EYSM - example from the article

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Introduction

In this repo there is an example presented in the EYSM short paper "Resampling Methods in Conditional Independence Testing". The details are described in the article.

- `resampling.R` - implementation of the resampling methods
- `simulation_function.R`, `simulations.R` - simulation-related files
- `model.R` - implementation of the model described in the section Model description
- `cpp_functions.cpp` - helper functions

Short description

Conditional mutual information is defined as

$$CMI(p) = I(X, Y|Z) = \sum_{x,y,z} p(x, y, z) \log \frac{p(x, y, z)p(z)}{p(x, z)p(y, z)}.$$

The estimator of a vector of probabilities $(p(x, y, z))_{x,y,z}$ is a vector of fractions $(\hat{p}(x, y, z))_{x,y,z} = (n(x, y, z)/n)_{x,y,z}$, where $n(x, y, z) = \sum_{i=1}^{n} \mathbb{I}(X_i = x, Y_i = y, Z_i = z)$. We estimate conditional mutual information using a plug-in estimator, namely

$$CMI(\hat{p}) = \sum_{x,y,z} \hat{p}(x, y, z) \log \frac{\hat{p}(x, y, z)\hat{p}(z)}{\hat{p}(x, z)\hat{p}(y, z)}.$$

Lemma 1

If $X \perp\!\!\!\perp Y|Z$ we have that $2nCMI(\hat{p}) \xrightarrow{d} \chi^2_{(|\mathcal{X}|-1)(|\mathcal{Y}|-1)|\mathcal{Z}|}$.

Theorem 2

If the null hypothesis $H_0 : X \perp\!\!\!\perp Y|Z$ holds, then

$$P\left(\frac{1 + \sum_{b=1}^{B} \mathbb{I}(T \leq T_b^*)}{1 + B} \leq \alpha\right) \leq \alpha,$$

where $T = T(\mathbf{X}_n, \mathbf{Y}_n, \mathbf{Z}_n)$ and $T_b^* = T(\mathbf{X}_{n,b}^*, \mathbf{Y}_{n,b}^*, \mathbf{Z}_{n,b}^*)$, $(\mathbf{X}_n, \mathbf{Y}_n, \mathbf{Z}_n)$ is a sample and $(\mathbf{X}_{n,b}^*, \mathbf{Y}_{n,b}^*, \mathbf{Z}_{n,b}^*)$ is a resampled sample (CR or CP scenarios).

Tests

We consider three tests for testing conditional independence $H_0 : X \perp\!\!\!\perp Y|Z$:

- `asymptotic` - a test based on Lemma 1,
- `exact` - a test based on Theorem 2,
- `df estimation` - a test that uses the chi-squared distribution as a benchmark distribution, but adjusts the number of degrees of freedom d based on resampled samples, $\hat{d} = \frac{1}{B} \sum_b \widehat{CMI}_b^*$.

Model description

Joint probability in the model is given as

$$p(x, y, z_1, z_2, z_3, z_4) = p(y)p(x|y) \prod_{s=1}^{4} p(z_i|y),$$

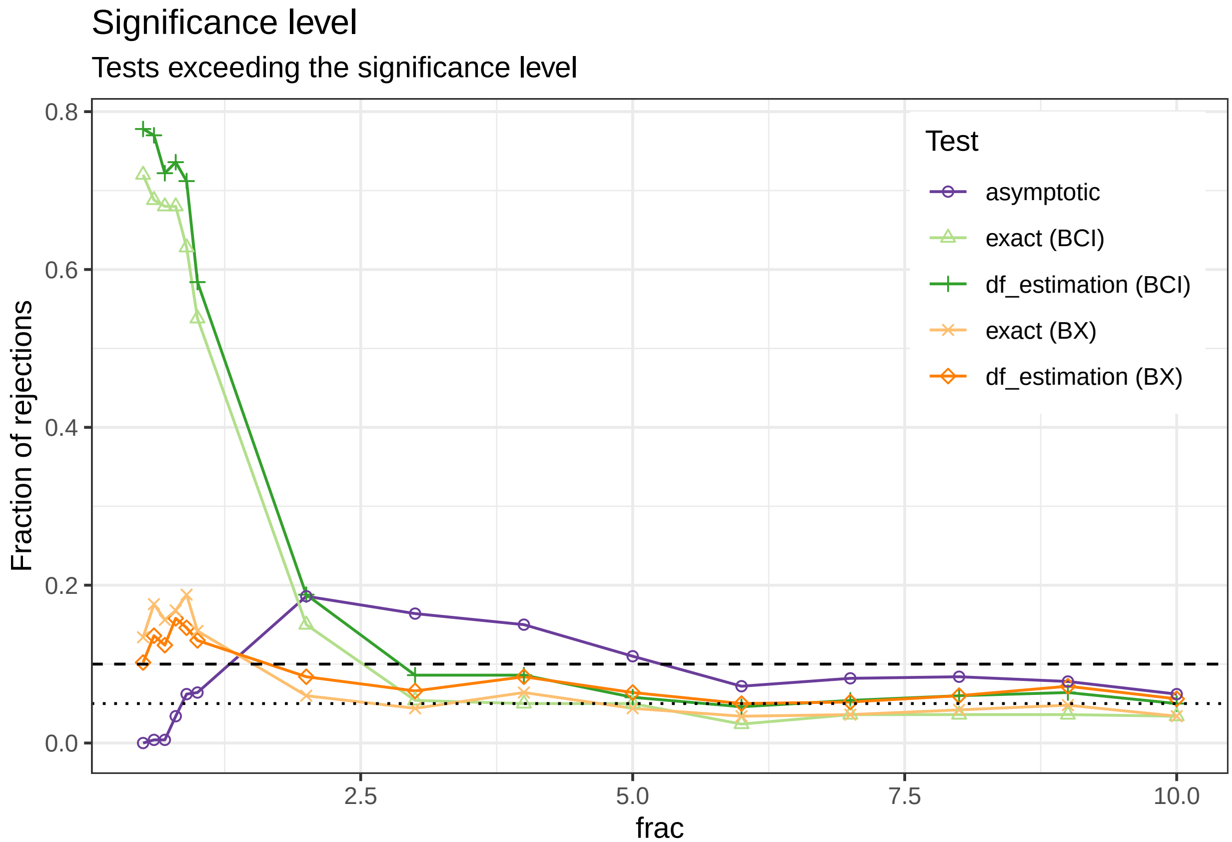
and is presented in Figure 1. Y is a Bernoulli random variable with probability of success equal to 0.5 and conditional distribution of $\tilde{X}$ and $\tilde{Z}_i$ for $i = 1, 2, 3, 4$ given $Y = y$ follows a normal distribution: $\tilde{X}|Y = y \sim N(y, 1)$, $\tilde{Z}_i|Y = y \sim N(\gamma^i y, 1)$ and $\gamma \in [0, 1]$ is a parameter. In this example, $\gamma = 0.5$. In order to obtain discrete variables from continuous $(\tilde{X}, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3, \tilde{Z}_4)$ we define

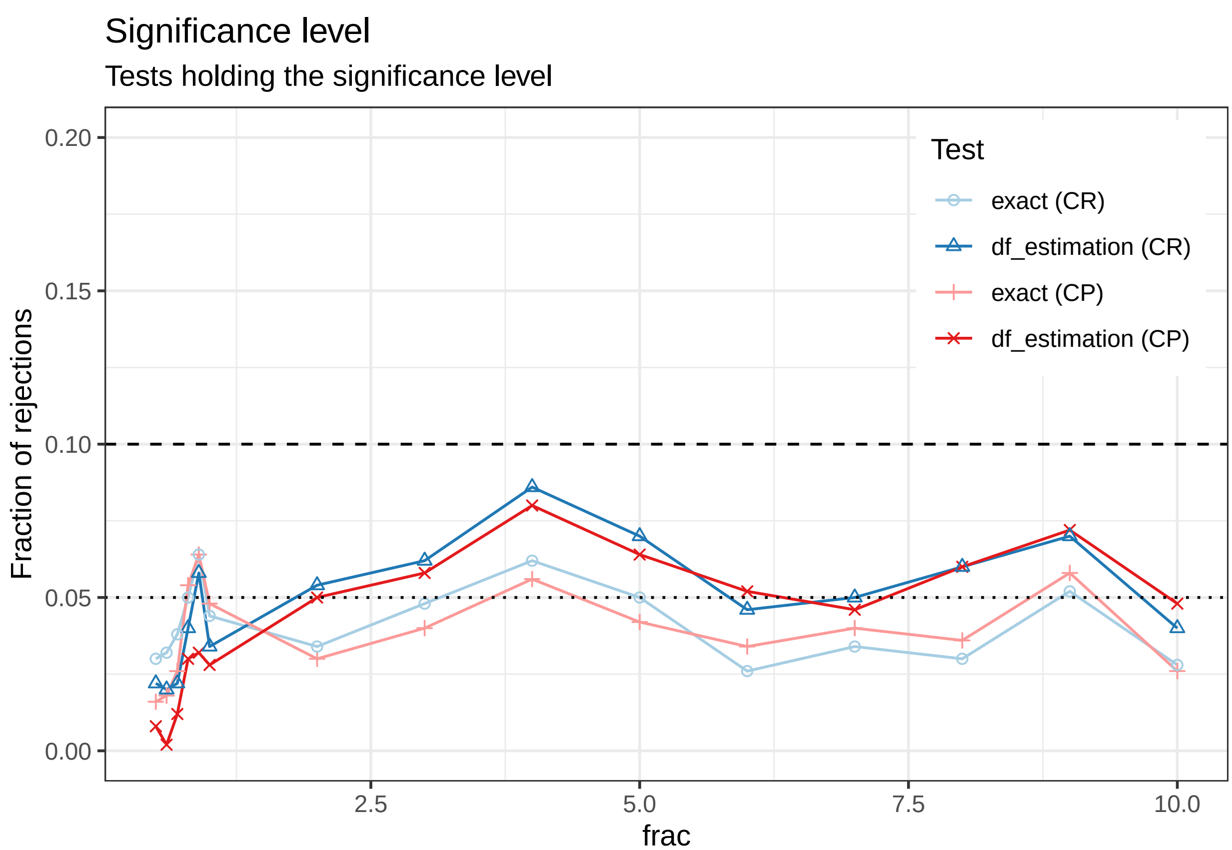
$$P(X = x|Y = y) = P\Big((-1)^x \tilde{X} \leq \frac{(-1)^x}{2} |Y = y\Big),$$

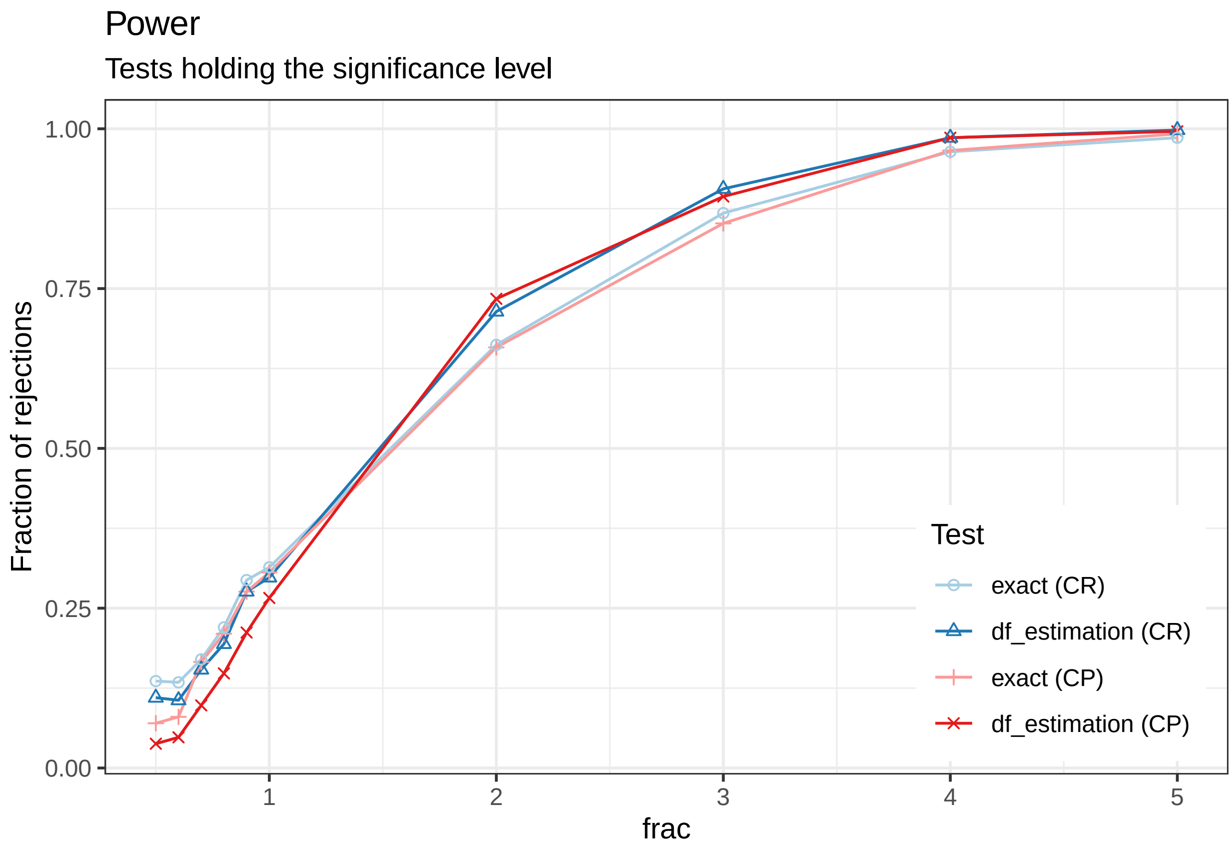
$$P(Z_i = z_i|Y = y) = P\Big((-1)^{z_i} \tilde{Z}_i \leq \frac{(-1)^{z_i}\gamma^i}{2} |Y = y\Big)$$

for $i = 1, 2, 3, , 4$, where $x, z_1, z_2, z_3, z_4 \in \{0, 1\}$. Thus $X|Y = y \sim Bern(\Phi((2y - 1)/2)$ and $Z_i|Y = y \sim Bern(\Phi(\gamma^i(2y - 1)/2))$. Variables X, Z_1, Z_2, Z_3, Z_4 are conditionally independent given Y but X an Y are not conditionally independent given Z_1, Z_2, Z_3, Z_4.

Plots


Significance level
Tests exceeding the significance level
Fraction of rejections
0.0
0.2
0.4
0.6
0.8
2.5
5.0
7.5
10.0
frac
Test
asymptotic
exact (BCI)
df_estimation (BCI)
exact (BX)
df_estimation (BX)


Significance level
Tests holding the significance level
Fraction of rejections
0.00
0.05
0.10
0.15
0.20
2.5
5.0
7.5
10.0
frac
Test
exact (CR)
df_estimation (CR)
exact (CP)
df_estimation (CP)


Power
Tests holding the significance level
1.00
0.75
0.50
0.25
0.00
Fraction of rejections
1
2
3
4
5
frac
Test
exact (CR)
df_estimation (CR)
exact (CP)
df_estimation (CP)