Abstract

We consider a finite relational vocabulary σ with a first order theory T written in σ composed of symmetry and anti-reflexivity axioms. We define a binomial random model of finite σ-structures that satisfy T and show that first order properties have well defined asymptotic probabilities in the sparse case. It is also shown that those limit probabilities are well-behaved with respect to some parameters that represent edge densities. An application of these results to the problem of random Boolean satisfability is presented afterwards. We show that there is no first order property of k-CNF formulas that implies unsatisfability and holds for almost all typical unsatisfable formulas when the number of clauses is linear.

Introduction

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0 Preliminaries

A vocabulary σ is a collection or constant symbols $(c_1, \dots, c_n, \dots)$ and relation symbols $(R_1, \dots, R_m, \dots)$ where each relation symbol has associated a natural number called its arity. A vocabulary is finite if both its sets of constant and relation symbols are finite and it is relational if it has no constant symbols.

A (σ-)structure $\mathfrak{A}$ is composed of (1) a set A, called the universe of $\mathfrak{A}$, (2) elements $c_i^{\mathfrak{A}} \in A$ for all constant symbols c_i in σ, and (3) sets of tuples $R_i^{\mathfrak{A}} \subseteq A^{a_i}$ for all relation symbols R_i in σ, where a_i is the arity of R_i. A structure is called finite if its universe is a finite set.

The first order language with signature σ deals with strings of symbols taken from the alphabet consisting of variable symbols $x_1, \dots, x_m, \dots$, the symbols in σ, the logical connectives $\neg, \wedge, \vee$, the universal $\forall$ and existential $\exists$ quantifiers, the equality symbol, and the parentheses $), ($. A term is either a variable symbol or a constant symbol from σ. A first order formula is a string obtained after applying the following set of rules a finite number of times:

(I) If t_1, t_2 are terms then $t_1 = t_2$ is a formula.

(II) If R_i is a relation symbol in σ with arity a_i, and $t_1, \dots, t_{a_i}$ are terms then $R_i(t_1, \dots, t_{a_i})$ is a formula.

(III) If φ and ψ are formulas, then both $(\varphi \wedge \psi)$ and $(\varphi \vee \psi)$ are formulas.

(IV) If φ is a formula then $\neq \varphi$ is also a formula.

(V) If φ is a formula and x is a variable then both $\forall x \varphi$ and $\exists x \varphi$ are formulas.

Let φ be a first order formula. An occurrence of a variable in φ is called bounded if it is within the scope of a quantifier binding it, and is called free otherwise. We will assume that the occurrences of any given variable in a first order sentence are either all free or all bounded. We will use the notation $\varphi(x_1, \dots, x_n)$ to indicate that $x_1, \dots, x_n$ are distinct and they are the free variables (i.e. the variables whose occurrences are all free) in φ. A formula with no free variables is called a sentence, and a formula whose variables are all free is called open.

Let $\mathfrak{A}$ be a σ-structure and $\varphi(x_1, \dots, x_n)$ be a first order sentence. Given a map α from the set of free variables $\{x_1, \dots, x_n\}$ to the universe A of $\mathfrak{A}$