Problem 1

Following §17.2 in *Numerical Recipes*, let us denote the exact solution for an advance from x to $x+h$ by $y(x+h)$ and the two approximate solutions by y_1 (one step of h) and y_2 (for two steps of $h/2$), where each solution is given by:

$$y(x+h) = y_1 + h^5\phi + O(h^6) \tag{1}$$

$$y(x+h) = y_2 + \frac{1}{16}h^5\phi + O(h^6). \tag{2}$$

The difference between the two numerical estimates is an indicator of the truncation error,

$$\Delta \equiv y_2 - y_1 = -\frac{1}{16}h^5\phi + h^5\phi = \frac{15}{16}h^5\phi. \tag{3}$$

We can improve our estimate of our calculated value by cancelling out the h^5 term by combining as follows:

$$y(x+h) = y_2 + \frac{\Delta}{15} + O(h^6) \tag{4}$$

where we now have leading order error $O(h^6)$.

Note that for `rk4_step`, each step requires 4 function evaluations. Contrastingly, `rk4_stepd` requires 11 (saved when evaluating $f(x, y)$). Since we skip the first point, the total number of function evaluations for `rk4_step` is:

$$4 \times 199 = 796. \tag{5}$$

Therefore, the number of points needed to generate 796 function evaluations using `rk4_stepd` is

$$n = \frac{796}{11} \approx 72. \tag{6}$$

See `problem_1.txt` for the printed output.

Problem 2

a)

See `problem_2.txt` for printed output. I used the *Radau* method to solve this problem since we have a set of stiff equations (huge difference in half-lives) and ran the simulation over one half life of U–238. Note that the sum of all products after allowing it to run is 1, which is what we'd expect given conservation of mass.

b)

Analytically, we would expect the ratio of products to follow:

$$\frac{\text{Decayed Product}}{\text{Starting Product}} = \frac{1 - \exp(-\lambda \mathrm{t})}{\exp(-\lambda \mathrm{t})} \tag{7}$$

where $\lambda = \frac{\ln(2)}{T_{half-life}}$.

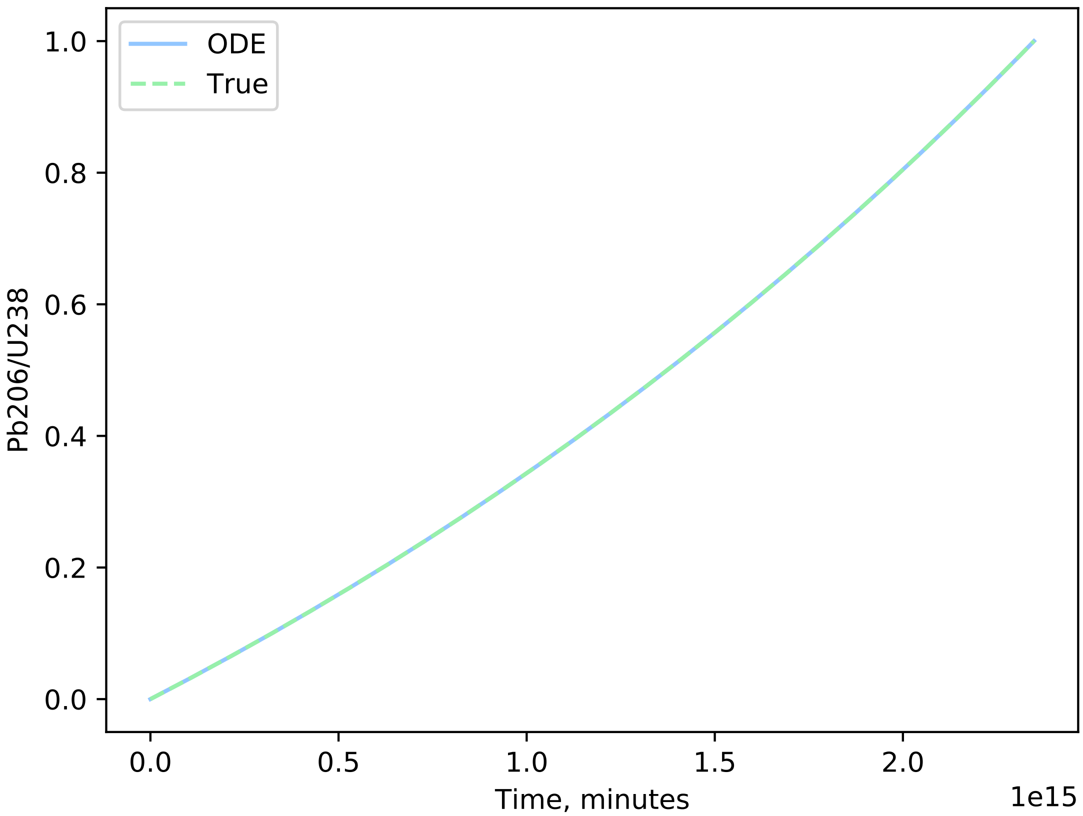

Figure 1: Ratio of Pb206 and U238 as a function of time, where the time elapsed is over one half-life of U238. Plot appears to agree with what we'd expect analytically, specifically, as we approach the half life of U238, the ratio approaches 1, which is what we'd expect to happen.

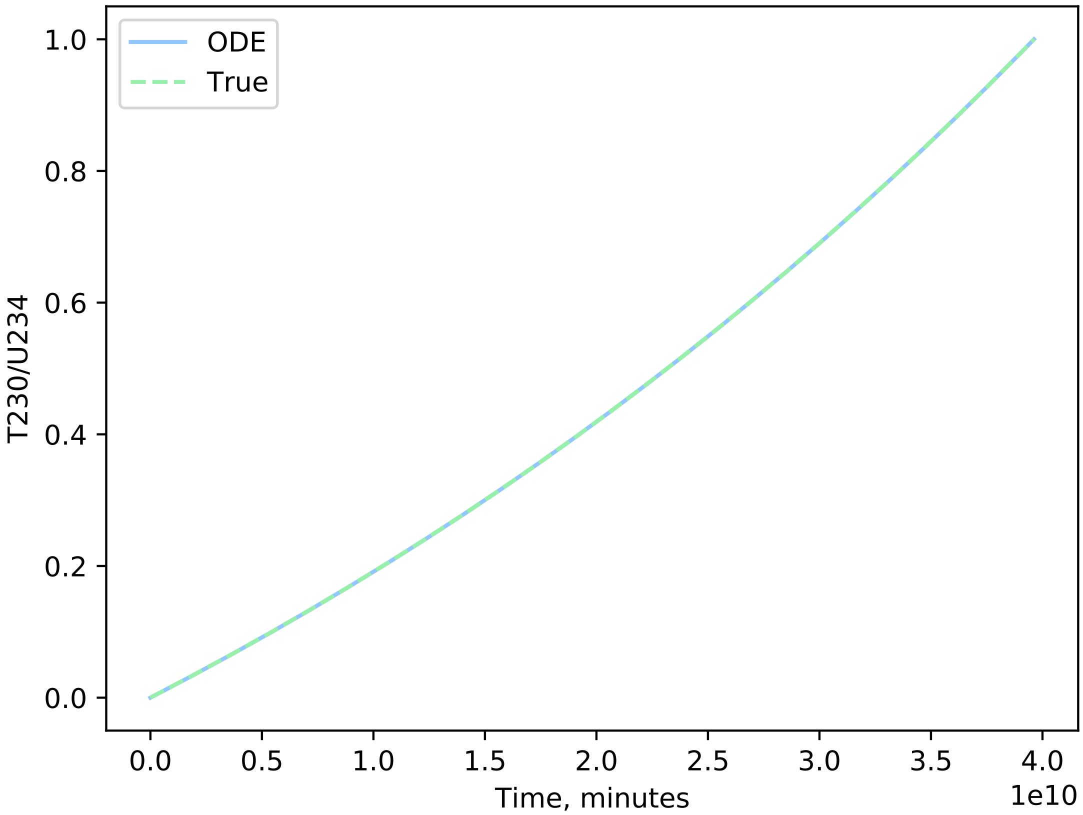

Figure 2: Ratio of Th230 and U234 as a function of time, where the time elapsed is over one half-life of U234. Notice that at t = $T_{U_{234}}$, the ratio of the products is 1, which is what we'd expect and the plot seems to agree with what we'd expect analytically. The plot is identical in shape as Figure 1, but the time scale is different.

Problem 3

a)

Define

$$z(x, y) = a((x - x_0)^2 + (y - y_0)^2) + z_0 \tag{8}$$

Expanding,

$$\begin{aligned} z(x, y) &= ax^2 - 2ax_0x + ax_0^2 + ay^2 - 2ay_0y + ay_0^2 + z_0 \\ &= a(x^2 + y^2) - 2ax_0x - 2ay_0y + ax_0^2 + ay_0^2 + z_0 \end{aligned}$$

Making the following substitutions:

$$B = -2ay_0,$$
$$C = -2ax_0,$$
$$D = ax_0^2 + ay_0^2 + z_0$$

the above becomes:

$$z(x, y) = a(x^2 + y^2) + By + Cx + D \tag{9}$$

where z is now linear in its parameters and as before, there are 4 of them, $a, B, C,$ and D.

b)

See printed output in `problem_3.txt` for best fit parameters.

c)

After determining the best fit parameters, I needed to estimate the noise matrix. For that reason, I plotted a histogram of the residuals to see how they were distributed. The results can be seen in Figure 3. Having now estimated the noise matrix, recall that the error in the model parameters can be easily computed using the following equation:

$$\sigma_m = \sqrt{\mathrm{diag}((A^T N^{-1} A)^{-1})} \tag{10}$$

Lastly, in order to get the focal length and its error, we note that by observation:

$$f = \frac{1}{4a} \tag{11}$$

where, by error propagation,

$$\sigma_f = f\frac{\sigma_a}{a}. \tag{12}$$

The results can be found in `problem_3.txt`.

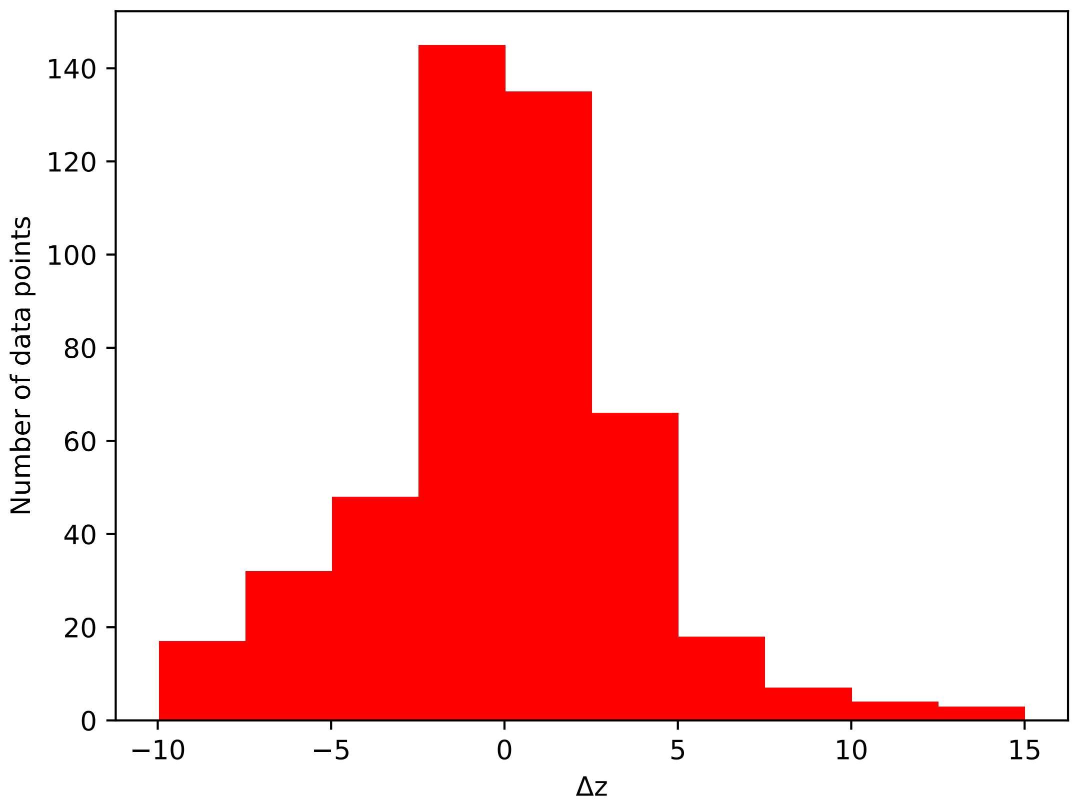

Figure 3: Distribution of residuals which appear to be normally distributed. Consequently, we can assume that the noise is uncorrelated, with variance equal to that of this histogram.