AULA 2

Software for Algebra and Geometry Experimentation - SAGE

Limit of a variable:

Limiting Value of a Function

The notation $\lim_{x\to x_0} f(x) = \ell$ is used to denote **the limiting value of a function** $f(x)$ **as** x **approaches the value** x_0**, but** $x \neq x_0$. Note that the limit statement $\lim_{x\to x_0} f(x)$ is dependent upon values of $f(x)$ for x near x_0, **but not for** $x = x_0$. One must examine the values of $f(x)$ both for x_0^+ values (values of x slightly greater than x_0) and for x_0^- values (values of x slightly less than x_0). These type of limiting statements are written

$$\lim_{x\to x_0^+} f(x) \qquad \text{and} \qquad \lim_{x\to x_0^-} f(x)$$

and are called **right-hand and left-hand limits** respectively. There may be situations where (a) $f(x_0)$ is not defined (b) $f(x_0)$ is defined but does not equal the limiting value ℓ (c) the limit $\lim_{x\to x_0} f(x)$ might become unbounded, in which case one can write a statement stating that "no limit exists as $x \to x_0$ ".

O cálculo diferencial e seus conceitos principais.

Consider the series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (\frac{-1}{2})^k + \cdots . \qquad (2.1)$$

The sum of any even number $(2n)$ of the first terms of this series is

$$\begin{aligned} S_{2n} &= 1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \cdots + \tfrac{1}{2^{2n-2}} - \tfrac{1}{2^{2n-1}} \\ &= \tfrac{\frac{1}{2^{2n}} - 1}{-\frac{1}{2} - 1} \\ &= \tfrac{2}{3} - \tfrac{1}{3 \cdot 2^{2n-1}}, \end{aligned} \qquad (2.2)$$

$$\begin{aligned} S_{2n+1} &= 1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \cdots - \tfrac{1}{2^{2n-1}} + \tfrac{1}{2^{2n}} \\ &= \tfrac{-\frac{1}{2^{2n+1}} - 1}{-\frac{1}{2} - 1} \\ &= \tfrac{2}{3} + \tfrac{1}{3 \cdot 2^{2n}}, \end{aligned} \qquad (2.3)$$

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Writing (2.2) and (2.3) in the forms

$$\frac{2}{3} - S_{2n} = \frac{1}{3 \cdot 2^{2n-1}}, \qquad S_{2n+1} - \frac{2}{3} = \frac{1}{3 \cdot 2^{2n}}$$

we have

$$\lim_{n\to\infty} \left(\frac{2}{3} - S_{2n}\right) = \lim_{n\to\infty} \frac{1}{3 \cdot 2^{2n-1}} = 0,$$

and

$$\lim_{n\to\infty} \left(S_{2n+1} - \frac{2}{3}\right) = \lim_{n\to\infty} \frac{1}{3 \cdot 2^{2n}} = 0.$$

```
sage: S = lambda n: add([(-1)^i*2^(-i) for i in range(n)])
sage: S
<function <lambda> at 0x6fecde0cc80>
sage: RR(S(1)); RR(S(2)); RR(S(5)); RR(S(10)); RR(S(20))
1.00000000000000
0.500000000000000
0.687500000000000
0.666015625000000
0.666666030883789
```

You can see from the Sage example that the limit does indeed seem to approach $2/3$.

O cálculo diferencial e seus conceitos principais.

Division by zero:

$$\frac{0}{0} = INDETERMINADO$$

Care should be taken not to divide by zero inadvertently. The following fallacy is an illustration. Assume that

$$a = b.$$

Then evidently

$$ab = a^2.$$

Subtracting b^2,

$$ab - b^2 = a^2 - b^2.$$

Factoring,

$$b(a-b) = (a+b)(a-b).$$

Dividing by $a - b$,

$$b = a + b.$$

But $a = b$, therefore $b = 2b$, or, $1 = 2$. The result is absurd, and is caused by the fact that we divided by $a - b = 0$, which is illegal.

O cálculo diferencial e seus conceitos principais.

Infinitesimal:

Definition 2.3.1. *A variable v whose limit is zero is called an infinitesimal*

This is written

$$\lim_{v=0}, \text{ or}, \lim_{v\to 0},$$

The concept of infinity (∞)

$$\lim_{v=+\infty}, \text{ or}, \lim_{v\to+\infty}, \text{ or}, v \to +\infty.$$

$$\lim_{v=-\infty}, \text{ or}, \lim_{v\to-\infty}, \text{ or}, v \to -\infty.$$

$$\lim_{v=\infty}, \text{ or}, \lim_{v\to\infty}, \text{ or}, v \to \infty.$$

O cálculo diferencial e seus conceitos principais.

Infinitesimal:

```
sage: 0/0

---------------------------------------------------------------------------
ZeroDivisionError                         Traceback (most recent call last)
<ipython-input-6-fb5081ece3ac> in <module>()
----> 1 Integer(0)/Integer(0)


/opt/sagemath-8.1/src/sage/rings/integer.pyx in
sage.rings.integer.Integer.__div__
(build/cythonized/sage/rings/integer.c:12947)()
   1858         if type(left) is type(right):
   1859             if mpz_sgn((<Integer>right).value) == 0:
-> 1860                 raise ZeroDivisionError("rational division by
zero")
   1861             x = <Rational> Rational.__new__(Rational)
   1862             mpq_div_zz(x.value, (<Integer>left).value,
(<Integer>right).value)


ZeroDivisionError: rational division by zero
```

O cálculo diferencial e seus conceitos principais.

The concept of infinity (∞)

$\lim_{t=\infty} 1/t = \lim_{t=-\infty} 1/t = 0.$

```
sage: t=var('t')
sage: limit(1/t, t=infifnity)
---------------------------------------------------------------------------
NameError Traceback (most recent call last)
<ipython-input-8-0065da7a6b0a> in <module>()
----> 1 limit(Integer(1)/t, t=infifnity)

NameError: name 'infifnity' is not defined
sage: limit(1/t, t=Infifnity)
---------------------------------------------------------------------------
NameError Traceback (most recent call last)
<ipython-input-9-5e7008034e96> in <module>()
----> 1 limit(Integer(1)/t, t=Infifnity)

NameError: name 'Infifnity' is not defined

sage: limit(1/t, t=Infinity)
0
sage: limit(1/t, t=-Infinity)
0
```

O cálculo diferencial e seus conceitos principais.

Limiting value of a function $\lim_{x \to a} f(x) = A.$

$$f(x) = \frac{x^2+2}{3+\sqrt{x^2+1}}$$

```
sage: limit((x^2+1)/(3+sqrt(x^2+1)), x=2)
5/(sqrt(5) + 3)
sage: limit((x^2+1)/(3+sqrt(x^2+1)), x=Infinity)
+Infinity
```

$$f(x) = \frac{x^2+1}{2+x+3x^2}$$

```
sage: limit((x^2+1)/(2+x+3*x^2),x=2)
5/16
sage: limit((x^2+1)/(2+x+3*x^2),x=infinity)
1/3
sage: limit((x^2+1)/(2+x+3*x^2),x=Infinity)
1/3
```

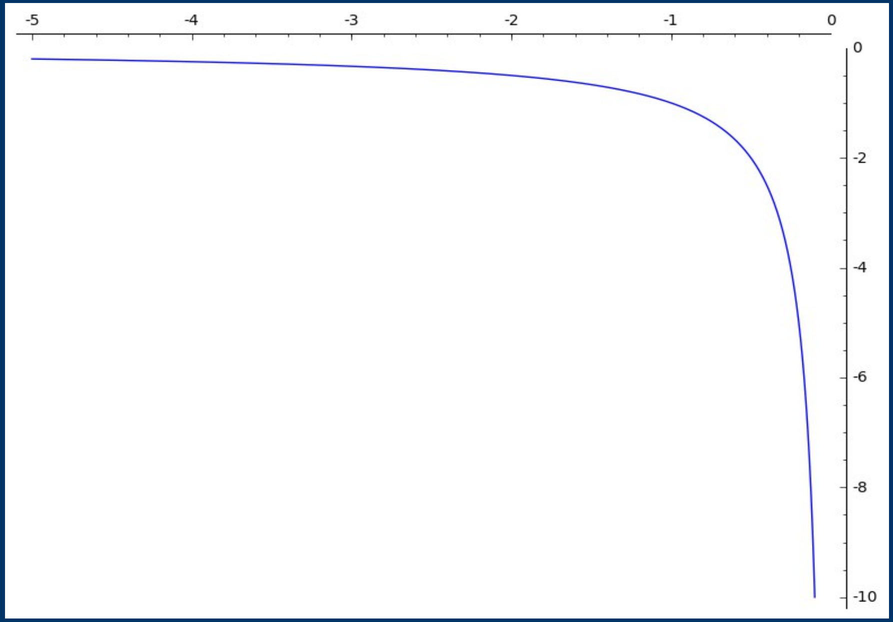
O cálculo diferencial e seus conceitos principais.

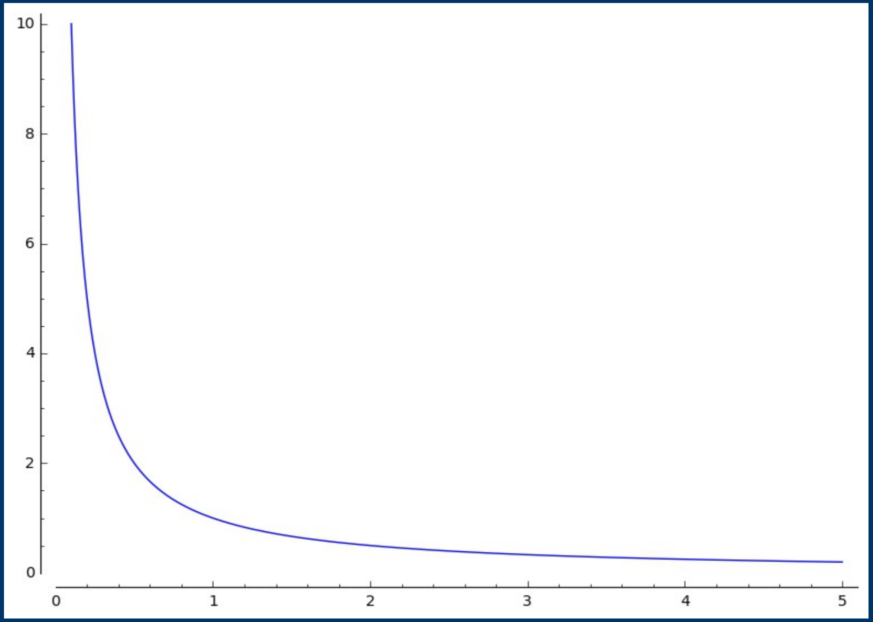
CONTINUOUS AND DISCONTINUOUS FUNCTIONS: the graph of a continuous function can have no “breaks.”

The function is said to be *discontinuous* for $x = a$ if this condition is not satisfied. For example, if

$$\lim_{x \to a} f(x) = \infty,$$

the function is discontinuous for $x = a$.





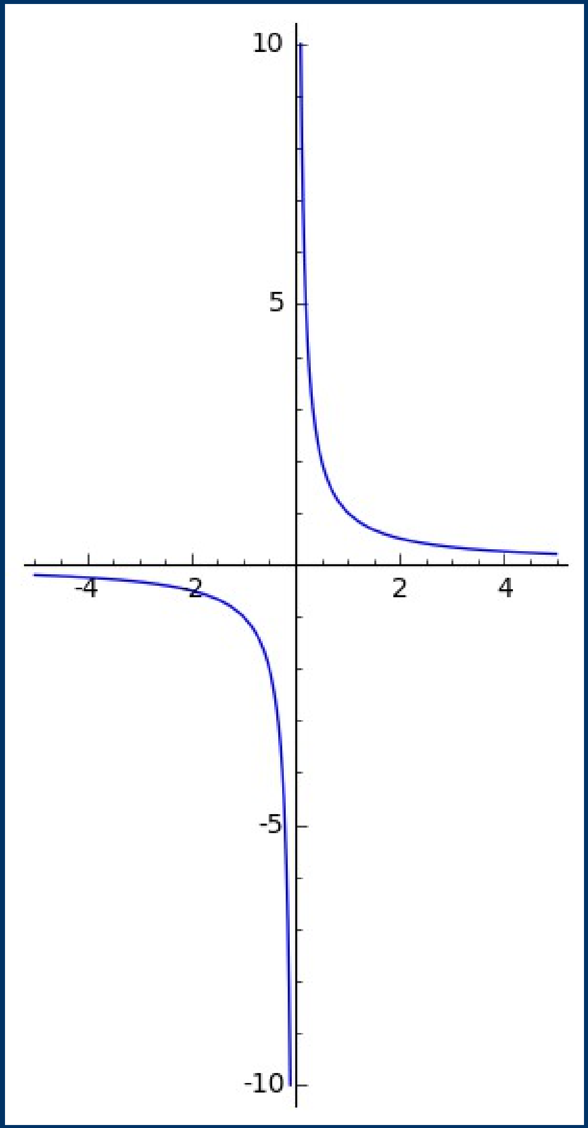
O cálculo diferencial e seus conceitos principais.

CONTINUOUS AND DISCONTINUOUS FUNCTIONS:

```
sage: t = var('t')
sage: P1 = plot(1/t, (t, -5, -0.1))
sage: P1
Launched png viewer for Graphics object consisting of 1
graphics primitive
sage: P2 = plot(1/t, (t, 0.1, 5))
sage: p2
-----------------------------------------------------
---------
NameError                            Traceback (most recent
call last)
<ipython-input-22-09e8d0db1c51> in <module>()
----> 1 p2

NameError: name 'p2' is not defined
sage: P2
Launched png viewer for Graphics object consisting of 1
graphics primitive
sage: show(P1+P2, aspect_ratio=1)
Launched png viewer for Graphics object consisting of 2
graphics primitives
sage: limit(1/t,t=0,dir="plus")
+Infinity
sage: limit(1/t,t=0,dir="minus")
-Infinity
sage: limit((x^2-4)/(x-2), x = 1)
3
sage: limit((x^2-4)/(x-2), x = 2)
4
```

$$f(x)=\frac{x^2-4}{x-2}$$



O cálculo diferencial e seus conceitos principais.

CONTINUOUS AND DISCONTINUOUS FUNCTIONS:

Eqn number	Written in the form of limits	Abbreviated form often used
(1)	$\lim_{x\to 0} \frac{c}{x} = \infty$	$\frac{c}{0} = \infty$
(2)	$\lim_{x\to\infty} cx = \infty$	$c \cdot \infty = \infty$
(3)	$\lim_{x\to\infty} \frac{x}{c} = \infty$	$\frac{\infty}{c} = \infty$
(4)	$\lim_{x\to\infty} \frac{c}{x} = 0$	$\frac{c}{\infty} = 0$
(5)	$\lim_{x\to-\infty} a^x, = +\infty$, when $a < 1$	$a^{-\infty} = +\infty$
(6)	$\lim_{x\to+\infty} a^x = 0$, when $a < 1$	$a^{+\infty} = 0$
(7)	$\lim_{x\to-\infty} a^x = 0$, when $a > 1$	$a^{-\infty} = 0$
(8)	$\lim_{x\to+\infty} a^x = +\infty$, when $a > 1$	$a^{+\infty} = +\infty$
(9)	$\lim_{x\to 0} \log_a \ x = +\infty$, when $a < 1$	$\log_a \ 0 = +\infty$
(10)	$\lim_{x\to+\infty} \log_a \ x = -\infty$, when $a < 1$	$\log_a(+\infty) = -\infty$
(11)	$\lim_{x\to 0} \log_a \ x = -\infty$, when $a > 1$	$\log_a \ 0 = -\infty$
(12)	$\lim_{x\to+\infty} \log_a \ x = +\infty$, when $a > 1$	$\log_a(+\infty) = +\infty$

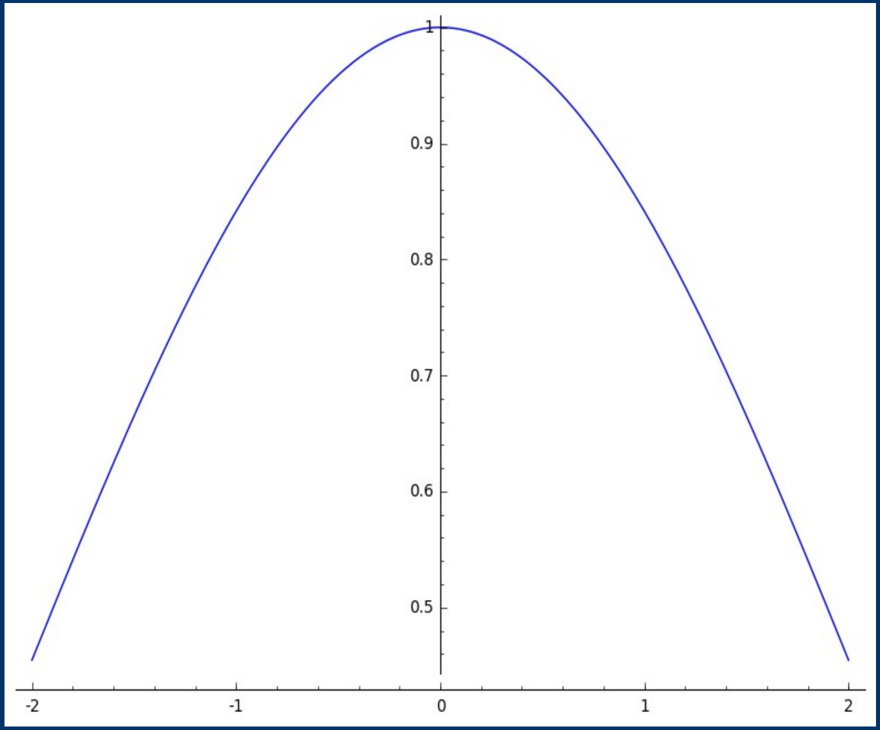
O cálculo diferencial e seus conceitos principais.

Show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

x	0.5000	0.2500	0.1250	0.06250	0.03125
$\frac{\sin(x)}{x}$	0.9589	0.9896	0.9974	0.9994	0.9998

```
sage: P = plot(sin(x)/x,-2,2)
sage: show(P)
Launched png viewer for Graphics object
consisting of 1 graphics primitive
sage: f = lambda x: sin(x)/x
sage: R = RealField(15)
sage: L = [1/2^i for i in range(1,6)]; L
[1/2, 1/4, 1/8, 1/16, 1/32]
sage: [R(x) for x in L]
[0.5000, 0.2500, 0.1250, 0.06250,
0.03125]
sage: [R(f(x)) for x in L]
[0.9589, 0.9896, 0.9974, 0.9994, 0.9998]
sage: limit(sin(x)/x,x=0)
32*sin(1/32)

sage: x = var('x')
sage: limit(sin(x)/x,x=0)
```



O cálculo diferencial e seus conceitos principais.

EXERCISES

$\lim_{x\to -2}\frac{x^2+1}{x+3} = 5.$

$\lim_{h\to 0}(3ax^2 - 2hx + 5h^2) = 3ax^2.$

$\lim_{x\to\infty}(ax^2 + bx + c) = \infty.$

$\lim_{k\to 0}\frac{(x-k)^2-2kx^3}{x(x+k)} = 1.$

$\lim_{x\to\infty}\frac{x^2+1}{3x^2+2x-1} = \frac{1}{3}.$

$\lim_{x\to\infty}\frac{3+2x}{x^2-5x} = 0.$

$\lim_{\alpha\to\frac{\pi}{2}}\frac{\cos(\alpha-a)}{\cos(2\alpha-a)} = -\tan\alpha.$

$\lim_{x\to\infty}\frac{ax^2+bx+c}{dx^2+ex+f} = \frac{a}{d}.$

$\lim_{z\to 0}\frac{a}{2}(e^{\frac{z}{a}} + e^{-\frac{z}{a}}) = a.$

$\lim_{x\to 0}\frac{2x^3+3x^2}{x^3} = \infty.$

References:

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