More proof Sketches Regarding the Closure of Limiting Probabilities in Sparse Random Hyper-graphs

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Preliminaries

$G^d(N,p)$ denotes the binomial model of random d-uniform hypergraphs. We will consider $d \geq 2$ fixed for the rest of this writing. We will deal with the case where $p = p(N) = c/N^{d-1}$ for some real number $c > 0$. For each $N \in \mathbb{N}$, we will use $G_N := G_N(c)$ to denote a random sample from $G^d(N,p)$. We will refer to d-uniform hypergraphs simply as hypergraphs.

Si no me equivoco todo lo de aquí sirve también para el caso en el que $p(N) \sim c/n^{d-1}$, no hace falta que sean iguales. Pongo igualdad porque es más fácil de escribir.

For each $n, m \in \mathbb{N}$ with $m \leq n$ we will denote the product $n(n-1)\cdots(n-m+1)$ by $(n)_m$.

Theorems

Theorem 0.1. *Let $0 < c < (d-2)!$. Then, a.a.s all the components of G_N, are either trees or unicycles.*

Proof. See [1] for $d = 2$ and [2] for the general case.

Está todo hecho en el modelo uniforme, pero la transferencia uniforme-> binomial es "sencilla"

□

Given an hypergraph G, we will call the **fragment** $F(G)$ of G to the union of its unicyclic components. We will abbreviate $F(G_N)$ as F_N.

Theorem 0.2. *Let $0 < c < (d-2)!$. For each $i \in \mathbb{N}$ let $X_{N,i}$ be the random variable that counts the number of unicyclic components in G_N that contain exactly i edges. Then $\lim_{N\to\infty} \mathrm{E}[e(F_N)]$ exists and is a finite quantity. Furthermore,*

$$\lim_{N\to\infty} \mathrm{E}[e(F_N)] = \sum_{i=1}^{\infty} i \lim_{N\to\infty} X_{N,i}.$$

Este teorema lo he visto para grafos en el artículo de Erdos de 1960 , y más completo en el libro de Karonski, por ejemplo. Para hypergrafos no lo he encontrado, pero igual es cosa de buscar mejor. Lo demuestro de todas formas.

Proof. Let $C_d(m)$ denote the number of connected labeled d-uniform hypergraphs with m edges and $n := (d-1)m$ vertices. By definition $e(F_N) = \sum_{i=1}^{\infty} i \cdot X_{N,i}$, and in consequence

$$\lim_{N\to\infty} \mathrm{E}[e(F_N)] = \lim_{N\to\infty} \sum_{i=1}^{\infty} i\mathrm{E}\big[X_{N,i}\big].$$

To prove the statement we have to show that we can exchange the limit and summation from the RHS of last equality. For that we will use the dominated convergence theorem. First, notice that for each $i \in \mathbb{N}$,

$$\begin{aligned}\lim_{N\to\infty} \mathrm{E}\big[X_{N,i}\big] &= \lim_{N\to\infty} \binom{N}{(d-1)i} \cdot \left(\frac{c}{N^{d-1}}\right)^i \cdot \left(1 - \frac{c}{N^{d-1}}\right)^{\binom{N}{d}-\binom{N-(d-1)i}{d}-i} \cdot C_d(i) \\ &= \left(ce^{-\frac{c}{(d-2)!}}\right)^i \cdot \frac{C_d(i)}{((d-1)i)!}.\end{aligned}$$

Here we have used that for any fixed i $\binom{N}{d} - \binom{N-(d-1)i}{d} - i \sim iN^{(d-1)}/(d-2)!$ as N tends to infinity, and it follows that

$$\lim_{n\to\infty} \left(1 - \frac{c}{N^{d-1}}\right)^{\binom{N}{d-1}-\binom{N-(d-1)i}{d-1}-i} = e^{-\frac{ic}{(d-2)!}}.$$

Now we have to dominate the terms $\mathrm{E}[X_{N,i}]$ by some sequence a_i such that $\sum_{i=1}^{\infty} ia_i \leq \infty$.

It has been proven (see [2]) that for big values of i

$$C_d(i) \leq \frac{((d-1)i)^{(d-1)i}}{e^{((d-2)i)}} \frac{1}{(d-2)!^i},$$

In consequence, if we name $j = (d-1)i$, for sufficiently large i we obtain

$$\begin{aligned}\mathrm{E}[X_{N,i}] &\leq \binom{N}{j} \cdot p(N)^i \cdot (1-p(N))^{\binom{N}{d}-\binom{N-j}{d}-i} \cdot \frac{j^j}{e^{((d-2)i)}} \frac{1}{(d-2)!^i} \\ &\leq \frac{N^j e^j}{j^j} \cdot p(N)^i \cdot e^{-p(N)\left[\binom{N}{d}-\binom{N-j}{d}-i\right]} \cdot \frac{j^j}{e^{(d-2)i}} \frac{1}{(d-2)!^i} \\ &= \left(\frac{c}{(d-2)!}\right)^i \cdot e^{j-p(N)\left[\binom{N}{d}-\binom{N-j}{d}-i\right]-(d-2)i} \qquad (1)\end{aligned}$$

Here we have used the bounds $1/j! \leq e^j/j^j$, $\quad (N)_j \leq N^j$, and $(1-p(N)) \leq e^{-p(N)}$. Operating on the exponent of e in the last term we get, for sufficiently large N uniformly in i,

$$\begin{aligned}&j - p(N)\left[\binom{N}{d} - \binom{N-j}{d} - i\right] - (d-2)i \leq \\ &-p(N)\left[\binom{N}{d} - \binom{N-j}{d}\right] + p(N)i + i \leq \\ &i\left(1 - \frac{c}{(d-2)!}\right) + c\end{aligned}$$

Here we have used that $\binom{N}{d} - \binom{N-j}{d} \leq N^{d-1}j/(d-1)!$, and also $p(N)i \leq c$ because we can suppose that $i << N^{d-1}$ (otherwise $\mathrm{E}[X_{N,i}] = 0$). Thus, substituting at the end of eq. (1) we obtain

$$\mathrm{E}[X_{N,i}] \leq \left(\frac{c}{(d-2)!} e^{1-\frac{c}{(d-2)!}}\right)^i \cdot e^c, \qquad (2)$$

for sufficiently large i and large N uniformly in i. One can easily check that xe^{1-x} grows monotonously from 0 to 1 as x goes from 0 to 1. Thus, $\frac{c}{(d-2)!}e^{1-\frac{c}{(d-2)!}} < 1$, and

$$\sum_{i=1}^{\infty} i \cdot \left(\frac{c}{(d-2)!} e^{1-\frac{c}{(d-2)!}}\right)^i \cdot e^c < \infty.$$

Hence, using eq. (2) we can apply the dominated convergence theorem obtaining

$$\lim_{N\to\infty}\sum_{i=1}^{\infty} i\mathrm{E}\big[X_{N,i}\big] = \sum_{i=1}^{\infty} i \lim_{N\to\infty} \mathrm{E}\big[X_{N,i}\big],$$

and both expressions are finite quantities, proving the theorem. □

Theorem 0.3. *Let* $0 < c < (d-2)!$*. For any* $k \in \mathbb{N}$*, let* $Y_{N,\geq k}$ *be the random variable that counts how many cycles with at least* k *edges lie in* G_N*. Then*

$$\lim_{N\to\infty} \mathrm{E}[\gamma_{N,\geq k}] = \sum_{l=k}^{\infty}\left(\frac{c}{(d-2)!}\right)^l \frac{1}{2l}$$

In particular, if Y_N *is the random variable that counts the cycles in* G_N *then:*

$$\lim_{N\to\infty} \mathrm{E}[Y_N] = \begin{cases} \frac{c}{2(d-2)!} + \ln\left(1 - \frac{c}{2(d-2)!}\right), & \text{if } d > 2 \\ \frac{c}{2} + \frac{c^2}{4}\ln\left(1-\frac{c}{2}\right), & \text{if } d = 2. \end{cases}$$

Esto está hecho en el artículo [1] para grafos, y para hypergrafos no lo he visto.

Proof. For $k \in \mathbb{N}$ let $Y_{N,k}$ be the random variable that counts the k-cycles that lie in G_N. A simple computation yields that for any $k \geq 2$ ($k \geq 3$ if $d = 2$), $\mathrm{E}[Y_{N,k}] = \frac{(N)_{k(d-1)}}{2k}\left(\frac{c}{N^{d-1}}\right)^k$, and $\lim_{N\to\infty} \mathrm{E}[Y_{N,k}] = \frac{c^k}{2k}$. Also, $\mathrm{E}[Y_{N,k}] \leq \frac{c^k}{2k}$ for all N. In consequence, applying the dominated convergence theorem we obtain

$$\lim_{N\to\infty}\sum_{l=k}^{\infty}\mathrm{E}[Y_{N,l}] = \sum_{l=k}^{\infty}\frac{c^l}{2l} \qquad \text{for any } k \in \mathbb{N}.$$

Using that $Y_{N,\geq k}$ is the sum of all the $Y_{N,l}$ for $l \geq k$ and the Taylor expansion of $ln(1-x)$ we obtain the desired result. □

Given $0 < c < (d-2)!$ we define the function $B_d(c)$ as the limit as N tends to infinity of the expected number of cycles in $G_N(c)$. Because of our last theorem

$$B_d(c) = \begin{cases} \frac{c}{2(d-2)!} + \ln\left(1 - \frac{c}{2(d-2)!}\right), & \text{if } d > 2 \\ \frac{c}{2} + \frac{c^2}{4}\ln\left(1-\frac{c}{2}\right), & \text{if } d = 2. \end{cases}$$

Corollary 0.1. *Let* $0 < c < (d-2)!$*. For each* $k \in \mathbb{N}$ *let* $X_{N,k}$ *be the random variable that counts the unicyclic components in* G_N *with exactly* k *edges. Then*

$$B_d(c) = \lim_{N\to\infty} \mathrm{E}\Big[\sum_{i=1}^{\infty} X_{N,i}\Big] = \sum_{i=1}^{\infty}\lim_{N\to\infty}\mathrm{E}\big[X_{N,i}\big].$$

Proof. The equality $B_d(c) = \lim_{N\to\infty} \mathrm{E}\big[\sum_{i=1}^{\infty}$ follows from the fact that the asymptotic number of expected cycles and the asymptotic number of expected unyciclic components in G_N coincide for $0 < c < (d-2)!$ because of theorem 0.1.

The equality $\lim_{N\to\infty} \mathrm{E}\big[\sum_{i=1}^{\infty} X_{N,i}\big] = \sum_{i=1}^{\infty} \lim_{N\to\infty} \mathrm{E}\big[X_{N,i}\big]$ follows from applying the dominated converge theorem as in theorem 0.2. □

A simple application of the factorial moments method proves the following theorem:

Theorem 0.4. *For each $k \in \mathbb{N}$, let $Y_{N,k}$ be the random variable that counts the number of k-cycles in G_N. Let $k_1, \ldots, k_j \geq 2$ (≥ 3 if $d = 2$). Then, as N tends to infinity the Y_{N,l_i}'s converge in distribution to independent Poisson variables with mean values $\lambda_i := \frac{c^{l_i}}{2l_i}$ respectively. That is, for any $a_1, \ldots, a_j \in \mathbb{N}$,*

$$\lim_{N\to\infty} \Pr\Big(\bigwedge_{i=1}^{r} Y_{N,l_i} = a_i \Big) = \prod_{i=1}^{j} e^{-\lambda_i} \frac{\lambda_i^{a_i}}{a_i!}$$

Esto está hecho en el libro de Bollobás de random graphs. Para hypergrafos no lo he encontrado, pero realmente es lo mismo. Si queréis lo hago.

Let H be an unicycle and let $i = e(H)$, $j = v(H) = i(d-1)$. Let $X_{N,H}$ be the random variable that counts the number of connected components in G_N isomorphic to H. Then

$$\mathrm{E}[X_{N,H}] = \frac{(N)_i}{|Aut(H)|} p(N)^i (1 - p(N))^{\binom{N}{d} - \binom{N-j}{d} - i}.$$

And substituting $p(N) = c/N^{d-1}$,

$$\lim_{N\to\infty} \mathrm{E}[X_{N,H}] = \frac{c^i}{|Aut(H)|} e^{\lim_{N\to\infty} c \frac{\binom{N}{d} - \binom{N-j}{d} - i}{N^{d-1}}} = \frac{c^i}{|Aut(H)|} e^{c \frac{i}{(d-2)!}}.$$

For convenience's sake we will sometimes use the auxiliary variable $s = \frac{c}{(d-2)!} e^{\frac{c}{(d-2)!}}$. We can rewrite last limit in terms of s as:

$$\lim_{N\to\infty} \mathrm{E}[X_{N,H}] = (s)^i \frac{(d-2)!^i}{|Aut(H)|}.$$

Another application of the factorial moments method proves the next theorem:

Theorem 0.5. *Let $H_1, \ldots, H_j$ be unicycles, and let $P(N) \sim c/N$. Then, as N tends to infinity, the X_{N,H_i}'s converge in distribution to independent Poisson variables with means $\lambda_i = s^{e(H_i)} \frac{(d-2)!^{e(H_i)}}{|Aut(H_i)|}$ respectively. That is, for any $a_1, \ldots, a_j \in \mathbb{N}$,*

$$\lim_{N\to\infty} \Pr\Big(\bigwedge_{i=1}^{r} X_{N,H_i} = a_i \Big) = \prod_{i=1}^{j} e^{-\lambda_i} \frac{\lambda_i^{a_i}}{a_i!}$$

Theorem 0.6. *Let $0 < c < (d-2)!$ and $p(N) \sim c/N$. For $i \in \mathbb{N}$ let $\lambda_i = \Big(\frac{c}{(d-2)!}\Big)^i \frac{1}{2i!}$. Let A_N be the event that $G^d(n, p(N))$ contains no cycles. Define $F_d := F_d(c)$ as*

$$F_d(c) = \begin{cases} e^{\sum_{i=2}^{\infty} \lambda_i} = e^{\frac{c}{2(d-2)!}} \sqrt{1 - \frac{c}{(d-2)!}} & \text{if } d > 2 \\ e^{\sum_{i=3}^{\infty} \lambda_i} = e^{\frac{c}{2} + \frac{c^2}{4}} \sqrt{1 - c} & \text{if } d = 2. \end{cases}$$

Then it is satisfied

$$\lim_{N\to\infty} \Pr(A_N) = F_d$$

Esto está hecho para grafos en [1].

Proof. For each $k \in \mathbb{N}$ define $F_{d,k} := F_{d,k}(c)$ as

$$F_{d,k}(c) = \begin{cases} e^{\sum_{i=2}^{k} -\lambda_i} & \text{if } d > 2 \\ e^{\sum_{i=3}^{k} -\lambda_i} & \text{if } d = 3. \end{cases}$$

A simple computation using the Taylor expansion of $\ln(1-x)$ shows that $\lim_{k\to\infty} F_{d,k} = F_d$. Fix an arbitrary $\epsilon > 0$. We show that there exists a constant k satisfying

$$\Big|\lim_{N\to\infty} \Pr\big(A_N\big) - F_{d,j}\Big| \leq \epsilon \text{ for any } j \geq k.$$

For each $k \in N$ let $A_{N,k}$ be the event that $G^d(N,p(N))$ contains no cycle with length at most k. Using theorem 0.4, we obtain

$$\lim_{N\to\infty} \Pr\big(A_{N,k}\big) = F_{d,k}.$$

For any $k \in \mathbb{N}$ let $\gamma_{N,\geq k}$ be the random variable that counts the number of cycles in $G^d(N,p(N))$ with length at least k. Then, using theorem 0.3 we obtain that $\lim_{k\to\infty}\lim_{N\to\infty} \mathrm{E}[\gamma_{N,\geq k}] = 0$.
Fix k such that for any $j \geq k$

$$\Big|\lim_{N\to\infty} \mathrm{E}[\gamma_{N,\geq k}]\Big| \leq \epsilon.$$

Notice that for any j, "A_N is true and $A_{N,j}$ is false" if and only if $\Pr\big(\gamma_{N,\geq j} \geq 1\big)$. Also, by Markov inequality, $\Pr\big(\gamma_{N,\geq j} \geq 1\big) \leq \mathrm{E}\big[\gamma_{N,\geq j}\big]$. In consequence, for any $j \geq k$

$$\lim_{N\to\infty} \Big|\Pr\big(A_N\big) - \Pr\big(A_{N,j}\big)\Big| \leq \epsilon.$$

In consequence, for any $j \geq k$

$$\Big|\lim_{N\to\infty} \Pr\big(A_N\big) - F_{d,j}\Big| \leq \Big|\lim_{N\to\infty} \Pr\big(A_{N,j}\big) - F_{d,j}\Big| + \epsilon = \epsilon,$$

and we are finished. □

Theorem 0.7. *Let $0 < c < (d-2)!$, and $p(N) \sim c/N^{d-1}$. Let H be an hypergraph whose components are all unicycles, and let $F_d := F_d(c)$ be as in last theorem. Then*

$$\lim_{N\to\infty} \Pr\Big(F_N \simeq H\Big) = F_d(c)\left(\frac{c}{(d-2)!}e^{\frac{c}{(d-2)!}}\right)^{e(H)} \frac{(d-2)!^{e(H)}}{|Aut(H)|}$$

Proof. Fix such H. Let $U_1, U_2, \dots, U_i, \dots$ be an enumeration of all unicycles in a way such that $e(U_i) \leq e(U_j)$ if $i \leq j$. For each $i \in \mathbb{N}$ let a_i be the number of connected components of H that are isomorphic to U_i, and let $X_{N,i}$ be the random variable that counts the number of connected components in $G^d(N,P(N))$ that are isomorphic to U_i. Clearly, $F_N \simeq H$ if and only if $X_{N,i} = a_i$ for all i. Thus,

$$\lim_{N\to\infty} \Pr\big(F_N \simeq H\big) = \lim_{N\to\infty} \Pr\big(\bigwedge_{i=1}^{\infty} X_{N,i} = a_i\big).$$

First, we are going to show that

$$\lim_{N\to\infty} \Pr\big(\bigwedge_{i=1}^{\infty} X_{N,i} = a_i\big) = \lim_{j\to\infty}\lim_{N\to\infty} \Pr\big(\bigwedge_{i=1}^{j} X_{N,i} = a_i\big).$$

Fix $\epsilon > 0$ an arbitrarily small real constant. We need to prove that there exists some $j_0 \in \mathbb{N}$ satisfying

$$\left|\lim_{N\to\infty} \Pr\big(\bigwedge_{i=1}^{j} X_{N,i} = a_i\big) - \lim_{j\to\infty}\lim_{N\to\infty} \Pr\big(\bigwedge_{i=1}^{j} X_{N,i} = a_i\big)\right| \leq \epsilon \text{ for all } j \geq j_0.$$

For each $k \in \mathbb{N}$ let $Y_{N,k}$ be the random variable that counts the uni-cyclic connected components of $G^d(N,p(N))$ with exactly k edges. By theorem 0.2 we have that for some $k_0 \in \mathbb{N}$

$$\lim_{N\to\infty} \sum_{l=k_0}^{\infty} \mathrm{E}\big[Y_{N,l}\big] \leq \epsilon$$

Let k_1 be the maximum number of edges in a connected component of H, and let $k = \max(k_0, k_1+1)$. Finally, fix j_0 such that $e(U_j) > k_1$ for any $j \geq j_0$.

Given any $j \geq j_0$, $F_N \simeq H$ if and only if

$$\Big(\bigwedge_{i=1}^{j} X_{N,i} = a_i\Big) \wedge \Big(\sum_{l=k}^{\infty} Y_{N,l} = 0\Big)$$

Using Markov inequality we get

$$\lim_{N\to\infty} \Pr\Big(\sum_{l=k}^{\infty} Y_{N,l} \geq 1\Big) \leq \epsilon$$

Thus,

$$\left| \lim_{N\to\infty} \Pr\Big(\bigwedge_{i=1}^{j} X_{N,i} = a_i\Big) - \lim_{j\to\infty}\lim_{N\to\infty} \Pr\Big(\bigwedge_{i=1}^{j} X_{N,i} = a_i\Big) \right| \leq \epsilon,$$

as we wanted to prove.

Using theorem 0.5 we get

$$\lim_{j\to\infty}\lim_{N\to\infty} \Pr\Big(\bigwedge_{i=1}^{j} X_{N,i} = a_i\Big) = \prod_{i=1}^{\infty} e^{-\lambda_i}\frac{\lambda_i^{a_i}}{a_i!},$$

where $\lambda_i = s^{e(U_i)}\frac{(d-2)!^{e(U_i)}}{|Aut(U_i)|} = \lim_{N\to\infty} \mathrm{E}[X_{N,i}]$, and $s = s(c)$ is defined as in the statement of the theorem. Using corollary 0.1 we obtain

$$\sum_{i=1}^{\infty} \lambda_i = B_d(c),$$

and in consequence

$$\prod_{i=1}^{\infty} e^{-\lambda_i} = e^{-B_d(c)} = A_d(c).$$

The following identities hold:

$$\sum_{i=1}^{\infty} e(U_i)a_i = e(H) \qquad \prod_{i=1}^{\infty} |Aut(U_i)^{a_i} a_i! = |Aut(H)|.$$

As a consequence we get

$$\prod_{i=1}^{\infty} \frac{\lambda_i^{a_i}}{a_i!} = \prod_{i=1}^{\infty} (s(d-2)!)^{e(U_i)a_i}\frac{1}{|aut(U_i)|^{a_i}a_i!} = s^{e(H)}\frac{(d-2)!^{e(H)}}{|Aut(H)|},$$

and the theorem follows. □

Let us denote by $\mathcal{U}$ the class of hypergraphs whose connected components are unicycles. For the case $d = 2$ the asymptotic distribution of the fragment F_N coincides with the Boltzmann-Poisson distribution of random graphs from $\mathcal{U}$ described in [3].

Tobias de una forma un poco críptica da a entender que usar el teorema 1.3 de [3] sirve para algo. Yo no he sabido cómo o para qué.

For completeness sake we give an argument of why the asymptotic distribution of F_N is indeed a probability distribution.

Theorem 0.8. *Let $0 < c < (d-2)!$ and let $p(N) \sim c/N^{d-1}$. Let $H_1, \dots H_k, \dots$ be an enumeration of all hypergraphs in $\mathcal{U}$. For each $i \in \mathbb{N}$ denote by p_i the limit $\lim_{N\to\infty} \Pr(F_N \simeq H_i)$. Then*

$$\sum_{i=1}^{\infty} p_i = 1.$$

Proof. Let $\epsilon > 0$ be an arbitrarily small real constant. We show that there exists some $j_0 \in \mathbb{N}$ such that

$$1 - \sum_{i=1}^{j} p_i \leq \epsilon \quad \text{for all } j \geq j_0.$$

Let $m = \lim_{N\to\infty} \mathrm{E}[e(F_N)]$. Notice that m exists by theorem 0.2. Define $M = m/\epsilon$. Then

$$\lim_{N\to\infty} \Pr\big(e(F_N) \geq M\big) \leq \epsilon.$$

Let j_0 be such that $e(H_i) \geq M$ for all $i \geq j_0$. Then, given any $j \geq j_0$,

$$1 - \sum_{i=1}^{j} p_i = \lim_{N\to\infty} \Pr\big(\bigwedge_{i=1}^{j} F_N \not\simeq H_i\big) \leq \lim_{N\to\infty} \Pr\big(e(F_N) \geq M\big) \leq \epsilon,$$

and the result follows. $\square$

Given $H \in \mathcal{U}$, the property $F(G) \simeq H$ cannot be expressed in FO logic. This is because one has to rule out the existence of arbitrarily long cycles in G. Hence, in the following proof it will be useful to consider "the fragment up to connected components with i edges". Given an hypergraph G and $i \in \mathbb{N}$, let $F_i(G)$ be the union of connected components in G that are unicycles with no more than i edges, and let $\mathcal{U}_i$ be the class of hypergraphs whose connected components are unicycles with at most i edges. Given any $H \in \mathcal{U}_i$, the property $F_i(G) \simeq H$, unlike $F(G) \simeq H$, is expressible in FO logic. We will abbreviate $F_i(G_N)$ as $F_{N,i}$.

Theorem 0.9. *Let $0 < c < (d-2)!$, and $p(N) \sim c/N^{d-1}$. Let $H_1, \dots, H_i, \dots$ be an enumeration of all hypergraphs in $\mathcal{U}$ and for each $i \in \mathbb{N}$ let $p_i = \lim_{N\to\infty} \Pr(F_N \simeq H_i)$. Consider the sets*

$$L_c := \{\lim_{N\to\infty} \Pr\big(P(G_N)\big) \,|\, P \text{ FO property}\},$$

and

$$S_c := \{\sum_{i\in T} p_i \,|\, T \subseteq \mathbb{N}\}.$$

Then it is satisfied that $\overline{L_c} = S_c$.

Proof. We will prove the statement by showing both $S_c \subseteq \overline{L_c}$ and $\overline{L_c} \subseteq S_c$.

We begin with $S_c \subseteq \overline{L_c}$. Let $T \subset \mathcal{U}$, and let $\epsilon > 0$ be an arbitrarily small real number. We show that there exists a first order property P such that

$$\Big|\lim_{N\to\infty} \Pr\big(P(G_N)\big) - \sum_{i\in T} p_i\Big| \leq \epsilon.$$

As $\sum_{i=1}^{\infty} p_i = 1$, there exists some j such that $\sum_{i=1}^{j} p_i \leq \epsilon/2$. Fix such j. Let $T' = T \cap \{1, \dots, j-1\}$. Suppose $T' = \{i_1, \dots, i_k\}$. Consider the properties Q and Q' defined as

$$Q(G) := \bigvee_{i\in T} F(G) \simeq H_i, \qquad Q'(G) := \bigvee_{x=1}^{k} F(G) \simeq H_{i_x}.$$

Then, it is satisfied

$$\lim_{N\to\infty} \Big|\Pr\big(Q(G_N)\big) - \Pr\big(Q'(G_N)\big)\Big| \leq \sum_{i=j}^{\infty} p_i \leq \frac{\epsilon}{2}.$$

Let l_1 be the maximum number of edges in a connected component belonging to any of $H_{i_1}, \dots, H_{i_k}$. Let $M = \lim_{N\to\infty} \mathrm{E}[e(F_N)]$, and let $l_2 = M/2k\epsilon$. Define $l = \max(l_1, l_2)$. The first order property P will be defined as

$$P(G) := \bigvee_{x=1}^{k} F_l(G) \simeq H_{i_x}.$$

As $l \geq l_1$, all $H_{i_1}, \dots, H_{i_k}$ belong to $\mathcal{U}_l$. For each i_x, we have that

$$\Pr\big(F_{N,l} \simeq H_{i_x} \wedge F_N \simeq H_{i_x}\big) \leq \Pr\big(F_N \notin \mathcal{U}_l\big) \leq \Pr\big(e(F_N) > l\big).$$

and because $l \geq l_2$,

$$\lim_{N\to\infty} \Pr\big(e(F_N) > l\big) \leq \frac{\epsilon}{2k}.$$

Thus,

$$\big|\lim_{N\to\infty} \Pr\big(P(G_N)\big) - \sum_{i\in T} p_i\big| = \lim_{N\to\infty} \big|\Pr\big(P(G_N)\big) - \Pr\big(Q(G_N)\big)\big| \leq \lim_{N\to\infty} \big|\Pr\big(P(G_N)\big) - \Pr\big(Q'(G_N)\big)\big| + \frac{\epsilon}{2}.$$

And because $P \subset Q'$,

$$\Pr\big(P(G_N)\big) - \Pr\big(Q'(G_N)\big) = \Pr\big(P(G_N) \wedge \neg Q'(G_N)\big).$$

It is also satisfied that

$$P(G_N) \wedge \neg Q'(G_N) \quad \iff \quad \bigvee_{x=1}^{k} F_{N,l} \simeq H_{i_x} \wedge F_N \not\simeq H_{i_x}$$

Taking into account that the events $F_{N,l} \simeq H_{i_x} \wedge F_N \not\simeq H_{i_x}$ are disjoint for each x we get

$$\Pr\big(\bigvee_{x=1}^{k} F_{N,l} \simeq H_{i_x} \wedge F_N \not\simeq H_{i_x}\big) = \sum_{x=1}^{k} \Pr\big(F_{N,l} \simeq H_{i_x} \wedge F_N \not\simeq H_{i_x}\big)$$

In consequence,

$$\big|\lim_{N\to\infty} \Pr\big(P(G_N)\big) - \sum_{i\in T} p_i\big| \leq \lim_{N\to\infty} \big|\sum_{x=1}^{k} \Pr\big(F_{N,l} \simeq H_{i_x} \wedge F_N \not\simeq H_{i_x}\big)\big| + \frac{\epsilon}{2} \leq \sum_{x=1}^{k} \frac{\epsilon}{2k} + \frac{\epsilon}{2} \leq \epsilon,$$

and we are finished.

Now it is left to prove that $\overline{L_c} \subseteq S_c$. It is a known fact [4], [5], [6] that S_c is a perfect set. In particular S_c is closed and $\overline{S_c} = S_c$. In consequence it suffices to show that $L_c \subset S_c$.

Let $H \in \mathcal{U}$. One can show via EF games (see [7]) that for any given FO property P,

$$\lim_{N\to\infty} \Pr\big(P(G_N)) \,|\, F_N \simeq H\big) = 0 \text{ or } 1.$$

We want to show that

$$\lim_{N\to\infty} \Pr\big(P(G_N))\big) = \sum_{i=1}^{\infty} \lim_{N\to\infty} \Pr\big(F_N \simeq H_i\big) \Pr\big(P(G_N)) \,|\, F_N \simeq H_i\big).$$

Fix an arbitrarily small real constant $\epsilon > 0$. We need to prove that there exists an index $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$

$$\big|\lim_{N\to\infty} \Pr\big(P(G_N)\big) - \sum_{i=1}^{j} \lim_{N\to\infty} \Pr\big(F_N \simeq H_i\big) \Pr\big(P(G_N) \,|\, F_N \simeq H_i\big)\big| \leq \epsilon.$$

Notice that the events $F_N \simeq H_i$ are disjoint for each i. So we obtain:

$$\lim_{N\to\infty} \Pr\big(P(G_N)\big) = \lim_{N\to\infty} \sum_{i=1}^{\infty} \Pr\big(F_N \simeq H_i\big) \Pr\big(P(G_N)) \,|\, F_N \simeq H_i\big).$$

Let $M = \lim_{N\to\infty} \mathrm{E}(e(F_N))$ and let $l = M/\epsilon$. There exists some $j_0 \in \mathbb{N}$ such that $e(H_i) \geq M$ for all $i \geq j_0$. In consequence, for all $j \geq j_0$,

$$\lim_{N\to\infty} \sum_{i=j}^{\infty} \Pr\big(F_N \simeq H_i\big) \leq \epsilon.$$

And we obtain

$$\big|\lim_{N\to\infty} \Pr\big(P(G_N)\big) - \sum_{i=1}^{j} \lim_{N\to\infty} \Pr\big(F_N \simeq H_i\big) \Pr\big(P(G_N) \,|\, F_N \simeq H_i\big)\big| = \lim_{N\to\infty} \sum_{i=j}^{\infty} \Pr\big(F_N \simeq H_i\big) \leq \epsilon,$$

as we wanted. This proves SADDA. Finally, let

$$T = \{ i \in \mathbb{N} \,|\, \lim_{N\to\infty} \Pr\big(P(G_N) \,|\, F_N \simeq H_i\big) = 1 \}$$

It is satisfied that

$$\sum_{i=1}^{\infty} \lim_{N\to\infty} \Pr\big(F_N \simeq H_i\big) \Pr\big(P(G_N)) \,|\, F_N \simeq H_i\big) = \sum_{i\in T} \lim_{N\to\infty} \Pr\big(F_N \simeq H_i\big) = \sum_{i \in T} p_i.$$

Finished!!! □

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