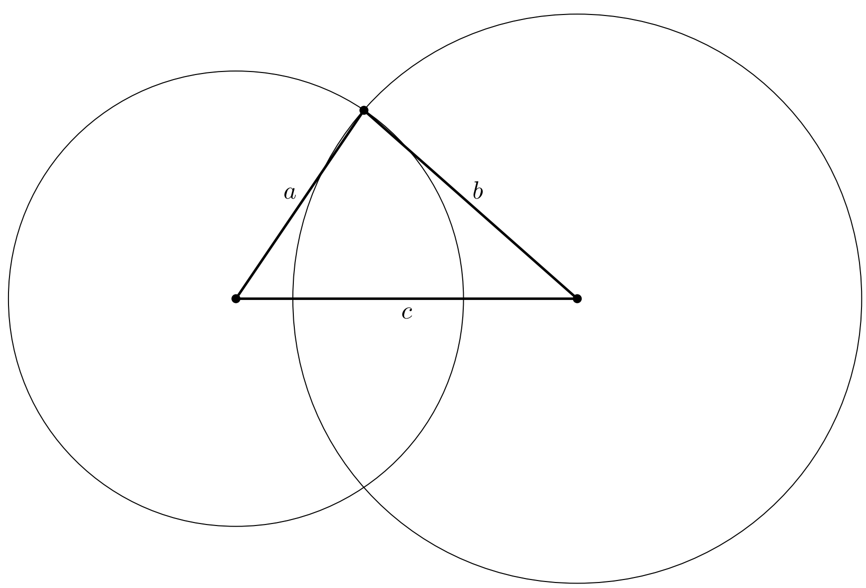
We shall use straight parenthesis to denote the length of a straight line segment. For example, the length of the side a is $|a|$.

Step 3. Now it is time to recall Definition 4.1. The points of the plane having the distance $|a|$ from the left end of the side c form a circumference of radius $|a|$ centered at the left node of the above picture. Similarly, the points having the distance $|b|$ from the right end of c belong to the circumference of radius $|b|$ centered at the right node.

[17] As we shall soon see, this isn't necessarily the case for a non-Euclidean plane.

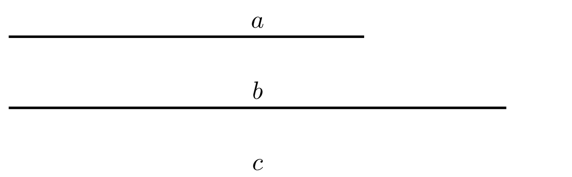


Step 4. The circumferences intersect at two points. Each of them has the distance $|a|$ from the left end of the side c and the distance $|b|$ from the right. We can pick either one as the third vertex of the triangle.



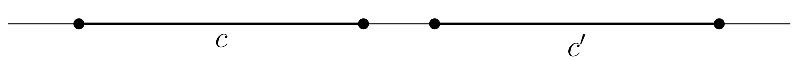
Proposition 4.4 *If a triangle in the Euclidean plane has its sides pairwise equal to the sides of another triangle in the plane, then the triangles are equal.*

Homework Problem 4.57 *Before we give this proposition a formal proof, let's check it out experimentally. Using a compass and a ruler, draw a triangle with the sides a, b, and c given below on two different sheets of paper. Cut both triangles out. Using the surface of a flat table as a model of the Euclidean plane, see if you can make the triangles coincide by moving them around the plane (and flipping over if necessary).*



Proof of Proposition 4.4 — Let two triangles in the Euclidean plane have the sides a, b, c and a' (reads "a prime"), b' (reads "b prime"), c' (reads "c prime") respectively. Let $|a| = |a'|$, $|b| = |b'|$, and $|c| = |c'|$. Let us move the "primed" triangle until its side c' coincides with side c of the other triangle.

Can we always do that? Yes, thanks to Axioms 3 and 5! Axiom 5 tells us that any straight line in a plane can be moved to the position of any other straight line in the same plane. So let us take the line side c' belongs to and move it until it coincides with the straight line passing through side c. This way, both sides become segments of one and the same straight line.



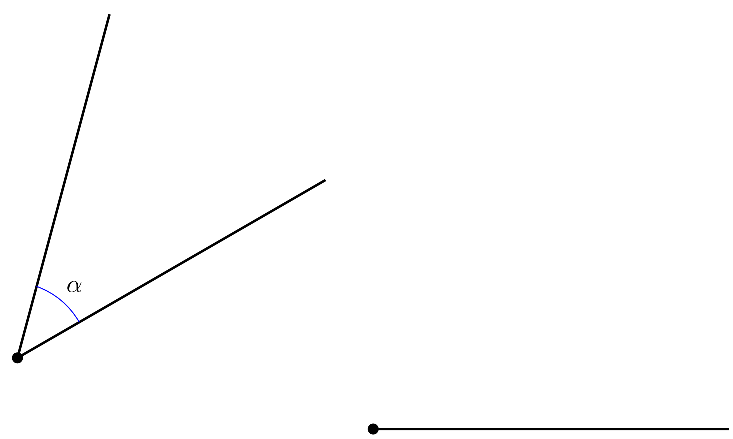
Axiom 3 tells us that any point of a straight line can be moved to any other point of the line by sliding the line along itself. Using a necessary translation, let us move the left end of the side c' to the left end of c. The symmetries used so far were all isometric. Since the sides have equal lengths, they must coincide.

Throughout the rest of the proof, you'll need to be constantly looking at the picture in Step 4 of Example 4.3.

Flipping the "primed" triangle over in 3D if necessary,[18] let us make sure that its a' side begins at the left end of the side c' (coinciding with c). Since $|a| = |a'|$, the other end of the side a' must lie on the circumference of radius $|a|$ centered at the left end of c.

The side b' must begin at the right end of the side c (coinciding with c'). Since $|b| = |b'|$, the other end of the side b' must lie on the circumference of radius $|b|$ centered at the right end of c. But these two circumferences meet at two points only! Flipping the "primed" triangle over its c' side if needed, we see that it coincides with the other one. Thus, they are equal. □

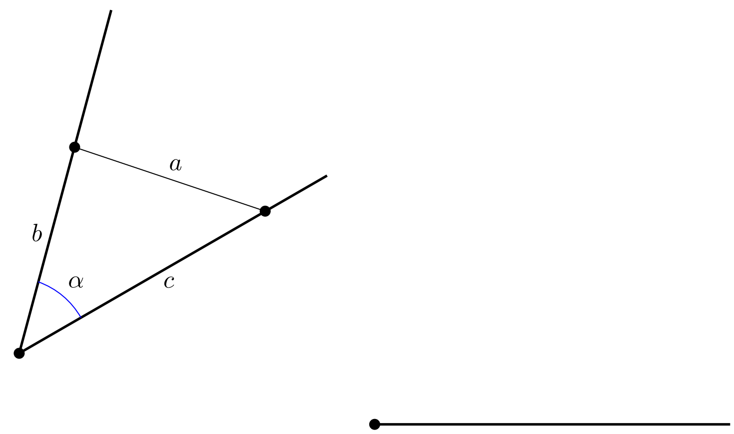
Example 4.4 *Using a compass and a ruler, draw a given angle α having a given ray as its side.*



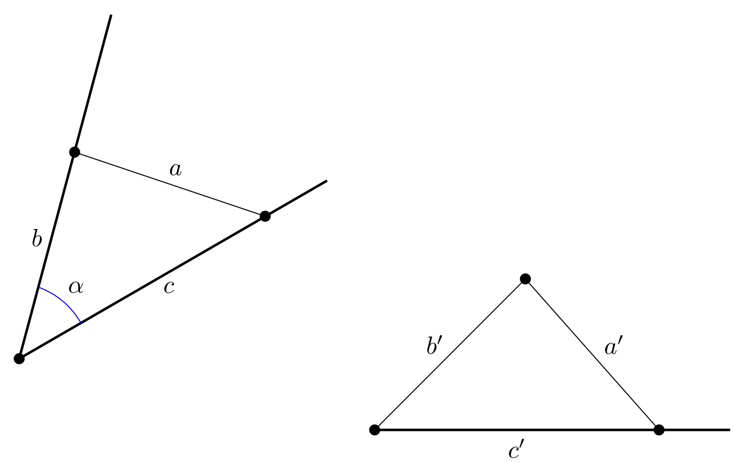
Step 1. Let us make an auxiliary[19] *triangle having α as its angle. Traditionally, the side opposite to α is called a.*

[18] This can also be done by another isometry of the plane, a reflection in the straight line containing the c segments. It will be useful to get back to this proof and discuss the detail when the child learns reflections.

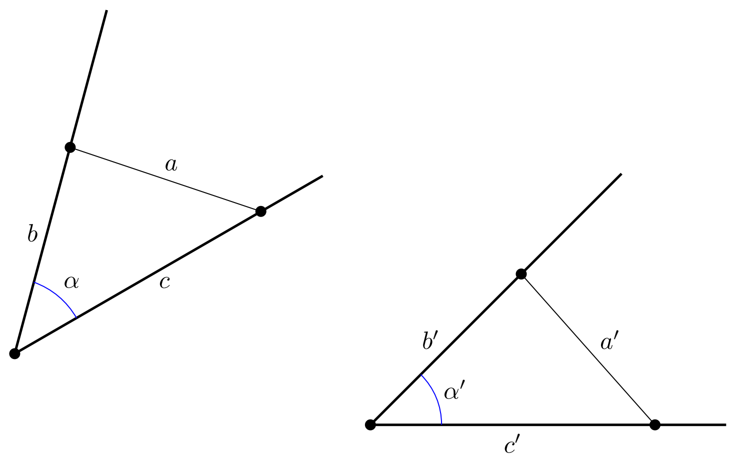
[19] Providing supplementary or additional help and support, as in an "auxiliary nanny", a nanny occasionally employed in addition to the main one. As many more words used in science, this one originates from Latin, the language of Ancient Rome. Its progenitor, the Latin word *auxilium* means "help".



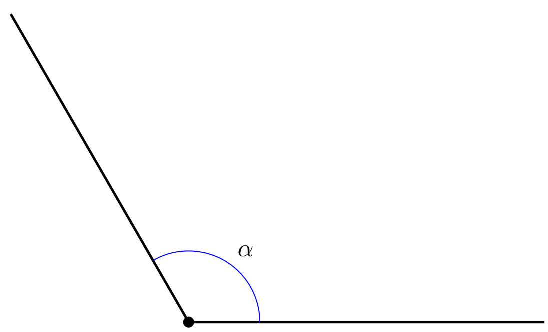
Step 2. Using the method of Example 4.3, let us construct the triangle having the sides a', b', and c' so that $|a'| = |a|$, $|b'| = |b|$, $|c'| = |c|$, c' goes along the given ray and the left end of c' coincides with the ray's vertex.



According to Proposition 4.4, the fact that $|a'| = |a|$, $|b'| = |b|$, and $|c'| = |c|$ implies that the triangles are equal. Thus, the angles α and α' opposite to the sides a and a' respectively are equal, too.

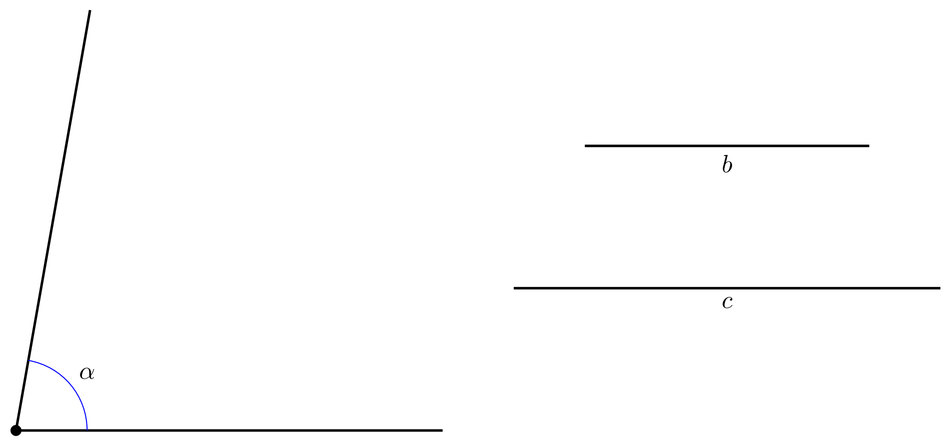


Homework Problem 4.58 *Using a compass and a ruler, draw an angle equal to the angle α given below on a separate sheet of paper.*



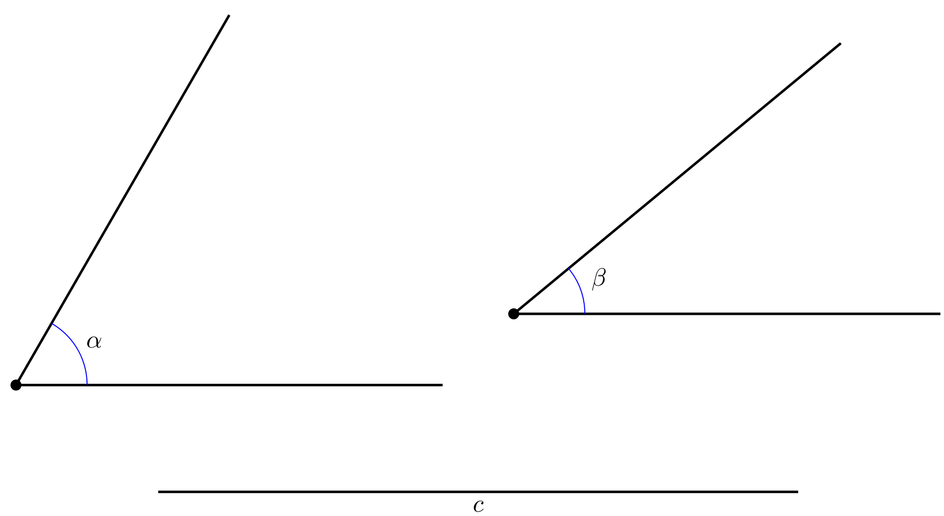
Homework Problem 4.59 *On a separate sheet of paper, draw a triangle with the angle α and adjacent*[20] *sides b and c given below. In this case, the word "adjacent" means that the vertex of α is an endpoint of b and c.*

[20] "Lying near to" as in "adjacent rooms" or "the houses adjacent to the park". Inherited from Latin without spelling change.



Hint: begin with drawing the angle. Note that constructing the latter, you can use not some arbitrary sides b and c of the auxiliary triangle, but the given sides b and c right away!

Homework Problem 4.60 *On a separate sheet of paper, draw a triangle with the side c and adjacent angles α and β given below. In this case, the word “adjacent” means that the endpoints of c are the vertices of α and β.*



Using Homework Problems 4.59 and 4.60, it is easy to extend Proposition 4.4 to the following important theorem.

Theorem 4.1 *Two triangles in the Euclidean plane are equal if either of the following holds.*

- *Their side lengths are pairwise equal.*

$$|a| = |a'|, \quad |b| = |b'|, \quad |c| = |c'|$$

- *They have an angle of equal size, and the lengths of the sides adjacent to the equal angles are pairwise equal.*

$$\alpha = \alpha', \quad |b| = |b'|, \quad |c| = |c'|$$

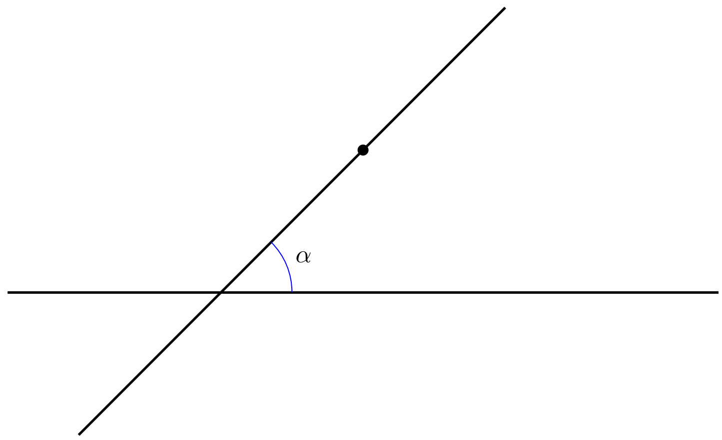
- *They have a side of equal length, and the adjacent angles are pairwise equal.*

$$|c| = |c'|, \quad \alpha = \alpha', \quad \beta = \beta'$$

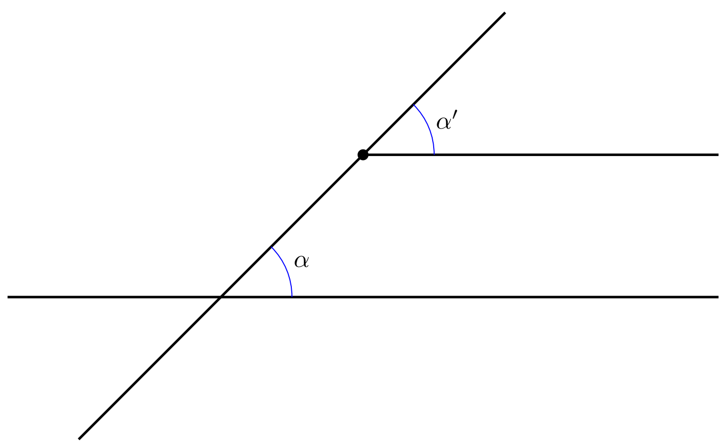
Homework Problem 4.61 *Study the proof of Proposition 4.4, then try to give a similar proof to the last two statements of the above theorem.*

Finally, the time has come to get back to Homework Problems 4.45 and 4.47 and achieve the final goal of this lesson. Consider the following picture.

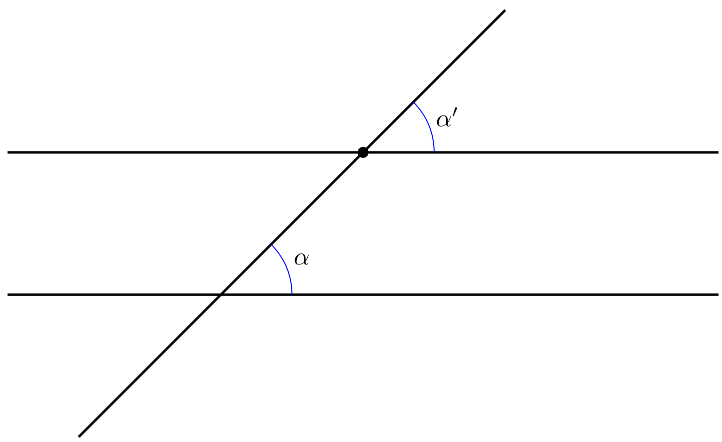
We want to draw a straight line parallel to the given one and passing through the given point away from the original line. The picture from Proposition 4.3 gives an idea. Let us first draw some line passing through the point and intersecting the given line. Let us call α the angle they form, as shown on the below picture.



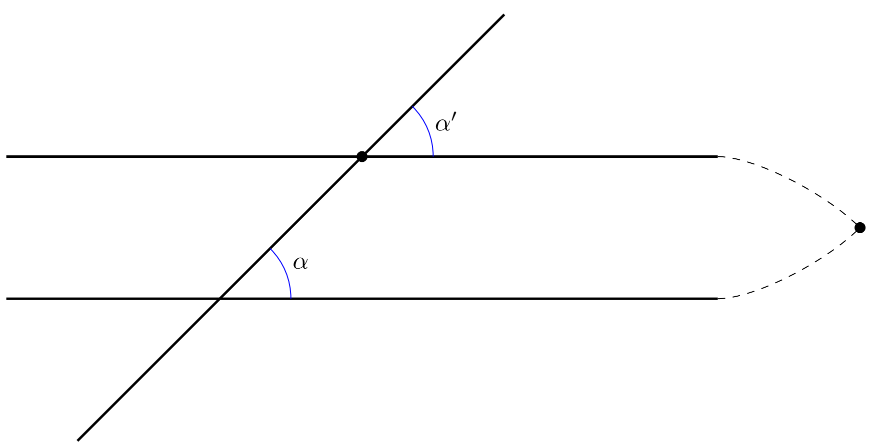
Now let us draw the angle α' equal to α, having the marked point as a vertex and the slanted upper ray as a side. Thanks to Example 4.4, we know how to do that.



It seems that the last thing left is to extend the horizontal side of the angle α' to the left.

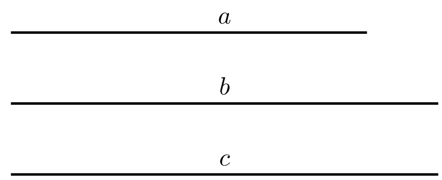


But do we get parallel lines this way? What if we don't and the lines that seem perfectly parallel in fact intersect, say, very far away on the right as shown on the below picture?



In Lesson 5.4, we shall prove that this is not the case. The above procedure indeed renders the parallel line we were looking for. Definition 4.6 guarantees its existence and uniqueness. Now we know how to construct it.

Homework Problem 4.62 *Using a compass and a ruler, draw a triangle with the given sides a, b, and c.*



Definition 4.13 *A triangle with two sides of equal length is called isosceles. A triangle with all sides of equal length is called equilateral.*

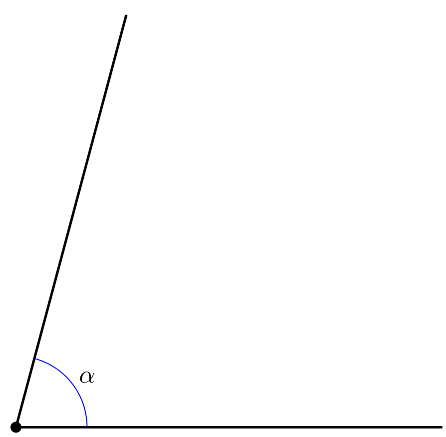
You may have noticed that $|b| = |c|$ in Homework Problem 4.62. The triangle you have constructed solving the problem was isosceles.

Homework Problem 4.63 *Using a compass and a ruler, construct an equilateral triangle with the side length 2".*

Some people define isosceles triangles as those having precisely two, but not three, sides of equal length. We do not follow this convention. According to Definition 4.13, an equilateral triangle is isosceles as well.

Homework Problem 4.64 *What's more out there, isosceles triangles or equilaterals? Isosceles triangles or all triangles? Why?*

Homework Problem 4.65 *Using a compass and a ruler, draw an isosceles triangle with the equal sides 5 cm long and with the angle between them equal to the angle α below.*



Homework Problem 4.66 *Prove that the angles the sides of equal length of an isosceles triangle form with the third side are equal. Hint: connect the common vertex of the equal sides to the center of the opposite side and see if Theorem 4.1 is of any help.*

Homework Problem 4.67 *Jack, Noah and Gregory have 12 toy race cars to play with. Can they split them into 3 equal teams? If positive, how many cars would there be in a team?*

Homework Problem 4.68 *What's more out there, cars or race cars? Why?*

Homework Problem 4.69 *Let x be a (positive integral) number such that*

$$12 < x < 19.$$

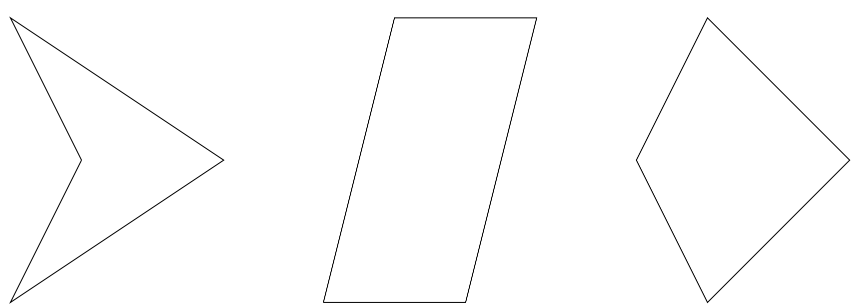
It is also known that x ends with 6. Find x.

4.9 Polygons in the Euclidean plane

Triangles in the Euclidean plane have many more important and fascinating properties we shall study a bit later. In this lesson, we shall meet polygons with more than three vertices.

Definition 4.14 *A quadrilateral, or a quad for short, is a polygon with four vertices, sides and angles such that no three vertices lie on a straight line.*

Here are a few quads.



Definition 4.15 *A quad is called a parallelogram, if its opposite sides are segments of parallel straight lines.*

The quad in the middle of the above picture is a parallelogram.

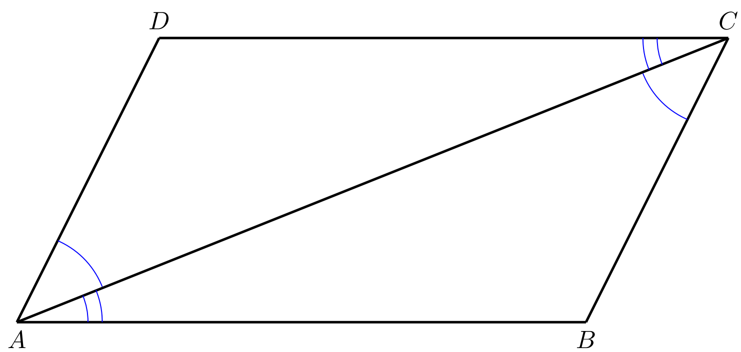
Definition 4.16 *A diagonal is a straight line segment connecting two non-neighboring vertices of a polygon or polyhedron.*

Theorem 4.2 • *Opposite sides of a parallelogram are of equal length.*

- *Opposite angles of a parallelogram have equal size.*

- *Diagonals of a parallelogram split each other in halves.*

Proof — Consider the following picture.



The lines AB and CD are parallel, so, according to Proposition 4.3,

$$\measuredangle CAB = \measuredangle ACD.$$

Similarly, AD is parallel to BC, implying

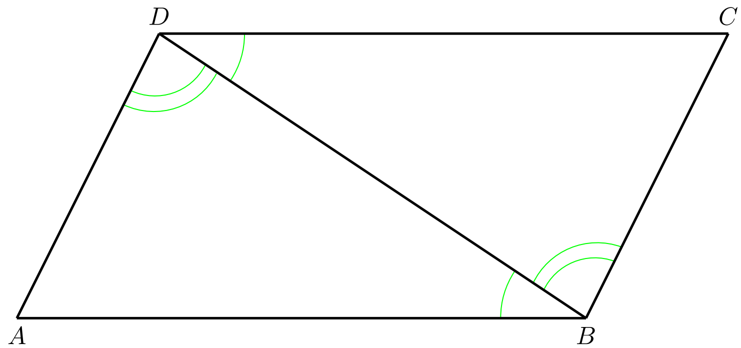
$$\measuredangle ACB = \measuredangle DAC.$$

Now consider the triangles ABC and ADC. They have a common side AC. The angles adjacent to it are pairwise equal, so, according to the third part of Theorem 4.1, the triangles are equal. Thus,

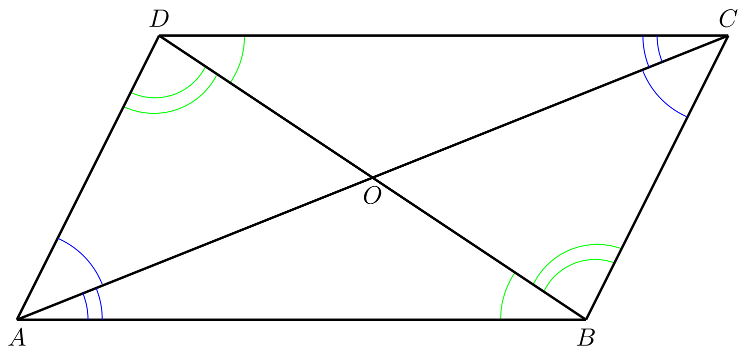
$$|AD| = |BC|, \quad |AB| = |DC|, \quad \measuredangle ABC = \measuredangle ADC.$$

A similar consideration of the triangles ABD and BCD proves that

$$\measuredangle DAB = \measuredangle BCD.$$



To prove the last statement of the theorem, let us consider triangles DOC and AOB on the following picture.



They have sides of equal length, $|DC| = |AB|$. The angles adjacent to the equal sides are pairwise equal as well, $\measuredangle CDB = \measuredangle ABD$, $\measuredangle DCA = \measuredangle CAB$. Thus, according to the third part of Theorem 4.1, the triangles are equal.

$$\triangle ODC = \triangle OAB.$$

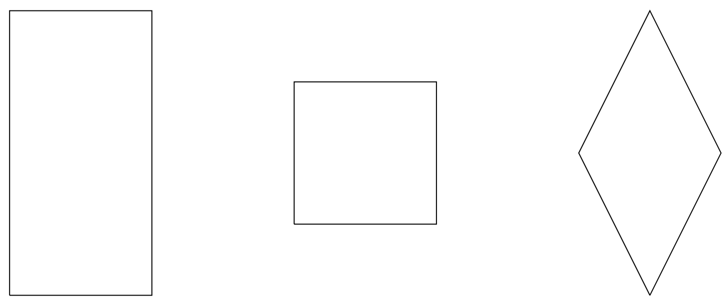
The latter implies

$$|OD| = |OB|, \quad |OC| = |OA|.$$

The intersection of the diagonals, point O, indeed lies at the center of the diagonals DB and AC. □

Definition 4.17

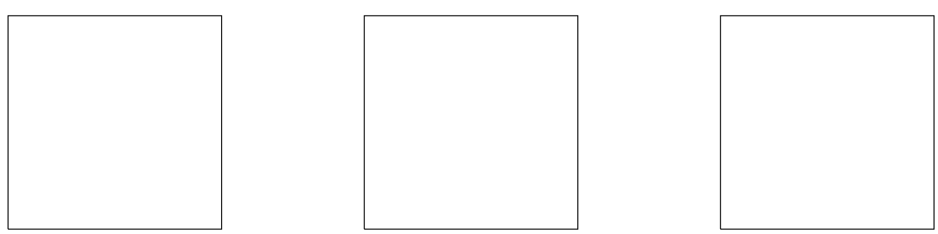
- *A rectangle is a parallelogram with all four angles equal to one another.*
- *A square is a rectangle with all four sides of equal length.*
- *A rhombus is a quad with all four sides of equal length.*



a rectangle   a square   a rhombus

A rhombus is sometimes called a *diamond*. A rectangle is occasionally called an *oblong*, the word generally meaning something that is longer than it is wide. Note that a square is a rectangle and a rhombus at the same time.

Homework Problem 4.70 *Draw a straight line that separates the below squares into a. two triangles; b. a triangle and a quadrilateral; c. two quads.*



Homework Problem 4.71 *Prove that opposite angles of a rhombus are equal. Hint: draw a diagonal, then use Theorem 4.1.*

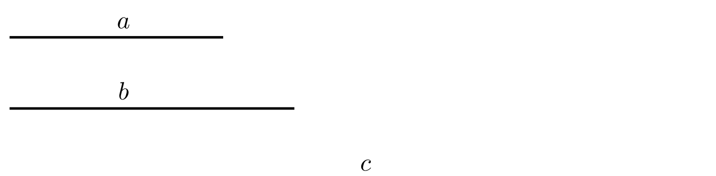
Homework Problem 4.72 *Prove that the diagonals of a rhombus split its angles in two equal parts. Hint: you will need Homework Problem 4.66 in addition to Theorem 4.1.*

Homework Problem 4.73 *Prove that the diagonals of a rhombus cut each other in halves.*

Homework Problem 4.74 *What's more out there, rectangles or squares?*

Homework Problem 4.75 *Why do they most often make windows of rectangular shape?*

Homework Problem 4.76 *Using a compass and a ruler, draw a triangle with the given sides a, b, and c.*



Can you? What's wrong?

Homework Problem 4.77 *What is a symmetry of a geometric object? Give an example.*

Homework Problem 4.78 *How does Euclid define a straight line? Has it anything to do with symmetries?*

Homework Problem 4.79 *Gregory's family are packing up for a trip. Gregory has five favorite toys, but he is only allowed to take two of them. All the toys are different. How many ways are there for Gregory to choose the toys?*

Homework Problem 4.80 *Let y be a (positive integral) number such that*

$$123 > y > 120.$$

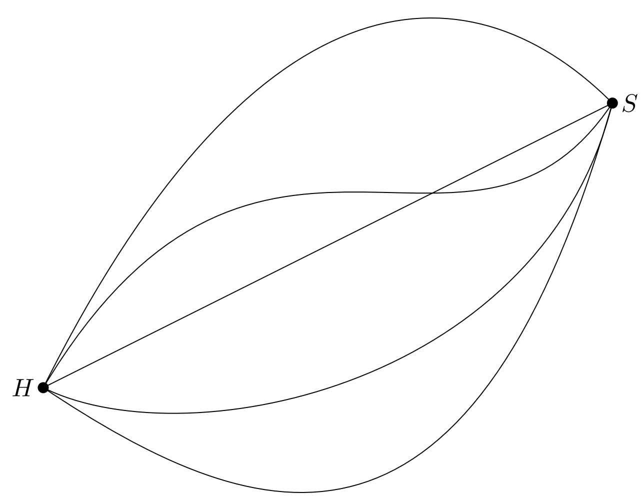
It is also known that y is even. What is y?

4.10 Straight lines on a cylinder

In this lesson, we shall see again the tiny bugs we have met in Lesson 4.5. The first insect family lives in the Euclidean plane. Their house is depicted as point H on the picture below. Point S is the school the bug family children attend. They need our help to find the shortest path from Home to School.

Definition 4.18 *The shortest path connecting two points on a surface is called a geodesic line.*

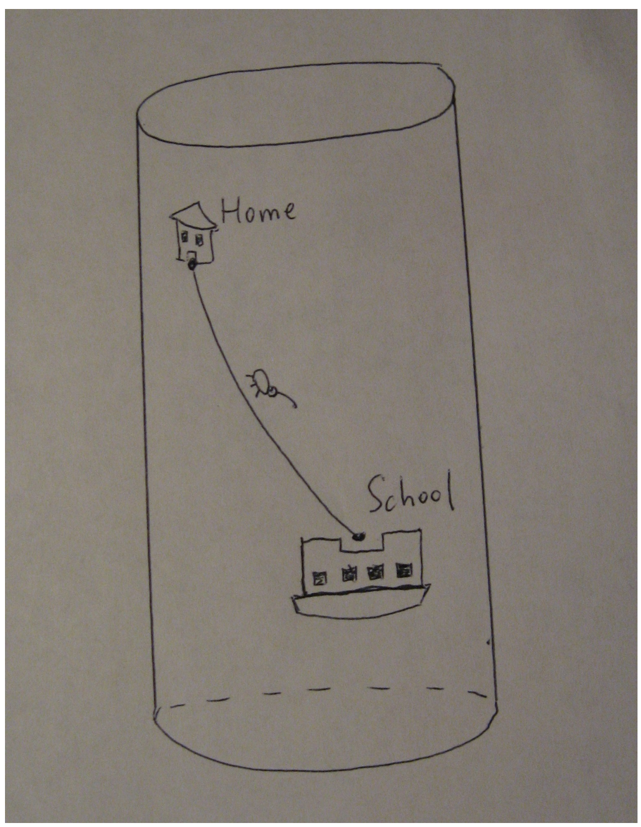
Homework Problem 4.81 *Please help the bug children, find the geodesic line from H to S among the lines below. Then take a ruler and see if the geodesic line is a segment of the straight line passing through the points H and S.*



Homework Problem 4.81 shows that straight line segments are the geodesic lines of the Euclidean plane. This is another feature that distinguishes the Euclidean plane from any other: the shortest path between any two different points of the Euclidean plane is always a segment of the unique straight line connecting them. Let us examine other 2D surfaces.

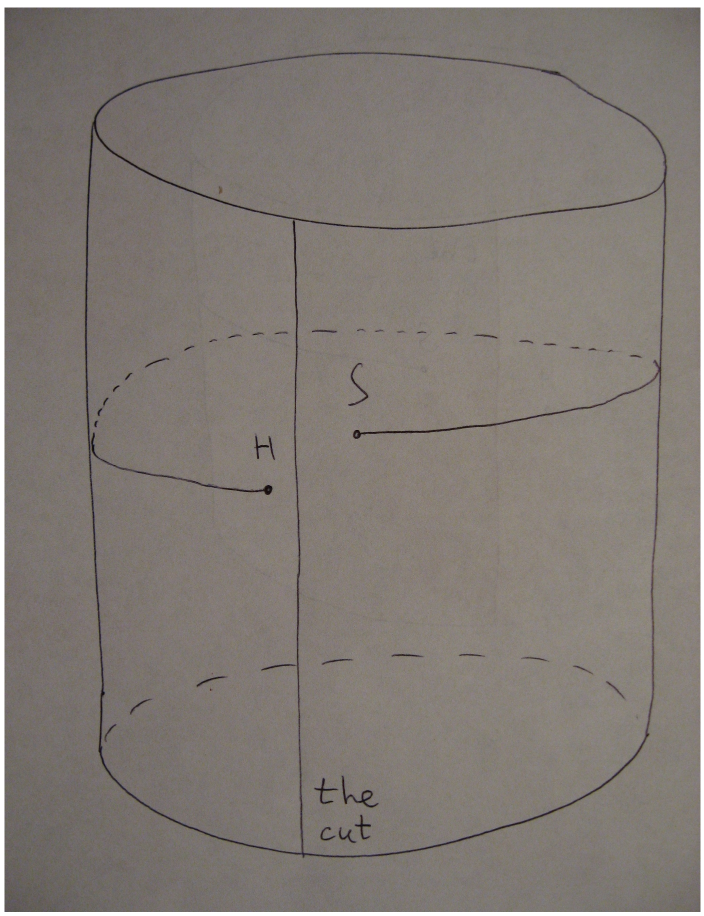
Homework Problem 4.82 *Take a sheet of paper and mark two points, H and S, on it. Then glue the sheet into a finite cylinder. Draw the shortest path for the young bugs living on the cylinder to follow going from Home to School.*

Tricky, isn't it? Here is an idea. As you have seen, gluing a sheet of paper into a cylinder can be done with no tearing or stretching. Thus, whatever length a line drawn on a flat sheet of paper had, it will retain the same length on the cylinder, and vice versa. So, we can cut the cylinder into a sheet again, put the sheet on a flat table surface, use a ruler to connect the points with a straight line segment, and finally glue the sheet back into the cylinder. Here comes the shortest path, or the geodesic line, connecting Home and School in the cylindrical world.

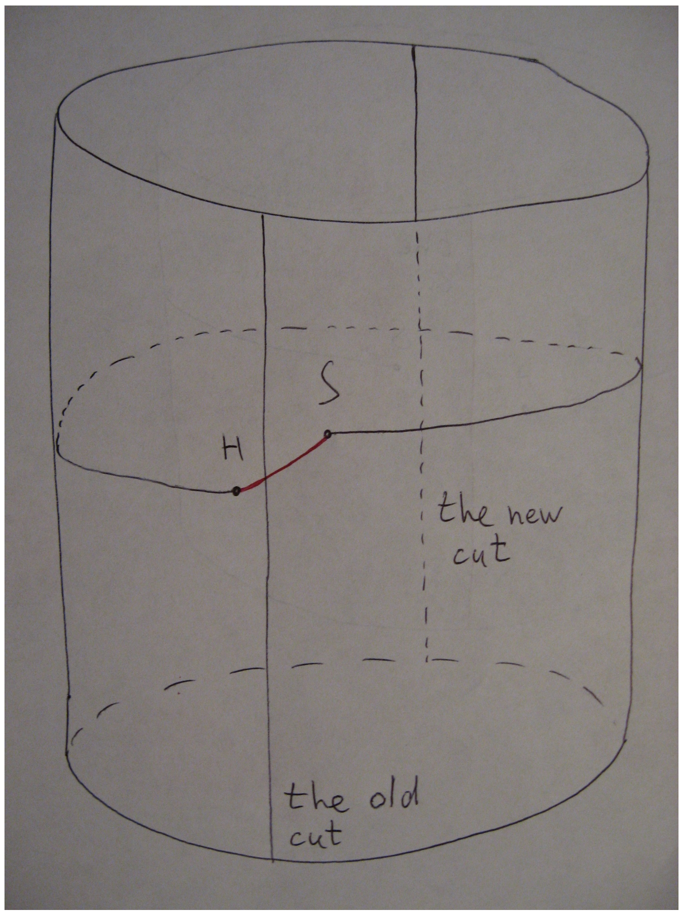


The geodesic line doesn't look straight on the cylinder, but it straightens up if we cut the surface into a flat sheet. Is it time to enlighten the bugs? Not yet! We were lucky to take the H and S points far enough from the cut. Let us see what happens if we take them close to the cut and at the opposite sides of it.

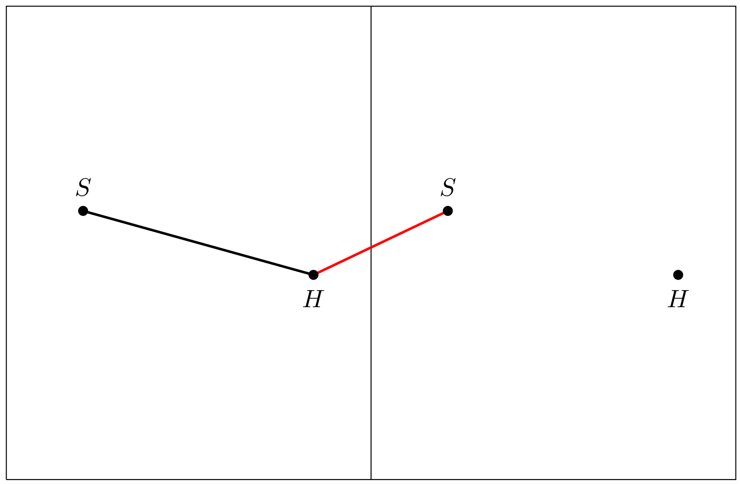
Let us start with a flat sheet of paper again. Let us mark point H not too far from the side we are about to glue to the opposite side of the sheet. Let us take point S close to the opposite side. Let us connect H to S by a straight line segment, glue the opposite sides of the sheet, turning the latter into a cylinder, and finally let us take a look. We'll see something like this:



The path that was the shortest on the flat sheet is obviously not the shortest on the cylinder! The way to remedy the problem is obvious: let us cut the cylinder again, this time not where we have initially glued it, but far away from that line. Then connect the points via a straight line segment, and glue the sheet back. Here comes the shortest path from Home to School, shown in red on the below picture, for the little bugs to take!

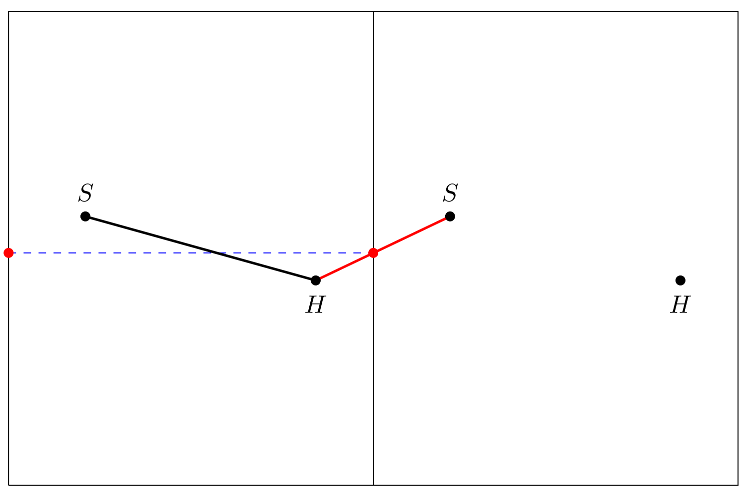


The only thing that still keeps the author of the book unhappy is the amount of cutting and gluing. Let us see, if we can do better. Let us draw two identical pictures of the bug world on different sheets of paper and let's put them next to each other as shown on the below picture.

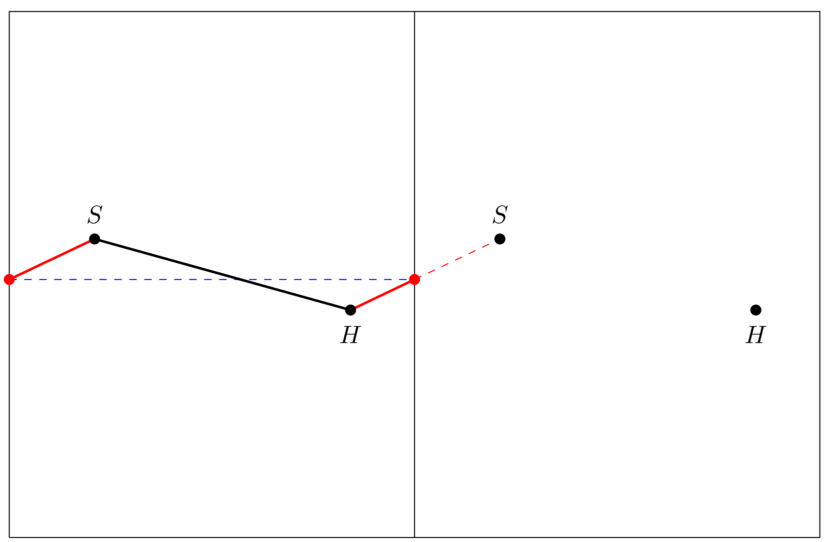


Homework Problem 4.83 *Which path on the above picture is shorter, the black one or the red one? Use a compass to compare.*

If the black path is shorter (which is not the case on the above picture), then we can glue the first sheet into a cylinder right away. If the red path is, then let us see how to draw it entirely on one sheet of paper, say, the first one. When we glue the longer sides of the first sheet together, the red line will appear on the opposite side.



Let us connect the point as it appears, red on the left hand side of the first sheet of paper on the picture above, to the bugs' School. Here comes our geodesic line, shown solid red on the picture below. Now we can help the bugs!

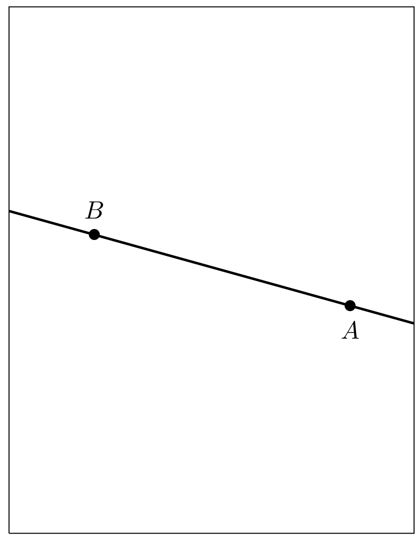


We can use a compass to parallel-transport the red dot from the right to the left. All we need is to measure the distance from the lower-right corner of the sheet to the red dot above it and then to mark the same distance from the lower-left corner along the left side of the sheet.

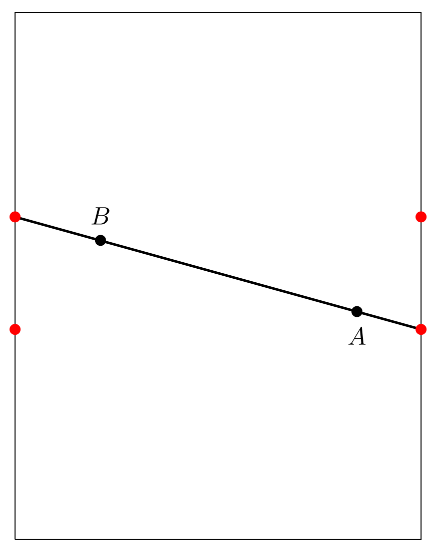
Let us recall that the main goal of this part of the book is to understand what a straight line is. We already know a few things about straight lines in the Euclidean plane. Now let's study straight lines on a cylinder. Let us take a sheet of paper and choose two random points, A and B on it. Let us draw a straight line passing through the points.

Homework Problem 4.84 *Does such a line always exist? Is there only one? Why?*

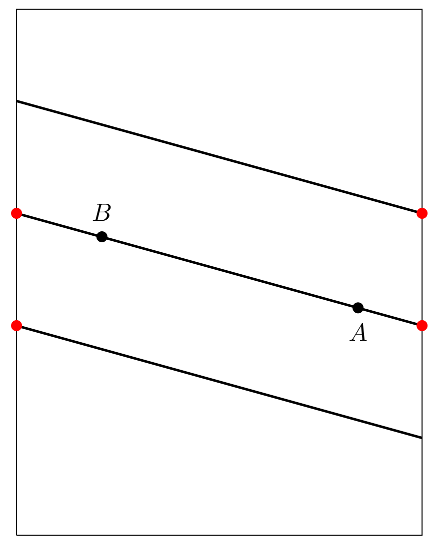
The moment the line reaches the sides of the sheet, we have to stop drawing and give it a thought.



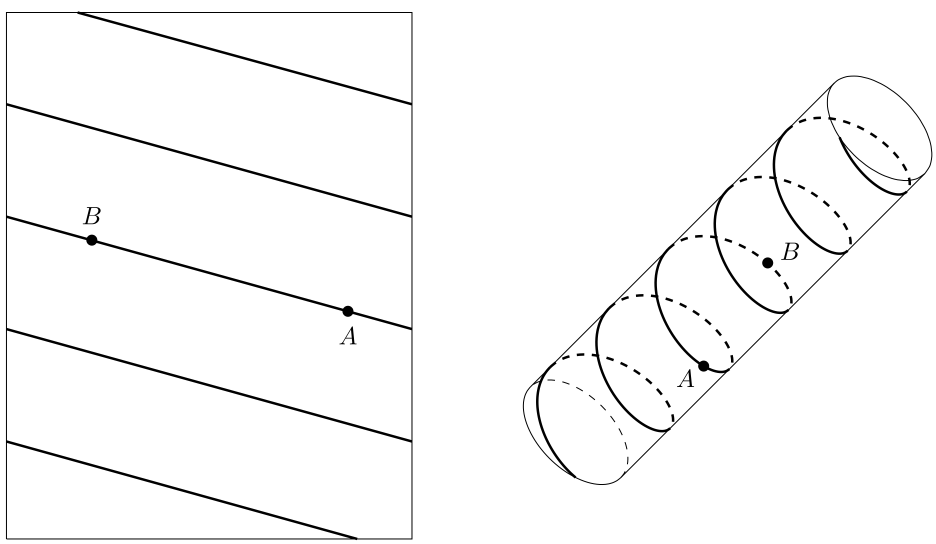
As we glue the sides together to form a cylinder, the left end of the segment will appear on the right hand side of the sheet while the right end will appear on the left.



The line will continue going past the cut until it reaches the sides of the sheet again.



There, it will jump to the opposite sides of the cut again. In the case of a finite cylinder, we end up with the following picture.



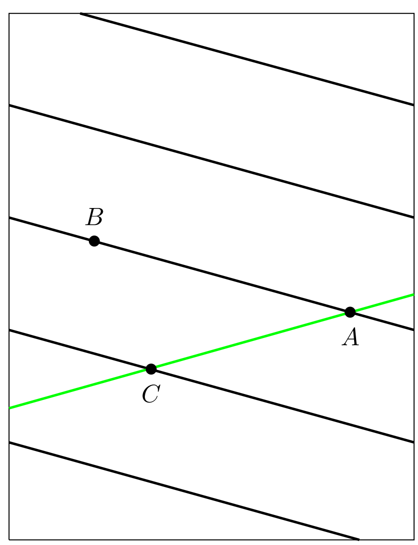
If the cylinder is finite, the line will seize up at the lower and upper ends. If the cylinder is infinite, it will wind around it to infinity. Note that the line, continuous on the cylinder, seems to be built of disjoint parallel equidistant segments when the cylinder is cut into a flat sheet.

Definition 4.19 *A straight line on a cylinder, winding around it as on the above picture, is called a helix.*

Homework Problem 4.85 *What kind of helical objects can you see in everyday life? Take a look at a few of them.*

Homework Problem 4.86 *Pick two random points on a sheet of paper to be glued into a finite cylinder. Construct a straight line on the cylinder passing through the points. Hint: use a compass to parallel-transport points from one side of the cut to the other.*

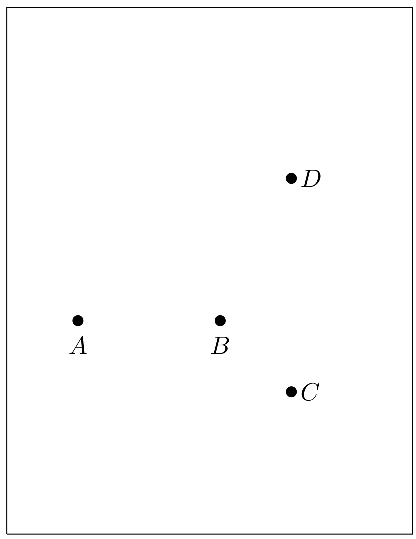
Although geometry of a cylinder borrows many features from the Euclidean plane, it is very different from it. For example, there is more then one straight line passing through a randomly chosen pair of points. Consider the following picture.



Point C belongs to the straight line AB, shown black on the above picture. At the same time, the green segment can be extended into a different straight line, AC, that also passes through A and C.

Homework Problem 4.87 *Is a cylinder a 2D plane? Hint: compare the above fact to Definition 4.4.*

Homework Problem 4.88 *Draw two straight lines, the first passing through the points A and B, the second through the points C and D on the picture below. What kind of lines do they become when you glue the below sheet into a cylinder?*

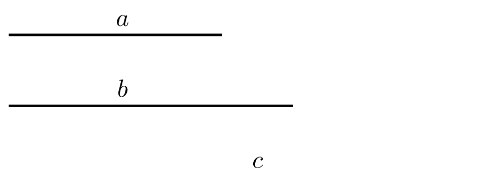


Homework Problem 4.88, in addition to our previous considerations, shows that there are three types of straight lines on a cylinder. A generic straight line on a cylinder is a helix. A straight line parallel to the generating circumference is a similar circumference. Finally, a straight line, parallel to the straight line generating the cylinder, is a similar straight line. Euclid would only have called the latter a straight line, although the circumference, as checked in Homework Problem 4.12, also satisfies his Axiom 3.

Homework Problem 4.89 *Pick two different points on the straight line drawn on the cylinder in Homework Problem 4.86. Imagine that the cylinder is infinite, not finite. Combining rotation and translation, can you slide one point to the position of the other by a move that preserves the helix as a whole? According to Euclid's Axiom 3, is a helix a straight line?*

Homework Problem 4.90 *Recall the definition of a 2D plane. Recall the definition of the Euclidean plane.*

Homework Problem 4.91 *Using a compass and a ruler, draw a triangle with the given sides a, b, and c.*



Compare your construction to those in Homework Problem 4.76 and Example 4.3. Why were you able to construct a triangle in Example 4.3, but neither in Homework Problem 4.76, nor in this one?

Homework Problem 4.92 *Prove the following theorem, known as the triangle inequality.*

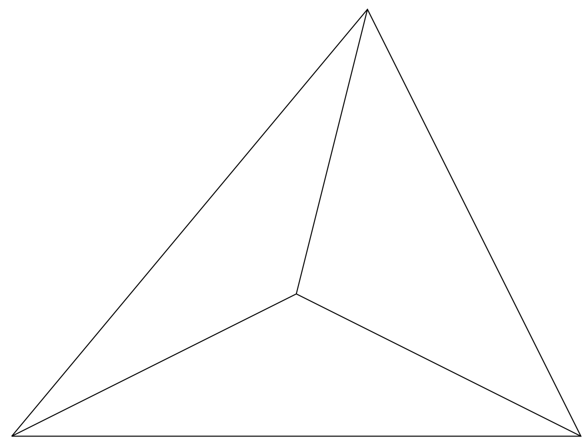
Theorem 4.3 *The length of a triangle side in the Euclidean plane is always less than the sum of the lengths of two other sides of the triangle.*

Homework Problem 4.93 *They have balloons of four colors at a children's party, the Blue, Green, Red, and Yellow. The kids are offered to take home as a party favor any three balloons of different colors. How many ways are there to arrange the balloons into different party favors?*

4.11 Triangles and circles on a cylinder

Inspired by the polygons we have seen in the Euclidean plane, let us try to define a polygon on a cylinder and generally on any 2D surface. A polygon should have a finite number of vertices. If it has infinitely many of them, then drawing such a figure would take infinite time. This is absolutely unacceptable for the very busy modern children, always in a hurry to get from school to their sports club, from the sports club to a play date, from the play date to a video game, and so on. They can only afford spending infinite amount of time on a meal they don't like, and even that they can afford not too often.

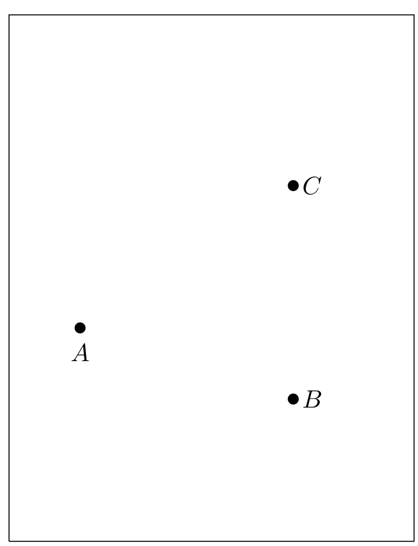
For the same time-saving reason, we want polygons to be of finite size. Each pair of the neighboring vertices should be joined by a geodesic line. To prevent the pictures similar to the one below, every vertex should have no more than two neighbors.



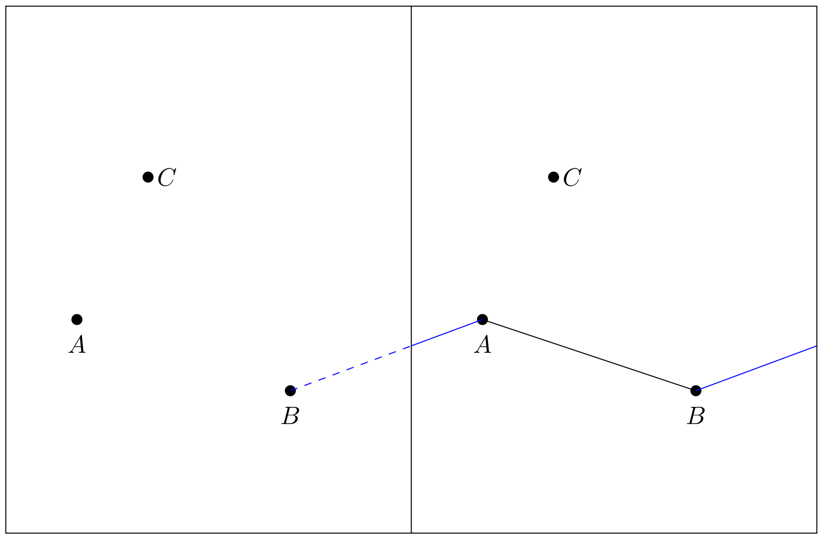
If we ever run in a complication drawing a polygon on some tricky surface, we can always modify the definition of a polygon, but for now this should do.

Definition 4.20 *A polygon on a 2D surface is a finite figure bounded by a finite number of points, called vertices, and by the geodesic lines joining the neighboring vertices, each vertex having no more than two neighbors.*

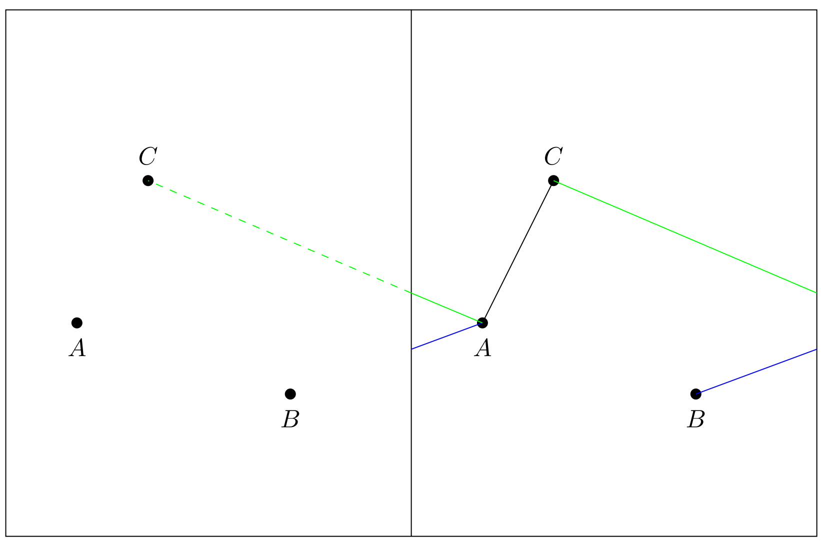
Example 4.5 *On a cylinder, draw a triangle having the below vertices.*



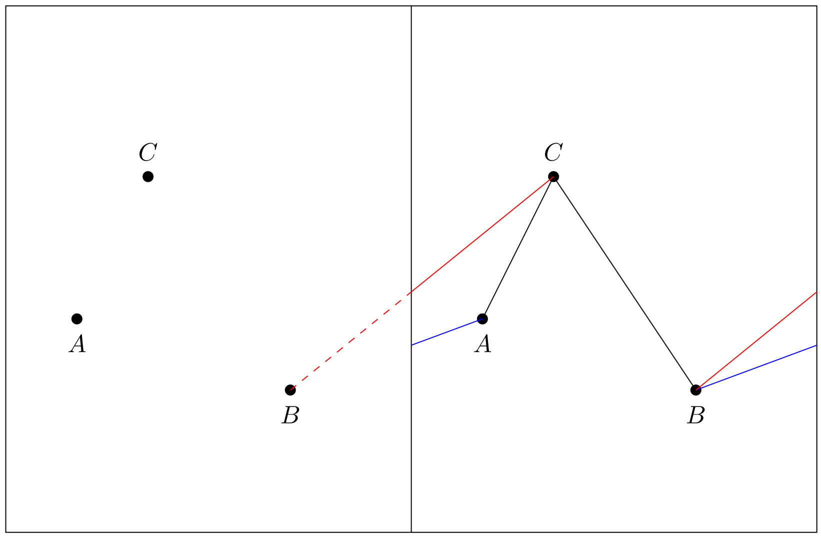
There are two candidates for the geodesic line connecting A and B, one crossing the cut, the other avoiding it.



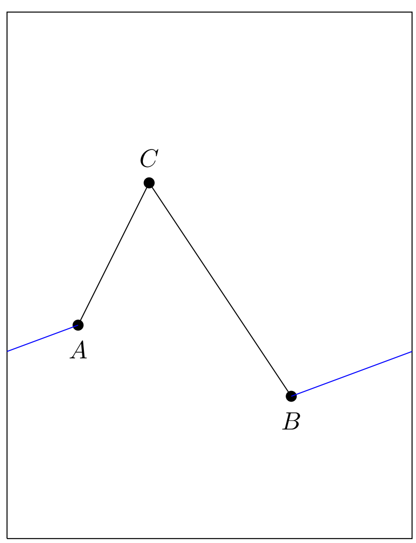
It is not visually clear which line is shorter, the blue or black one. A compass measurement shows that it's the blue line we are looking for. It is visually clear that the shortest path connecting A and C doesn't cross the cut.



It is more or less clear that the red line connecting B and C is longer than the black one.

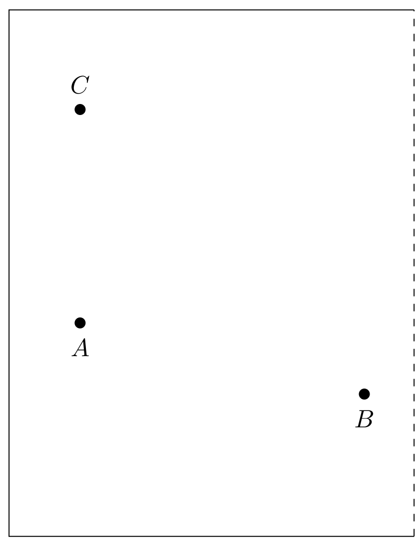


Double-checking with a compass confirms that there is no mistake. Finally, here comes the triangle.



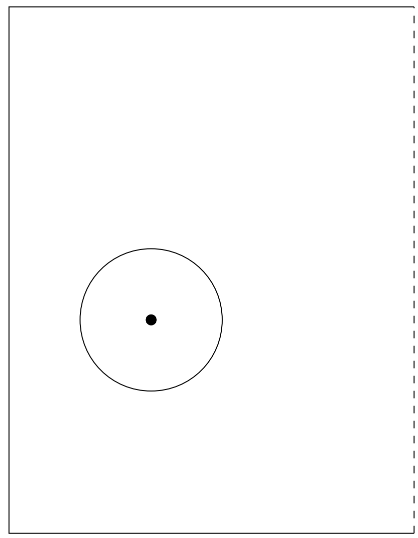
It doesn't look like a triangle on a flat paper sheet, but becomes visually recognizable when we glue the sheet into a cylinder (do it!).

Homework Problem 4.94 *The dashed line on the below picture represents the side to be glued to the opposite one. Take three points on a cylinder as shown on the picture, two of them close to the cut and at the opposite sides of it, similar to the points A and B.*

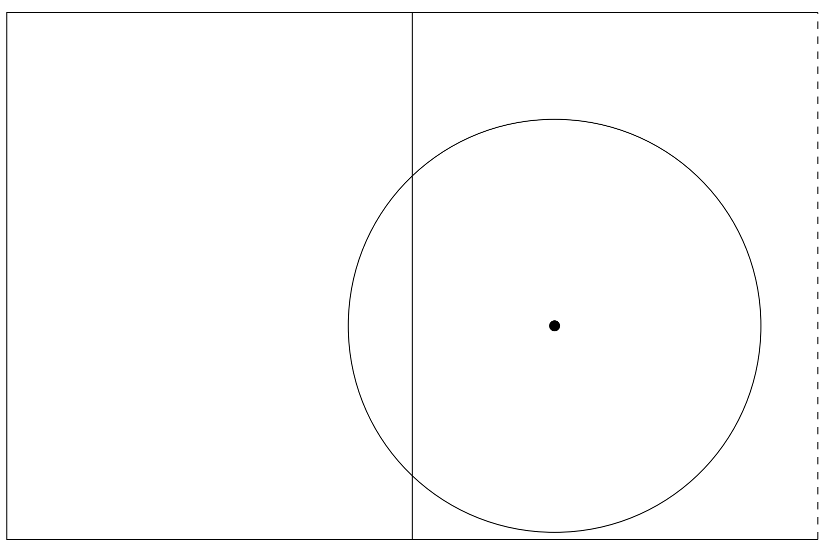


On the cylinder, draw a triangle having the points A, B and C as its vertices.

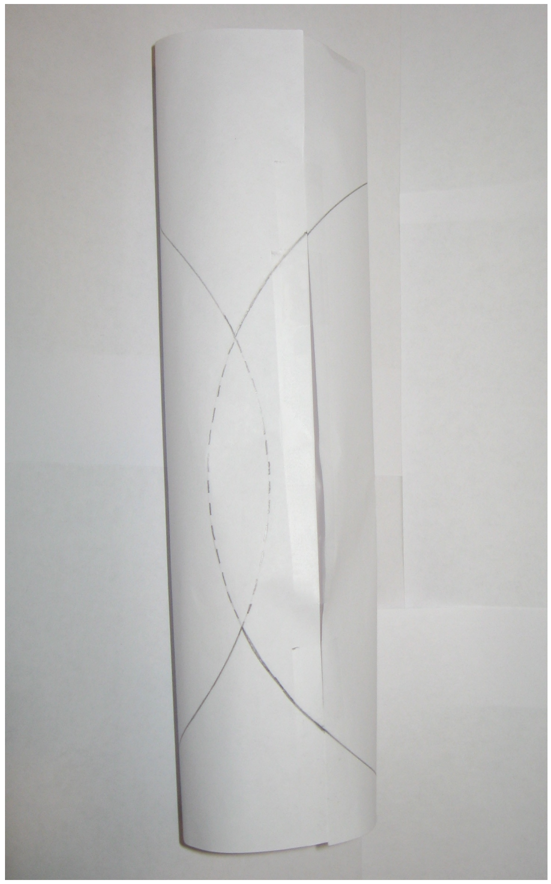
Similar to Definition 4.1, let us define a circumference on a cylinder as the set of all the points of the cylinder having an equal distance, called the radius, from a special point, called the center. Like a straight line, we can draw a circumference on a flat sheet and then glue the latter into a cylinder. As long as the radius is shorter than a half of the sheet's width, we get a familiar picture.



Suppose that the radius is longer than the half of the sheet's width. To better see what happens at both sides of the cut, let us again put together two copies of the cylinder.



Transporting the "sticking out" part to the opposite side of the cut and gluing the sheet into a cylinder gives us the following picture.



The points lying on the dashed lines must be thrown out as their distance to the center is shorter than the radius. Unlike the previous circumference on the cylinder, this one doesn't look like its Euclidean plane sibling. It looks more like number 8 with the removed middle part.

Definition 4.21 *A line is called connected, if we can travel between any two distinct points of the line by moving along the line. The line is called disconnected otherwise.*

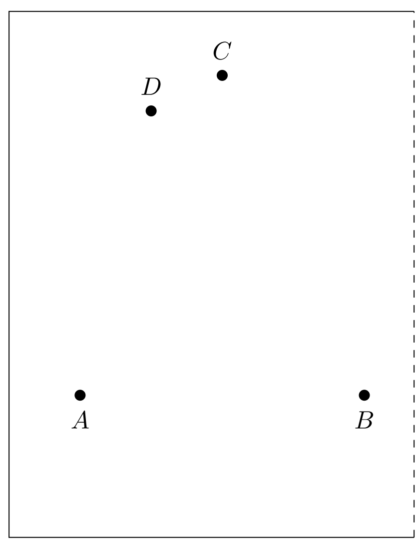
For example, a straight line and a circumference on the Euclidean plane are connected.

Let us call a circumference on a cylinder small, if it looks like the one at the top of page 83. Let us call it large, if looks like the one on page 84.

Homework Problem 4.95 *Is a small circumference on a cylinder connected? Is a large one? How many connected components has the latter?*

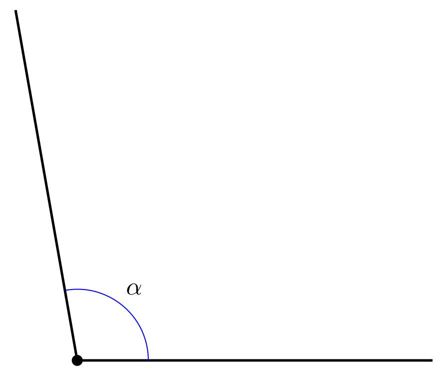
Homework Problem 4.95 shows that, unlike the Euclidean plane, large circumferences on a cylinder have two connected components, not one.

Homework Problem 4.96 *Let A, B, C, and D be the points located on a cylinder similar to the below picture.*



On the cylinder, draw the quad $ABCD$. Hint: do the sides AB and CD cross the cut?

Homework Problem 4.97 *Using a compass and a ruler, draw an angle equal to the angle α given below on a flat sheet of paper.*



Draw the angle opposite to α.

Homework Problem 4.98 *On a flat sheet of paper, draw an equilateral triangle with the side length equal to 9 cm.*

Homework Problem 4.99 *On a cylinder made of a standard letter-size paper sheet, draw an equilateral triangle with the side length equal to 3".*

4.12 Geodesics on a sphere

Our world has three dimensions, length, width, and height. Time is sometimes counted as the fourth dimension, but geometrically it is very different from the three spatial dimensions. While the latter stretch infinitely both ways, time has the beginning, the moment when our universe exploded from a point about 13.7 billion years ago in the event known as the Big Bang. The three spacial axes are straight lines, whereas time is a ray.

As we have mentioned in Lesson 4.3, heavy masses warp the space-time around them. In our everyday life however, we can safely consider our 3D spacial world as Euclidean, or simply flat. Let us compare the features of the Euclidean 3D space to those of the Euclidean 2D plane.

2D	3D
For any two distinct points, there exists one and only one straight line passing through them.	For any three distinct points not lying on a straight line, there exists one and only one Euclidean 2D plane passing through them.
Two different straight lines are called parallel, if they have no common points.	Two different 2D planes are called parallel, if they have no common points.
For a straight line and a point away from it, there exists a unique straight line passing through the point parallel to the original line.	For a 2D plane and a point away from it, there exists a unique 2D plane passing through the point parallel to the original plane.
The Euclidean plane is highly symmetric. Any straight line can be moved to the position of any other by symmetries of the plane, rotations and translations.	The Euclidean 3D space is highly symmetric. Any 2D plane can be moved to the position of any other by symmetries of the space, rotations and translations.
The shortest path between any two distinct points is a segment of the unique straight line connecting them.	The shortest path between any two distinct points is a segment of the unique straight line connecting them.

Homework Problem 4.100 *Let two sheets of paper be the models of two Euclidean planes imbedded in the 3D space. Ask you parent/teacher to hold one of the planes still. Check if you can, using rotations and translations, move the other plane to the position of the first one.*

The 2D surfaces we have seen so far, planes and cylinders, "live" naturally in the 3D Euclidean space. So does the one we shall study next, the 2D sphere.

Definition 4.22 *A sphere is the set of all the points in a Euclidean space that have an equal distance, called the radius, from a given point, called the center.*

Note that the above definition doesn't specify the number of dimensions of the Euclidean space.

Homework Problem 4.101 *Is a circumference a 1D sphere? Hint: compare Definition 4.22 to Definition 4.1.*

Geometry of a 2D sphere is very different from the geometries of a 2D Euclidean plane and a 2D cylinder. For example, a sphere is home not to a single straight line. So what would be the shortest path between two distinct points in the spherical world? To answer this question, let us first observe that any cross-section of a sphere by a plane is a circle.

The following homework problem will help us understand the previous statement better. To solve the problem, you will need either an orange, or a watermelon, or any other fruit of spherical shape. Your will need your parent/teacher's assistance in slicing the fruit with a knife having a long, wide and flat blade, a blade without grooves or serrations. Finally, you'll need a minor mental effort. The surface of a fruit is not a perfect sphere. You'll have to imagine that it is. You will also have to imagine that the surface of the blade is perfectly flat, like that of a 2D Euclidean plane. Then the 1D boundary line of the slice will be the cross-section of the 2D sphere by the 2D plane.

Homework Problem 4.102

- *Ask your parent/teacher to cut a few slices. Keep imagining that the fruit has a perfect spherical form. What shape has the boundary line of the slices?*

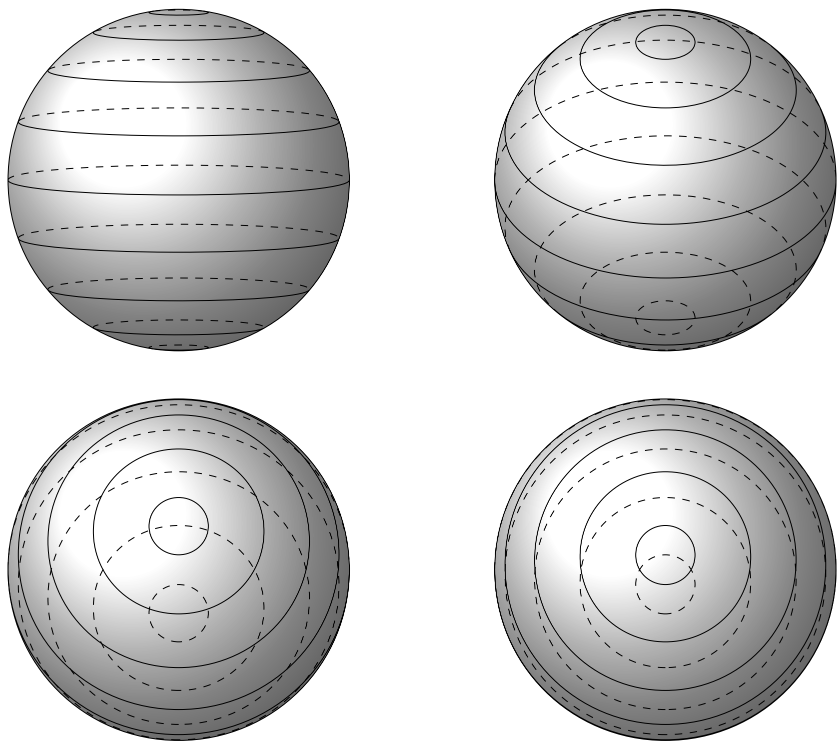
- *Choose a few yummiest slices and eat them.*

- *Get a scoop that can make round balls of ice-cream. Redo the experiment with various sorts of ice-cream instead of fruits.*

- *Which cross-section do you like more, chocolate or vanilla?*

- *Repeat the experiments until you get completely certain in the outcome.*

If you are still not convinced, please take a look at the following pictures showing parallel slices of a sphere at various angles.[21]



As we move the cutting edge through the sphere, the radius of the cross-section circumference grows. It reaches its maximum when the plane passes through the center of the sphere. At this moment, the radius of the circumference equals to that of the sphere. Then the radius begins to decrease.

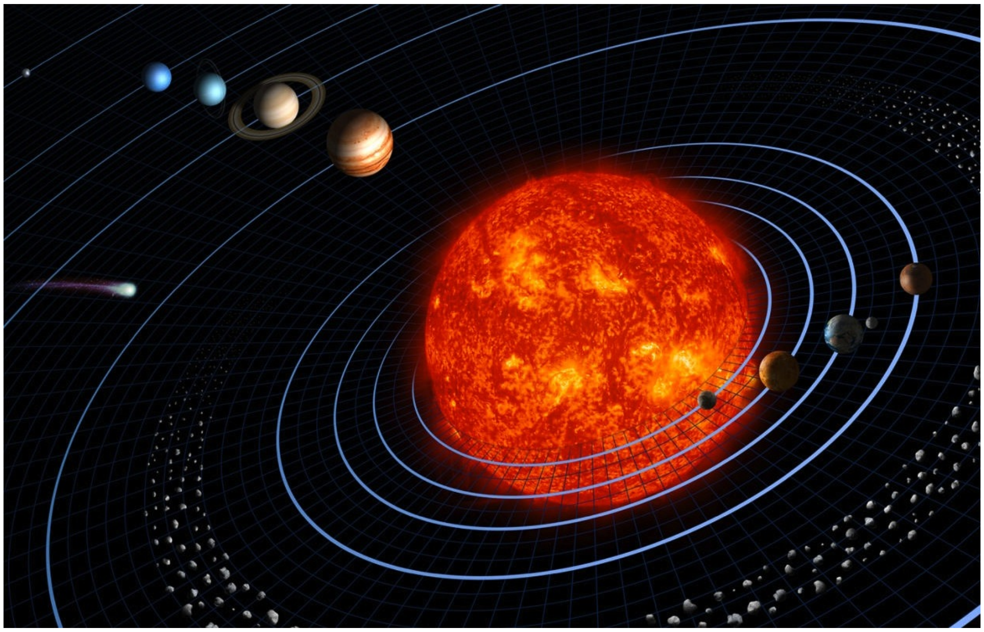
Definition 4.23 *A circular slice of a sphere bounded by a circumference having the same center as the sphere, and thus having the maximal possible radius, is called a great circle.*

Homework Problem 4.103 *Find a great circle at the above pictures of the sliced sphere.*

[21]The TikZ code generating the pictures was written by Tomasz M. Trzeciak, downloaded from `http://www.texample.net/tikz/examples/map-projections/` and slightly modified by the author.

The most important of all the spheres is the one we live on, the globe. We shall need a model of the globe for our spherical geometry studies. The following was bought at "K-mart" for $18.

All the planets of our Solar System orbit the Sun in one plane.



From the center outward: the Sun, Mercury, Venus, the Earth and its Moon, Mars, the asteroid belt, Jupiter, Saturn, Uranus, Neptune, Pluto. A comet is also seen on the left.[22]

The Earth dashes around our home star covering on average 19 miles every second (18.64, to be precise). This is many times faster than a bullet from the most powerful rifle people can make! While zipping around the Sun, the Earth also rotates around an imaginary line, called the *axis of rotation*. The duration of one full revolution of our planet around its axis is called a *day*. The Earth makes one full circle around the Sun in one *year*. There are 365 days and a quarter of a day in a year. It is more convenient to have years with a whole number of days. That is why we conventionally count 365 days in every three years out of four. Every forth year, called the *leap year*, has one extra day.

Homework Problem 4.104 *How many days are there in a leap year?*

[22] Downloaded from `http://upload.wikimedia.org/wikipedia/commons/c/c2/Solar_sys.jpg`.

The extra day the leap year has, but the rest of the years have not, is February 29. There are only 28 days in February of a common year. A person born on February 29 is sometimes called a "leapling" or a "leap year baby".

Homework Problem 4.105 *How often would a leapling have a birthday party? Can you suggest a way to improve this dismally unfair situation?*

The points where the axis intersects the globe are called the *poles*, the North Pole and the South Pole.

Homework Problem 4.106 *Find the poles on your model globe.*

The line equidistant from the poles is the most important great circle on the globe, called the *Eqautor*.

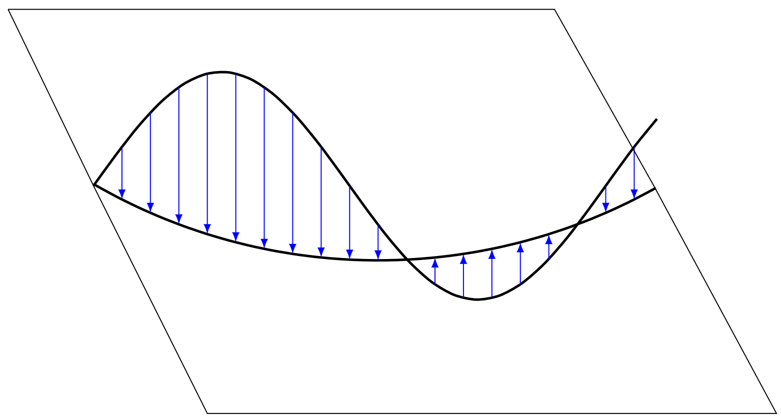
Homework Problem 4.107 *Find the Equator on the globe.*

The Equator divides the Earth into two halves, the Northern Hemisphere and the Southern Hemisphere.

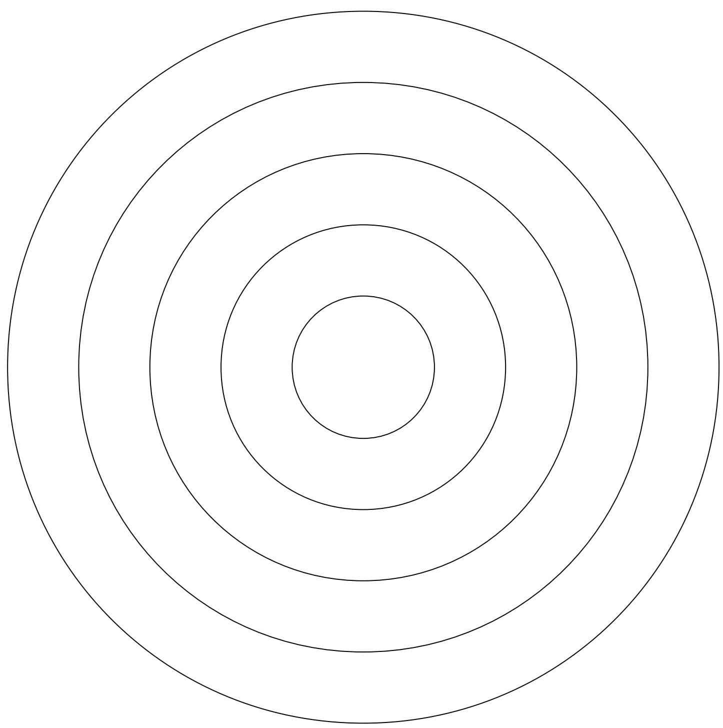
Homework Problem 4.108 *Find the Northern and Southern Hemispheres on the globe. What hemisphere do you live in?*

Let us get back to figuring out the shape of geodesic lines on a 2D sphere. The sphere we consider naturally "lives", or is imbedded, in our 3D Euclidean space. The shortest path connecting two points in this space is a segment of the unique straight line passing through them. However, unlike the 2D cylinder, there is no obvious way to place a segment of a straight line on a sphere. Let us search for the best possible approximation, the straightest, or flattest, line connecting two different points on a sphere. The closest in shape to a segment of a straight line, it must be the geodesic line we look for.

To be the flattest, the desired line must lie entirely in a 2D plane. Otherwise, it can be flattened, and shortened, by squeezing it into the plane.



But, as proven in our fruit and ice-cream experiments, a circumference is the only possible shape for an intersection of a sphere with a plane. Thus, among all the circumferences passing through the two given points on a sphere, we have to choose the flattest.



Homework Problem 4.109

- *Which of the above circumferences is the flattest?*
- *Which of the above circumferences is the curviest?*
- *Which of the above circumferences has the biggest radius?*
- *Which of the above circumferences has the smallest radius?*

Homework Problem 4.109 shows that the larger the radius, the flatter a circumference is. Thus, to get the straightest possible path from one point of a sphere to another, we have to connect the points by an arc of a great circle. We just have proven[23] the following

Theorem 4.4 *Geodesic lines on a 2D sphere are segments of great circles.*

Definition 4.24 *Two points on a sphere are called antipodes, if they belong to a straight line passing through the sphere's center.*

For example, the North and South Poles on the globe are antipodes.

Suppose that the two points on a sphere we consider are not antipodal and do not coincide. The comparison table on page 87 states that any three different points in the Euclidean 3D space uniquely define a 2D plane passing through them. Let us add the center of the sphere to the two points we are trying to connect by a geodesic line. The triple uniquely defines a 2D plane. The plane passes through the center of the sphere. The cross-section is a great circle passing through the given pair of points on the sphere. The points divide the circle into two arches. Choosing the shorter one gives us the geodesic line we were looking for. Moreover, the line is unique!

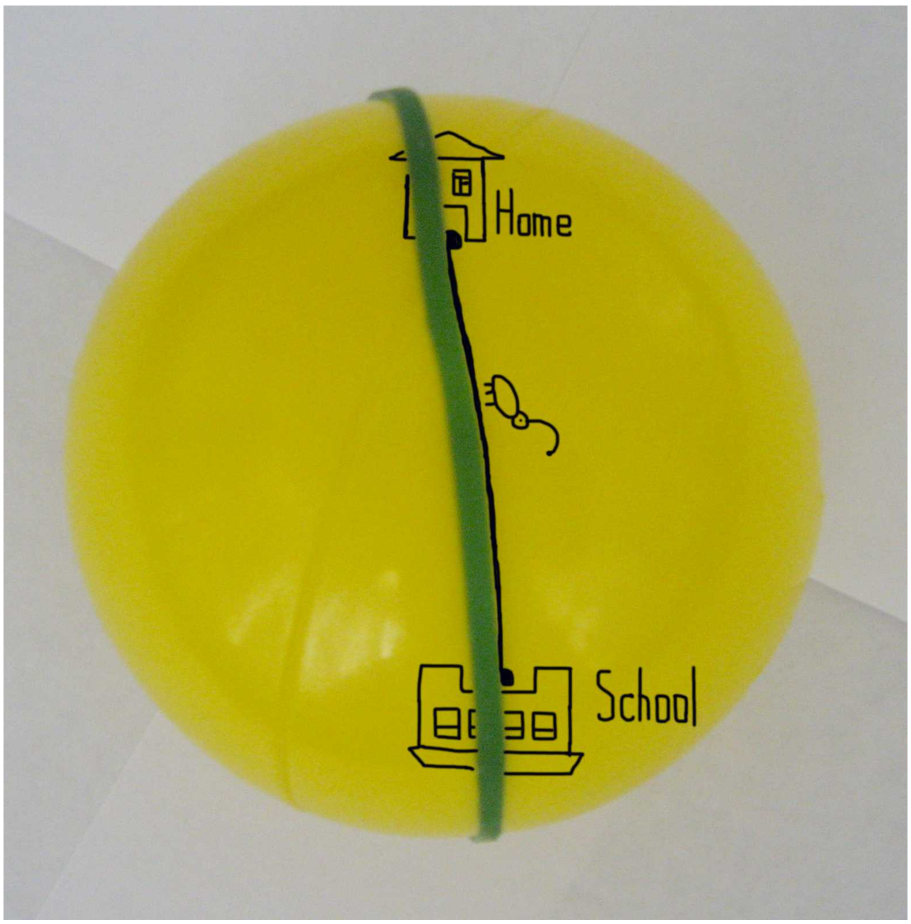
A practical way to draw geodesic lines on the surface of a plastic ball representing a 2D sphere was suggested to the author by Igor Pak.

An asparagus bunch typically comes from a supermarket held together by a flat rubber band. The band "feels" the curvature of an object we try to

[23]The above argument can be turned into a proof rigorous by the modern standards.

pull it on. It will hold on to a sphere well only if set along a great circle.

Similar to Homework Problem 4.82, let us imagine once again the tiny bugs, this time living on a huge sphere. Using our understanding of spherical geometry and the rubber band technique, we can help the bugs' children find the shortest path from Home to School!



For any point next to a bug, the antipodal point is literally at the opposite end of the world. From their everyday experience the bugs would eventually notice that for any two points they take, there exists one and only one shortest path connecting them. They may call this path a “straight line”. Intelligent creatures, like us, the bugs could one day define the 2D plane as in Definition 4.4. Just like us humans, they would only be able to realize that their world is not flat when they venture far away from home.

Homework Problem 4.110 *Pick two different points that are not antipodes on the surface of a plastic ball. Using the rubber band technique, draw the geodesic line connecting them.*

If we start with a pair of antipodes, then the center of the sphere belongs to the straight line passing through them. This triple doesn’t define the cutting plane uniquely. Any one containing the line will do. For such a pair, the shortest path connecting the points is not unique. We can take either half of any great circle connecting them.

Let us take another look at the model globe. If you don’t have one yet, please look at the picture on page 90. As you can see, the surface of the globe is covered with two sorts of lines. Full circles parallel to the Equator are called the *latitude lines*, or the *parallels*. Halves of great circles connecting the poles are called the *longitude lines*, or the *meridians*. Since each meridian is a half of a great circle, they all have the same length. Each of them is a geodesic line connecting the poles. We shall call the special meridians and parallels marked on the globe the *coordinate lines* on a sphere, or the *grid*.

Homework Problem 4.111 *On the globe, find the grid meridian and parallel closest to your home.*

Arcs of great circles are the geodesic lines on a 2D sphere, similar to straight line segments of the Euclidean plane. Any point on a great circle can be moved to the position of any other point on the circle by turning the sphere in a way that keeps the circle in place as a whole. Any great circle can be moved to the position of any other great circle by an appropriate rotation of the sphere. According to Euclid’s Axioms 3 and 5, the sphere is a plane and great circles are the straight lines in this plane. Even Definition 4.4 is satisfied if we do not consider pairs of points positioned at the opposite ends of the world, the antipodal points. Locally, a 2D sphere is a plane. Globally however, it isn’t due to the fact that there are infinitely many “straight lines” passing through every pair of antipodes. Spherical geometry is in no way Euclidean. Any two different great circles intersect precisely at two points. "Straight lines" cannot be parallel in the spherical world.

Understanding spherical geometry is very important for navigation, the process of steering a ship or an airplane from one place on the globe to another or finding your way through the backcountry, away from roads and people.

Homework Problem 4.112 *Using a rubber band to model a great circle on the globe, find the shortest flight path between Moscow, Russia and New York City, the US.*

As we start seeing, borders between various sciences, geometry and geography in this particular case, are rather artificial. We need to learn coordinates on the globe to better understand geometry of a sphere. The latter is indispensable in helping us ... to navigate the globe! And this is tricky indeed, because the globe is not flat. We have to be very careful, even with the notions as elementary as "left" and "right". When you start going right on the Euclidean 2D plane, you just keep going. You never come back. Let us see what happens on a 2D sphere.

Homework Problem 4.113 *On the globe, find the grid latitude line passing closest to your home. See if you can get back home by always going right along the parallel.*

Homework Problem 4.113 shows that by going to the right along a latitude line we come back to our starting point ... from the left!

For the globe having the North Pole at the top and the South Pole at the bottom as shown on page 90, going right along a latitude line is called going East. Going along a parallel in the opposite direction is called going West. Moving up/down along the meridian passing through the point of your location is called moving North/South.

Homework Problem 4.114 *Which way, left to right or right to left, does the Earth spin around its axis? Hint: the Sun rises in the East.*

Homework Problem 4.115 *You are in New York City and want to go West. Show the direction on the globe. Now you are on the North Pole and want to go West. Which way do you go?*

Homework Problem 4.116 *You are in New York City and want to go South. Show the direction on the globe. Now you are on the North Pole and want to go South. Which way do you go?*

Homework Problem 4.117 *Using a rubber band to model a great circle on the globe, find the shortest flight path between Tokyo, Japan and Los Angeles, the US.*

Homework Problem 4.118 *Mercury is smaller than Mars. Mars is smaller than Earth. Which planet is larger, Mercury or Earth? Solve the problem without looking at the picture on page 91.*

Homework Problem 4.119 *Using the picture on page 91, learn the names of the Solar System planets.*

Homework Problem 4.120 *Pinpoint your home on the model globe. What is the "up" direction at the point? At the North Pole? At the South Pole? At some point on the Equator?*

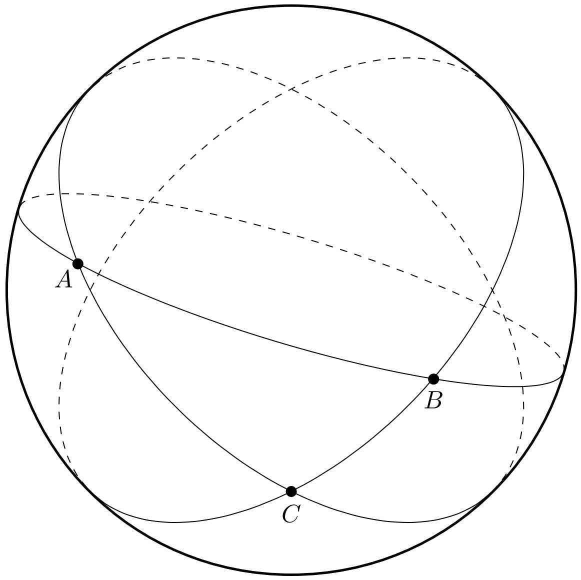
Homework Problem 4.121 *Find Australia on the globe. Do people in Australia walk upside down?*

Homework Problem 4.122 *Watch a documentary about manned space flight. What is the "up" direction in space?*

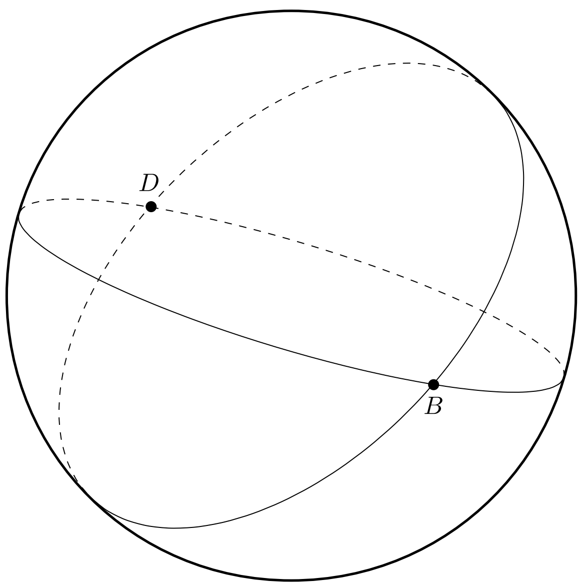
4.13 Spherical polygons

Recalling Definition 4.20, let us meet some simpler polygons on a 2D sphere.

Let us start out with a triangle. Let us mark three generic points, A, B and C, on the surface of a ball and use rubber bands to draw a great circle through each pair. Here comes a spherical triangle ABC.



Let us see what happens, if we remove one rubber band.



The two remaining great circles intersect at the antipodal points B and D. The four halves of great circles connecting the points are all of equal length, so they all are geodesics. The meridians split the sphere into four spherical polygons, called biangles, each having only two vertices and angles, an impossibility on the Euclidean plane!

Homework Problem 4.123 *Find a similar picture formed by the grid lines of you model globe. What are the points B and D?*

Homework Problem 4.124

1. *Define a straight angle on a sphere.*

2. *Is there a way to split a sphere into two biangles?*

Homework Problem 4.125 *Find the midpoint of a straight line segment in the Euclidean plane. Hint: use Homework Problem 4.73.*

4.14 Straight lines in physics

In the previous lessons, we have looked at the straight line from the point of view of geometry. This time we shall learn its physical nature. For that, we must understand the notion of speed first.

We say that a car travels at the speed of 60 miles per hour, or 60 mph, if it would cover 60 miles in one hour going as fast as it does at the moment, without accelerating or braking down.

Homework Problem 4.126 *A car travels at the speed of 60 mph. What distance would it cover in 3 hours? Hint: 60 is six tens. You can use a Young diagram to find the value of* $6+6+6$.

Project 4.1 *Ask the child to walk from you to a standing out object, like a TV set, in five seconds. Then ask her/him to take the same amount of time walking from you to a more remote object, say, a window. Next, ask the child to run to an even further object, say, her/his bed in a different room, in the same five seconds time interval. Ask the child where she/he had to walk at a greater speed and why.*

Recall that the Earth revolves around the Sun covering, on average, 19 miles every second. In one hour, it makes about 68,400 miles. Your family car, traveling at the common highway speed of 60 mph, needs 47 and a half days to cover this distance.

The Earth's Equator stretches for 24,860 miles. Our planet makes one full revolution around its axis in 24 hours. This way, a point on the Equator moves at the speed of 1,036 mph. For comparison, the fastest serially produced aircraft, the famous SR-71 Blackbird, has set a speed record flying at 2,193 mph.

SR-71 Blackbird[24]

Homework Problem 4.127 *Find a grid parallel on the model globe not too far from a pole. What latitude line is longer, this one or the Equator?*

Homework Problem 4.128 *As the Earth rotates around its axis, what points move faster, the ones closer to the pole or these closer to the Equator? Hint: compare this problem to Project 4.1.*

[24]Downloaded from **here**.

Homework Problem 4.129 *Why do they build space flight launching pads as close to the Equator as possible?*

If you don't have *Google Earth*, a geographic information program created by Keyhole, Inc., a company acquired by Google in 2004, please install it on your computer now. It is freely available to non-commercial users and can be downloaded from here.

Homework Problem 4.130 *Find the US Kennedy Space Center on the globe. For that, type "Kennedy Space Center" in the search box located in the upper-left corner of the Google Earth program, and hit "Enter". Zoom in and out until you can pinpoint the center's location on your model globe. Is Kennedy Space Center close to the Equator?*

A space shuttle blasts off from Kennedy Space Center.

The world's first and largest operational space launch facility, Baikonur Cosmodrome is located in the country of Kazakhstan and leased by Russia.[25] The first Earth-orbiting artificial satellite, Sputnik 1, was launched from Baikonur in 1957. The first manned flight in the history of this planet was also launched from Baikonur in 1961.

Yuri Gagarin, the first man in space.

Homework Problem 4.131 *Use Google Earth to find Baikonur Cosmodrome on the globe. What is closer to the Equator, Baikonur or Kennedy Space Center? Suppose that you have a choice to launch a rocket into space from either place. Which one would you choose? Why?*

Of all the known physical objects, light is the fastest. In the vacuum of space, a particle of light, called a *photon*, covers over 186,282 miles every second. It would take your family car eight and a half years to travel this distance at the speed of 60 mph.

[25]Currently until 2050. Russia and Kazakhstan were once parts of one country, the Soviet Union. The Soviet Union broke apart in 1991.

Homework Problem 4.132 *Traveling at a constant speed, a car covered 210 miles in three hours. What was the speed of the vehicle? Hint: use an appropriate Young diagram to see if you can break 21 into three equal parts.*

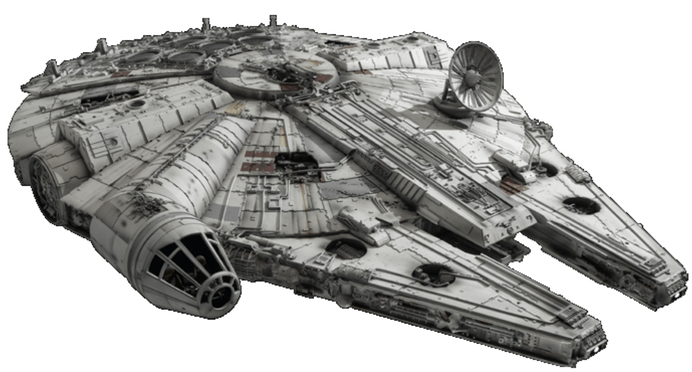
Out of all the great minds humanity has produced over the known history, there are two clearly standing out, Sir Isaac Newton and Albert Einstein. Modern math begins with Newton and so does modern physics. It were Newton's three laws of motion, together with his universal gravitation law, that turned a loose collection of more or less understood natural phenomena into an all-encompassing science based on a small number of clear general principles.

Sir Isaac Newton, a 1702 portrait by Godfrey Kneller.[26]

[26]Downloaded from **here**.

Newton's laws of motion govern mechanics of a flat, or Euclidean, universe. It was the main achievement of Einstein to notice that the universe is not flat and that Newton's laws work only for relatively light objects moving at the speeds low compared to the speed of light.

The first of Newton's laws claims, "In the absence of force, a body is either at rest or moves in a straight line with constant speed." Suppose that Captain Solo's[27] ship, *Millennium Falcon*, chased by Imperial fighters, takes a desperate jump through space, turning up at the fringe of the universe. There are no stars, black holes, planets, asteroids, and spaceships in sight. The *Falcon* is surrounded by nothing but dark void. The ship itself is a massive (versus zero-mass) body, so it bends the space-time in its own vicinity, but, since its mass is small compared to that of a star, we can disregard the General Relativity effects and think that the *Falcon* moves in the Euclidean 3D space.



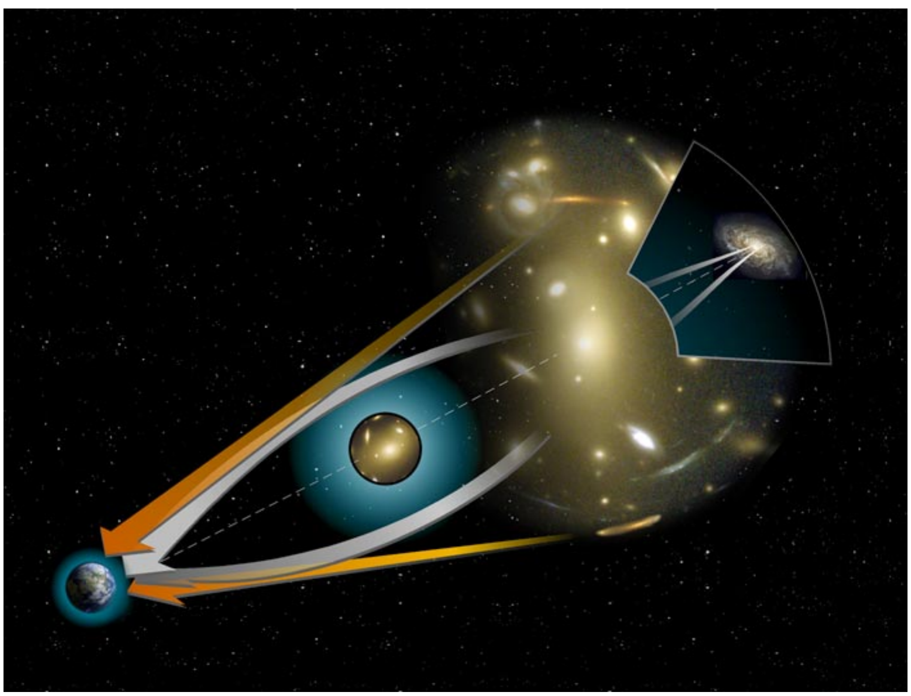
Millenium Falcon

Tired from the battle, Captain Solo shuts down the engines, turns on his favorite music, grabs a can of the *Galactic Stout* ale from the ship's fridge, and relaxes. The universe around the ship is nearly Euclidean, so, according to Newton's first law, as Captain Solo rests, his ship drifts along a straight line at a constant speed (to be precise, the ship's center of gravity does).

[27]One of the main characters of the popular *Star Wars* movies.

Another physical realization of the straight line is provided by light. Beams of light travel along the geodesic lines of our universe. Away from large masses, the universe is almost Euclidean, so light travels along straight lines. Near large masses however, it gets bent, giving rise to the gravitational lensing effect.

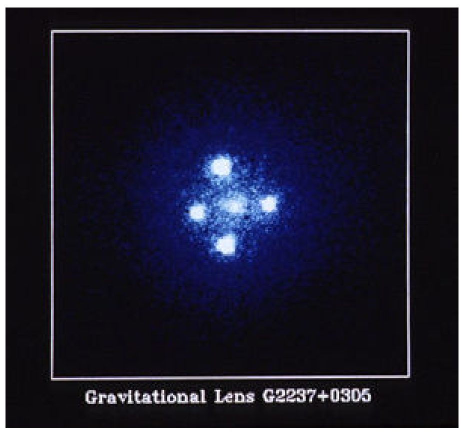
At the picture below, beams of light are emitted by a distant source, say, a very bright star in the center of a remote galaxy. On their way to our eyes, they pass next to a very massive object, say, a galaxy cluster million times the size of our Milky Way, represented by the spherical collection of galaxies surrounded by the blue cloud of interstellar hydrogen on the picture below. The cluster bends the beams in such a way that we see two light sources instead of one. Moreover, we perceive the sources to be far away from each other, emitting the light along the straight orange arrows, whereas in fact the light has traveled along the bent white ones.



A gravitational lens.[28]

[28]Downloaded from **here**.

The above is an artist's rendering of the effect. The below is an actual NASA photograph of the formation known as Einstein's Cross, named after the inventor of the relativity theory, the man sharing with Sir Isaac Newton the distinction of being the greatest scientist of all times, Albert Einstein. On the photograph, four images of the same distant quasar (a quasi-stellar radio source, a very energetic distant galaxy with an active galactic nucleus) appear around a foreground galaxy due to strong gravitational lensing.



Einstein's Cross.[29]

Finally, it's time to put together everything we know about straight lines. A straight line is

1. a flat line, or a 1D Euclidean space;

2. a place where the (real) numbers live, or the number line;

3. a 1D object so symmetric that any point of it can be moved to the position of any other point by a move preserving the line as a whole. Numerically, this is expressed by the fact that for any two numbers x and z, there always exists a number y such that $x + y = z$;

[29]Downloaded from `http://en.wikipedia.org/wiki/File:Einstein_cross.jpg`.

4. a geodesic line, or the shortest path connecting any two distinct points in the Euclidean space of dimension 2 and higher;

5. the path of a moving body not affected by any force acting on it in the Euclidean world;

6. the path of a ray of light in the Euclidean universe.

Not bad for an object that seemed so intuitively clear! To complete this chapter, we have to take a closer look at two more notions we have already touched upon, infinity and dimensions higher than three. We shall do so in the subsequent lessons. In the meantime, here are a few more problems to solve. To better understand the following definition and homework problem, please recall Definition 4.9.

Definition 4.25 *An angle that is a half of the straight angle is called the right angle.*

Homework Problem 4.133 *Using a compass and a ruler, construct a right angle. Hint: Take another look at the two auxiliary triangles from Homework Problem 4.66. Have they equal angles adding up to a straight angle? You can use your solution of Homework Problem 4.125 to find the middle point (or you can use mine, provided on page 176).*

Homework Problem 4.134 *You take 10 steps due South, then 10 more due East, then 10 more due North and end up at your starting point. Where is the starting point on the globe?*

Homework Problem 4.135 *What is the simplest way to determine the "down" direction anywhere on Earth?*

To solve the following two problems, consider what happens if Captain Solo drops his beer mug.

Homework Problem 4.136 *As Captain Solo drifts at a constant speed along a straight line in his spaceship, what is the "up" direction on board of the vessel? Is there any?*

Homework Problem 4.137 *The rest is over, it's time to save the world! Captain Solo turns on the engine. What is the "up" direction on board of Millennium Falcon now?*

4.15 The first look at infinity

In a universe with infinitely many inhabited worlds, you are a manager of a space hotel with infinitely many suites. The suites, numbered 1, 2, 3, and so on to infinity, are all full. There flies in a new customer and asks for a suite. You find one for him, the task impossible for a full hotel with finitely many units. That's how you do it: you ask the guest residing in suite 1 to move to suite 2, the guest residing in suite 2 – to suite 3, the guest residing in suite 3 – to suite 4, and so on. This way you free suite 1 for the new customer while still providing accommodations to all others.

Homework Problem 4.138 *There fly in five more customers. Can you accommodate them as well? How?*

Our son's initial solution was, "Question the visitors if they like staying here. Ask the five guys that like it the least to leave."

The next day, members of the inter-galactic Mathematical Society come in for a three-day conference. There are infinitely many of them and they all want to stay in your famous hotel. That's how you make room for them: you ask the person from suite 1 to move to suite 2, the person from suite 2 – to suite 4, the person from suite 3 – to suite 6, the person from suite 4 – to suite 8, and so on. Whatever number you look at, you double it. Doubling means adding the number to itself (you can use the corresponding two-row Young diagram if you need to). Then ask the guest to move to the corresponding unit. This way, all the old customers end up in the units with even numbers (remember, we have defined an even number as the number of boxes in the corresponding two-row Young diagram with equal amount of boxes in each row), while all the units with odd (remember, a number is called odd, if it's not even) numbers are vacated. There are infinitely many of the latter, so you can accommodate all the new guests as well. As you can see, infinity is tricky!

There exist a special symbol for infinity: ∞.

Homework Problem 4.139 *What is a half of ∞? Hint: are there infinitely many odd numbers? How about even? Do the odd and even numbers interlace on the number line? (Take another look at the picture on page 34.)*

Homework Problem 4.140 *Does there exist the largest number? If you think that it does, would it be possible to add one to it? Note: ∞ is greater than any number, however, it's not a number!*

To solve the following problem, please recall Definition 4.25 of the right angle and the way to construct it developed in Homework Problem 4.133. You will also need the following definiton.

Definition 4.26 *A triangle is called a right triangle, if one of its angles is right.*

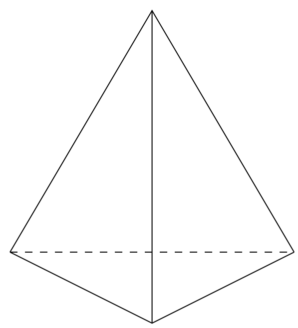
Homework Problem 4.141 *Using a compass and a ruler, construct an isosceles right triangle with the 2" long sides of equal length.*

4.16 4D and more

The most typical question the author has heard about dimensions higher than three was, "Can you show one to me?" The short answer is, "Yes and no." The longer one goes, "No, I cannot point my finger in the direction of the fourth spatial dimension. Yes, I can show you mathematical models of some processes very important for our well-being, from weather forecasts to portfolio optimization (maximizing profit while simultaneously minimizing risk of your investment), that naturally "live" in spaces with dimensions far more than three." Even more, mathematical models of quantum particles, in particular the ones that make up atoms, are straight lines in a Euclidean space of infinitely many dimensions. The next question usually is, "If we can't see higher dimensions, how can we study them?" To get a partial answer to the latter, let us take a journey through polygonal and polyhedral shapes, from a 2D triangle to a 4D cube.

As we already know, the most basic polygon in the Euclidean plane is a triangle. The triangle is fundamental, because any two different points in a 2D plane belong to a straight line. We take the third point away from the line – and get the simplest possible polygon, the triangle.

The simplest 3D polyhedron, the three-dimensional analogue of the triangle, is constructed similarly. Every three points in 3D, such that they all do not lie on a straight line, define a 2D plane. In the plane, they define a triangle. Let us take the fourth point away from the plane and connect the vertices. We get a 3D *simplex*, also known as either the *tetrahedron* or the *triangular pyramid*. It has four vertices, each joined by an edge to all others.

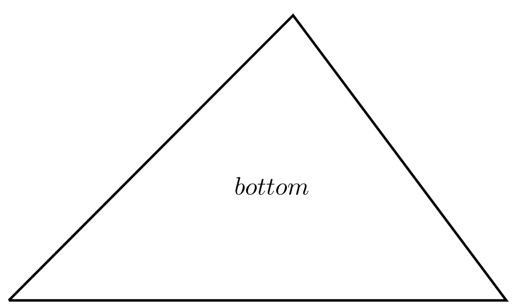


Homework Problem 4.142 *How many 2D faces has a tetrahedron?*

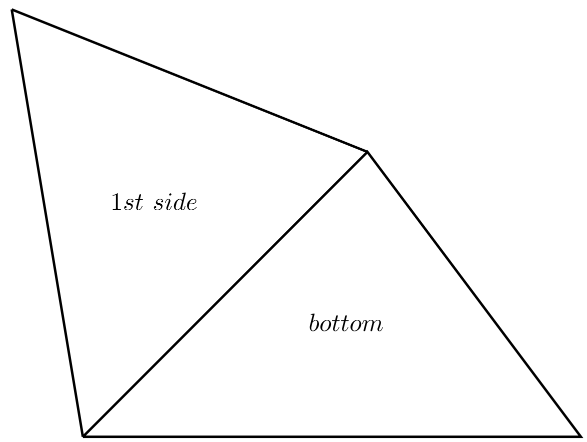
Homework Problem 4.143 *Draw a triangular pyramid facing you a. with a vertex; b. with an edge; c. with a side. What vertices, edges and faces of the tetrahedron are visible in each case? Use solid lines to draw the visible edges and dashed lines for the invisible ones. Hint: looking at an actual triangular pyramid will help a lot.*

Definition 4.27 *A tetrahedron having all edges of equal length is called regular. It is called irregular otherwise.*

Example 4.6 *Let us see how to make a tetrahedron out of paper. Let us first draw one of its faces, say, the bottom. It can be any triangle.*



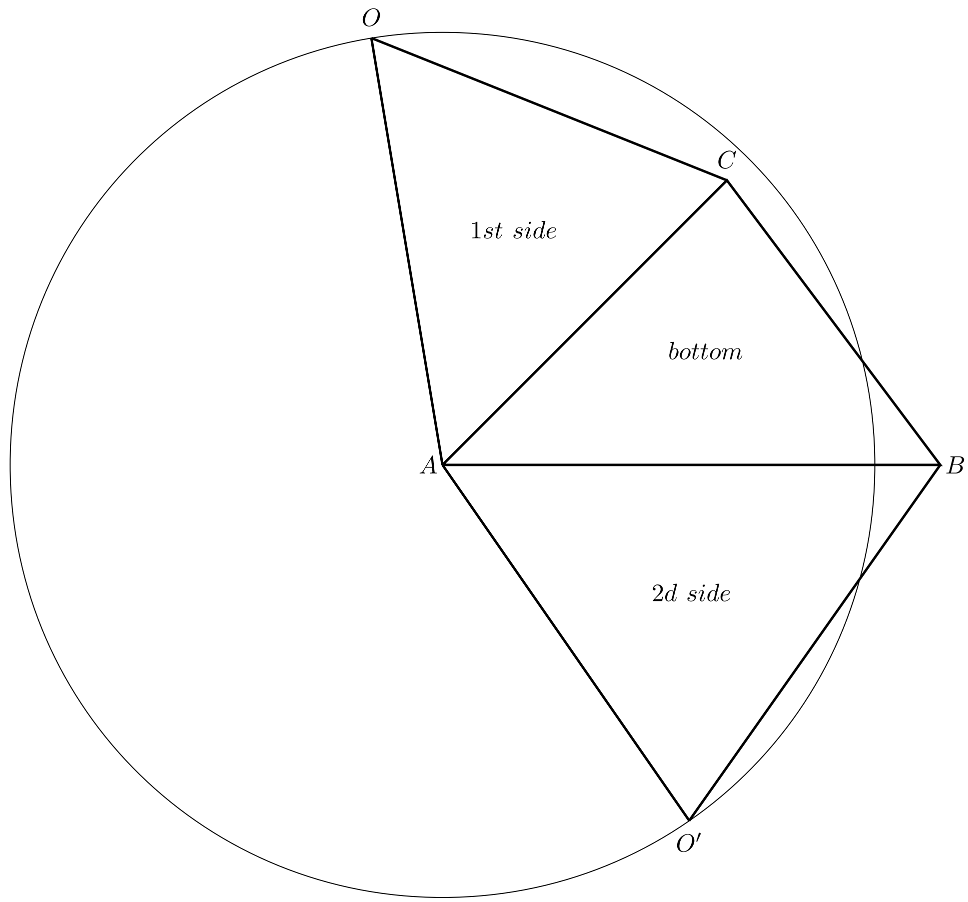
We can also take any triangle for its first side.



The situation becomes trickier when we attempt to draw the second side. Since we would eventually attach the first side to the second one by means of gluing together the segments AO and AO', they must be of the same length.

$$|AO| = |AO'|$$

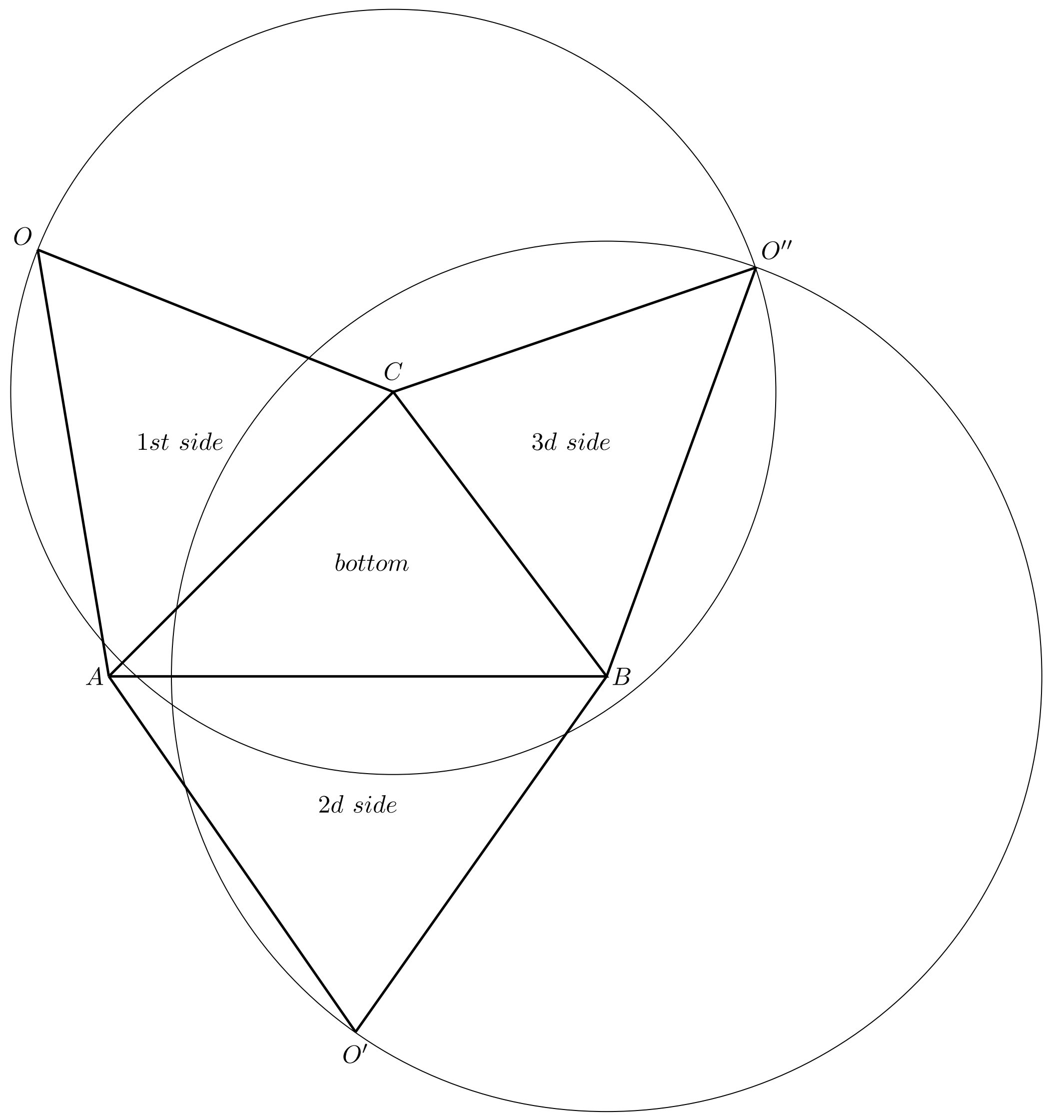
The point O' cannot be taken arbitrarily, it must lie on the circumference of radius $|AO|$ centered at A (why?).



Similarly, we must have

$$|BO'| = |BO''| \;\; \textit{and} \;\; |CO| = |CO''|$$

for the third side. The first condition means that O'' lies on a circumference of radius $|BO'|$ centered at B. The second means that the point belongs to the circumference of radius $|CO|$ centered at C.

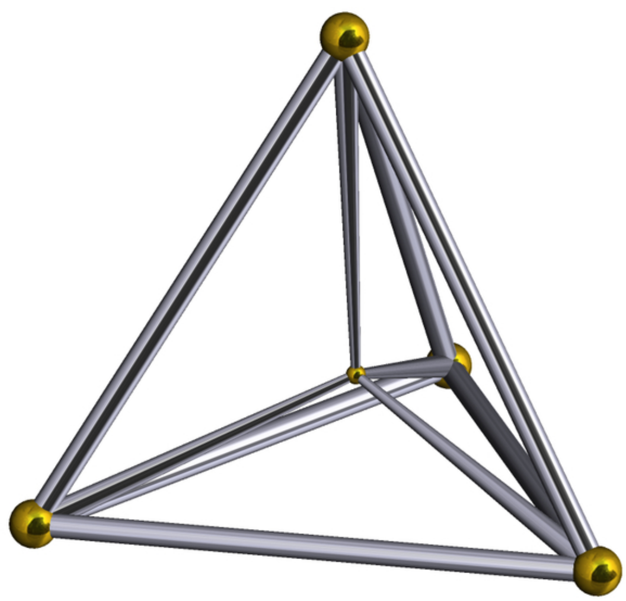


Now we can cut out the figure $OAO'BO''C$, glue the appropriate sides with sticky tape, and here comes the tetrahedron!

Homework Problem 4.144 *Make an irregular tetrahedron out of paper.*

Homework Problem 4.145 *Make a regular tetrahedron out of paper. Hint: since all the edges of the tetrahedron are of equal length, each face is an equilateral triangle.*

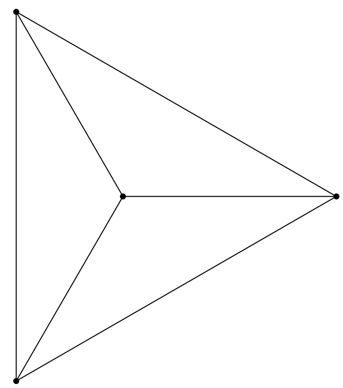
Just like every three points not lying on a straight line uniquely define a 2D plane in 3D, four points not lying in a 2D plane uniquely define a 3D plane in 4D. A plane having one dimension less than the Euclidean world it lives in is often called a *hyperplane*. A 3D plane, for example, is a hyperplane in the 4D Euclidean space, but not in 5D and higher. The above four points not only define the hyperplane, they also give rise to a 3D simplex in it. Now take the fifth point away from the hyperplane, connect the vertices of the simplex to it and you get the simplest possible 4D polyhedron, the 4D simplex, or, in other words, a 4-dimensional pyramid with a tetrahedral base also called the *pentachoron*.



A 3D wire model of the pentachoron.[30]

To better understand the above wire model, let us take another look at a picture from Homework Problem 4.143, the one of a tetrahedron facing us with a vertex.

[30]This is a Wikipedia image downloaded from **here**.



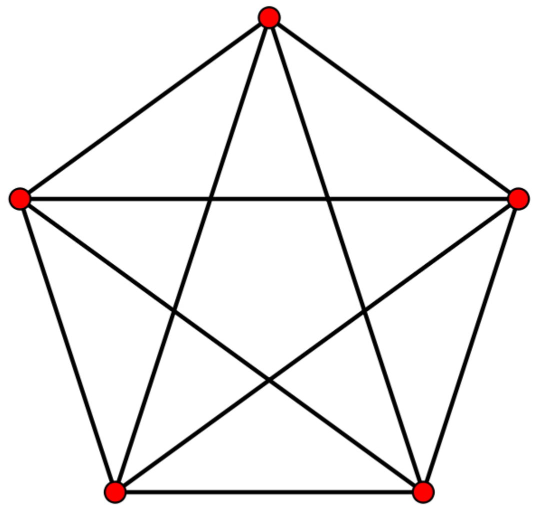
We see a 2D object, a triangle, a point inside the triangle connected to its three vertices, and the three smaller triangles the inner vertex and edges split the outside triangle into. In the 3D reality however, the “inner” vertex is not inside the “larger” triangle, but sticks outside of its plane in the third dimension. Some of the three “inner” triangles, faces of the original tetrahedron, can actually be larger than the “outer” one, the base.

Similarly, the above wire diagram is a 3D model, or projection, of a 4D object. In 4D, the “inner” vertex sticks not inside the “larger” tetrahedron, but outside of it in the fourth dimension. Some of the four “inner” tetrahedrons can actually be larger than the “outer” one. Together, they form the five 3D faces of the pentachoron, hence the name.[31]

Homework Problem 4.146 *Find all the five 3D faces of the pentachoron on the above wire diagram. How many edges has a 4D simplex? How many 2D faces?*

Although wire models of the 4D polyhedra can be built in 3D, for many practical purposes it suffices to look at a 2D projection, i.e. a picture. There exists a picture of the pentachoron different from the above that is very important for ... magic! Called the pentagram, it looks like this:

[31] In Ancient Greek, “pente” meant “five”, giving rise to the modern words such as the pentachoron, pentagon, a polygon with five angles, and so on.



A pentagram.[32]

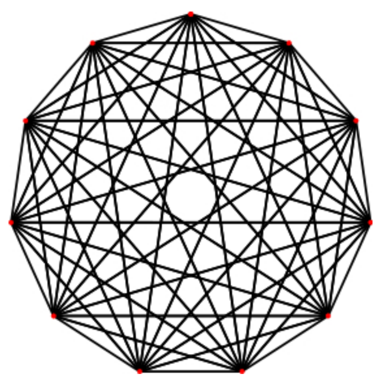
Indeed, the pentachoron has five vertices, all joined to one another by an edge, just like the tetrahedron having four vertices all connected by edges. The above projection is chosen in such a way that the vertices of the pentachoron form a pentagon, the 2D polygon with five vertices mentioned in footnote 31. Then the vertices are connected the way they are in 4D, and here comes the pentagram.

People used to believe that if you draw a pentagram on the floor, stand in its center and declaim a proper spell, a magic spirit would descend upon you. From watching the numerous movies where they have tried the trick, the author of this book came to the conclusion that it's better not to do it. According to the Hollywood experts, the spooks always come to harm you one way or another. Here is my guess: the 4D space is inhabited by some shortsighted 4D creatures. Just like in our 3D, many hotels in their world have large revolving doors, but since their world is rather different from ours, their revolving doors are shaped as pentachorons. In their language, the spell probably sounds like, "Welcome to our hotel, Sir (Madam)!" They hear it, look at the pentagram, mistake it for a hotel door and try to enter. As they find out that they were wickedly duped by some 3D tricksters, a joke that

[32]Downloaded from `http://en.wikipedia.org/wiki/File:Complete_graph_K5.svg`.

flat makes them very seriously mad!

Taking another look at the pentagram, we can understand why they need spaces of many dimensions in math and physics. On the one hand, the pentagram looks somewhat complicated. On the other hand, it's a 2D projection of the simplest possible 4D polyhedron, the pentachoron. Take one vertex less, and there would be no 4D solid at all, only a 3D pyramid, the tetrahedron. A simple object up in 4D gives us something complicated down here, on a 2D sheet of paper. Likewise, many objects that seem very complicated in 2D and 3D can be unraveled into something rather elementary, but residing in a space of many more dimensions. If you are not yet convinced, please take a look at this:

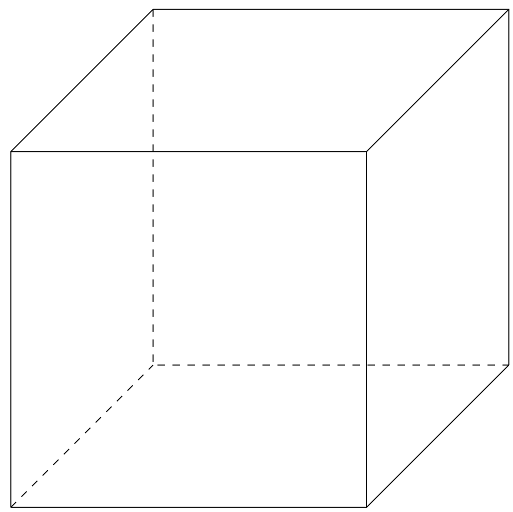


A 2D projection of a 10D simplex.[33]

The above is a picture of the most elementary 10D polyhedron, a 10D simplex. It probably is elementary up in 10D, but down here on a 2D sheet of paper, it looks anything but elementary, doesn't it?

The 3D analogue of a square is a *cube*, a polyhedron such that all of its six 2D faces are squares.

[33]Downloaded from `http://en.wikipedia.org/wiki/10-simplex`.



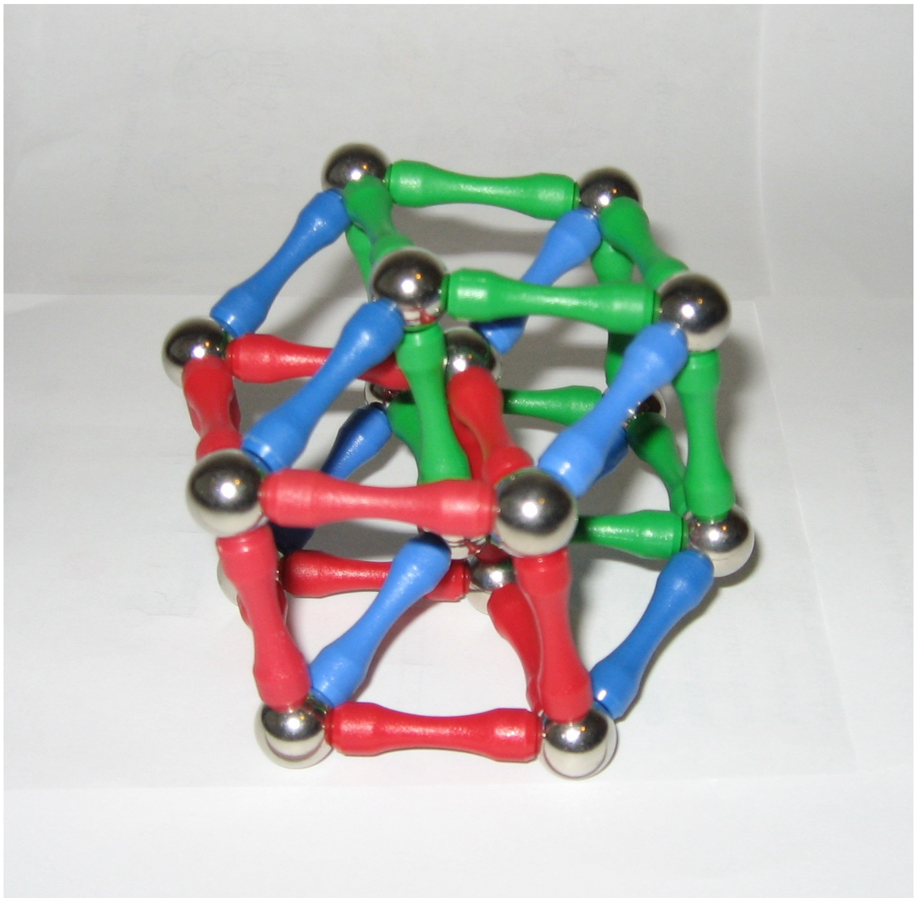
Let us inspect the way a 3D cube is drawn on a 2D sheet of paper. First we draw a 2D cube, the square completely made of solid lines on the above picture. Then we choose a direction different from those of the sides of the square. Pretending that this is the extra dimension we need, the third one, we drag the square along and get the desired 2D projection of the 3D cube.

Homework Problem 4.147 *How many edges has a 3D cube? How many vertices?*

Homework Problem 4.148 *Draw a cube facing you a. with a vertex; b. with an edge; c. with a face. Use solid lines to draw the visible edges and dashed lines of the invisible ones.*

Homework Problem 4.149 *Make a cube out of paper.*

We construct a 3D projection of a 4D cube, also known as the *tesseract*, exactly the same way we have drawn a 2D projection of a 3D cube. Let us first build a 3D cube out of red magnetic pegs as edges and magnetic balls as vertices (see below). Let us choose a direction different from those of the edges of the red cube and pretend that this is the extra dimension we need, the fourth one. The "extra" dimension is represented by the blue pegs on the picture. Then we drag our 3D cube along the "extra" dimension, getting the green cube as a result. Here is the desired 3D projection (the cubes came out a bit skewed due to the poor craftsmanship of the author).

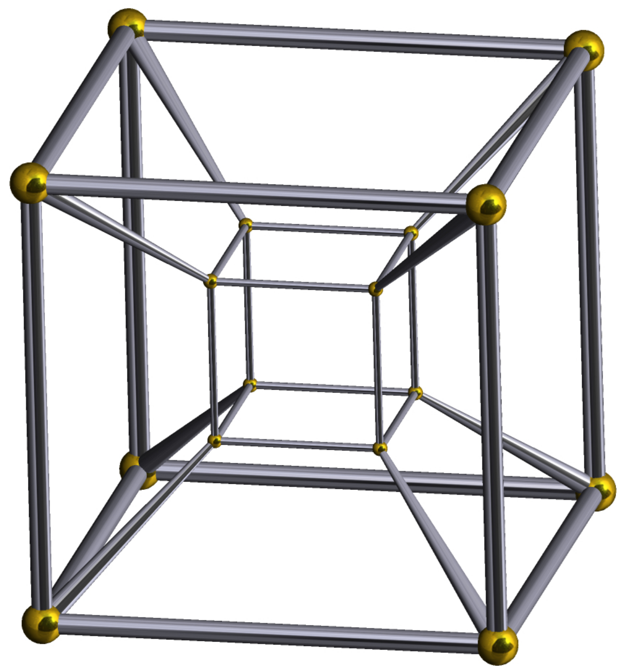


A 3D projection of the 4D cube.

In 3D, the red and green cubes intersect, which wouldn't have been the case in 4D. Despite this minor deficiency, our model is quite easy to work with: we can count the 3D faces of the 4D cube (the 3D cubes), the 2D faces (the squares), the edges, the vertices, see their relative positions, etc.

Homework Problem 4.150 *How would you go about drawing a 5D cube? Don't do it, just give an idea. How about the 6D cube and so on?*

Homework Problem 4.151 *How many 3D faces (3D cubes) has the tesseract? (To solve this problem, you will either need to build a 3D model similar to the above, or to look at the below picture.)*



Another 3D projection of a 4D cube.[34]

The inside cube at the above picture is a face of the tesseract as well as the outside one. In 4D, they have the same size. The "smaller" cube sticks not inside the "larger" one, but in the fourth dimension. All the six polyhedra connecting the 2D faces of the inner cube to the corresponding 2D faces of the outer one are also cubes of equal size in 4D. This observation is crucial for solving Homework Problem 4.151.

Homework Problem 4.152 *Suppose that you witness a 3D sphere, the one that "lives" in the 4D Euclidean space, passing through our 3D world. What would you see? Hint: you have seen what happens when a 2D plane passes through a 2D sphere in Homework Problem 4.102.*

Closing this chapter, the author would like to encourage the parent/teacher to read the book [1] to her/his little one(s). First published in 1884, the adventures of Mr. Square, an inhabitant of a two-dimensional country called

[34]This is a Wikipedia image downloaded from **here**.

Flatland, still are, according to one of the most prominent sci-fi writers of the 20th century, Isaac Asimov, "the best introduction one can find into the manner of perceiving dimensions". The author of this book wholeheartedly agrees with the statement.

5 Time and space

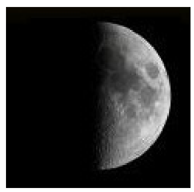
We had started learning time a couple of months before they did it in our son's first grade. As a rule, the author tries not to jump ahead of school, so that the child doesn't get bored too much with the school math. However, the way we measure time is closely related to the topic we have already touched upon in the book, that is the globe rotation. Also, our son, a great fan of LEGO, has earned a LEGO watch for his gymnastics achievements and he very much wanted to know how to use the new piece of hardware. So here comes this chapter...

Time, like money, is a human-invented concept. The money allows us to compare values of different goods without resorting to barter. Just imagine trying to figure out the price of a car in terms of swimming trunks or the value of your favorite toy in terms of that of the Thanksgiving turkey directly, without using the money. Similarly, time serves as a universal measure allowing us to compare durations of various happenings, from swimming a lap to a galaxy formation. The currency unit here is the duration of a very specific astronomic event, one full rotation of our planet, the Earth, around its axis, the imaginary straight line passing through the North and South Pole. We call the latter period one day and compare all other durations to it. For shorter happenings, we use one hour, the time it takes the Earth to spin 1/24 part of the full circle. One hour is further divided into 60 minutes and a minute into 60 seconds.

There are 24 hours in a day. However, if we take a look at a clock or a watch, we see that its circular face is divided into 12 hours, not 24! Thousands of years ago, when people knew very little astronomy, they have defined one hour as 1/12 of the time between the sunrise and sunset. According to the original definition, the length of an hour varied depending on the observer's location on the Earth's surface, time of the year, and even local weather! This *temporal*, or *seasonal*, hour was used by all the known ancient cultures that have left written time-keeping records, the Chinese, Jews, Greeks, and Romans.

As the Earth orbits the Sun, the planet's biggest satellite, the Moon, orbits the Earth reflecting sunlight to our planet. Depending on the relative position of the celestial bodies, we see a part of the Moon darkened by the Earth's shadow. The first visible crescent appearing from the dark is called the New Moon. The following phases go after it.

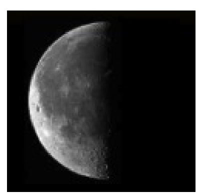

Waxing crescent

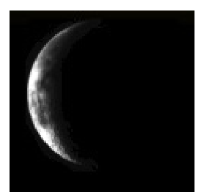

First quarter Moon


Waxing gibbous Moon


Full Moon


Waning gibbous Moon


Last quarter Moon


Waning crescent

Finally, the Moon vanishing in the dark of the Earth's shade is called the Dark Moon. One full sequence of the lunar phases is called a *lunar cycle* or *lunar month*. There are twelve lunar months is a year. This observation has prompted the ancients to similarly divide the time between the sunrise and sunset into 12 equal parts. Honoring the millennia-long tradition, our clocks and watches have 12-hour faces, although there are 24 hours in a full day!

5.1 Adding and subtracting multi-digit numbers

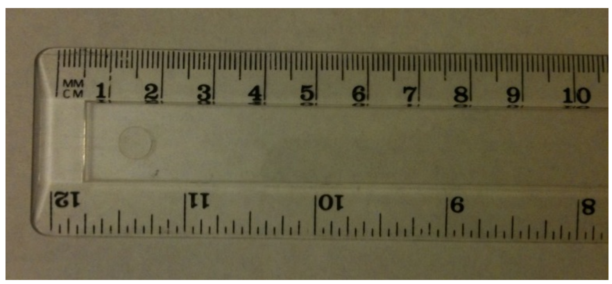
As a necessary technical tool, we need to teach the child to add and subtract multi-digit numbers. Alas, this means jumping ahead of school again :(

Let us check out various ways to compute $17 + 19$.

1. We can draw a two-row Young diagram, 19 boxes in the lower row and 17 in the upper. To find the answer, all that is left to do is to count them. (Do it!)

2. We can draw the number line (with sufficiently many numbers on it), find the number 17, take 19 more steps to the right, and see where we end up. (Do it!)

3. We can slightly modify the previous method, using the ruler instead of the number line. Let us further familiarize the child with this useful gadget.



A part of a ruler.

The upper part of the ruler above is marked with *centimeters*, the lower, going in the opposite direction, with *inches*. You can see that one inch is a bit longer than two and a half centimeters (1"= 2.54cm, to be precise).

The idea was to use the ruler as the number line. Unfortunately, it's not long enough to count 17 and then 19 more steps in centimeters, not to mention inches! But let's take a better look: there are short lines at the top of the ruler dividing each centimeter into 10 *millimeters* (mm). The good thing is that there are enough millimeter marks for us to solve the problem. The bad thing is that millimeters are not marked with numbers, only the centimeters are. So we first take 17 one-millimeter-long steps to the right, then 19 more. Then we remember the final mark and count the total number of steps toward it. (Do it!)

4. Let us take another look at the above procedure. Isn't it true that 17 millimeters is nothing else but 1 centimeter and 7 more millimeters? It is!

$$17mm = 1cm + 7mm$$

Similarly,

$$19mm = 1cm + 9mm.$$

Idea: we can add centimeters and millimeters separately! Then

$$17mm + 19mm = 1cm + 7mm + 1cm + 9mm = 1cm + 1cm + 7mm + 9mm.$$

To compute $7 + 9$, we again can use either the proper Young diagram or the number line. With the help of the latter, we see that

$$7mm + 9mm = 16mm = 1cm + 6mm.$$

So, our problem boils down to the following computation:

$$17mm + 19mm = 2cm + 7mm + 9mm = 2cm + 1cm + 6mm.$$

Finally,

$$17mm + 19mm = 3cm + 6mm = 36mm.$$

But there were no millimeters in the original setting! Erasing them, we get the answer:

$$17 + 19 = 36.$$

5. Actually, do we need millimeters and centimeters in the approach above? No! $17 = 10 + 7$ and $19 = 10 + 9$ without any millimeters, centimeters, meters, or miles! We can simply do the following:

$$17+19 = 10+7+10+9 = 10+10+7+9 = 20+16 = 20+10+6 = 30+6 = 36.$$

Thus, the short and efficient way to sum up big numbers is to add tens to tens and single digits to their like.

Example 5.1

$$25 + 53 = 20 + 5 + 50 + 3 = 20 + 50 + 5 + 3 = 70 + 8 = 78$$

Adding 50 to 20, we in fact add 5 to 2, but in the units 10 times larger than the original ones (centimeters vs. millimeters, if you wish).

Homework Problem 5.1 *Using the new method, find*

$$23+74=$$

$$31+46=$$

$$64+17=$$

$$81+5=$$

The method works not only for two summands, but also for three and more. Let us try it with three.

$$23 + 74 + 31 = 20 + 3 + 70 + 4 + 30 + 1 = 20 + 70 + 30 + 3 + 4 + 1.$$

Once again, to sum up $20 + 70 + 30$, let us think of them in centimeters vs. the original millimeters. Then all we need is to find the value of $2+7+3$. We can do it with the help of the corresponding Young diagram, using the number line, or just in our heads. The most efficient way is to notice that $7+3 = 10$. Then $2+7+3 = 2+10 = 12$. But that's in centimeters whereas we need the answer in millimeters. There are 10 millimeters in a centimeter, so $20+70+30 = 120$. The last number, called *one hundred twenty*, means 12 tens (12cm = 12 tens of millimeters). The number 100, *one hundred*, means ten tens. 200 means 20 tens, 300 means 30 tens, and so on. Getting back to our computation,

$$23 + 74 + 31 = 20 + 70 + 30 + 3 + 4 + 1 = 120 + 8 = 128.$$

Homework Problem 5.2 *Compute the following:*

$32+47+11=$

$16+26+36=$

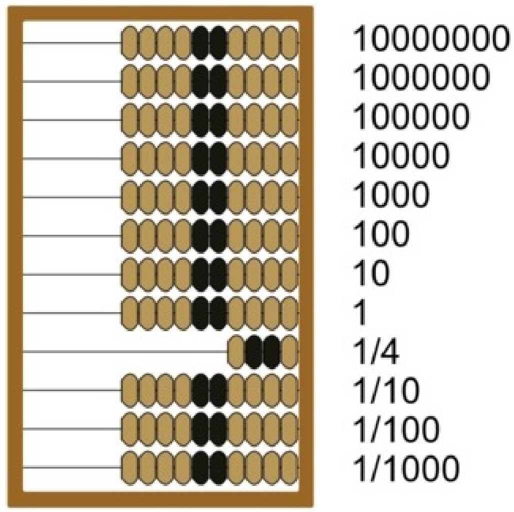
$10+40+25=$

$51+22+3=$

$60+30+90 \ =$

$120+37+23=$

A great tool to teach children addition and subtraction of multi-digit numbers is the Russian *abacus*, a deck with multiple wires and ten beads on each wire except for one which only has four as on the picture below.



The Russian abacus.[35]

The four-bead wire is used to count quarters of a ruble[36] or a dollar. The ten beads above it are used for counting single-digit numbers, the next wire – for counting tens, the next one – for hundreds, and so on.

The abaci were used in the societies predating Ancient Greeks, such as the Sumerian[37], Egyptian, Hebrew, and others. These computers of the ancient world were first made as sand grooves filled with stones for counting, hence the name (the word *abacus* originates from the Hebrew *ābāq*, meaning *dust*).

Forgotten in Europe by the end of the 16th century, the abacus was reintroduced by the French mathematician Jean-Victor Poncelet.

[35]Downloaded from `http://en.wikipedia.org/wiki/File:Russian_abacus.png`.

[36]Russian currency.

[37]Sumer, a civilization and historical region in southern Mesopotamia, modern Iraq, is the earliest known civilization in the history of this planet.

Jean-Victor Poncelet.[38]

A military engineer in the Napoleon army, Monsieur Poncelet was captured by the Russian troops in 1812 and released in 1814.[39] He brought the abacus to France from his Russian captivity and began using it as a teaching tool. In Russia itself, the abacus was a standard computational device on every counter, from a bank to a grocery store, until the middle of the 1980s, when it eventually got replaced by electronic calculators.

[38]Downloaded from **here**.

[39]Napoleonic Wars were the biggest military conflict of the 19th century, similar to World War II of the 20th. In both conflicts, it were the Russians who defeated the bulk of the opposing coalition of forces, the forces led by the French in the first case and by the Germans in the second.

A Russian saleslady using the abacus.

Let us see how we can use the abacus to solve the problem we have started this lesson with. We want to find the $17 + 19$ sum. First, we move to the left one bead on the wire that corresponds to tens (the second above the four-bead wire). Then we move seven beads to the left on the lower wire, the one representing single digit numbers. This combination of beads visualizes number 17. We need to add number 19 to it. For that, we move one more bead to the left on the upper wire. We would also like to move nine beads to the left on the lower one, but there are only three left. So we move these three first. As all the ten lower wire beads are on the left now, we move one more bead to the left on the upper wire and all the ten lower beads – back to the right. This way, we add ten to the number we are figuring out and immediately subtract it, keeping the result unchanged. There are six more beads to move to the left on the lower wire (remember, we have only moved three out of nine). Finally, here comes the answer, three beads on the upper wire and six more – on the lower.

$$17 + 19 = 36$$

Homework Problem 5.3 *Redo Homework Problems 5.1 and 5.2 using the abacus.*

Similarly, we can add three-digit numbers. This time, we need to separately add hundreds, tens, and single digits. Let us compute $152 + 263$. 152 is represented by a single bead moved to the left on the third wire (up from the four-bead one), five more on the second, and two more on the first. To add 263, we first move two more beads to the left of the third wire. Next, we need to move six beads to the left of the second, but there are only five available. Let's move these five. Now we have all the ten second wire beads on the left. But ten second wire bids are equivalent to one third wire bead, just like 10mm=1cm, so we move the second wire beads back to the right as we move one more third wire bead to the left instead. Remember, we only have moved five beads out of six to the left on the second wire. Now we can move the last one. Finally, 2+3 is easy to add up on the first wire. The result of our computation appears as four beads on the third wire, one bead on the second, and five beads on the first.

$$152 + 263 = 415$$

Homework Problem 5.4 *Use the abacus to compute the following:*

35+427=

162+263=

254+7=

223+3210=

$$100+200+300 =$$

Subtraction of multi-digit numbers works along the same lines. Consider the following example:

$$35 - 17.$$

First, we represent 35 on the abacus as three beads on the second wire and five more – on the first. To subtract 17, we move one of the three beads on the upper wire to the right. Then we need to move seven beads to the right of the lower wire. But we only have five! We move these five to the right and we still have to move two more. We do it the following way. We move one of the remaining two beads on the upper wire to the right and all the ten beads on the lower wire to the left. This way, we subtract and simultaneously add ten to the result of our computation. Now we can move the remaining two beads on the lower wire to the right! The result appears in the form of one bead left on the upper wire and eight more beads on the lower one.

$$35 - 17 = 18$$

The numerical equivalent of this computation is:

$$35-17 = 30+5-10-7 = 30-10+5-7 = 20-2 = 10+10-2 = 10+8 = 18.$$

Homework Problem 5.5 *Solve the following problems with and without the abacus:*

$$43\text{-}8=$$

$$92\text{-}67=$$

$$54\text{-}12=$$

$$100\text{-}50=$$

Homework Problem 5.6 *The first president of the United States, George Washington, was born in 1732 and died in 1799. How long did he live? How old would he have been, if he was still alive this day?*

Homework Problem 5.7 *Put the correct sign, $>$ or $<$, in the box between the numbers. Remember, the sign opens up towards the greater number.*

$$704 \square 740 \quad 200 \square 20 \quad 1000 \square 999$$

Homework Problem 5.8 *We have noticed that adding and subtracting multi-digit numbers, we can perform the operations separately with single digits, tens, hundreds, and so forth. Does it matter what to add or subtract first?*

Practice addition and subtraction of multi-digit numbers until the child becomes fluent at it, both with the abacus and without it.

NO ELECTRONIC CALCULATORS PLEASE!!!

5.2 Choosing coordinates on a straight line and on a circle

Let us recall the procedure from Subsection 4.4 we have used to turn a straight line into the number line. First, we have to choose the starting point, mark it on the line and call it zero. Second, we have to choose the unit step. Any one will do for using the number line to clarify geometric properties of addition and subtraction. However, a finite part of the number line can also be used as a ruler for taking measurements. In this case, using the standard unit steps such as the historic one inch (1"), one foot (1'), one yard (1 yd), one mile (1 mi); or the more modern one centimeter (1 cm), one meter (1 m), one kilometer (1 km), and so on, will make it possible to communicate the results of your measurements to other people.

There are 12 inches in a foot and 36 inches in a yard.

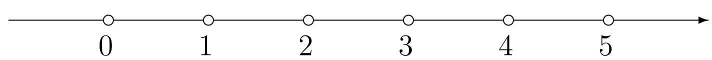
Homework Problem 5.9 *How many feet are there in one yard? Hint: compute $12 + 12 + 12$. Use the abacus first, then check the answer with the proper Young diagram.*

There are 100 centimeters in a meter and 1000 meters in a kilometer. In fact, the word “kilo” means 1000 in Ancient Greek, the language spoken by Euclid and his contemporaries.

Homework Problem 5.10 *Let us use the first abacus wire above the four-bead one for counting millimeters. 1cm=10mm, so we can use the next wire to count centimeters. What wire should we use for meters? Kilometers?*

Homework Problem 5.11 *A boy went for a hike with his parents. They walked for 2 km and 300 m before taking a snack break. Then they walked some 3 km and 700 m more. What distance did they cover?*

Once we mark down zero on the straight line and choose the unit step, 1”, 1 cm, or any other, all that is left to do is to take more steps and to count them in the process.

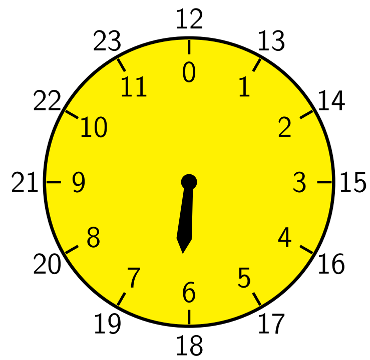


The situation is trickier with a circle. You can mark zero wherever you like. However, taking an arbitrary unit step will result in never getting back to the zero mark.

Homework Problem 5.12 *Draw a circle with a compass. Mark a point on the circle as zero. Choose the unit step at random. Using the compass to mark the steps, wind around the circle trying to come back to the starting point until your hands get tired. Have you eventually hit zero? Why not? (A precise answer to the latter question is not expected at this point.)*

The above exercise most likely has convinced the reader that the unit step on a circle must be chosen with care. The most natural way to do it is to take a positive integer n and to divide the circle into n equal parts. You will see some standard choices of n below.

1. The circle is divided into twelve equal parts. In this case, one step is usually called one hour.



0 coincides with 12. The hour hand moves from 0 to 1, from 1 to 2, ... from 11 to 12 just as it would have on the number line. However, 12 equals 0 on this circle, so there it goes again, from 1 to 2, and so on. We write down the fact that 12 equals 0 as

$$12 \equiv 0 \ (mod \ 12) \tag{5.1}$$

and read it as *12 is equivalent to 0 modulo 12*. The usual "=" sign is reserved for the number line; we use "≡" on the circle instead. The *mod* 12 symbol tells us that the circle is divided into 12 equal parts, so 12 coinsides with 0, 13 – with 1, 14 – with 2, and so on. Or in the new notations,

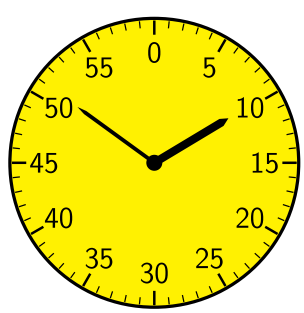
$$13 \equiv 1 \ (mod\ 12), \ \ 14 \equiv 2 \ (mod\ 12), \ldots, 23 \equiv 11 \ (mod\ 12), \ \ 24 \equiv 12 \equiv 0 \ (mod\ 12).$$

Homework Problem 5.13 *What is* $27 \ (mod \ 12)$*?*

Let us compare the above picture to the photograph of our son's LEGO watch:

The hour hand is always the shortest. It is also often made thicker than the other two hands, the minute and second ones. The number of hours the watch shows is equal to the nearest number the hour hand has passed. It's 10 on the above watch, for example.

2. Another standard way is to divide the circle into 60 equal parts. Depending on the situation, the unit step is called either a minute or a second.



All the numbers living on this circle are considered modulo 60. In particular, $60 \equiv 0 \pmod{60}$. There are 60 minutes in an hour. Our watches and clocks are made in such a way that by the time the hour hand takes a unit step, that is moves for one hour along the 12-hour circle, the minute hand makes a full circle from 0 to 60 along the 60-minutes-long circle of its own. Unlike the above picture, the number of minutes is sometimes not given on the face of a watch or a clock. In this case, you have to find it yourself.

To distinguish from the hour hand, the minute hand is made longer and often thinner. It also moves faster, making 720 full revolutions in the time the hour hand makes one. The time the minute hand shows equals to the nearest number of minutes it has already passed. For example, the minute hand of the above LEGO watch shows 10 minutes (past ten).

Homework Problem 5.14 *What is* $72 \pmod{60}$*?*

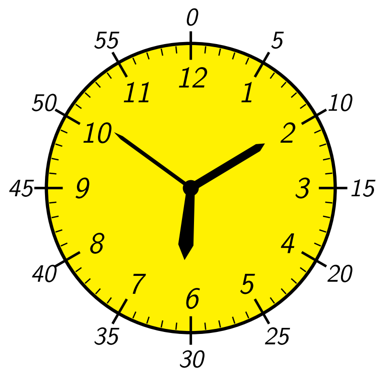
There are 60 seconds in one minute. The second hand is the longest, thinnest, and fastest. By the time the minute hand makes one full circle, one hour that is, the second hand makes 3600 circles! It's so fast that it seems to be the only hand moving. If we look carefully, we can see the motion of the minute hand as well, very slow compared to that of it's skinny sibling. Just like the minute hand, the number of seconds the second hand shows equals to the nearest number it just has passed. However, it moves so fast that by the time we finish our observation, the second hand moves further ahead, so when telling time, people often skip seconds.

Let us take another look at the LEGO watch on page 135. Now we can tell the time it shows. It is

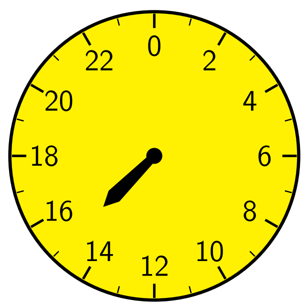
$$10 : 10 : 00$$

or 10 hours, 10 minutes, and 0 seconds. It's more customary to say, "It's ten minutes past ten now."

Homework Problem 5.15 *What is the time on the clock below?*



3. There are 24 hours in a day, so another standard way is to divide the circle into 24 equal parts. The US military use the 24-hour clock. They would say that the clock below shows *fifteen hundred hours* (15:00), whereas our sun's LEGO watch on page 135 shows *ten hundred ten*.



The following is a photograph of the 24-hour clock from the USS (United States Ship) *Mullinnix*, the last "all gun" US Navy destroyer in the Pacific, decommissioned in 1982.[40]

[40]See its homepage at `http://www.ussmullinnix.org/`

USS *Mullinnix* 24-hour clock.[41]

It shows *zero five hundred twenty five*, five hours and twenty five minutes A.M. in the civilian language.

Homework Problem 5.16 *The following computations will be needed for what follows.*

$$12 + 12 =$$

$$24 + 24 + 12 =$$

It follows from the first part of Homework Problem 5.16 that 12 is a half

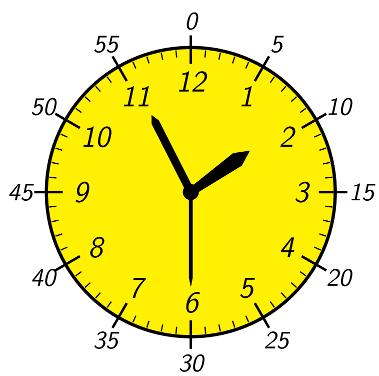
[41] Downloaded from `http://www.ussmullinnix.org/MuxMemorabilia.html`

of 24, or $12 = 24/2$. It follows from the second part that

$$\frac{60}{24} = \frac{24+24+12}{24} = 2 + \frac{12}{24} = 2 + \frac{1}{2} = 2\frac{1}{2}.$$

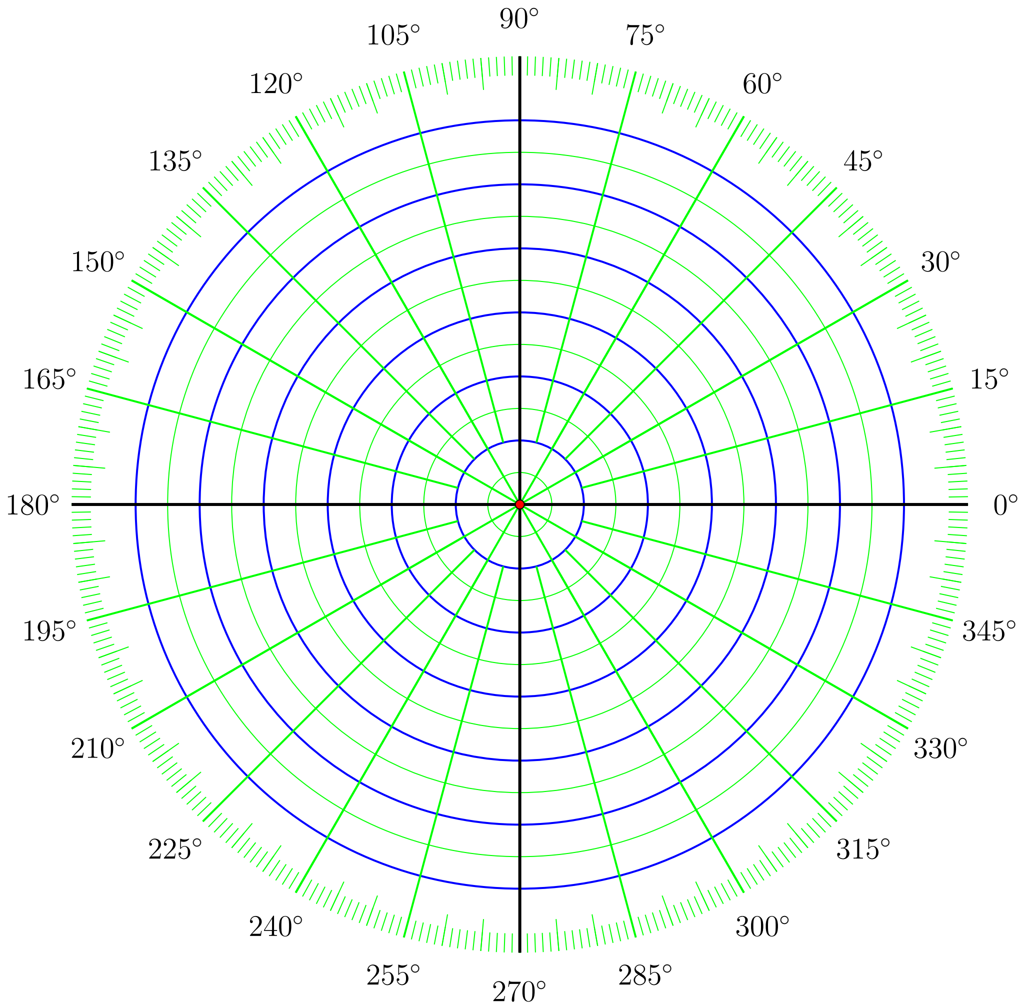
Since $60/24$ is not a whole number, we can't use the same marks on the face of a 24-hour clock for minutes and hours (to better see this, please find the minute and hour marks on the face of the USS *Mullinnix* clock). $60/12 = 5$, so this inconvenience doesn't exist for the clocks and watches we are used to. On the other hand, to disambiguate between, say, 1 o'clock night time and 1 o'clock afternoon, we have to use the A.M./P.M. notation not needed in the military. In their language, 1 o'clock P.M. is 13:00 hours, plain and simple.

Homework Problem 5.17 *What is the time on the clock below?*



Homework Problem 5.18 *Gregory wakes up at 7:30 A.M. It takes him half an hour to eat his breakfast and to get dressed for school. It takes his mom another 15 minutes to drive him to school. What time does Gregory come to school?*

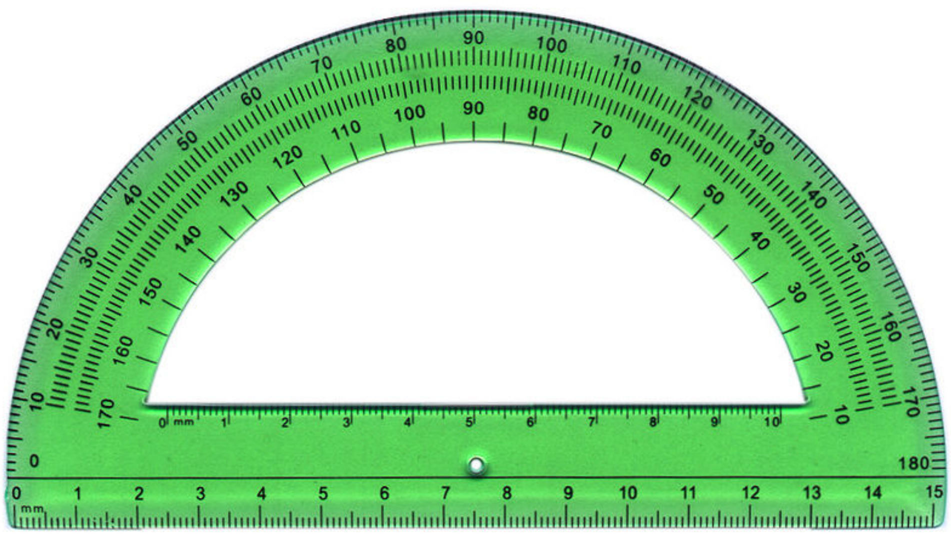
4. The last standard way to introduce coordinates on a circle we consider in this book is related to measuring not time, but *angles*. For the purpose of measuring angles, it is customary to divide the circle into 360 equal parts, each called 1° (one degree). The device we use to measure angles is called a *protractor*. It is to an angle what a ruler is to a straight line.



The TikZ code generating the above picture was written by Zoran Nikolic, downloaded from `http://www.texample.net/tikz/examples/polar-coordinates-template`, and slightly modified by the author.

To measure an angle by means of the above circular protractor, one needs to print it out on a transparent sheet of plastic, place the angle's vertex in the center of the measuring circle so that one of the rays forming the angle coincides with the ray going through the 0° mark. Then see what mark the second ray passes through.

The circular protractor is more convenient for measuring angles, but the standard protractor, like the one below is better for drawing them.

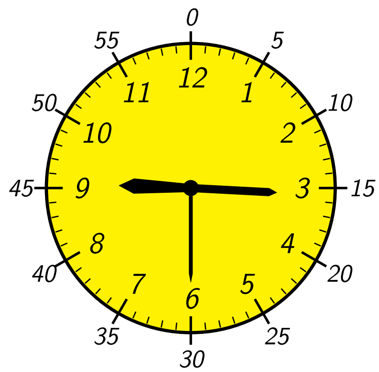


To draw an angle of a particular size, choose its vertex first. Then take a ruler (you can use the one at the bottom of the protractor) and draw one of the two side rays. Place the vertex at the center of the protractor's semicircle (the small hole at the bottom of the protractor) so that the ray goes to the right along the horizontal line next to the hole. Mark the needed size of the angle using the protractor's inner scale (the outer goes in the opposite direction). Use the ruler to draw the second ray from the vertex to the marked point and beyond. Voilà!

Homework Problem 5.19 *Draw the* $75°$*angle.*

Homework Problem 5.20 *Measure the angle on page 44.*

Homework Problem 5.21 *What is the time on the clock below?*



Suppose that this is the time P.M. How would the military call it?

Homework Problem 5.22 *Draw the civilian clock showing 1:35:47. Draw the military clock showing the same time P.M. (Some parental help may be needed.)*

From this moment on, keep asking the child to tell the time a few times a day, using any time peace available. Gradually decrease your help until the little one is able to tell the time by her/himself.

Homework Problem 5.23 *Use a protractor to measure the blue and green angles on page 44. Add up the numbers. Then measure the sum of the angles on the picture. Do you get the same number?*

Homework Problem 5.24 *There are 1760 yards in a mile. How many yards are there in two miles?*

Homework Problem 5.25 *There are approximately 1609 meters (1,609.344 to be precise) in a mile. How many meters are there in two miles?*

Homework Problem 5.26 *What distance is greater, 2 mi or 3 km?*

Homework Problem 5.27 *Jason's school is over at 2:45 P.M. His school day is 6 hours and 25 minutes long. What time do Jason's classes begin?*

5.3 Using a latitude line as a clock

Recalling the material from page 96, let us take another look at the grid lines covering the model globe. The circles of various sizes winding around the planet are the latitude lines, or the parallels. The halves of the great circles connecting the poles are the longitude lines, or the meridians. Any half of a great circle connecting the poles is a meridian, but only 24 of them, all equally spaced, are shown on the globe. Why?

Most likely, the child has guessed it right. Because there are 24 hours in a day! It takes the Earth an hour to move one of the 24 special meridians to the position of the next one. The marks the 24 special meridians make on any of the latitude circles turn the latter into a 24-hour clock. One only has to choose the zero mark.

The Royal Observatory in Greenwich, the United Kingdom, was commissioned in 1675 by King Charles II. At the International Meridian Conference of 1884, the meridian passing through the observatory was universally accepted as the zero longitude line. It is the intersection of the Greenwich meridian with a latitude circle that should be marked as zero, if we intend to use our planet as a clock.

The English word "meridian" originates from the Latin "*meridies*". The meaning of the latter is "midday". Moving around the sky, the Sun crosses the meridian line in the middle of the day at that longitude, that is at 12:00 noon local geographic time. Thus the A.M. notation, an abbreviation of "Ante Meridian", or "before midday". Similarly P.M., or "Past Meridian", means "after midday".

Suppose that it is 12:00 noon in Greenwich now. The Sun is in its highest point in the sky for this day. Five hours later, as the Earth spins, the Sun will be at the top of the sky over New York City, the United States. It will be 12:00 in Big Apple and it will be 5:00 P.M. in Greenwich as well as in the neighboring capital of the UK, the city of London. Hence, the time difference between London and New York is 5 hours.

Greenwich time is known as the GMT (Greenwich Mean Time) and as the UTC (Universal Time Coordinated). The standard way to say that the

New York time is five hours behind the time in Greenwich is

$$NY\ time = UTC - 5.$$

The Royal Observatory, Greenwich, the UK.[42]

To make sure that soldiers understand their radio communications through the noise of battle, the US military use special phonetic names for the alphabet letters. In their language, A is “Alpha”, B – “Bravo”, C – “Charlie”, and so on. In particular, Z is “Zulu”. Greenwich time is the time at the zero

[42]The above image was downloaded from **here**.

meridian, abbreviated to the letter Z in the military. This way, the UTC is called the "Zulu time" in the US armed forces.

Example 5.2 *What is the time difference between Los Angeles and New York City?*

Let us take a look at the globe. Los Angeles is to the West of Big Apple, so the Sun comes there later. For example, when the Sun is at the top of the sky in New York, it's still going up in Los Angeles. Both cities are located near the grid meridians, Los Angeles three of them away to the West. Thus

$$LA\ time = NY\ time - 3.$$

If it is, say, 5 P.M. in New York City, it's 2 P.M. in Los Angeles.

Homework Problem 5.28 *It's 2 P.M. in Los Angeles. What time is it in London, the UK? In Moscow, Russia?*

Geographic time changes smoothly around the Earth. For example, there exists the meridian 5 hours 37 minutes and 51 seconds to the West of Greenwich. When the Sun is at its highest point in the sky over that meridian, the clocks in London show 05:37:51 P.M. civilian time (17:37:51 Zulu). However, using geographic time is extremely inconvenient. Just imagine having one time at home, a different time at school, both different from the time at Mom's work, and all three of them different from that in Dad's office!

To avoid this horrible mess, the Earth is split into time zones, each roughly equal to the stripe between the neighboring main meridians. Time is considered the same within each time zone and changes by an hour from one zone to the next one.

Continental US, that is the part of the country excluding Alaska, Hawaii, and other territories disconnected from the mainland, is split into four time zones as shown below.

The continental US time zones.[43]

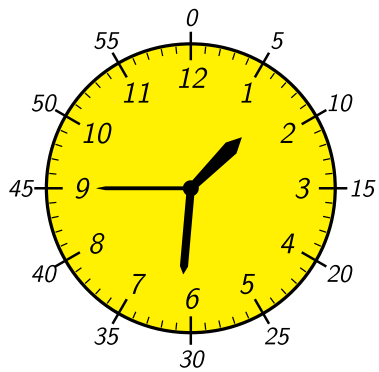
The East of the country, including the US capital city of Washington, D.C., New York City, Boston, etc., belongs to the Eastern Standard Time zone, or the EST. The Central Standard Time zone, or the CST, contains, among others, the cities of Chicago, Houston and Dallas. The Mountain Standard Time zone (MST) to the left of it covers the Rocky Mountains (hence the name) as well as the major cities of Denver and Phoenix. Finally, the Pacific Standard Time zone, the PST, is located on the country's Western coast, including the cities of Seattle, San Francisco, and Los Angeles. As the above chart shows,

$$6\ P.M.\ EST = 5\ P.M.\ CST = 4\ P.M.\ MST = 3\ P.M.\ PST.$$

Homework Problem 5.29 *Write down the formulae relating the EST, CST, MST, and PST to the Greenwich time, the UTC. Use parents'/teacher's help, if needed.*

[43]Downloaded from `http://en.wikipedia.org/wiki/File:US-Timezones.svg`

Homework Problem 5.30 *What is the time on the clock below?*



Suppose that this is the time A.M. PST. What is the time in Greenwich, the UK? In Tokyo, Japan?

Homework Problem 5.31 *Use a protractor to measure the blue and green angles on page 45. Subtract the size of the green angle from the size of the blue one. Then measure the difference of the angles. Do the numbers match?*

Homework Problem 5.32 *What is the size of the straight angle in degrees? (See Definition 4.9, if needed.)*

Homework Problem 5.33 *What is the value, in degrees, of the sum of two supplementary angles? (See Definition 4.10.)*

5.4 Angles and 2D polygons

Let us take another look at the protractor on page 141. We see that there are five special angles of sizes 0°, 90°, 180°, 270°, and 360°. The rays forming the 0° angle coincide, looking like one single ray. There are no points in between the side rays. The 0° angle is just one ray.

The sides of the 180° angle are opposite to each other forming a straight line. This way, the 180° angle is merely a half-plane, the upper half-plane on page 141, for example.

The 360° angle is the whole plane. One of the rays sweeps around it until it joins the other one passing through the 0° mark.

Homework Problem 5.34 *Compute* $180° + 180°$*.*

Geometric expression of the answer to the above problem is the fact that the sum of two complementary half-planes is the whole plane. Here we see for the first time that one and the same mathematical statement can be expressed in the forms that seem very different for the first look. It is this richness of structure that makes math so attractive for an inquisitive mind.

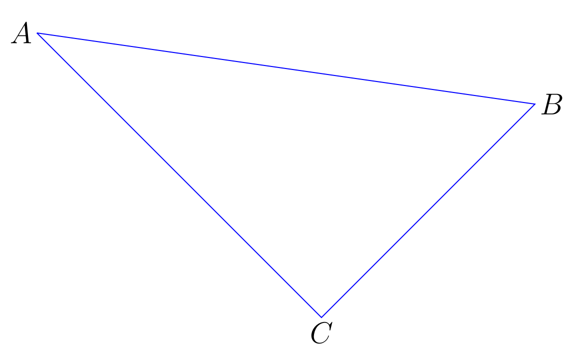
Homework Problem 5.35 *Recall the definition of the right angle (see Definition 4.25 on page 108, if needed.)*

Homework Problem 5.36 *Compute* $90° + 90°$*. What is the size of the right angle in degrees?*

Homework Problem 5.37 *How many right angles do you need to cover the plane? Hint: the plane is covered by a* $360°$ *angle.*

Homework Problem 5.38 *Using a protractor, measure the angles* $\angle A$*,* $\angle B$ *and* $\angle C$ *of the below triangle. Find the sum*

$$\angle A \ + \ \angle B \ + \ \angle C.$$



Note 5.1 *Please do not tell the child at this point that the sum of the angles of any triangle in the Euclidean plane is always equal to* $180°$*! We shall first discover this fact experimentally as the Ancient Greeks have.*

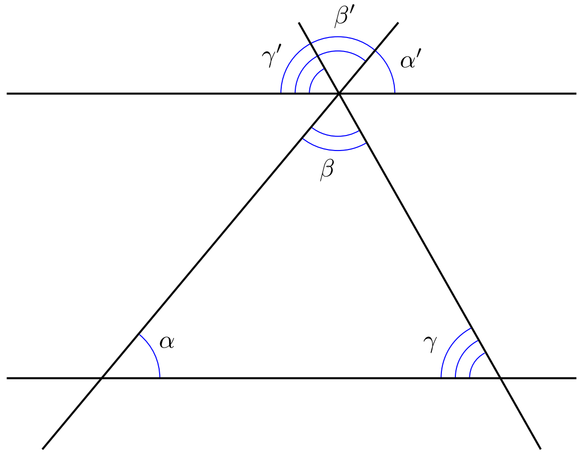
The observed sum may be different from $180°$ *due to the measurement mistakes. The* $1°$ *marks on the protractor are fairly small, so a* $1°$ *mistake is* *quite likely. In the worst case, all three of them can accumulate to the total mistake of* $3°$*. Let us give the child one more degree as a bonus. If the sum* $\angle A + \angle B + \angle C$ *differs from* $180°$ *by more than* $4°$*, please ask the little one to redo the measurements with greater care.*

Project 5.1 *Ask the child to draw a triangle every day, measure its angles and add them up. Make her/him do it until the child can perform the whole procedure without help. This can take a week or two. In addition to geometry, this is a good exercise in summing up two- and three-digit numbers.*

Keep the results. Review them at the end of the project. Discuss the idea of measurement mistakes with the child. Point out to her/him that the sums in question never deviate too far from $180°$*. Out of all the angles,* $176°$*,* $177°$*,* $178°$*,* $179°$*,* $180°$*,* $181°$*,* $182°$*,* $183°$*, and* $184°$*, that occur as the possible outcomes, only one, the* $180°$*, looks nice. Could it be that the rest are merely the measurement mistakes?*

Theorem 5.1 *For any triangle in the Euclidean plane, the sum of its angles always equals* $180°$*.*

Proof— For an arbitrary triangle, let us pick a vertex and draw a straight line passing through the vertex parallel to the opposite side of the triangle as on the picture below.



Let us recall Definition 4.6: *A 2D plane is called Euclidean if for any line in the plane and for any point in the plane not lying on the line, there exists one and only one straight line passing through the point parallel to the original line.* We consider our Mr. Triangle living in the Euclidean 2D (next to Mr. Square, the main character of the book [1].) Thus the line we need exists and there is only one such.

According to Proposition 4.3,

$$\alpha = \alpha', \quad \text{and} \quad \gamma = \gamma'.$$

The angles β and β' are opposite. According to Proposition 4.2,

$$\beta = \beta'.$$

Thus $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma'$. But the latter three angles add up to a straight angle. So,

$$\alpha + \beta + \gamma = 180^\circ.$$

□

The above feature distinguishes the Euclidean plane from all other 2D surfaces. Let us take a look at the globe. We can see that any meridian intersects the Equator at the right angle (90°). Arcs of great circles, meridians and the Equator in particular, are the geodesic lines, the straight lines of the spherical world. Let us take two meridians intersecting at the right angle at the poles. For example, we can take the Greenwich meridian and the grid meridian next to Chicago, six hours to the West of Greenwich. The upper halves of the above meridians, together with the quarter of the Equator they cut off, form a triangle on the surface of the Earth. All the angles of this triangle are right (please, see it on the globe).

Homework Problem 5.39 *Compute* $90^\circ + 90^\circ + 90^\circ$
a. using the abacus;
b. numerically on a sheet of paper;
c. taking three 90° *steps on a circle (you can use the protractor on page 141).*

We see that the sum of the angles of the above triangle is way more than 180°.

Homework Problem 5.40 *How much more?*

The sum of the angles of a spherical triangle is always greater than $180°$. Unlike the Euclidean plane, the sum depends on the size of the triangle.[44] The smaller a triangle, the closer the sum of its angles approaches the Euclidean limit. If we take a truly minuscule spherical triangle, the sum of its angles will only be slightly more than $180°$.

Let us take a look at the situation from the point of view of the tiny sphere inhabitants we have met before. Being very small, they may not notice the curvature of their world and may think that it is flat, or Euclidean. They may even prove Theorem 5.1, just like we have, thinking that it describes their world, not ours. Of course, once they venture far away from home, they'll realize their mistake. Although locally almost Euclidean, globally the spherical world is so different from the Euclidean plane that a triangle is not even the simplest possible polygon over there. There exist polygons with only two vertices, sides and angles on a sphere, as shown on page 99.

Back to the Euclidean plane, here is an idea: if the angles of every triangle add up to $180°$ over there, maybe a similar property holds for polygons with more angles. How about the one with four angles, the quadrilateral? Let us take another look at the quadrilaterals on page 63.

Homework Problem 5.41 *Divide each of the quadrilaterals on page 63 into two triangles.*

We see that it is always possible to divide a quadrilateral into two triangles. We also see that the sum of the angles of the quadrilateral is equal to the sum of the angles of the triangles it is divided into.

Homework Problem 5.42 *Compute* $180° + 180°$

a. using the abacus;
b. numerically on a sheet of paper;
c. taking two $180°$ *steps on a circle (you can use the protractor on page 141).*

Note that we just have proven the following

[44] $\alpha + \beta + \gamma = 180° \left(1 + \frac{S}{\pi r^2}\right)$, where r is the radius of the sphere, S is the area of the triangle and α, β, and γ are the sizes of its angles measured in degrees.

Theorem 5.2 *For any quad in the Euclidean 2D, the sum of its angles always equals* $360°$.

Recall that Definition 4.17 defines a rectangle as a quad with four equal angles.

Homework Problem 5.43 *Compute* $90° + 90° + 90° + 90°$
a. using the abacus;
b. numerically on a sheet of paper;
c. taking four $90°$ *steps on a circle (you can use the protractor on page 141).*

Let us bring the above three facts together:

a. all the angles of a rectangle are equal to one another by definition;

b. for any quadrilateral in the Euclidean plane, the sum of its angles is always $360°$;

c. $90° + 90° + 90° + 90° = 360°$.

We just have proven another

Theorem 5.3 *All the angles of any rectangle in the Euclidean plane are equal to* $90°$.

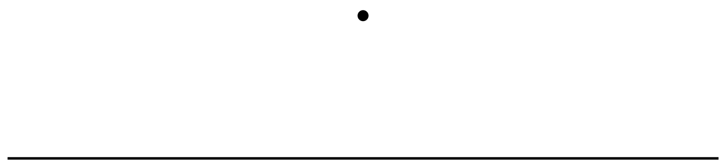
Traditionally, a rectangle is defined as a quadrilateral with four right angles. The author prefers Definition 4.17 because it is more visual and does not involve any numbers. Proving Theorem 5.3 is the price.

Homework Problem 5.44 *Using the method to construct a right angle with a compass and ruler developed in Homework Problem 4.133, draw a rectangle having sides 2" and 3" long.*

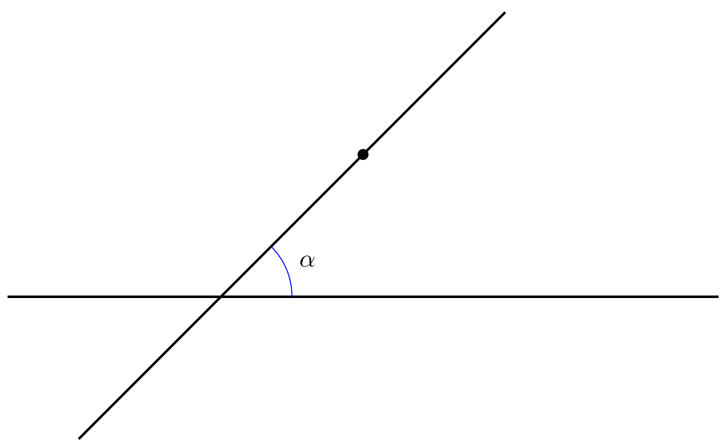
Homework Problem 5.45 *Prove that a rectangle is a parallelogram. Hint: extend its sides to straight lines and take another look at Proposition 4.3.*

Homework Problem 5.46 *Prove that a rhombus is also a parallelogram.*

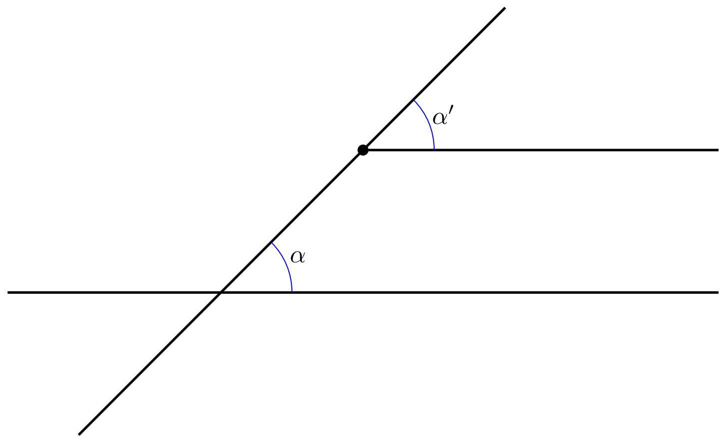
Theorem 5.1 gives us the tool needed to prove that the construction at the end of Lesson 4.8 indeed renders a straight line parallel to the given one and passing through the given point.



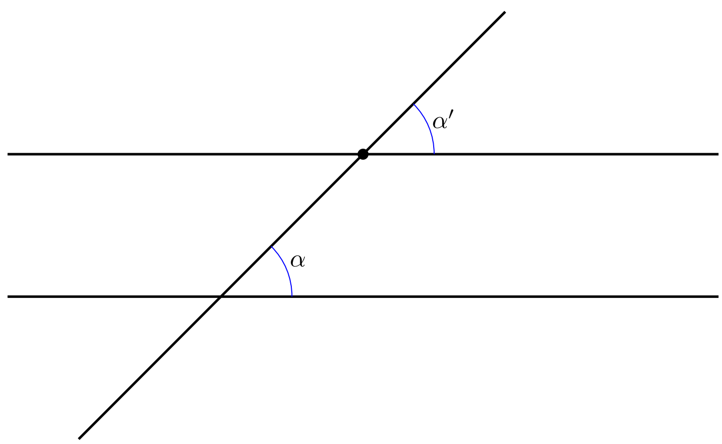
Recall that we first draw a straight line that passes through the given point and intersects the original line. We call α the angle they form.



At the next step, we use the technique developed in Example 4.4 to draw the angle α' equal to α, having the marked point as a vertex and the slanted upper ray as a side.

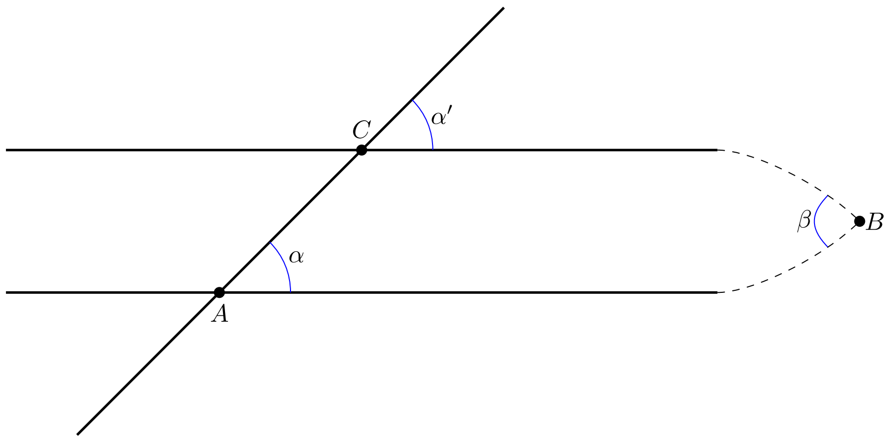


Finally, we extend the horizontal side of the angle α' to the left, getting the line our intuition is eager to accept as the one we were looking for.



However, the question if we really get parallel lines this way remains open. That is, until now.

Suppose that the straight lines in consideration are not really parallel. Then they do intersect, although possibly very far away, say, on the right, as shown on the below picture. Let us call the angle they form β. Let us also call the intersection points A, B, and C.



Consider the triangle ABC. The angle ACB, the one having point C as a vertex, is supplementary to the angle α' (recall Definition 4.10, if needed). Thus, $\measuredangle ACB + \alpha' = 180°$. But $\alpha' = \alpha$. So,

$$\measuredangle ACB = 180° - \alpha.$$

As we can see on the above picture, $\measuredangle BAC = \alpha$, $\measuredangle CBA = \beta$. The sum of the angles

$$\measuredangle BAC + \measuredangle CBA + \measuredangle ACB = 180°,$$

because, as proven in Theorem 5.1, it's true for any triangle in the Euclidean plane. Thus,

$$\alpha + \beta + 180° - \alpha = 180°.$$

Subtracting α from α yields 0, so we are left with the following equation:

$$\beta + 180° = 180°.$$

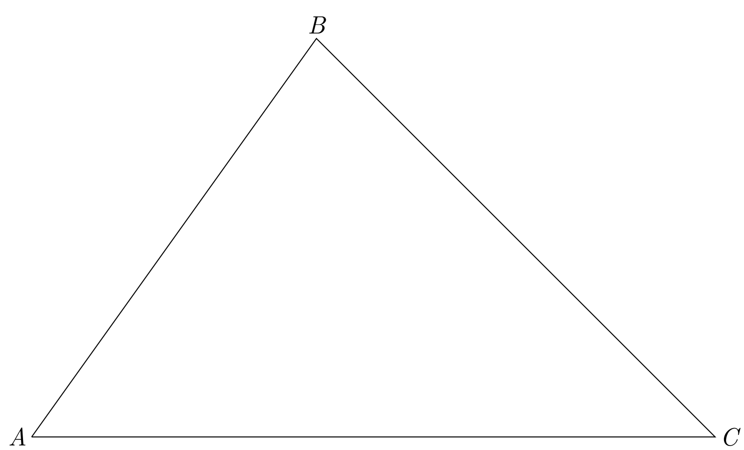
This can hold only if $\beta = 0$. However, $\beta = 0$ means that the sides of the angle CBA coincide. If the rays forming the angle coincide, then the straight lines they span coincide as well. This means that point C belongs to the line AB, because the latter coincides with the line CB. But point C doesn't belong to the line AB! This contradiction proves that our hypothesis was wrong in the first place. Thus, the lines we have constructed are parallel indeed. □

We shall get back to triangles, quadrilaterals and other polygons on the Euclidean plane, on the sphere and elsewhere later. They have many more fascinating properties for us to discover.

Homework Problem 5.47 *Without looking in the book, try to prove Theorem 5.1.*

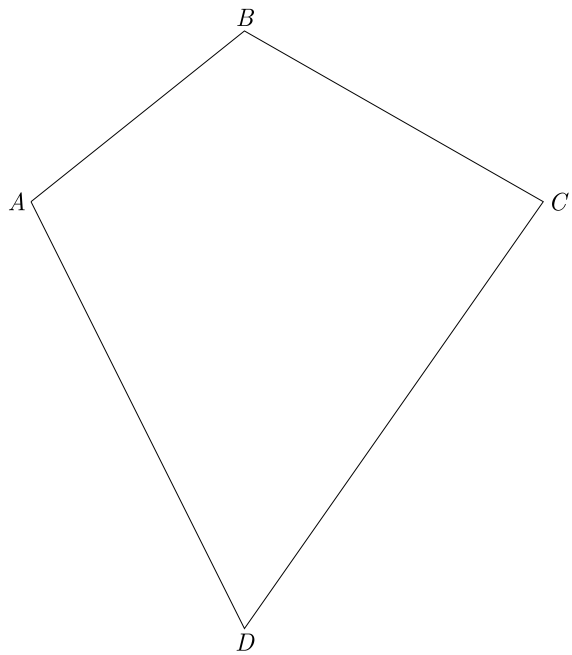
Homework Problem 5.48 *Without looking in the book, try to prove Theorem 5.2.*

Homework Problem 5.49 *Using a protractor, measure the angles $\angle A$ and $\angle B$ of the following triangle.*



Knowing that $\angle A + \angle B + \angle C = 180°$, find $\angle C$ without measuring it.

Homework Problem 5.50 *Using a protractor, measure the angles $\angle A$, $\angle B$ and $\angle C$ of the following quadrilateral.*



Knowing that $\angle A + \angle B + \angle C + \angle D = 360^\circ$*, find* $\angle D$ *without measuring it.*

Project 5.2 *For this project, you would need a grown-up relative or friend living in a time zone different from yours. Check with your parents what part of the day is good to give her/him a call. Determine the local time in your area. Counting meridians, find the time difference between your place and the one where the relative/friend lives. Does she/he live to the East or to the West of you? To figure out her/his time, do you need to add the time difference to your local time or to subtract it? Figure out the time at your relative/friend's place. Give her/him a call to check out the correctness of your computation.*

Homework Problem 5.51 *Can a triangle in the Euclidean plane have two right angles? How about a spherical triangle?*

Homework Problem 5.52 *Prove that diagonals split a rhombus into four right triangles of equal size.*

Homework Problem 5.53 *What is* $\infty + 10$*? Hint: can you accommodate 10 new visitors in a full hotel with infinitely many suites?*

5.5 Navigating the globe

Knowing your coordinates on the globe means knowing where you are on the Earth's surface. It is especially important at high seas for there are no highways, roads or streets, and very few, if any, passersby you could ask for directions.

Neglecting elevation, we can consider the globe as a 2D sphere. To pinpoint a location on a 1D surface, we need only one number. It's the distance from zero (positive or negative) on a straight line. On a circle, it's the angle formed by the rays originating at the center of the circle and passing through the zero mark and the point in consideration. On a 2D surface we need two numbers. The ones standardly used on a sphere are the longitude and latitude. The longitude tells you what meridian passes through the point where you are currently located. The latitude specifies what circle parallel to the equatorial plane passes through that point. The intersection of the meridian semi-circle with the latitude circle is your location.

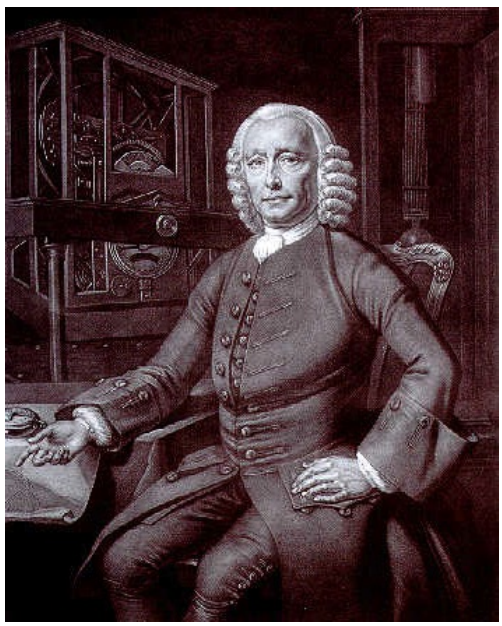
Nowadays, the longitude and latitude are instantly determined by the GPS, the satellite-based Global Positioning System. All you need is an unobstructed line of sight to four or more GPS satellites. Developed and built by the US Department of Defense (DOD) in 1973, the system is currently available for free to any person in the world equipped with a GPS receiver. It is the purpose of this lesson to give the child the first look at how navigation was done prior to the GPS arrival and how it can still be done if the GPS is unavailable.

Project 5.3 *Your home address and telephone number are the vital pieces of information the child must know by heart. If she/he doesn't know them yet, now is the time to learn. We'll need the address for this project anyways.*

Please find the search ("Fly to") box in the upper-left corner of the Google Earth program window. Ask the child to type in your home address and then press the "Enter" button. The program will zoom in on your home building. The coordinates will appear at the bottom of the picture in the $40°41'19''$ *N* $74°02'45''$ *W form. It's this pair of numbers we want to*
1. understand the meaning of and
2. learn how to find without the GPS or Google help.

Homework Problem 5.54 *What famous object is located at* $40°41'19'' N \quad 74°02'45'' W$*? Hint: use Google Earth to find out.*

Figuring out one's longitude used to be a hard problem that had impeded marine travel for a few thousand years. In fact, the problem was considered so hard that the British Parliament offered a prize of £20, 000 (comparable to $5 million in modern currency) for the solution. The riddle was solved in the 18th century by John Harrison[45], the English clockmaker who invented the first functioning *marine chronometer*, a very precise timepiece designed to keep its precision regardless the air temperature fluctuations, humidity, and mechanical interference caused by the waves tossing the ship.



John Harrison.[46]

Imagine that you have a chronometer set to the GMT, the Greenwich Mean Time (a.k.a. the UTC and Zulu time). If you need to figure out your longitude during daytime, all you have to do is to see what time your chronometer shows when the Sun is at the highest point in the sky for the

[45] 1693-1776

[46] Downloaded from **here**.

day. Suppose that it shows 3 P.M. GMT at the time of your local geographic noon. That means it takes the Earth three hours to spin the Greenwich meridian to the place of your current location. Thus your longitude is the third grid meridian line to the West of Greenwich!

Simple, isn't it? However, there exists a complication we have to overcome. Let us take another look at the globe. As you can see, the grid meridians are not marked with hours, but with degrees, like angles. The coordinates from Project 5.3 and Homework Problem 5.54 also use degrees and some other notations.

Historically, there have appeared tons of different notations, some using the same symbols for marking different objects, some others using different ways to call one and the same thing. Since their appearance is firmly imbedded in the history of human science, engineering, geographic discoveries, and warfare, the only option we have is to learn them all, although they are a mess!

A full circle is 360° (please take a look at the picture on page 141). It takes the Earth 24 hours to make one complete spin.

Homework Problem 5.55 *Compute* $\underbrace{15^\circ + \ldots + 15^\circ}_{\textit{24 times}}$

1. *using the abacus,*
2. *taking twenty four* 15° *steps on a circle (use the picture on page 141).*

Once the computation is finished, let the child know that there exists an easy way, called multiplication, to carry it out. We shall learn it a bit later.

The above exercise shows that to move a grid meridian in the place of the neighboring one, the Earth must rotate by the 15° angle. Here is another way to look at it.

Homework Problem 5.56 *Take any of the yellow clock pictures you have seen in this book and print out the corresponding page. Draw two rays originating at the center of the face of the clock and passing through two neighboring hour marks. Measure the angle you have constructed with a protractor.*

The above measurement gives you the angular value of one hour for a civilian clock. Military clocks divide the circle into 24 parts instead of 12.

Homework Problem 5.57 *Use the number you have obtained in Homework Problem 5.56 to find the angular value of one hour for a military clock without measuring the corresponding angle.*

The Greenwich meridian is marked with $0°$. The twelve grid meridians to the West of it are marked with the $15°W$, $30°W$, ... , $165°W$, and $180°$ signs (note that the last angle is not followed by a letter). The twelve grid meridians to the East of Greenwich are marked as $15°E$, $30°E$, ... , $165°E$, and $180°$. Since $180° + 180° = 360°$ (see Homework Problem 5.42),

$$180°W = 180°E.$$

For this reason, the $180°$ meridian, the one passing through the Wrangel Island in the Arctic Ocean, doesn't need a letter mark. Opposite to the Greenwich meridian, this one is as important for timekeeping. This meridian is chosen to be the International Date Line, or the IDL.

Imagine that you fly in an airplane from the East to the West along a latitude line. Every $15°$ of your flight, you need to wind your watch back by an hour. $360°$ or one full circle around the Earth later, you unwind your watch by 24 hours, or one full day, back. If you keep flying without any adjustments to your calendar, you will find yourself traveling "back in time". To prevent this phenomenon from turning the date count into an uncontrollable mess, people have decided to choose a line on the globe such that when you cross it from the East to the West you add a day to your calendar as a compensation. If you travel in the opposite direction, every $15°$ of your flight add an hour to your time. Then crossing the line, the International Date Line, unwinds your watch twenty four hours back, preventing you from getting into the "future". As we have mentioned above, the IDL largely coincides with the $180°$ meridian.

An airplane crossing the IDL between Alaska and Russia.[47]

Homework Problem 5.58 *Imagine that you are standing on an ice sheet facing the North Pole a few steps away from it in such a way that your feet are at the opposite sides of the* 180° *meridian. What is the time difference between your left and right foot?*

Homework Problem 5.59 *Would you like to get your birthday presents twice a year instead of just once? Check the IDL location on the globe to find out where you need to live for that.*

Homework Problem 5.60 *Japan is often called the Land of the Rising Sun, because it's a major country not too far to the West from the IDL. What countries have better grounds to claim the title?*

Homework Problem 5.61 *It takes the globe one hour to rotate* 15° *(see Homework Poblem 5.55). How long does it take the globe to rotate* 5°*?*

Homework Problem 5.62 *Suppose that at the time of your local geographic noon, your GMT-set chronometer shows 10:20 A.M. What is your longitude?*

[47]Downloaded from **here**.

The 1° arch of the equatorial circle is 69 miles long (69.17 to be precise). To determine our position on the planet with more precision, we need some smaller measuring units. They are called the *angular minute* (1') and *angular second* (1"). There are 60 angular minutes in the 1° angle and there are 60 angular seconds in the 1' angle.

$$1^\circ = 60' \qquad\qquad 1' = 60''$$

The 1" arch of the equatorial circle is 34 (33.8 to be precise) yards long. This precision is good enough for navigation, so we do not need smaller units. Unfortunately, the names of the units as well as their notations and values are rather inconvenient. The *minute* and *second* names are already taken by the time measuring units. The fact that there are 60 minutes in an hour and 60 seconds in a minute is similar to the corresponding angular formulae. However, the angular value of one hour is 30° for the 12-hour clock and 15° for the 24-hour one. Moreover, the notations 1' and 1" are taken, too. 1' is one foot, 1" is one inch. These are the units for measuring distance, not angles.

Minutes and seconds trace back to the counting system of Sumer, the first civilization in human history that appeared more than 6,000 BC. Just like ours, their numeral system was *place-value*, meaning that the value of a digit depended not only on the digit itself, but also on its position in a number. Our system is base ten, theirs was base sixty, or *sexagesimal*, hence 60 seconds in a minute, 60 minutes in 1° and one hour. As inconvenient as it currently seems, discarding this ancient way of counting in favor of a more modern one would be a disgrace to the history of human culture.

One more unit closely related to our discussion is the *nautical mile* (1 NM). 1 NM is the length of the 1' arch along any meridian. Let us compare the nautical mile to the "ground" one:

1 NM	=	1,852 m	=	2,025 yd	=	6,076'
1 mi	=	1,609 m	=	1,760 yd	=	5,280'

Homework Problem 5.63 *In the above table, what does the ' sign mean, feet or angular minutes? Why?*

Homework Problem 5.64 *Can we use the length of the 1' arch along a latitude line as a standard unit for measuring length? Why?*

Homework Problem 5.65 *What distance is greater, 2 NM or 4 km? How much greater?*

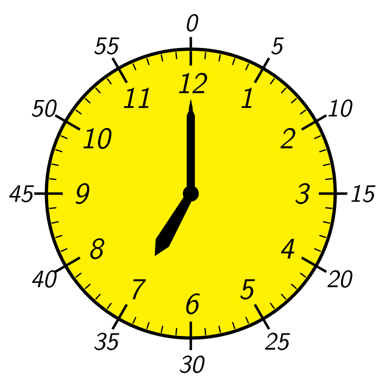
Let us take another look at Project 5.3 and the subsequent Homework Problem 5.54 on page 159. Now we understand what the notation

$$74°02'45''\ W$$

means. It means that the object we are trying to locate lies on the meridian 74 degrees, 2 angular minutes and 45 angular seconds to the West of Greenwich.

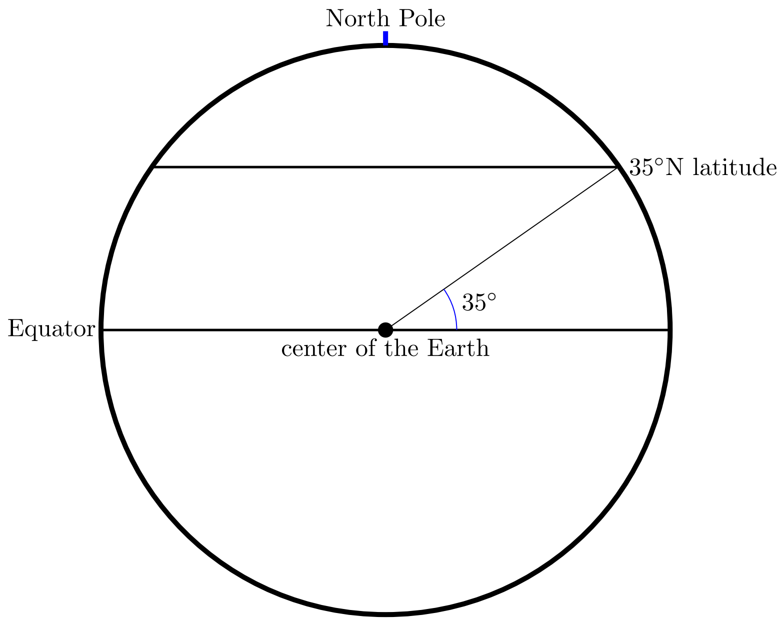
Homework Problem 5.66 *What big US city does the meridian pass through? Hint: the* 75° *meridian is a grid line.*

Homework Problem 5.67 *At your local geographic noon (the moment when the Sun is at its highest point in the sky) you GMT-set chronometer shows the following time A.M.*



(Both the minute and second hands point at zero minutes/seconds sharp.) What meridian passes through the point of your location? Hint: recall that one hour time difference corresponds to the 15° *angle between the corresponding meridians.*

The latitude is the angle between the direction to the center of the Earth and the equatorial plane.



Note that the horizontal lines on the above picture are actually circles on the globe, the lower one – the Equator and the upper one – the 35°N parallel. Furthermore, since the Equator is a circle on the Earth's surface, the center of the Earth doesn't lie on it; it just looks so on our 2D projection.

Homework Problem 5.68 *Find the above lines on the globe.*

The letter N in the above notations denotes the Northern Hemisphere. A similar parallel in the Southern Hemisphere has the 35°S latitude.

Homework Problem 5.69 *What is the radius of the* $90° S$ *latitude circle?*

Homework Problem 5.70 *What is the radius of the* $0°$ *latitude circle? Hint: the Earth's radius is approximately 3,959 mi.*

Now we understand what the notations of Project 5.3 and Homework Problem 5.54 mean. The $40°41'19'' N$ parallel is a circle on the Earth's surface located in the Northern Hemisphere such that for any point of this circle the angle between the direction to the center of the Earth and the equatorial plane is 40 degrees, 41 angular minutes, and 19 angular seconds. The $74°02'45'' W$ meridian is a half of the Great Circle connecting the North and

South Poles lying to the West of Greenwich such that the angle between it and the Greenwich meridian is 74°, 2 angular minutes, and 45 angular seconds. The coordinate lines in consideration intersect at one point, precisely locating the object of our interest.

Homework Problem 5.71 *Give a similar interpretation to the coordinates of your home.*

Homework Problem 5.72 *What is the difference between your local time and the GMT? How does that correspond to your home's longitude?*

Homework Problem 5.73 *Use Google Earth to find the coordinates of the Kennedy Space Center and Baikonur Cosmodrome. Use the coordinates to find the spaceports on the globe. Hint: find the nearest grid lines first.*

The author has seen the following problem in one of Martin Gardner's books. It was decades ago, so unfortunately it is now hard to recall which book it was. A great problem about higher dimensions, here it is the way the author remembers it.

Homework Problem 5.74 *Recall that a point can be considered as a zero-dimensional space (see Axiom 1 and Proposition 4.1, if needed). A point is flat, so we can consider it as a zero-dimensional Euclidean space.*

A one-dimensional sphere, that is a circumference, naturally rotates around a zero-dimensional Euclidean space, it's center, in 2D (see the picture on page 26).

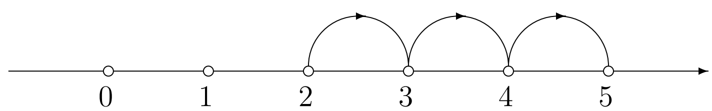
A 2D sphere, such as the surface of the globe, naturally rotates around a 1D Euclidean space, a straight line, in 3D (see the picture on page 90).

Can you guess the dimension of the Euclidean space a 3D sphere naturally rotates around in 4D?

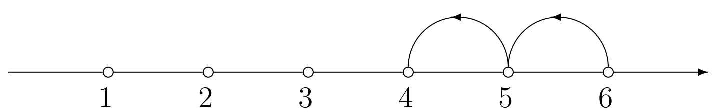
5.6 Arithmetic on a circle

Geometric foundation of addition and subtraction is the number line. Once we choose zero and one on a straight line, all we need to do to add an integer a to an integer b is to find a on the line and to make b unit steps to the right.

Let us recall the $2 + 3 = 5$ computation from Subsection 4.4.

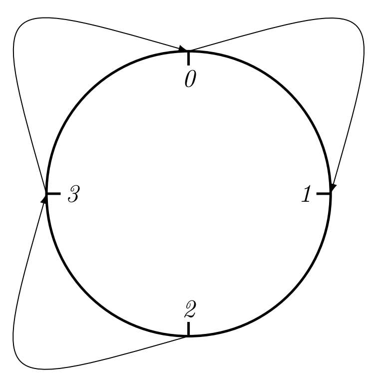


Similarly, subtracting b is equivalent to taking b unit steps to the left, like in the following $6 - 2 = 4$ example also considered in Subsection 4.4.



As we have noticed on page 135 and further in Subsection 5.2, various computations involving time also boil down to counting steps, this time not on a straight line, but on a circle divided into 12, 24, or 60 equal parts. In this subsection, we shall study the *mod* n arithmetic, that of a circle divided into n equal parts.

Example 5.3 *Let us compute* $2 + 3$ $(mod\ 4)$*. To do that, we start at* 2 *and take* 3 *steps in the clockwise direction, that is in the direction a clock's hands move.*



We end up at 1*, so* $2 + 3 \equiv 1$ $(mod\ 4)$*. Remember, the answer reads as "two plus three is equivalent to one modulo four."*

Homework Problem 5.75 *Use the face of a clock for counting steps in the first four problems. Draw the "5-hour" face to solve the last two.*

$$7+7 \equiv \qquad (mod\ 12)$$

$$3-6 \equiv \qquad (mod\ 12)$$

$$15-20 \equiv \qquad (mod\ 60)$$

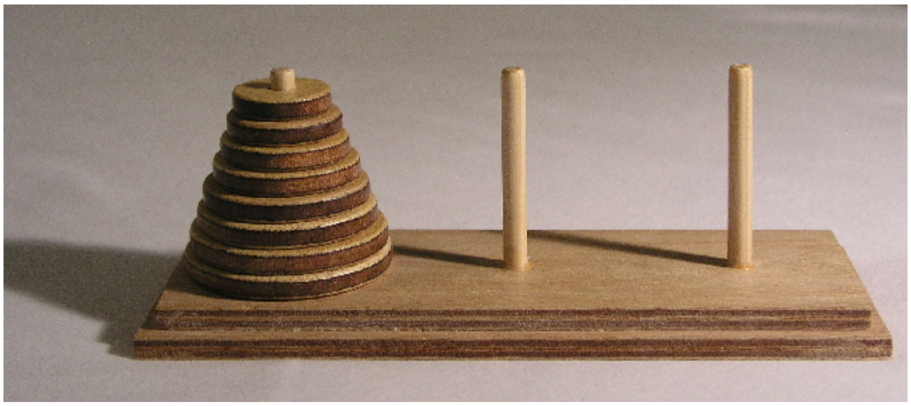
$$40+25 \equiv \qquad (mod\ 60)$$

$$2+3 \equiv \qquad (mod\ 5)$$

$$2-3 \equiv \qquad (mod\ 5)$$

6 The Hanoi tower

Hidden in the jungle near Hanoi, the capital city of Vietnam, there exists a Buddhist monastery where monks keep constantly moving golden disks from one diamond rod to another. There are 64 disks, all of different sizes, and three rods. Only one disk can be moved at a time and no larger disk can be placed on the top of a smaller one. Originally, all the disks were on one rod, say, the left one. At the end, they all must be moved to the right rod. When all the disks are moved, the world will come to an end. (No worries here, it will take the monks a few hundred billion years to complete the task.)



The Hanoi tower with eight disks.[48]

This tale was created by the French mathematician Édouard Lucas to promote the puzzle he had invented. Called the Hanoi tower, it's a great way to show the child an algorithm, an effective method for solving a problem expressed as a finite sequence of steps.

Project 6.1 *Buy the Hanoi tower puzzle. (We've got ours on the Internet for about $10, including shipping.) Let the child play with the puzzle for a week or two. Begin with two disks. Each time the little one learns how to move the previous number of disks, add one more. Get back to the lesson when the child can solve the puzzle with four or five disks without your help.*

Consider the original arrangement with all the disks on the left rod. Using a marker, number the disks 1 through n from the top to the bottom. Mark the rods A, B, and C from the left to the right. Let us first examine the situation $n = 2$. By this time, the child has already figured out the algorithm: the first disk goes from rod A to rod B, then the second disk is moved from rod A to rod C, and finally the first disk goes from rod B to rod C. Let us write the algorithm down as follows:

$$n = 2. \;\; 1AB \; 2AC \; 1BC$$

[48]Downloaded from `http://en.wikipedia.org/wiki/File:Tower_of_Hanoi.jpeg`.

This algorithm not only works, it works the best way possible. Any other algorithm for moving two disks from rod A to rod C will obviously take more moves. The child has come up with an optimal algorithm!

Consider the $n = 3$ case. To move the third disk from rod A to rod C, we first need to move the first two disks from rod A to rod B. But this is the previous problem with rod B replacing rod C!

Previous problem:	move two disks from A to C.
This problem:	move two disks from A to B.

The algorithm solving the latter problem is the one solving the former, but with the letters B and C replacing each other.

$$1AC\ 2AB\ 1CB$$

Once the first two disks are moved from rod A to rod B, the third disk goes from rod A to rod C.

$$1AC\ 2AB\ 1CB\ 3AC$$

The remaining problem is to move the first two disks from rod B to rod C. This is again done by our algorithm for $n = 2$, this time with the letters A and B switched (why?).

$$n = 3.\ \ 1AC\ 2AB\ 1CB\ 3AC\ 1BA\ 2BC\ 1AC$$

Quite obviously, our algorithm for $n = 3$ is as optimal as it was for $n = 2$.

Homework Problem 6.1 *Using the actual puzzle, check that the algorithm works for* $n = 3$*.*

Homework Problem 6.2 *Write the algorithm for* $n = 4$*. Use the puzzle to see if you got it right. Do the same for* $n = 5$ *and* 6*.*

The above is an example of a *recursive algorithm.* In order to solve the puzzle with n disks, we need to apply our solution procedure to the problem with $n - 1$ disks. To solve the latter, we run the algorithm for $n - 2$ disks, and so forth.

Let us see how many moves it takes to shift n disks from the left rod to the right. We needed three moves for $n = 2$.

$$N(2) = 3$$

For $n = 3$, we need $N(2)$ moves to shift the first two disks from A to B, then one more move to shift the third disk from A to C, and finally $N(2)$ more moves to shift the remaining two disks from B to C.

$$N(3) = N(2) + 1 + N(2) = 7$$

Homework Problem 6.3 *Find $N(4)$, $N(5)$, and $N(6)$.*

As we can see, in general

$$N(n) = N(n-1) + 1 + N(n-1)$$

Note 6.1 *The above formula holds when $N(n) = 2^n - 1$. Suppose that the monks, taking shifts to eat, sleep, etc., move a disk every second. They will need $2^{64} - 1$ seconds to complete their task. $2^{64} - 1$ seconds is a bit more than $584,542,046,090$ years. Buddhism is a religion founded by Siddhartha Gautama, who lived some time between 600 and 400 BC. According to the legend, the end of this world will not come in quite some time!*[49]

Homework Problem 6.4 *Find the optimal algorithm to solve the Hanoi tower puzzle with eight disks having the following initial arrangement: the eighth disk is on rod B, the seventh – on rod A, the first six disks – on rod C.*

Consider the Hanoi power puzzle with four rods (also known as Reve's puzzle). We can easily come with the following winning algorithm for n disks.

- Choose $0 < k < n$.

[49]The aging Sun will swell up to the current Earth's orbit much sooner, in 4 to 5 billion years from now. Although, due to the Sun's loss of mass, the actual Earth's orbit will move further away from the dying star, the rising temperature will still make our planet inhospitable to life. If we are still around as a species, we better find ourselves another abode!

- Use the optimal algorithm for three rods and k disks to move the first k disks to any rod except for the last one.

- Keeping the first k disks still, move the remaining $n-k$ disks to the last rod by using the optimal algorithm for three rods and $n-k$ disks.

- The problem is reduced to a similar one, but with k disks instead of n.

For any $k = 1, 2, \ldots, n-1$, the algorithm solves the puzzle, but is any of the above $n-1$ algorithms optimal? We don't know! The problem of finding an optimal solution for the Hanoi tower puzzle with more than three rods is still open. Quite often, a path from an elementary school problem to the frontier of the ongoing research is not too long.

Homework Problem 6.5 *Using a pencil as the forth rod, solve the Hanoi tower puzzle with four rods and five disks.*

Homework Problem 6.6 *Draw a clock with an appropriate number of "hours" to solve the following problems.*

$$3+3 \equiv \qquad (mod\ 4)$$

$$1-3 \equiv \qquad (mod\ 4)$$

$$3+3 \equiv \qquad (mod\ 6)$$

$$1-3 \equiv \qquad (mod\ 6)$$

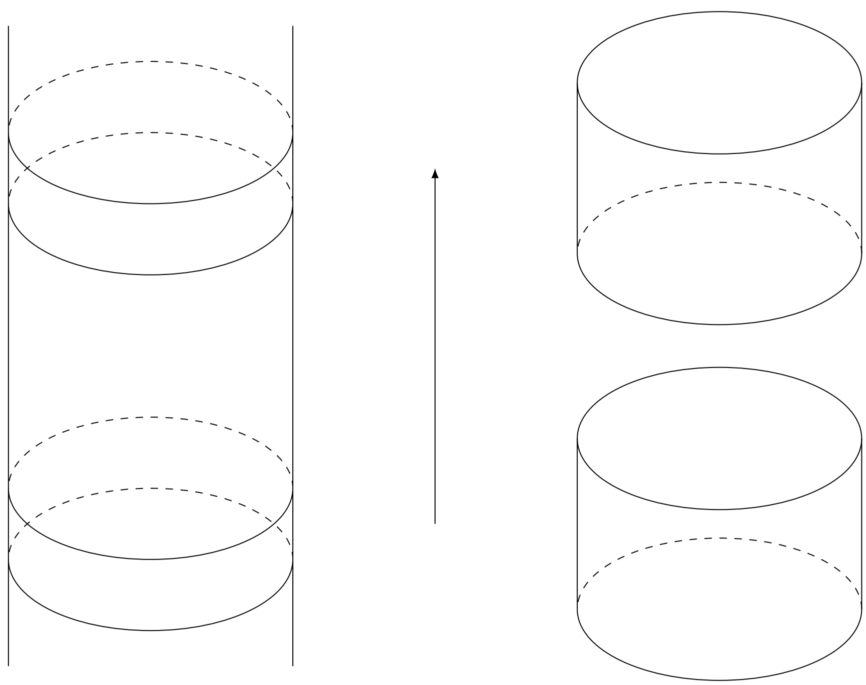
Homework Problem 6.7 *Go through the book one more time. Interrupt this algorithm when you feel that the little one has learned enough. Beware of infinite loops!*

7 Answers to some homework problems

3.2 The boy grew taller.

4.27 If they don't venture far away from home, they would. They can notice that their worlds are not flat, if they start from home along the generating circumference and keep traveling until they get back to the starting point moving in one and the same direction all the way.

4.28 A cylinder will stay in place, a finite cylinder will move as on the picture below.



a cylinder

a finite cylinder

4.49 Add a vertex, a point on the line that will split the line into two rays.

4.67 Realize number 12 as a volume of a Young diagram with three rows. If you manage to find such a diagram with equal number of boxes in each row, then the answer is positive. The diagram will also give the needed partition, 12=4+4+4.

4.79 Let us call the toys A, B, C, D, and E. A choice of two toys out of five is given by a two-letter combination, say, AB. The order of toys doesn't matter, so AB = BA. Let us agree to always write down letters in the alphabetical order. Here are the combinations: AB, AC, AD, AE, BC, BD, BE, CD, CE, and DE, ten altogether.

4.84 Recall Definition 4.4

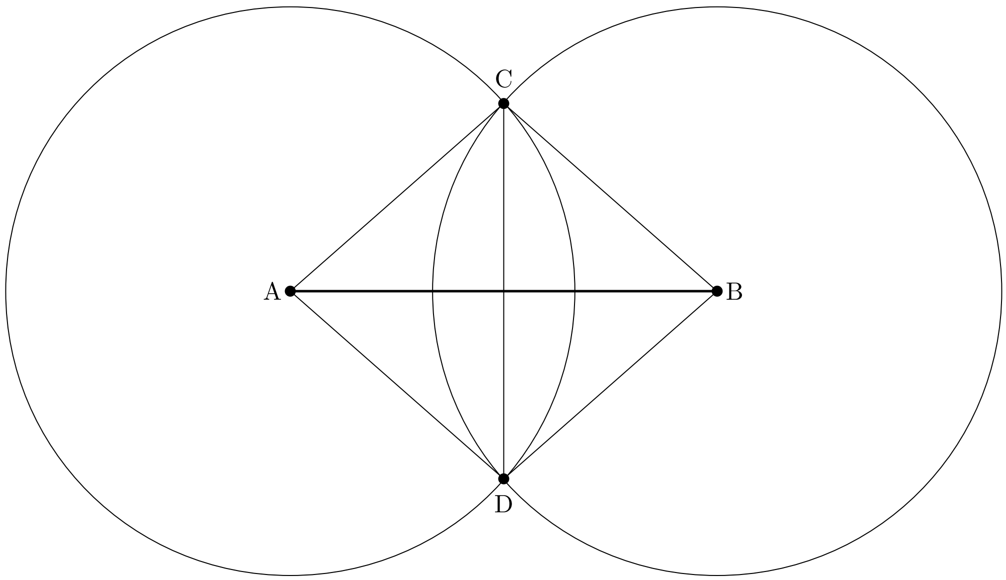
4.85 Screws, corkscrews, spiral staircases.

4.89 According to Euclid's Axiom 3, a helix is a straight line. It has enough symmetries, rotations combined with translations, to move any point of it to the position of any other.

4.115 You can't go West. No matter where you go from the North Pole, you go South.

4.121 From the point of view of the people living in the Northern Hemisphere, yes they do, although most likely they don't feel that way.

4.125 Let us take a compass and draw two circumferences of equal radius centered at the opposite ends of the given segment as on the picture below. Let us connect the points where the circumferences intersect each other to the endpoints of the segment.



Since $|AC| = |CB| = |BD| = |DA|$, we get a rhombus. According to Homework Problem 4.73, diagonals of a rhombus cut each other in halves.

4.131 You should launch the rocket from Kennedy Space Center because it is closer to the Equator than the Baikonur Cosmodrome. Using the Earth's rotation to propel rockets into space saves a lot of precious fuel.

4.135 Drop something.

4.151 Eight.

5.29

$$\text{EST} = \text{UTC - 5}$$

$$\text{CST} = \text{UTC - 6}$$

$$\text{MST} = \text{UTC - 7}$$

$$\text{PST} = \text{UTC - 8}$$

5.54 Statue of Liberty.

5.60 Russia and New Zealand.

6.2 $n = 4$. $1AB$ $2AC$ $1BC$ $3AB$ $1CA$ $2CB$ $1AB$ $4AC$ $1BC$ $2BA$ $1CA$ $3BC$ $1AB$ $2AC$ $1BC$

6.4 Use the optimal algorithm for moving six disks found in Homework Problem 6.2 to move the first six disks from rod C to rod A. Then move the eighth disk from rod B to rod C. Now the problem is reduced to the standard Hanoi tower puzzle with seven disks. To solve the latter, move the first six disks from rod A to rod B. Then move the seventh disk from rod A to rod C. Finally, employ the optimal algorithm for moving six disks found in Homework Problem 6.2 one last time, this time moving the disks from rod B to rod C.

References

[1] E. A. Abbott, *Flatland, A Romance of Many Dimensions.*

[2] A. Zvonkin, *Math for little Ones*, in Russian.