First Order Logic of Sparse Random Hyper-Graphs

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1. First Order Logic of Sparse Graphs

2. General relational structures

Contents

The first order language of graphs.

- Variables $x_1, \ldots, x_n, \ldots$
- Connectives $\wedge, \vee$, equality symbol $=$, and negation symbol $\neg$.
- Quantifiers $\forall, \exists$.
- A binary relation symbol R

- Vertices.
- "And", "or", "equals", "not".
- "For all", "there exists".
- Edges $x \sim y$.

$$\forall x_1, x_2\, R(x_1, x_2) \implies \exists x_3 (\neg(x_3 = x_1) \wedge \neg(x_3 = x_2) \wedge R(x_1, x_3))$$

The binomial model

The binomial model of random graphs $G(n, p)$ is a discrete probability space where we assign to each graph $G = ([n], E)$ the probability

$$\Pr(G) = p^{|E|} \cdot (1-p)^{\binom{n}{2}-|E|}.$$

Lynch's theorem

Theorem (Lynch, 1992)

Let φ be a sentence in the F.O. language of graphs. Then the map $\mathrm{F}_\varphi : [0,\infty) \to \mathbb{R}$ given by

$$\mathrm{F}_\varphi(\beta) = \lim_{n\to\infty} \mathrm{Pr}(\, G(n, \beta/n) \models \varphi \,)$$

is well defined and admits an analytic extension to $\mathbb{C}$.

Overview of the proof

Some properties of $G(n, \beta/n)$:

- The number of cycles of length $3, 4 \ldots, r$ are asymptotically distributed like independent Poisson variables.
- Small cycles are a.a.s far away.
- Fixed vertices are a.a.s far away.
- The ball of a given radius centered in fixed vertex is a.a.s a tree. Any tree occurs with a positive probability.

Overview of the proof

For each fixed quantifier rank k:

(1) It is given a finite classification of "small" uni-cycles.

(2) It is shown that the rank k type of random graph G in $G(n, \beta/n)$ a.a.s depends exclusively on the number of "small" uni-cycles belonging to each class.

(3) The asymptotic distribution of those quantities is obtained.

Contents

Edge sets

Definition

Fix

- natural numbers $n, a \in \mathbb{N}$, with $a \geq 2$
- a subgroup of the symmetric group $\Phi \leq S_a$
- and a subset of pairs $A \subset ([a] \times [a]) \setminus \Delta_a$.

The **total edge set** $\mathcal{H}_{(a,\Phi,A)}(n)$ of size a, symmetry group Φ and restrictions A, on n elements is the set

$$\mathcal{H}_{(a,\Phi,A)}(n) = ([n]^a/\Phi) \ \setminus R,$$

where

$$R = \{ [x_1, \ldots, x_a] \in [n]^a/\Phi \mid x_i = x_j \text{ for some } (i,j) \in A\}$$

Graphs

Definition

An (hyper)-graph $([n], H_1, \dots, H_c)$ with edge colors $1, \dots, c$, sizes $a_1, \dots, a_c$, symmetry groups $\Phi_1, \dots, \Phi_c$ and restrictions $A_1, \dots, A_c$ consists of

- The vertex set $[n]$ for some natural number n.
- For $i = 1, \dots, c$, a "colored" edge set $H_i \subseteq \mathcal{H}_{(a_i, \Phi_i, A_i)}(n)$ whose elements have color i.

The first order language

Consider the first order purely relational language $\mathcal{L}$ whose signature consists of the relation symbols $R_1, \dots, R_c$ with arities $a_1, \dots, a_c$.

A graph $G = ([n], H_1, \dots, H_c)$ is a $\mathcal{L}$-structure in the following way:

- The universe of G is its vertex set, $[n]$.
- For each $1 \leq i \leq c$,

$$(x_1, \dots, x_{a_i}) \in R_i^G \iff [x_1, \dots, x_{a_i}] \in H_i.$$

The first order language

By definition, a graph $G = ([n], H_1, \ldots, H_n)$ satisfies, for each $1 \le i \le c$:

- Symmetry formulas:

$$S_g := \big(R_i(x_1 \ldots, x_{a_i}) \iff R_i(x_{g(1)} \ldots, x_{g(a_i)})\big),$$

  where g is an element from Φ_i.

- Anti-reflexivity formulas:

$$A_{i,(j,l)} := \big(R_i(x_1 \ldots, x_{a_i}) \implies \neg(x_j = x_l)\big),$$

  where $(j, l) \in A_i$.

The random model

The random model $HG(n, p_1, \ldots, p_c)$ is a discrete probability space where for each graph $G = ([n], H_1, \ldots, H_c)$,

$$\Pr(G) = \prod_{i=1}^{c} p_i^{|H_i|} \cdot (1 - p_i)^{|\mathcal{H}_{(a_i, \Phi_i, A_i)}(n)| - |H_i|}.$$

We consider the case where for each $1 \leq i \leq c$, $p_i(n) = \beta_i / n^{a_i - 1}$.

Let $\beta = (\beta_1, \ldots, \beta_c) \in [0, \infty)^c$. We abbreviate

$$HG(n, p(n, \beta)) := HG(n, \beta_1 / n^{a_1 - 1}, \ldots, \beta_c / n^{a_c - 1}).$$

The theorem

For each $1 \leq i \leq c$ let $p_i(n, \beta_i) = \beta_i / n^{a_i - 1}$. Let $\beta = (\beta_1, \ldots, \beta_c)$ and $p(n, \beta) = (p_1(n, \beta_i), \ldots, p_c(n, \beta_c)$.