First Order Logic of Sparse Random Graphs

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Preliminaries: The first order logic (FO) of graphs

Variables $x_1, \ldots, x_n, \ldots \rightarrow$ Vertices
A binary relation symbol $E \rightarrow$ Edges
Boolean connectives $\wedge, \vee, \neg, \ldots$ and equality symbol $=$.
Quantifiers $\forall, \exists$.

Can express the existence of a given subgraph, the existence of a covering set of size $k \ldots$

Cannot express connectivity, k-colorability, existence of a Hamiltonian path...

Example: $\exists x \exists y (Exy \wedge \forall z (Ezy \iff Ezx))$.

Preliminaries: The binomial model of random graphs $G(n, p)$

Start with vertex set $[n] = \{1, \ldots, n\}$ and each edge is added with probability p independently. Given $G = ([n], E)$,

$$\Pr(G) = p^{|E|} \cdot (1-p)^{\binom{n}{2} - |E|}.$$

We are interested in the asymptotic behavior of $G(n, p)$ when $p = p(n) = c/n$.

Preliminaries: The binomial model of random graphs $G(n, p)$

Theorem (Erdös, Rényi, 1960)

For $c < 1$, a.a.s the connected components in G_n have size $O(\log(n))$ and are either trees or unicycles.
For $c > 1$, a.a.s there is a unique large complex component in G_n of size $O(n)$.

Theorem (Lynch, 1960)

Let P be a F.O. property of graphs. Then, the function

$$F_P(c) = \lim_{n\to\infty} \Pr\big(P(G_n(c))\big)$$

is well defined and analytic.

The problem

F.O. properties cannot individually detect the phase transition at $c = 1$.

What if we consider the set $\overline{L(c)}$? Where

$$L(c) = \{\lim_{n\to\infty} \Pr\big(P(G_n(c))\big) \mid P \text{ is a F.O. property .}\}$$

Previous related work: *Logical limit laws for minor-closed classes of graphs* by P. Heinig, T. Muller, M. Noy, A. Taraz.

Supercritical phase

$X_{\leq k}(G) := \#$ of cycles of length at most k in G.

Theorem

$$X_{\leq k}(G_n) \xrightarrow[n\to\infty]{D} \mathrm{Pois}(\lambda_k),$$

where

$$\lambda_k = \sum_{i=3}^{k} \frac{c^i}{2i}.$$

For $c \geq 1$, $\lambda_k \xrightarrow[k\to\infty]{} \infty$.

The property $X_{\leq k}(G) = a$ can be written in F.O. logic.

Supercritical phase

Because of the Central Limit Theorem one can show that

$$\lim_{x\to\infty} \Pr\bigl(\mathrm{Pois}(x) \leq x + y\sqrt{x}\bigr) = \Phi(y),$$

where $\Phi(y)$ is the CDF of $\mathrm{N}(0,1)$.

Theorem

For $c \geq 1$, $\overline{L(c)} = [0,1]$.

Subcritical phase.

$F(G) :=$ Union of unicyclic components of G.

$\mathcal{U} :=$ Class of graphs whose components are unicycles.

Let P be a F.O. property. One can show via combinatorial games that for any $H \in \mathcal{U}$

$$\lim_{n\to\infty} \Pr\big(P(G_n) \,|\, F(G_N) \simeq H_i\big) = 0 \text{ or } 1.$$

At least one gap for $c < c_0$

$$A(c) := \lim_{n\to\infty} \Pr\big(G_n \text{ is acyclic }\big)$$

Theorem (Erdös, Rényi, 1960)

$$A(c) = e^{\frac{2c+c^2}{4}}\sqrt{1-c}.$$

Let c_0 be the only solution to $A(c_0) = 1/2$ in $[0,1]$.

Theorem

If $c < c_0$ then $1/2 \notin \overline{L(c)}$.

The set of limit probabilities of the fragment

Consider $H_1, \dots, H_i, \dots$ an enumeration of $\mathcal{U}$.

Define $p_i := \lim_{n\to\infty} \Pr\Big(F(G_n) \simeq H_i\Big)$, and $S(c) := \{\sum_{i\in T} p_i(c) \mid T \subset \mathbb{N}\}$.

Theorem

$\overline{L(c)} = S(c)$.

We show an sketch of the proof.

- $\overline{L(c)} \subseteq S(c)$

Let P be a F.O. property. Recall that for any H_i

$$\lim_{n\to\infty} \Pr\big(P(G_n) \,|\, F(G_N) \simeq H_i\big) = 0 \text{ or } 1.$$

In consequence, if T is the set of i's that make last limit 1,

$$\lim_{n\to\infty} \Pr\big(P(G_n)\big) := \sum_{i\in T} p_i$$

- $\overline{L(c)} \supseteq S(c)$

The property $F(G) \simeq H_i$ cannot be expressed in F.O.

$F_k(G) :=$ Union of the unicyclic components of G whose size is at most k.

The property $F_k(G) \simeq H_i$ can be expressed in F.O., and

$$\lim_{k \to \infty} \lim_{n \to \infty} \Pr\big(F_k(G_n) \sim H_i\big) = p_i$$

Kakeya criterion for partial sums

Fix $0 \leq c < 1$. Suppose that $p_1 \leq p_2 \leq \cdots \leq p_i \leq \ldots$

Theorem (J. E. Nymann, R.A Sáenz, 2000)

If $p_i \leq \sum_{j=i+1}^{\infty}$ *for all* i *then* $S(c) = [0,1]$.

If $p_i \leq \sum_{j=i+1}^{\infty}$ *for all sufficiently large* i *then* $S(c)$ *is a finite disjoint union of closed intervals.*

If $p_i > \sum_{j=i+1}^{\infty}$ *for all sufficiently large* i *then* $S(c)$ *is homeomorphic to the Cantor set.*

Probability distribution of the fragment

Theorem

Let $H \in \mathcal{U}$. Then

$$\lim_{n\to\infty} \Pr\big(F(G_n) \simeq H\big) = A(c) \cdot \frac{s(c)^{v(H)}}{|Aut(H)|},$$

where $s(c) = ce^{-c}$, and

$$A(c) = \lim_{n\to\infty} \Pr\big(G_n \text{ is acyclic }\big) = e^{\frac{2c+c^2}{4}}\sqrt{1-c}.$$

Theorem

$\overline{L(c)}$ *is always a finite union of intervals*

If $A \cdot s^{k-1} \geq p_i > A \cdot s^k$ we can prove that

$$\sum_{j=i+1}^{\infty} p_j \leq A \cdot s^{k+1} \frac{k-3}{2}.$$

Thus, if $\frac{k-3}{2} \geq \frac{1}{s}$, then

$$p_i \geq \sum_{j=i+1}^{\infty} p_j.$$

Theorem

For $c \geq c_0$
$\overline{L(c)} = [0, 1]$

Using our previous bound and $s(c) > 1/3$ we get that if $A \cdot s^{k-1} \geq p_i > A \cdot s^k$ for $k \geq 9$, then

$$p_i \geq \sum_{j=i+1}^{\infty} p_j.$$

After this we have to check only finitely many p_i's.

More results

One can show that the number of gaps in $\overline{L(c)}$ tends to infinity as c goes to zero.

These results (with the appropriate changes) also hold in the setting of random uniform hypergraphs.

Recap

For $c \geq 1$ all probabilities can be approximated with statements of the type "G_n contains less than k cycles of order less than l".

For $c < 1$ we can approximate F.O. probabilities using sums of fragment probabilities.

Using the Kakeya criterion one can show that there is a gap in $\overline{L(c)}$ only when the asymptotic probability of G_n being acyclic is greater than $1/2$.