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Zero-one k-law

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ABSTRACT

In this article we study asymptotical behavior of the probabilities of some properties of Erdős–Rényi random graphs $G(N, p)$. We consider the first-order properties and the probabilities $p = N^{-\alpha}$ for rational α. The zero-one law in ordinary sense for these graphs doesn't hold. We weakened the law by considering the formulas with quantifier depth bounded by a fixed number. To prove our results we used theorems on estimates for the number of extensions of small subgraphs in the random graph. Such an approach was first used by Spencer and Shelah in 1988. We also used our recent results from this area.

1. History of the problem

We study zero–one laws for random graphs. This problem relates to logic and graph theory.

Let us recall a definition of *the Erdős–Rényi graphs*. The random graph in this model is the probabilistic space

$$G(N, p) = (\Omega_N, \mathcal{F}_N, \mathsf{P}_{N,p}),$$

where N is a natural number, $0 \le p \le 1$, $\Omega_N = \{\mathcal{G} = (\mathcal{V}_N, \mathcal{E})\}$ is the set of all undirected graphs without loops and multiple edges with the set of vertices $\mathcal{V}_N = \{1, 2, \ldots, N\}$,

$$\mathcal{F}_N = 2^{\Omega_N}, \qquad \mathsf{P}_{N,p}(\mathcal{G}) = p^{|\mathcal{E}|}(1-p)^{C_N^2 - |\mathcal{E}|}.$$

In other words, any two different vertices of a graph in $G(N, p)$ are connected with probability p independently of all other pairs of vertices (see [1,3,13]).

First-order properties of graphs (classes $\mathcal{C} \subseteq \Omega_N$) are defined by first-order formulas (see [1,13,17,16,6,2,18,11,9,21]), which are built of

- predicate symbols $\sim$, $=$;
- logical connectivities $\neg$, $\Rightarrow$, $\Leftrightarrow$, $\vee$, $\wedge$;
- variables x, y, $x_1 \ldots$;
- quantifiers $\forall$, $\exists$.

The relation symbol $\sim$ expresses the property of two vertices being adjacent. The event "*graph G satisfies the property L*" is both an element of $\mathcal{F}_N$ and a set of all graphs in Ω_N, that satisfy the property L (we write $G \vDash L$). The random graph satisfies the property L *almost surely* if

$$\lim_{N\to\infty} \mathsf{P}_{N,p}(G \vDash L) = 1.$$

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It was proved in 1969 by Glebskii et al. in [11] (and independently in 1976 by Fagin in [9]) that for any first order property L either "almost all" graphs satisfy this law as N tends to infinity or "almost all" graphs don't satisfy the law. In other words, if p doesn't depend on N, then for any first-order property L either

$$\lim_{N\to\infty} \mathsf{P}_{N,p}(G \models L) = 1$$

or

$$\lim_{N\to\infty} \mathsf{P}_{N,p}(G \models L) = 0$$

(if for any property one of these equalities holds, then the random graphs is said to *follow zero–one law* (see [1,17,11,9,21])). Moreover, the following statement holds.

Theorem 1. *Let a function $p = p(N)$ satisfy two properties: $pN^{\alpha} \to \infty$ when $N \to \infty$ and $(1-p)N^{\alpha} \to \infty$ when $N \to \infty$ for any $\alpha > 0$. Then the random graph $G(N, p)$ follows the zero–one law.*

Shelah and Spencer (see [17]) expanded the class of functions $p(N)$ "that follow the zero–one law".

Theorem 2. *Let $p(N) = N^{-\alpha}$ and α be an irrational number such that $0 < \alpha < 1$. Then the random graph $G(N, p)$ follows the zero–one law.*

There are a lot of other works that relate to zero–one law (see [14,10,4,12,5]). In the article [14] zero–one laws for K_{l+1}-free graphs were obtained. In [17] some laws for the random graph $G(N, p)$ and functions $p(N)$ satisfying the property $p = o(N^{-\alpha})$ for any $\alpha \in (0, 1)$ were stated. In 2007 Gilman et al. Miasnikov proved the zero–one law for a graph with infinite number of vertices (see [10]).

Until this moment we have considered the class of all first-order properties expressed by the sentences of a finite length. The formulas containing an infinite number of conjunctions and disjunctions were also studied. In 1997 McArthur (see [16]) considered such formulas with quantifier depth bounded by a fixed number. In her work zero–one laws for the random graph $G(N, p)$ with $p = N^{-\alpha}$ and some rational α from $(0, 1]$ were obtained.

Let us sum up. The random graph $G(N, p)$ doesn't follow zero–one law for the properties from the class of all first-order properties expressed by the sentences of a finite length if $p = N^{-\alpha}$ and α from $(0, 1]$ is rational. But for some rational α it follows zero–one law when the sentences can be of an infinite length and quantifier depth is bounded. So it is natural to consider sentences of a finite length with bounded quantifier depth. The following question arises. Which rational α is required for the random graph $G(N, N^{-\alpha})$ to follow zero–one law (of course, if the random graph follows zero–one from the work of McArthur, then it follows "our law")? The aim of this work is to answer this question.

2. Main result

Let k be a natural number. Denote by $\mathcal{L}_k$ the class of the first-order properties of graphs defined by formulas with quantifier depth bounded by the number k (the sentences are of a finite length). Let us say that the random graph satisfies *zero–one k-law*, if for any first-order property $L \in \mathcal{L}_k$ one of the following holds:

$$\lim_{N\to\infty} \mathsf{P}_{N,p}(\mathcal{G} \models L) = 0,$$
$$\lim_{N\to\infty} \mathsf{P}_{N,p}(\mathcal{G} \models L) = 1.$$

Let us state a result on the truth of the zero–one k-law when $\alpha < \frac{1}{k-2}$.

Theorem 3. *Let $p = N^{-\alpha}$, $0 < \alpha < \frac{1}{k-2}$. Then the random graph $G(N, p)$ satisfies the zero–one k-law.*

When α is irrational the statement of this theorem follows immediately from Theorem 2. We will prove this statement for rational α. When $\alpha = \frac{1}{k-2}$ we will disprove the k-law.

Theorem 4. *Let $p = N^{-\alpha}$, $\alpha = \frac{1}{k-2}$. Then the random graph $G(N, p)$ doesn't satisfy the zero–one k-law.*

If $\alpha > \frac{1}{k-2}$ and α is rational, then there exist some α such that k-law holds and some other α such that k-law doesn't hold. However, we have not still obtained a general result for all rational $\alpha > \frac{1}{k-2}$.

In the next section we will prove these theorems. We will start the proof from describing auxiliary definitions, statements and constructions. Notably, we need the definition of extensions and the theorems on the extensions (Section 3.1). Also a theorem about the Ehrenfeucht game (see Section 3.2) is necessary for the proof. In Sections 3.4 and 4 we will prove Theorems 4 and 5 directly.

3. Proofs of Theorems 3 and 4

Let us introduce some additional constructions.

3.1. Extensions

It is necessary to consider different pairs of graphs H and G, $H \subseteq G$, to formulate definitions of extensions, and to give estimates on the number of extensions of subgraphs in the random graph. We will describe these constructions in Section 3.1. In particular, we will formulate theorems of Spencer and Shelah (see [1,13,19]) on the number of extensions. In Section 3.2 we will formulate a theorem about the number of maximal extensions, which is a combination of our recent results (see [22]) and some results of Spencer and Shelah (see [1]).

Moreover, we will formulate in Section 3.1.3 and prove in Section 3.1.4 a theorem on the property of containing a maximal subgraph. This result will help us to prove Theorem 4.

3.1.1. Bounding the number of extensions

Let us consider graphs H with $V(H) = \{x_1, \ldots, x_k\}$ and G with $V(G) = \{x_1, \ldots, x_l\}$. Let $H \subset G$. A graph $\widetilde{G}$ with $V(\widetilde{G}) = \{\widetilde{x}_1, \ldots, \widetilde{x}_l\}$ is called a (G, H)-*extension of a graph* $\widetilde{H}$ with $V(\widetilde{H}) = \{\widetilde{x}_1, \ldots, \widetilde{x}_k\}$, if $\widetilde{H} \subset \widetilde{G}$ and

$$\{x_i, x_j\} \in E(G) \setminus E(H) \Rightarrow \{\widetilde{x}_i, \widetilde{x}_j\} \in E(\widetilde{G}) \setminus E(\widetilde{H}).$$

If relation

$$\{x_i, x_j\} \in E(G) \setminus E(H) \Leftrightarrow \{\widetilde{x}_i, \widetilde{x}_j\} \in E(\widetilde{G}) \setminus E(\widetilde{H})$$

holds, then the extension is called *strict*. Set

$$v(G, H) = |V(G) \setminus V(H)|, \qquad e(G, H) = |E(G) \setminus E(H)|.$$

Let $\alpha > 0$ be a fixed number. Denote

$$f(G, H) = v(G, H) - \alpha \cdot e(G, H).$$

Moreover, denote the number of vertices in a graph G by $v(G)$ and the number of edges by $e(G)$. If for any graph S such that $H \subset S \subseteq G$ the inequality $f(S, H) > 0$ holds, then the pair (G, H) is called α-*safe* (see [1,13,19]). If for any graph S such that $H \subseteq S \subset G$, the inequality $f(G, S) < 0$ holds, then the pair (G, H) is called α-*rigid* (see [1,13]).

Let us return to the random graphs. Let us call the function $p_0(n)$ *threshold* for a property $\mathcal{A}$, if $p \gg p_0$ implies

$$\lim_{N\to\infty} \mathsf{P}_{N,p}(G \vDash \mathcal{A}) = 1$$

and $p \ll p_0$ implies

$$\lim_{N\to\infty} \mathsf{P}_{N,p}(G \vDash \mathcal{A}) = 0$$

(or vice-versa). Here and after we will write $f(N) \gg g(N)$ ($f(N) \ll g(N)$) if there exists a function $\alpha(N)$ such that $\alpha(N) \to \infty$ when $N \to \infty$ and $f(N) \geq \alpha(N)g(N)$ ($g(N) \geq \alpha(N)f(N)$) for all N.

Let us consider

$$G(N, p(N)) = (\Omega_N, \mathcal{F}_N, \mathsf{P}_{N,p})$$

and two graphs H and G, $H \subset G$. Let $V(H) = \{x_1, \ldots, x_k\}$, $V(G) = \{x_1, \ldots, x_l\}$ and $\widetilde{x}_1, \ldots, \widetilde{x}_k \in \mathcal{V}_N$. Let us consider a random variable $N_{(G,H)}(\widetilde{x}_1, \ldots, \widetilde{x}_k)$ on the probability space $G(N, p(N))$: to each graph $\mathcal{G}$ from Ω_N it assigns the number of all unlabeled (G, H)-extensions of the subgraph induced on $\{\widetilde{x}_1, \ldots, \widetilde{x}_k\}$ in $\mathcal{G}$. In other words let $W \subset \mathcal{V}_N \setminus \{\widetilde{x}_1, \ldots, \widetilde{x}_k\}$ be a set of cardinality $|W| = l - k$. If the elements from the set W can be numerated by the numbers $k+1, k+2, \ldots, l$ so that the graph $\mathcal{G}|_{\{\widetilde{x}_1,\ldots,\widetilde{x}_l\}}$ is a (G, H)-extension of the graph $\mathcal{G}|_{\{\widetilde{x}_1,\ldots,\widetilde{x}_k\}}$, then we set $I_W(\mathcal{G}) = 1$. Otherwise, we set $I_W(\mathcal{G}) = 0$. Then the random variable $N_{(G,H)}(\widetilde{x}_1, \ldots, \widetilde{x}_k)$ is defined by the following equality:

$$N_{(G,H)}(\widetilde{x}_1, \ldots, \widetilde{x}_k) = \sum_{W \subset \mathcal{V}_N \setminus \{\widetilde{x}_1,\ldots,\widetilde{x}_k\},\ |W| = l-k} I_W.$$

In 1990 Spencer proved the following theorem (see [1,19]).

Theorem 5. *Let $p = N^{-\alpha}$, a pair (G, H) is α-safe. Then for any $\varepsilon > 0$ the equality*

$$\lim_{N\to\infty} \mathsf{P}_{N,p}\left(\forall \widetilde{x}_1, \ldots, \widetilde{x}_k \; \left|N_{(G,H)}(\widetilde{x}_1, \ldots, \widetilde{x}_k) - \mathsf{E}_{N,p} N_{(G,H)}(\widetilde{x}_1, \ldots, \widetilde{x}_k)\right| \leq \varepsilon \mathsf{E}_{N,p} N_{(G,H)}(\widetilde{x}_1, \ldots, \widetilde{x}_k)\right) = 1$$

holds. Here $\mathsf{E}_{N,p}$ is the expectation. Moreover, $\mathsf{E}_{N,p} N_{(G,H)}(\widetilde{x}_1, \ldots, \widetilde{x}_k) = \Theta(N^{f(G,H)})$.

In fact, the statement of this theorem means that almost surely for any vertices $\widetilde{x}_1, \ldots, \widetilde{x}_k$ the relation

$$N_{(G,H)}(\widetilde{x}_1, \ldots, \widetilde{x}_k) \sim \mathsf{E}_{N,p} N_{(G,H)}(\widetilde{x}_1, \ldots, \widetilde{x}_k)$$

holds, i.e. almost surely

$$\frac{N_{(G,H)}(\widetilde{x}_1, \ldots, \widetilde{x}_k)}{\mathsf{E}_{N,p} N_{(G,H)}(\widetilde{x}_1, \ldots, \widetilde{x}_k)} \to 1, \quad N \to \infty.$$

In what follows in such cases we will use this kind of notation.

3.1.2. Bounding the number of maximal extensions

Let $0 < \alpha \leq 1$ be a rational number. Let us consider two graphs H and G. Let $H \subset G$. Let each vertex of the graph H be adjacent to some vertex of $V(G) \setminus V(H)$. Assume that for any graph S such that $H \subset S \subset G$, the inequality

$$v(S, H) - \alpha \cdot e(S, H) > 0$$

holds, but

$$v(G, H) - \alpha \cdot e(G, H) = 0.$$

In this case let us call the pair (G, H) *α-neutral*. For example, if G is a complete graph, $V(G) = \{x_1, x_2, x_3, x_4\}$, then the pair $(G, G|_{\{x_1,x_2\}})$ is 2/5-neutral. Note that the condition $0 < \alpha \leq 1$ is essential. It is necessary for the existence of a neutral pair. Indeed, let a pair (G, H) be α-neutral. In addition, let for some vertices $y \in V(G) \setminus V(H)$, $x \in V(H)$ the relation $y \sim x$ hold. The definition of a neutral extension implies the existence of such vertices. Let S be the subgraph induced on the set of vertices $\{y\} \cup V(H)$ of the graph G. Then $v(S, H) = 1$, $e(S, H) \geq 1$. But

$$\alpha \leq \frac{v(S, H)}{e(S, H)}.$$

So $\alpha \leq 1$.

Let $\widetilde{H} \subset \widetilde{G} \subset \Gamma$, $T \subset K$ and $|V(T)| \leq |V(\widetilde{G})|$. Let us call the pair $(\widetilde{G}, \widetilde{H})$ *(K, T)-maximal in Γ*, if for any subgraph $\widetilde{T}$ of the graph $\widetilde{G}$ such that $|V(\widetilde{T})| = |V(T)|$ and $\widetilde{T} \cap \widetilde{H} \neq \widetilde{T}$ and for any strict (K, T)-extension $\widetilde{K}$ of $\widetilde{T}$ in $\Gamma \setminus (\widetilde{G} \setminus \widetilde{T})$ there exists a vertex from $V(\widetilde{K}) \setminus V(\widetilde{T})$ adjacent to some vertex from $V(\widetilde{G}) \setminus V(\widetilde{T})$. This definition is similar to the definition of the *generic extension* defined by Spencer in his book [20]. Our definition makes it possible to separate (K, T)-maximal pairs for an α-neutral pair (K, T) and (K, T)-maximal pairs for an α-rigid pair (K, T). Actually let the pair (K, T) be α-neutral and there exist subgraph $\widetilde{T}$ of the graph $\widetilde{G}$ and graph $\widetilde{K}$ such that $|V(\widetilde{T})| = |V(T)|$, $\widetilde{T} \cap \widetilde{H} \neq \widetilde{T}$ and in $\Gamma \setminus (\widetilde{G} \setminus \widetilde{T})$ there exists a vertex from $V(\widetilde{K}) \setminus V(\widetilde{T})$ adjacent to some vertex from $V(\widetilde{G}) \setminus V(\widetilde{T})$. Then the pair $(\widetilde{G}, \widetilde{H})$ can be (K, T)-maximal despite the fact that inequality $f(\widetilde{G} \cup \widetilde{K}, \widetilde{G}) < 0$ holds. In the Section 3.2 we define Ehrenfeucht game. The definitions of maximal extension and generic extension are necessary to describe a winning strategy of the second player. Zero-one laws follow from the existence of such a strategy. Let us give an example of (K, T)-maximal pair. Let $V(K) = \{x_1, x_2, x_3\}$, $V(T) = \{x_1, x_2\}$. Let K be a complete graph. Let $V(\Gamma) = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5\}$, $V(\widetilde{G}) = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3\}$, $V(\widetilde{H}) = \{\widetilde{x}_1, \widetilde{x}_2\}$, $E(\Gamma) = \{\{\widetilde{x}_1, \widetilde{x}_2\}, \{\widetilde{x}_1, \widetilde{x}_3\}, \{\widetilde{x}_1, \widetilde{x}_4\}, \{\widetilde{x}_2, \widetilde{x}_3\}, \{\widetilde{x}_2, \widetilde{x}_4\}, \{\widetilde{x}_3, \widetilde{x}_4\}, \{\widetilde{x}_1, \widetilde{x}_5\}\{\widetilde{x}_2, \widetilde{x}_5\}, \{\widetilde{x}_3, \widetilde{x}_5\}\}$. Let $\widetilde{G}, \widetilde{H}$ be induced subgraphs of Γ. Then the pair $(\widetilde{G}, \widetilde{H})$ is (K, T)-maximal in Γ.

Let a pair (G, H) be α-safe, $V(H) = \{x_1, \dots, x_k\}$, $V(G) = \{x_1, \dots, x_l\}$. Let $\Sigma^{\mathrm{neutral}}(r)$ be the set of all α-neutral pairs (K_i, T_i), where

$$|V(T_i)| \leq |V(G)|, \qquad |V(K_i) \setminus V(T_i)| \leq r.$$

Let $\Sigma^{\mathrm{rigid}}(r)$ be the set of all α-rigid pairs (K_i, T_i), where

$$|V(T_i)| \leq |V(G)|, \qquad |V(K_i) \setminus V(T_i)| \leq r.$$

Let us consider again the random graph $G(N, p(N)) = (\Omega_N, \mathcal{F}_N, \mathsf{P}_{N,p})$ and some vertices $\widetilde{x}_1, \dots, \widetilde{x}_k \in \mathcal{V}_N$. Define a random variable $\widehat{N}^{\mathrm{neutral}}_{(G,H),r}(\widetilde{x}_1, \dots, \widetilde{x}_k)$ (a random variable $\widehat{N}^{\mathrm{rigid}}_{(G,H),r}(\widetilde{x}_1, \dots, \widetilde{x}_k)$) which assigns to each graph $\mathcal{G} \in \Omega_N$ the number of all strict (G, H)-extensions $\widetilde{G}$ of the graph $\widetilde{H} = \mathcal{G}|_{\{\widetilde{x}_1,\dots,\widetilde{x}_k\}}$ such that for each pair $(K_i, T_i) \in \Sigma^{\mathrm{neutral}}(r)$ (pair $(K_i, T_i) \in \Sigma^{\mathrm{rigid}}(r)$) the pair $(\widetilde{G}, \widetilde{H})$ is (K_i, T_i)-maximal in $\mathcal{G}$. For these random variables a result similar to Theorem 5 holds. We have already noticed that this result is a consequence of both our work (see [22]) and a work by Spencer and Shelah (see [1,17]).

Theorem 6. *Almost surely for all vertices $\widetilde{x}_1, \dots, \widetilde{x}_k$, the following relations hold:*

$$\widehat{N}^{\mathrm{neutral}}_{(G,H),r}(\widetilde{x}_1, \dots, \widetilde{x}_k) \sim \mathsf{E}_{N,p}\widehat{N}^{\mathrm{neutral}}_{(G,H),r}(\widetilde{x}_1, \dots, \widetilde{x}_k) = \Theta\left(N^{f(G,H)}\right),$$
$$\widehat{N}^{\mathrm{rigid}}_{(G,H),r}(\widetilde{x}_1, \dots, \widetilde{x}_k) \sim \mathsf{E}_{N,p}\widehat{N}^{\mathrm{rigid}}_{(G,H),r}(\widetilde{x}_1, \dots, \widetilde{x}_k) \sim N_{(G,H)}(\widetilde{x}_1, \dots, \widetilde{x}_k) = \Theta\left(N^{f(G,H)}\right).$$

3.1.3. Bounding the number of subgraphs

A statement similar to Theorem 5 can be formulated for subgraphs. To formulate this statement it is necessary to introduce the notion of a *balanced graph*.

Erdős and Rényi were the first to estimate the number of copies of a graph in the random graph (see [8]). Let us consider a graph G with v vertices and e edges. Let us call the fraction $\rho(G) = \frac{e}{v}$ *the density of* G. The graph G is called *balanced*, if for any subgraph $H \subseteq G$, the inequality $\rho(H) \leq \rho(G)$ holds. The graph G is *strictly balanced* if for any $H \subset G$, the strict inequality $\rho(H) < \rho(G)$ holds. Let us now formulate a theorem (see [1,3,13]) about the number of copies of a strictly balanced graph. Let $G(N, p)$ be the random graph, N_G be the number of copies of G in the random graph, $a(G)$ be the number of automorphisms of the graph G. Set

$$\rho^{\max}(G) = \max\{\rho(H) : H \subseteq G\}.$$

Theorem 7. *Let G be an arbitrary graph. The function $p = N^{-1/\rho^{\max}(G)}$ is the threshold function for the property of containing a copy of G.*

If G is a strictly balanced graph and $p \gg N^{-1/\rho(G)}$, then almost surely

$$N_G \sim \mathsf{E}_{N,p} N_G.$$

If $p = N^{-1/\rho(G)}$, then

$$\lim_{N\to\infty} \mathsf{P}_{N,p}(N_G = 0) = e^{-1/a(G)}.$$

Let $\widetilde{G} \subset \mathcal{G}$ and $T \subset K$. Graph $\widetilde{G}$ is called (K, T)-*maximal in* $\mathcal{G}$, if for any subgraph $\widetilde{T}$ of the graph $\widetilde{G}$ such that $|V(T)| = |V(\widetilde{T})|$ and for any (K, T)-extension $\widetilde{K}$ of $\widetilde{T}$ there exists a vertex from $V(\widetilde{K}) \setminus V(\widetilde{T})$ adjacent to some vertex from $V(\widetilde{G}) \setminus V(\widetilde{T})$. For example, let $V(\mathcal{G}) = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4\}$, $V(\widetilde{G}) = \{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3\}$, $V(K) = \{x_1, x_2, x_3\}$, $V(T) = \{x_1, x_2\}$. Let K, T, $\mathcal{G}$ and $\widetilde{G}$ be complete graphs. Then the graph $\widetilde{G}$ is (K, T)-maximal in $\mathcal{G}$.

We say that a pair (G, H) is an α-*neutral chain*, if there exist graphs $K_1, \ldots, K_r, T_1, \ldots, T_{r-1}$ such that $H = K_1 \subset K_2 \subset \ldots \subset K_r = G$, for each $i \in \{1, \ldots, r-1\}$ the relation $T_i \subseteq K_i$ holds, the pair $((K_{i+1} \setminus K_i) \cup T_i, T_i)$ is α-neutral, and there are no edges which connect the vertices from $K_{i+1} \setminus K_i$ and the vertices from $K_i \setminus T_i$.

Let us state our theorem on the property of containing a maximal copy of a graph. We will use this theorem to disprove the k-law when $\alpha = \frac{1}{k-2}$. Let G be some graph. Let graph H be a strictly balanced subgraph of the graph G. Let either the graph G coincide with graph H or the pair (G, H) be α-neutral chain. Let s be a natural number greater than or equal to $|V(G) \setminus V(H)|$. Denote by $\mathcal{S}$ the set of all α-neutral pairs (K, T) such that

$$|V(K) \setminus V(T)| \le s, \qquad |V(T)| \le |V(G)|.$$

Consider the property $\mathcal{L}_G^s$ of containing a (K, T)-maximal copy of G for each pair $(K, T) \in \mathcal{S}$. In other words, $\mathcal{L}_G^s$ is an element of the set $\mathcal{F}_N$ with the following property. For each graph $\mathcal{G} \in \mathcal{L}_G^s$, there exists a collection of vertices $\widetilde{x}_1, \ldots, \widetilde{x}_v$ such that the graph $\widetilde{G} = \mathcal{G}|_{\{\widetilde{x}_1, \ldots \widetilde{x}_v\}}$ is a copy of the graph G (that is $\{x_i, x_j\} \in E(G)$ implies $\{\widetilde{x}_i, \widetilde{x}_j\} \in E(\widetilde{G})$) and the graph $\widetilde{G}$ is (K, T)-maximal in $\mathcal{G}$ for each pair $(K, T) \in \mathcal{S}$. Set

$$\frac{v(G)}{e(G)} = \alpha.$$

Theorem 8. *Let $G(N, p)$ be the random graph with $p = N^{-\alpha}$. Let the graph G follow the properties described above. Then there exists a number $0 < \xi < 1$ such that the relation*

$$\lim_{N\to\infty} \mathsf{P}_{N,p}\left(\mathcal{L}_G^s\right) = \xi$$

holds.

3.1.4. Proof of Theorem 8

As before let there be v vertices and e edges in the graph G, a be the number of automorphisms of the graph G. Let us consider all ordered collections of v vertices chosen from $\mathcal{V}_N = \{1, \ldots, N\}$. Let M be a subset of the set of all such collections, and let all different unordered collections be in M. Let in addition $(\widetilde{x}_{i_1}, \ldots, \widetilde{x}_{i_v})$ and $(\widetilde{x}_{j_1}, \ldots, \widetilde{x}_{j_v})$ be differently numerated collections consisting of the same vertices (the sets $\{\widetilde{x}_{i_1}, \ldots, \widetilde{x}_{i_v}\}$ and $\{\widetilde{x}_{j_1}, \ldots, \widetilde{x}_{j_v}\}$ are equal). Let $(\widetilde{x}_{i_1}, \ldots, \widetilde{x}_{i_v}) \in M$ and $\widetilde{G}$ be a graph on the set of vertices $\{\widetilde{x}_{i_1}, \ldots, \widetilde{x}_{i_v}\}$ such that the graphs G and $\widetilde{G}$ are isomorphic. Then if the graph constructed from $\widetilde{G}$ by the permutation $\begin{pmatrix} i_1 & \cdots & i_v \\ j_1 & \cdots & j_v \end{pmatrix}$ is isomorphic to G, then M doesn't contain the collection $(\widetilde{x}_{j_1}, \ldots, \widetilde{x}_{j_v})$. Otherwise, this collection is in M. In M there are no collections except for the described ones. Set $|M| = m$ (the equality $m = C_N^v \frac{v!}{a}$ holds). Let us numerate the elements of this set by the numbers $1, \ldots, m$. Consider events $\mathcal{B}_1, \ldots, \mathcal{B}_m$. The event $\mathcal{B}_i$ is that the subgraph on the i-th collection from M is a (K, T)-maximal copy of the graph G for each pair $(K, T) \in \mathcal{S}$. Let X_i be the indicator of the event $\mathcal{B}_i$. Consider the random variable $X = \sum_{i=1}^m X_i$ equal to the number of all (K, T)-maximal copies of G for each pair $(K, T) \in \mathcal{S}$. The following equality holds.

$$\begin{aligned}\mathsf{P}_{N,p}(X = 0) = {} & 1 - \sum_{i=1}^m \mathsf{P}_{N,p}(\mathcal{B}_i) + \sum_{\{i_1,i_2\}\in\{1,\ldots,m\}} \mathsf{P}_{N,p}(\mathcal{B}_{i_1} \wedge \mathcal{B}_{i_2}) \\ & + \cdots + (-1)^n \sum_{\{i_1,i_2,\ldots,i_n\}\in\{1,\ldots,m\}} \mathsf{P}_{N,p}(\mathcal{B}_{i_1} \wedge \mathcal{B}_{i_2} \wedge \cdots \wedge \mathcal{B}_{i_n}) + \cdots. \end{aligned} \quad (1)$$

We need to prove the equality

$$\lim_{N\to\infty} \mathsf{P}_{N,p}(\mathcal{L}_G^s) = \lim_{N\to\infty} (1 - \mathsf{P}_{N,p}(X = 0)) = \xi.$$

Therefore, it is necessary to prove the asymptotics

$$\mathsf{P}_{N,p}(X=0)\sim 1-\xi.$$

Note that the first sum in this equality is the expectation of the random variable X. Denote by $\phi_1(N)$ the conditional probability of a subgraph induced on a particular collection of v vertices to be (K,T)-maximal for each pair $(K,T)\in\mathcal{S}$ under the condition that this induced subgraph is a copy of G. Then

$$\sum_{i=1}^{m}\mathsf{P}_{N,p}(\mathcal{B}_i)=\mathsf{E}_{N,p}(X)=\frac{C_N^v v!\phi_1(N)p^e}{a}\sim\frac{v!\phi_1(N)}{v!a}=\frac{\phi_1(N)}{a}.$$

So, we get an estimate for the first term of the series from the equality (1). It is necessary to estimate all the remaining terms. Consider the n-th term. It is equal to $\sum_{i_1,i_2,\dots,i_n=1}^{m}\mathsf{P}_{N,p}(\mathcal{B}_{i_1}\wedge\mathcal{B}_{i_2}\wedge\cdots\wedge\mathcal{B}_{i_n})$. We write $i\sim j$ when $i,j\in\{1,\dots,m\}$, $i\neq j$ and the collections corresponding to these numbers have common vertices. Denote the sum with intersecting collections of vertices by $r(n,N)$, that is

$$\sum_{\{i_1,i_2,\dots,i_n\}\in\{1,\dots,m\}}\mathsf{P}_{N,p}(\mathcal{B}_{i_1}\wedge\mathcal{B}_{i_2}\wedge\cdots\wedge\mathcal{B}_{i_n})-\sum_{i_1,i_2,\dots,i_n:\,\forall l\neq j\in\{1,\dots,n\}\,i_l\nsim i_j}\mathsf{P}_{N,p}(\mathcal{B}_{i_1}\wedge\mathcal{B}_{i_2}\wedge\cdots\wedge\mathcal{B}_{i_n})=r(n,N).$$

Let $\mathbf{X}_1,\dots,\mathbf{X}_n$ be some disjoint subsets of $\mathcal{V}_N$ such that $|\mathbf{X}_1|=\cdots=|\mathbf{X}_n|=v$. Let $\mathcal{X}_n(N)$ be the set of all graphs $\widetilde{G}$ in Ω_N such that the induced subgraphs $\widetilde{G}|_{\mathbf{X}_1},\dots,\widetilde{G}|_{\mathbf{X}_n}$ are (K,T)-maximal for each pair $(K,T)\in\mathcal{S}$. Let $\mathcal{Y}_n(N)$ be the set of all graphs $\widetilde{G}$ in Ω_N such that the induced subgraphs $\widetilde{G}|_{\mathbf{X}_1},\dots,\widetilde{G}|_{\mathbf{X}_n}$ are copies of G. Let $\phi_n(N)$ be the conditional probability $\mathsf{P}_{N,p}(\mathcal{X}_n(N)|\mathcal{Y}_n(N))$. It is easy to see that $\phi_n(N)$ doesn't depend on the choice of the sets $\mathbf{X}_1,\dots,\mathbf{X}_n$. Then

$$\sum_{\{i_1,i_2,\dots,i_n\}\in\{1,\dots,m\}}\mathsf{P}_{N,p}(\mathcal{B}_{i_1}\wedge\mathcal{B}_{i_2}\wedge\cdots\wedge\mathcal{B}_{i_n})=C_N^{nv}\frac{(nv)!}{(v!)^n n!}\left(\frac{v!}{a}\right)^n\phi_n(N)p^{ne}+r(n,N)\sim\frac{\phi_n(N)}{n!}\left(\frac{1}{a}\right)^n+r(n,N).$$

We will prove the following statement below.

Statement 1. *The equality*

$$r(n,N)=o(1)$$

holds.

To clear up the behavior of terms of the considered series it is necessary to describe the behavior of the quantities $\phi_n(N)$ for each n. We are going to formulate the statement which describes it.

Statement 2. *There exists a constant* $0<\zeta<1$ *such that*

$$\phi_1(N)\sim\zeta,\qquad \phi_n(N)\sim\zeta^n.$$

Before proving Statements 1 and 2 we formulate one more statement that finishes the proof of the theorem.

Statement 3. *Let $\{a_n(N)\}_{n\in\mathbb{N}}$ be a sequence of functions such that there exists a sequence of numbers $\{b_n\}_{n\in\mathbb{N}}$ such that $a_n(N)\sim b_n$ for each $n\in\mathbb{N}$. Let $b_n\to 0$ when $n\to\infty$. Let the series $\sum_{n=1}^{\infty}b_n$ converge to some number b. Let for each $N\in\mathbb{N}$ the series $\sum_{n=1}^{\infty}a_n(N)$ converge, and let for each $s\in\mathbb{N}$ and each $N\in\mathbb{N}$ the inequality*

$$\sum_{n=1}^{2s+1}a_n(N)\le\sum_{n=1}^{\infty}a_n(N)\le\sum_{n=1}^{2s}a_n(N)$$

hold. Then $\sum_{n=1}^{\infty}a_n(N)\sim b$.

Indeed, it follows from the formulated statements that

$$\mathsf{P}_{N,p}(X=0)\to e^{-\xi}$$

when $N\to\infty$, where $\xi=\frac{\zeta}{a}$.

Proof of Statement 3. Let ε be an arbitrary positive number. Under the assumptions of the statement, there exists an $s_0=s_0(\varepsilon)$ such that for every $s>s_0$, the following inequalities hold:

$$|b_s|<\frac{\varepsilon}{3},\qquad\left|\sum_{i=1}^{s}b_i-b\right|<\frac{\varepsilon}{3}.$$

Let $s > s_0$. There exists an $N_0 = N_0(s, \varepsilon)$ such that for each $N > N_0$ and each $i \in \{1, 2, \dots, 2s, 2s+1\}$, the inequality

$$|a_i(N) - b_i| < \frac{\varepsilon}{3(2s+1)}$$

holds. Therefore, we have

$$-\frac{\varepsilon}{3} + \sum_{i=1}^{2s+1} b_i \leq \sum_{i=1}^{\infty} a_i(N) \leq \frac{\varepsilon}{3} + \sum_{i=1}^{2s} b_i.$$

Let's apply all the considered inequalities:

$$-\frac{\varepsilon}{3} - \frac{\varepsilon}{3} + b < \sum_{i=1}^{\infty} a_i(N) < \frac{\varepsilon}{3} + b + \frac{\varepsilon}{3},$$

$$\left| \sum_{i=1}^{\infty} a_i(N) - b \right| < \varepsilon.$$

The statement is proved. □

Proof of Statement 1. Consider two graphs G_1 and G_2 isomorphic to G with non-empty intersection. Let in addition the graphs G_1 and G_2 be strict (G, H)-extensions of the graphs H_1 and H_2 respectively. Then H_i, $i \in \{1, 2\}$, is a subgraph of G_i with density $\frac{e}{v}$ such that the number of its vertices is minimal. Set $G_{12} = G_1 \cap G_2$.

Consider some graph $\mathcal{G} \in \Omega_N$. Let us suppose that there are subgraphs $\widetilde{G}_1, \widetilde{G}_2$ in $\mathcal{G}$ and an isomorphism

$$\varphi : G_1 \cup G_2 \to \widetilde{G}_1 \cup \widetilde{G}_2$$

such that $\varphi(G_1) = \widetilde{G}_1$, $\varphi(G_2) = \widetilde{G}_2$, and the graphs $\widetilde{G}_1, \widetilde{G}_2$ are (K, T)-maximal for each pair $(K, T) \in \mathcal{S}$ in $\mathcal{G}$.

Let us show that the inequality

$$\rho(\widetilde{G}_1 \cup \widetilde{G}_2) > \frac{1}{\alpha}$$

holds. Consider three cases:

(1) the set $V(H_1) \cap V(G_{12})$ is not empty and $H_1 \cap G_{12}$ is a proper subgraph of H_1;
(2) the set $V(H_1) \cap V(G_{12})$ is empty;
(3) the equality $H_1 \cap G_{12} = H_1$ holds.

In the first case, under the assumptions of the statement, the relations

$$f(G_1, H_1 \cup G_{12}) \leq 0, \qquad v(G_2) - \alpha \cdot e(G_2) = 0$$

hold. On the other hand, the graph H_1 is strictly balanced, so

$$\frac{e(H_1) - e(H_1, G_{12} \cap H_1)}{v(H_1) - v(H_1, G_{12} \cap H_1)} = \frac{e(G_{12} \cap H_1)}{v(G_{12} \cap H_1)} < \frac{1}{\alpha}.$$

Therefore,

$$f(H_1 \cup G_{12}, G_{12}) \leq f(H_1, G_{12} \cap H_1) < 0.$$

So, the following inequalities hold:

$$e(G_1 \cup G_2) \geq e(G_2) + e(G_1, H_1 \cup G_{12}) + e(H_1 \cup G_{12}, G_{12}) > \frac{1}{\alpha} v(G_1 \cup G_2)$$

and

$$\rho(\widetilde{G}_1 \cup \widetilde{G}_2) = \rho(G_1 \cup G_2) > \rho(G) = \frac{1}{\alpha}.$$

Consider the second case ($H_1 \cap G_{12} = \emptyset$). The definition of an α-neutral chain implies the inequality $v(G_{12}) - \alpha \cdot e(G_{12}) > 0$. Therefore, the relations

$$e(G_1 \cup G_2) \geq e(G_2) + e(G_1, G_{12}) = e(G_2) + e(G_1) - e(G_{12}) > \frac{1}{\alpha}(v(G_2) + v(G_1) - v(G_{12})) = \frac{1}{\alpha} v(G_1 \cup G_2)$$

hold, that is again $\rho(\widetilde{G}_1 \cup \widetilde{G}_2) > \frac{1}{\alpha}$.

Finally, let $G_{12} \supseteq H_1$. Then either $f(G_{12}, H_1) = 0$ or $f(G_{12}, H_1) > 0$. If $f(G_{12}, H_1) = 0$, then the pair $(G_2 \cup G_1, G_2)$ is an α-neutral chain. Actually, if this pair is not an α-neutral chain, then there exists a graph $X \subseteq G_2 \setminus G_{12}$ such that

$$f(X \cup G_{12}, G_{12}) = 0, \qquad E(X \cup G_{12}) \setminus (E(G_{12}) \cup E(X)) = \emptyset,$$

that is $v(X) - \alpha \cdot e(X) = 0$. But this fact contradicts the definition of the α-neutral chain (G_1, H_1). That is the graph $\widetilde{G}_2$ is not "maximal" in $\mathcal{G}$. If $f(G_{12}, H_1) > 0$, then $f(G_1, G_{12}) < 0$ and the following inequalities hold:

$$e(G_1 \cup G_2) \geq e(G_2) + e(G_1, G_{12}) > \frac{1}{\alpha} v(G_1 \cup G_2).$$

Therefore, the inequality $\rho(\widetilde{G}_1 \cup \widetilde{G}_2) > \frac{1}{\alpha}$ holds again.

Consider n graphs $\widetilde{G}_1, \ldots, \widetilde{G}_n$ isomorphic to G and being (K, T)-maximal in $\mathcal{G}$ for each pair $(K, T) \in \mathcal{S}$. Assume there exist two of these n graphs that have non-empty intersection. Let us prove by induction the inequality

$$\rho(\widetilde{G}_1 \cup \cdots \cup \widetilde{G}_n) > \frac{1}{\alpha}.$$

When $n = 2$ we have already proved this inequality. Suppose it holds for $n - 1$ graphs. Consider n graphs. We can choose some $n - 1$ of them two of which intersect. Let $\widetilde{G}_1, \ldots, \widetilde{G}_{n-1}$ be such graphs. The inequality

$$\rho(\widetilde{G}_1 \cup \cdots \cup \widetilde{G}_{n-1}) > \frac{1}{\alpha}$$

holds. The graph $\bigcup_{i=1}^{n-1} \widetilde{G}_i \cap \widetilde{G}_n$ is a subgraph of the graph $\widetilde{G}_n$. Therefore,

$$\rho\left(\bigcup_{i=1}^{n-1} \widetilde{G}_i \cap \widetilde{G}_n\right) \leq \frac{1}{\alpha}.$$

So, the following relations hold:

$$\rho\left(\bigcup_{i=1}^{n} \widetilde{G}_i\right) = \frac{e\left(\bigcup_{i=1}^{n} \widetilde{G}_i\right)}{v\left(\bigcup_{i=1}^{n} \widetilde{G}_i\right)} = \frac{e\left(\bigcup_{i=1}^{n-1} \widetilde{G}_i\right) + e(\widetilde{G}_n) - e\left(\bigcup_{i=1}^{n-1} \widetilde{G}_i \cap \widetilde{G}_n\right)}{v\left(\bigcup_{i=1}^{n-1} \widetilde{G}_i\right) + v(\widetilde{G}_n) - v\left(\bigcup_{i=1}^{n-1} \widetilde{G}_i \cap \widetilde{G}_n\right)} > \frac{1}{\alpha}. \tag{2}$$

Let us define a set of graphs $\mathcal{Q}$ in the following way. A graph Q is in the set $\mathcal{Q}$ if and only if $v(Q) < nv$ and $Q = G_1 \cup G_2 \cup \cdots \cup G_n$, where each of the graphs G_i is isomorphic to G. Let $Q \in \mathcal{Q}$ and $G_1 \cup \cdots \cup G_n$ be its decomposition into graphs isomorphic to G. Consider a graph $\widetilde{Q} \subset \mathcal{G}$ isomorphic to Q with the following property. Let $\varphi : Q \to \widetilde{Q}$ be an isomorphism. That is for $x, y \in V(Q), \widetilde{x}, \widetilde{y} \in V(\widetilde{Q})$ and $\varphi(x) = \widetilde{x}, \varphi(y) = \widetilde{y}$ the condition $x \sim y \Leftrightarrow \widetilde{x} \sim \widetilde{y}$ holds. Let $\varphi(G_i) = \widetilde{G}_i$ for each $i \in \{1, \ldots, n\}$. Let the graphs $\widetilde{G}_i$ be (K, T)-maximal in $\mathcal{G}$ for each pair $(K, T) \in \mathcal{S}$. Denote by $N_Q^{\max}(\mathcal{G})$ the number of such subgraphs of $\mathcal{G}$. Then the inequality

$$r(n, N) \leq \sum_{Q \in \mathcal{Q}} \mathsf{E}_{N,p} N_Q^{\max}$$

holds. It follows from the inequality (2) that for each graph $Q \in \mathcal{Q}$ there exists positive constants $C(Q), \mu(Q)$ such that the inequalities

$$\mathsf{E}_{N,p} N_Q^{\max} < C(Q) N^{v(Q) - \alpha \cdot e(Q)} < C(Q) N^{-\mu(Q)}$$

hold. Therefore, there exist positive constants C, μ such that $r(n, N) < CN^{-\mu}$. So, $r(n, N) = o(1)$. The statement is proved. $\square$

Proof of Statement 2. Let

$$\mathcal{S} = \{(K_1, T_1), \ldots, (K_r, T_r)\}, \qquad t_i = v(K_i, T_i), \qquad b_i = v(T_i).$$

For each pair of graphs K, T such that $T \subseteq K$ denote by $a(K, T)$ the number of automorphisms of the set $E(K) \setminus E(T)$. For some graph $G = (V, E)$ we say that a bijection $\lambda : V \to V$ is an *automorphism of the set of edges* $E = \{\{x_i, x_j\}, (i, j) \in L\}, L \subseteq \{1, 2, \ldots, |V|\}^2$, if

$$\{x_i, x_j\} \in E \Leftrightarrow \{\lambda(x_i), \lambda(x_j)\} \in E.$$

When we say that λ is an automorphism of the set of edges $E(K) \setminus E(T)$ we mean that λ is an automorphism of the set of edges of the graph $(V(K), E(K) \setminus E(T))$. Let n graphs induced on a particular pairwise disjoint collections each with v vertices be (K, T)-maximal for each pair $(K, T) \in \mathcal{S}$ with the probability $\varphi_n(N)$ (absolute probability unlike

$\phi_n(N)$). Moreover, let $\{\widetilde{x}_1^1,\ldots,\widetilde{x}_v^1\}$, $\{\widetilde{x}_1^2,\ldots,\widetilde{x}_v^2\}$, ..., $\{\widetilde{x}_1^i,\ldots,\widetilde{x}_v^i\}$ be pairwise disjoint subsets of $\mathcal{V}_N$. Denote the collection $\widetilde{x}_1^1,\ldots,\widetilde{x}_v^1,\widetilde{x}_1^2,\ldots,\widetilde{x}_v^2,\ldots,\widetilde{x}_1^i,\ldots,\widetilde{x}_v^i$ by $\widetilde{\mathbf{x}}_v^i$. Introduce the following additional notations:

- $\Phi_i(\widetilde{\mathbf{x}}_v^i)$ is a subset of Ω_N consisting of graphs $\mathcal{G}$ such that the induced subgraphs $\mathcal{G}|_{\{\widetilde{x}_1^1,\ldots,\widetilde{x}_v^1\}}, \mathcal{G}|_{\{\widetilde{x}_1^2,\ldots,\widetilde{x}_v^2\}},\ldots,\mathcal{G}|_{\{\widetilde{x}_1^i,\ldots,\widetilde{x}_v^i\}}$ are (K,T)-maximal for each pair $(K,T)\in\mathcal{S}$;
- $\widehat{\Phi}_i(\widetilde{\mathbf{x}}_v^i)$ is a subset of Ω_N consisting of graphs $\mathcal{G}$ such that the induced subgraphs $\mathcal{G}|_{\{\widetilde{x}_1^1,\ldots,\widetilde{x}_v^1\}}, \mathcal{G}|_{\{\widetilde{x}_1^2,\ldots,\widetilde{x}_v^2\}},\ldots,\mathcal{G}|_{\{\widetilde{x}_1^i,\ldots,\widetilde{x}_v^i\}}$ are (K,T)-maximal in $\mathcal{G}\setminus\mathcal{G}|_{\{\widetilde{x}_1^1,\ldots,\widetilde{x}_v^1,\widetilde{x}_1^2,\ldots,\widetilde{x}_v^2,\ldots,\widetilde{x}_1^i,\ldots,\widetilde{x}_v^i\}}$ for each pair $(K,T)\in\mathcal{S}$;
- $\mathcal{A}_i(\widetilde{\mathbf{x}}_v^i)$ is a subset of Ω_N consisting of graphs $\mathcal{G}$ such that the induced subgraphs $\mathcal{G}|_{\{\widetilde{x}_1^1,\ldots,\widetilde{x}_v^1\}}, \mathcal{G}|_{\{\widetilde{x}_1^2,\ldots,\widetilde{x}_v^2\}},\ldots,\mathcal{G}|_{\{\widetilde{x}_1^i,\ldots,\widetilde{x}_v^i\}}$ are isomorphic to G.

The following relations hold:

$$\begin{aligned}\phi_n(N) &= \frac{\mathsf{P}_{N,p}(\Phi_n(\widetilde{\mathbf{x}}_v^i)\wedge\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))}{\mathsf{P}_{N,p}(\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))} = \frac{\mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n)\setminus(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n)\setminus\Phi_n(\widetilde{\mathbf{x}}_v^n))\wedge\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))}{\mathsf{P}_{N,p}(\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))}\\ &= \frac{\mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n)\wedge\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))\setminus((\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n)\setminus\Phi_n(\widetilde{\mathbf{x}}_v^n))\wedge\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))}{\mathsf{P}_{N,p}(\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))} = \frac{\mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n))\mathsf{P}_{N,p}(\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))}{\mathsf{P}_{N,p}(\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))}\\ &\quad - \frac{\mathsf{P}_{N,p}((\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n)\setminus\Phi_n(\widetilde{\mathbf{x}}_v^n))\wedge\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))}{\mathsf{P}_{N,p}(\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))} = \mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n)) - \zeta(\widetilde{\mathbf{x}}_v^n),\end{aligned}$$

where

$$\zeta(\widetilde{\mathbf{x}}_v^n) = \frac{\mathsf{P}_{N,p}((\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n)\setminus\Phi_n(\widetilde{\mathbf{x}}_v^n))\wedge\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))}{\mathsf{P}_{N,p}(\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))}.$$

Moreover, $\varphi_n(N)$ satisfies the following relations:

$$\varphi_n(N) = \mathsf{P}_{N,p}(\Phi_n(\widetilde{\mathbf{x}}_v^n)) = \mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n)) - \mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n)\setminus\Phi_n(\widetilde{\mathbf{x}}_v^n)).$$

In what follows, we prove that $\varphi_1(N)$ tends to some constant which is not equal to one and zero when $N\to\infty$ and the following equalities hold:

$$\begin{aligned}&\mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n)) = (\varphi_1(N))^n(1+o(1)),\\ &\mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n)\setminus\Phi_n(\widetilde{\mathbf{x}}_v^n)) = o(\mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n))),\\ &\zeta(\widetilde{\mathbf{x}}_v^n) = o(\mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n))).\end{aligned}$$

After that we may conclude that $\varphi_n(N)\sim\phi_n(N)\sim(\phi_1(N))^n$.

First, let us estimate $\varphi_1(N)$. For each pair $(K,T)\in\mathcal{S}$ define some set of indexes $J^{(K,T)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)$, $|J^{(K,T)}| = \nu(K,T)$, such that $\nu(K,T)$ is the greatest number that satisfies the following property. Let $|V(K)\setminus V(T)| = t$, $|V(T)| = b$, $\widetilde{x}_{i_1},\ldots,\widetilde{x}_{i_b}\in\{\widetilde{x}_1,\ldots,\widetilde{x}_v\}$. Each index $j\in J^{(K,T)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)$ can be assigned to an ordered collection of vertices $\widetilde{x}_{v+1}^j,\ldots,\widetilde{x}_{v+t}^j\in\mathcal{V}_N\setminus\{\widetilde{x}_1,\ldots,\widetilde{x}_v\}$ with the following unique restriction. Let $i,j\in J^{(K,T)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)$, $i\neq j$, let the collections $\widetilde{x}_{v+1}^i,\ldots,\widetilde{x}_{v+t}^i$ and $\widetilde{x}_{v+1}^j,\ldots,\widetilde{x}_{v+t}^j$ coincide (they consist of the same vertices), and let for some graphs $\mathcal{G}_1,\mathcal{G}_2\in\Omega_N$ the graphs $\widetilde{G}_1 = \mathcal{G}_1|_{\{\widetilde{x}_{i_1},\ldots,\widetilde{x}_{i_b},\widetilde{x}_{v+1}^i,\ldots,\widetilde{x}_{v+t}^i\}}$, $\widetilde{G}_2 = \mathcal{G}_2|_{\{\widetilde{x}_{i_1},\ldots,\widetilde{x}_{i_b},\widetilde{x}_{v+1}^j,\ldots,\widetilde{x}_{v+t}^j\}}$ be strict (K,T)-extensions of the graphs $\widetilde{T}_1 = \mathcal{G}_1|_{\{\widetilde{x}_{i_1},\ldots,\widetilde{x}_{i_b}\}}$ and $\widetilde{T}_2 = \mathcal{G}_2|_{\{\widetilde{x}_{i_1},\ldots,\widetilde{x}_{i_b}\}}$, respectively. Then after renumbering the vertices $\widetilde{x}_{v+1}^i,\ldots,\widetilde{x}_{v+t}^i$ (which saves the edges and corresponds these vertices to $\widetilde{x}_{v+1}^j,\ldots,\widetilde{x}_{v+t}^j$) the set of edges obtained from the set $E(\widetilde{G}_1)\setminus E(\widetilde{T}_1)$ doesn't coincide with $E(\widetilde{G}_2)\setminus E(\widetilde{T}_2)$. Moreover, let for each pair (K_{i_1},T_{i_1}), $(K_{i_2},T_{i_2})\in\mathcal{S}$ the equality $J^{(K_{i_1},T_{i_1})}\cap J^{(K_{i_2},T_{i_2})} = \emptyset$ hold. Clearly, $\nu(K,T) = C_{N-v}^t\frac{t!}{a(K,T)}$, where $a(K,T)$ is the number of automorphisms of the set $E(K)\setminus E(T)$.

Let again $\widetilde{x}_{i_1},\ldots,\widetilde{x}_{i_b}\in\{\widetilde{x}_1,\ldots,\widetilde{x}_v\}$, and let $\mathcal{B}_j^{(K,T)}(\widetilde{x}_{i_1},\ldots,\widetilde{x}_{i_b})\subset\Omega_N$, $j\in J^{(K,T)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)$, be the sets of graphs $\mathcal{G}$ such that the induced subgraphs $\mathcal{G}|_{\{\widetilde{x}_{i_1},\ldots,\widetilde{x}_{i_b},\widetilde{x}_{v+1}^j,\ldots,\widetilde{x}_{v+t}^j\}}$ are strict (K,T)-extensions of the induced subgraphs $\mathcal{G}|_{\{\widetilde{x}_{i_1},\ldots,\widetilde{x}_{i_b}\}}$.

Define some set of indexes $L^{(K,T)}(\widetilde{x}_1,\ldots,\widetilde{x}_v) = \{1,\ldots,\theta(K,T)\}$ where $\theta(K,T)$ is the greatest number such that the following property holds. Let $\widetilde{x}_{v+1},\ldots,\widetilde{x}_{v+t}\in\mathcal{V}_N\setminus\{\widetilde{x}_1,\ldots,\widetilde{x}_v\}$. Each index $l\in L^{(K,T)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)$ can be assigned to an ordered collection of vertices $\widetilde{x}_1^l,\ldots,\widetilde{x}_b^l\in\{\widetilde{x}_1,\ldots,\widetilde{x}_v\}$ with the following unique restriction. Let $l_1,l_2\in L^{(K,T)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)$, $l_1\neq l_2$, let the collections $\widetilde{x}_1^{l_1},\ldots,\widetilde{x}_b^{l_1}$ and $\widetilde{x}_1^{l_2},\ldots,\widetilde{x}_b^{l_2}$ coincide (they consist of the same vertices), and let for some graphs $\mathcal{G}_1,\mathcal{G}_2\in\Omega_N$ the graphs $\widetilde{G}_1 = \mathcal{G}_1|_{\{\widetilde{x}_1^{l_1},\ldots,\widetilde{x}_b^{l_1},\widetilde{x}_{v+1},\ldots,\widetilde{x}_{v+t}\}}$, $\widetilde{G}_2 = \mathcal{G}_2|_{\{\widetilde{x}_1^{l_2},\ldots,\widetilde{x}_b^{l_2},\widetilde{x}_{v+1},\ldots,\widetilde{x}_{v+t}\}}$ be strict (K,T)-extensions of the graphs $\widetilde{T}_1 = \mathcal{G}_1|_{\{\widetilde{x}_1^{l_1},\ldots,\widetilde{x}_b^{l_2}\}}$ and $\widetilde{T}_2 = \mathcal{G}_2|_{\{\widetilde{x}_1^{l_1},\ldots,\widetilde{x}_b^{l_2}\}}$, respectively. Then after renumbering the vertices $\widetilde{x}_1^{l_1},\ldots,\widetilde{x}_b^{l_1}$ (which saves the edges and assigns these vertices to $\widetilde{x}_1^{l_2},\ldots,\widetilde{x}_b^{l_2}$), the set of edges obtained from the set $E(\widetilde{G}_1)\setminus E(\widetilde{T}_1)$ doesn't coincide with $E(\widetilde{G}_2)\setminus E(\widetilde{T}_2)$.

Moreover, define for each $j \in J^{(K,T)}(\widetilde{x}_1, \ldots, \widetilde{x}_v)$ the set $[\mathcal{B}]_j^{(K,T)}(\widetilde{x}_1, \ldots, \widetilde{x}_v) \subset \Omega_N$ by the equality

$$[\mathcal{B}]_j^{(K,T)}(\widetilde{x}_1, \ldots, \widetilde{x}_v) = \bigcup_{l \in L^{(K,T)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)} \mathcal{B}_j^{(K,T)}(\widetilde{x}_1^l, \ldots, \widetilde{x}_b^l).$$

Let

$$\{\mathcal{D}_i\}_{i\in I} = \bigcup_{(K,T)\in\mathcal{S}} \left\{[\mathcal{B}]_j^{(K,T)}(\widetilde{x}_1, \ldots, \widetilde{x}_v)\right\}_{j\in J^{(K,T)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)},$$

that is

$$I = \bigsqcup_{(K,T)\in\mathcal{S}} J^{(K,T)}(\widetilde{x}_1, \ldots, \widetilde{x}_v).$$

Set $h = |I|$. We write $i \sim j$ if $i, j \in I$, $i \neq j$, and the corresponding collections of vertices intersect. For each $i \in I$ consider random variables $Y_i = \mathbb{I}(\mathcal{D}_i)$ and $Y = \sum_{i\in I} Y_i$.

Settle some index $j_1 \in J^{(K_1,T_1)}(\widetilde{x}_1, \ldots, \widetilde{x}_v)$. Let $j_2 \in J^{(K_2,T_2)}(\widetilde{x}_1, \ldots, \widetilde{x}_v)$, ..., $j_r \in J^{(K_r,T_r)}(\widetilde{x}_1, \ldots, \widetilde{x}_v)$ be indexes such that the ordered collections corresponding to indexes $j_1, j_2, \ldots, j_r$ coincide (for each index $j_1 \in J^{(K_1,T_1)}(\widetilde{x}_1, \ldots, \widetilde{x}_v)$ the sets $J^{(K_2,T_2)}(\widetilde{x}_1, \ldots, \widetilde{x}_v), \ldots, J^{(K_r,T_r)}(\widetilde{x}_1, \ldots, \widetilde{x}_v)$ measured up to our definition can be chosen so that the described indexes $j_2, \ldots, j_r$ exist). Let us define for each $i \in L^{(K,T)}(\widetilde{x}_1, \ldots, \widetilde{x}_v)$ and each $u \in \{1, \ldots, r\}$ random variables $W_i^{(K_u,T_u)}$ by the equality

$$W_i^{(K_u,T_u)} = \mathbb{I}(\mathcal{B}_{j_u}^{(K_u,T_u)}(\widetilde{x}_1^i, \ldots, \widetilde{x}_{b_u}^i)).$$

Set

$$W^{(K_u,T_u)} = \sum_{i\in L^{(K,T)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)} W_i^{(K_u,T_u)}.$$

Let us find the probability $\mathsf{P}_{N,p}(W^{(K_u,T_u)} = 0)$. From the inclusion–exclusion principle it follows that

$$\begin{aligned}\mathsf{P}_{N,p}(W^{(K_u,T_u)} = 0) = 1 &- \sum_{i\in L^{(K_u,T_u)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)} \mathsf{P}_{N,p}\left(\mathcal{B}_{j_u}^{(K_u,T_u)}(\widetilde{x}_1^i, \ldots, \widetilde{x}_{b_u}^i)\right) \\ &+ \sum_{i_1,i_2\in L^{(K_u,T_u)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)} \mathsf{P}_{N,p}\left(\mathcal{B}_{j_u}^{(K_u,T_u)}(\widetilde{x}_1^{i_1}, \ldots, \widetilde{x}_{b_u}^{i_1}) \wedge \mathcal{B}_{j_u}^{(K_u,T_u)}(\widetilde{x}_1^{i_2}, \ldots, \widetilde{x}_{b_u}^{i_2})\right) \\ &+ \cdots + (-1)^s \sum_{i_1,i_2,\ldots,i_s\in L^{(K_u,T_u)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)} \mathsf{P}_{N,p} \\ &\times \left(\mathcal{B}_{j_u}^{(K_u,T_u)}(\widetilde{x}_1^{i_1}, \ldots, \widetilde{x}_{b_u}^{i_1}) \wedge \cdots \wedge \mathcal{B}_{j_u}^{(K_u,T_u)}(\widetilde{x}_1^{i_s}, \ldots, \widetilde{x}_{b_u}^{i_s})\right) + \cdots.\end{aligned}$$

Hereinafter all the sums are taken over sets of pairwise different numbers $i_1, \ldots, i_j$. Note that when $s \neq 1$ the equality

$$\sum_{i_1,i_2,\ldots,i_s\in L^{(K_u,T_u)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)} \mathsf{P}_{N,p}\left(\mathcal{B}_{j_u}^{(K_u,T_u)}(\widetilde{x}_1^{i_1}, \ldots, \widetilde{x}_{b_u}^{i_1}) \wedge \cdots \wedge \mathcal{B}_{j_u}^{(K_u,T_u)}(\widetilde{x}_1^{i_s}, \ldots, \widetilde{x}_{b_u}^{i_s})\right) = o\left(\mathsf{P}_{N,p}\left(\mathcal{B}_{j_u}^{(K_u,T_u)}(\widetilde{x}_1^i, \ldots, \widetilde{x}_{b_u}^i)\right)\right)$$

holds and when $s = 1$ the equality

$$\sum_{i\in L^{(K_u,T_u)}(\widetilde{x}_1,\ldots,\widetilde{x}_v)} \mathsf{P}_{N,p}\left(\mathcal{B}_{j_u}^{(K_u,T_u)}(\widetilde{x}_1^i, \ldots, \widetilde{x}_{t_u}^i)\right) = \theta(K_u, T_u)\mathsf{P}_{N,p}\left(\mathcal{B}_{j_u}^{(K_u,T_u)}(\widetilde{x}_1^i, \ldots, \widetilde{x}_{b_u}^i)\right)$$

holds. Therefore, by Statement 3 for each pair $(K_u, T_u) \in \mathcal{S}$ the following relation holds:

$$\mathsf{P}_{N,p}(W^{(K_u,T_u)} = 0) \sim 1 - \theta(K_u, T_u)p^{e(K_u,T_u)}. \tag{3}$$

Let us turn to the random variable Y.

Let $\widetilde{G}$ be some graph, $v(\widetilde{G}) = v$, $(K_{i_1}, T_{i_1}) \in \mathcal{S}$, $(K_{i_2}, T_{i_2}) \in \mathcal{S}$, ..., $(K_{i_l}, T_{i_l}) \in \mathcal{S}$; $T_{i_1} \subseteq \widetilde{G}$, $T_{i_2} \subseteq \widetilde{G}$, ..., $T_{i_l} \subseteq \widetilde{G}$; $K_{i_1} \cap \widetilde{G} = T_{i_1}$, $K_{i_2} \cap \widetilde{G} = T_{i_2}$, ..., $K_{i_l} \cap \widetilde{G} = T_{i_l}$. Moreover, let for some $j_1, j_2 \in \{1, 2, \ldots, l\}$ the set $V((K_{i_{j_1}} \setminus T_{i_{j_1}}) \cap (K_{i_{j_2}} \setminus T_{i_{j_2}}))$ be not empty. As in Statement 1 it can be proved by induction that the following relations hold

$$\frac{e\left(\bigcup_{j=1}^{l} K_{i_j} \cup \widetilde{G}, \widetilde{G}\right)}{v\left(\bigcup_{j=1}^{l} K_{i_j} \cup \widetilde{G}, \widetilde{G}\right)} = \frac{e\left(\bigcup_{j\in\{1,\ldots,l\}\setminus\{j_2\}} K_{i_j} \cup \widetilde{G}, \widetilde{G}\right) + e\left(\bigcup_{j\in\{1,\ldots,l\}\setminus\{j_2\}} K_{i_j} \cup \widetilde{G} \cup K_{i_{j_2}}, \bigcup_{j\in\{1,\ldots,l\}\setminus\{j_2\}} K_{i_j} \cup \widetilde{G}\right)}{v\left(\bigcup_{j\in\{1,\ldots,l\}\setminus\{j_2\}} K_{i_j} \cup \widetilde{G}, \widetilde{G}\right) + v\left(\bigcup_{j\in\{1,\ldots,l\}\setminus\{j_2\}} K_{i_j} \cup \widetilde{G} \cup K_{i_{j_2}}, \bigcup_{j\in\{1,\ldots,l\}\setminus\{j_2\}} K_{i_j} \cup \widetilde{G}\right)}$$

$$\geq \frac{e\left(\bigcup_{j\in\{1,\dots,l\}\setminus\{j_2\}} K_{i_j}\cup\widetilde{G},\widetilde{G}\right)+e(K_{i_{j_2}}\cup\widetilde{G},\widetilde{G})-e\left(\left(\bigcup_{j\in\{1,\dots,l\}\setminus\{j_2\}} K_{i_j}\cap K_{i_{j_2}}\right)\cup\widetilde{G},\widetilde{G}\right)}{v\left(\bigcup_{j\in\{1,\dots,l\}\setminus\{j_2\}} K_{i_j}\cup\widetilde{G},\widetilde{G}\right)+v(K_{i_{j_2}}\cup\widetilde{G},\widetilde{G})-v\left(\left(\bigcup_{j\in\{1,\dots,l\}\setminus\{j_2\}} K_{i_j}\cap K_{i_{j_2}}\right)\cup\widetilde{G},\widetilde{G}\right)}$$

$$> \frac{1}{\alpha}. \tag{4}$$

Consider the functions $\sigma_s : I^s \to \{0,1\}$. Let $\sigma_s(i_1,\dots,i_s)=1$ if and only if the indexes $i_1,\dots,i_s$ are different in pairs and there exists $\mu,\nu\in\{1,\dots,s\}$ such that $i_\mu\sim i_\nu$. Consider the sum $\sum_{i_1,\dots,i_s\in I:\,\sigma_s(i_1,\dots,i_s)=1}\mathsf{P}_{N,p}(\mathcal{D}_{i_1}\wedge\cdots\wedge\mathcal{D}_{i_s})$, (we summarize over those collections of indexes that correspond to intersecting collections). It is easy to see (similarly to Statement 1) that the inequality (4) yields

$$\sum_{i_1,\dots,i_s\in I:\,\sigma_s(i_1,\dots,i_s)=1}\mathsf{P}_{N,p}(\mathcal{D}_{i_1}\wedge\cdots\wedge\mathcal{D}_{i_s})=o(1).$$

The equality $\varphi_1(N)=\mathsf{P}_{N,p}(Y=0)$ holds. On the other hand, from the inclusion–exclusion principle it follows that

$$\begin{aligned}\mathsf{P}_{N,p}(Y=0) &= 1-\sum_{i\in I}\mathsf{P}_{N,p}(\mathcal{D}_i)+\sum_{i_1,i_2\in I}\mathsf{P}_{N,p}(\mathcal{D}_{i_1}\wedge\mathcal{D}_{i_2})\\ &\quad+\cdots+(-1)^s\sum_{i_1,i_2,\dots,i_s\in I}\mathsf{P}_{N,p}(\mathcal{D}_{i_1}\wedge\mathcal{D}_{i_2}\wedge\cdots\wedge\mathcal{D}_{i_s})+\cdots.\end{aligned}$$

Therefore, Statement 3 and relation (3) implies the following series of relations:

$$\begin{aligned}\varphi_1(N) &\sim 1-\sum_{i\in I}\mathsf{P}_{N,p}(\mathcal{D}_i)+\cdots+(-1)^s\sum_{i_1,i_2,\dots,i_s\in I}\;\sum_{\forall i\neq j\in\{i_1,\dots,i_s\}\,\sigma(i,j)=0}\mathsf{P}_{N,p}(\mathcal{D}_{i_1}\wedge\cdots\wedge\mathcal{D}_{i_s})+\cdots\\ &\sim 1-\sum_{i=1}^{r}C_N^{t_i}\frac{t_i!}{a(K_i,T_i)}\theta(K_i,T_i)p^{e(K_i,T_i)}+\cdots+(-1)^s\sum_{i_1,\dots,i_r=0;\;i_1+\cdots+i_r=s}^{s}(C_N^{t_1})^{i_1}\dots(C_N^{t_r})^{i_r}\frac{1}{i_1!\dots i_r!}\\ &\quad\times\left(\frac{t_1!}{a(K_1,T_1)}\theta(K_1,T_1)\right)^{i_1}\dots\left(\frac{t_r!}{a(K_r,T_r)}\theta(K_r,T_r)\right)^{i_r}p^{i_1e(K_1,T_1)+\cdots+i_re(K_r,T_r)}\\ &\quad+\cdots\varphi_1(N)\sim 1-\sum_{i=1}^{r}\frac{\theta(K_i,T_i)}{a(K_i,T_i)}+\frac{1}{2}\left(\sum_{i=1}^{r}\frac{\theta(K_i,T_i)}{a(K_i,T_i)}\right)^2+\cdots+(-1)^s\frac{1}{s!}\left(\sum_{i=1}^{r}\frac{\theta(K_i,T_i)}{a(K_i,T_i)}\right)^s+\cdots\\ &= \exp\left(-\sum_{i=1}^{r}\frac{\theta(K_i,T_i)}{a(K_i,T_i)}\right).\end{aligned}$$

Set

$$\eta=\sum_{i=1}^{r}\frac{\theta(K_i,T_i)}{a(K_i,T_i)}.$$

The described argument can be drawn for bounding the value $\mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{x}_1^1,\dots,\widetilde{x}_v^1,\dots,\widetilde{x}_1^n,\dots,\widetilde{x}_v^n))$ by defining a set $\{\mathcal{D}_i\}_{i\in I^n}$ not for collections from the set $\mathcal{V}_N\setminus\{\widetilde{x}_1,\dots,\widetilde{x}_v\}$, but for collections from the set $\mathcal{V}_N\setminus\{\widetilde{x}_1^1,\dots,\widetilde{x}_v^1,\dots,\widetilde{x}_1^n,\dots,\widetilde{x}_v^n\}$. Moreover, each event $\mathcal{D}_i$ corresponds to some graph on the set of vertices from $\mathcal{V}_N$ and some pair $(K,T)\in\mathcal{S}$. Let us multiply the number of events by n counting every such graph n times. That is for every such graph $\mathcal{H}$ we get n events $\mathcal{D}_{i_1},\dots,\mathcal{D}_{i_n}$. Let an event $\mathcal{D}_{i_j}$ be such that there exists a subgraph $\mathcal{U}\subseteq\mathcal{G}|_{\{\widetilde{x}_1^j,\dots,\widetilde{x}_v^j\}}$ such that the graph $\mathcal{H}\cup\mathcal{U}$ is a strict (K,T)-extension of the graph $\mathcal{U}$. Then for the corresponding random variable Y^n the following relation holds:

$$\begin{aligned}\mathsf{P}_{N,p}(Y^n=0) &\sim 1-\sum_{i=1}^{r}\frac{n\theta(K_i,T_i)}{a(K_i,T_i)}+\frac{1}{2}\left(\sum_{i=1}^{r}\frac{n\theta(K_i,T_i)}{a(K_i,T_i)}\right)^2+\cdots+(-1)^s\frac{1}{s!}\left(\sum_{i=1}^{r}\frac{n\theta(K_i,T_i)}{a(K_i,T_i)}\right)^s+\cdots\\ &\sim \exp\left(-n\sum_{i=1}^{r}\frac{\theta(K_i,T_i)}{a(K_i,T_i)}\right)=\exp(-n\eta).\end{aligned}$$

Therefore, the equality

$$\mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n))=(\varphi_1(N))^n(1+o(1))$$

holds.

Return to the random variables $\zeta(\widetilde{\mathbf{x}}_v^n)$ and $\mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n) \setminus \Phi_n(\widetilde{\mathbf{x}}_v^n))$. Let $(K,T) \in \mathcal{S}$ and $|V(K) \setminus V(T)| = t$. Define the set $\widehat{J}^{(K,T)}(\widetilde{\mathbf{x}}_v^n) \subset J^{(K,T)}(\widetilde{x}_1, \dots, \widetilde{x}_v)$ in the following way:

$$j \in \widehat{J}^{(K,T)}(\widetilde{x}_1^1, \dots, \widetilde{x}_v^1, \dots, \widetilde{x}_1^n, \dots, \widetilde{x}_v^n) \Leftrightarrow \exists i \in \{2, \dots, v\} \qquad |\{\widetilde{x}_1^j, \dots, \widetilde{x}_t^j\} \cap \{\widetilde{x}_1^i, \dots, \widetilde{x}_v^i\}| \geq 2.$$

Let

$$\{\widehat{\mathcal{D}}_i\}_{i \in \widehat{I}} = \bigcup_{k \in \{1,\dots,n\}} \bigcup_{(K,T) \in \mathcal{S}} \left\{[\mathcal{B}]_j^{(K,T)}(\widetilde{x}_1, \dots, \widetilde{x}_v)\right\}_{j \in \widehat{J}^{(K,T)}(\widetilde{x}_1^k, \dots, \widetilde{x}_v^k, \widetilde{x}_1^1, \dots, \widetilde{x}_v^1, \dots, \widetilde{x}_1^{k-1}, \dots, \widetilde{x}_v^{k-1}, \widetilde{x}_1^{k+1}, \dots, \widetilde{x}_v^{k+1}, \dots, \widetilde{x}_1^n, \dots, \widetilde{x}_v^n)},$$

that is

$$\widehat{I} = \bigsqcup_{k \in \{1,\dots,n\}} \bigsqcup_{(K,T) \in \mathcal{S}} \widehat{J}^{(K,T)}(\widetilde{x}_1^k, \dots, \widetilde{x}_v^k, \widetilde{x}_1^1, \dots, \widetilde{x}_v^1, \dots, \widetilde{x}_1^{k-1}, \dots, \widetilde{x}_v^{k-1}, \widetilde{x}_1^{k+1}, \dots, \widetilde{x}_v^{k+1}, \dots, \widetilde{x}_1^n, \dots, \widetilde{x}_v^n).$$

Set $\widehat{h} = |\widehat{I}|$. Consider for each $i \in \widehat{I}$ random variables

$$\widehat{Y}_i = \mathbb{I}(\widehat{\mathcal{D}}_i), \qquad \widehat{Y} = \sum_{i \in \widehat{I}} \widehat{Y}_i$$

and

$$\widehat{Z}_i = \mathbb{I}(\widehat{\mathcal{D}}_i \wedge \mathcal{A}_n(\widetilde{x}_1^1, \dots, \widetilde{x}_v^1, \dots, \widetilde{x}_1^n, \dots, \widetilde{x}_v^n)), \qquad \widehat{Z} = \sum_{i \in \widehat{I}} \widehat{Z}_i.$$

Then

$$\zeta(\widetilde{\mathbf{x}}_v^n) \leq \frac{\mathsf{P}_{N,p}(\widehat{Z} > 0)}{\mathsf{P}_{N,p}(\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n))},$$
$$\mathsf{P}_{N,p}(\widehat{\Phi}_n(\widetilde{\mathbf{x}}_v^n) \setminus \Phi_n(\widetilde{\mathbf{x}}_v^n)) \leq \mathsf{P}_{N,p}(\widehat{Y} > 0).$$

The relations

$$\mathsf{P}_{N,p}(\widehat{Y} > 0) \leq \sum_{i \in \widehat{I}} \mathsf{P}_{N,p}(\widehat{\mathcal{D}}_i) = \Theta\left(N^{-2}\right) = o(1)$$

hold, because the number of ways of choosing two vertices from $\widetilde{x}_1^1, \dots, \widetilde{x}_v^1, \dots, \widetilde{x}_v^n, \dots, \widetilde{x}_v^n$ is independent of N and the quantity $C_N^{t-2} p^{\frac{t}{\alpha}}$ equals $\Theta(N^{-2})$;

$$\mathsf{P}_{N,p}(\widehat{Z} > 0) \leq \sum_{i \in \widehat{I}} \mathsf{P}_{N,p}(\widehat{\mathcal{D}}_i \wedge \mathcal{A}_n(\widetilde{\mathbf{x}}_v^n)) = O\left(N^{-\alpha n e - \gamma}\right) = o(\mathcal{A}_n(\widetilde{\mathbf{x}}_v^n)),$$

where $\gamma > 0$ is some constant (because if (K,T) is an α-neutral pair and $X \subseteq K \setminus T$ then $f(K,T) - (v(X) - \alpha \cdot e(X)) < 0$). The statement is proved. $\square$

3.2. Ehrenfeucht game

Let us define the game EHR(G, H, k) with two graphs G and H, two players (Spoiler and Duplicator) and a fixed number of rounds k. It is called *the Ehrenfeucht game* (see [1,13,17,16,18,11,9,21,7]). At the ν-th round ($1 \leq \nu \leq k$) Spoiler chooses either a vertex $x_{j_\nu} \in V(G)$ or a vertex $y_{j'_\nu} \in V(H)$. Duplicator chooses a vertex of the other graph. If Spoiler chooses at the μ-th round the vertex $x_{j_\mu} \in V(G)$, $j_\mu = j_\nu$ $(\nu < \mu)$, then Duplicator must choose $y_{j'_\nu} \in V(H)$. If at this round Spoiler chooses a vertex $x_{j_\mu} \in V(G)$, $j_\mu \notin \{j_1, \dots, j_{\mu-1}\}$, then Duplicator must choose a vertex $y_{j'_\mu} \in V(H)$, $j'_\mu \notin \{j'_1, \dots, j'_{\mu-1}\}$. If Duplicator cannot do it then Spoiler wins the game. At the end of the game the vertices $x_{j_1}, \dots, x_{j_k} \in V(G)$; $y_{j'_1}, \dots, y_{j'_k} \in V(H)$ are chosen. Let $x_{h_1}, \dots, x_{h_l}$ and $y_{h'_1}, \dots, y_{h'_l}$, $l \leq k$, be different vertices from the sets $\{x_{j_1}, \dots, x_{j_k}\}$ and $\{y_{j'_1}, \dots, y_{j'_k}\}$. Duplicator wins if and only if

$$G|_{\{x_{h_1}, \dots, x_{h_l}\}} \cong H|_{\{y_{h'_1}, \dots, y_{h'_l}\}},$$

that is the graphs induced on the sets $\{x_{h_1}, \dots, x_{h_l}\}$ and $\{y_{h'_1}, \dots, y_{h'_l}\}$ are isomorphic.

Consider the probability space

$$G(N, p(N)) \times G(M, p(M)) = \left(\Omega_N \times \Omega_M, \mathcal{F}_N \times \mathcal{F}_M, \mathsf{P}_{N,M,p}\right),$$

where $\mathsf{P}_{N,M,p}(\mathcal{G}_N, \mathcal{G}_M) = \mathsf{P}_{N,p}(\mathcal{G}_N)\mathsf{P}_{M,p}(\mathcal{G}_M)$ for $\mathcal{G}_N \in \Omega_N$, $\mathcal{G}_M \in \Omega_M$.

The proofs of the theorems will be based on the following statement (see [1,13,17,21]).

Theorem 9. *The equality*

$$\lim_{N,M\to\infty} \mathsf{P}_{N,M,p}\ (\textit{Duplicator wins the game}\ \mathrm{EHR}(G,H,k)) = 1$$

holds if and only if the random graph $G(N,p)$ follows the zero–one k-law.

3.3. Proof of Theorem 3

Let us consider the sets $\left(0,\frac{1}{k-1}\right)$, $\left\{\frac{1}{k-1}\right\}$, $\left(\frac{1}{k-1},\frac{2}{2k-3}\right)$, $\left\{\frac{2}{2k-3}\right\}$, $\left(\frac{2}{2k-3},\frac{1}{k-2}\right)$ and divide the proof of the theorem into cases depending on which set α is in. In fact, some of the cases can be included into the others. However, we'll consider all of them to make a simpler understanding of the proof.

Hereinafter let us suppose that $\mathcal{G}\in\Omega_N$ and $\mathcal{H}\in\Omega_M$ are the graphs from the game, the variables x_i are the vertices from the set $V(\mathcal{G})$, and the variables y_i are the vertices from the set $V(\mathcal{H})$.

(1) Let $\alpha<\frac{1}{k-1}$, $r\in\mathbb{N}$, $\mathcal{E}_r\in\mathcal{F}_N$ be the set of graphs satisfying the property that for any integer non-negative numbers a,b such that $a+b\le r$ and any vertices $x_1,\ldots,x_a,y_1,\ldots,y_b$, there exists a vertex z adjacent to $x_1,\ldots,x_a$ and non-adjacent to $y_1,\ldots,y_b$. Such property is called a *full level r extension property*. Let us show that $\lim_{N\to\infty}\mathsf{P}_{N,p}(\mathcal{E}_{k-1})=1$. The following relations hold:

$$\mathsf{P}_{N,p}(\Omega_N\setminus\mathcal{E}_{k-1})\le(k-1)^2N^{k-1}(1-p^{k-1})^N\le(k-1)^2N^{k-1}\exp\left(-N^{1-(k-1)\alpha}\right)\to 0,$$

that is, indeed, $\lim_{N\to\infty}\mathsf{P}_{N,p}(\mathcal{E}_{k-1})=1$. But if the full level $k-1$ extension property holds almost surely, then, obviously, Duplicator has a winning strategy in the game on $k-1$ rounds almost surely. Thus, let Spoiler at the k-th round choose a vertex $x_k\in V(G)$. The vertices $x_1,\ldots,x_{k-1}\in V(G)$, $y_1,\ldots,y_{k-1}\in V(H)$ are already chosen. The pair $(\mathcal{G}|_{\{x_1,\ldots,x_k\}},\mathcal{G}|_{\{x_1,\ldots,x_{k-1}\}})$ is α-safe (it follows from the inequality $\alpha<\frac{1}{k-1}$). Therefore, from Theorem 5 it follows that almost surely Duplicator is able to find a vertex $y_k\in V(\mathcal{H})$ such that the graph $\mathcal{H}|_{\{y_1,\ldots,y_k\}}$ is a strict $(\mathcal{G}|_{\{x_1,\ldots,x_k\}},\mathcal{G}|_{\{x_1,\ldots,x_{k-1}\}})$-extension of the graph $\mathcal{H}|_{\{y_1,\ldots,y_{k-1}\}}$. Then at the k-th round Duplicator chooses the vertex y_k and wins. Therefore, it follows from Theorem 9 that the zero–one k-law holds.

(2) Let $\alpha=\frac{1}{k-1}$. Obviously, it follows from the previous case that the full level $k-2$ extension property holds. Therefore, Duplicator has a winning strategy for the game on $k-2$ rounds almost surely. Thus, let Spoiler choose a vertex x_{k-1}. Duplicator must choose some vertex y_{k-1}.

Note that if $H\subset G$ and $v(G,H)=1$, $v(H)=k-2$, then the pair (G,H) is α-safe. Indeed, $e(G,H)\le k-2$ and $1-\alpha(k-2)>0$. If $v(G,H)=1$, $v(H)=k-1$, then the pair (G,H) is either α-safe or α-neutral. It is α-neutral if and only if $e(G,H)=k-1$.

Let (G,H) be an α-neutral pair such that $v(G,H)=1$, $v(H)=k-1$. The chosen graph $\mathcal{G}|_{\{x_1,\ldots,x_{k-1}\}}$ either has a (G,H)-extension in $\mathcal{G}$ or doesn't have it. If an extension exists (suppose a vertex x_k forms it), then the pair $(\mathcal{G}|_{\{x_1,\ldots,x_k\}},\mathcal{G}|_{\{x_1,\ldots,x_{k-2}\}})$ is α-safe. Therefore, it follows from Theorem 5 that almost surely Duplicator is able to find vertices $y_{k-1},y_k\in V(\mathcal{H})$ such that the graph $\mathcal{H}|_{\{y_1,\ldots,y_k\}}$ is a strict $(\mathcal{G}|_{\{x_1,\ldots,x_k\}},\mathcal{G}|_{\{x_1,\ldots,x_{k-2}\}})$-extension of the graph $\mathcal{H}|_{\{y_1,\ldots,y_{k-2}\}}$. Then at the $k-1$-th round Duplicator chooses the vertex y_{k-1}. If after that Spoiler chooses a vertex adjacent to each of the $k-1$ chosen vertices, then Duplicator is able to find a vertex adjacent to each of the vertices chosen in his graph. If he chooses, say, a vertex y_k non-adjacent to some vertex from $y_1,\ldots,y_{k-1}$, then the pair $(\mathcal{H}|_{\{y_1,\ldots,y_k\}},\mathcal{H}|_{\{y_1,\ldots,y_{k-1}\}})$ is α-safe and Duplicator wins by Theorem 5.

If a (G,H)-extension doesn't exist, then by Theorem 6 Duplicator is able to chose a *suitable* vertex y_{k-1} (that is $\forall i\in\{1,2,\ldots,k-2\}\ x_i\sim x_{k-1}\Leftrightarrow y_{k-1}\sim y_i$) such that the graph $\mathcal{H}|_{\{y_1,\ldots,y_{k-1}\}}$ doesn't have (G,H)-extension in $\mathcal{H}$. Therefore, in this case almost surely Duplicator has a winning strategy too.

(3) Let $\frac{1}{k-1}<\alpha<\frac{2}{2k-3}$. The full level $k-2$ extension property holds. If there are $k-2$ vertices in $V(H)$, then for any graph G with k vertices containing H the pair (G,H) is α-safe, and for any graph G with $k-1$ vertices containing H the pair (G,H) is α-safe too. If there are $k-1$ vertices in $V(H)$, then there exists graph G with k vertices containing H such that the pair (G,H) is α-rigid (the pair (G,H) is α-rigid if and only if $e(G,H)=k-1$).

Let Spoiler choose a vertex x_{k-1} at the $k-1$-th round. Vertices $x_1,\ldots,x_{k-2},y_1,\ldots,y_{k-2}$ are already chosen. Let us consider graphs H,G, $H\subset G$, $v(H)=k-1$, $v(G,H)=1$, such that the pair (G,H) is α-rigid. Then the graph $\mathcal{G}|_{\{x_1,\ldots,x_{k-1}\}}$ either has a (G,H)-extension in $\mathcal{G}$ or doesn't have it. If a (G,H)-extension doesn't exist, then by Theorem 6 Duplicator is able to choose suitable a vertex y_{k-1} (that is $\forall i\in\{1,2,\ldots,k-2\}\ x_i\sim x_{k-1}\Leftrightarrow y_{k-1}\sim y_i$) such that there is no (G,H)-extension of $\mathcal{H}|_{\{y_1,\ldots,y_{k-1}\}}$ in $\mathcal{H}$. Then Spoiler chooses the last vertex. By Theorem 5 Duplicator is able to find a suitable vertex and to win.

If the (G,H)-extension exists, then the pair $(\mathcal{G}|_{\{x_1,\ldots,x_k\}},\mathcal{G}|_{\{x_1,\ldots,x_{k-2}\}})$ is α-safe. Therefore, Duplicator is able to find vertices y_{k-1},y such that the vertex y_{k-1} satisfies the rules of the game and the pair $(\mathcal{H}|_{\{x_1,\ldots,x_k\}},\mathcal{H}|_{\{x_1,\ldots,x_{k-1}\}})$ is α-rigid. Subsequent arguments are similar to the ones in the case (2).

(4) Let $\alpha=\frac{2}{2k-3}$. The full level $k-2$ extension property holds. If there are $k-2$ vertices in $V(H)$, then for any graph G with k vertices containing H the pair (G,H) is either α-safe or α-neutral (the pair (G,H) is α-neutral if and only if $e(G,H)=2k-3$),

and for any graph G with $k-1$ vertices containing H the pair (G, H) is α-safe. In the case when $v(H) = k-1$, $v(G) = k$ the pair (G, H) is either α-safe or α-rigid (the pair (G, H) is α-rigid if and only if $e(G, H) = k-1$).

Let us define graphs H_1, H_2, G_1, G_2. Let $H_1 \subset G_1, H_2 \subset G_2$. Moreover, let $v(H_1) = k-2$, $v(G_1, H_1) = 2$, $v(H_2) = k-1$, $v(G_2, H_2) = 1$ and the pair (G_1, H_1) be α-neutral, the pair (G_2, H_2) be α-rigid.

Let Spoiler choose a vertex x_{k-2}. Vertices $x_1, \ldots, x_{k-3}, y_1, \ldots, y_{k-3}$ are already chosen. The graph $\mathcal{G}|_{\{x_1,\ldots,x_{k-2}\}}$ either has a (G_1, H_1)-extension in $\mathcal{G}$ or doesn't have it. If a (G_1, H_1)-extension doesn't exist then by Theorem 6 Duplicator is able to choose a suitable vertex y_{k-2} such that there is no (G_1, H_1)-extension of $\mathcal{H}|_{\{y_1,\ldots,y_{k-2}\}}$ in $\mathcal{H}$. Then Spoiler chooses the $k-1$-th vertex. If in the corresponding graph there is a (G_2, H_2)-extension of the subgraph induced on the set of the chosen $k-1$ vertices, then this $k-1$-th vertex and the k-th vertex that forms the extension make together with the $k-2$ chosen vertices an α-safe pair. Therefore, by Theorem 5 Duplicator is able to choose a suitable $k-1$-th vertex (which has (G_2, H_2)-extension too) and to win. If the (G_2, H_2)-extension doesn't exist, then Duplicator is able to find a suitable vertex by Theorem 6.

If there is a (G_1, H_1)-extension $\mathcal{G}|_{\{x_1,\ldots,x_k\}}$ of $\mathcal{G}|_{\{x_1,\ldots,x_{k-2}\}}$, then the pair $(\mathcal{G}|_{\{x_1,\ldots,x_k\}}, \mathcal{G}|_{\{x_1,\ldots,x_{k-3}\}})$ is α-safe. Therefore, Duplicator is able to find vertices y_{k-2}, y_{k-1}, y such that the vertex y_{k-2} *leads him to the victory* (that is $\forall i \in \{1, 2, \ldots, k-3\}$ $x_i \sim x_{k-2} \Leftrightarrow y_{k-2} \sim y_i$) and the pair $(\mathcal{H}|_{\{y_1,\ldots,y_k\}}, \mathcal{H}|_{\{y_1,\ldots,y_{k-2}\}})$ is α-neutral. Subsequent arguments are similar to the ones in the case when a (G_1, H_1)-extension doesn't exist.

(5) Let $\frac{2}{2k-3} < \alpha < \frac{1}{k-2}$. The full level $k-2$ extension property holds. If $v(H) = k-2$, then for any graph G with k vertices containing H the pair (G, H) is either α-safe or α-rigid (the pair (G, H) is α-rigid if and only if $e(G, H) = 2k-3$), and for any graph G with $k-1$ vertices containing H the pair (G, H) is α-safe. If $v(H) = k-1$, $v(G) = k$, then the pair (G, H) is either α-safe, or α-rigid (the pair (G, H) is α-rigid if and only if $e(G, H) = k-1$).

Consider the graphs from the case (4): H_1, H_2, G_1, G_2.

Let Spoiler choose a vertex x_{k-2}. Vertices $x_1, \ldots, x_{k-3}, y_1, \ldots, y_{k-3}$ are already chosen. The graph $\mathcal{G}|_{\{x_1,\ldots,x_{k-2}\}}$ either has a (G_1, H_1)-extension in $\mathcal{G}$ or doesn't have it. If a (G_1, H_1)-extension doesn't exist then by Theorem 6 Duplicator is able to choose a suitable vertex y_{k-2} such that there is no (G_1, H_1)-extension of $\mathcal{H}|_{\{y_1,\ldots,y_{k-2}\}}$ in $\mathcal{H}$. Then Spoiler chooses the $k-1$-th vertex. If in the corresponding graph there is a (G_2, H_2)-extension of the subgraph induced on the set of the chosen $k-1$ vertices, then this $k-1$-th vertex and the k-th vertex that form the extension make together with the $k-2$ chosen vertices an α-safe pair. Therefore, by Theorem 5 Duplicator is able to choose a suitable $k-1$-th vertex (which has (G_2, H_2)-extension too) and to win. If a (G_2, H_2)-extension doesn't exist, then Duplicator is able to find a suitable vertex by Theorem 6.

If there is a (G_1, H_1)-extension, then the pair $(\mathcal{G}|_{\{x_1,\ldots,x_k\}}, \mathcal{G}|_{\{x_1,\ldots,x_{k-3}\}})$ is α-safe. Therefore Duplicator is able to find vertices y_{k-2}, y_{k-1}, y such that the vertex y_{k-2} *leads him to the victory* and the pair $(\mathcal{H}|_{\{y_1,\ldots,y_k\}}, \mathcal{H}|_{\{y_1,\ldots,y_{k-2}\}})$ is α-rigid. Subsequent arguments are similar to the ones in the case when there is no rigid extension.

The theorem is proved.

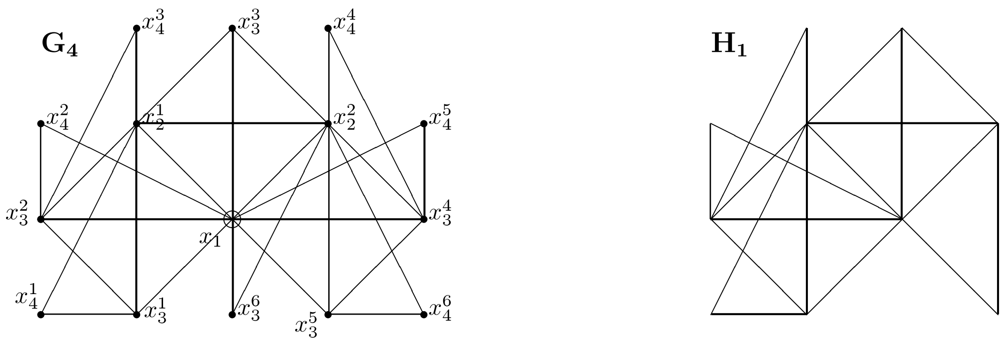
3.4. *Proof of Theorem 4*

Let us consider the cases $k = 4$, $k = 5$, $k = 6$ and $k > 6$.

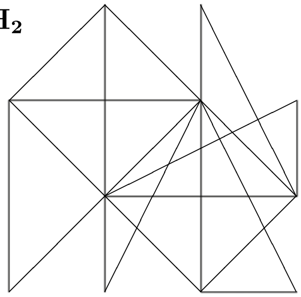
(1) Consider the graph $G_4 = (V_4, E_4)$ drawn on the picture. Write out all the elements of the sets V_4 and E_4:

$$V_4 = \{x_1, x_2^1, x_2^2, x_3^1, x_3^2, x_3^3, x_3^4, x_3^5, x_3^6, x_4^1, x_4^2, x_4^3, x_4^4, x_4^5, x_4^6\};$$

$$E_4 = \Big\{\{x_1, x_2^1\}, \{x_1, x_2^2\}, \{x_1, x_3^1\}, \{x_1, x_3^2\}, \{x_1, x_3^3\}, \{x_1, x_3^4\}, \{x_1, x_3^5\}, \{x_1, x_3^6\}, \{x_1, x_4^2\}, \{x_1, x_4^5\}, \{x_2^1, x_2^2\},$$
$$\{x_2^1, x_3^1\}, \{x_2^1, x_3^2\}, \{x_2^1, x_3^3\}, \{x_2^1, x_4^1\}, \{x_2^1, x_4^3\}, \{x_2^2, x_3^3\}, \{x_2^2, x_3^4\}, \{x_2^2, x_3^5\}, \{x_2^2, x_3^6\}, \{x_2^2, x_4^4\}, \{x_2^2, x_4^6\},$$
$$\{x_3^1, x_3^2\}, \{x_3^1, x_4^1\}, \{x_3^2, x_4^2\}, \{x_3^2, x_4^3\}, \{x_3^4, x_3^5\}, \{x_3^4, x_4^4\}, \{x_3^4, x_4^5\}, \{x_3^5, x_4^6\}\Big\}.$$



$\mathbf{H_2}$



Let $H_1 \subset G_4$ be the subgraph induced on the set

$$\{x_1, x_2^1, x_2^2, x_3^1, x_3^2, x_3^3, x_3^5, x_4^1, x_4^2, x_4^3\};$$

let $H_2 \subset G_4$ be the subgraph induced on the set

$$\{x_1, x_2^1, x_2^2, x_3^1, x_3^3, x_3^4, x_3^5, x_3^6, x_4^4, x_4^5, x_4^6\}.$$

Moreover, assume that $X \supseteq G_4$ and there is no vertex in $X \setminus G_4$ adjacent (by edges from $E(X)$) to at least 2 vertices of G_4. Let X and Y be the graphs from the game. Let in the first round Spoiler choose a vertex from X and Duplicator choose a vertex from Y. Suppose that Duplicator has a winning strategy. Let us study some properties of the graph Y.

In the first round Spoiler chooses the vertex x_1, Duplicator chooses a vertex y_1. Let in the second round Spoiler choose either the vertex x_2^1 or the vertex x_2^2. If Duplicator can find such a strict $(H_2, \{x_1\})$-extension of the vertex y_1, then Spoiler chooses the vertex x_2^1. Otherwise, Spoiler chooses the vertex x_2^2. So, we can already say that in the graph Y there should be two adjacent vertices.

If there is a strict $(H_2, \{x_1\})$-extension of y_1 in Y, then we say that a *property* $\mathrm{Prop}_0(y_1, H_2, Y)$ *holds*. Suppose that Spoiler chooses x_2^2. If in Y there is no complete graph with 4 vertices containing y_1, then Duplicator, obviously, loses. Suppose that such complete graph exists. Duplicator must choose the second vertex y_2 such that the vertices y_1, y_2 are in this complete graph. In this case if in every complete graph H with 4 vertices containing y_1, y_2 there is no a vertex y such that $y \sim y_1, y \sim y_2$ and $y \notin V(H)$, then Spoiler chooses the vertex x_3^6. Then Duplicator must choose a vertex y_3 such that y_1, y_2, y_3 are in complete graph on 4 vertices. After that Spoiler chooses a vertex $y_4 \in V(Y)$ such that the vertices y_1, y_2, y_3, y_4 are pairwise adjacent. There is no such vertex in X. Thus, Duplicator loses. So, if in Y there is no copy of H_2, then there should be a complete graph on 4 vertices, two of which (y_1, y_2) are adjacent to the 5-th vertex. Denote this property by $\exists y_1, y_2 \in V(Y)\ \mathrm{Prop}_1(y_1, y_2, Y)$.

Suppose that for the graph Y this property holds. We say that *two vertices of a triangle are extended*, if there is a vertex adjacent to theses two vertices but non-adjacent to the third vertex of the triangle. Let us consider the following property of the vertices y_1, y_2 of the graph Y: there exist vertices $y_3^1, y_3^2, y_3^3, y_3^4$ adjacent to each of y_1, y_2 such that the vertices y_3^1, y_1 of the triangle on the vertices y_3^1, y_1, y_2 are extended, the vertices y_3^2, y_2 of the triangle on the vertices y_3^2, y_1, y_2 are extended, the vertices y_3^3, y_1 and the vertices y_3^3, y_2 of the triangle on the vertices y_3^3, y_1, y_2 are extended, the vertices y_3^1, y_2 of the triangle on the vertices y_3^1, y_1, y_2 aren't extended, the vertices y_3^2, y_1 of the triangle on the vertices y_3^2, y_1, y_2 aren't extended, the vertices y_3^4, y_1 and the vertices y_3^4, y_2 of the triangle on the vertices y_3^4, y_1, y_2 aren't extended. In other words there are all *types of extensions of triangles* constructed on the vertices y_1, y_2. Suppose that the graph Y doesn't satisfy this property. Then there is a type of extension which is not realized for the triangles constructed on y_1, y_2. Suppose, for example, that there isn't a vertex y such that $y \sim y_1, y \sim y_2$ and the vertices y, y_1 and the vertices y, y_2 of the triangle on the vertices y, y_1, y_2 aren't extended. Then Spoiler chooses the vertex x_3^3 and at the 4-th round chooses a vertex which extends some pair of the vertices of the triangle chosen by Duplicator. Thus, some vertices y_1, y_2 of Y should satisfy the described property of extending. Denote this property by $\exists y_1, y_2 \in V(Y)\ \mathrm{Prop}_2(y_1, y_2, Y)$. Therefore, for the graph Y the following property holds:

$$(\forall y \in V(Y)\ \neg\mathrm{Prop}_0(y, H_2, Y)) \Rightarrow (\exists y_1, y_2 \in V(Y)\ \mathrm{Prop}_1(y_1, y_2, Y) \wedge \mathrm{Prop}_2(y_1, y_2, Y)).$$

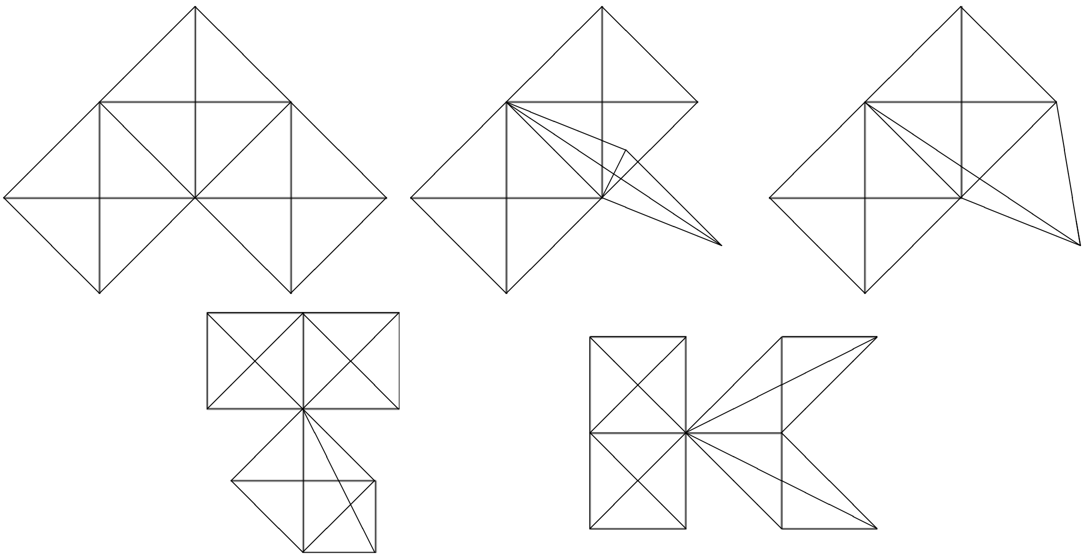
Suppose that the graph Y satisfies this property. Then there should be the vertices $y_3^1, y_3^2, y_3^3, y_3^4$ giving all types of extensions such that for each y_3^i there exists y adjacent to each of y_1, y_2, y_3^i. Otherwise, Spoiler at the third round chooses a vertex in the graph G_4 which gives a type of extension j such that for each vertex y_3^j in Y giving the same type of extension there is no vertex y adjacent to each of y_1, y_2, y_3^j. Duplicator must choose the vertex y_3 giving the type of extension j. At the 4-th round Spoiler chooses a vertex adjacent to each of the three chosen vertices and wins.

Thus, if for any vertex $y \in V(Y)$ the property $\neg\mathrm{Prop}_0(y, H_2, Y)$ holds, then in the graph Y there should exist adjacent vertices y_1, y_2 with the following properties: there are vertices $y_3^0, y_3^1, y_3^2, y_3^3, y_3^4$ with the property $\forall i \in \{0, 1, 2, 3, 4\}$ $(y_3^i \sim y_1, y_2)$ in Y; there is no vertex adjacent to at least 2 of y_1, y_2, y_3^0; for each $i \in \{1, 2, 3, 4\}$ there is a vertex adjacent to each of y_1, y_2, y_3^i; the vertices $y_3^1, y_3^2, y_3^3, y_3^4$ give all possible types of extensions of pairs of vertices in the triangles constructed on the vertices y_1, y_2. Denote this property of vertices y_1, y_2 of the graph Y by $\mathcal{H}_2(y_1, y_2, Y)$.

Suppose that Spoiler chooses x_2^1, that is $\exists y \in V(Y)$ $\mathrm{Prop}_0(y, H_2, Y)$. Then by the similar arguments, in the graph Y there should be a vertex y_2 adjacent to y_1 with the following properties: there are the vertices $y_3^1, y_3^2, y_3^3, y_3^4$ with the property $\forall i \in \{1, 2, 3, 4\}$ $(y_3^i \sim y_1, y_2)$ in Y; for each $i \in \{1, 2, 3, 4\}$ there is a vertex adjacent to each of y_1, y_2, y_3^i; the vertices $y_3^1, y_3^2, y_3^3, y_3^4$ give all possible types of extensions of pairs of vertices in the triangles constructed on the vertices y_1, y_2. Otherwise, suppose there is a vertex y adjacent to each of y_1, y_2 in Y such that there is no vertex adjacent to each of y_1, y_2, y. Then at the third round Spoiler chooses the vertex y and Duplicator chooses a vertex in X adjacent to each of x_1, x_2^1. Unfortunately, for every such vertex there should be the fourth vertex adjacent to each of the three chosen vertices. Spoiler chooses in the fourth round such fourth vertex. Therefore, in the graph Y there should be no vertex y adjacent to each of y_1, y_2 such that there is no fourth vertex adjacent to each of y_1, y_2, y. Denote the described property by $\mathcal{H}_1(y_1, y_2, Y)$.

Thus, in Y there should be either a vertex y_1 such that there is a strict $(H_2, \{x_1\})$-extension and a vertex y_2 with the property $\mathcal{H}_1(y_1, y_2, Y)$ or a vertex y_1 such that there is no strict $(H_2, \{x_1\})$-extension, but there is a vertex y_2 with the property $\mathcal{H}_2(y_1, y_2, Y)$.

In the first case in Y there should be a copy of H_2 $(x_1 \to y)$, a union of two complete graphs (not from H_2) with 4 vertices with one common edge and common vertex y. In the second case in Y there should be "nearly" a copy of H_2 that doesn't contain copies of the edges $\{x_2^1, x_3^3\}$, $\{x_3^5, x_3^4\}$. Four triangles of this copy should be extended to the complete graphs with four vertices. Therefore there should be either 4 additional vertices and 12 additional edges, or 3 additional vertices and 10 additional edges, or 2 additional vertices and 7 additional edges, or an additional vertex and 5 additional edges, or just 2 additional edges. In the last case the graph Y contains a copy of H_2. Therefore, if Duplicator has a winning strategy, then the graph Y should contain one of the graphs drawn on the picture.



The number of automorphisms of each of these graphs is greater than or equal to 8. The density of each of these graphs equals $\frac{1}{2}$. As the expectation of the number of copies of a graph with density ρ and a automorphisms in the random graph $G(N, N^{-\rho})$ tends to $\frac{1}{a}$ when $N \to \infty$, the expectation of the number of copies of the graphs represented on the picture in the random graph $G(N, N^{-1/2})$ for large enough N is less than or equal to $\frac{5}{8} + \varepsilon$. The probability that a random graph contains one of these graphs is less than or equal to this expectation. Therefore, for $N \geq N_0$ this probability is less than or equal to $\frac{3}{4}$.

Consider a set $\mathcal{S}$ consisting of one α-neutral pair (K, T) (there are three vertices in the graph K and two vertices in the graph T). Moreover, let $[\mathcal{S}]$ be a set of pairs $(\widehat{K}, \widehat{T})$ such that $\widehat{K} \supset K, \widehat{T} \supset T, \widehat{T} \setminus T = \widehat{K} \setminus K$, and the vertices of the graph $\widehat{T} \setminus T$ are adjacent to the vertex of the graph $K \setminus T$. Then by Theorem 8 the probability of containing $(\widehat{K}, \widehat{T})$-maximal for each pair $(\widehat{K}, \widehat{T}) \in [\mathcal{S}]$ copy of G_4 tends to some constant $\xi \in (0, 1)$. Therefore, when N is sufficiently large Spoiler wins with probability greater than or equal to $\frac{\xi}{4}$.

Before considering the remaining cases let us introduce some necessary definitions. Consider a complete graph with vertices $y_1, y_2, \ldots, y_h$ and a vertex y_{h+1} adjacent to each of $y_1, y_2, \ldots, y_h$. Let a vertex z be adjacent to a vertex y_{h+1} and to $h-1$ vertices from $y_1, y_2, \ldots, y_h$. In this case we say that the vertex z *extends the vertex y_{h+1} of the graph on the vertices* $y_1, \ldots, y_h, y_{h+1}$. Let it be adjacent to $y_{h+1}, y_{i_1}, \ldots, y_{i_{h-1}}$. Then let us call the collection $\{i_1, \ldots, i_{h-1}\}$ *a way of extending*. Let $\mathcal{J}^h$ be the set of all ways of extending the vertex y_{h+1} of the graph on the vertices $y_1, y_2, \ldots, y_{h+1}$. We call a subset $J^h \subseteq \mathcal{J}^h$ *a type of extension*. Let us order all types of extension $2^{\mathcal{J}^h} = \{J_1^h, J_2^h, \ldots, J_{2^h}^h\}$. We say that the vertex y_{h+1} *is of type j relative to the vertices* $y_1, y_2, \ldots, y_h$, if the collections from J_j^h are the ways of extending the vertex y_{h+1} of the graph on the vertices $y_1, \ldots, y_h, y_{h+1}$, but any collection $i \in \mathcal{J}^h \setminus J_j^h$ is not a way of extending the vertex y_{h+1} of the graph on the vertices $y_1, \ldots, y_h, y_{h+1}$.

(2) Let us again consider some graph whose copy is in the random graph with a probability from the interval (0, 1). Denote this graph by $G_5 = (V_5, E_5)$. We didn't draw it on a picture, because a large number of vertices and edges makes the picture not obvious. Describe a subgraph H_5 of G_5. In this subgraph there are 16 vertices, two of which x_1, x_2 are adjacent to all remaining vertices and $x_1 \sim x_2$. For the vertex x_3^1 in the graph H_5 there is no vertex y adjacent to x_3^1 such that there is no vertex z adjacent to y and x_3^1 both. For the vertex x_3^2 there exists the only such vertex x_4^{12}. The vertices x_3^1 and x_3^2 are adjacent. Moreover, the vertex x_3^1 is adjacent to each of the vertices $x_4^1, x_4^2, x_4^3, x_4^4, x_4^5, x_4^6, x_4^7$ and the vertex x_3^2 is adjacent to each of the vertices $x_4^7, x_4^8, x_4^9, x_4^{10}, x_4^{11}$. Besides the described edges, the set $E(H_5)$ contains the edges $\{x_4^1, x_4^2\}, \{x_4^3, x_4^4\}, \{x_4^5, x_4^6\}, \{x_4^8, x_4^9\}, \{x_4^{10}, x_4^{11}\}$. There are no other edges in this set. (The graph $H_5|_{V(H_5)\setminus\{x_1,x_2,x_4^{12}\}}$ is represented on the schematic picture below.)

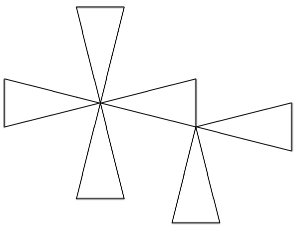
Let us return to the graph G_5. Let for each $j \in \{1, 2, \dots, 8\}$ there be a vertex from $x_4^1, x_4^2, x_4^3, x_4^4, x_4^5, x_4^6, x_4^7, x_3^2$ of the type j relative to the vertices x_1, x_2, x_3^1. Moreover, let the vertices $x_4^7, x_4^8, x_4^9, x_4^{10}, x_4^{11}, x_3^1$ be of pairwise different types relative to the vertices x_1, x_2, x_3^2. Let these types form the set $I \subset \{1, 2, \dots, 8\}$. There are no other types in this set. The graph G_5 doesn't contain the vertices and the edges except the described ones (we also count the edges and the vertices which form the extensions). When two vertices are extended by some two other vertices these two extending vertices are different. It can be shown that the number of vertices of the graph G_5 is in the segment [31, 37]. It varies subject to the kind of the elements of the set I.

Now we can describe the strategies of the players. Let $X \supseteq G_5$ and there is no vertex in $X \setminus G_5$ adjacent to at least 3 vertices of G_5 in X. Let X and Y be the graphs from the game, in the first round Spoiler chooses a vertex from X, Duplicator chooses a vertex from Y. Suppose that Duplicator has a winning strategy. Let us study the properties of the graph Y. In the first two rounds Spoiler chooses the vertices $x_1, x_2 \in V(H_5)$, Duplicator chooses vertices $y_1, y_2 \in V(Y)$. Consider two cases.

(2.1) In the graph Y there is no vertex y adjacent to the vertices y_1, y_2 with the two following properties. The first property: there is no vertex z, $z \sim y$, $z \sim y_1$, $z \sim y_2$, such that there is no vertex adjacent to each of the vertices y, y_1, y_2, z. The second property: there are vertices $z_1, z_2, \dots, z_8$, $z_1 \sim z_2$, $z_3 \sim z_4$, $z_5 \sim z_6$, $z_7 \sim z_8$, adjacent to each of the vertices y_1, y_2, y such that any two of them are of different types relative to the vertices y, y_1, y_2. Denote the conjunction of these two properties by $\mathrm{Prop}_{2.1}(y_1, y_2, Y)$.

(2.2) In the graph Y there is no vertex y adjacent to the vertices y_1, y_2 with the two following properties. The first property: there is a vertex z, $z \sim y$, $z \sim y_1$, $z \sim y_2$, such that there is no vertex adjacent to each of the vertices y, y_1, y_2, z. The second property: there are vertices $z_1, z_2, z_3, z_4, z_5, z_6$, $z_1 \sim z_2$, $z_3 \sim z_4$, $z_5 \sim z_6$, adjacent to each of the vertices y_1, y_2, y such that each of them is of a type from the set I relative to the vertices y, y_1, y_2 and any two of them are of the different types relative to the vertices y, y_1, y_2. Denote the conjunction of these two properties by $\mathrm{Prop}_{2.2}(y_1, y_2, Y)$.

Hereinafter we have no need to give further arguments because they are similar to the ones from the case (1). So, we can conclude that in Y there should be two vertices y_1, y_2 with the property $\mathrm{Prop}_{2.1}(y_1, y_2, Y) \wedge \mathrm{Prop}_{2.2}(y_1, y_2, Y)$.

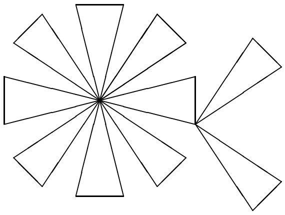


Therefore, in Y there should be either a subgraph with density greater than 3 or the subgraph represented on the schematic drawing. We didn't represent the vertices y_1, y_2. It means that these two vertices are adjacent to each of the remaining thirteen and $y_1 \sim y_2$. This subgraph is strictly balanced and its density equals 3. Let $\mathcal{X} \in \mathcal{F}_N$ be a set of graphs which don't contain such subgraph. Then by Theorem 7 the equality $\lim_{N\to\infty} \mathsf{P}_{N,p}(\mathcal{X}) = \exp\left(-\frac{1}{a}\right)$ holds, where a is the number of automorphisms of this subgraph.

Consider now a set $\mathcal{S}$ consisting of one α-neutral pair (K, T) (there are four vertices in the graph K and three vertices in the graph T). Moreover, let $[\mathcal{S}]$ be a set of pairs $(\widehat{K}, \widehat{T})$ such that $\widehat{K} \supset K$, $\widehat{T} \supset T$, $\widehat{T} \setminus T = \widehat{K} \setminus K$, and the vertices of the graph $\widehat{T} \setminus T$ are adjacent to the vertex of the graph $K \setminus T$. Then by Theorem 8 the probability of containing $(\widehat{K}, \widehat{T})$-maximal for each pair $(\widehat{K}, \widehat{T}) \in [\mathcal{S}]$ copy of G_5 tends to some constant $\xi \in (0, 1)$. Therefore, when N is sufficiently large Spoiler wins with probability greater than or equal to $\frac{\xi}{4}$.

(3) Let us describe a graph $G_6 = (V_6, E_6)$. For this purpose consider a subgraph H_6 of the graph G_6. In this subgraph there are 25 vertices three of which x_1, x_2, x_3 are adjacent to all remaining vertices and to each other. For the vertex x_4^1 in the graph H_6 there is no vertex y adjacent to x_4^1 such that there is no vertex z adjacent to y and x_4^1 both. For the vertex x_4^2 there exists the only such vertex x_5^{20}. The vertices x_4^1 and x_4^2 are adjacent. Moreover, the vertex x_4^1 is adjacent to each of the vertices $x_5^1, x_5^2, \dots, x_5^{15}$ and the vertex x_4^2 is adjacent to each of the vertices $x_5^{15}, x_5^{16}, x_5^{17}, x_5^{18}, x_5^{19}$. Besides the described edges, the set $E(H_6)$ contains the edges $\{x_5^1, x_5^2\}, \{x_5^3, x_5^4\}, \{x_5^5, x_5^6\}, \{x_5^8, x_5^9\}, \{x_5^{10}, x_5^{11}\}, \{x_5^{12}, x_5^{13}\}, \{x_5^{14}, x_5^{15}\}$. There are no other edges in this set. (The graph $H_6|_{V(H_6)\setminus\{x_1,x_2,x_3,x_5^{20}\}}$ is represented in the schematic drawing below.)

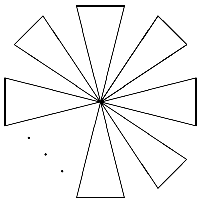
Let us return to the graph G_6. Let for each $j \in \{1, 2, \dots, 16\}$ there be a vertex from $x_5^1, x_5^2, \dots, x_5^{15}, x_4^2$ of type j relative to the vertices x_1, x_2, x_3, x_4^1. Moreover, let the vertices $x_5^{15}, x_5^{16}, x_5^{17}, x_5^{18}, x_5^{19}, x_4^1$ be of pairwise different types relative to the vertices x_1, x_2, x_3, x_4^2. The graph G_6 doesn't contain the vertices and the edges except the described ones (when two vertices are extended by some two other vertices these two extending vertices are different). It can be shown again that the number of vertices of the graph G_6 is in the segment [59, 69].



We are not going to describe the strategy of Spoiler in detail again, because the necessary arguments are identical to the corresponding ones for the graph G_5. So, let X ($X \supseteq G_6$) and Y are the graphs from the game and there is no vertex in $X \setminus G_6$ adjacent to at least 4 vertices of G_6 in X. Suppose Duplicator wins. Then in Y there should be either a subgraph with density greater than 4 or the subgraph represented on the schematic picture. We didn't represent the vertices y_1, y_2, y_3. It means that these three vertices are adjacent to each of the remaining vertices and to each other. This subgraph is strictly balanced and its density equals 4. Let $\mathcal{X} \in \mathcal{F}_N$ be a set of graphs which doesn't contain such subgraph. Then by Theorem 7 the equality $\lim_{N\to\infty} \mathsf{P}_{N,p}(\mathcal{X}) = \exp\left(-\frac{1}{a}\right)$ holds, where a is the number of automorphisms of this subgraph.

Consider now a set $\mathcal{S}$ consisting of one α-neutral pair (K, T) (there are five vertices in the graph K and four vertices in the graph T). Moreover, let $[\mathcal{S}]$ be a set of pairs $(\widehat{K}, \widehat{T})$ such that $\widehat{K} \supset K, \widehat{T} \supset T, \widehat{T} \setminus T = \widehat{K} \setminus K$, and the vertices of the graph $\widehat{T} \setminus T$ are adjacent to the vertex of the graph $K \setminus T$. Then by Theorem 8 the probability of containing $(\widehat{K}, \widehat{T})$-maximal for each pair $(\widehat{K}, \widehat{T}) \in [\mathcal{S}]$ copy of G_6 tends to some constant $\xi \in (0, 1)$. Therefore, when N is sufficiently large Spoiler wins with probability greater than or equal to $\frac{\xi}{4}$.

(4) Consider, finally, the graph $G_k = (V_k, E_k)$, $k > 6$. We are going to describe all vertices and edges of this graph. When we say that two vertices of two subgraphs are extended by some two other vertices, then, critically, these two extending vertices are different. Let the vertices $x_1, x_2, \dots, x_{k-2}$ of G_k be adjacent to each other and to each of the vertices $x_{k-1}^1, x_{k-1}^2, \dots, x_{k-1}^{(k-1)(k-2)}$ of G_k. Moreover, let the vertices $x_{k-1}^1, x_{k-1}^2, \dots, x_{k-1}^{(k-1)(k-2)}$ be of pairwise different types relative to the vertices $x_1, x_2, \dots, x_{k-2}$. This graph exists, because the number of elements in the set $2^{\mathcal{I}^{k-2}}$ equals 2^{k-2}, and when $k > 6$ the inequality $2^{k-2} > (k-1)(k-2)$ holds. Let these types form the set $I \subset \{1, 2, \dots, 2^{k-2}\}$.



As in the previous cases, suppose Duplicator wins. Then in Y there should exist the vertices $y_1, y_2, \dots, y_{k-2}$ with the following property. There are the vertices $z_1, z_2, \dots, z_{(k-1)(k-2)}$ adjacent to each of $y_1, y_2, \dots, y_{k-2}$ such that for each vertex z_i of $z_1, z_2, \dots, z_{(k-1)(k-2)}$ there exists a vertex adjacent to it and to $y_1, y_2, \dots, y_{k-2}$; the type of z_i relative to the vertices $y_1, y_2, \dots, y_{k-2}$ is in I; any two vertices of $z_1, z_2, \dots, z_{(k-1)(k-2)}$ are of different types relative to the vertices $y_1, y_2, \dots, y_{k-2}$.

Therefore, in Y there should be either a subgraph with density greater than $k-2$ or the subgraph represented on the schematic picture. We didn't represent the vertices $y_1, y_2, \dots, y_{k-3}$. It means that these vertices are adjacent to each of the remaining vertices and to each other. This subgraph is strictly balanced and its density equals $k-2$. Let $\mathcal{X} \in \mathcal{F}_N$ be a set of graphs which don't contain such subgraph. Then by Theorem 7 the equality $\lim_{N\to\infty} \mathsf{P}_{N,p}(\mathcal{X}) = \exp\left(-\frac{1}{a}\right)$ holds, where a is the number of automorphisms of this subgraph.

Consider now a set $\mathcal{S}$ consisting of one α-neutral pair (K, T) (there are $k-1$ vertices in the graph K and $k-2$ vertices in the graph T). Moreover, let $[\mathcal{S}]$ be a set of pairs $(\widehat{K}, \widehat{T})$ such that $\widehat{K} \supset K, \widehat{T} \supset T, \widehat{T} \setminus T = \widehat{K} \setminus K$, and the vertices of the graph $\widehat{T} \setminus T$ are adjacent to the vertex of the graph $K \setminus T$. Then by Theorem 8 the probability of containing $(\widehat{K}, \widehat{T})$-maximal for each pair $(\widehat{K}, \widehat{T}) \in [\mathcal{S}]$ copy of G_k tends to some constant $\xi \in (0, 1)$. Therefore, when N is sufficiently large Spoiler wins with probability greater than or equal to $\frac{\xi}{4}$.

4. Future research

Let us present open problems of our work and future research directions.

- To find all rational $\alpha > \frac{1}{k-2}$ such that the random graph $G(N, p)$ satisfies the zero–one k-law.

– To resolve the question of whether all properties from $\mathcal{L}_k$ satisfy a *convergence law* when $\alpha = 1/(k-2)$. A property L satisfies a convergence law if there exists $\lim_{N\to\infty} \mathsf{P}_{N,p}(\mathcal{G} \models L)$. Convergence laws for other values of α were proven in [15].
– To find all rational $\alpha > \frac{1}{k-2}$ such that all properties from $\mathcal{L}_k$ satisfy a *convergence law*.

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