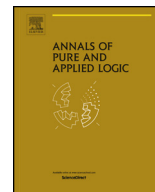




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Monadic second-order properties of very sparse random graphs[☆]L.B. Ostrovsky, M.E. Zhukovskii^{*}

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ABSTRACT

We study asymptotical probabilities of first order and monadic second order properties of Bernoulli random graph $G(n, n^{-a})$. The random graph obeys FO (MSO) zero-one k -law (k is a positive integer) if, for any first order (monadic second order) formulae with quantifier depth at most k , it is true for $G(n, n^{-a})$ with probability tending to 0 or to 1. Zero-one k -laws are well studied only for the first order language and $a < 1$. We obtain new zero-one k -laws (both for first order and monadic second order languages) when $a > 1$. Proofs of these results are based on the earlier studies of first order equivalence classes and our study of monadic second order equivalence classes. The respective results are of interest by themselves.

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1. Logic of the random graph

In 1959, P. Erdős and A. Rényi, and independently E. Gilbert, introduced two closely related random graph models. While a seminal paper of Erdős and Rényi [7] gave birth to rapidly developing area of mentioned random graphs, the more popular Bernoulli model $G(n, p)$ was proposed by Gilbert in [8] (but actually this model is much older). In the paper, we study asymptotical probabilities of first order and monadic second order properties of the random graph $G(n, p)$ (see also [3,11,15,21]). Recall that edges in this graph on the set of vertices $V_n = \{1, \dots, n\}$ appear independently with probability p (i.e., for each undirected graph $H = (V_n, E)$ without loops and multiple edges, the equality $P(G(n, p) = H) = p^{|E|}(1-p)^{\binom{n}{2}-|E|}$ holds).

Formulae in the first order language of graphs (first order formulae) [1,5,21,17,15] are constructed using relational symbols \sim (the symbol of adjacency) and $=$; logical connectivities \neg , \Rightarrow , \Leftrightarrow , \vee , \wedge ; variables

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x, y, x_1, \dots (that express vertices of a graph); and quantifiers \forall, \exists . Monadic second order formulae [9,16] are built of the above symbols of the first order language and variables X, Y, X_1, \dots that express unary predicates. Following [1,5,21,17], we call a number of nested quantifiers in the longest chain of nested quantifiers of a formula ϕ the *quantifier depth* $q(\phi)$. For example, the formula

$$(\forall X ([\exists x_1 \exists x_2 (X(x_1) \wedge (\neg(X(x_2))))] \Rightarrow [\exists y \exists z (X(y) \wedge (\neg(X(z))) \wedge (y \sim z)])))$$

has quantifier depth 3 and expresses the property of being connected. It is known that this property is not expressed by a first order formula (to the best of our knowledge, the first proof of this fact appears in [10] for sentences with quantifier depth 4; the general proof can be found in, e.g., [21]).

We say that $G(n, p)$ obeys FO zero-one law (MSO zero-one law) if, for any first order formula (monadic second order formula), it is either true asymptotically almost surely (a.a.s.) or false a.a.s. (as $n \rightarrow \infty$). Note that we use the phrase “asymptotically almost surely” when the considered property of the random graph holds with asymptotical probability 1 (as $n \rightarrow \infty$). In some papers on random graphs, the phrase “almost surely” (a.s.) is used instead. However, in probability theory the latter phrase normally means that the considered event holds with probability 1 (this is why we choose the first phrase).

In 1988, S. Shelah and J. Spencer [14] proved the following zero-one law for the random graph $G(n, n^{-\alpha})$.

Theorem 1. *Let $\alpha > 0$. The random graph $G(n, n^{-\alpha})$ does not obey FO zero-one law if and only if either $\alpha \in (0, 1] \cap \mathbb{Q}$ or $\alpha = 1 + 1/l$ for some integer l .*

Obviously, there is no MSO zero-one law when even FO zero-one law does not hold. In 1993, J. Tyszkiewicz [16] proved that $G(n, n^{-\alpha})$ does not obey MSO zero-one law for irrational $\alpha \in (0, 1)$ also. When $\alpha > 1$ and does not equal to $1 + 1/l$ for any positive integer l , MSO zero-one law holds. The last statement simply follows from standard arguments from the theory of logical equivalence. We believe this result is known. Unfortunately, we did not find it in the related papers. So we give the proof in Section 4.2. Below, we state the general result on MSO zero-one law for $G(n, n^{-\alpha})$.

Theorem 2. *Let $\alpha > 0$. The random graph $G(n, n^{-\alpha})$ does not obey MSO zero-one law if and only if either $\alpha \in (0, 1]$ or $\alpha = 1 + 1/l$ for some integer l .*

For a formula ϕ consider the set $S(\phi)$ of α such that $G(n, n^{-\alpha})$ does not obey the zero-one law for the fixed formula ϕ . Neither theorem gives an explanation of how the set $S(\phi)$ depends on ϕ (or even on a quantifier depth of this formula). However, better insight into an asymptotical behavior of probabilities of the properties expressed by first order and monadic second order formulae is given by zero-one k -laws (k is a positive integer upper bound for quantifier depths of formulae, see Section 3), which are well studied only for the first order language and $\alpha \leq 1$ (see, e.g., [21]). In this paper, we obtain new zero-one k -laws (both for first order and monadic second order languages) when $\alpha > 1$ and give their statements in Section 3. Proofs of these results are based on the earlier study of first order equivalence classes and our study of monadic second order equivalence classes (see Section 2). The respective results are of interest by themselves.

2. Logical equivalence

For two graphs G and H and any positive integer k , the notation $G \equiv_k^{\text{FO}; \text{graphs}} H$ denotes that any first order formula ϕ with $q(\phi) \leq k$ is true on both G and H or false on both G and H . The notation $G \equiv_k^{\text{MSO}; \text{graphs}} H$ is defined similarly. Obviously, $\equiv_k^{\text{FO}; \text{graphs}}$ and $\equiv_k^{\text{MSO}; \text{graphs}}$ are both equivalence relations on the set of all graphs. Moreover, for every k there are only finitely many equivalence classes (see, e.g., [5]) and an upper bound for the cardinality $r_k^{\text{FO}; \text{graphs}}$ of the set of all $\equiv_k^{\text{FO}; \text{graphs}}$ -equivalence classes $\mathcal{R}_k^{\text{FO}; \text{graphs}}$

is known [13] and given below. Let $T(s)$ be the tower function: $T(s) = 2^{T(s-1)}$, $T(1) = 2$. Let $\log^*(k) = \min\{i : T(i) \geq k\}$.

Theorem 3.

$$r_k^{\text{FO; graphs}} \leq T(k + 2 + \log^*(k)) + O(1).$$

Similarly, $r_k^{\text{MSO; graphs}}$ and $\mathcal{R}_k^{\text{MSO; graphs}}$ are defined. In this paper, we prove a similar result for $\equiv_k^{\text{MSO; graphs}}$ -equivalence (the proof is given in Section 4.3).

Theorem 4. For any positive integer k ,

$$r_k^{\text{MSO; graphs}} \leq T(k + 2 + \log^*(k)).$$

Note that this result is stronger than Theorem 3.

In order to prove the results on zero-one laws from Section 3, we also need an extension of the above theory to the case of rooted trees. Recall that a *rooted tree* T_R is a tree with one distinguished vertex R , which is called *the root*. If R, \dots, x, y is a simple path in T_R , then x is called a *parent* of y and y is called a *child* of x . The first order language for rooted trees has a constant symbol R (for the root) and the parent–child relation $P(x, y)$. For two rooted trees T_R and $T'_{R'}$ and any positive integer k , the notations $T_R \equiv_k^{\text{FO; trees}} T'_{R'}$, $T_R \equiv_k^{\text{MSO; trees}} T'_{R'}$, $r_k^{\text{FO; trees}}$, $r_k^{\text{MSO; trees}}$, $\mathcal{R}_k^{\text{FO; trees}}$, $\mathcal{R}_k^{\text{MSO; trees}}$ are defined in the same way as for graphs. The following result is proven in [13] (hereinafter, we denote by $V(G)$ the set of vertices of a graph G).

Theorem 5.

$$r_k^{\text{FO; trees}} \leq T(k + 2 + \log^*(k)) + O(1).$$

For any $A \in \mathcal{R}_k^{\text{FO; trees}}$, the following inequality holds:

$$\min_{T_R \in A} |V(T_R)| \leq T(k + 4 + \log^*(k)) + O(1).$$

In this paper, we prove a similar result for $\equiv_k^{\text{MSO; trees}}$ -equivalence (the proof is given in Section 4.4).

Theorem 6. Let $k \geq 4$ be an integer. Then

$$r_k^{\text{MSO; trees}} \leq T(k + 2 + \log^*(k)). \quad (1)$$

For any $A \in \mathcal{R}_k^{\text{MSO; trees}}$, the following inequality holds:

$$\min_{T_R \in A} |V(T_R)| \leq T(k + 3 + \log^*(k + 1)).$$

Note that this result is stronger than Theorem 5.

3. Zero-one k -laws

By Theorem 1, for any rational $\alpha \in (0, 1]$ there is a first order formula ϕ whose asymptotic probability on $G(n, p)$ either does not exist or does not equal to 0 or 1. This statement is not true for formulae with bounded quantifier depth (see, e.g., [12, 18]). We say that $G(n, p)$ *obeys FO zero-one k -law* (*MSO zero-one k -law*) if, for any first order formula (monadic second order formula) with quantifier depth at most k , it is either true a.a.s. or false a.a.s. (as $n \rightarrow \infty$). In [19, 20, 18], the following zero-one k -laws are proven.

Theorem 7. For any $k \geq 3$ and any $\alpha \in (0, 1/(k-2))$ the random graph $G(n, n^{-\alpha})$ obeys FO zero-one k -law. If $\alpha = 1/(k-2)$, then $G(n, n^{-\alpha})$ does not obey FO zero-one k -law. For any $k \geq 4$ and any $\alpha \in (1 - 1/(2^k - 2), 1)$ the random graph $G(n, n^{-\alpha})$ obeys FO zero-one k -law. If $\alpha = 1 - 1/(2^k - 2)$, then $G(n, n^{-\alpha})$ does not obey FO zero-one k -law.

In this paper, we consider the very sparse case $\alpha > 1$. From Theorems 1, 2, $G(n, n^{-\alpha})$ obeys both zero-one k -laws if $\alpha \neq 1 + 1/l$ for any positive integer l . In Sections 4.5, 4.6 we prove the following result.

Theorem 8. Let l be positive integer, $\alpha = 1 + 1/l$.

- Let $k \geq 4$ be an arbitrary integer. If $l \geq T(k + \log^*(k+1) + 3)$, then the random graph $G(n, n^{-\alpha})$ obeys MSO zero-one k -law.
- Let $k \geq 7$ be an arbitrary integer. If $l \leq 2T(k-4)$, then the random graph $G(n, n^{-\alpha})$ does not obey FO zero-one k -law.

4. Proofs

We start from notations and auxiliary statements (Section 4.1). Recall that the simple case ($\alpha > 1$) of the statement of Theorem 2 is known. We give its proof in Section 4.2, because we did not find it in the related papers. Theorem 8 is proven in Sections 4.5, 4.6. These proofs are based on Theorems 4, 6. Proofs of the latter results can be found in Sections 4.3, 4.4 respectively.

4.1. Preliminaries

Throughout this paper, if G is a graph then we denote its vertex set by $V(G)$ and its edge set by $E(G)$ (i.e. $G = (V(G), E(G))$). The *distance* between vertices u and v in a connected graph is the minimum length of a path connecting u and v . It is denoted by $d(u, v)$. For a disconnected graph, the distance between vertices in different components equals ∞ . The *eccentricity* of a vertex v is $e(v) = \max_{u \in V(G)} d(v, u)$. The *diameter* and the *radius* of G are $d(G) = \max_{v \in V(G)} e(v)$ and $r(G) = \min_{v \in V(G)} e(v)$ respectively. A vertex v is *central* if $e(v) = r(G)$.

For a rooted tree T_R , we call the eccentricity of its root the *depth* of T_R . The relation of being a *descendant* is the transitive and reflexive closure of the relation of being a child. If $v \in V(T_R)$, then $T_R(v)$ denotes the subtree of T_R spanned by the set of all descendants of v and rooted at v .

For a graph G and a formula ϕ , we write $G \models \phi$ if ϕ is true on G .

In this section, we review well-known statements (and prove new, see Section 4.1.2, Lemma 1 and Lemma 2) from the random graph theory (Section 4.1.1) and the model theory (Section 4.1.2), which are exploited in our proofs.

4.1.1. Small subgraphs of the random graph

Consider a graph G on v vertices and e edges. Denote the number of automorphisms of G by $a(G)$. The fraction $\rho(G) = \frac{e}{v}$ is called the *density* of G . The graph G is called *strictly balanced* if for any proper subgraph $H \subset G$ the inequality $\rho(H) < \rho(G)$ holds. Let N_G be the number of (not necessarily induced) copies of G in $G(n, p)$. Let G be strictly balanced. In [7], a threshold probability for the property of containing G was obtained.

Theorem 9. If $p \gg n^{-1/\rho(G)}$, then a.a.s. in $G(n, p)$ there is a copy of G (not necessarily induced). Moreover, for any $\varepsilon > 0$, a.a.s. $\left| \frac{N_G}{\varepsilon n^{\rho(G)}} - 1 \right| < \varepsilon$. If $p \ll n^{-1/\rho(G)}$, then a.a.s. in $G(n, p)$ there are no copies of G .

Moreover, in [2] an asymptotical distribution of N_G in the threshold was found.

Theorem 10. If $p = cn^{-1/\rho(G)}$, then N_G converges in distribution to a random variable with the distribution $\text{Pois}(c^e/a(G))$ (as $n \rightarrow \infty$).

From these results it follows that for any positive integer l and any α such that $1 + 1/(l+1) < \alpha < 1 + 1/l$ the following three properties hold.

- T1 The random graph $G(n, n^{-\alpha})$ is a forest a.a.s.
- T2 A.a.s. any component of $G(n, n^{-\alpha})$ has at most $l + 1$ vertices.
- T3 For any integer K , a.a.s. for any tree T on at most $l + 1$ vertices there are at least K components in $G(n, n^{-\alpha})$ which are isomorphic to T .

Moreover, for any positive integer l and $\alpha = 1 + 1/l$ the properties T1 and T2 hold. Moreover,

- T4 For any integer K , a.a.s. for any tree T on at most l vertices there are at least K components in $G(n, n^{-\alpha})$ which are isomorphic to T . For any tree T on $l + 1$ vertices, the probability of containing T tends to $1 - e^{-1/a(T)}$.

4.1.2. Ehrenfeucht game

The main tool of all the above results is Ehrenfeucht game [5,6,9–15,17,21]. We start from the general first order theory on arbitrary finite structures. Consider the first order language consisting of arbitrary relational symbols P_1, \dots, P_m of arities a_1, \dots, a_m respectively and constant symbols R_1, \dots, R_s . The game $\text{EHR}^{\text{FO}}(A, B, k)$ is played on structures A (with distinguished elements R_1^A, \dots, R_s^A) and B (with distinguished elements R_1^B, \dots, R_s^B) of the above vocabulary. There are two players (Spoiler and Duplicator) and a fixed number of rounds k . At the ν -th round ($1 \leq \nu \leq k$) Spoiler chooses either an element x_ν of A or an element y_ν of B . Duplicator chooses an element of the other structure. In the case of monadic second order logic, players can choose subsets of A and B as well: at the ν -th round ($1 \leq \nu \leq k$) of the game $\text{EHR}^{\text{MSO}}(A, B, k)$ Spoiler chooses either structure A or B . Say, he chooses A . Then he either chooses an element x_ν or a subset X_ν of A . If an element is chosen, Duplicator chooses an element of B . Otherwise, Duplicator chooses a subset of B .

In EHR^{FO} , at the end of the game the elements x_1, \dots, x_k of A , y_1, \dots, y_k of B are chosen. Denote $x_{k+1} = R_1^A, y_{k+1} = R_1^B, \dots, x_{k+s} = R_s^A, y_{k+s} = R_s^B$. Duplicator wins if and only if the following property holds.

- For any $i \in \{1, \dots, m\}$ and $\nu_1, \dots, \nu_a \in \{1, \dots, k + s\}$ (where $a = a_i$), $P_i(x_{\nu_1}, \dots, x_{\nu_a}) \Leftrightarrow P_i(y_{\nu_1}, \dots, y_{\nu_a})$.

In EHR^{MSO} , at the end of the game elements x_1, \dots, x_t of A , y_1, \dots, y_t of B (for any $j \in \{1, \dots, t\}$, the elements x_j, y_j are chosen in the same round $i_j \in \{1, \dots, k\}$) and subsets X_1, \dots, X_{k-t} of A , Y_1, \dots, Y_{k-t} of B (for any $j \in \{1, \dots, k - t\}$, the elements X_j, Y_j are chosen in the same round $\ell_j \in \{1, \dots, k\}$) are chosen. In the same way, we denote $x_{k+1} = R_1^A, y_{k+1} = R_1^B, \dots, x_{k+s} = R_s^A, y_{k+s} = R_s^B$. Duplicator wins if and only if the following two properties hold.

- For any $i \in \{1, \dots, m\}$ and $\nu_1, \dots, \nu_a \in \{1, \dots, t, k + 1, \dots, k + s\}$ (where $a = a_i$), $P_i(x_{\nu_1}, \dots, x_{\nu_a}) \Leftrightarrow P_i(y_{\nu_1}, \dots, y_{\nu_a})$.
- For any $\nu \in \{1, \dots, t, k + 1, \dots, k + s\}$ and $\mu \in \{1, \dots, k - t\}$, $x_\nu \in X_\mu \Leftrightarrow y_\nu \in Y_\mu$.

Further, we follow definitions and notations from [13] (see Section 3). Here, we reformulate these definitions for the monadic second order theory on arbitrary structures. Let t, l be nonnegative integers, $t + l \leq k$.

Consider two structures A and B , elements u_1, \dots, u_t of A , v_1, \dots, v_t of B and subsets U_1, \dots, U_l of A , V_1, \dots, V_l of B . Let elements $u_1, \dots, u_t, v_1, \dots, v_t$ be chosen in the first t rounds of $\text{EHR}^{\text{MSO}}(A, B, k)$, sets $U_1, \dots, U_l, V_1, \dots, V_l$ be chosen in the next l rounds (the definition does not depend on the order of choices of elements and sets, it is essential here that in the same round either two vertices with the same indexes or two sets with the same indexes are chosen). Denote $\bar{u} = (u_1, \dots, u_t)$, $\bar{v} = (v_1, \dots, v_t)$, $\bar{U} = (U_1, \dots, U_l)$, $\bar{V} = (V_1, \dots, V_l)$. We write $(A, \bar{u}, \bar{U}) \equiv_k^{\text{MSO}} (B, \bar{v}, \bar{V})$ if Duplicator has a winning strategy in the remaining $k - t - l$ rounds. Obviously, this \equiv_k^{MSO} is an equivalence relation (the \equiv_k^{FO} -equivalence is defined in the same way, see [13]). The k -Ehrenfeucht value of (A, \bar{u}, \bar{U}) is the \equiv_k^{MSO} -equivalence class it belongs to. We let $\text{EHR}(k, t, l)$ denote the set of all k -Ehrenfeucht values for structures of the considered vocabulary with marked t elements and l subsets. Set $\text{EHR}(k) = \text{EHR}(k, 0, 0)$.

Our proofs of Theorems 4, 6 (see Sections 4.3, 4.4 respectively) are based on an extension of Lemma 3.2 from [13] to the monadic second order language, which is stated below.

Lemma 1. *Consider two cases.*

- If $t + l = k$, then

$$|\text{EHR}(k, t, l)| \leq 2^{\sum_{i=1}^m a_i! \binom{s+t}{a_i}} 2^{(s+t)l}. \quad (2)$$

- If $t + l < k$, then

$$|\text{EHR}(k, t, l)| \leq 2^{|\text{EHR}(k, t+1, l)| + |\text{EHR}(k, t, l+1)|}. \quad (3)$$

Proof. Let $t + l = k$. Then the \equiv_k^{MSO} -class of A with marked \bar{u}, \bar{U} is determined by memberships of $u_1, \dots, u_t, R_1, \dots, R_s$ in U_1, \dots, U_l , and the induced substructure $A|_{\{u_1, \dots, u_t, R_1, \dots, R_s\}}$. Therefore, we get (2).

If $t + l < k$, then $(A, \bar{u}, \bar{U}) \equiv_k^{\text{MSO}} (B, \bar{v}, \bar{V})$ if and only if

$$\forall u \in A \exists v \in B \quad (A, (u_1, \dots, u_t, u), \bar{U}) \equiv_k^{\text{MSO}} (B, (v_1, \dots, v_t, v), \bar{V}), \quad (4)$$

$$\forall U \subset A \exists V \subset B \quad (A, \bar{u}, (U_1, \dots, U_s, U)) \equiv_k^{\text{MSO}} (B, \bar{v}, (V_1, \dots, V_s, V)), \quad (5)$$

and vice versa (for every $v \in B$, there exists $u \in A$ such that the relation in (4) holds, and for every $V \subset B$ there exists $U \subset A$ such that the relation in (5) holds). Obviously the latter conditions hold if and only if

- the set of k -Ehrenfeucht values of the structure A with marked elements u_1, \dots, u_t, u and subsets U_1, \dots, U_l over all $u \in A$ and the set of k -Ehrenfeucht values of the structure B with marked elements v_1, \dots, v_t, v and subsets V_1, \dots, V_l over all $v \in B$ coincide;
- the set of k -Ehrenfeucht values of the structure A with marked elements u_1, \dots, u_t and subsets U_1, \dots, U_l, U over all $U \subset A$ and the set of k -Ehrenfeucht values of the structure B with marked elements v_1, \dots, v_t and subsets V_1, \dots, V_l, V over all $V \subset B$ coincide.

Therefore, the k -Ehrenfeucht value of the structure A with marked elements u_1, \dots, u_t and subsets U_1, \dots, U_l is defined by the set of k -Ehrenfeucht values of the structure G with marked elements u_1, \dots, u_t, u and subsets U_1, \dots, U_l over all $u \in A$ and the set of k -Ehrenfeucht values of the graph G with marked elements u_1, \dots, u_t and subsets U_1, \dots, U_l, U over all $U \subset A$. This leads to Equation (3). \square

For graphs (vocabulary consists of two relational symbols $\sim, =$ and no constant symbols, see Section 1) and rooted trees (two relational symbols $P, =$ and one constant symbol R are considered, see Section 2), the bound (2) can be strengthened.

Lemma 2. Let $t + l = k$. For graphs, $|\text{EHR}(k, t, l)| \leq 2^{k^2 - k}$. For rooted trees and $k \geq 5$, $\log_2 |\text{EHR}(k, t, l)| \leq 2^k - 2$. For rooted trees and $k = 4$, $|\text{EHR}(4, t, l)| \leq 3 \cdot 2^{13}$.

Proof. As relations $\sim, =$ are symmetric, for graphs,

$$|\text{EHR}(k, t, l)| \leq 2^{2\binom{t}{2} + tl} = 2^{t(k-1)} \leq 2^{k(k-1)}.$$

For rooted trees, we consider two cases: vertices u_1, \dots, u_t, R are either pairwise distinct or not. Note that there are at most $(t+1)^{t-1} 3^t$ directed forests on $t+1$ labeled vertices (there are $(t+1)^{t-1}$ trees, and each edge can be either removed or directed in one of two ways). Therefore, in the first case (for distinct u_1, \dots, u_t, R), the number of k -Ehrenfeucht values of (A, \bar{u}, \bar{U}) is at most

$$(t+1)^{t-1} 3^t 2^{(t+1)l} = 2^{(t-1)\log_2(t+1) + t\log_2 3 + (t+1)l} \leq 2^{g_1(k)},$$

where $g_1(k) = \left(\frac{k+1}{2}\right)^2 + (k-1)\log_2(k+1) + k\log_2 3$. As the number of ways of determining the equality relation on u_1, \dots, u_t, R is at most $2^{\binom{t+1}{2}}$, in the second case (u_1, \dots, u_t, R are not pairwise distinct), the number of k -Ehrenfeucht values is less than

$$2^{\binom{t+1}{2}} t^{t-2} 3^{t-1} 2^{tl} \leq 2^{(\frac{t+1}{2}) + (t-2)\log_2 t + (t-1)\log_2 3 + tl} \leq 2^{g_2(k)},$$

where $g_2(k) = \frac{(k+1)k}{2} + (k-2)\log_2 k + (k-1)\log_2 3$. Therefore, for $k \geq 5$, we get the following bound:

$$|\text{EHR}(k, t, l)| \leq 2^{g_1(k)} + 2^{g_2(k)} < 2^{2^k - 2}.$$

The last inequality holds because, first, for $k = 5$, both $g_1(k), g_2(k)$ are less than $2^k - 3$, and, second, $2^k - 3$ grows faster than $g_1(k)$ and $g_2(k)$ for $k > 5$.

Let $k = 4$. In the first case (for distinct u_1, \dots, u_t, R), the number of k -Ehrenfeucht values of (A, \bar{u}, \bar{U}) is at most

$$\max_{t \in \{0, 1, 2, 3, 4\}} (t+1)^{t-1} 3^t 2^{(t+1)(4-t)} = 3^4 5^3.$$

There are exactly $F(t) = 1, 3, 19, 201$ directed forests on $t = 1, 2, 3, 4$ labeled vertices respectively. Moreover, for $t = 1, 2, 3, 4$, the number of ways of determining the equality relation on u_1, \dots, u_t, R in such a way that these vertices are not pairwise distinct equals $E(t) = 1, 4, 14, 51$ respectively. Finally, if u_1, \dots, u_t, R are not pairwise distinct, then, for $t = 1, 2, 3, 4$, the number of ways of determining memberships of u_1, \dots, u_t, R in U_1, \dots, U_{4-t} is at most $M(t) = 8, 16, 8, 1$ respectively. Therefore, in the second case (u_1, \dots, u_t, R are not pairwise distinct), the number of k -Ehrenfeucht values is at most

$$\max_{t \in \{0, 1, 2, 3, 4\}} F(t)E(t)M(t) = 201 \cdot 51.$$

Finally,

$$|\text{EHR}(4, t, l)| \leq 3^4 5^3 + 201 \cdot 51 = 20376 < 3 \cdot 2^{13}. \quad \square$$

For graphs and vocabulary consisting of two relational symbols $\sim, =$ and no constant symbols, we denote the games $\text{EHR}^{\text{FO}}(G, H, k)$, $\text{EHR}^{\text{MSO}}(G, H, k)$ on graphs G, H by

$$\text{EHR}^{\text{FO}; \text{graphs}}(G, H, k), \quad \text{EHR}^{\text{MSO}; \text{graphs}}(G, H, k)$$

respectively. In 1960 [6], A. Ehrenfeucht proved that there is the following connection between Ehrenfeucht games and logical equivalences (we give the statement confirming to the case of graphs).

Theorem 11. *For any positive integer k and any graphs G, H ,*

$$\begin{aligned} G \equiv_k^{\text{FO}; \text{graphs}} H &\Leftrightarrow \text{Duplicator has a winning strategy in } \text{EHR}^{\text{FO}; \text{graphs}}(G, H, k), \\ G \equiv_k^{\text{MSO}; \text{graphs}} H &\Leftrightarrow \text{Duplicator has a winning strategy in } \text{EHR}^{\text{MSO}; \text{graphs}}(G, H, k). \end{aligned}$$

We need the following well-known corollary from this theorem (see, e.g., [9,11,15,21]).

Corollary 1. *Let k be any positive integer. The random graph $G(n, p)$ obeys FO zero-one k -law if and only if asymptotically almost surely Duplicator has a winning strategy in $\text{EHR}^{\text{FO}; \text{graphs}}(G(n, p), G(m, p), k)$ as $n, m \rightarrow \infty$. The random graph $G(n, p)$ obeys MSO zero-one k -law if and only if asymptotically almost surely Duplicator has a winning strategy in $\text{EHR}^{\text{MSO}; \text{graphs}}(G(n, p), G(m, p), k)$ as $n, m \rightarrow \infty$.*

The following two standard facts about $\equiv_k^{\text{MSO}; \text{graphs}}$ -equivalence are essential tools in our arguments. They follow from Theorem 11 (see the proofs in [4], Theorems 2.2 and 2.3). Denote by $A \sqcup B$ a disjoint union of graphs A, B : $A \sqcup B = (V(A) \sqcup V(B), E(A) \sqcup E(B))$.

Lemma 3. *Let k be a positive integer. If $H_1 \equiv_k^{\text{MSO}; \text{graphs}} H_2$, $G_1 \equiv_k^{\text{MSO}; \text{graphs}} G_2$, then $H_1 \sqcup G_1 \equiv_k^{\text{MSO}; \text{graphs}} H_2 \sqcup G_2$. The same holds for $\equiv_k^{\text{FO}; \text{graphs}}$ -equivalence.*

Denote by aG a disjoint union of a copies of a graph G .

Lemma 4. *For any positive integer k there is a positive integer a such that for any integer $b \geq a$ and any graph G we have $bG \equiv_k^{\text{MSO}; \text{graphs}} aG$.*

In the proofs, we also exploit Ehrenfeucht game on rooted trees. We denote the games $\text{EHR}^{\text{FO}}(T_R, T'_{R'}, k)$, $\text{EHR}^{\text{MSO}}(T_R, T'_{R'}, k)$ on rooted trees $T_R, T'_{R'}$ by $\text{EHR}^{\text{FO}; \text{trees}}(T_R, T'_{R'}, k)$, $\text{EHR}^{\text{MSO}; \text{trees}}(T_R, T'_{R'}, k)$ respectively (two relational symbols $P, =$ and one constant symbol R are considered). Another particular case of Ehrenfeucht Theorem is stated below.

Theorem 12. *For any positive integer k and any rooted trees $T_R, T'_{R'}$,*

$$\begin{aligned} T_R \equiv_k^{\text{FO}; \text{trees}} T'_{R'} &\Leftrightarrow \text{Duplicator has a winning strategy in } \text{EHR}^{\text{FO}; \text{trees}}(T_R, T'_{R'}, k), \\ T_R \equiv_k^{\text{MSO}; \text{trees}} T'_{R'} &\Leftrightarrow \text{Duplicator has a winning strategy in } \text{EHR}^{\text{MSO}; \text{trees}}(T_R, T'_{R'}, k). \end{aligned}$$

4.2. Proof of Theorem 2: $\alpha > 1$

Let l be a positive integer.

We start from $\alpha \in (1 + 1/(l + 1), 1 + 1/l)$ and $p = n^{-\alpha}$. Let k be a positive integer. Consider $a = a(k)$ from Lemma 4. Let \mathcal{T} be the set of all pairwise nonisomorphic trees on at most $l + 1$ vertices. Consider the forest

$$T_0 = \bigsqcup_{T \in \mathcal{T}} aT.$$

From Properties T1, T2, T3 (see Section 4.1.1) it follows that a.a.s. there exist K_T for every $T \in \mathcal{T}$ such that

$$G(n, p) = \bigsqcup_{T \in \mathcal{T}} K_T T$$

and $K_T > a$ for any $T \in \mathcal{T}$. From [Lemmas 3 and 4](#), a.a.s. $G(n, p) \equiv_k^{\text{MSO}; \text{graphs}} T_0$. This means that a.a.s. for any monadic second order ϕ with $q(\phi) = k$ a.a.s. $G(n, p) \models \phi$ if and only if $T_0 \models \phi$. Thus, $G(n, p)$ obeys MSO zero-one k -law. As k is arbitrary, MSO zero-one law holds.

If $\alpha = 1 + 1/l$ and $p = n^{-\alpha}$, then $G(n, p)$ does not obey the FO zero-one law. Therefore, MSO zero-one law fails as well.

If $\alpha > 2$, then a.a.s. in $G(n, n^{-\alpha})$ there are no edges (i.e. $G(n, n^{-\alpha})$ is the union of isolated vertices). From [Lemma 4](#), in this case MSO zero-one law holds.

4.3. Proof of [Theorem 4](#)

Fix a positive integer k . To avoid trivialities, we assume $k \geq 2$. Set $\tilde{T}(1) = 2^{k^2-k}$, $\tilde{T}(i) = 2^{2\tilde{T}(i-1)}$, $\hat{T}(1) = 2^{2^k}$, $\hat{T}(i) = 2^{\hat{T}(i-1)}$. As $2^k \geq k^2$ and, for a positive x , $1 + 2^x < 2^x + 2^x = 2^{x+1}$, we get the inequalities $\tilde{T}(i) \leq \hat{T}(i)$, $i \in \{1, \dots, k+1\}$. Moreover, $\hat{T}(1) \leq 2^{2^{T(\log^*(k))}} = T(2 + \log^*(k))$. For any nonnegative integer β such that $\beta \leq k$ set $f(k, \beta) = \max_{t \in \{0, 1, \dots, \beta\}} |\text{EHR}(k, t, \beta - t)|$. By [Lemma 1](#) and [Lemma 2](#),

$$|\text{EHR}(k)| \leq 2^{2f(k, 1)} \leq 2^{2 \cdot 2^{2f(k, 2)}} \leq \dots \leq \tilde{T}(k+1) \leq \hat{T}(k+1) = T(k+2 + \log^*(k)).$$

Finally, by [Theorem 11](#), we get $r_k^{\text{MSO}; \text{graphs}} = |\text{EHR}(k)| \leq T(k+2 + \log^*(k))$.

4.4. Proof of [Theorem 6](#)

Fix an integer $k \geq 5$. Set $\tilde{T}(1) = 2^{k^2-2}$, $\tilde{T}(i) = 2^{2\tilde{T}(i-1)}$, $\hat{T}(1) = k - 1 + 2^{2^k-1}$, $\hat{T}(i) = 2^{\hat{T}(i-1)}$. As in the previous section, for any $i \in \{1, \dots, k+1\}$, we have $\tilde{T}(i) \leq \hat{T}(i)$. Moreover, $\hat{T}(1) < 2^{2^k} \leq 2^{2^{T(\log^*(k))}} = T(2 + \log^*(k))$. Therefore, by [Lemma 1](#) and [Lemma 2](#),

$$|\text{EHR}(k)| \leq \tilde{T}(k+1) \leq \hat{T}(k+1) < T(k+2 + \log^*(k)).$$

Let $k = 4$. By [Lemma 1](#) and [Lemma 2](#),

$$\log_2 \log_2 |\text{EHR}(k)| \leq 1 + 2^{1+2^{1+6 \cdot 2^{13}}} < 2^{2^{3+6 \cdot 2^{13}}} < 2^{2^{2^{16}}} = T(6) = T(k + \log^*(k)).$$

By [Theorem 12](#), for any $k \geq 4$ we get $r_k^{\text{MSO}; \text{trees}} = |\text{EHR}(k)| \leq T(k+2 + \log^*(k))$.

The second statement of [Theorem 6](#) follows the statement about representatives of $\equiv_k^{\text{MSO}; \text{trees}}$ -classes which is an extension of [Lemma 8.6](#) and [Lemma 8.7](#) of [\[13\]](#) to the monadic second order language and stated below. Set $z = 2^k f(k, 1) \dots f(k, k)$ (see the notation in [Section 4.3](#)).

Lemma 5. *Let k be a positive integer. Fix any $A \in \mathcal{R}_k^{\text{MSO}; \text{trees}}$. Let $T_R \in A$ be a rooted tree with a minimal order over all rooted trees in A . Then*

- each vertex of T_R has at most $zf(k, 0)$ children;
- the depth of T_R is at most $f(k, 0)$.

Proof. For \equiv_k^{MSO} -equivalence class, we say that it has *type m* , if all of its representatives have t chosen vertices and l chosen subsets where $t + l = m$ (see [Section 4.1.2](#)). For all $m \in \{0, 1, \dots, k\}$, set $z(k, m) = 1$. Moreover, for all $i \in \{1, \dots, k\}$, $m \in \{0, 1, \dots, i-1\}$, set

$$z(i-1, m) = z(i, m) + z(i, m+1)f(k, m+1). \quad (6)$$

By downwards recursion on i , for all $i \in \{0, 1, \dots, k-1\}$, $m \in \{0, 1, \dots, i\}$,

$$z(i, m) \leq 2^{k-i} f(k, m+1) \dots f(k, k).$$

Therefore, $z(0, 0) \leq z$.

Consider an arbitrary vertex y of T_R . Let w_1, \dots, w_s be children of y . Let k -Ehrenfeucht values of $T_R(w_1), \dots, T_R(w_s)$ equal a_1, \dots, a_s respectively. Suppose that for some $i \in \{1, \dots, s\}$ (say, $i = 1$) more than z values of a_1, \dots, a_s equal a_i . Let $w_1, \dots, w_{z'}$ be all children of y such that k -Ehrenfeucht values of $T_R(w_1), \dots, T_R(w_{z'})$ equal a_1 . Consider the rooted tree T_R^- which is obtained from T_R by removing all but z subtrees rooted at children of y such that their k -Ehrenfeucht values equal a_1 (say, the subtrees $T_R(w_{z+1}), \dots, T_R(w_{z'})$).

We prove that $T_R \equiv_k^{\text{MSO}; \text{trees}} T_R^-$. By Theorem 11, the trees are equivalent if and only if Duplicator has a winning strategy in $\text{EHR}^{\text{MSO}; \text{trees}}(T_R, T_R^-, k)$. We do not consider choices of vertices and subsets of

$$V(T_R) \setminus V(T_R(w_1) \cup \dots \cup T_R(w_{z'}))$$

by Spoiler, because for such choices the strategy of Duplicator is trivial. Set

$$\mathcal{C}_{0;1}(0) = \{T_R(w_1), \dots, T_R(w_{z'})\}, \quad \mathcal{C}_{0;1}^-(0) = \{T_R^-(w_1), \dots, T_R^-(w_z)\}.$$

After each round i of the game, the equivalence classes of type $m \in \{0, \dots, i\}$ with trees from $\mathcal{C}_{0;1}(0)$ (from $\mathcal{C}_{0;1}^-(0)$) are denoted by $\mathcal{C}_{i,j}(m)$ (by $\mathcal{C}_{i,j}^-(m)$), where $j = 1, \dots, \gamma_i(m)$.

Consider **the first round** of $\text{EHR}^{\text{MSO}; \text{trees}}(T_R, T_R^-, k)$.

If Spoiler chooses a vertex $u \neq y$ (say, in a tree $T_R(w_i)$), then Duplicator chooses any vertex v such that $(T_R(w_i), u) \equiv_k^{\text{MSO}} (T_R^-(w_j), v)$ (say, in a tree $T_R^-(w_j)$). He can do this, because all trees from $\mathcal{C}_{0;1}^-(0)$ and $\mathcal{C}_{0;1}(0)$ have the same k -Ehrenfeucht value. Set

$$\begin{aligned} \mathcal{C}_{1;1}(1) &= \{(T_R(w_i), u)\}, & \mathcal{C}_{1;1}(0) &= \{T_R(w_1), \dots, T_R(w_{z'})\} \setminus \{T_R(w_i)\}, \\ \mathcal{C}_{1;1}^-(1) &= \{(T_R^-(w_j), v)\}, & \mathcal{C}_{1;1}^-(0) &= \{T_R^-(w_1), \dots, T_R^-(w_z)\} \setminus \{T_R^-(w_j)\}, \\ \gamma_1(0) &= 1, \gamma_1(1) = 1. \end{aligned}$$

If Spoiler chooses a subset U (say, $U = U_1 \cup \dots \cup U_{z'}$, U_i is (maybe, empty) subset of $V(T_R(w_i))$), then $\mathcal{C}_{1;1}(1), \dots, \mathcal{C}_{1;\gamma_1(1)}$ are \equiv_k^{MSO} -equivalence classes on pairs $(T_R(w_i), U_i)$ such that U_i are not empty, and $\mathcal{C}_{1;1}(0)$ is the \equiv_k^{MSO} -equivalence class (maybe, empty) on $T_R(w_i)$ such that U_i are empty. Obviously, the number of equivalence classes $\mathcal{C}_{1;j}(1)$ equals $\gamma_1(1) \leq f(k, 1)$, the number of equivalence classes $\mathcal{C}_{1;1}(0)$ equals $\gamma_1(0) \leq 1$ ($\gamma_1(0) = 0$ if $\mathcal{C}_{1;1}(0)$ is empty). Duplicator constructs a subset V and classes of subtrees of T_R^- in the following way.

If $\mathcal{C}_{1;1}(0)$ is not empty and $|\mathcal{C}_{1;1}(0)| \leq z(1, 0)$, Duplicator considers first $|\mathcal{C}_{1;1}(0)|$ trees of $T_R^-(w_1), \dots, T_R^-(w_z)$ and put them into the set $\mathcal{C}_{1;1}^-(0)$:

$$\mathcal{C}_{1;1}^-(0) = \{T_R^-(w_1), \dots, T_R^-(w_{|\mathcal{C}_{1;1}(0)|})\}.$$

If $|\mathcal{C}_{1;1}(0)| > z(1, 0)$, then Duplicator considers first *at least* $z(1, 0)$ trees of $T_R^-(w_1), \dots, T_R^-(w_z)$ and put them into the set $\mathcal{C}_{1;1}^-(0)$. If $\mathcal{C}_{1;1}(0)$ is empty, then $\mathcal{C}_{1;1}^-(0)$ is empty as well.

After that, for each j from 1 to $\gamma_1(1)$, if $|\mathcal{C}_{1;j}(1)| \leq z(1, 1)$, then Duplicator considers next $|\mathcal{C}_{1;j}(1)|$ unconsidered trees of $T_R^-(w_1), \dots, T_R^-(w_z)$ and chooses their subsets such that each tree with the respective

subset is in the same \equiv_k^{MSO} -equivalence class with all representatives of $\mathcal{C}_{1;j}(1)$. If $|\mathcal{C}_{1;j}(1)| > z(1, 1)$, then Duplicator considers next *at least* $z(1, 1)$ trees and their sets of vertices in the same way. As

$$z' > z \geq z(0, 0) = z(1, 0) + z(1, 1)f(k, 1),$$

the subset V exists.

Consider **the i -th round**, $i \in \{2, \dots, k\}$. Let, after the previous $i - 1$ st round, nonnegative integers $\gamma_{i-1}(0), \dots, \gamma_{i-1}(i - 1)$ and \equiv_k^{MSO} -equivalence classes $\mathcal{C}_{i-1;j}(m)$, $\mathcal{C}_{i-1;j}^-(m)$ (where $m \in \{0, \dots, i - 1\}$, $j \in \{1, \dots, \gamma_{i-1}(m)\}$) be chosen (where each class consists of rooted trees from $\mathcal{C}_{0;1}(0)$ and $\mathcal{C}_{0;1}^-(0)$ respectively with chosen m vertices and subsets). For any $m \in \{0, \dots, i - 1\}$ and any $j \in \{1, \dots, \gamma_{i-1}(m)\}$, all elements of $\mathcal{C}_{i-1;j}^-(m)$ and $\mathcal{C}_{i-1;j}(m)$ have the same k -Ehrenfeucht value. Finally, for any $m \in \{0, \dots, i - 1\}$ and any $j \in \{1, \dots, \gamma_{i-1}(m)\}$,

$$\text{either } |\mathcal{C}_{i-1;j}^-(m)| = |\mathcal{C}_{i-1;j}(m)|, \text{ or } \min\{|\mathcal{C}_{i-1;j}^-(m)|, |\mathcal{C}_{i-1;j}(m)|\} \geq z(i - 1, m). \quad (7)$$

If Spoiler chooses a vertex $u \neq y$ (say, in a tree $T_R(w_i)$ with chosen $\hat{m} \in \{0, \dots, i - 1\}$ vertices \bar{u} and subsets \bar{U} such that $(T_R(w_i), \bar{u}, \bar{U}) \in \mathcal{C}_{i-1;1}(\hat{m})$), then Duplicator finds $\tilde{i} \in \{1, \dots, z\}$ such that $(T_R^-(w_{\tilde{i}}), \bar{v}, \bar{V})$ is from $\mathcal{C}_{i-1;1}^-(\hat{m})$ and chooses a vertex v such that $(T_R(w_i), \bar{u}, u, \bar{U}) \equiv_k^{\text{MSO}} (T_R^-(w_{\tilde{i}}), \bar{v}, v, \bar{V})$. Set

$$\begin{aligned} \mathcal{C}_{i;1}(\hat{m}) &= \mathcal{C}_{i-1;1}(\hat{m}) \setminus \{(T_R(w_i), \bar{u}, \bar{U})\}, & \mathcal{C}_{i;j}(\hat{m}) &= \mathcal{C}_{i-1;j}(\hat{m}), & j &\in \{2, \dots, \gamma_{i-1}(\hat{m})\}, \\ \mathcal{C}_{i;1}(\hat{m} + 1) &= \{(T_R(w_i), \bar{u}, u, \bar{U})\}, & \mathcal{C}_{i;j+1}(\hat{m} + 1) &= \mathcal{C}_{i-1;j}(\hat{m} + 1), & j &\in \{1, \dots, \gamma_{i-1}(\hat{m} + 1)\}, \\ \mathcal{C}_{i;j}(m) &= \mathcal{C}_{i-1;j}(m), & m &\in \{0, \dots, i - 1\} \setminus \{\hat{m}, \hat{m} + 1\}, & j &\in \{1, \dots, \gamma_{i-1}(m)\}. \end{aligned}$$

In the same way, the classes $\mathcal{C}_{i;j}^-(m)$ are defined. Moreover, $\gamma_i(m) = \gamma_{i-1}(m)$ for all $m \in \{0, \dots, i\} \setminus \{\hat{m} + 1\}$ and $\gamma_i(\hat{m} + 1) = \gamma_{i-1}(\hat{m} + 1) + 1$ (here, we assume $\gamma_{i-1}(i) = 0$).

If Spoiler chooses a subset U (say, $U = U_1 \cup \dots \cup U_{z'}$, U_l is (maybe, empty) subset of $V(T_R(w_l))$), then classes $\mathcal{C}_{i;1}(m + 1), \mathcal{C}_{i;2}(m + 1), \dots$ are obtained by dividing each $\mathcal{C}_{i-1;j}(m)$ into at most $f(k, m + 1)$ \equiv_k^{MSO} -equivalence classes with respect to nonempty subsets U_l . Elements of a class $\mathcal{C}_{i-1;j}(m)$ with empty subsets U_l (if such elements exist) form a class of the same type m . Duplicator constructs a subset V in the following way.

She starts from $m = 0$, $j = 1$ and finishes at $m = i$, $j = \gamma_i(m)$. At each step, she considers new (unconsidered previously) trees from $\mathcal{C}_{0;1}^-(0)$. Suppose that Duplicator has already visited all pairs up to (m, j) , $m \in \{0, \dots, i\}$, $j \in \{1, \dots, \gamma_i(m)\}$ (she visited all pairs (m_0, j_0) such that $m_0 \in \{0, \dots, m - 1\}$, $j_0 \in \{1, \dots, \gamma_i(m_0)\}$ and all pairs (m, j_0) such that $j_0 \in \{1, \dots, j - 1\}$). At the step (m, j) , she finds $\hat{m} \in \{m - 1, m\}$, $\hat{j} \in \{1, \dots, \gamma_{i-1}(\hat{m})\}$ (only one such pair exists) such that $\mathcal{C}_{i-1;\hat{j}}(\hat{m}) \supset \mathcal{C}_{i;j}(m)$. If $|\mathcal{C}_{i;j}(m)| \leq z(i, m)$, then Duplicator considers the next $|\mathcal{C}_{i;j}(m)|$ elements of the class $\mathcal{C}_{i-1;\hat{j}}^-(\hat{m})$ and chooses (maybe, empty) subsets of the respective rooted trees such that each tree with chosen subsets (maybe, accounting for the new one which is nonempty) and vertices is in the same \equiv_k^{MSO} -equivalence class with all representatives of $\mathcal{C}_{i;j}(m)$. If $|\mathcal{C}_{i;j}(m)| > z(i, m)$, then Duplicator considers next *at least* $z(i, m)$ elements of the class $\mathcal{C}_{i-1;\hat{j}}^-(\hat{m})$ in the same way. The subset V exists, because (6) with $m = \hat{m}$ and (7) hold.

Obviously, this strategy of Duplicator is winning. Therefore, $T_R^- \in A$ and $|V(T_R^-)| < |V(T_R)|$, that contradicts with the assumptions. As the number of different classes among a_1, \dots, a_s is at most $f(k, 0)$, the first statement of Lemma 5 is proven.

Suppose there is a vertex $y \in V(T_R)$ such that $d(R, y) > f(k, 0)$. Consider the path R, \dots, y . By the definition of $f(k, 0)$, in this path there are distinct nonroot vertices w_1, w_2 such that $T_R(w_1) \equiv_k^{\text{MSO}} T_R(w_2)$. Assume $d(R, w_1) < d(R, w_2)$. Consider a tree T_R^- rooted at R which is obtained from T_R by the result of replacing $T_R(w_1)$ by $T_R(w_2)$ (the root w_1 is replaced by the root w_2). It is easy to see that $T_R \equiv_k^{\text{MSO}} T_R^-$.

By [Theorem 12](#), $T_R \equiv_k^{\text{MSO}; \text{trees}} T_R^-$. Therefore, $T_R^- \in A$ and $|V(T_R^-)| < |V(T_R)|$, that contradicts with the assumptions. \square

Fix $A \in \mathcal{R}_k^{\text{MSO}; \text{trees}}$. From [Lemma 5](#) and the first statement of [Theorem 6](#), we get

$$\min_{T_R \in A} |V(T_R)| \leq \sum_{i=0}^{f(k,0)} (zf(k,0))^i < (zf(k,0))^{f(k,0)+1} \leq T(k+3+\log^*(k+1)).$$

Theorem is proven.

4.5. Proof of [Theorem 8](#): MSO k -law

Let $k \geq 4$, $l \geq T(k+\log^*(k+1)+3)$, $\alpha = 1 + \frac{1}{l}$, $p = n^{-\alpha}$. Consider $a = a(k)$ from [Lemma 4](#). Let \mathcal{T} be the set of all pairwise nonisomorphic trees on at most l vertices and \mathcal{T}^+ be the set of all pairwise nonisomorphic trees on exactly $l+1$ vertices. Consider the forest

$$T_0 = \bigsqcup_{T \in \mathcal{T}} aT.$$

From Properties T1, T2, T4 (see [Section 4.1.1](#)) it follows that a.a.s. there exist K_T for every $T \in \mathcal{T} \cup \mathcal{T}^+$ such that

$$G(n, p) = \bigsqcup_{T \in \mathcal{T} \cup \mathcal{T}^+} K_T T$$

and $K_T > a$ for any $T \in \mathcal{T}$.

For any $T \in \mathcal{T} \cup \mathcal{T}^+$, fix a nonnegative integer K_T such that $K_T \geq a$ for all $T \in \mathcal{T}$. Consider the forest

$$F = \bigsqcup_{T \in \mathcal{T} \cup \mathcal{T}^+} K_T T.$$

From [Theorem 6](#), for any $\hat{T} \in \mathcal{T}^+$ there exists $T(\hat{T}) \in \mathcal{T}$ such that $T(\hat{T}) \equiv_k^{\text{MSO}; \text{graphs}} \hat{T}$. Denote $\mathcal{T}(\mathcal{T}^+)$ the set of all $T \in \mathcal{T}$ such that there exists $\hat{T} \in \mathcal{T}^+$ with $T = T(\hat{T})$. For any $T \in \mathcal{T}(\mathcal{T}^+)$, denote $\mathcal{T}^+(T)$ the set of trees \hat{T} from \mathcal{T}^+ such that $T = T(\hat{T})$. For any $T \in \mathcal{T}(\mathcal{T}^+)$, set $\tilde{K}_T = K_T + \sum_{\hat{T} \in \mathcal{T}^+(T)} \min\{K_{\hat{T}}, a\}$. For all the others $T \in \mathcal{T}$ set $\tilde{K}_T = K_T$.

From [Lemma 4](#), $\tilde{K}_T T \equiv_k^{\text{MSO}; \text{graphs}} aT$ for any $T \in \mathcal{T}$ and $\min\{K_{\hat{T}}, a\}T \equiv_k^{\text{MSO}; \text{graphs}} K_{\hat{T}}T$ for all $T \in \mathcal{T}(\mathcal{T}^+)$ and $\hat{T} \in \mathcal{T}^+(T)$. From [Lemma 3](#),

$$\begin{aligned} T_0 &\equiv_k^{\text{MSO}; \text{graphs}} \bigsqcup_{T \in \mathcal{T}} \tilde{K}_T T = \left(\bigsqcup_{T \in \mathcal{T}} K_T T \right) \sqcup \left(\bigsqcup_{T \in \mathcal{T}(\mathcal{T}^+)} \left(\sum_{\hat{T} \in \mathcal{T}^+(T)} \min\{K_{\hat{T}}, a\} \right) T \right) \equiv_k^{\text{MSO}; \text{graphs}} \\ &\left(\bigsqcup_{T \in \mathcal{T}} K_T T \right) \sqcup \left(\bigsqcup_{T \in \mathcal{T}(\mathcal{T}^+)} \bigsqcup_{\hat{T} \in \mathcal{T}^+(T)} K_{\hat{T}} T \right) \equiv_k^{\text{MSO}; \text{graphs}} \left(\bigsqcup_{T \in \mathcal{T}} K_T T \right) \sqcup \left(\bigsqcup_{T \in \mathcal{T}(\mathcal{T}^+)} \bigsqcup_{\hat{T} \in \mathcal{T}^+(T)} K_{\hat{T}} \hat{T} \right) \\ &= \left(\bigsqcup_{T \in \mathcal{T}} K_T T \right) \sqcup \left(\bigsqcup_{\hat{T} \in \mathcal{T}^+} K_{\hat{T}} \hat{T} \right) = F. \end{aligned}$$

Therefore, a.a.s. $G(n, p) \equiv_k^{\text{MSO}; \text{graphs}} T_0$. This means that a.a.s. for any monadic second order ϕ with $q(\phi) = k$ a.a.s. $G(n, p) \models \phi$ if and only if $T_0 \models \phi$. Thus, $G(n, p)$ obeys MSO zero-one k -law.

4.6. Proof of Theorem 8: FO k -law

In the proof, we follow definitions and notations from [13], Section 6. We give them below. If w is a child of $u \in V(T_v)$, then $T_v(w)$ is a u -branch of T_v . A rooted tree T_v is called *diverging* if, for every vertex u , all u -branches of T_v are pairwise nonisomorphic. A tree is called *diverging* if, for every central vertex v (see Section 4.1), the rooted tree T_v is diverging. We say that rooted trees T_v and S_u are *isomorphic* if there is an isomorphism between the respective unrooted trees T and U which maps v to u .

We use the following result, which is proved in [13] (see Lemma 6.13).

Lemma 6. *Let $i \geq 3$. For every n such that $2i + 2 \leq n \leq 2T(i - 1) + 1$ there exists a diverging tree of order n and radius $i + 1$.*

For each $l \in \{1, \dots, 8\}$, it is easy to construct a first order formula with quantifier depth at most 7 which is true on $G(n, n^{-1-1/l})$ with asymptotical probability in $(0, 1)$ (using the properties T1 and T4, see Section 4.1.1). Indeed, for $l \in \{1, \dots, 6\}$, such a formula $\exists x_1 \dots \exists x_{l+1} (P_l(x_1, \dots, x_{l+1}))$, where

$$P_{l+1} = (x_1 \sim x_2) \wedge \dots \wedge (x_l \sim x_{l+1}) \wedge \left(\bigwedge_{1 \leq i < j \leq l+1} (x_i \neq x_j) \right),$$

has $l + 1$ existential quantifiers in the beginning and express the property of containing a simple path on $l + 1$ vertices. For $l = 7$ and $l = 8$,

$$\begin{aligned} & \exists x_1 [(\exists x_2 \exists x_3 [(x_1 \sim x_2) \wedge (x_1 \sim x_3) \wedge (x_2 \neq x_3)]) \wedge (\exists x_2 \dots \exists x_7 [P_l(x_1, \dots, x_7)])], \\ & \exists x_1 [(\exists x_2 \exists x_3 \exists x_4 [(x_1 \sim x_2) \wedge (x_1 \sim x_3) \wedge (x_1 \sim x_4) \wedge (x_2 \neq x_3) \wedge (x_2 \neq x_4) \wedge (x_3 \neq x_4)]) \wedge \\ & \quad \wedge (\exists x_2 \dots \exists x_7 [P_l(x_1, \dots, x_7)])] \end{aligned}$$

are such formulae respectively.

Let $l \geq 9$. Fix $k \geq 7$. We start from $2T(k - 4) \geq l \geq 2k - 5$. Set $p = n^{-\alpha}$, $\alpha = 1 + 1/l$. By Lemma 6, there exists a diverging tree S of order $l + 1$ and radius $k - 2$. By T1, T2 and T4, with some asymptotical probability $c \in (0, 1)$ in $G(n, p)$ there is a component isomorphic to S . Moreover, with asymptotical probability $1 - c$ in $G(n, p)$ there is no copy of S .

Consider two forests A and B such that there is a component S^A in A isomorphic to S , and there is no copy of S in B . Let us prove that Spoiler has a winning strategy in $\text{EHR}^{\text{FO}; \text{graphs}}(A, B, k)$. In the first round, Spoiler chooses a central vertex x_1 in S^A . Duplicator chooses a vertex y_1 in a component S^B of B .

Suppose that $d(S^A) < d(S^B)$ (if $d(S^A) > d(S^B)$, Spoiler applies the same strategy). In the next two rounds, Spoiler chooses y_2 and y_3 (not necessary different from y_1) in S^B such that $d(y_2, y_3) = d(S^A) + 1$ and $d(y_1, y_2) \leq d(S^A)$. Duplicator chooses vertices x_2, x_3 . If $x_2 \notin V(S^A)$, then there is a winning strategy of Spoiler in next $\lceil \log_2(d(S^A)) \rceil \leq \lceil \log_2(2r(S^A)) \rceil = 1 + \lceil \log_2(k - 2) \rceil \leq k - 3$ rounds (see, e.g., [13], Lemma 6.4). If $x_2 \in V(S^A)$, then $d_{S^A}(x_2, x_3) \neq d(y_2, y_3) = d(S^A) + 1$. Therefore, there is a winning strategy of Spoiler in next $\lceil \log_2(d(S^A) + 1) \rceil \leq \lceil \log_2(2k - 3) \rceil \leq k - 3$ rounds.

Let $d(S^A) = d(S^B)$. Further, we apply the strategy of Spoiler from Lemma 6.7 and Lemma 6.8 in [13]. Note that this strategy is winning in a game on two trees. However, in the main part of this strategy, Spoiler in each round chooses a vertex which is adjacent to one of the vertices chosen in the previous rounds. Therefore, Duplicator can not change a tree (i.e., this strategy is also winning in a game on forests). To make our proof self-contained, we sketch this strategy in the paragraph below.

If y_1 is not a central vertex of S^B , then as in the previous cases Spoiler has a winning strategy in next at most $1 + \lceil r(S^A) \rceil < k - 1$ rounds. Let y_1 be a central vertex of S^B . Suppose that the tree S^B is diverging.

Let us prove that for any $i \in \{1, \dots, k\}$ either Spoiler wins in the i -th round or chosen vertices x_1, x_2, \dots, x_i and y_1, y_2, \dots, y_i form simple paths, and the rooted trees $S_{x_1}^A(x_i)$ and $S_{y_1}^B(y_i)$ are not isomorphic. For $i = 1$, this is already proven. Assume that for $i \in \{1, \dots, k\}$ this statement is also true. Consider the round $i + 1$. If only one vertex of x_i, y_i is a leaf, Spoiler wins. Assume x_i, y_i are not leaves. As the rooted trees $S_{x_1}^A$ and $S_{y_1}^B$ are diverging, x_i -branches of $S_{x_1}^A$ are pairwise nonisomorphic (y_i -branches of $S_{y_1}^B$ are pairwise nonisomorphic as well), see [13], Lemma 6.2. Without loss of generality, there is a vertex x_{i+1} which is a child of x_i such that there is no y_i -branch isomorphic to $S_{x_{i+1}}^A$. Spoiler chooses x_{i+1} , and the statement is proven. Therefore, Spoiler wins in at most $r(S^A) + 1 = k - 1$ rounds. If the tree S^B is not diverging, consider a vertex t of S^B such that $S_{y_1}^B(t)$ is not a diverging tree but any its t -branch is diverging. Let two isomorphic t -branches be rooted at z_1, z_2 . Spoiler selects the path y_1, y_2, \dots, t, z_1 , Duplicator's response is x_1, x_2, \dots, x_i . If depths of $S_{y_1}^B(z_1)$ and $S_{x_1}^A(x_i)$ are distinct, Spoiler prolongs a path corresponding to a smaller depth and wins in at most $r(S^A) + 1 \leq k - 1$ rounds. If depths are equal and $S_{y_1}^B(z_1), S_{x_1}^A(x_i)$ are not isomorphic, then the winning strategy of Spoiler in $r(S^A) + 1$ rounds is given above. Finally, if $S_{y_1}^B(z_1), S_{x_1}^A(x_i)$ are isomorphic, in the round $i + 1$ Spoiler chooses z_2 and Duplicator responds with x_{i+1} , a child of x_{i-1} . As $S_{x_1}^A(x_{i-1})$ is a diverging tree, the trees $S_{x_1}^A(x_{i+1})$ and $S_{y_1}^B(z_2)$ are not isomorphic. Analogously, Spoiler wins in at most $r(S^A) + 2 \leq k$ rounds.

From the above,

$$\liminf_{n, m \rightarrow \infty} \mathbb{P}(\text{Spoiler has a winning strategy in } \text{EHR}(G(n, p(n)), G(m, p(m)), k)) \geq c(1 - c).$$

By Theorem 1, FO zero-one k -law fails when $2k - 5 \leq l \leq 2T(k - 4)$ and $k \geq 7$. Therefore, FO zero-one k -law fails for all $l \in \{9, \dots, 2k - 6\}$ as well. Theorem is proven.

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