Convergence Law for Random Graphs With Specified Degree Sequence

JAMES F. LYNCH
Clarkson University

The degree sequence of an n-vertex graph is $d_0, \ldots, d_{n-1}$, where each d_i is the number of vertices of degree i in the graph. A random graph with degree sequence $d_0, \ldots, d_{n-1}$ is a randomly selected member of the set of graphs on $\{1, \ldots, n\}$ with that degree sequence, all choices being equally likely. Let $\lambda_0, \lambda_1, \ldots$ be a sequence of nonnegative reals summing to 1. A class of finite graphs has degree sequences approximated by $\lambda_0, \lambda_1, \ldots$ if, for every i and n, the members of the class of size n have $\lambda_i n + o(n)$ vertices of degree i. Our main result is a convergence law for random graphs with degree sequences approximated by some sequence $\lambda_0, \lambda_1, \ldots$. With certain conditions on the sequence $\lambda_0, \lambda_1, \ldots$, the probability of any first-order sentence on random graphs of size n converges to a limit as n grows.

Categories and Subject Descriptors: F.4.1 [**Mathematical Logic and Formal Languages**]: Mathematical Logic—*Model theory*; G.2.1 [**Discrete Mathematics**]: Combinatorics—*Counting problems*; *generating functions*; G.2.2 [**Discrete Mathematics**]: Graph Theory

General Terms: Theory

Additional Key Words and Phrases: Finite model theory, random graphs

1. INTRODUCTION

Random graphs have been used as models of networks in diverse areas of science, engineering, and sociology. These include ecological food webs [Cohen et al. 1990; Williams and Martinez 2000], epidemiology [Anderson and May 1995; Ball et al. 1997; Keeling 1999; Kretschmar and Morris 1996; Sattenspiel and Simon 1988], metabolic pathways [Bhalla and Iyengar 1999; Bower and Bolouri 2001; Derrida and Pomeau 1986; Kauffman 1984, 1993; Lynch 2002], electric power grids [Kosterev et al. 1999], telephone call networks [Abello et al. 1998], networks of social contacts and scientific collaboration [Glance and Huberman 1993; Granovetter 1978; Newman 2001; Valente 1996], and, of particular interest to computer science, the internet [Adamic and Huberman 1999; Barabási et al. 1999; Broder et al. 2000; Cohen et al. 2000;

Author's address: Department of Mathematics and Computer Science, Box 5815, Clarkson University, 8 Clarkson Avenue, Potsdam, NY 13699-5815.

Faloutsos et al. 1999; Kumar et al. 2000]. Many additional citations to these topics may be found in Newman et al. [2001] and the review article [Strogatz 2001].

A common feature of all these examples is that they evolve through unpredictable processes of creation and deletion of nodes and their connections. In some cases, such as the Internet or the spread of infectious diseases, the time scale of these changes is rather rapid. In others, such as the growth of a power grid, it is measured in years rather than days, and it can even be many orders of magnitude slower, as in the case of metabolic pathways that evolved over millions of years.

The theory of random graphs, initiated by Erdős and Rényi [1959, 1960] and Gilbert [1961], is now a well-developed branch of combinatorics. Many of its results pertain to the statistical behavior of properties such as path lengths, cycle sizes, and component sizes, that are of interest to practitioners in the fields mentioned above. Most of these theoretical results are based on the independent edge model. Here, the number of vertices n of the graph and an edge probability p are specified. Then, independently for each pair of vertices, there is an edge between them with probability p. In the other widely studied random graph model, the number of vertices and number of edges M of the graph are specified, and all such graphs are equally likely. This was actually the type studied in the original papers of Erdős and Rényi [1959, 1960]. These two models are essentially the same when M is close to $pn(n-1)/2$, the expected number of edges in the first type of random graph, and since the first type is generally more tractable, researchers have focused on it. (See Bollobás [1985] for a precise statement relating these two types of random graphs.)

There is, however, a growing awareness of fundamental drawbacks to the independent edge model. For example, three groups of researchers [Barabási et al. 1999; Broder et al. 2000; Faloutsos et al. 1999] have independently concluded, based on empirical studies, that the internet does not behave like this type of random graph. The degree distribution in an independent edge probability random graph is Poisson: the probability that a vertex has degree d is asymptotic to

$$\frac{(pn)^d \exp(-pn)}{d!}.$$

But the degree distribution of nodes in the internet appears to follow a power law: the probability that a vertex has degree d is proportional to d^{-c} for some $c > 0$. (Of course, this is only an asymptotic approximation. Also, the constant of proportionality and the exponent c may depend on n.) Newman et al. [2001] and Strogatz [2001] give other examples of networks whose degree distributions seem to obey power laws. The distinction between the two types of random graphs is not trivial. Poisson distributions decay much more rapidly than power laws when p is small, and this leads to very different structural properties. For historical reasons, networks satisfying power law degree distributions are often called scale-free networks.

Realistic models of evolving networks are important for developing better simulations and more relevant theories. These considerations have stimulated research into new classes of random graphs. Aiello et al. [2001] showed how the size of the largest component in a power law graph depends on the exponent c of the power law distribution d^{-c}. In Aiello et al. [2002], the same authors proved that several classes of randomly evolving graphs satisfy power laws.

A natural generalization of power law distributions is to take an arbitrary sequence $d_0, \ldots, d_{n-1}$ of natural numbers whose sum is n, and consider the probability space of all graphs with d_i vertices of degree i, for $i = 0, \ldots, n-1$, where all such graphs are equally likely. Łuczak [1992] and Molloy and Reed [1995, 1998] proved results relating the size of the largest component to the degree distribution in such graphs.

To our knowledge, the logic of random graphs with specified degree distributions has not been investigated. In this article, we study them from the viewpoint of first-order logic. With suitable uniformity conditions imposed on the degree sequences as n gets large, we show that every first-order sentence about graphs has probability converging to a limit. We cannot claim any immediate applications, but one possibility might be to randomly evolving databases that satisfy a power law. Our methods could help to characterize the probability of queries on such databases, and lead to fast average time algorithms for query evaluation, as developed for other classes of random graphs [Abiteboul et al. 1992; Lifschitz and Vianu 1998]. Formal analysis of models of complex systems such as food webs, biochemical networks, and the internet is also a potential application.

2. DEFINITIONS

By graph, we mean a structure $\langle V, E\rangle$ where E is a symmetric, irreflexive binary relation on V. A graph on $n \in \omega$ is a graph where $V = \{1, 2, \ldots, n\}$. The first-order logic of graphs is defined in the usual way, where we use E to represent both a binary relation symbol and its interpretation in graphs. The depth of a formula is its maximum nesting of quantifiers. That is, atomic formulas have depth 0; if formulas α and β have depths k_1 and k_2 respectively, then the depths of $\neg\alpha$, $\alpha \vee \beta$, and $\exists x\alpha$ are k_1, $\max(k_1, k_2)$, and $k_1 + 1$ respectively. For any $k \in \omega$, two graphs G and H are k-equivalent, written $G \equiv_k H$, if they agree on every sentence of depth k.

The degree sequence of a graph $\langle V, E\rangle$ is the sequence $d_0, d_1, \ldots$ where $|\{ v \in V : \deg(v) = i \}| = d_i$ for $i = 0, 1, \ldots$. Of course, for a finite graph on n, $d_i = 0$ for $i \geq n$, and we can truncate the sequence to its first n terms. The sequence $d_0, \ldots, d_{n-1}$ is *feasible* if there exists a graph on n with that degree sequence. Given a feasible degree sequence of length n, a random graph with that specified degree sequence is a uniformly random member of the collection of graphs on n with that degree sequence.

Since we are interested in arbitrarily large finite graphs with a given degree sequence, we will use a sequence of degree sequences $\mathcal{D} = (d_0(n), d_1(n), \cdots : n \in \omega)$ where $d_0(n), \ldots, d_{n-1}(n)$ is feasible for each $n \in \omega$. Note that this implies $d_i(n) = 0$ for $i \geq n$ and $\sum_{i=0}^{n-1} d_i(n) = n$. $\mathcal{D}$ is said to be an asymptotic degree sequence. A random graph with specified asymptotic degree sequence $\mathcal{D}$ is a random graph on n with degree sequence $d_0(n), \ldots, d_{n-1}(n)$ for some $n \in \omega$. If

$\mathcal{P}$ is a property of graphs, $\text{pr}(\mathcal{P}, n, \mathcal{D})$ is the probability that a random graph on n with degree sequence $d_0(n), \ldots, d_{n-1}(n)$ has $\mathcal{P}$. When $\mathcal{D}$ is understood, we just write $\text{pr}(\mathcal{P}, n)$. If $\mathcal{P}$ is definable by a first-order sentence σ, we put $\text{pr}(\sigma, n)$.

Since we will be analyzing the asymptotic probabilities of large random graphs with specified degree sequences, we need some uniformity conditions on asymptotic degree sequences. We will also need some additional restrictions required by the combinatorial methods we will use. The following conditions are sufficient for our purposes.

(1) For each $i \in \omega$, there exists $\lambda_i \in [0, 1]$ such that
   (a) $\lim_{n\to\infty} d_i(n)/n = \lambda_i$ uniformly in i. That is, the proportion of vertices of degree i is approximately λ_i.
   (b) When $\lambda_i = 0$, $d_i(n) = 0$ for all n.
(2) (a) $\sum_{i=0}^{\infty} \lambda_i = 1$.
   (b) $\sum_{i=1}^{\infty} i\lambda_i = \Lambda$ is finite.
(3) $\lim_{n\to\infty} \sum_{i=1}^{n-1} id_i(n)/n = \Lambda$.
(4) There exists a constant $a < 1/4$ such that for all n and $i > n^a$, $d_i(n) = 0$.

These or similar conditions appear to be common assumptions for this subject. For example, Molloy and Reed [1995, 1998] refer to asymptotic degree sequences satisfying condition (1)(a) as *smooth*, those satisfying (2)(b) as *sparse*, and those satisfying a uniformity condition similar to (3), as *well-behaved*. Condition (1)(b) prevents cases such as $d_1(n) = 2$ for even n and $d_1(n) = 0$ for odd n, which clearly violate any convergence law. Weaker variations of this condition would suffice, but they would require more complicated versions of Condition (1)(a). The fourth is a technical condition known as a cutoff. It will allow us to apply a theorem of McKay [1985] (described below). Other papers on random graphs with specified degree distributions such as Aiello et al. [2001], Molloy and Reed [1995, 1998], and Newman et al. [2001] also use cutoffs of the form $n^{O(1)}$. In fact, it is claimed in Newman et al. [2001] that many real-world networks exhibit such cutoffs.

Conditions (1)–(4) are satisfied by degree sequences of regular graphs, graphs of bounded degree, the graphs in Molloy and Reed [1995, 1998] in the critical region where the giant component appears, and graphs having degree distributions approximated by power laws with exponent greater than 2. In particular, the empirical studies of the internet indicate that its degree distribution obeys a power law with exponent between 2 and 3. From a theoretical and possibly practical standpoint, it would be interesting to analyze the logical properties of random graphs when the degree sequence is not sparse.

Our convergence law also extends the convergence law for classical random graphs with edge probability c/n, c constant [Lynch 1992] or with specified number of edges $cn/2$. Without going into details, this is because the degree sequences of these random graphs are approximated with sufficient accuracy by the sequence $\lambda_0, \lambda_1, \ldots$ where $\lambda_i = c^i e^{-c}/i!$. Not surprisingly, there are similarities between the proofs of the main theorems of this paper and [Lynch 1992].

A final condition on our degree sequences is

(5) $\lambda_0 = 0$.

This will simplify our proofs, but is not a loss of generality, as shown by the following lemma.

LEMMA 2.1. *For any asymptotic degree sequence $\mathcal{D}$ and any sentence σ, there exist an asymptotic degree sequence $\mathcal{D}'$ and a sentence σ' such that $d'_0(n) = 0$, and for sufficiently large n,* $\mathrm{pr}(\sigma, n, \mathcal{D}) = \mathrm{pr}(\sigma', n - d_0(n), \mathcal{D}')$.

PROOF. For every $n \in \omega$, let $n' = n - d_0(n)$ and define

$$d'_0(n') = 0 \quad \text{and}$$
$$d'_i(n') = d_i(n) \quad \text{for} \quad i > 0.$$

Let $I = \{n' : n \in \omega\}$. We can assume I is infinite, because if not, then by condition 1. (b), $d_0(n) = n$ for sufficiently large n, and our convergence law (Theorem 3.1 below) is trivial. We define the asymptotic degree sequence $\mathcal{D}' = (d'_0(n), d'_1(n), \cdots : n \in I)$ on I.

Let k be the depth of σ. For any n and graph $G = \langle V, E\rangle$ where $V = \{1, 2, \ldots, n\}$, let $G' = \langle V', E \restriction V'\rangle$ where $V' = \{i \in V : \deg(i) \neq 0\}$. By an easy application of Gaifman's characterization of first-order definable properties [Gaifman 1982] or the Ehrenfeucht game [Ehrenfeucht 1961], there is a sentence σ' (of depth k) such that for all sufficiently large n, $G \models \sigma$ if and only if $G' \models \sigma'$. Therefore, $\mathrm{pr}(\sigma, n, \mathcal{D}) = \mathrm{pr}(\sigma', n', \mathcal{D}')$. □

If $\mathcal{D}$ satisfies all the conditions (1)–(4) above, then so does $\mathcal{D}'$ on I. Our convergence law is stated for asymptotic degree sequences defined on all $n \in \omega$, but it extends immediately to any degree sequence defined on infinite subsequences of ω. That is,

$$\lim_{\substack{n\to\infty \\ n\in I}} \mathrm{pr}(\sigma', n, \mathcal{D}')$$

exists. Therefore, so does $\lim_{n\to\infty} \mathrm{pr}(\sigma, n, \mathcal{D})$.

3. MAIN THEOREM

From this point on, $\mathcal{D}$ is some fixed asymptotic degree sequence satisfying conditions (1)–(5) above.

THEOREM 3.1. *For any first-order sentence σ about graphs,* $\lim_{n\to\infty} \mathrm{pr}(\sigma, n)$ *exists.*

3.1 Configurations

Although there is ample evidence for the existence of scale-free networks, they are not easy to generate directly. However, another closely related type of structure, known as a configuration, can be randomly generated by a very simple procedure, and we can translate results about random configurations to results about random graphs with specified degree sequence.

Definition 3.2

(1) A configuration is a structure $\langle U, \equiv, M\rangle$ where $\equiv$ is an equivalence relation on U, and $\langle U, M\rangle$ is a matching, that is, a graph with node set U and edge set M where each node has degree one. We refer to the elements of a configuration as nodes and elements of a graph as vertices, and we use infix notation, for example, xMy, to indicate the presence of an edge between nodes x and y.
(2) The quotient of $\langle U, \equiv, M\rangle$ is the multigraph $\langle U/\equiv, M/\equiv\rangle$. That is, $U/\equiv$ is the set of equivalence classes of $\equiv$, and, letting $[x]$ denote the equivalence class of $x \in U$, for any x, $y \in U$, there are distinct edges between $[x]$ and $[y]$ for every pair $x' \equiv x$ and $y' \equiv y$ such that $x'My'$.
(3) A property of graphs is said to hold for a configuration if the quotient of the configuration has that property.

Just as we fixed the universes of our random graphs to be some $n \in \omega$, we need to fix the universes of our random configurations. A configuration on n will have $n \equiv$ classes labeled $1, 2, \ldots, n$ and $m = \sum_{i=1}^{n-1} i d_i(n)$ nodes labeled $1, 2, \ldots, m$. A random configuration on n is a uniformly random member of the set of configurations on n that have $d_i(n)$ classes of size i for $i = 1, \ldots, n-1$. A random configuration on n is generated by randomly choosing a partitioning of $\{1, \ldots, m\}$ as just described, and then randomly assigning a matching on the nodes. If $\mathcal{P}$ is a property of configurations, we put $\mathrm{cpr}(\mathcal{P}, n)$ for the probability that a random configuration on n satisfies $\mathcal{P}$.

Our method of constructing a random graph on n is to first construct a random configuration on n and then take its quotient. Note that the resulting structure need not be a graph. It may contain self-loops and multiple edges, that is, it is a multigraph. However, any two graphs with the same degree sequence are quotients of the same number of configurations. Thus, probabilities about random graphs can be defined in terms of probabilities about random configurations. Specifically, let $\mathcal{P}$ be any property of graphs and let $\mathcal{Q}$ be the property that holds for configurations whose quotient is a graph. Then

$$\mathrm{pr}(\mathcal{P}, n) = \frac{\mathrm{cpr}(\mathcal{P} \wedge \mathcal{Q}, n)}{\mathrm{cpr}(\mathcal{Q}, n)}.$$

(The graph property $\mathcal{P}$ holds for a configuration if it holds for the quotient of the configuration.) Further, $\mathcal{Q}$ is first-order definable, and as shown by McKay [1985], $\mathrm{cpr}(\mathcal{Q}, n)$ is asymptotic to a positive constant as $n \to \infty$. This will enable us to derive convergence laws for graphs from convergence laws for configurations. Thus, our main theorem is a corollary of the following:

THEOREM 3.3. *For any first-order sentence σ about configurations, $\lim_{n\to\infty} \mathrm{cpr}(\sigma, n)$ exists.*

3.2 Neighborhoods

Our proof that $\mathrm{cpr}(\sigma, n)$ converges for all σ actually establishes a stronger result: for every $k \in \omega$, we define an equivalence relation $\sim_k$ that refines $\equiv_k$ and show that the probability that a random configuration belongs to a given $\sim_k$

class converges. The relation $\sim_k$ is determined by local properties of the configuration, and our proof is essentially a logical and combinatorial analysis of these properties. By a local property we mean a property of a neighborhood of a given size around a given element of the model.

As shown by Gaifman [1982], local properties determine the first-order properties of a model. Specifically, the type and number of disjoint bounded neighborhoods around elements of the model determine whether a given sentence is true. Another well-known method for analyzing the expressive power of first-order logic is the Ehrenfeucht game [Ehrenfeucht 1961]. Our approach is a hybrid of these two methods. Given $k \in \omega$, we first define $\sim_k$ on neighborhoods that have at most one cycle. Thus, they can be characterized as acyclic or unicyclic. We use the Ehrenfeucht game to show that any two such neighborhoods in the same $\sim_k$ class are in the same $\equiv_k$ class. We then extend $\sim_k$ to configurations that have only acyclic or unicyclic small neighborhoods. Two such configurations are $\sim_k$ if they have the same number (up to k) of such neighborhoods in each $\sim_k$ class. We show that any two such configurations in the same $\sim_k$ class are in the same $\equiv_k$ class.

The rest of the proof is combinatorial. We show that almost all configurations have no small neighborhoods with more than one cycle. Further, almost all configurations have more than k (in fact an unbounded number of) neighborhoods in each acyclic $\sim_k$ class, and for each unicyclic $\sim_k$ class of neighborhoods and natural number j, the probability that a configuration has exactly j neighborhoods in that class approaches a limit.

Gaifman's methods are based on a combinatorial metric, which in the case of graphs is the usual distance function on pairs of vertices. We will use δ to denote Gaifman's metric on the nodes of a configuration.

Definition 3.4. Let $\langle U, \equiv, M\rangle$ be a configuration and $x, y \in U$.

(1) A walk of length d from x to y is a sequence $x_0, x_1, \dots, x_d$ where $x = x_0$, $y = x_d$, and $x_i \equiv x_{i+1}$ or $x_i M x_{i+1}$ for $0 \le i < d$. The walk is closed if $x = y$.
(2) $\langle U, \equiv, M\rangle$ is connected if there is a walk between every pair of nodes in U.
(3) $\delta(x, y)$ is the length of the shortest walk from x to y. For two subsets X and Y of U, $\delta(X, Y) = \min(\delta(x, y) : x \in X, y \in Y)$, and $\delta(X, x) = \delta(X, \{x\})$.
(4) The subconfiguration of $\langle U, \equiv, M\rangle$ induced by $W \subseteq U$ is $\langle W, \equiv\restriction W, M \restriction W\rangle$. We will sometimes abbreviate it as $\langle W, \equiv, M\rangle$. Note that it may be a partial configuration, that is, it may have nodes that are not matched with any node.
(5) A path is a walk such that all $x_0, \dots, x_d$ are distinct, and in the subconfiguration induced by $\{x_0, \dots, x_d\}$, $|[x_i]| \le 2$ for $0 \le i \le d$.
(6) A cycle is a closed walk where all $x_0, \dots, x_{d-1}$ are distinct and, in the subconfiguration induced by $\{x_0, \dots, x_{d-1}\}$, $|[x_i]| = 2$ for $0 \le i < d$. Also, when $d = 2$, $x_0 M x_1$.
(7) A (partial) configuration is a tree if it is connected and has no cycles.
(8) A (partial) configuration is unicyclic if it has exactly one cycle.
(9) For any $R \subseteq U$ and $r \ge 0$, the shell of radius r around R, $S(R; r)$, is $\{y \in U : \delta(R, y) = r\}$.

(10) The neighborhood of radius r around R, $N(R;r)$, is $\bigcup_{t\leq r} S(R;t)$.

The Ehrenfeucht game is a two-player game of perfect information played for some number k of rounds on two structures of the same type, in our case two (partial) configurations $\mathcal{U}^i = \langle U^i, \equiv^i, M^i\rangle$ for $i = 1, 2$. In each round j, a new constant, a^i_j, is added to each $\mathcal{U}^i$. We assume that a^1_j and a^2_j interpret the same constant symbol a_j. Player I begins each round j by choosing i and a^i_j. Player II responds by choosing a^{3-i}_j. A popular way of describing the game is to imagine that there are two sets of pebbles $\{a^i_1, \ldots, a^i_k\}$, $i = 1, 2$, and at each round j Player I places the pebble a^i_j on some node in U^i, and then Player II responds in kind on the other configuration.

After k rounds, Player II wins if and only if the two subconfigurations induced by $\{a^i_1, \ldots, a^i_k\}$ with constants $a^i_1, \ldots, a^i_k$, $i = 1, 2$, are isomorphic. That is, for any $h, j \in \{1, \ldots, k\}$,

$$a^1_h = a^1_j \quad \text{if and only if} \quad a^2_h = a^2_j \tag{1}$$

$$a^1_h \equiv^1 a^1_j \quad \text{if and only if} \quad a^2_h \equiv^2 a^2_j \tag{2}$$

$$a^1_h M^1 a^1_j \quad \text{if and only if} \quad a^2_h M^2 a^2_j \tag{3}$$

The Ehrenfeucht game provides a very useful method for showing that two structures are $\equiv_k$. We say that Player II has a winning strategy for the k-round game if, no matter how Player I moves, Player II can always respond with a move that can lead to a win.

THEOREM 3.5 [EHRENFEUCHT 1961]. *Player II has a winning strategy for the k-round game on $\mathcal{U}^1$ and $\mathcal{U}^2$ if and only if $\mathcal{U}^1 \equiv_k \mathcal{U}^2$.*

In describing our winning strategy, it will be helpful to extend our models with an additional unary relation R. That is, we consider models $\langle U, \equiv, M, R\rangle$ where $\langle U, \equiv, M\rangle$ is a configuration and $R \subseteq U$. We refer to this as a rooted configuration and the elements in R as the roots. We put $\mathcal{N}(R;r)$ for the subconfiguration induced by $N(R;r)$ with roots R. $\langle U, \equiv, M, R\rangle$ is a rooted tree if $\langle U, \equiv, M\rangle$ is a tree and $|R| = 1$. A rooted forest is the disjoint union of a collection of rooted trees. The depth of a node in a rooted forest is the length of the shortest path from the node to R. The depth of a rooted forest is the maximum depth of any of its nodes. If $\mathcal{F}$ is a rooted forest and $r \geq 0$, we put $\mathcal{F} \restriction r$ for the subconfiguration of $\mathcal{F}$ induced by the set of nodes of depth at most r. Note that $\mathcal{F} \restriction r$ is also a rooted forest.

Definition 3.6. Fixing k, we define the equivalence relation $\sim_k$ on rooted forests by induction on their depth. At each stage r of the induction, we first define $\sim_k$ on rooted trees of depth r in terms of $\sim_k$ on rooted forests of depth at most $r - 1$. We then define $\sim_k$ on rooted forests of depth r in terms of $\sim_k$ on rooted trees of depth at most r.

For convenience, we define a rooted forest of depth -1 to be the null forest, that is, the forest with no trees. There is only one of these, so trivially all rooted forests of depth -1 are $\sim_k$.

Assume $\sim_k$ has been defined on rooted forests of depth at most $r-1, r \geq 0$. For $i = 1, 2$, let $\mathcal{U}^i = \langle U^i, \equiv^i, M^i, \{x_i\}\rangle$ be a rooted tree of depth r. For $y \in S(x_i; 1)$, let $\mathcal{V}^i_y$ be the subtree of $\mathcal{U}^i$ induced by the set of nodes in U^i that are descendants of y, with root y, and let $\mathcal{F}^i$ be the forest of such trees where $\neg x_i M^i y$. If there exists $y \in U^i$ such that $x_i M^i y$ (it must be unique), let $\mathcal{T}^i = \mathcal{V}^i_y$; otherwise let $\mathcal{T}^i$ be the null tree. Then $\mathcal{U}_1 \sim_k \mathcal{U}_2$ if

$$\mathcal{T}_1 \sim_k \mathcal{T}_2 \quad \text{and} \quad \mathcal{F}_1 \sim_k \mathcal{F}_2.$$

Now let $\mathcal{U}^i, i = 1, 2$, be a rooted forest of depth r. For each $\sim_k$ class τ of rooted trees of depth at most r, let $t_{i,\tau}$ be the number of rooted trees of type τ in $\mathcal{U}^i$. Then $\mathcal{U}^1 \sim_k \mathcal{U}^2$ if and only if for all τ

$$t_{1,\tau} = t_{2,\tau} \quad \text{or} \quad t_{1,\tau}, t_{2,\tau} \geq k.$$

LEMMA 3.7. *If $\mathcal{U}^1$ and $\mathcal{U}^2$ are rooted forests such that $\mathcal{U}^1 \sim_k \mathcal{U}^2$ then $\mathcal{U}^1$ and $\mathcal{U}^2$ have the same depth, and for any s and $j \leq k$, $\mathcal{U}^1 \restriction s \sim_j \mathcal{U}^2 \restriction s$*

PROOF. This follows by induction on the maximum depth of $\mathcal{U}^1$ and $\mathcal{U}^2$. □

We next extend $\sim_k$ to a larger class of rooted partial configurations that includes all the bounded neighborhoods found in almost all random configurations. A centered configuration is a partial configuration with roots R where every node has exactly one path to $\bigcup_{x\in R}[x]$. This is a path beginning at the node and ending at some node in $\bigcup_{x\in R}[x]$, and no other node in the path is in $\bigcup_{x\in R}[x]$. A consequence is that all cycles are contained in $\bigcup_{x\in R}[x]$. A centered configuration can be decomposed as follows.

Definition 3.8. Let $\mathcal{U} = \langle U, \equiv, M, R\rangle$ be a centered configuration and $\mathcal{C} = \mathcal{U} \restriction R$. For every $x \in R$ let $\mathcal{V}_x$ be the tree induced by the set V_x of nodes whose path to R ends in $[x]$, with root x, and all edges in $M \restriction [x]$ deleted. Then $V_x \cap R = [x] \cap R$ and $V_x \cap V_y = \emptyset$ for all $x \not\equiv y$. $(\mathcal{C}, \mathcal{V}_x : x \in R)$ is the canonical decomposition of $\mathcal{U}$. The depth of $\mathcal{U}$ is the maximum depth of any $\mathcal{V}_x$.

Definition 3.9. For $i = 1, 2$, let $\mathcal{U}_i$ be a connected centered configuration with canonical decomposition $(\mathcal{C}^i, \mathcal{V}^i_x : x \in R^i)$. Then $\mathcal{U}^1 \sim_k \mathcal{U}^2$ if there is an isomorphism f from $\mathcal{C}^1$ to $\mathcal{C}^2$ such that for all $x \in R^1$,

$$\mathcal{V}^1_x \sim_k \mathcal{V}^2_{f(x)}.$$

If $\mathcal{U}^1$ and $\mathcal{U}^2$ are components in two configurations on which the Ehrenfeucht game is being played, then we also require that f preserve the pebbled nodes. That is, a^1_j is in $\mathcal{U}^1$ if and only if a^2_j is in $\mathcal{U}^2$, and if a^1_j is in $\mathcal{U}^1$ then $f(a^1_j) = a^2_j$.

Extending $\sim_k$ to arbitrary centered configurations $\mathcal{U}^i$, $i = 1, 2$, for each $\sim_k$ class τ of connected centered configurations, let $t_{i,\tau}$ be the number of components of type τ in $\mathcal{U}^i$. Then $\mathcal{U}^1 \sim_k \mathcal{U}^2$ if and only if for all such τ

$$t_{1,\tau} = t_{2,\tau} \quad \text{or} \quad t_{1,\tau}, t_{2,\tau} \geq k.$$

Finally we extend $\sim_k$ to a class of configurations that includes almost all configurations. We say that a configuration $\mathcal{U} = \langle U, \equiv, M\rangle$ is k-rich if, for every

$r \leq 3^{k-1}$, every rooted tree $\mathcal{T}$ that occurs as an r-neighborhood in some configuration, and every set $S \subseteq U$ of $k-1$ nodes, there is some $x \in U$ such that $\delta(S,x) > 2 \cdot 3^{k-1}$, $\delta(C,x) > 2 \cdot 3^{k-1}$ for every cycle $C \subseteq U$ of size at most $2 \cdot 3^{k-1}$, and $\mathcal{N}(x;r) \sim_k \mathcal{T}$. We say that $\mathcal{U}$ is k-simple if it does not have two cycles of size at most 3^k within $2 \cdot 3^{k-1}$ of each other.

Definition 3.10. For $i = 1,2$ let $\mathcal{U}^i$ be a k-rich and k-simple configuration. For each class τ of unicyclic centered configurations of cycle size at most $2 \cdot 3^{k-1}$ and depth at most 3^{k-1}, let $t_{i,\tau}$ be the number of cycles C in $\mathcal{U}^i$ such that $\mathcal{N}(C;3^{k-1}) \in \tau$. Then $\mathcal{U}_1 \sim_k \mathcal{U}_2$ if and only if for all such τ

$$t_{1,\tau} = t_{2.\tau} \quad \text{or} \quad t_{1,\tau}, t_{2,\tau} \geq k.$$

We next show that $\sim_k$ refines $\equiv_k$. The proof is an application of Ehrenfeucht's theorem (3.5). Just as we defined $\sim_k$ on rooted forests and then generalized it to centered configurations and finally to k-rich and k-simple configurations, we follow the same pattern in describing a winning strategy for Player II.

Let $\mathcal{U}^i = \langle U^i, \equiv^i, M^i, R^i \rangle$, $i = 1,2$, be centered configurations. For $j = 0,\dots,k$ let

$$\mathcal{U}^i_j = \langle U^i, \equiv^i, M^i, R^i_j, a^i_1, \dots, a^i_j \rangle \quad \text{where}$$

$$R^i_j = R^i \cup \left\{ x \in U_i | x \text{ is on the path from some } a^i_h \text{ to } \bigcup_{x \in R^i} [x] \right\}.$$

Then $\mathcal{U}^1$ and $\mathcal{U}^2$ are centered configurations. Player II's strategy is to maintain

$$\mathcal{U}^1_j \sim_{k-j} \mathcal{U}^2_j \tag{4}$$

after each round j. $\mathcal{U}^1 \sim_k \mathcal{U}^2$ is equivalent to (4) for $j = 0$ and (4) for $j = k$ implies Player II has won. Therefore we will be done if we can show that if (4) holds for j then Player II can move so that (4) holds for $j+1$.

LEMMA 3.11. *Let $\mathcal{U}^1 \sim_k \mathcal{U}^2$ be rooted forests and $0 \leq j < k$. If* (4) *holds after round j of the Ehrenfeucht game, then Player II can move so that* (4) *holds after round $j+1$.*

PROOF. Following the same outline as in Definition 3.6, we use induction on r, the depth of $\mathcal{U}^1$ and $\mathcal{U}^2$. The lemma holds trivially for rooted forests of depth -1.

Now assume the lemma holds for pairs of rooted forests of depth less than r, $r \geq 0$, and let $\mathcal{U}^1$ and $\mathcal{U}^2$ be rooted trees of depth r. We use the notation of Definition 3.6. There are two cases, depending on Player I's choice for a^i_{j+1}:

(1) a^i_{j+1} is in $\mathcal{T}^i$. By Definition 3.6, $\mathcal{T}^1 \sim_k \mathcal{T}^2$, and by (4), $\mathcal{T}^1_j \sim_{k-j} \mathcal{T}^2_j$. Since $\mathcal{T}^1$ and $\mathcal{T}^2$ have depth less than r, by induction, Player II can choose a^{3-i}_{j+1} in $\mathcal{T}^{3-i}$ so that $\mathcal{T}^1_{j+1} \sim_{k-j-1} \mathcal{T}^2_{j+1}$. By Lemma 3.7, this implies that (4) holds after round $j+1$.
(2) a^i_{j+1} is in $\mathcal{F}^i$. The same strategy works here, with $\mathcal{T}^i$ replaced by $\mathcal{F}^i$.

Now let $\mathcal{U}^i$, $i = 1, 2$, be a rooted forest of depth r. A tree in $\mathcal{U}^i$ is active if it contains a pebbled node. There are two cases:

(1) a^i_{j+1} is in $\mathcal{T}^i$ for some inactive tree $\mathcal{T}^i$ of type τ. Since $t_{1,\tau} = t_{2,\tau}$ or $t_{1,\tau}, t_{2,\tau} \geq k$, there must be an inactive $\mathcal{T}^{3-i}$ of type τ in $\mathcal{U}^{3-i}$. Thus $\mathcal{T}^1 \sim_k \mathcal{T}^2$, and using the strategy for rooted trees, Player II can move so that $\mathcal{T}^1_1 \sim_{k-1} \mathcal{T}^2_1$. Therefore by Lemma 3.7, (4) holds after round $j + 1$.

(2) a^i_{j+1} is in $\mathcal{T}^i$ for some active tree $\mathcal{T}^i$. Let $\mathcal{T}^{3-i}$ be the corresponding active rooted tree in $\mathcal{U}^{3-i}$. Then since (4) holds, $\mathcal{T}^1_j \sim_{k-j} \mathcal{T}^2_j$, and using the strategy for rooted trees, Player II can move so that $\mathcal{T}^1_{j+1} \sim_{k-j-1} \mathcal{T}^2_{j+1}$, and again (4) holds after round $j + 1$. □

LEMMA 3.12. *For $i = 1, 2$, let $\mathcal{U}^i$ be a centered configuration with canonical decomposition $(\mathcal{C}^i, \mathcal{V}^i_x : x \in R^i)$. If $\mathcal{U}^1 \sim_k \mathcal{U}^2$ and (4) holds after round j of the Ehrenfeucht game, then Player II can move so that (4) holds after round $j + 1$.*

PROOF. Assuming first that $\mathcal{U}^1$ and $\mathcal{U}^2$ are connected, let $f : R^1 \to R^2$ be the isomorphism of Definition 3.9. If Player I chooses a node in $\mathcal{V}^i_x$, then Player II responds with a node in $\mathcal{V}^{3-i}_{f(x)}$, using the strategy of Lemma 3.11.

For arbitrary centered configurations, the strategy is similar to that for rooted forests. A component of $\mathcal{U}^i$ is active if it contains a pebbled node, and the two cases depend on whether or not Player I chooses a node in an active component. □

Lastly, we describe the winning strategy on k-rich and k-simple configurations $\mathcal{U}^1$ and $\mathcal{U}^2$. For $i = 1, 2$ let

$$A^i = \bigcup\{C \subseteq U^i : C \text{ is a cycle and } |C| \leq 2 \cdot 3^{k-1}\},$$

for $j = 0, \ldots, k$ let

$$U^i_j = N(\{a^i_1, \ldots, a^i_j\} \cup A^i; 3^{k-j}),$$
$$R^i_j = \{x \in U^i_j | x \text{ is on a path between some } a^i_g \text{ and } a^i_h \text{ or } a^i_g \text{ and some } y \in A^i\},$$
$$\mathcal{U}^i_j = \langle U^i_j, \equiv^i, M^i, R^i_j, a^i_1, \ldots, a^i_j \rangle.$$

Since $\mathcal{U}^i_j$ is k-simple, it is a centered configuration. Player II's strategy is to maintain condition (4), but now in the context of the redefined $\mathcal{U}^i_j$.

LEMMA 3.13. *Let $\mathcal{U}^1 \sim_k \mathcal{U}^2$ be k-rich and k-simple configurations and $0 \leq j < k$. If (4) holds after round j of the Ehrenfeucht game, then Player II can move so that (4) holds after round $j + 1$.*

PROOF. Again, there are several cases, depending on Player I's move:

(1) $N(a^i_{j+1}; 3^{k-j-1}) \not\subseteq U^i_j$. Then $N(a^i_{j+1}; 3^{k-j-1}) \cap N(\{a^i_1, \ldots, a^i_j\} \cup A^i; 3^{k-j-1}) = \emptyset$ and $\mathcal{N}(a^i_{j+1}; 3^{k-j-1})$ is acyclic. Let τ be the $\sim_k$ type of $\mathcal{N}(a^i_{j+1}; 3^{k-j-1})$. Since $\mathcal{U}^{3-i}$ is k-rich, there exists $a^{3-i}_{j+1} \in U^{3-i}$ such that $\mathcal{N}(a^{3-i}_{j+1}; 3^{k-j-1})$ is of type τ and $N(a^{3-i}_{j+1}; 3^{k-j-1}) \cap N(\{a^{3-i}_1, \ldots, a^{3-i}_j\} \cup A^{3-i}; 3^{k-j-1}) = \emptyset$. Therefore by Lemma 3.7, (4) holds after round $j + 1$.

(2) $N(a^i_{j+1}; 3^{k-j-1}) \subseteq N(C^i; 3^{k-j})$ where C^i is a cycle in an inactive component of $\mathcal{U}^i_j$. Then $N(\{a^i_1, \dots, a^i_j\}; 3^{k-j-1}) \cap N(C^i; 3^{k-j-1}) = \emptyset$. Since $\mathcal{U}^1 \sim_k \mathcal{U}^2$ are k-simple and $\mathcal{U}^1_j \sim_{k-j} \mathcal{U}^2_j$, there exists a cycle C^{3-i} in $\mathcal{U}^{3-i}$ such that $\mathcal{N}(C^1; 3^{k-j-1}) \sim_k \mathcal{N}(C^2; 3^{k-j-1})$ and $N(\{a^{3-i}_1, \dots, a^{3-i}_j\}; 3^{k-j-1}) \cap N(C^{3-i}; 3^{k-j-1}) = \emptyset$. By Lemma 3.12, there exists $a^{3-i}_{j+1} \in N(C^{3-i}; 3^{k-j-1})$ such that $\langle N(C^1; 3^{k-j-1}), \equiv^1, M^1, R^1\rangle \sim_{k-1} \langle N(C^2; 3^{k-j-1}), \equiv^2, M^2, R^2\rangle$ where $R^i = C^i \cup P^i$ and P^i is the path from a^i_{j+1} to $\bigcup_{x \in C^i}[x]$. By Lemma 3.7 (4) holds after round $j+1$.

(3) $N(a^i_{j+1}; 3^{k-j-1}) \subseteq W^i$ for some active component W^i of $\mathcal{U}^i$. Let W^{3-i} be the corresponding active component of $\mathcal{U}^{3-i}$. For $i = 1, 2$ let $\overline{b^i}$ be the subsequence of $a^i_1, \dots, a^i_j$ consisting of those $a^i_h \in W^i$, and let $\mathcal{W}^i = \langle W^i, \equiv^i, M^i, R^i_j, \overline{b^i}\rangle$. By (4), $\mathcal{W}^1 \sim_{k-j} W^2$. Therefore by Lemma 3.12, Player II can choose $a^{3-i}_{j+1} \in W^{3-i}$ so that $\langle W^1, \equiv^1, M^1, R^1 \cup P^1, \overline{b^1}, a^1_{j+1}\rangle \sim_{k-j-1} \langle W^2, \equiv^2, M^2, R^2 \cup P^2, \overline{b^2}, a^2_{j+1}\rangle$ where P^i is the path from a^i_{j+1} to $\bigcup_{x \in R^i}[x]$. Therefore by Lemma 3.7, (4) holds after round $j+1$. $\square$

3.3 Tree Processes

To summarize the results of the previous subsection, the $\sim_k$ type of a k-rich and k-simple configuration is determined by the numbers of neighborhoods in each small unicyclic $\sim_k$ type. The rest of the proof of Theorem 3.3 is an analysis of the probability that a random configuration has specified numbers of small acyclic and unicyclic neighborhoods. We start by analyzing the probability that the neighborhood of a given set of nodes in a random configuration is isomorphic to a given centered configuration. This will be facilitated by applying basic results about branching processes. We will summarize the results that we need. For more information on branching processes, see Harris [1989].

A branching process abstracts a certain type of population growth process that we will call a tree process. The process may continue indefinitely, although in our applications we are interested in only a finite number of generations in such a process. The tree process generates a type of random rooted tree or forest of rooted trees. We will refer to the elements in such a structure as individuals. Generation r consists of all individuals of depth r. Generation 0 is fixed. The number of children of any individual is independent of the number of children of any other individual, but the probability of having a certain number of children is the same for all individuals. Thus the probability space of a tree process is determined by a probability function $p\colon \mathbb{N} \mapsto [0, 1]$ where $p(j)$ is the probability that an individual has j children.

The probability measure on this space will be denoted by tpr. In describing events in this space, $\mathcal{P}$ will denote a tree process. For $r \geq 0$, $\mathcal{P} \restriction r$ will be the finite tree or forest which is $\mathcal{P}$ restricted to its first r generations.

The branching process determined by the tree process is the sequence $Z_0, Z_1, \dots$ where Z_r is the size of generation r. The basic result about branching processes that we will need is the following. Let $\mu = \sum_{j=1}^{\infty} j p(j)$ be the expected number of offspring of an individual. Then

$$\mathbf{E}(Z_r) = Z_0 \mu^r. \tag{5}$$

The next series of definitions and lemmas will be used to describe the relationship between exposure of neighborhoods and tree processes. Recall that $m = \sum_{i=1}^{n-1} id_i(n) = n(\Lambda + o(1))$. Let $\mathcal{U} = \langle U, \equiv, M\rangle$ be a random configuration on n, $r \geq 0$, $R \subseteq U$, $\mathcal{R} = \langle R, \equiv, M\rangle$, and $\mathcal{N} = \mathcal{N}(R; r)$. Let $\mathcal{W} = \langle W, \equiv', M', S\rangle$ be a centered configuration with canonical decomposition $(\mathcal{S}, \mathcal{W}_x : x \in S)$. In the cases we will be considering, $\mathcal{W}$ is either a rooted forest or unicyclic with cycle S. If $f : S \underset{\text{onto}}{\overset{\text{1-1}}{\to}} R$ we put $\mathcal{S} \overset{f}{\cong} \mathcal{R}$ if f is an isomorphism from $\mathcal{S}$ to $\mathcal{R}$, and $\overline{\mathcal{W}} \overset{f}{\to} \overline{\mathcal{N}}$ if $\mathcal{N}$ is centered with canonical decomposition $(\mathcal{R}, \mathcal{V}_x : x \in R)$ and f can be extended to W so that $f \restriction W_x$ is an isomorphism from $\mathcal{W}_x$ to $\mathcal{V}_{f(x)}$ for all $x \in S$. Then $\mathcal{W} \cong \mathcal{N}$ if there is $f : S \underset{\text{onto}}{\overset{\text{1-1}}{\to}} R$ such that $\mathcal{S} \overset{f}{\cong} \mathcal{R}$ and $\overline{\mathcal{W}} \overset{f}{\to} \overline{\mathcal{N}}$.

Fixing $f : S \underset{\text{onto}}{\overset{\text{1-1}}{\to}} R$, we will find an expression for the probability that $\mathcal{S} \overset{f}{\cong} \mathcal{R}$. Then we analyze the probability that $\overline{\mathcal{W}} \overset{f}{\to} \overline{\mathcal{N}}$, conditioned on $\mathcal{S} \overset{f}{\cong} \mathcal{R}$. This will be done by exposing the extension of f one $\equiv$ class at a time. Exposure of $[f(x)]$ consists of two steps, which can be performed in either order: labeling $[f(x)]$ with an equivalence class in $\{1, \ldots, n\}$, and assigning nodes to $[f(x)]$ so that $|[f(x)]| = |[x]|$. However, the order in which the $[f(x)]$ are exposed is important. We use an ordering $x_1, x_2, \ldots, x_w \in W$ so that $x_1, \ldots, x_s \in S$, all $[x_i]$ are distinct, $W = \bigcup_{i=1}^{w}[x_i]$, $S \subseteq \bigcup_{i=1}^{s}[x_i]$, and if $[x_i]$ is an ancestor of $[x_j]$ in $\mathcal{W}/\equiv'$, then $i \leq j$. With such an ordering, the growth of f can be approximated by a tree process where the individuals in the process correspond to the $\equiv$ classes. In the following, for $1 \leq i \leq s$ let

$$b_i = |[x_i] \cap S|,$$
$$c_i = |\{y \in [x_i] \cap S : (\exists z \in S)(yM'z)\}|, \quad \text{and}$$
$$C = \sum_{j=1}^{s} c_j/2.$$

That is, C is the number of edges of M' joining nodes in S. For $1 \leq i \leq w$ let

$$a_i = \big|[x_i]\big| \quad \text{and}$$
$$A_i = \sum_{j=1}^{i} a_j.$$

By conditions (1) (a) and (3), there is a function $\rho : \mathbb{N} \to \mathbb{R}^+$ such that for all $i, n > 0$, $\lim_{n\to\infty} \rho(n) = 0$, $|d_i(n)/n - \lambda_i| \leq \rho(n)$, $|\sum_{i=1}^{n-1} id_i(n)/n - \Lambda| \leq \rho(n)$, and $\rho(n) \geq n^{-1/2}$.

Lemma 3.14. *If* $|W| \leq 1/\sqrt{\rho(n)}$*, then*

$$\operatorname{cpr}(\mathcal{S} \overset{f}{\cong} \mathcal{R}, n) = \left[\prod_{i=1}^{s} \sum_{j=b_i}^{\infty} \frac{(j-1)!\lambda_j}{(j-b_i)!}\right] \times \frac{1 + O(\sqrt{\rho(n)})}{m^{|S|+C-s}}.$$

Proof. We expose the equivalence classes $[f(x_i)]$, $1 \leq i \leq s$, one at a time. Let $j_i = |[f(x_i)]|$ and $J_i = \sum_{h=1}^{i} j_h + s - i$. Then $j_i \geq b_i$, and by Condition

(4), $J_i = o(n^{1/4}/\sqrt{\rho(n)})$. Since the number of $h < i$ such that $|[f(x_h)]| = j_i$ is between 0 and $i-1$, the probability that $|[f(x_i)]| = j_i$ is between

$$\frac{d_{j_i}(n) - i + 1}{n - i + 1} \quad \text{and} \quad \frac{d_{j_i}(n)}{n - i + 1}.$$

Thus it is $\lambda_{j_i}(1 + O(\rho(n) - 1/(n\sqrt{\rho(n)})) = \lambda_{j_i}(1 + O(\rho(n)))$.

Since $\sum_{h=1}^{i-1} j_h + 1$ nodes have already been assigned to the equivalence classes $[f(x_1)], \ldots, [f(x_i)]$ and $f(x_{i+1}), \ldots, f(x_s)$ will be assigned to other equivalence classes, there are

$$\binom{m - J_{i-1}}{j_i - 1}$$

ways of assigning the remaining nodes to $[f(x_i)]$. But the b_i nodes in $f([x_i] \cap S)$ must be assigned to $[f(x_i)]$, so there are only

$$\binom{m - J_{i-1} - b_i + 1}{j_i - b_i}$$

ways of choosing the remaining nodes. The probability that $j_i - b_i$ other nodes have been assigned to $[f(x_i)]$ is therefore

$$\begin{aligned}
\frac{\binom{m-J_{i-1}-b_i+1}{j_i-b_i}}{\binom{m-J_{i-1}}{j_i-1}} &= \frac{(j_i - 1)!}{(j_i - b_i)! m^{b_i-1}(1 - O(J_i/m))^{b_i-1}} \\
&= \frac{(j_i - 1)!(1 + o(1/(n^{3/4}\sqrt{\rho(n)})))^{b_i-1}}{(j_i - b_i)! m^{b_i-1}}.
\end{aligned}$$

Therefore the probability that f preserves $\equiv'$ from S to R is

$$\begin{aligned}
&\prod_{i=1}^{s} \sum_{j=b_i}^{\infty} \frac{(j-1)!\lambda_j(1 + O(\rho(n)) + o(1/(n^{3/4}\sqrt{\rho(n)})))^{b_i-1}}{(j - b_i)! m^{b_i-1}} \\
&= \left[\prod_{i=1}^{s} \sum_{j=b_i}^{\infty} \frac{(j-1)!\lambda_j}{(j-b_i)!}\right] \times \left[\frac{1 + O(\rho(n))}{m}\right]^{|S|} \times m^s \\
&= \left[\prod_{i=1}^{s} \sum_{j=b_i}^{\infty} \frac{(j-1)!\lambda_j}{(j-b_i)!}\right] \times \frac{1 + O(\sqrt{\rho(n)})}{m^{|S|-s}}.
\end{aligned}$$

The probability that f preserves M' from S to R is

$$\begin{aligned}
\frac{1}{\prod_{j=1}^{C}(m - 2j + 1)} &= \frac{1}{m^C(1 - O(C/m))^C} \\
&= \frac{1}{m^C(1 - O(1/(n\sqrt{\rho(n)})))^{O(1/\sqrt{\rho(n)})}} \\
&= \frac{1 + O(\rho(n))}{m^C},
\end{aligned}$$

and the lemma follows. □

The conditional probability that $\overline{\mathcal{W}} \xrightarrow{f} \overline{\mathcal{N}}$, given that $\mathcal{S} \overset{f}{\cong} \mathcal{R}$, is analyzed by exposing the equivalence classes $[f(x_i)]$ in the order described above. Beginning with the equivalence classes in generation 0, that is, $[f(x_i)]$ for $1 \leq i \leq s$, we will show that for any $j \geq b_i - c_i$, conditioned on $\mathcal{S} \overset{f}{\cong} \mathcal{R}$, the probability that $[f(x_i)]$ has j children in $\mathcal{V}_{f(x_i)}/\equiv$ is approximately λ_{j+c_i}. For $[f(y)]$ in a later generation of $\mathcal{V}_{f(x_i)}/\equiv$, the probability that $[f(y)]$ has j children is approximately λ_{j+1}.

Thus our tree process $\mathcal{P}$ has s individuals in generation 0, which are the roots of trees $\mathcal{P}_1, \ldots, \mathcal{P}_s$. The root of $\mathcal{P}_i$ has branching probability

$$p_i(j) = \lambda_{j+c_i}.$$

Individuals in later generations have branching probability

$$p(j) = \lambda_{j+1}.$$

This is a slight generalization of the basic branching process because the probabilities of generation 0 are different than the probabilities of later generations. But a version of Formula (5) still holds: the expected number of individuals in generation $t > 0$ is

$$\sum_{i=1}^{s} \sum_{j=1}^{\infty} j p_i(j) \left(\sum_{j=1}^{\infty} j p(j) \right)^{t-1}. \tag{6}$$

If $\overline{\mathcal{W}} \xrightarrow{f} \overline{\mathcal{N}}$ then the maximum depth of any node in $\mathcal{W}$ is r. Such a node belongs to an $\equiv'$ class of depth $\lceil r/2 \rceil$ or $\lfloor r/2 \rfloor$ in $\mathcal{W}/\equiv'$. Assuming without loss of generality that r is even, this means that the process of exposing f is approximated by the first $r/2$ generations of the tree process. We will use $\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2$ to denote the tree process event that $\mathcal{W}_{x_i}/\equiv'$ is isomorphic to $\mathcal{P}_i \restriction r/2$ for $1 \leq i \leq s$. We use $\mathrm{cpr}(\phi|\psi, n)$ to denote the probability of ϕ conditioned on ψ.

Lemma 3.15. *If* $|W| \leq 1/\sqrt{\rho(n)}$, *then* $\mathrm{cpr}(\overline{\mathcal{W}} \xrightarrow{f} \overline{\mathcal{N}} | \mathcal{S} \overset{f}{\cong} \mathcal{R}, n) = \mathrm{tpr}(\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2)(1 + O(\sqrt{\rho(n)}))$.

Proof. Beginning with generation 0, let $1 \leq i \leq s$. Then $[f(x_i)]$ must satisfy $|[f(x_i)]| = a_i$ and $f([x_i] \cap S) \subseteq [f(x_i)]$. Using the same ideas as in the proof of Lemma 3.14, the probability that $|[f(x_i)]| = a_i$ and all $y \in [f(x_i)]$ not matched by M to some $z \in R$ are matched to some $z \notin \bigcup_{j=1}^{i} [f(x_j)]$ is $\lambda_{a_i}(1 + O(\rho(n)))$. Since $[x_i]$ has $a_i - c_i$ children in $\mathcal{W}/\equiv'$, the probability that f has been extended to $[x_i]$ in a way that is consistent with $\overline{\mathcal{W}} \xrightarrow{f} \overline{\mathcal{N}}$ is $p_i(a_i - c_i)(1 + O(\rho(n)))$.

Continuing with later generations of $\mathcal{W}/\equiv'$, for $i > s$ $|[f(x_i)]| = a_i$ and all $y \in [f(x_i)]$ not matched by M to some $z \in \bigcup_{j=1}^{i-1} [f(x_j)]$ are matched to some $z \notin \bigcup_{j=1}^{i} [f(x_j)]$. Using computations similar to the above, this event has probability $\lambda_{a_i}(1 + O(\rho(n)))$, and the probability that f has been extended to $[x_i]$ in a way that is consistent with $\overline{\mathcal{W}} \xrightarrow{f} \overline{\mathcal{N}}$ is $p(a_i - 1)(1 + O(\rho(n)))$.

Eventually, when all of f has been exposed, the probability that $\overline{\mathcal{W}} \xrightarrow{f} \overline{\mathcal{N}}$ conditioned on $\mathcal{S} \overset{f}{\cong} \mathcal{R}$ is

$$\begin{aligned}
&\prod_{i=1}^{s} p_i(a_i - c_i)(1 + O(\rho(n))) \times \prod_{i=s+1}^{w} p(a_i - 1)(1 + O(\rho(n)))\\
&= \prod_{i=1}^{s} p_i(a_i - c_i) \times \prod_{i=s+1}^{w} p(a_i - 1) \times (1 + O(\rho(n)))^{w}\\
&= \mathrm{tpr}(\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2)(1 + O(\sqrt{\rho(n)})). \qquad \square
\end{aligned}$$

COROLLARY 3.16. *There is a constant α (depending on $\mathcal{S}$) such that*

$$\mathrm{cpr}(\mathcal{N}(R;r) \cong \mathcal{W}, n) = \alpha \left[\prod_{i=1}^{s} \sum_{j=b_i}^{\infty} \frac{(j-1)!\lambda_j}{(j-b_i)!}\right] \times \frac{1 + O(\sqrt{\rho(n)})}{m^{|S|+C-s}} \times \mathrm{tpr}(\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2).$$

PROOF. A permutation ϕ on S is said to be an automorphism on $\mathcal{S}$ if, for all $x, y \in S$, $x \equiv' y$ if and only if $\phi(x) \equiv' \phi(y)$, and $xM'y$ if and only if $\phi(x)M'\phi(y)$. Let β be the number of automorphisms on $\mathcal{S}$. Any two functions $f_i : S \underset{\text{onto}}{\overset{\text{1-1}}{\to}} R$, $i = 1, 2$, are said to be equivalent if $f_1 \circ f_2^{-1}$ is an automorphism. Letting α be the number of these equivalence classes, $\alpha = s!/\beta$, and taking any $f : S \underset{\text{onto}}{\overset{\text{1-1}}{\to}} R$,

$$\mathrm{cpr}(\mathcal{N}(R;r) \cong \mathcal{W}, n) = \alpha\, \mathrm{cpr}(\mathcal{S} \overset{f}{\cong} \mathcal{R}, n) \times \mathrm{cpr}(\overline{\mathcal{W}} \xrightarrow{f} \overline{\mathcal{N}} | \mathcal{S} \overset{f}{\cong} \mathcal{R}, n),$$

and the corollary follows by Lemmas 3.14 and 3.15. $\square$

COROLLARY 3.17. $\lim_{n\to\infty} \mathrm{cpr}(\mathcal{U}$ *is k-rich*$, n) = 1$.

PROOF. Taking $r = 3^k$, by Lemma 3.7, it suffices to show that, with probability asymptotic to 1, for any tree $\mathcal{T}$ that occurs as the r-neighborhood of a node in some configuration, there are at least k disjoint r-neighborhoods in $\mathcal{U} \sim_k$ to $\mathcal{T}$. Since there are only finitely many $\sim_k$ classes of trees of depth at most r, we can take $\mathcal{T}$ fixed, and we will actually show that, with probability asymptotic to 1, there are at least k disjoint r-neighborhoods isomorphic to $\mathcal{T}$.

Let $R \subseteq U$, $|R| \le 1/\sqrt[4]{\rho(n)}$, $\mathcal{N} = \mathcal{N}(R;r)$, and f be the identity function on R. Let $\mathbf{A}$ be the collection of all centered configurations $\mathcal{W} = \langle W, \equiv', M', R\rangle$ of depth $\le r$ where $|W| \le 1/\sqrt{\rho(n)}$, $\equiv' \restriction R$ and $M' \restriction R$ are empty, whose canonical decomposition is $(\langle R, \emptyset, \emptyset\rangle, \mathcal{W}_x : x \in R)$. Let $\mathbf{B}$ be the subcollection of $\mathbf{A}$ consisting of those $\mathcal{W}$ such that there exist at least k nodes $x \in R$ where $\mathcal{W}_x \cong \mathcal{T}$.

The probability that there are at least k disjoint neighborhoods isomorphic to $\mathcal{T}$ is greater than or equal to

$$\begin{aligned}
&\mathrm{cpr}\left((\exists x_1, \ldots, x_k \in R)\left(\bigwedge_{i\ne j} N(x_i : r) \cap N(x_j; r) = \emptyset \wedge \bigwedge_{1\le i\le k} \mathcal{N}(x_i; r) \cong \mathcal{T}\right), n\right)\\
&\ge \mathrm{cpr}((\exists \mathcal{W} \in \mathbf{B})(\overline{\mathcal{W}} \xrightarrow{f} \overline{\mathcal{N}}), n)\\
&= \mathrm{cpr}((\exists \mathcal{W} \in \mathbf{A})(\overline{\mathcal{W}} \xrightarrow{f} \overline{\mathcal{N}}), n) - \mathrm{cpr}((\exists \mathcal{W} \in \mathbf{A} - \mathbf{B})(\overline{\mathcal{W}} \xrightarrow{f} \overline{\mathcal{N}}), n).
\end{aligned}$$

By Lemma 3.15,

$$\begin{aligned}\mathrm{cpr}((\exists\mathcal{W}\in\mathbf{A})(\overline{\mathcal{W}}\xrightarrow{f}\overline{\mathcal{N}}),n) &= \mathrm{tpr}((\exists\mathcal{W}\in\mathbf{A})(\overline{\mathcal{W}}\to\mathcal{P}\restriction r/2)(1+O(\sqrt{\rho(n)}))\\ &\geq \mathrm{tpr}(|\mathcal{P}\restriction r/2|\leq 1/(2\sqrt{\rho(n)}))(1+O(\sqrt{\rho(n)}))\end{aligned}$$

because for every tree process $\mathcal{P}$ with t individuals in its first $r/2$ generations, there is some centered configuration $\mathcal{W}$ such that $|W|\leq 2t$ and $\overline{\mathcal{W}}\to\mathcal{P}\restriction r/2$. By Formula (6),

$$\begin{aligned}\mathbf{E}(|\mathcal{P}\restriction r/2|) &\leq (r/2)|R|\sum_{j=1}^{\infty} j\lambda_j\left(\sum_{j=1}^{\infty} j\lambda_{j+1}\right)^{r/2-1}\\ &< (r/2)|R|\Lambda^{r/2}\\ &= O(1/\sqrt[4]{\rho(n)}).\end{aligned}$$

By Markov's inequality,

$$\mathrm{tpr}(|\mathcal{P}\restriction r/2| > 1/(2\sqrt{\rho(n)}))\leq O(\sqrt[4]{\rho(n)})$$

and therefore

$$\mathrm{cpr}((\exists\mathcal{W}\in\mathbf{A})(\overline{\mathcal{W}}\xrightarrow{f}\overline{\mathcal{N}}),n) = 1-O(\sqrt[4]{\rho(n)}).$$

Using Lemma 3.15 again,

$$\begin{aligned}&\mathrm{cpr}((\exists\mathcal{W}\in\mathbf{A}-\mathbf{B})(\overline{\mathcal{W}}\xrightarrow{f}\overline{\mathcal{N}}),n)\\ &\leq \mathrm{tpr}(|\{x\in R:\mathcal{T}\to\mathcal{P}_x\restriction r/2\}|<k)(1+O(\sqrt{\rho(n)})).\end{aligned}$$

Using induction on r and Condition (1)(b), it can be shown that $\mathrm{tpr}(\mathcal{T}\to\mathcal{P}_x\restriction r/2)>0$. Since the events $\mathcal{T}\to\mathcal{P}_x\restriction r/2$ are independent for $x\in R$,

$$\begin{aligned}\mathbf{E}(|\{x\in R:\mathcal{T}\to\mathcal{P}_x\restriction r/2\}|) &= \Theta(1/\sqrt[4]{\rho(n)})\text{ and}\\ \mathrm{var}(|\{x\in R:\mathcal{T}\to\mathcal{P}_x\restriction r/2\}|) &= \Theta(1/\sqrt[4]{\rho(n)}).\end{aligned}$$

Therefore by Chebyshev's inequality,

$$\begin{aligned}\mathrm{tpr}(|\{x\in R:\mathcal{T}\to\mathcal{P}_x\restriction r/2\}|<k) &= O(\sqrt[4]{\rho(n)}),\\ \mathrm{cpr}((\exists\mathcal{W}\in\mathbf{A}-\mathbf{B})(\overline{\mathcal{W}}\xrightarrow{f}\mathcal{N}),n) &= O(\sqrt[4]{\rho(n)}),\end{aligned}$$

and the corollary follows. □

COROLLARY 3.18. $\lim_{n\to\infty}\mathrm{cpr}(\mathcal{U}\textit{ is }k\textit{-simple})=1$.

PROOF. It suffices to show that, for any fixed s and r,

$$\lim_{n\to\infty}\mathrm{cpr}(\exists R(R\text{ is a cycle of size }2s\wedge\mathcal{N}(R;r)\text{ is not unicyclic}),n)=0.$$

Let $\mathbf{A}$ be the collection of centered configurations $\mathcal{W}=\langle W,\equiv',M',S\rangle$ of depth $\leq r$ where $|W|\leq 1/\sqrt{\rho(n)}$, and S is a fixed cycle of size $2s$. Taking $f:S\overset{1-1}{\to}U$

and letting $R = f(S)$, $\mathcal{S} = \langle S, \equiv', M' \rangle$, $\mathcal{R} = \langle R, \equiv, M \rangle$, and $\mathcal{N} = \mathcal{N}(R; r)$,

$$\mathrm{cpr}(\mathcal{S} \overset{f}{\cong} \mathcal{R} \wedge \mathcal{N} \text{ is not unicyclic}, n) \leq \mathrm{cpr}(\mathcal{S} \overset{f}{\cong} \mathcal{R}, n) - \mathrm{cpr}(\mathcal{S} \overset{f}{\cong} \mathcal{R} \wedge (\exists \mathcal{W} \in \mathbf{A})(\overline{\mathcal{W}} \overset{f}{\to} \overline{\mathcal{N}}), n).$$

Now

$$\begin{aligned} &\mathrm{cpr}(\mathcal{S} \overset{f}{\cong} \mathcal{R} \wedge (\exists \mathcal{W} \in \mathbf{A})(\overline{\mathcal{W}} \overset{f}{\to} \overline{\mathcal{N}}), n) \\ &= \mathrm{cpr}(\mathcal{S} \overset{f}{\cong} \mathcal{R}, n) \times \mathrm{cpr}((\exists \mathcal{W} \in \mathbf{A})(\overline{\mathcal{W}} \overset{f}{\to} \overline{\mathcal{N}}) | \mathcal{S} \overset{f}{\cong} \mathcal{R}, n). \end{aligned}$$

By Lemma 3.14

$$\begin{aligned} \mathrm{cpr}(\mathcal{S} \overset{f}{\cong} \mathcal{R}, n) &= \left[\sum_{j=2}^{\infty} (j-1)\lambda_j \right]^s \times \frac{1 + O(\sqrt{\rho(n)})}{m^{2s}} \\ &= \frac{(\Lambda - 1)^s + O(\sqrt{\rho(n)})}{m^{2s}}. \end{aligned}$$

By Lemma 3.15

$$\begin{aligned} &\mathrm{cpr}((\exists \mathcal{W} \in \mathbf{A})(\overline{\mathcal{W}} \overset{f}{\to} \overline{\mathcal{N}}) | \mathcal{S} \overset{f}{\cong} \mathcal{R}, n) \\ &= \mathrm{tpr}((\exists \mathcal{W} \in \mathbf{A})(\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2))(1 + O(\sqrt{\rho(n)})) \\ &= 1 - O(\sqrt[4]{\rho(n)}), \end{aligned}$$

using methods similar to those in the proof of Corollary 3.17. Therefore

$$\begin{aligned} &\mathrm{cpr}(\mathcal{S} \overset{f}{\cong} \mathcal{R} \wedge \mathcal{N} \text{ is not unicyclic}, n) \\ &= \frac{O(\sqrt[4]{\rho(n)})}{m^{2s}}, \end{aligned}$$

and since there are no more than m^{2s} ways of choosing f,

$$\begin{aligned} &\mathrm{cpr}(\exists R (R \text{ is a cycle of size } 2s \wedge \mathcal{N} \text{ is not unicyclic}), n) \\ &= O(\sqrt[4]{\rho(n)}). \end{aligned}$$

□

Corollary 3.19. *Let $r \geq 0$ and τ be a unicyclic $\sim_k$ type of depth $\leq r$. Then*

$$\lim_{n \to \infty} \mathbf{E}(|\{R \subseteq U : \mathcal{N}(R; r) \in \tau\}|) = \gamma$$

for some constant γ.

Proof. Let $2s$ be the size of the cycles in the members of τ, $\mathbf{A}$ the collection of all $\mathcal{W} = \langle W, \equiv', M', S \rangle$ of depth $\leq r$ such that $|W| \leq 1/\sqrt{\rho(n)}$ and S is a fixed cycle of size $2s$, and $\mathbf{B}$ the collection of configurations with center S that belong to τ. Then for fixed $R \subseteq U$ such that $|R| = 2s$,

$$\mathrm{cpr}(\mathcal{N}(R; r) \in \tau, n) = \mathrm{cpr}((\exists \mathcal{W} \in \mathbf{A} \cap \mathbf{B})(\mathcal{N}(R; r) \cong \mathcal{W}), n) + \mathrm{cpr}((\exists \mathcal{W} \in \mathbf{B} - \mathbf{A})(\mathcal{N}(R; r) \cong \mathcal{W}), n).$$

By Corollary 3.16

$$\mathrm{cpr}((\exists \mathcal{W} \in \mathbf{A} \cap \mathbf{B})(\mathcal{N}(R;r) \cong \mathcal{W}), n) = (2s-1)!(\Lambda-1)^s \times \frac{1+O(\sqrt{\rho(n)})}{m^{2s}} \times \mathrm{tpr}((\exists \mathcal{W} \in \mathbf{A} \cap \mathbf{B})(\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2)),$$

$$\mathrm{tpr}((\exists \mathcal{W} \in \mathbf{A} \cap \mathbf{B})(\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2)) = \mathrm{tpr}((\exists \mathcal{W} \in \mathbf{B})(\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2)) - \mathrm{tpr}((\exists \mathcal{W} \in \mathbf{B} - \mathbf{A})(\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2)), \text{ and}$$

$$\mathrm{tpr}((\exists \mathcal{W} \in \mathbf{B} - \mathbf{A})(\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2)) = O(\sqrt[4]{\rho(n)})$$

using methods similar to those in the proof of Corollary 3.17. Also,

$$\mathrm{cpr}((\exists \mathcal{W} \in \mathbf{B} - \mathbf{A})(\mathcal{N}(R;r) \cong \mathcal{W}), n) \leq \mathrm{cpr}(\mathcal{S} \cong \mathcal{R}, n) - \mathrm{cpr}((\exists \mathcal{W} \in \mathbf{A})(\mathcal{N}(R;r) \cong \mathcal{W}), n).$$

Applying Corollary 3.16,

$$\mathrm{cpr}(\mathcal{S} \cong \mathcal{R}, n) = (2s-1)!(\Lambda-1)^s \times \frac{1+O(\sqrt{\rho(n)})}{m^{2s}} \text{ and}$$

$$\mathrm{cpr}((\exists \mathcal{W} \in \mathbf{A})(\mathcal{N}(R;r) \cong \mathcal{W}), n) = (2s-1)!(\Lambda-1)^s \times \frac{1+O(\sqrt{\rho(n)})}{m^{2s}} \times \mathrm{tpr}((\exists \mathcal{W} \in \mathbf{A})(\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2)).$$

Again

$$\mathrm{tpr}((\exists \mathcal{W} \in \mathbf{A})(\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2)) = 1 - \frac{O(\sqrt[4]{\rho(n)})}{m^{2s}}$$

by the argument used in the proof of Corollary 3.17. Therefore

$$\mathrm{cpr}(\mathcal{N}(R;r) \in \tau, n) = (2s-1)!(\Lambda-1)^s \times \frac{1+O(\sqrt{\rho(n)})}{m^{2s}} \times \mathrm{tpr}((\exists \mathcal{W} \in \mathbf{B})(\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2)).$$

There are $\binom{m}{2s}$ ways to choose R, so, taking $\gamma = (\Lambda-1)^s\, \mathrm{tpr}((\exists \mathcal{W} \in \mathbf{B})(\overline{\mathcal{W}} \to \mathcal{P} \restriction r/2))/(2s)$, the corollary follows. □

We will use a generalization of the inclusion-exclusion principle and Bonferroni's inequalties to finish the proof of Theorem 3.3. Complete details on this method are given in [Lynch 1985, 1992]. Let τ_i, $i \in I$, be an indexing of all unicyclic $\sim_k$ classes where members of τ_i have cycle size $s_i \leq 2 \cdot 3^{k-1}$ and depth $r_i \leq 3^{k-1}$, and let γ_i be the constant that Corollary 3.19 associates with τ_i. Then, for any k-rich and k-simple configuration, its $\sim_k$ type is described by the vector $(u_i : i \in I)$ where $0 \leq u_i \leq k$ is the number (up to k) of neighborhoods of type τ_i. That is, if $u_i = k$ then there are at least k neighborhoods of type τ_i. In the following, $\mathcal{U}$ will be a configuration on n with universe $U = \{1, \ldots, m\}$. For $i \in I$ we put $C_i(\mathcal{U}) = \{R \subseteq U : \mathcal{N}(R; 3^{k-1}) \in \tau_i\}$ and $\overline{C(\mathcal{U})} = (C_i(\mathcal{U}) : i \in I)$.

Let $\overline{S} = (S_i : i \in I)$ where each $S_i \subseteq \{R \subseteq U : |R| = s_i\}$. We define

$$E^{\geq}(\overline{S}) = \{\mathcal{U} : C_i(\mathcal{U}) \supseteq S_i \text{ for } i \in I\}$$

$$E^{=}(\overline{S}) = \{\mathcal{U} : \overline{C(\mathcal{U})} = \overline{S}\}.$$

That is, $E^{\geq}(\overline{S})$ is the set of configurations on n where every $R \in S_i$ is a cycle in a 3^{k-1}-neighborhood of type τ_i, and $E^{=}(\overline{S})$ is the set of configurations on n where S_i is precisely the set of cycles belonging to 3^{k-1}-neighborhoods of type τ_i. Given a vector $\overline{u} = (u_i : i \in I)$, let

$$L(\overline{u}, n) = \sum_{\overline{S}:|S_i|=u_i} \operatorname{cpr}(E^{\geq}(\overline{S}), n)$$

and for $J \subseteq I$ let

$$M(J, \overline{u}, n) = \bigcup_{\substack{\overline{S}:|S_i|=u_i \text{ for } i\in J,\\ |S_i|\geq u_i \text{ for } i\in I-J}} E^{=}(\overline{S}).$$

That is, $M(J, \overline{u}, n)$ is the set of configurations on n with exactly u_i neighborhoods of type τ_i for $i \in J$ and at least u_i neighborhoods of type τ_i for $i \in I - J$. Thus we need only show that $\operatorname{cpr}(M(J, \overline{u}, n), n)$ converges for large n. For any I-vector $\overline{v}$, we put $\overline{v} \geq \overline{u}$ if $v_i \geq u_i$ for all $i \in I$.

Using the methods of [Lynch 1985, 1992],

$$\lim_{n\to\infty} \operatorname{cpr}(M(J, \overline{u}, n), n) = \sum_{\overline{v}\geq\overline{u}} (-1)^{\sum(v_i-u_i)} \prod_{i\in J} \binom{v_i}{u_i} \times \prod_{i\in I-J} \binom{v_i-1}{u_i-1} \times \lim_{n\to\infty} L(\overline{v}, n)$$

and

$$\lim_{n\to\infty} L(\overline{v}, n) = \prod_{i\in I} \frac{\gamma_i^{v_i}}{v_i!}.$$

Therefore

$$\lim_{n\to\infty} \operatorname{cpr}(M(J, \overline{u}, n), n) = \prod_{i\in J} \sum_{v\geq u_i} (-1)^{v-u_i} \frac{\gamma_i^v}{u_i!(v-u_i)!} \times \prod_{i\in I-J} \sum_{v\geq u_i} (-1)^{v-u_i} \frac{\gamma_i^v}{v(u_i-1)!(v-u_i)!}.$$

For $i \in J$,

$$\sum_{v\geq u_i} (-1)^{v-u_i} \frac{\gamma_i^v}{u_i!(v-u_i)!} = \frac{\gamma_i^{u_i} e^{-\gamma_i}}{u_i!}.$$

For $i \in I - J$,

$$\begin{aligned}\sum_{v\geq u_i} (-1)^{v-u_i} \frac{\gamma_i^v}{v(u_i-1)!(v-u_i)!} &= \sum_{v\geq u_i} (-1)^{v-u_i} \left[\sum_{w<u_i} (-1)^{u_i-w-1} \binom{v}{w}\right] \frac{\gamma_i^v}{v!} \\ &= 1 - \sum_{v<u_i} \left[\sum_{w\leq v} (-1)^{v-w} \binom{v}{w}\right] \frac{\gamma_i^v}{v!} \\ &\quad - \sum_{v\geq u_i} (-1)^{v-u_i} \left[\sum_{w<u_i} (-1)^{u_i-w} \binom{v}{w}\right] \frac{\gamma_i^v}{v!}\end{aligned}$$

$$= 1 - \sum_{w<u_i} \left[\sum_{v \geq w} (-1)^{v-w} \binom{v}{w} \frac{\gamma_i^v}{v!} \right]$$
$$= 1 - \sum_{w<u_i} \frac{\gamma_i^w e^{-\gamma_i}}{w!}.$$

This completes the proof of Theorem 3.3.

4. CONCLUSIONS

Some open problems suggested by Theorem 3.1 are:

(1) What are the possible values of $\lim_{n\to\infty} \mathrm{pr}(\sigma, n)$?
(2) What is the complexity of computing this limit?
(3) If $\Lambda = \infty$, does the convergence law still hold?
(4) Does the convergence law hold for any logical languages more powerful than first-order logic?
(5) Of particular interest to internet applications, physics, and biology, is there a general characterization of random graph processes that result in power law distributions?

REFERENCES

ABELLO, J., BUCHSBAUM, A., AND WESTBROOK, J. 1998. A functional approach to external graph algorithms. In *Lecture Notes Computer Science*, vol. 1461. Springer-Verlag, New York, 332–343.

ABITEBOUL, S., COMPTON, K., AND VIANU, V. 1992. Queries are easier than you thought, probably. In *Proceedings of the ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems*. ACM, New York, 23–32.

ADAMIC, L. A. AND HUBERMAN, B. A. 1999. Growth dynamics of the world wide web. *Nature 401*, 131.

AIELLO, W., CHUNG, F., AND LU, L. 2001. A random graph model for power law graphs. *Experimental Math 10*, 53–66.

AIELLO, W., CHUNG, F., AND LU, L. 2002. Random evolution of massive graphs. In *Handbook on Massive Data Sets*, J. Abello, P. M. Pardalos, and M. G. C. Resonde, Eds. Kluwer Academic Press, 97–122.

ANDERSON, R. M. AND MAY, R. M. 1995. Susceptible-infectious-recovered epidemic models with dynamic partnerships. *J. Math. Biology 33*, 661–675.

BALL, F., MOLLISON, AND SCALIA-TOMBA, G. 1997. Epidemics with two levels of mixing. *Ann. Appl. Probab. 7*, 46–89.

BARABÁSI, A., ALBERT, R., AND JEONG, H. 1999. Scale-free characteristics of random networks: The topology of the world wide web. *Physica A 272*, 173–187.

BHALLA, U. S. AND IYENGAR, R. 1999. Emergent properties of networks of biological signalling pathways. *Science 283*, 381–387.

BOLLOBÁS, B. 1985. *Random Graphs*. Academic Press, London.

BOWER, J. M. AND BOLOURI, H. 2001. *Computational Modeling of Genetic and Biochemical Networks*. The MIT Press, Cambridge, MA.

BRODER, A., KUMAR, S. R., MAGHOUL, F., RAGHAVAN, P., RAJAGOPALAN, S., STATA, R., TOMPKINS, A., AND WIENER, J. 2000. Graph structure in the web. *Comput. Netw. 33*, 209–321.

COHEN, J. E., BRIAND, E., AND NEWMAN, C. M. 1990. *Community Food Webs: Data and Theory*. Springer-Verlag, Berlin.

COHEN, R., EREZ, K., BEN AVRAHAM, D., AND HAVLIN, S. 2000. Resilience of the Internet to random breakdowns. *Phys. Rev. Lett. 85*, 4626–4628.

DERRIDA, B. AND POMEAU, Y. 1986. Random networks of automata: A simple annealed approximation. *Europhys. Lett. 1*, 45–49.

EHRENFEUCHT, A. 1961. An application of games to the completeness problem for formalized theories. *Fund. Math. 49*, 129–141.

ERDŐS, P. AND RÉNYI, A. 1959. On random graphs. *Pub. Math 6*, 290–297.

ERDŐS, P. AND RÉNYI, A. 1960. On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutato Int. Kozl. 5*, 17–61.

FALOUTSOS, M., FALOUTSOS, P., AND FALOUTSOS, C. 1999. On power-law relationships of the internet topology. *Comput. Comm. Rev. 29*, 251–262.

GAIFMAN, H. 1982. On local and non-local properties. In *Proceedings of the Herbrand Symposium Logic Colloquium '81*. North Holland.

GILBERT, E. 1961. Random plane networks. *J. SIAM 9*, 533–543.

GLANCE, N. S. AND HUBERMAN, B. A. 1993. The outbreak of cooperation. *J. Math. Soc. 17*, 281–302.

GRANOVETTER, M. 1978. Threshold models of collective behavior. *Am. J. Soc. 83*, 1420–1443.

HARRIS, T. E. 1989. *The Theory of Branching Processes*. Dover Publications, Inc., New York.

KAUFFMAN, S. A. 1984. Emergent properties in random complex automata. *Physica D 10*, 145–156.

KAUFFMAN, S. A. 1993. *The Origins of Order: Self-Organization and Selection in Evolution*. Oxford University Press, New York.

KEELING, M. J. 1999. The effects of local spatial structure on epidemiological invasions. *Proc. Royal Soc. Lond. B 266*, 859–867.

KOSTEREV, D. N., TAYLOR, C. W., AND MITTELSTADT, W. A. 1999. A model validation for the August 10, 1996 WSCC system outage. *IEEE Trans. Power Systems 14*, 967–979.

KRETSCHMAR, M. AND MORRIS, M. 1996. Measures of concurrency in networks and the spread of infectious disease. *Math. Biosciences 133*, 165–195.

KUMAR, R., RAGHAVAN, P., RAJAGOPALAN, S., SIVAKUMAR, D., TOMKINS, A., AND UPFAL, E. 2000. Stochastic models for the web graph. In *Proceedings of the 41st IEEE Symposium on FOCS*. IEEE Computer Society Press, Los Alamitos, CA, 57–65.

LIFSCHITZ, S. AND VIANU, V. 1998. A probabilistic view of datalog parallelization. *Theoretical Computer Science 190*, 211–239.

ŁUCZAK, T. 1992. Sparse random graphs with a given degree sequence. In *Random Graphs, Vol. 2*, A. Freize and T. Łuczak, Eds. Wiley, New York, 165–182.

LYNCH, J. F. 1985. Probabilities of sfirst-order sentences about unary functions. *Trans. AMS 287*, 543–568.

LYNCH, J. F. 1992. Probabilities of sentences about very sparse random graphs. *Random Struct. Alg. 3*, 33–53.

LYNCH, J. F. 2002. Critical points for random boolean networks. *Physica D 172*, 49–64.

MCKAY, B. D. 1985. Asymptotics for symmetric 0-1 matrices with prescribed row sums. *Ars Combinatorica 19A*, 15–25.

MOLLOY, M. AND REED, B. 1995. A critical point for random graphs with a given degree sequence. *Random Struct. Alg. 6*, 161–179.

MOLLOY, M. AND REED, B. 1998. The size of the giant component of a random graph with a given degree sequence. *Combinat. Prob. Comput. 7*, 295–305.

NEWMAN, M. E. J. 2001. The structure of scientific collaboration networks. *Prod. Nat. Acad. Sci. 98*, 404–409.

NEWMAN, M. E. J., STROGATZ, S. H., AND WATTS, D. J. 2001. Random graphs with arbitrary degree distributions and their applications. *Phys. Rev. E 64*, 026118.

SATTENSPIEL, L. AND SIMON, C. P. 1988. The spread and persistence of infectious diseases in structured populations. *Math. Biosci. 90*, 367–383.

STROGATZ, S. H. 2001. Exploring complex networks. *Nature 410*, 268–276.

VALENTE, T. W. 1996. Social network thresholds in the diffusion of innovations. *Social Netw. 18*, 69–89.

WILLIAMS, R. J. AND MARTINEZ, N. D. 2000. Simple rules yield complex food webs. *Nature 404*, 180–183.

Received November 2003; revised June 2004; accepted June 2004