

Two Optimization Problems for Unit Disks

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Abstract

We present an implementation of a recent algorithm to compute shortest-path trees in unit disk graphs in $O(n \log n)$ worst-case time, where n is the number of disks.

In the minimum-separation problem, we are given n unit disks and two points s and t , not contained in any of the disks, and we want to compute the minimum number of disks one needs to retain so that any path connecting s to t intersects some of the retained disks. We present a new algorithm solving this problem in $O(n^2 \log^3 n)$ worst-case time and its implementation.

1 Introduction

In this paper we consider two geometric optimization problems in the plane where unit disks play a prominent role. For both problems we discuss efficient algorithms to solve them, provide an implementation of these algorithms, and present experimental results on the implementation.

The first problem we consider is computing a *shortest-path tree* in the (unweighted) intersection graph of unit disks. The input to the problem is a set \mathcal{D} of n disks of the same size, each disk represented by its center. The corresponding unit disk (intersection) graph has a vertex for each disk, and an edge connecting two disks D and D' of \mathcal{D} whenever D and D' intersect. An alternative, more convenient point of view, is to take as vertex set the set of centers of the disks, denoted by P , and connecting two points p and q of P whenever the Euclidean length $|pq|$ is at most the diameter of a disk. Given a root $r \in P$, the task is to compute a shortest-path tree from r in this graph. See Figure 1.

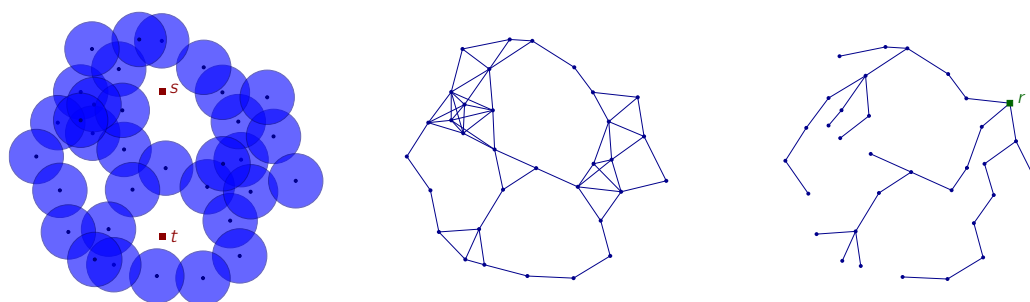


Figure 1 Left: unit disks and two additional points s and t . Middle: intersection graph of the disks. Right: a shortest-path tree in the graph.

The second problem we consider is the *minimum-separation problem*. The input is a set \mathcal{D} of n unit disks in the plane and two points s and t not covered by any disks of \mathcal{D} . We say that \mathcal{D} *separates* s and t if each curve in the plane from s to t intersects some disk of

^{*} Supported by the Slovenian Research Agency, program P1-0297 and project L7-5459.

\mathcal{D} . The task is to find the minimum cardinality subset of \mathcal{D} that separates s and t . See the right of Figure 1 for an example. Formally, we want to solve

$$\begin{aligned} \min \quad & |\mathcal{D}'| \\ \text{s.t.} \quad & \mathcal{D}' \subset \mathcal{D} \text{ and } \mathcal{D}' \text{ separates } s \text{ and } t. \end{aligned}$$

Unit disks are the most standard model used for wireless sensor networks; see for example [7, 9, 17]. Often the model is referred as UDG. This model provides an appropriate trade off between simplicity and accuracy. Other models are more accurate, as for example discussed in [11, 14], but obtaining efficient algorithm for them is much more difficult.

While unit disks give a simple model, exploiting the geometric features of the model is often challenging. Shortest paths in unit disk graphs are essential for routing and are a basic subroutine for several other more complex tasks. A somehow unexpected application of shortest paths in unit-disk graphs to boundary recognition is given in [16]. The minimum-separation problem and variants thereof have been considered in [3, 8, 15]. The problem is dual to the barrier-resilience problem considered in [1, 10, 12, 13]. It is not obvious that the minimum-separation problem can be solved optimally in polynomial time, and the known algorithm for this uses as a subroutine shortest paths in unit disk graphs. Thus, both problems considered in this paper are related and it is worth to consider them together.

Our contribution. We are aware of two algorithms to compute shortest-path trees in unit disk graphs in $O(n \log n)$ worst-case time: one by Cabello and Jeřič [4] and one Efrat, Itai and Katz [6]. Here we report on an implementation of a modification of the algorithm in [4], and compare it against two obvious alternatives. The only complex ingredient in the algorithm is computing the Delaunay triangulation, but efficient libraries are available for this. The algorithm of [6] would be substantially harder to implement and it has worse constants hidden in the O -notation.

As mentioned before, it is not obvious that the minimum-separation problem can be solved in polynomial time. A 2-approximation algorithm is given by Gibson, Kanade, and Varadarajan [8]. Cabello and Giannopoulos [3] provide an exact algorithm that takes $O(n^3)$ worst-case time and works for arbitrary shapes, not just disks. In this paper we improve this last algorithm to near-quadratic time for the special case of unit disks. The basic principle of the algorithm is the same, but several additional tools from Computational Geometry have to be employed to reduce the worst-case running time. We implement a variant of the new, near-quadratic-time algorithm and report on the experiments.

Assumptions. We will assume that *unit disk* means that it has radius $1/2$. Up to scaling the input data, this choice is irrelevant. However, it is convenient for the exposition because then the disks intersect whenever the distance between their centers is 1. The implementation and the experiments also make this assumption.

Henceforth P will be the set of centers of \mathcal{D} . All the computation will be concentrated on P . In particular, we assume that P is known. (For the shortest path problem, one could possibly consider weaker models based on adjacencies.)

We will work with the graph $G_{\leq 1}(P)$ with vertex set P and an edge between two points $p, q \in P$ whenever their Euclidean distance $|pq|$ is at most 1. In the notation we remove the dependency on P and on the distance. Thus we just use G instead of $G_{\leq 1}(P)$. For simplicity of the theoretical exposition we will sometimes assume that G is connected. It is trivial to adapt to the general case, for example treating each connected component separately. The implementation does not make this assumption.

67 **Organization of the paper.** In Section 2 we discuss the theoretical algorithms for both
 68 problems and their guarantees. In Section 3 we discuss the implementations and in Section 4
 69 we present our experimental results.

70 **2 Description of algorithms**

71 **2.1 Shortest-path tree in unit-disk graphs**

72 We describe here the algorithm of Cabello and Jejčič [4] to compute a shortest path tree in
 73 G from a given root point $r \in P$. As it is usually done for shortest path algorithms we use
 74 tables $\text{dist}[\cdot]$ and $\pi[\cdot]$ indexed by the points of P to record, for each point $p \in P$, the distance
 75 $d_G(s, p)$ and the ancestor of p in a shortest (s, p) -path.

76 The pseudocode of the algorithm, which we call UNWEIGHTEDSHORTESTPATH, is in
 77 Figure 3. We explain the intuition, taken almost verbatim from [4]. We start by computing
 78 the Delaunay triangulation $DT(P)$ of P . We then proceed in rounds for increasing values of
 79 i , where at round i we find the set W_i of points at distance exactly i in G from the source r .
 80 We start with $W_0 = \{r\}$. At round i , we use $DT(P)$ to grow a neighbourhood around the
 81 points of W_{i-1} that contains W_i . More precisely, we consider the points adjacent to W_{i-1}
 82 in $DT(P)$ as candidate points for W_i . For each candidate point that is found to lie in W_i ,
 83 we also take its adjacent vertices in $DT(P)$ as new candidates to be included in W_i . For
 84 checking whether a candidate point p lies in W_i we use a data structure to find a nearest
 85 neighbour of p in W_{i-1} . If the distance from p to its nearest neighbour w in W_{i-1} is smaller
 86 than 1, then the shortest path tree is extended by connecting p to w .

87 Cabello and Jejčič [4] show that the algorithm correctly computes the shortest-path tree
 88 from r . If for nearest neighbors we use a data structure that, for n points, has construction
 89 time $T_c(n)$ and query time $T_q(n)$, and the Delaunay triangulation is computed in $T_{DT}(n)$ time,
 90 then the algorithm takes $O(T_{DT}(n) + T_c(n) + nT_q(n))$ time. Standard tools in Computational
 91 Geometry imply that $T_{DT}(n) = O(n \log n)$, $T_c(n) = O(nq \log n)$ and $T_q(n) = O(\log n)$. This
 92 leads to the following.

93 ► **Theorem 2.1** (Cabello and Jejčič [4]). *Let P be a set of n points in the plane and let r
 94 be a point from P . In time $O(n \log n)$ we can compute a shortest path tree from s in the
 95 unweighted graph $G_{\leq 1}(P)$.*

96 It is clear that, when computing the shortest path tree from several sources, we only need
 97 to compute the Delaunay triangulation once.

98 **2.2 Minimum separation with unit-disk**

99 Cabello and Giannopoulos [3] present an algorithm for the minimum separation problem
 100 that in the worst-case runs in cubic-time. The algorithm has one feature that is both an
 101 advantage and a disadvantage: it works for any reasonable shapes, like segments or ellipses,
 102 and not just unit disks. This means that it is very generic, which is good, but it cannot
 103 exploit any properties of unit disks.

104 In this section we are going to describe an algorithm to solve the minimum separation
 105 problem *for unit disks* in roughly quadratic time. The improvement is based on 3 ingredients.
 106 The first ingredient is a reinterpretation of the algorithm of [3] for disks. In the original
 107 algorithm, we had to select a point inside each shape. For disks there is a natural, obvious
 108 choice, the center of the disk. This allows for a simpler description and interpretation of the
 109 algorithm. We provide the description in Section 2.2.1

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UNWEIGHTEDSHORTESTPATH( $P, r$ )
1  build the Delaunay triangulation  $DT(P)$ 
2  for  $p \in P$ 
3       $dist[p] = \infty$ 
4       $\pi[p] = \text{NIL}$ 
5   $dist[r] = 0$ 
6   $W_0 = \{r\}$ 
7   $i = 1$ 
8  while  $W_{i-1} \neq \emptyset$ 
9      build data structure for nearest neighbour queries in  $W_{i-1}$ 
10      $Q = W_{i-1}$  // candidate points
11      $W_i = \emptyset$ 
12     while  $Q \neq \emptyset$ 
13          $q$  an arbitrary point of  $Q$ 
14         remove  $q$  from  $Q$ 
15         for  $qp$  edge in  $DT(P)$ 
16              $w =$  nearest neighbour of  $p$  in  $W_{i-1}$ 
17             if  $dist[p] = \infty$  and  $|pw| \leq 1$ 
18                  $dist[p] = i$ 
19                  $\pi[p] = w$ 
20                 add  $p$  to  $Q$ 
21                 add  $p$  to  $W_i$ 
22      $i = i + 1$ 
23 return  $dist[\cdot]$  and  $\pi[\cdot]$ 

```

■ **Figure 2** Algorithm from [4] to compute a shortest path tree in the unweighted case.

110 The second ingredient is the efficient algorithm for shortest-path trees for the graph
 111 G . The third ingredient is a compact treatment of the edges of G using a few tools from
 112 Computational Geometry, namely range trees, point-line duality, and nearest-neighbour
 113 searches. This is explained in Section 2.2.2.

114 2.2.1 Generic algorithm specialized for unit disks

115 Let us first introduce some notation. Recall that s and t are the two points to separate.
 116 Each walk W in the graph $G = G_{\leq 1}(P)$ defines a planar curve in the obvious way: we
 117 connect the points of P with segments in the order given by W . We will relax the notation
 118 slightly and denote also by W the curve itself. For any spanning tree T of G and any edge
 119 $e \in E(G) \setminus E(T)$, let $cycle(T, e)$ be the unique cycle in $T + e$. Finally, for any walk in $G(P)$,
 120 let $\text{cr}_2(st, W)$ be the modulo 2 value of the number of crossings between the segment st and
 121 (the curve defined by) W .

122 The following property is implicit in [3] and explicit in [5]:

123 Let T be any spanning tree of G . The set of unit disks with centers in P separate s and t
 124 if and only if there exists some edge $e \in E(G) \setminus E(T)$ such that $\text{cr}_2(st, cycle(T, e)) = 1$.

A consequence of this is that finding a minimum separation amounts to finding a shortest

cycle G that crosses the segment st an odd number of times. Moreover, one can show that we can restrict our search to a very concrete family cycles, as follows. For each root r let us fix a shortest-path tree T_r from r in G . When there are many, the choice of T_r is irrelevant. Then, we can restrict our search to

$$\{cycle(T_r, e) \mid r \in P, e \in E(G) \setminus E(T_r)\}.$$

125 This follows from the co-called 3-path condition; see [3] for the ideas, and [appendix ??](#) for a
126 self-contained proof.

127 The values $\text{cr}_2(st, cycle(T_r, e))$ can be computed in constant amortized time per edge
128 with some easy bookkeeping. Consider a fixed tree T_r . For each point $p \in P$ we store $N[p]$
129 as the parity of the number of crossings of the path in T_r from r to p . When p is not the
130 root, the value $N[p]$ can be computed from the value of its parent $\pi[p]$ in T_r using that
131 $N[p] = N[\pi[p]] + \text{cr}_2(st, p\pi[p])$. In the algorithm we have written it this way (lines 4–6), but
132 one can of course compute the values at the time of computing the shortest path tree T_r .

We then have for each T_r

$$\begin{aligned} \forall pq \in E(G) \setminus E(T_r) : \quad & \text{cr}_2(st, cycle(T_r, pq)) = N[p] + N[q] + \text{cr}_2(st, pq) \pmod{2} \\ \forall pq \in E(T_r) : \quad & 0 = N[p] + N[q] + \text{cr}_2(st, pq) \pmod{2} \end{aligned}$$

133 because crossings that are counted twice cancel out modulo 2. (In particular, the path in T_r
134 from r to the lowest common ancestor of p and q is counted twice.) This implies that we can
135 just check for *all* edges pq of G whether the sum $N[p] + N[q] + \text{cr}_2(st, pq)$ is 0 modulo 2. The
136 final resulting algorithm, denoted as **GENERICMINIMUMSEPARATION**, is given in Figure 4.

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GENERICMINIMUMSEPARATION( $P, s, t$ )
1   $best = \infty$  // length of the best separation so far
2  for  $r \in P$ 
3      ( $dist[\ ], \pi[\ ]$ ) = shortest path tree from  $r$  in  $G(P)$ 
      // Compute  $N[\ ]$ 
4       $N[r] = 0$ 
5      for  $p \in P \setminus \{r\}$  in non-decreasing values of  $dist[p]$ 
6           $N[p] = N[\pi[p]] + \text{cr}_2(st, p\pi[p]) \pmod{2}$ 
7      for  $pq \in E(G(P))$ 
8          if  $N[p] + N[q] + \text{cr}_2(pq, st) \pmod{2} = 1$ 
9               $best = \min\{best, dist[p] + dist[q] + 1$ 
10 return  $best$ 

```

■ **Figure 3** Adaptation of the generic algorithm to compute the minimum separation for unit disks.

137 Let us look into the time complexity of the algorithm. For each point $r \in P$ we have to
138 compute a shortest-path tree in G . This can be done in $O(n \log n)$ in our case, as discussed
139 in Section 2.1. Then, for each edge pq of G some constant amount of work is done. Thus for
140 each point r we spend $O(n \log n + |E(G)|)$. This is cubic in the worst-case. We could get an
141 improved running time if we can treat all the edges of G compactly. This is what we explain
142 next.

2.2.2 Compact treatment of edges

From now on we will assume that s is the origin and t is the point $(0, \tau)$, with $\tau \geq 0$. Thus, the segment st is vertical and t is above s . The implementation also makes this assumption. A simple rigid transformation can be applied to the input to get to this setting.

We will use the data structure in the following lemma. It is essentially a multi-level data structure consisting of a 2-dimensional range tree T with a data structure for nearest neighbour at each node of the secondary structure of T .

► **Lemma 2.2.** *Let B be a set of n points with positive x -coordinates. We can preprocess B in $O(n \log^3 n)$ time such that, for any query point a with negative x -coordinate, we can decide in $O(\log^3 n)$ time whether the set $\{b \in B \mid ab \text{ intersects } \sigma \text{ and } |ab| \leq 1\}$ is empty. A similar data structure with the same guarantees handles queries to know whether $\{b \in B \mid ab \text{ does not intersect } \sigma \text{ and } |ab| \leq 1\}$ is empty.*

Proof. We are going to use point-line duality and range trees. These are standard concepts in Computational Geometry; see for example [2, Chapters 5 and 8]. We assume that the reader is familiar with the topic.

We use the following precise point-line duality: the non-vertical line $\ell \equiv y = mx + c$ is mapped to the point $\ell^* = (m, -c)$ and vice-versa. Let \mathbb{L} be the set of non-vertical lines. Let σ be the line segment st . Let σ^* be the set of points dual to non-vertical lines that intersect σ . Thus

$$\sigma^* = \{\ell^* \mid \ell \in \mathbb{L}, \ell \cap \sigma \neq \emptyset\}.$$

Since we assumed that $s = (0, 0)$ and $t = (0, \tau)$, in the dual space the set σ^* is the horizontal slab

$$\sigma^* = \{(m, -c) \in \mathbb{R}^2 \mid 0 \leq c \leq \tau\}.$$

For every point $p \in \mathbb{R}^2$, outside the y -axis, let L_p^* be the set of points dual to the lines through p that intersect σ :

$$L_p^* = \{\ell^* \mid \ell \in \mathbb{L}, p \in \ell, \text{ and } \sigma \cap \ell \neq \emptyset\}.$$

In the dual space, L_p^* is a segment with endpoints $(\varphi_1(p), 0)$ and $(\varphi_2(p), \tau)$, for some values $\varphi_1(p)$ and $\varphi_2(p)$ that are easily computable. Namely, $\varphi_1(p)$ is the slope of the line through p and $(0, 0)$ while $\varphi_2(p)$ is the slope of the line through p and $(0, \tau)$. The segment L_p^* is contained in the slab σ^* and has the endpoints on different boundaries of σ^* . Finally, define the mapping $\varphi(p) = (\varphi_1(p), \varphi_2(p))$. Thus, φ maps points in the plane with nonzero x -coordinate to points in the plane.

Let a be any point to the left of the y -axis and let b be a point to the right of the y -axis. The segment ab intersects σ if and only if L_a^* intersects L_b^* . Namely, an intersection of L_a^* and L_b^* is dual to the line through a and b . The segments L_a^* and L_b^* intersect if and only if the order of their endpoints on the boundaries of σ^* are reversed. Moreover, since a is to the left of the y -axis and b is to the right of the y -axis, if the segment ab intersects σ , then $\varphi_1(a)$, the slope of the line through a and $(0, 0)$, is smaller than $\varphi_1(b)$, the slope of the line through b and $(0, 0)$. Thus we have the following property:

$$ab \cap \sigma \neq \emptyset \iff \varphi_1(a) \leq \varphi_1(b) \text{ and } \varphi_2(a) \geq \varphi_2(b).$$

Given a point a to the left of the y axis, the set of points $b \in B$ with the property that ab intersects σ corresponds to the points b with $\varphi(b)$ in the bottom-right quadrant with apex $\varphi(a)$.

167 We can use a 2-dimensional range tree to store the point set $\varphi(B)$, where each point
 168 $b \in B$ is identified with its image $\varphi(b)$. Moreover, for each node v in the secondary level of
 169 the range tree, we store a data structure for nearest neighbours for the canonical set $P(v)$ of
 170 points that are stored below v in the secondary structure.

For any query $a \in A$, the points $b \in B$ such that ab intersects σ are obtained by querying the 2-dimensional range tree for the points of $\varphi(B)$ contained in the quadrant

$$\{(x, y) \mid \varphi_1(a) \leq x \text{ and } \varphi_2(a) \geq y\}.$$

171 This means that we get the set $\{b \in B \mid ab \text{ intersects } \sigma\}$ as the union of canonical subsets
 172 $P(v_1), \dots, P(v_k)$ for $k = O(\log^2 n)$ nodes in the secondary structure of the 2-dimensional
 173 range tree. For each such canonical subset $P(v_i)$ we query for the nearest neighbour of a .
 174 If for some v_i we get a nearest neighbour at distance at most 1 from a , then we know that
 175 $\{b \in B \mid ab \text{ intersects } \sigma \text{ and } |ab| \leq 1\}$ is non-empty. Otherwise the set is empty.

The construction time of the 2-dimensional range tree is $O(n \log n)$. Each point appears in $O(\log^2 n)$ canonical subsets $P(v)$. This means that $\sum_v |P(v)| = O(n \log^2 n)$, where the sum iterates over all nodes v in the secondary data structure. Since for each node v in the secondary level we build a data structure for nearest neighbours, which takes $O(|P(v)| \log |P(v)|)$, the total construction time is $O(n \log^3 n)$. For the query time, the standard 2-dimensional range tree takes $O(\log^2 n)$ time to find the $O(\log^2 n)$ nodes v_1, \dots, v_k such that

$$\bigcup_{i=1}^k P(v_i) = \{b \in B \mid ab \text{ intersects } \sigma\},$$

176 and then we need additional $O(\log n)$ time per node to query for a nearest neighbor.

177 Answering the queries for $\{b \in B \mid ab \text{ does not intersect } \sigma \text{ and } |ab| \leq 1\}$ is done similarly
 178 (and the same data structure works), we just have to query for 2 of the other quadrants. (We
 179 do not need to query for the other 3 quadrants because one of them is always empty.) ◀

From the theoretical perspective it would be more efficient to compute the union of the $|B|$ regions

$$\{(x, y) \in \mathbb{R}^2 \mid x < 0, |(x, y)b| \leq 1, (x, y) \text{ intersects } \sigma\}, \quad b \in B$$

180 and make point location there. Since the regions cannot have many crossings, good as-
 181 symptotic bounds can be obtained. However, such approach seems to be only of theoretical
 182 interest and the improvement on the overall result is rather marginal.

Consider now a fixed root r . Assume that we have computed the shortest path tree T_r and the corresponding tables $\pi[\]$, $\text{dist}[\]$ and $N[\]$, as discussed in Section 2.2.1. We group the points by their distance from r :

$$W_i = \{p \in P \mid \text{dist}[p] = i\}, \quad i = 0, 1, \dots, n$$

183 A standard property of BFS trees, that also holds here, is that all the distances from the
 184 root for any two adjacent vertices differ by at most 1. That is, the neighbours of a point
 185 $p \in P$ in G are contained in $W_{\text{dist}[p]-1} \cup W_{\text{dist}[p]} \cup W_{\text{dist}[p]+1}$. We will exploit this property.

We make groups L_i^j and R_i^j (where L stands for left and R for right) defined by

$$L_i^j = \{p \in P \mid \text{dist}[p] = i, p.x < 0, N[p] = j\}, \quad \text{where } j = 0, 1 \text{ and } i = 0, 1, \dots$$

$$R_i^j = \{p \in P \mid \text{dist}[p] = i, p.x > 0, N[p] = j\}, \quad \text{where } j = 0, 1 \text{ and } i = 0, 1, \dots$$

186 We are interested in edges pq of G such that $N[p] + N[q] + \text{cr}_2(st, pq) = 1 \pmod{2}$. Up
 187 to symmetry (exchanging p and q), this is equivalent to pairs of points (p, q) in one of the
 188 following two cases:

189 ■ for some $i \in \mathbb{N}$ and some $j \in \{0, 1\}$, we have $p \in L_i^j \cup R_i^j$, $q \in L_i^{1-j} \cup R_i^{1-j} \cup L_{i-1}^{1-j} \cup R_{i-1}^{1-j}$,
 190 $|pq| \leq 1$, and pq does not cross st ;

191 ■ for some $i \in \mathbb{N}$ and some $j \in \{0, 1\}$, we have $p \in L_i^j \cup R_i^j$, $q \in L_i^j \cup R_i^j \cup L_{i-1}^j \cup R_{i-1}^j$,
 192 $|pq| \leq 1$, and pq crosses st .

193 Each one of these cases can be solved efficiently. Up to symmetry, we have the following
 194 cases:

195 ■ If we want to search the candidates $(p, q) \in L_i^j \times L_{i'}^{1-j}$ (that cannot cross st since they
 196 are on the same side of the y -axis), we first preprocess $L_{i'}^{1-j}$ for nearest neighbours. Then,
 197 for each point p in L_i^j , we query the data structure to find its nearest neighbour q_p in
 198 $L_{i'}^{1-j}$. If for some p we get that $|pq_p| \leq 1$, then we have obtained a closed walk through
 199 r of length $i + i' + 1$ crossing st an odd number of times. The overall running time, if
 200 $m = |L_i^j| + |L_{i'}^{1-j}|$, is $O(m \log m)$.

201 ■ If we want to search the candidates $(p, q) \in L_i^j \times R_{i'}^j$, such that pq crosses st , we first
 202 preprocess $R_{i'}^j$ as discussed in Lemma 2.2 into a data structure. Then, for each point
 203 $p \in L_i^j$ we query the data structure (for crossing st). If we get some nonempty set, then
 204 we get a closed walk through r of length $i + i' + 1$ that crosses st an odd number of times.
 205 The overall running time, if $m = |L_i^j| + |R_{i'}^j|$, is $O(m \log^3 m)$.

206 ■ If we want to search the candidates $(p, q) \in L_i^j \times R_{i'}^{1-j}$ such that pq does not cross st , we
 207 first preprocess $R_{i'}^{1-j}$ as in Lemma 2.2 into a data structure. Then, for each point $p \in L_i^j$
 208 we query the data structure (for not crossing st). If we get some nonempty set, then we
 209 get a cycle of length (at most) $i + i' + 1$ that crosses st an odd number of times. The
 210 overall running time, if $m = |L_i^j| + |R_{i'}^{1-j}|$, is $O(m \log^3 m)$.

211 We conclude that each of the cases can be done in $O(m \log^3 m)$ time, where m is the number
 212 of points involved in the case. Iterating over all possible values i , it is now easy to convert
 213 this into an algorithm that spends $O(n \log^3 n)$ time per root r . We summarize the result we
 214 have obtained. This improves for the case of unit disks the previous, generic algorithm.

215 ► **Theorem 2.3.** *The minimum-separation problem for n unit disks can be solved in $O(n^2 \log^3 n)$
 216 time.*

217 **Proof.** Let P be the centers of the disks and, as before, consider the graph $G = G_{\leq 1}(P)$.
 218 For each root $r \in P$ we build the shortest-path tree and the sets $W_i, L_i^0, L_i^1, R_i^0, R_i^1$ in
 219 $O(n \log n)$ time. We then have at most n iterations where, in iteration i we spend $O(|W_i \cup$
 220 $W_{i-1}| \log^3 |W_i \cup W_{i-1}|)$ time. Since the sets W_i are disjoint, adding over i , this means that
 221 we spend $O(n \log^3 n)$ time per root $r \in P$.

222 Correctness follows from the foregoing discussion and the fact that the algorithm is
 223 computing the same as the generic algorithm. ◀

224 The work for each root is described schematically in Figure 5. A slightly more precise
 225 version of the algorithm is given as SEPARATIONUNITDISKS COMPACT in Figure 8 of the
 226 Appendix. As before, the variable *best* stores the length of the shortest cycle (or actually
 227 rooted closed walk) that we have found so far. We can start setting *best* = $n + 1$ at start. If
 228 eventually we finish with the value *best* = $n + 1$, it means that there is no feasible solution
 229 for the separation problem. When we consider a root r we are interested in closed walks
 230 rooted at r and length at most *best*. Since any closed walk through a vertex of W_i has length
 231 at least $2i$, we only need to consider indices i such that $2i < \text{best}$. Moreover (and this is
 232 not described in the algorithm), we can consider first the pairs that give walks for length $2i$
 233 first, like for example $L_i^0 \times L_{i-1}^1$ and then the ones that give length $2i + 1$, like for example
 234 $L_i^0 \times L_i^1$. If we use this order, as soon as we find a closed walk of inside the while-loop we
 235 can finish the work for the root r , and move onto the next root.


```

SEPARATIONUNITDISKSCOMPACTROOT( $P, s, t, r, best$ )
1  ( $dist[\ ], \pi[\ ]$ ) = shortest path tree from  $r$  in  $G_{\leq 1}(P)$ 
2  Compute the levels  $W_0, W_1, \dots$ 
3  for  $i = 0 \dots n$ 
4      Compute  $N[p]$  for each  $p$  in  $W_i$ 
5      Compute  $L_i^0, L_i^1, R_i^0, R_i^1$ 
6   $i = 1$ 
7  while  $2i < best$ 
    // within each side of the  $y$ -axis
8      search candidates in
         $L_i^0 \times L_{i-1}^1, L_i^1 \times L_{i-1}^0, R_i^0 \times R_{i-1}^1, R_i^1 \times R_{i-1}^0, L_i^0 \times L_i^1$  and  $R_i^0 \times R_i^1$ 
        // across  $y$ -axis crossing  $\sigma$ 
9      search candidates crossing  $\sigma$  in
         $L_i^0 \times R_{i-1}^0, L_i^1 \times R_{i-1}^1, L_{i-1}^0 \times R_i^0, L_{i-1}^1 \times R_i^1, L_i^0 \times R_i^0$  and  $L_i^1 \times R_i^1$ 
        // across  $y$ -axis not crossing  $\sigma$ 
10     search candidates not crossing  $\sigma$  in
         $L_i^0 \times R_{i-1}^1, L_i^1 \times R_{i-1}^0, L_{i-1}^1 \times R_i^0, L_{i-1}^0 \times R_i^1, L_i^0 \times R_i^1$  and  $L_i^1 \times R_i^0$ 
11      $i = i + 1$ 

```

■ **Figure 4** Work for each vertex in the news algorithm for minimum separation with unit disks.

3 Implementation and experiments

We have implemented the algorithms described using CGAL (.version

Computer, environment.

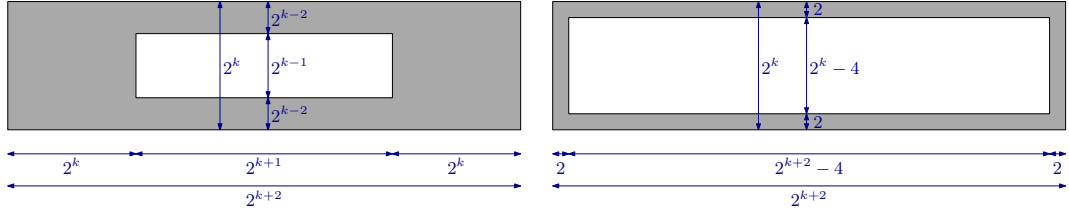
3.1 Shortest-path tree in unit-disk graphs

Alternative constructions. Let us mention two obvious alternatives that we use in our comparison.

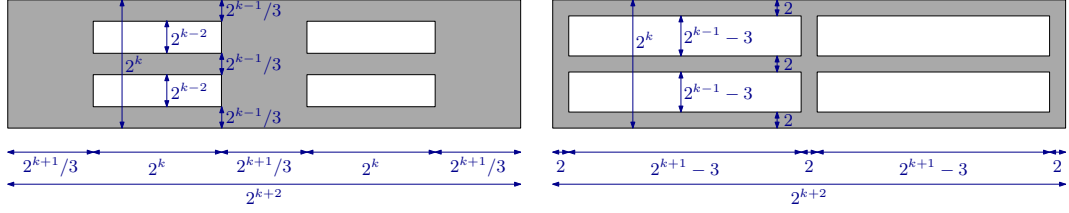
The first alternative is to build the graph $G = G_{\leq 1}(P)$ explicitly. Thus, for each pair of points p, q we check whether their distance is at most once and add an edge to a graph data structure. We can then use breadth-first-search (BFS) from the given root r . The preprocessing is quadratic, and the time spent to compute a shortest-path tree depends on the density of the graph G .

The second option we consider is to use a unit-length grid. Two points (x, y) and (x', y') are in the same grid cell if and only if $(\lfloor x \rfloor, \lfloor y \rfloor) = (\lfloor x' \rfloor, \lfloor y' \rfloor)$. We store all the points of a grid cell c in a list $\ell(c)$. The non-empty lists $\ell(c)$ are stored in a dictionary, where the bottom-left corner of the cell is used as key. We can then run some sort of BFS using this data. When processing a point p in a cell c , we have to treat all the points in c and its adjacent cells as candidate points. The preprocessing is linear, and the time spent to compute a shortest-path tree depends on the density of the graph G . The number of candidates that are checked is proportional to the number of edges of the graph.

Are perhaps
points deleted
from the list
once they are
level?



■ **Figure 5** Data generation with a small hole (left) and a large hole (right).



■ **Figure 6** Data generation with four small holes (left) and four large holes (right).

3.2 Minimum separation with unit-disk

4 Experimental results

Data was generated uniformly at random in polygons...

4.1 Shortest-path tree in unit-disk graphs

4.2 Minimum separation with unit-disk

5 Conclusions

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12 Two Optimization Problems for Unit Disks

300 **A** Missing proofs

301 3-path condition

302 **B** Full new algorithm for separation with unit disks

```

SEPARATIONUNITDISKSCOMPACT( $P, s, t$ )
1   $best = n + 1$  // length of the best separation so far
2  for  $r \in P$ 
3      ( $dist[\ ], \pi[\ ]$ ) = shortest path tree from  $r$  in  $G_{\leq 1}(P)$ 
      // Compute the levels  $W_i$ 
4      for  $i = 0 \dots n$ 
5           $W_i =$  new empty list
6      for  $p \in P$ 
7          add  $p$  to  $W_{dist[p]}$ 
      // Compute  $N[\ ]$  for the elements of  $W_i$  and
      // and construct  $L_i^0, L_i^1, R_i^0, R_i^1$ 
8       $N[r] = 0$ 
9      for  $i = 1 \dots n$ 
10         for  $p \in W_i$ 
11              $N[p] = N[\pi[p]] + cr_2(st, p\pi[p]) \pmod{2}$ 
12             if  $p$  to the left of the  $y$ -axis
13                 add  $p$  to  $L_i^{N[p]}$ 
14             if  $p$  to the right of the  $y$ -axis
15                 add  $p$  to  $R_i^{N[p]}$ 
16             Preprocess  $R_i^0$  and  $R_i^1$  as in Lemmas ??
17             Preprocess  $L_i^1$  and  $R_i^1$  for nearest neighbours
18      $i = 1$ 
19     while  $2i < best$ 
20         // length  $2i$ ; within each side of the  $y$ -axis
21         search candidates in  $L_i^0 \times L_{i-1}^1$ 
22         search candidates in  $L_i^1 \times L_{i-1}^0$ 
23         search candidates in  $R_i^0 \times R_{i-1}^1$ 
24         search candidates in  $R_i^1 \times R_{i-1}^0$ 
25         // length  $2i$ ; across  $y$ -axis crossing  $\sigma$ 
26         search candidates in  $L_i^0 \times R_{i-1}^0$  crossing  $\sigma$ 
27         search candidates in  $L_i^1 \times R_{i-1}^1$  crossing  $\sigma$ 
28         search candidates in  $L_{i-1}^0 \times R_i^0$  crossing  $\sigma$ 
29         search candidates in  $L_{i-1}^1 \times R_i^1$  crossing  $\sigma$ 
30         // length  $2i$ ; across  $y$ -axis not crossing  $\sigma$ 
31         search candidates in  $L_i^0 \times R_{i-1}^1$  not crossing  $\sigma$ 
32         search candidates in  $L_i^1 \times R_{i-1}^0$  not crossing  $\sigma$ 
33         search candidates in  $L_{i-1}^0 \times R_i^0$  not crossing  $\sigma$ 
34         search candidates in  $L_{i-1}^1 \times R_i^1$  not crossing  $\sigma$ 
35         // length  $2i + 1$ ; within each side of the  $y$ -axis
36         search candidates in  $L_i^0 \times L_i^1$ 
37         search candidates in  $R_i^0 \times R_i^1$ 
38         // length  $2i + 1$ ; across  $y$ -axis crossing  $\sigma$ 
39         search candidates in  $L_i^0 \times R_i^0$  crossing  $\sigma$ 
40         search candidates in  $L_i^1 \times R_i^1$  crossing  $\sigma$ 
41         // length  $2i + 1$ ; across  $y$ -axis not crossing  $\sigma$ 
42         search candidates in  $L_i^0 \times R_i^1$  not crossing  $\sigma$ 
43         search candidates in  $L_i^1 \times R_i^0$  not crossing  $\sigma$ 
44      $i = i + 1$ 
45 return  $best$ 

```

■ **Figure 7** New algorithm for minimum separation with unit disks.