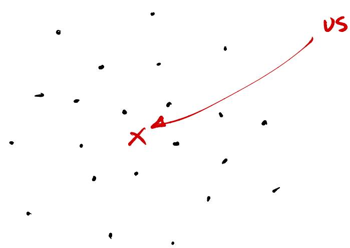
Problem 4

(a) Suppose we have a uniform distribution of stars of equal luminosity and we are located at the center of the region:



Recall that the flux observed by us a distance d away from a source w/ luminosity L is

$$S = \frac{L}{4\pi r^2}$$

Suppose we now make our telescope a factor of 4 more sensitive. At what distance could we now see sources that we couldn't see before compared to our original sensitivity?

$$\frac{S_{old}}{S_{new}} = \frac{\frac{L}{4\pi r_{old}^2}}{\frac{4L}{4\pi r_{new}^2}} = \frac{r_{new}^2}{4 r_{old}^2} = 1$$

another way to 1

$$\Rightarrow r_{new}^2 = 4 r_{old}^2$$

$$\Rightarrow r_{new} = 2 r_{old}$$

The number of sources compared to old is simply the ratio of volumes, e.g.

$$\frac{N_{new}}{N_{old}} \propto \frac{V_{new}}{V_{old}} = \frac{r_{new}^3}{r_{old}^3} = \frac{8 r_{old}^3}{r_{old}^3} = 8$$

$\Rightarrow$ $N_{new} = 8 N_{old}$, e.g. we can see 8x as many sources then before.

In general, we know that

$$S \propto \frac{1}{r^2} . \Rightarrow r \propto S^{-1/2}$$

However, from the above, we also know that

$$N \propto r^3$$

$$\Rightarrow N \propto \left(S^{-1/2}\right)^3 = S^{-3/2}$$

$$\boxed{\therefore \; N \propto S^{-3/2}}$$

(b) For the sake of this problem, let

$$N\,(S > 1\text{mJy}) = 100\, S_{\text{mJy}}^{-3/2} / \text{deg}^2$$

$$= \frac{180^2}{\pi^2} \cdot 100\, S_{\text{mJy}}^{-3/2}\ \text{sr}^{-1}$$

Recall: $\text{Sr} = \text{rad}^2$

$$= 328280\ S_{\text{mJy}}^{-3/2}\ \text{sr}^{-1}$$

We hit the confusion limit when there is $N = 1$ visible source every 30 beams. Recall that the beam is given by

$$\theta_{FWHM} = \frac{\lambda}{D} = \frac{c}{\nu D}$$

The above condition can be written as:

$$N \leq \frac{1}{30\Omega},$$

where $\Omega = \pi \left(\frac{\theta_{FWHM}}{2}\right)^2$ is the solid angle of the telescopes beam.

If N exceeds this limit, there will be too many sources in the beam and we'll get confused. If we take it one step further, we can set the minimum flux density that a source must have to be able to resolve it.

$$N \leq \frac{1}{30\Omega} = \frac{2}{15\pi\theta_{FWHM}^2} = \frac{2\nu^2 D^2}{15\pi c^2}$$

$$\Rightarrow 328280 \, S_{mJy}^{-3/2} \, Sr = \frac{2\nu^2 D^2}{15\pi c^2}$$

$$\Rightarrow S_{mJy} \geq \left(\frac{4.06\times10^{-7} \, \nu^2 D^2}{\pi c^2} \right)^{-2/3}$$

See Notebook for values.

(c) Using the radiometer equation

$$\sigma_c = \frac{SEFD}{\sqrt{\Delta\nu \tau}} \Rightarrow \boxed{\tau = \frac{SEFD^2}{\sigma_c^2 \Delta\nu}}$$

, SEFD is the system equivalent flux density, obtained by converting T_{sys} into a flux density.

SEFD is related to the telescopes system temp and gain via

$$T_{sys,j} = SEFD_j \cdot G_i \cdot \frac{A_j}{A_i}$$

$$\Rightarrow SEFD_j = \frac{T_{sys,j}}{G_i} \cdot \frac{A_i}{A_j}$$

What have I done here? I'm saying the the SEFD of the j^{th} telescope scales relative to the gain of the i^{th} telescope and ratio of effective areas. The reason I've done it this way is that we know GBT has $G_{GBT} = 2\ K/Jy$ w/ $A_{GBT} = 5500 m^2$.

From this, we can easily compute the SEFD of each array since we know $T_{sys} = 25\ K$ and can compute the effective area.

$$SEFD_j = \frac{5500\ m^2}{A_{e,j}} \cdot \frac{25\ K}{2\ K/Jy} \cdot 1\times10^3 \frac{mJy}{Jy} = \frac{6.875\times10^7}{\left(\frac{A_{e,j}}{m^2}\right)} mJy$$

where I've converted to mJy since those are the units I calculated in (b). Lastly,

$$\boxed{\tau = \frac{SEFD^2}{\sigma_c^2\ \Delta\nu} = \frac{4.72\times10^{15}}{\left(\frac{A_e}{m^2}\right)^2 \sigma_c^2} \Delta\nu^{-1}}$$

where σ_c is what we calculated in (b).

See Jupyter Notebook for values.

(d) We assume all sources have a spectral index of -0.8

$$\Rightarrow S_{8GHz} = \left(\frac{8}{1.4}\right)^{-0.8} S_{1.4GHz}$$

In (b), we derived an equation for the minimum flux density required to resolve a source. Since sources get dimmer at higher frequencies, we can scale this value down according to the above.

$$\Rightarrow S_{mJy,\,8GHz} \geq \left(\frac{8}{1.4}\right)^{-0.8} S_{mJy,\,1.4GHz}$$

From here, I'll just code up the results we're looking for. See notebook.