RANGE AND DEGREE OF REALIZABILITY OF FORMULAS IN THE RESTRICTED PREDICATE CALCULUS

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The following numerical characteristics for formulas of the restricted predicate calculus are considered: the range of realizability, i.e., the number of all n-models (models with n individuals) that satisfy a given formula, and the degree of realizability, i.e., the ratio of the range of realizability to the number of all possible n-models. Analogous characteristics have been defined by Kemeny [1]. A general computational algorithm is given, and it is shown that the function of n that represents the range of realizability (which we shall refer to as the range) is Kalmár-elementary (Section 1.1). Next (Section 1.2) we consider certain special cases of formulas for which the range algorithm is less cumbersome. We then show (Section 2) that for all formulas in the restricted predicate calculus the degree of realizability (which we shall refer to as the degree) has a limit at $n \to \infty$. When the formula is closed and contains no propositional variables, this limit will be 0 or 1.*

1. ON A CLASS OF FUNCTIONS EXPRESSING THE RANGE OF REALIZABILITY OF LOGICAL FORMULAS

1.1. Let A be a closed formula in the restricted predicate calculus and let $F_1^{m_1}, F_2^{m_2}, \ldots, F_k^{m_k}$ be the complete list of predicate variables occurring in A. Some of the $m_1, m_2, \ldots, m_k$ may equal zero; then the corresponding $F_i^{m_i}$ ($m_i = 0$) will be propositional variables.

The sequence $\langle F_1^{m_1}, F_2^{m_2}, \ldots, M_k^{m_k} \rangle$, where the $F_i^{m_i}$, ($i = 1, 2, \ldots, k$) are m_i-place predicates defined on the set of all natural numbers from 1 to n, inclusive, is called an n-model of A.

By the range of realizability of A we mean the number $v_n A$ of different n-models that turn A into a true statement. To specify an n-model, we need only give the characteristic functions of the predicate $F_1^{m_1}, F_2^{m_2}, \ldots, F_k^{m_k}$. We represent these as $\varphi_1^{m_1}, \varphi_2^{m_2}, \ldots, \varphi_k^{m_k}$.

We consider the prenex normal form A' of A. It has the form $[(\Pi_1 x_1)(\Pi_2 x_2) \ldots (\Pi_s x_s) B]$, where the $[(\Pi_i x_i)$ ($i = 1, 2, \ldots, s$)] are the universal or existential quantifiers, and B is a formula containing no quantifiers.

We assume that B is in the full disjunctive normal form (fdnf) B'; we expand into the fdnf in terms of the elementary parts of B. In B', we replace each symbol $F_i^{m_i}$ by $\varphi_i^{m_i}$, or $(1-\varphi_i^{m_i})$, respectively; the conjunction symbols are then replaced by the multiplication sign, and the disjunction symbols by the plus sign. The result will be the arithmetic expression $a(x_1, x_2, \ldots, x_s)$. We consider the expression:

$$R_1 x_1 [R_2 x_2 [\ldots [R_s x_s (\mathrm{sg}\, a(x_1; x_2, \ldots, x_s))] \ldots]]. \tag{1}$$

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* This last result was reported to the seminar on algebra and logic of the Institute of Mathematics of the Siberian Division, AS USSR in April 1966 and August 23, 1966, during the Fourth International Mathematical Congress.

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Here $R_i x_i$ is $\prod_{x_i=1}^{n}$, if $\Pi_i x_i$ is the universal quantifier, and $\sum_{x_i=1}^{n}$, if $\Pi_i x_i$ is the existential quantifier.

Expression (1) is a function of n, $\varphi_1^{m_1}$, $\varphi_2^{m_2}$, . . . , $\varphi_k^{m_k}$. We represent it by $U_A(n, \varphi_1^{m_1}, \dots, \varphi_k^{m_k})$. For a fixed n-model, i.e., for fixed n, $\varphi_1^{m_1}$, $\varphi_2^{m_1}$, . . . , $\varphi_k^{m_k}$, the expression U_A will equal 1 or 0: 1 if A is true on the given n-model and 0 if it is not.

The characteristic function $\varphi_i^{m_i}$, i.e., the predicate $F_i^{m_i}$, can be selected in $2^{n^{m_i}}$ different ways. Thus, the total number of different n-models of A is $2^{n^{m_1}+n^{m_2}+\dots+n^{m_k}}$.

We now number the ways of specifying the functions $\varphi_i^{m_i}$ (see Table 1); there will be k tables of this ind, since each function will have its own table).

As the table shows, the numbering algorithm permits a natural extension of the table if we go to a larger domain of argument values. If we go to a smaller domain, however, we should use a particular part of the given table (the upper left-hand corner).

If we assume that the k tables of the above type are unbounded (i.e., they specify the numbering of functions for any natural n), then the functions that they specify become everywhere defined, and we represent them as $\tilde{\varphi}_i^{m_i}$

THEOREM 1. Every function $\tilde{\varphi}_i^{m_i}$ is Kalmár-elementary [2].

This is trivially proven if any function $\tilde{\varphi}_i^{m_i}$ is nonzero at only a finite set of points.

Let some function $\varphi_i^{m_i}$ have the number y, in accordance with Table 1; we then represent it as $\varphi_{i,y}^{m_i}$. We introduce the functions

$$f_i^{m_i+1}(y, x_{i_1}, x_{i_2}, \dots, x_{i_{m_i}}) = \varphi_{i,y}^{m_i}(x_{i_1}, x_{i_2}, \dots, x_{i_{m_i}}), \quad i = 1, 2, \dots, k.$$

TABLE 1. Numbering of Ways of Specifying the Functions $\varphi_i^{m_i}$

Argument $x_1\, x_2 \dots x_{m_i-1}\, x_{m_i}$	№ 1	№ 2	№ 3	. . .	№ $2^{2^{m_i}}$	№ $2^{2^{m_i}}+1$	. . .	№ $2^{3^{m_i}}$	. . .	№ $2^{k^{m_i}}$		№ $2^{n^{m_i}}$
1 1 . . . 1 1	0	1	0	. . .	1	0	. . .	1	. . .	1	. . .	1
1 1 . . . 1 2	0	0	1		1	0		1		1		1
1 1 . . . 2 1	0	0	0	. . .	1	0	. . .	1	. . .	1	. . .	1
. . . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
2 2 . . . 2 1	0	0	0	0	1	0	. . .	1	. . .	1	. . .	1
2 2 . . . 2 2	0	0	0	0	1	0		1		1		1
1 1 . . . 1 3	0	0	0	. . .	0	1	. . .	1	. . .	1	. . .	1
1 1 . . . 2 3	0	0	0		0	0		1		1		1
1 1 . . . 3 1	0	0	0		0	0		1		1		1
1 1 . . . 3 2	0	0	0		0	0		1		1		1
1 1 . . . 3 3	0	0	0	. . .	0	0	. . .	1	. . .	1	. . .	1
. . . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
3 3 . . . 3 3	0	0	0		0	0		1		1		1
. . . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
1 1 . . . 1 k	0	0	0		0	0		0		1		1
1 1 . . . 2 k	0	0	0		0	0		0		1		1
. . . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
1 1 . . . k—1 k	0	0	0	-	0	0		0		1		1
1 1 . . . k 1	0	0	0	. . .	0	0	. . .	0	. . .	1	. . .	1
1 1 . . . k 2	0	0	0		0	0		0		1		1
. . . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
k k . . . k k	0	0	0		0	0		0		1		1
. . . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
n n n n—1	0	0	0		0	0		0		0		1
n n n n	0	0	0		0	0		0		0		1

For fixed n, the first argument of $f_i^{m_i+1}$ takes on values from 1 to $2^{n^{m_i}}$, inclusive; the remaining arguments take on values on the natural-number set 1, 2, ..., n. If the k numbered tables of the Table-1 type are taken to be unbounded, then the functions $f_i^{m_i+1}$ become everywhere defined; in such cases, we represent them as $\tilde{f}^{m_i+1}$.

THEOREM 2. The functions $\tilde{f}^{m_i+1}$ belong to the class F_1 in the Ritchie hierarchy [2], and are Kalmár-elementary.

The proof that $\tilde{f}^{m_i+1}$ belongs to F_1 consists in constructing a Turing machine that computes $\tilde{f}_i^{m_i+1}$ so that the amount of tape required to compute $\tilde{f}_i^{m_i+1}$ $(y, x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ is majorized by a certain linear function $l_i(y, x_{i_1}, x_{i_2}, \ldots, x_{i_m})$. We shall not describe such a machine, since the process is very cumbersome. By virtue of Theorem 3 of [2], all of the functions $\tilde{f}_i^{m_i+1}$ are elementary since they belong to F_1.

To specify an n-model of A means to give the $F_1^{m_1}, F_2^{m_2}, \ldots, F_k^{m_k}, \varphi_1^{m_1}, \varphi_2^{m_2}, \ldots, \varphi_k^{m_k}$, which is equivalent to specifying certain $f_1^{m_1+1}, f_2^{m_1+1}, \ldots, f_k^{m_k+1}$ for particular values of the first arguments $y_1, y_2, \ldots, y_k$, where $y_i \in \{1, 2, \ldots, 2^{n^{m_i}}\}$ $(i = 1, 2, \ldots, k)$.

We consider the function

$$V_A(n, y_1, y_2, \ldots, y_k) = U_A(n, \tilde{f}_1^{m_1+1}, \tilde{f}_2^{m_2+1}, \ldots, \tilde{f}_k^{m_k+1}).$$

Then the range of A is

$$v_n A = \sum_{y_1=1}^{2^{n^{m_1}}} \sum_{y_2=1}^{2^{n^{m_2}}} \cdots \sum_{y_k=1}^{2^{n^{m_k}}} V_A(n, y_1, y_2, \ldots, y_k). \tag{2}$$

THEOREM 3. The function $v_n A$ is Kalmár-elementary.

The function $V_A(n, y_1, y_2, \ldots, y_k)$ is elementary since $\tilde{f}_i^{m_i+1}$ $(i = 1, 2, \ldots, k)$ is elementary and the class of elementary functions is closed under the operations of permutation, summation, and multiplication. Thus, $2^{n^{m_1}}, 2^{n^{m_2}}, \ldots, 2^{n^{m_k}}$ are elementary, and so is $v_n A$. We can show that in the Ritchie hierarchy, the function $V_A(n, y_1, y_2, \ldots, y_k)$ belongs to F_1, while $v_n A$ belongs to F_2. We let W represent the class of functions, each of which is the range of a certain formula A, and is specified by an expression of the type (2). We then have the following theorem.

THEOREM 4. The question of the equality of the elements of W is undecidable.

This theorem is a consequence of a result due to Trakhtenbrot [4].

1.2. Relationship (2) from Section 1.1 permits us to calculate, for any fixed n, the range of an arbitrary formula of the restricted predicate calculus. This algorithm, however, is a method for examining all possible n-models. For certain classes of formulas it is possible, when determining the range, to avoid inspecting all possible n-models, and to find the way in which the range depends on n in a more direct way than in Section 1.1.

Here we shall only consider closed formulas from the singulary calculus $F^{1,1}$,* and we shall show that if a formula

$$A(P_1, P_2, \ldots, P_k) \in F^{1,1}$$

(where $P_1, P_2, \ldots, P_k$ is the list of all single-placed predicate variables occurring in $A(P_1, P_2, \ldots, P_k)$), then the range is

$$v_n A = \sum_{s=1}^{t} \alpha_s l_s^n, \tag{3}$$

where α_s and l_s are constant integers determined by the form of $A(P_1, P_2, \ldots, P_k)$ and $l_s \geq 0$ [compare (1)]. We can arrive at this result by the following argument.

* See [3, p. 165].

Let $A(P_1, P_2, \ldots, P_k) \in F^{1.1}$. By a finite number of applications of certain transformation rules (which can be found, for example, in [3], pp. 199–204), we can transform $A(P_1, P_2, \ldots, P_k)$ to a formula $B(P_1, P_2, \ldots, P_k)$ such that: a) $\vdash A \equiv B$; b) the formula $B(P_1, P_2, \ldots, P_k)$ will have the following form:

$$\bigvee_{i=1}^{s} (x)\, C_0^i(P_i, \ldots, P_k) \,\&\, (\exists x)\, C_1^i(P_1, \ldots, P_k) \,\&\ldots\&\, (\exists x)\, C_{l_i}^i(P_1, \ldots, P_k),$$

where all of the $C_j^i(P_1, \ldots, P_k)$ are quantifier-free formulas with a single free variable x (in particular, $C_0^i(P_1, \ldots, P_k)$ can be an identically true formula).

Let us determine the range of the formula $B(P_1, \ldots, P_k)$ (by virtue of the condition a): $v_nA(P_1, \ldots, P_k = v_nB(P_1, \ldots, P_j)$. Letting D_i represent each disjunctive term in $B(P_1, \ldots, P_k)$, and using the principle of inclusion and exclusion [5], we find that

$$v_nB = v_n\left(\bigvee_{i=1}^{s} D_i\right) = \sum_{r=1}^{s} \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant s} (-1)^{r+1} \times v_n(D_{i_1} \& \ldots \& D_{i_r}). \tag{4}$$

It is easy to see, however, that every formula occurring on the right-hand side of (4) can be reduced to the form

$$(x)\, C_0(P_1, \ldots, P_k) \,\&\, (\exists x)\, C_1(P_1, \ldots, P_k) \,\&\ldots\&\, (\exists x)\, C_j(P_1, \ldots, P_k), \tag{5}$$

where all of the $C_i(P_1, \ldots, P_k)$ are quantifier-free formulas with predicate variables $P_1, \ldots, P_k$ and free individual variable x. Thus to find the v_nB we need only be able to find the range of formulas in $F^{1.1}$ that have the form (5).

Let $j = 0$, i.e., we have

$$(x)\, C_0(P_1, \ldots, P_k). \tag{6}$$

We expand $C_0(P_1, \ldots, P_k)$ into the fdnf, assuming that the $P_1(x), P_2(x), \ldots, P_k(x)$ are propositional variables. We then have

$$(x)\, C_0(P_1, \ldots, P_k) \equiv (x)\left[\bigvee_{(\sigma_1, \ldots, \sigma_k)} P_1^{\sigma_1}(x) \& \ldots \& P_k^{\sigma_k}(x)\right].$$

It is simple to show that $v_n[(x)C_0(P_1, \ldots, P_k)] = l_0^n$ is the number of disjunctive terms in the fdnf of of C_0.

Now let $j = 1$; we then have E

$$(x)\, C_0(P_1, \ldots, P_k) \,\&\, (\exists x)\, C_1(P_1, \ldots, P_k).$$

But

$$(x) C_0(P_1, \ldots, P_k) \equiv (x)\, C_0(P_1, \ldots, P_k) \,\&\, (\exists x)\, C_1(P_1, \ldots, P_k) \vee (x)\, C_0(P_1, \ldots, P_k) \,\&\, (x) \overline{C}_1(P_1, \ldots, P_k),$$

so that

$$v_n[(x)\, C_0(P_1, \ldots, P_k) \,\&\, (\exists x)\, C_1(P_1, \ldots, P_k)] = v_n(x)\, C_0 - v_n(x)\,(C_0 \& \overline{C}_1).$$

Since $(x)(C_0$ and $C_1)$ has the form (6), $v_n[(x)C_0$ and $(Ex)C_1] = l_0^n$, where l_0 is the number of different disjunctive terms in the fdnf of $Co(P_1, \ldots, P_k)$, while l_{01} is the number of common disjunctive terms in the expansions of the formulas $C_0(P_1, \ldots, P_k)$ and $C_1(P_1, \ldots, P_k)$ into the fdnf with respect to the variables $P_1(x), P_2(x), \ldots, P_k(x)$.

Arguing similarly, and using induction on j, we find that formulas in $F^{1.1}$ of form (5) have the following range of realizability:

$$v_n[(x)\, C_0 \& (\exists x)\, C_1 \& (\exists x)\, C_2 \& \ldots \& (\exists x)\, C_j] = l_0^n - \sum_{i_1=1}^{j} l_{0i_1}^n + \ldots + (-1)^r \times \sum_{1 \leqslant i_1 < i_2 \ldots < i_r \leqslant j} l_{0i_1i_2i_r}^n + \ldots + (-1)^j l_{012\ldots j}^n,$$

where $l_{0i_1i_2\ldots i_r}$ is the number of common disjunctive terms in the expansions of the formulas $C_0, \overline{C}_{i_1}, \overline{C}_{i_2}, \ldots, \overline{C}_{i_r}$ into the fdnf with respect to the variables $P_1(x), P_2(x), \ldots, P_k(x)$.

Using this scheme to calculate the range for each formula of (4), we arrive at the desired relationship (3).

2. ASYMPTOTIC BEHAVIOR OF DEGREE OF REALIZABILITY FOR FORMULAS OF THE RESTRICTED PREDICATE CALCULUS

2.1. We consider an arbitrary formula A of the restricted predicate calculus. Let

$$q_1, q_2, \ldots, q_l \tag{7}$$

be a finite list of propositional variables such that it contains all propositional variables of the formula A, let

$$F_1^{m_1}, F_2^{m_2}, \ldots, F_k^{m_k} \tag{8}$$

be a finite list of predicate variables containing all predicate variables of A, and let

$$y_1, y_2, \ldots, y_m \tag{9}$$

be a finite list of individual variables containing all free individual variables of A.

We take the set U_n, consisting of the n objects 1, 2, ..., n, as the domain of interpretation of the individual variables.

Definition. The degree of realizability of A in U_n with respect to the fixed set of values $y_1^0, y_2^0, \ldots, y_m^0$ for list (9) is defined as the quantity

$$\delta_n A(y_1^0, y_2^0, \ldots, y_m^0) = \frac{v_n A(y_1^0, y_2^0, \ldots, y_m^0)}{2^l \cdot 2^{n^{m_1}+n^{m_2}+\ldots+n^{m_k}}}, \tag{10}$$

where $v_n A(y_1^0, y_2^0, \ldots, y_m^0)$ is the range of realizability of the closed formula obtained from A if we replace the free individual variables by corresponding values from the set $y_1^0, y_2^0, \ldots, y_m^0$; $2^{n^{m_1}+n^{m_2}+\ldots+n^{m_k}}$ is the number of all possible n-models for list (8); and 2^l is the number of all possible sets of values for list (7).

Remark 1. It is easy to see that $\delta_n A(y_1^0, y_2^0, \ldots, y_m^0)$ depends solely on the variables from (7)-(9) that occur in A. Thus, we shall represent the degree of the formula A with free individual variables $x_1, x_2, \ldots, x_r$ by $\delta_n A(x_1^0, x_2^0, \ldots, x_r^0)$, where $x_1^0, x_2^0, \ldots, x_r^0$ are the values of the variables $x_1, x_2, \ldots, x_r$ in the set $y_1^0, y_2^0, \ldots, y_m^0$.

Remark 2. Since the domain U_n of objects is finite, we can define degree of realizability in another manner if we recall that every formula A in the restricted predicate calculus can be treated as the propositional formula A' obtained from A by replacing all universal quantifiers by n-conjunctions and all existential quantifiers by n-disjunctions. The propositional variables in A' will consist of all of the propositional variables of A together with expressions obtained from the elementary parts $F_i^{m_i}(x_i, \ldots, x_{m_i})$ of A by substitution of 1, 2, ..., n for the individual variables. If we let s be the number of different propositional variables in A', and l the number of different sets of values of the propositional variables of A' for which A' becomes "true," then obviously

$$\delta_n A(x_1^0, x_2^0, \ldots, x_r^0) = \frac{l}{2^s}.$$

Let $x_1^0, x_2^0, \ldots, x_r^0$ and $x'_1, x'_2, \ldots, x'_r$ be two sets of values for the free individual variables of A, such that for any pair i, j, $x_i^0 = x_j^0$ will imply $x'_i = x'_j$, and $x'_i = x'_j$ will imply $x_i^0 = x_j^0$. Then on the basis of of Remark 2 we can show that

$$\delta_n A(x_1^0, x_2^0, \ldots, x_r^0) = \delta_n A(x'_1, x'_2, \ldots, x'_r).$$

We can establish a one-to-one mapping φ of U_n onto itself such that $x_i^{\prime} = \varphi x_i^0$ and $x_i^0 = \varphi x_i^{\prime}$, $i = 1, 2, \ldots, r$, while the remaining elements stay fixed. Let $F_1^0, F_2^0, \ldots, F_k^0$ be a certain n-model from list (8) that makes the formula $A(x_1^{\prime}, x_2^{\prime}, \ldots, x_r^{\prime})$ true. We define the n-model $F_1^{\prime}, F_2^{\prime}, \ldots, F_k^{\prime}$ from the same list as follows:

$$F_i^{\prime}(x_1, x_2, \ldots, x_r) = F_i^0(\varphi^{-1}x_1, \varphi^{-1}x_2, \ldots, \varphi^{-1}x_r).$$

Clearly, this n-model will make the formula $A(x'_1, x'_2, \ldots, x'_r)$ true. The one-to-one relationship thus established between the n-models shows that

$$\delta_n A(x_1^0, x_2^0, \ldots, x_r^0) = \delta_n A(x_1^{\prime}, x_2^{\prime}, \ldots, x_r^{\prime}).$$

These properties of the degree of realizability permit us to give the following definition.

Definition. We let $\gamma_n A = \delta_n A(x_1^0, x_2^0, \ldots, x_r^0)$ be the normal degree of the formula A when $x_i^0 \neq x_j^0$, provided that $i \neq j$.

Remark 3. We note that for closed formulas of the restricted predicate calculus, the concepts of degree of realizability and normal degree coincide. It is also clear that for any set of values $x_1^0, x_2^0, \ldots, x_r^0$, the degree of realizability $\delta_n A(x_1^0, x_2^0, \ldots, x_r^0)$ can be treated as the normal degree of the formula A' obtained from A by identification of the free variables whose values coincide in the set $x_1^0, x_2^0, \ldots, x_r^0$.

The following relationships, which connect the degrees of realizability for different formulas, follow directly from the definition:

2.1.1. if $\vdash A(x_1, \ldots, x_r) \equiv B(y_1, \ldots, y_t)$, * then $\delta_n A(x_1^0, \ldots, x_r^0) = \delta_n B(y_1^0, \ldots, y_t^0)$.

2.1.2. if $\vdash A(x_1, \ldots, x_r) \supset B(y_1, \ldots, y_t)$, then $\delta_n A(x_1^0, \ldots, x_2^0) \leqslant \delta_n B(y_1^0, \ldots, y_t^0)$.

2.1.3. $\delta_n \overline{A}(x_1^0, \ldots, x_r^0) = 1 - \delta_n A(x_1^0, \ldots, x_r^0)$.

2.1.4. if $\vdash \overline{A(x_1, \ldots, x_r) \& B(y_1, \ldots, y_t)}$, then

$$\delta_n [A(x_1^0, \ldots, x_r^0) \vee B(y_1^0, \ldots, y_t^0)] = \delta_n A(x_1^0, \ldots, x_r^0) + \delta_n B(y_1^0, \ldots, y_t^0).$$

2.1.5. $\delta_n [A(x_1^0, \ldots, x_r^0) \vee B(y_1^0, \ldots, y_t^0)] \geqslant \delta_n A(x_1^0, \ldots, x_r^0)$.

2.1.6. $\delta_n [A(x_1^0, \ldots, x_r^0) \vee B(y_1^0, \ldots, y_t^0)] \leqslant \delta_n A(x_1^0, \ldots, x_r^0) + \delta_n B(y_1^0, \ldots, y_t^0)$.

2.1.7. If A and B contain no common propositional or predicate variables, then for any set $x_1^0, \ldots, x_r^0, y_1^0, \ldots, y_r^0$,

$$\delta_n [A(x_1^0, \ldots, x_r^0) \& B(y_1^0, \ldots, y_t^0)] = \delta_n A(x_1^0, \ldots, x_r^0) \times \delta_n B(y_1^0, \ldots, y_t^0).$$

All of these relationships hold for normal degrees as well.

The relationships 2.1.1-2.1.7, together with 2.1.8-2.1.10, give a simple algorithm for determining the degree of a formula, which can be represented so as to satisfy the following conditions:

a) each predicate and propositional variable has no more than one occurrence;

b) each predicate variable occurring in the range of a quantifier $(\forall x)$ or $(\exists x)$ depends on x.

2.1.8. If the formula A has the form $F(x_1, \ldots, x_m)$ (where F is a predicate variable and the $x_1, x_2, \ldots, x_m$ are individuals) or appears as a separate propositional variable, then $\delta A = 1/2$.

2.1.9. If the formula A satisfies conditions a) and b), and has the form $B \vee C$, then $\delta_n A = \delta_n B + \delta_n C - \delta_n B \cdot \delta_n C$.

2.1.10. If the formula A satisfies the conditions a) and b), and has the form $(x)B$ (or $(\exists x)B$), then $\delta_n A = (\delta_n B)^n$ (or $\delta_n A = 1-(1-\delta_n B)^n$).

2.2. We extend the language of the restricted predicate calculus as follows. If the $x_1, \ldots, x_l$ are individual variables, we use the symbol combination $(x/x_1, \ldots, x_l)$ as exclusive universal quantifier (read as "for all x other than $x_1, \ldots, x_l$"). We use the combination $(\exists x/x_1, \ldots, x_l)$ as the exclusive existential quantifier (read as "there exists an x other than $x_1, x_2, \ldots, x_l$") (compare with [6] and [7]).

* If A is a formula, then $\vdash$ A means that A is valid.

We supplement the rules for formula formation in the restricted predicate calculus as follows.

If $A(x, x_1, \ldots, x_r)$ is a rule for constructing a formula with free variables $x, x_1, \ldots, x_r$ and $y_1, y_2, \ldots, y_l$ is a finite list containing the free individual variables of A, except for x, then the formulas $(x/y_1, y_2, \ldots, y_l) A(x, x_1, \ldots, x_r)$ and $(\exists x/y_1, y_2, \ldots, y_l) A(x, x_1, \ldots, x_r)$ will be a rule constructed with free variables $y_1, y_2, \ldots, y_l$. We shall say that the formula $(x/y_1, y_2, \ldots, y_l)\ A(x, x_1, \ldots, x_r)$ for some system of values $y_1^0, y_2^0, \ldots, y_r^0$ $(u_i^0 \in U_n)$ of its free variables $y_1, y_2, \ldots, y_l$ assumes the value "true" on a certain n-model if for all values of the variable x from W_n different from $y_1^0, y_2^0, \ldots, y_l^0$ the formula $A(x, x_1^0, \ldots, x_r^0)$ is true on this n-model. If this is not the case, it assumes the value "false" on this n-model.

For values $y_1^0, y_2^0, \ldots, y_l^0$ $(y_i^0 \in U_n)$, the formula $(\exists x/y_1, y_2, \ldots, y_l)\ A(x, x_1, \ldots, x_r)$ becomes "true" on a certain n-model if there exists a value $x^0 \in U_n$ for the variable x such that $x^0 \neq y_1^0, x^0 \neq y_2^0, \ldots, x^0 \neq y_l^0$ and the formula $A(x^0, x_1^0, \ldots, x_r^0)$ is true on this n-model. If there is no such x^0, the formula is "false" on this n-model (compare [6] and [7]). Formulas that contain no (conventional) universal and existential quantifiers are said to be Γ-formulas.

Since

$$(x)\, A(x, x_1, \ldots, x_r) \equiv (x/y_1, \ldots, y_l)$$

$$A(x, x_1, \ldots, x_r) \,\&\, \mathop{\&}_{i=1}^{l} A(y_i, x_1, \ldots, x_r) \qquad (\exists x) A(x, x_1, \ldots, x_r) \equiv (\exists x/y_1, \ldots, y_l) A(x, x_1, \ldots, x_r) \vee \bigvee^{l} A(y_i, x_1, \ldots, x_r)$$

we can make the following assertion.

2.2.1. For every formula in the restricted predicate calculus there exists an equivalent Γ-formula.

In view of this, we shall only consider Γ-formulas.

If the list $y_1, y_2, \ldots, y_l$ contains all free variables of A other than x, and contains no other variables, then we write $(\dot{x})$ A in place of $(x/y_1, \ldots, y_l) A(x, x_1, \ldots, x_r)$ and $(\exists \dot{x})$ A in place of $(\exists x/y_1, \ldots, y_l)$ $A(x, x_1, \ldots, x_r)$.

If, for formulas of the restricted predicate calculus in the definition of degree of realizability or normal degree, we substitute "Γ-formula" for "formula," we obtain the definition of degree of realizability and normal degree of Γ-formulas.

In certain special cases, the normal degree of a Γ-formula can be evaluated from the form of the formula itself, and it is unnecessary to adjust all of the n-models. We shall show, for example, how to determine the normal degree of a closed Γ-formula A of the form

$$(\dot{x}_1)(\dot{x}_2)\ldots(\dot{x}_l)\,[A_1 \,\&\, [\ldots A_{k-1} \,\&\, [(\dot{x}_{l_{k-1}+1})(\dot{x}_{l_{k-1}+2})\ldots(\dot{x}_{l_k}) A_k]\ldots]] \tag{10}$$

or

$$(\exists \dot{x}_1)(\exists \dot{x}_2)\ldots(\exists \dot{x}_{l_1})\,[A_1 \vee [\ldots A_{k-1} \vee [(\exists \dot{x}_{l_{k-1}+1})\ldots(\exists \dot{x}_{l_k}) A_k]\ldots]], \tag{11}$$

where A_i $(i = 1, 2, \ldots, k)$ is a quantifier-free formula, each of whose predicate variables depends on all of the individual variables $x_1, x_2, \ldots, x_{l_i}$.

Let $B(x_1, x_2, \ldots, x_r)$ be a quantifier-free formula that depends on the individual variables $x_1, x_2, \ldots, x_r$. We let $B^{\&}$ represent the formula $\mathop{\&}\limits_{\alpha_1 \ldots, \alpha_r} B(x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_r})$ where the conjunction is taken over all permutations $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ of the r elements 1, 2, ..., r. Similarly, we let $B^{\vee}$ represent the formula

$$\bigvee_{\alpha_1, \alpha_2, \ldots, \alpha_r} B(x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_r}).$$

Now, looking at Remark 2 of Section 2.1, it is easy to see that for a Γ-formula A represented in the form (10),

$$\gamma_n A = \prod_{i=1}^{k} (\gamma_n A^{\&})^{\binom{n}{l_i}},$$

while for a Γ-formula of the form (11),

$$\gamma_n A = 1 - \prod_{i=1}^{k} (1 - \gamma_n A_i^{\vee})^{\binom{n}{l_i}};$$

in turn, $\gamma_n A_i^{\&}$ and $\gamma_n A_i^{\vee}$ are easily found, and are constant.

We now set up several relationships, some of which are not true for conventional formulas of the restricted predicate calculus:

$$\text{2.2.2. } \overline{(x/y_1, \ldots, y_l) A(x, x_1, \ldots, x_r)} \equiv (\exists x/y_1, \ldots, y_l) \bar{A}(x, x_1, \ldots, x_r).$$

$$\text{2.2.3. } \overline{(\exists x/y_1, \ldots, y_l) A(x, x_1, \ldots, x_r)} \equiv (x/y_1, \ldots, y_l) \bar{A}(x, x_1, \ldots, x_r).$$

$$\text{2.2.4 } (x/y_1, y_2, \ldots, y_l)(A \circ B) \equiv (x/y_1, \ldots, y_l) A \circ B,$$

$$(\exists x/y_1, \ldots, y_l)(A \circ B) \equiv (\exists x/y_1, \ldots, y_l) A \circ B,$$

if x is not a free variable in B; by the symbol "∘" we mean & or ∨ .

2.2.5. If U_n is such that $n \geq l$, then

$$\gamma_n [(\exists x/y_1, \ldots, y_l) A(x, x_1, \ldots, x_r)] \geqslant \gamma_n A(x, x_1, \ldots, x_r).$$

This inequality follows from 2.1.5.

$$\text{2.2.6 } \gamma_n [(\exists x/y_1, \ldots, y_l) A(x, x_1, \ldots, x_r)] \leqslant n \gamma_n A(x, x_1, \ldots, x_r).$$

Let us fix a certain set $y_1^0, y_2^0, \ldots, y_l^0$ such that $y_i^0 \neq y_j^0$ if $i \neq j$. Then

$$(\exists x/y_1^0, \ldots, y_l^0) A(x, x_1, \ldots, x_r) \equiv \bigvee_{\substack{s=1 \\ s \neq y_i^0,\ i=1,\ldots,l}}^{n} A(s, x_1^0, \ldots, x_r^0).$$

But

$$\delta_n \Big(\bigvee_{\substack{s=1 \\ s \neq y_i^0}} A(s, x_1^0, \ldots, x_r^0) \Big) \leqslant \sum_{\substack{s=1 \\ s \neq y_i^0}}^{n} \delta_n A(s, x_1^0, \ldots, x_r^0).$$

Since on the left-hand side $s \neq x_1^0$, $s \neq x_2^0$, ..., $s \neq x_r^0$, and $x_i^0 \neq x_j^0$ for all x_i^0, x_j^0, we have $\delta_n A(s, x_1^0, \ldots, x_r^0) = \gamma_n A$.

Thus,

$$\gamma_n [(\exists x/y_1, \ldots, y_l) A(x, x_1, \ldots, x_r)] \leqslant (n - l) \gamma_n A \leqslant n \gamma_n A.$$

It is important to note that in general, for formulas of the form $(\exists x) A(x, x_1, \ldots, x_r)$ we will not obtain this result, as will be clear from the following example. We take the formula

$$(\exists y)(x)[F(x, y, z) \vee \bar{F}(x, z, y)].$$

On the basis of Remark 2 it is easy to calculate that

$$\gamma_n ((x)[F(x, y, z) \vee \bar{F}(x, z, y)]) = \left(\frac{3}{4}\right)^n.$$

For the formula

$$(\exists y)(x)[F(x, y, z) \vee \bar{F}(x, z, y)]$$

we now construct the equivalent Γ-formula

$$(\exists y/z)(x)[F(x, y, z) \vee \bar{F}(x, z, y)] \vee (x)[F(x, z, z) \vee \bar{F}(x, z, z)].$$

Its normal degree equals 1 for any n, since $(x)[F(x, z, z) \vee \bar{F}(x, z, z)]$ is valid. But beginning with a certain n, $n\left(\frac{3}{4}\right)^n$ will be less than 1.

Definition. We say that the occurrence of an elementary part $F_i(x_1, x_2, \ldots, x_{m_i})$ in the Γ-formula A is free if none of the variables $x_1, x_2, \ldots, x_{m_i}$ are bound in A.

2.2.7. Any Γ-formula A can be represented as $\bigvee_{i=1}^{s} B_i \& C_i$, where the C_i are Γ-formulas containing no free elementary parts, and in which the number of all possible quantifiers equals the number of quantifiers in A; the B_i are quantifier-free formulas.

In fact, we can use relationship 2.2.4 to remove from quantification all elementary parts whose occurrences in A are free. We then use different propositional symbols $p_1, p_2, \ldots, p_f$ for all different elementary parts. We also use the propositional symbols $q_1, q_2, \ldots, q_t$ to represent all well-formed parts that do not occur in the range of any quantifier. Writing the resultant propositional formula in fdnf and replacing p_i and q_i by their originals, we obtain the required representation.

2.2.8. The Γ-formula

$$(x/y_1, \ldots, y_l)[A \vee B] \supset (x/y_1, \ldots, y_l) A \vee (\exists x/y_1, \ldots, y_l) B$$

is valid.

For a certain set of values $y_1^0, y_2^0, \ldots, y_r^0$ and a fixed n-model let the left-hand side of the implication be true. If A is true for all values of the variable x other than y_i^0, then $(x/y_1^0, \ldots, y_l^0)$ A will also be true. If A is false for a certain x^0, $x^0 \neq y_i^0$, then B must be true for this x^0; this means, however, that the formula $(\exists x/y_1^0, \ldots, y_l^0)$ B is true. This completes the proof.

2.3. Definition. The sequence $\{\alpha_n\}$ of real numbers is said to be 0-admissible (1-admissible) if $0 \leq \alpha_n \leq 1$, beginning with a certain n, and $\lim_{n \to \infty} n^k \alpha_n = 0$ ($\lim_{n \to \infty} n^k(1-\alpha_n) = 0$) for any k = 1, 2, ...

We note the following relationships.

2.3.1. If the sequence $\{\alpha_n\}$ is 0-admissible, then the sequence $\{1-\alpha_n\}$ is 1-admissible.

2.3.2. If $0 \leq \alpha_n \leq \beta_n$ and $\{\beta_n\}$ is 0-admissible, then $\{\alpha_n\}$ is 0-admissible.

2.3.3. If $1 \geq \alpha_n \geq \beta_n$ and $\{\beta_n\}$ is 1-admissible, then $\{\alpha_n\}$ is 1-admissible.

2.3.4. The sequence $\{\alpha_n + \beta_n\}$ is 0-admissible if $\{\alpha_n\}$ and $\{\beta_n\}$ are 1-admissible.

2.3.5. The sequence $\{\alpha_n \cdot \beta_n\}$ is 1-admissible if $\{\alpha_n\}$ and $\{\beta_n\}$ are 1-admissible.

In fact,

$$n^k(1-\alpha_n\beta_n) = n^k(1-\alpha_n+\alpha_n-\alpha_n\beta_n) = n^k(1-\alpha_n) + \alpha_n n^k(1-\beta_n) < n^k(1-\alpha_n) + n^k(1-\beta_n) \to 0.$$

2.3.6. If the sequence $\{\alpha_n\}$ is 0-admissible and $c > 0$ then $\{c\alpha_n\}$ is 0-admissible.

2.3.7. If the sequence $\{\alpha_n\}$ is 0-admissible then $\{n\alpha_n\}$ is 0-admissible.

2.3.8. If $0 \leq \lambda < 1$, then $\{\lambda^n\}$ is 1-admissible.

2.4. Definition. A formula A (which we can assume is in a language with exclusive quantifiers) is said to be 0-admissible (or 1-admissible) if the sequence $\{\gamma_n A\}$ is 0-admissible (or 1-admissible).

The formula A is said to be admissible if it is 0-admissible or 1-admissible.

2.4.1. If A is 0-admissible, then $\bar{A}$ is 1-admissible.

2.4.2. The formula $A \vee B$ is 1-admissible if at least one of the formulas A or B is 1-admissible. If both A and B are 0-admissible, $A \vee B$ is 0-admissible.

2.4.3. The formula A & B is 0-admissible if either A or B is 0-admissible.

If A and B are 1-admissible, then A & B is 1-admissible.

2.4.4. If $A(x, x_1, \ldots, x_r)$ is 1-admissible, then $(\exists x/y_1, \ldots, y_l)\, A(x, x_1, x_r)$ is 1-admissible. This follows from 2.2.5.

Our aim is to show that any closed formula of the restricted predicate calculus is admissible.

LEMMA 1. Let $A(x, x_1, \ldots, x_r)$ be a quantifier-free formula, each of whose elementary parts includes a variable x, while the list $y_1, y_2, \ldots, y_l$ contains all $x_1, x_2, \ldots, x_r$; then the formula

$$(x/y_1, \ldots, y_l)\, A(x, x_1, \ldots, x_r)$$

is 1-admissible if A is identically true, and 0-admissible otherwise.

Proof. Let $y_1^0, y_2^0, \ldots, y_l^0$ be fixed values for $y_1, y_2, \ldots, y_l$ and $y_i^0 \neq y_j^0$ when $i \neq j$; $y_i^0 \in U_n$. We represent A in fdnf form, assuming the different elementary parts to be different propositional variables. Then

$$(x/y_1^0, \ldots, y_l^0) A \equiv (x/y_1^0, \ldots, y_l^0) \Big(\bigvee_{(\sigma_1, \ldots, \sigma_m)} P_1^{\sigma_1} \& P_2^{\sigma_2} \& \ldots \& P_m^{\sigma_m} \Big).$$

Let s be the number of different disjunctive terms in the fdnf of formula A. Making use of Remarks 2, 2.1.4, and 2.1.8, we obtain

$$\gamma_n (x/y_1, \ldots, y_l)\, A = \left(\frac{s}{2^m}\right)^{n-l} = \left(\frac{s}{2^m}\right)^{n} \left(\frac{s}{2^m}\right)^{-l}.$$

If $s = 2^m$ (and this can only occur if A is identically true), then $(x/y_1, y_2, \ldots, y_l)$ is 1-admissible. If $s < 2^m$, then $(x/y_1, \ldots, y_l)$ is 0-admissible by virtue of 2.3.6 and 2.3.8.

COROLLARY 1. If A satisfies the hypothesis of Lemma 1, then the formula $(\exists x/y_1, \ldots, y_l)$ is 1-admissible if A is not identically false and 0-admissible if A is identically false.

Proof. We use negation and Lemma 1.

LEMMA 2. The formula $(x/y_1, y_2, \ldots, y_l)\,(B \vee C)$ is 0-admissible if $(x/y_1, \ldots, y_l)$ is 0-admissible and C is 0-admissible.

Proof. By virtue of 2.1.2, 2.2.8, and 2.1.6, we have

$$\gamma_n (x/y_1, \ldots, y_l)(B \vee C) \leqslant \gamma_n [(x/y_1, \ldots, y_l) B \vee (\exists x/y_1, \ldots, y_l) C] \leqslant \gamma_n (x/y_1, \ldots, y_l) B + \gamma_n (\exists x/y_1 \ldots, y_l)\, C.$$

By 2.2.6, however, $\gamma_n(\exists x/y_1, \ldots, y_l) C \leq n\gamma_n C$, and by 2.3.7, the formula $(\exists x/y_1, \ldots, y_l)C$ is 0-admissible.

It follows from 2.3.4 and 2.3.2 that $(x/y_1, \ldots, y_l) \times (B \vee C)$ is 0-admissible.

COROLLARY 2. The formula $(\exists x/y_1, \ldots, y_l)(B \& C)$ is 1-admissible if $(\exists x/y_1, y_2, \ldots, y_l)B$ is 1-admissible and C is 1-admissible.

We use negation for the proof.

THEOREM 5. Every Γ-formula with no free elementary parts of propositional variables is admissible.

Proof. We use mathematical induction on m, the number of occurrences of exclusive quantifiers in the Γ-formula. When m = 1, the validity of the theorem follows from Lemma 1 and Corollary 1.

Assume that the theorem is valid for Γ-formulas where the number of exclusive quantifiers is m or less. We consider a Γ-formula A in which there are m + 1 exclusive quantifiers. If A is constructed from simpler formulas with the aid of the improper symbols V and &, then the theorem follows for A from 2.4.1-2.4.3. Thus, we need only consider the case in which A has one of the following forms:

$$(x/y_1, \ldots, y_l)\, B(x, x_1, \ldots, x_r) \text{ or } (\exists x/y_1, \ldots, y_l)\, B(x, x_1, \ldots, x_r),$$

where the list $y_1, \ldots, y_r$, as we have already said, includes all free variables of A, except x. Let us look at the formula

$$(\exists x/y_1, y_2, \ldots, y_l) B(x, x_1, \ldots, x_r).$$

If A satisfies the hypothesis of the theorem, all free elementary parts of B will depend on x. By virtue of 2.2.7, B can be represented as

$$B \equiv \bigvee_{i=1}^{s} C_i \& D_i,$$

where the C_i are quantifier-free formulas all of whose elementary parts depend on x, while the formulas D_i contain no free elementary parts and no D_i contains more than m exclusive quantifiers. By virtue of our assumption, the theorem is valid for all D_i.

Case 1. For all $i = 1, 2, \ldots, s$ the following condition is satisfied: C_i is identically false or D_i is 0-admissible. By virtue of 2.2.6, $\gamma_n(\exists x/y_1, \ldots, y_l)\ B \leq n\gamma_n B$, but

$$\gamma_n B = \gamma_n \left(\bigvee_{i=1}^{s} C_i \& D_i\right) \leqslant \sum_{i=1}^{s} \gamma_n (C_i \& D_i).$$

If C_i is identically false, then $C_i \& D_i$ is also identically false; if, however, D_i is 0-admissible, then by virtue of 2.4.3 the formula $C_i \& D_i$ is also 0-admissible. Consequently, $\sum_{i=1}^{s} \gamma_n (C_i \& D_i)$ is in this case a 0-admissible sequence. From 2.3.7 and 2.3.2 we find that $\{\gamma_n(\exists x/y_1, \ldots, y_l)B\}$ is 0-admissible.

Case 2. There exists an $i = i_0$ such that C_{i_0} is not identically false, while D_{i_0} is 1-admissible. Here,

$$\gamma_n \left[(\exists x/y_1, \ldots, y_l) \left(\bigvee_{i=1}^{s} C_i \& D_i\right)\right] = \gamma_n \left[\bigvee_{i=1}^{s} (\exists x/y_1, \ldots, y_l)(C_i \& D_i)\right].$$

Using 2.1.5, we find that

$$\gamma_n (\exists x/y_1, \ldots, y_l)\, B \geqslant \gamma_n (\exists x/y_1, \ldots, y_l)\, (C_{i_0} \& D_{i_0}). \tag{12}$$

By assumption, D_{i_0} is 1-admissible, while C_{i_0} is a quantifier-free formula that is not identically false, and each of whose elementary parts contains the variable x.

By Corollary 1, $(\exists x/y_1, \ldots, y_l)\, C_{i_0}$ is a 1-admissible formula. Using Corollary 2, we find that $(\exists x/y_1, \ldots, y_l)(C_{i_0} \& D_{i_0})$ is 1-admissible. By virtue of (12), however, $\{\gamma_n(\exists x/y_1, \ldots, y_l)B\}$ is also a 1-admissible sequence.

Cases 1 and 2 are clearly exhaustive, so the theorem is valid for any formula of the form $(\exists x/y_1, \ldots, y_l)B$.

The proof for formulas of the form $(x/y_1, \ldots, y_l)B(x, x_1, \ldots, x_r)$ is obtained by negation.

Thus, we have proved the theorem completely.

COROLLARY 3. If the formula $(\exists x/y_1, \ldots, y_l)A(x, x_1, \ldots, x_r)$ satisfies the hypothesis of Theorem 1 and is 0-admissible or 1-admissible, so will be the formula $(\exists x/y_1, \ldots, y_l, z)A(x, x_1, \ldots, x_r)$. Actually, since $(\exists x/y_1, \ldots, y_l)A(x, x_1, \ldots, x_r)$ satisfies Theorem 1, it is also valid for the formula $(\exists x/y_1, \ldots, y_l, z)\, A$, and this is either 0-admissible or 1-admissible.

Let $(\exists x/y_1, \ldots, y_l)A$ be 1-admissible and $(\exists x/y_1, \ldots, y_l, z)\, A$ be 0-admissible. We use the following equivalence:

$$(\exists x/y_1, \ldots, y_l) A (x, x_1, \ldots, x_r) \equiv (\exists x/y_1, \ldots, y_l, z) A (x, x_1, \ldots, x_r) \vee A (z, x_1, \ldots, x_r). \tag{13}$$

It follows from the relationship $\gamma_n(\exists x/y_1, \dots, y_l, z)\, A(x, x_1, \dots, x_r) \geq \gamma_n A(x, x_1, \dots, x_r)$ that $\gamma_n A(x, x_1, \dots, x_r)$ is a 0-admissible sequence. It is also clear, however, that

$$\gamma_n A(z, x_1, \dots, x_r) = \gamma_n A(x, x_1, \dots, x_r),$$

so $A(z, x_1, \dots, x_r)$ is 0-admissible. It follows from (13) and 2.4.2 that $(\exists x/y_1, \dots, y_l)\, A(x, x_1, \dots, x_r)$ is also 0-admissible. But this contradicts our assumption. The second part of the corollary is trivial by virtue of (13).

COROLLARY 4. If $(x\, y_1, \dots, y_l)\, A(x, x_1, \dots, x_r)$ satisfies the hypothesis of Theorem 1 and is 1-admissible (0-admissible), so will be the formula

$$(x/y_1, \dots, y_l, z)\, A(x, x_1, \dots, x_r),$$

where z is a free variable $y_1, y_2, \dots, y_l$.

THEOREM 6. (The "0 or 1" theorem.) If A is an arbitrary closed formula containing no propositional variables in the restricted predicate calculus, then $\lim_{n \to \infty} \gamma_n A$ exists and equals either 0 or 1.

Proof. This follows directly from the fact that for every closed formula A with no propositional variables we can find the equivalent Γ-formula B with no free elementary parts. The following procedure can be used to find such a formula: let $(\Pi_l x_l)(\Pi_{l-1} x_{l-1}) \dots (\Pi_1 x_1) S$ be the prenex normal form for formula A [here $\Pi_i x_i$ represents either (x_i) or $(\exists x_i)$. Using 2.2.1, we replace every quantifier $(\Pi_i x_i)$, beginning at $i = 1$, by an exclusive quantifier; each time, we eliminate only the free variables of the new formulas. After a finite number of such steps, we arrive at a Γ-formula B containing no free occurrences of elementary parts.

THEOREM 7. If A is an arbitrary formula of the restricted predicate calculus, then $\lim_{n \to \infty} \gamma_n A$ exists and equals $l/2^s$, where l and s are positive integers determined by the form of A.

Proof. Let $A(q_1, \dots, q_t, F_1, \dots, F_k, y_1, \dots, y_m)$ be an arbitrary formula of the restricted predicate calculus (here q_i, F_i, y_i are lists of propositional, predicate, and free individual variables). We consider the prenex normal form of A,

$$(\Pi_f x_f)(\Pi_{f-1} x_{f-1}) \dots (\Pi_1 x_1)\, B.$$

Proceeding as in the proof of Theorem 6, we reduce A to a Γ-formula C equivalent to A. Now, however, C can contain free elementary parts in addition to the propositional variables. We let $p_1, \dots, p_r$ represent the elementary parts. Clearly, each of the p_i has the form $E_j(y_{k_1}, y_{k_2}, \dots, y_{k_{m_j}})$ where all the y_{k_i} are from the list $y_1, \dots, y_l$. In C, we remove all the q_i and p_j from quantification ranges, and let $r_1, r_2, \dots, r_f$ represent all distinct well-formed parts of C' that neither contain p_j and q_j nor occur within the range of any exclusive quantifier. We now expand C' into the fdnf with respect to the variables q_i, p_j, r_k, and then replace the r_k by their originals; this yields the formula D, which has the form

$$\bigvee_{i=1}^{d} q_1^{\sigma_1^i} \,\&\, \dots \,\&\, q_t^{\sigma_t^i} \,\&\, p_1^{\sigma_{t+1}^i} \,\&\, \dots \,\&\, p_r^{\sigma_{t+r}^i} \,\&\, B_i,$$

$$(\sigma_j^i = 0 \quad \text{or} \quad 1),$$

and is equivalent to A. We find $\gamma_n D = \gamma_n A$. By virtue of 2.1.4,

$$\gamma_n A = \sum_{i=1}^{d} \gamma_n [q_1^{\sigma_1^i} \,\&\, \dots \,\&\, q_t^{\sigma_t^i} \,\&\, p_1^{\sigma_{t+1}^i} \,\&\, \dots \,\&\, p_r^{\sigma_{t+r}^i} \,\&\, B_i].$$

It is clear from the construction of D that the formulas $q_1^{\sigma_1^i} \,\&\, \dots \,\&\, p^{\sigma_{t+r}^i}$ and B_i are formally independent, so that

$$\gamma_n [q_1^{\sigma_1^i} \,\&\, \dots \,\&\, q_t^{\sigma_t^i} \,\&\, p_1^{\sigma_{t+1}^i} \,\&\, \dots \,\&\, p_r^{\sigma_{t+r}^i} \,\&\, B_i] = \gamma_n B_i \cdot \frac{1}{2^t} \cdot \frac{1}{2^r} = \frac{1}{2^s} \gamma_n B_i$$

here s = t + r. Thus $\gamma_n A = \sum_{i=1}^{d} \frac{1}{2^s} \gamma_n B_i$. Since the B_i are formulas containing no free elementary parts, however, Theorem 5 holds for them. Consequently,

$$\lim_{n\to\infty} \gamma_n A = \frac{1}{2^s} \sum_{i=1}^{d} \lim_{n\to\infty} \gamma_n B_i = \frac{m}{2^s} \left(m = \sum_{i=1}^{d} \lim_{n\to\infty} \gamma_n B_i \right).$$

Thus, we have proved Theorem 7.

The following question naturally arises in connection with the "0 or 1" theorem. Assume that A and B are two arbitrary closed formulas of the restricted predicate calculus. How will $\gamma_n(A/B) = \frac{\gamma_n(A \& B)}{\gamma_n B}$ for $n \to \infty$.

As we see from the following example, in the general case such a "0 or 1" theorem is not regular. Let A be $(x)[P_1(x) P_2(x) \lor \overline{P}_1(x) \overline{P}_2(x)]$ and be $(x)[P_1(x) P2(x) \lor \overline{P}_1(x) \overline{P}_2(x)] \lor (x)[\overline{P}_1(x) P_2(x) \lor P_1(x) \overline{P}_2(x)]$. From the argument of Section 1.2 it is clear that $\gamma_n A = (1/2)^n$, $\gamma_n B = 2(1/2)^n$, but A & B = A, so that $\gamma_n(A/B) = 1/2$.

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