# Proof Sketches Regarding the Closure of Limiting Probabilities in Sparse Random Hyper-graphs

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## Preliminaries

Let $G^d(n,p)$ denote the binomial random $d$-uniform hyper-graph on $n$ vertices where each edge has probability $p$. We are interested in the case where $p = \frac{c}{n^{d-1}}$ for some non-negative real constant $c$.

For the remainder of this writing we will work with $d$-uniform hyper-graphs with $d \geq 3$ some fixed number. We will abbreviate "$d$-uniform hyper-graph" as "graph".

We will call the **fragment** $F_n$ to the union of the unicyclic components in $G^d(n, c/n^{d-1})$. In this setting, a unicycle is a connected graph $H$ such that $(d-1)e(H) - v(H) = 0$.

We can show that for $0 \leq c \leq (d-2)!$

$$\lim_{n\to\infty} \Pr\big(F_n \text{ is empty }\big) = e^{\frac{c}{2(d-2)!}} \sqrt{1 - c/(d-2)!}.$$

Instead of working with $c$ it is more convenient to use $r := c/(d-2)!$ as our parameter. We will denote the RHS of last equation as $A(r)$. The limit probability $A(r)$ that a graph is acyclic reaches 1/2 when $r = 0.898172...$.

Given any graph $H$ which is a disjoint union of unicycles, for $0 \leq r < 1$ it is satisfied

$$\lim_{n\to\infty} \Pr\big(F_n \simeq H\big) = A(r)(re^{-r})^{e(H)} \frac{(d-2)!^{e(H)}}{|Aut(H)|}, \tag{1}$$

where $Aut(H)$ denotes the group of automorphisms of $H$.

## For $0.898172... \leq r < 1$ there are no gaps

Fix $0.898172... \leq r < 1$. Let $H_0, H_1, \dots$ be an enumeration of all graphs with unicyclic components such that $p_0 \leq p_1 \leq \dots$, where $p_i$ is the limit probability that the fragment $F_n$ is isomorphic to $H_i$, given by eq. (1). We want to show that

$$p_i \leq \sum_{j>i} p_j \qquad \forall i. \tag{2}$$

Let $q_i = p_i/A(r)$ and let $s = re^{-r}$. Then

$$q_i = s^{e(H_i)} \frac{(d-2)!^{e(H_i)}}{|Aut(H_i)|},$$

and eq. (2) holds iff

$$q_i \leq \sum_{j>i} q_j \qquad \forall i. \tag{3}$$

We will call a vertex $v \in V(H)$ **free** if it belongs to exactly one edge in $H$. Free vertices inside an edge are indistinguishable, so

$$\prod_{h \in E(H)} |Free(h)|! \leq |Aut(H)|,$$

where $Free(h)$ denotes the number of free vertices in the edge $h$.

We call an edge $e \in E(H)$ a **leaf** if it only contains one non-free vertex.

The following lemma is not strictly necessary. It will be only used once and its use could have been avoided in exchange of doing a more exhaustive enumeration of graphs so maybe that is the preferable option.

**Lemma.** For any graph $H$ whose connected components are unicycles,

$$\frac{(d-2)!^{e(H)}}{|Aut(H)|} \leq \frac{(d-2)^2}{(d-1)^2}.$$

*Proof.* It suffices to prove the statement for unicycles, because

$$\frac{(d-2)!^{e(H)}}{|Aut(H)|} \leq \prod_i \frac{(d-2)!^{e(H_i)}}{|Aut(H_i)|},$$

where the $H_i$'s are the connected components of $H$.

Let $\lambda = \lambda(H)$ be the number of leaves in $H$. We show by induction that

$$\prod_{h \in E(H)} \frac{(d-2)!}{|Free(h)|!} \leq \left(\frac{d-2}{d-1}\right)^{\lambda}. \tag{4}$$

If $\lambda = 0$ then $H$ is a cycle and each one of its edges contains exactly $d-2$ free vertices, so

$$\prod_{h \in E(H)} \frac{(d-2)!}{|Free(h)|!} = 1 = \left(\frac{d-2}{d-1}\right)^{0}.$$

And $H$ satisfies eq. (4).

Now let $H$ be an unicycle satisfying eq. (4). Add a new edge $h'$ to $H$ to obtain another unicycle $H'$. Then $h'$ intersects $H$ in only one vertex $v$. In consequence $h'$ is a leave of $H'$.

Consider the case $\lambda(H') = \lambda(H)$, where no new leaves are created with the addition of $h'$. This means that $v$ is a free vertex in one leaf $f$ of $H$ (that is, $h'$ "grows" out of $f$), and

$$\prod_{h \in E(H')} \frac{(d-2)!}{|Free(h)|!} = \prod_{h \in E(H)} \frac{(d-2)!}{|Free(h)|!}.$$

Otherwise $\lambda(H') = \lambda(H) + 1$. In this case $h'$ intersects an edge of $H$ that is not a leaf. Here, the case that maximizes $\prod_{h \in E(H')} \frac{(d-2)!}{|Free(h)|!}$ is the one where $h'$ grows out of a free vertex of an edge in $H$ with exactly $d-2$ free vertices. In this case

$$\prod_{h \in E(H')} \frac{(d-2)!}{|Free(h)|!} = \frac{d-2}{d-1} \prod_{h \in E(H)} \frac{(d-2)!}{|Free(h)|!},$$

and $H'$ satisfies eq. (4) as well.

Finally, as all unicycles can be obtained adding edges to a cycle successively, eq. (4) holds for all unicycles.

To prove the original statement consider the cases $\lambda = 0, \lambda = 1$ and $\lambda \geq 2$. If $\lambda = 0$ then $H$ is a cycle of length $l \geq 2$ and $|Aut(H)| = (d-2)!^l 2l$, yielding

$$\frac{(d-2)!^{e(H)}}{|Aut(H)|} = \frac{1}{2l} \leq \frac{(d-2)^2}{(d-1)^2},$$

as $1/2l \leq 1/4 \leq (d-2)^2/(d-1)^2$ for all $l \geq 2, d \geq 3$.

If $\lambda = 1$ then $H$ is a cycle with a path attached to it. In this case, there is a reflection of the cycle of $H$ in $Aut(H)$, and in consequence $2\prod_{h \in E(H)} |Free(h)|! \leq |Aut(H)|$. Using this and eq. (4) we get

$$\frac{(d-2)!^{e(H)}}{|Aut(H)|} \leq \frac{1}{2} \prod_{h \in E(H)} \frac{(d-2)!}{|Free(h)|!} \leq \frac{1}{2}\left(\frac{d-2}{d-1}\right) \leq \left(\frac{d-2}{d-1}\right)^2,$$

as we wanted.

Finally, for the case $\lambda \geq 2$ just eq. (4) suffices, as

$$\prod_{h \in E(H)} \frac{(d-2)!}{|Free(h)|!} \leq \left(\frac{d-2}{d-1}\right)^\lambda \leq \left(\frac{d-2}{d-1}\right)^2.$$

□

The bound in last lemma can be reached. Consider the family of graphs $(T_{\alpha,\beta})$, for $\alpha, \beta > 0$, where $T_{\alpha,\beta}$ denotes the graph consisting of a triangle with two paths of length $\alpha$ and $\beta$ respectively attached to two of its free vertices, each one from a different edge. One can check

$$\frac{(d-2)!^{e(T_{\alpha,\beta})}}{|Aut(T_{\alpha,\beta})|} = \frac{(d-2)!^{\alpha+\beta+3}}{|Aut(T_{\alpha,\beta})|} = \begin{cases} \left(\frac{d-2}{d-1}\right)^2 & \text{for } \alpha \neq \beta, \text{ or} \\ \frac{1}{2}\left(\frac{d-2}{d-1}\right)^2 & \text{otherwise.} \end{cases}$$

Then, for $k \geq 4$, if we sum $\frac{(d-2)!^k}{|Aut(H)|}$ for all different graphs with $k$ edges of the form $T_{\alpha,\beta}$ we obtain

$$\sum_{\alpha=1}^{\lfloor \frac{k-3}{2} \rfloor} \frac{(d-2)!^k}{|Aut(T_{\alpha,k-3-\alpha})|} = \frac{k-4}{2}\left(\frac{d-2}{d-1}\right)^2 \tag{5}$$

using considerations similar to the ones in the sketches of Tobias.

We introduce now another two families of graphs.

The graph $B_{\alpha,\beta}$, for $\alpha, \beta > 0$, is the one consisting of a two-cycle with two paths of length $\alpha$ and $\beta$ respectively attached to two of its free vertices, each one from a different edge. In this case

$$\frac{(d-2)!^{e(B_{\alpha,\beta})}}{|Aut(B_{\alpha,\beta})|} = \frac{(d-2)!^{\alpha+\beta+2}}{|Aut(B_{\alpha,\beta})|} = \begin{cases} \frac{1}{2}\left(\frac{d-2}{d-1}\right)^2 & \text{for } \alpha \neq \beta, \text{ or} \\ \frac{1}{4}\left(\frac{d-2}{d-1}\right)^2 & \text{otherwise.} \end{cases}$$

Summing $\frac{(d-2)!^k}{|Aut(H)|}$ for all different graphs of the form $B_{\alpha,\beta}$ with $k \geq 3$ edges yields

$$\sum_{\alpha=1}^{\lfloor \frac{k-2}{2} \rfloor} \frac{(d-2)!^k}{|Aut(T_{\alpha,k-2-\alpha})|} = \frac{k-3}{4}\left(\frac{d-2}{d-1}\right)^2 \tag{6}$$

We define the graph $O_{\alpha,\beta}$, with $\alpha > 1, \beta > 0$, as the one formed by attaching a path of length $\beta$ to a free vertex of a cycle of length $\alpha$. One can check that $e(O_{\alpha,\beta}) = \alpha+\beta$ and that $\frac{(d-2)!^{\alpha+\beta}}{|Aut(O_{\alpha,\beta})|} = \frac{1}{2}\left(\frac{d-2}{d-1}\right)$. In consequence, summing $\frac{(d-2)!^k}{|Aut(H)|}$ for all different graphs of the form $O_{\alpha,\beta}$ with $k \geq 2$ edges we obtain

$$\sum_{\alpha=2}^{k-1} \frac{(d-2)!^k}{|Aut(O_{\alpha,k-\alpha})|} = \frac{k-2}{2}\left(\frac{d-2}{d-1}\right). \tag{7}$$

For any $i \in \mathbb{N}$, let $k = k(i)$ be the natural number such that

$$s^{k-1}\left(\frac{d-2}{d-1}\right)^2 \geq q_i > s^k\left(\frac{d-2}{d-1}\right)^2$$

Notice that because of our previous lemma $k(i) - 1 \geq e(H_i)$.

Let $\mathcal{U}_k$ be the set of unlabeled graphs with $k$ edges whose components are unicycles. We will show that eq. (3) is satisfied for the different possible values of $k(q_i)$.

If $k \geq 4$ then

$$s^k \sum_{H \in U_k} \frac{(d-2)!^k}{|Aut(H)|} \geq s^k\left[\frac{k-4}{2}\left(\frac{d-2}{d-1}\right)^2 + \frac{k-2}{2}\left(\frac{d-2}{d-1}\right)\right] \geq s^k(k-3)\left(\frac{d-2}{d-1}\right)^2.$$

This is obtained taking into account the graphs of the form $T_{\alpha,\beta}$ and $O_{\alpha,\beta}$ in $\mathcal{U}_k$ and using eq. (5) and eq. (5).

Using last inequality and that $1/3 < s$, if $k = k(i) \geq 6$ then

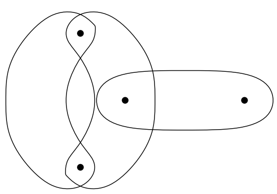
$$\sum_{j>i}^{q_i} > s^k \sum_{H \in U_k} \frac{(d-2)!^k}{|Aut(H)|} > s^k(k-3)\left(\frac{d-2}{d-1}\right)^2 > s^{k-1}\left(\frac{d-2}{d-1}\right)^2 > q_i.$$

If $k = k(i) = 5$ then

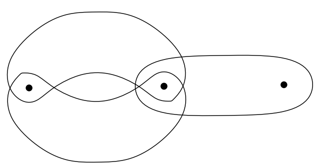
$$\sum_{j>i}^{q_i} > s^5 \sum_{H \in U_5} \frac{(d-2)!^5}{|Aut(H)|} + s^6 \sum_{H \in U_6} \frac{(d-2)!^6}{|Aut(H)|} > (s^5 2 + s^6 3)\left(\frac{d-2}{d-1}\right)^2 > s^4\left(\frac{d-2}{d-1}\right)^2 \geq q_i.$$

The only cases left are $k(i) = 4$ and $k(i) = 3$. We will study them together.

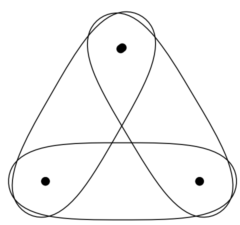
Assume $k(i) \leq 4$. Then $e(H_i) \leq 3$ (this is the only place where we really use the lemma). If $e(H_i) = 3$ then an enumeration of all unicycles with three edges gives $q_i \leq s^3 \frac{1}{2}\left(\frac{d-2}{d-1}\right)$.



(a) $\frac{(d-2)!^3}{|Aut(H)|} = \frac{1}{2}\left(\frac{d-1}{d-2}\right)$.



(b) $\frac{(d-2)!^3}{|Aut(H)|} = \frac{1}{2}\left(\frac{1}{d-2}\right)$.



(c) $\frac{(d-2)!^3}{|Aut(H)|} = \frac{1}{6}$.

Using eq. (7) we can bound $\sum_{H \in U_l} \frac{(d-2)!^6}{|Aut(H)|}$ by $\frac{l-2}{2}\left(\frac{d-2}{d-1}\right)$ for $l = 4, 5$, and we get

$$\sum_{j>i}^{q_i} > s^4 \sum_{H \in U_4} \frac{(d-2)!^5}{|Aut(H)|} + s^5 \sum_{H \in U_5} \frac{(d-2)!^6}{|Aut(H)|} > \left(\frac{s^4 2 + s^5 3}{2}\right)\left(\frac{d-2}{d-1}\right) > \frac{s^3}{2}\left(\frac{d-2}{d-1}\right) \geq q_i.$$

Finally if $H_i$ has only two edges, then $H_i = C_2$, (here $C_2$ denotes the cycle of length two, as usual) and $q_1 = s^2\frac{1}{4}$. One can also check, using that $s \leq 1/e$ and the previous enumeration of unicycles with three edges that $q_i = q_1$. That is, for any other graph $H \in \mathcal{U}$, $s^{e(H)}\frac{(d-2)!^{e(H)}}{|Aut(H)|} < \frac{s^2}{4}$. Let us call $\mathcal{T}$, $\mathcal{B}$ and $\mathcal{O}$ to the sets of graphs of the form $T_{\alpha,\beta}$, $B_{\alpha,\beta}$ and $O_{\alpha,\beta}$ respectively. We have:

$$\sum_{j>1} q_j > s^3\frac{(d-2)!^3}{|Aut(C_3)|} + \sum_{H\in\mathcal{B}} s^{e(H)}\frac{(d-2)!^{e(H)}}{|Aut(H)|} + \sum_{H\in\mathcal{O}} s^{e(H)}\frac{(d-2)!^{e(H)}}{|Aut(H)|}. \tag{8}$$

We can bound each of the terms in the LHS of last inequality. Using eq. (6) we obtain

$$\sum_{H\in\mathcal{B}} s^{e(H)}\frac{(d-2)!^{e(H)}}{|Aut(H)|} = \sum_{k=4}^{\infty} s^k\frac{k-3}{4}\left(\frac{d-2}{d-1}\right)^2.$$

And using that $s > 1/3$, and the fact that for $a, b, c, x \in \mathbb{R}$ and $|x| < 1$ we have

$$\sum_{n=0}^{\infty} c(a+nb)x^n = \frac{ac}{1-x} + \frac{bcx}{(1-x)^2},$$

we can get

$$\sum_{H\in\mathcal{B}} s^{e(H)}\frac{(d-2)!^{e(H)}}{|Aut(H)|} = \sum_{k=5}^{\infty} s^k\frac{k-4}{4}\left(\frac{d-2}{d-1}\right)^2 > s^4\left(\frac{d-2}{d-1}\right)^2\frac{9}{16} > s^3\left(\frac{d-2}{d-1}\right)^2\frac{3}{16}. \tag{9}$$

Analogously as before, using eq. (7) and the formula for the arithmetico-geometric series we get

$$\sum_{H\in\mathcal{O}} s^{e(H)}\frac{(d-2)!^{e(H)}}{|Aut(H)|} = \sum_{k=3}^{\infty} s^k\frac{k-2}{4}\left(\frac{d-2}{d-1}\right). > s^3\left(\frac{d-2}{d-1}\right)\frac{9}{8} \tag{10}$$

Now, using eq. (9) and eq. (10) in eq. (8) an the fact that $\frac{(d-2)!^3}{|Aut(C_3)|} = \frac{1}{6}$ we have

$$\sum_{j>1} q_j > s^3\frac{1}{6} + s^3\left(\frac{d-2}{d-1}\right)^2\frac{3}{16} + s^3\left(\frac{d-2}{d-1}\right)\frac{9}{8}.$$

Finally, substituting $\left(\frac{d-2}{d-1}\right) \geq \frac{1}{2}$, we obtain

$$\sum_{j>1} q_j > s^3\frac{1}{6} + s^3\frac{3}{64} + s^3\frac{9}{16} = s^3\frac{32+9+108}{192} > s^3\frac{3}{4} > s^2\frac{1}{4} = q_1,$$

as we wanted.