More proof Sketches Regarding the Closure of Limiting Probabilities in Sparse Random Hyper-graphs

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1 Preliminaries

$G^d(N,p)$ denotes the binomial model of random d-uniform hyper-graphs. Consider $d \geq 2$ fixed for the rest of this writing.

Theorem 1.1. *Let $c < (d-2)!$. Then, a.a.s all the components of $G^d(N, p(N))$, where $p(N) \sim c/N^{d-1}$, are either trees or unicycles.*

Proof. See [1] for $d = 2$ and [2] for the general case.

Está todo hecho en el modelo uniforme, pero la transferencia uniforme-> binomial es "sencilla"

□

Theorem 1.2. *Let $c < (d-2)!$ and let $p(N) \sim c/N^{d-1}$. Let δ_N be the random variable that counts the vertices in $G^d(N, p(N))$ belonging to some unicyclic component. Then there is some constant C such that $\mathrm{E}[\delta_n]$ is smaller than C for any N.*

Este teorema lo he visto para grafos en el artículo de Erdos de 1960 , y más completo en el libro de Karonski, por ejemplo. Para hypergrafos no lo he encontrado, pero igual es cosa de buscar mejor. Lo demuestro de todas formas.

Proof. Let $C_d(m)$ denote the number of connected labeled d-uniform hypergraphs with m edges and $n := (d-1)m$ vertices. It is satisfied that for big values of m:

$$C_d(m) \leq \frac{n^n}{e^{(m-n)}} \frac{1}{(d-2)!^m},$$

where the constant hidden by the O-notation depends only on d. Let $X_{N,m}$ be the random variable that counts the number of unicyclic components in $G^d(N, c/N^{d-1})$ with exactly m edges. Then we have $\delta_N = \sum_{m=2}^{N/(d-1)} (d-1)m \cdot X_{N,m}$. Also,

$$\begin{aligned}
\mathrm{E}[X_{N,m}] &\leq \binom{N}{n} p(N)^m (1-p(N))^{\binom{N}{d} - \binom{N-n}{d} - m} \frac{n^n}{e^{(n-m)}} \frac{1}{(d-2)!^m} \\
&\leq \frac{N^n e^n}{n^n} e^{-\frac{n(n-1)}{2N}} p(N)^m e^{-p(N)\left[\binom{N}{d} - \binom{N-n}{d} - m\right]} \frac{n^n}{e^{(n-m)}} \frac{1}{(d-2)!^m} \\
&= \left(\frac{c}{(d-2)!}\right)^m e^{n - \frac{n(n-1)}{2N} - p(N)\left[\binom{N}{d} - \binom{N-n}{d} - m\right] + (m-n)} \qquad (1)
\end{aligned}$$

Operating on the exponent of e in the last term:

$$
\begin{aligned}
&n - \frac{n(n-1)}{2N} - p(N)\left[\binom{N}{d} - \binom{N-n}{d} - m\right] + (m-n) \leq \\
&-\frac{n(n-1)}{2N} - p(N)\left[\binom{N}{d} - \binom{N-n}{d}\right] + p(N)m + m \leq \\
&m\Big(-\frac{(d-1)}{2} - \frac{c}{(d-2)!} + p(N) + 1\Big) \leq \\
&m\Big(1 - \frac{c}{(d-2)!}\Big).
\end{aligned}
$$

Thus, substituting in eq. (1) we obtain

$$
\mathrm{E}[X_{N,m}] \leq \left(\frac{c}{(d-2)!}e^{1-\frac{c}{(d-2)!}}\right)^m \tag{2}
$$

One can easily check that xe^{1-x} grows monotonously from 0 to 1 as x goes from 0 to 1. Thus, $\frac{c}{(d-2)!}e^{1-\frac{c}{(d-2)!}} < 1$.

By definition it is satisfied

$$
\mathrm{E}[\delta_N] = \sum_{m=2}^{N/(d-1)} m\mathrm{E}[X_{N,m}],
$$

except for the case $d = 2$, where the sum starts at $m = 3$ instead of $m = 2$. This is because there are no connected graphs with 2 vertices and 2 edges. Because of eq. (2), for sufficiently large values of m the terms $m\mathrm{E}[X_{N,m}]$ are bounded by $m(\frac{c}{(d-2)!}e^{1-\frac{c}{(d-2)!}})^m$ uniformly for all values of N. Otherwise, for small values of m, the $m\mathrm{E}[X_{N,m}]$ are bounded as well because

$$
\lim_{N\to\infty} m\mathrm{E}[X_{N,m}] = \left(ce^{\frac{c}{(d-2)!}}\right)^m \frac{C_d(m)}{(m\cdot(d-1))!}
$$

In consequence, using the dominated converge theorem we can conclude that

$$
\lim_{N\to\infty} \mathrm{E}[\delta_N] = \sum_{m=2}^{N/(d-1)} m\mathrm{E}[X_{N,m}]
$$

exists and is not infinite. $\square$

Theorem 1.3. *Let $c < (d-2)!$, and let $p(N) \sim c/N^{d-1}$. For any $k \geq 2$, let $\gamma_{N,\geq k}$ be the random variable that counts how many cycles with at least k edges lie in $G^d(N, p(N))$. Then*

$$
\lim_{N\to\infty} \mathrm{E}[\gamma_{N,\geq k}] = \sum_{l=k}^{\infty} \left(\frac{c}{(d-2)!}\right)^l \frac{1}{2l}
$$

In particular, if γ_N is the random variable that counts the cycles in $G^d(N, p(N))$ then, if $d > 2$:

$$
\lim_{N\to\infty} \mathrm{E}[\gamma_N] = \begin{cases} \frac{c}{2(d-2)!} + \ln\left(1 - \frac{c}{2(d-2)!}\right), & \text{if } d > 2 \\ \frac{c}{2} + \frac{c^2}{4}\ln\left(1 - \frac{c}{2}\right), & \text{if } d = 2. \end{cases}
$$

Esto está hecho en el artículo de Erdos para grafos, y para hypergrafos no lo he visto.

Proof. For $k \geq 2$ ($k \geq 3$ if $d = 2$), let $\gamma_{N,k}$ be the random variable that counts the k-cycles that lie in $G^d(N, p(N))$. A simple computation yields $\mathrm{E}[\gamma_{N,k}] = \frac{(N)_{k(d-1)}}{2k}\left(\frac{c}{N^{d-1}}\right)^k$, and $\lim_{N\to\infty} \mathrm{E}[\gamma_{N,k}] = \frac{c^k}{2k}$.

Not only this, but also $\mathrm{E}[\gamma_{N,k}] \leq \frac{c^k}{2k}$ for all N. In consequence, applying the dominated convergence theorem we obtain

$$\lim_{N\to\infty} \sum_{l=k}^{\infty} \mathrm{E}[\gamma_{N,l}] = \sum_{l=k}^{\infty} \frac{c^l}{2l}.$$

Using that $\gamma_{N,\geq k}$ is the sum of all the $\gamma_{N,l}$ for $l \geq k$ and the Taylor expansion of $ln(1-x)$ yields the desired results. □

Given $0 < c < (d-2)!$ we define the function $B_d(c)$ as the limit as N tends to infinity of the expected number of cycles in $G^d(N, c/N^{d-1})$. Because of our last theorem

$$B_d(c) = \begin{cases} \frac{c}{2(d-2)!} + \ln\left(1 - \frac{c}{2(d-2)!}\right), & \text{if } d > 2 \\ \frac{c}{2} + \frac{c^2}{4} \ln\left(1 - \frac{c}{2}\right), & \text{if } d = 2. \end{cases}$$

A simple application of the factorial moments method proves the following theorem:

Theorem 1.4. *Let $k_1, \ldots, k_j \geq 2$ (≥ 3 if $d = 2$). Then, as N tends to infinity the γ_{N,l_i}'s converge in distribution to independent Poisson variables with mean values $\lambda_i := \frac{c^{l_i}}{2l_i}$ respectively. That is, for any $a_1, \ldots, a_j \in \mathbb{N}$,*

$$\lim_{N\to\infty} \Pr\Big(\bigwedge_{i=1}^{r} \gamma_{N,l_i} = a_i\Big) = \prod_{i=1}^{j} e^{-\lambda_i} \frac{\lambda_i^{a_i}}{a_i!}$$

Esto está hecho en el libro de Bollobás de random graphs. Para hypergrafos no lo he encontrado, pero realmente es lo mismo. Si queréis lo hago.

Let H be an unicycle and let $m = |E(H)|$, $n = |V(H)| = m(d-1)$. Let $X_{N,H}$ be the random variable that counts the number of connected components in $G^d(N, p(N))$ isomorphic to H. Then

$$\mathrm{E}[X_{N,H}] = \frac{(N)_n}{|Aut(H)|} p(N)^m (1 - p(N))^{\binom{N}{d} - \binom{N-n}{d} - m}.$$

And if $p(N) \sim c/N$,

$$\lim_{N\to\infty} \mathrm{E}[X_{N,H}] = \frac{c^m}{|Aut(H)|} e^{\lim\limits_{N\to\infty} c \frac{\binom{N}{d} - \binom{N-n}{d} - m}{N^{d-1}}} = \frac{c^m}{|Aut(H)|} e^{c \frac{m}{(d-2)!}}.$$

For convenience's sake we will often use the auxiliary variable $s = \frac{c}{(d-2)!} e^{\frac{c}{(d-2)!}}$. We can rewrite last limit in terms of s as:

$$\lim_{N\to\infty} \mathrm{E}[X_{N,H}] = (s)^m \frac{(d-2)!^m}{|Aut(H)|}.$$

Another application of the factorial moments method proves the next theorem:

Theorem 1.5. *Let $H_1, \ldots, H_j$ be unicycles, and let $P(N) \sim c/N$. Then, as N tends to infinity, the X_{N,H_i}'s converge in distribution to independent Poisson variables with means $\lambda_i = s^{|E(H_i)|} \frac{(d-2)!^{|E(H_i)|}}{|Aut(H_i)}$ respectively. That is, for any $a_1, \ldots, a_j \in \mathbb{N}$,*

$$\lim_{N\to\infty} \Pr\Big(\bigwedge_{i=1}^{r} X_{N,H_i} = a_i\Big) = \prod_{i=1}^{j} e^{-\lambda_i} \frac{\lambda_i^{a_i}}{a_i!}$$

Next we show some stuff

Theorem 1.6.

References

[1] Paul Erdős and Alfréd Rényi. On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci*, 5(1):17–60, 1960.

[2] Michał Karoński and Tomasz Łuczak. The phase transition in a random hypergraph. *Journal of Computational and Applied Mathematics*, 142(1):125–135, 2002.