An Introduction to Synthetic Differential Geometry

Lázaro Alberto Larrauri Borroto

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What is Synthetic Differential Geometry?

Given two different points there exists only one line incident to both of them.

Through a point not in a line, only one parallel line to the given one can be drawn.

Euklid.jpg

But what are points and lines?

What is Synthetic Differential Geometry?

It is an axiomatic theory that deals with space forms in terms of their structure.

It allows for rigorous reasoning with nilpotent infinitesimals.

Lie.jpg

Where does it take place?

We work in an ambient category $\mathcal{E}$ composed of "smooth spaces and morphisms".

Synthetic differential geometry has no models in the category of sets. It has to be interpreted over a **topos**.

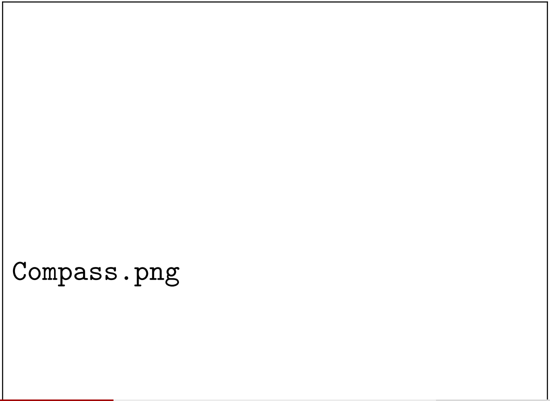
> Cartesian closed category with sub-object classifier.
> $A \times B$, A^B, $A \cup B$, $A \cap B$, $P(A)$ ...

Basic structure of the geometric line.

The geometric line R satisfies:

Axiom

R is a non-trivial $\mathbb{Q}$-algebra.



The Kock-Lawvere axiom.

$$f(x) \simeq f(0) + f'(0)x$$

Is there any x such that we can substitute $\simeq$ with $=$ for all f's?

No, we would need x "so small" that $x^2 = 0$.

The Kock-Lawvere axiom.

Let

$$D := \{\, d \in R \mid d^2 = 0 \,\}$$

Axiom (Kock-Lawvere axiom)

For any $f : D \to R$ there exists a unique $b \in R$ such that

$$\forall d \in D : \quad f(d) = f(0) + db.$$

Wait, are we safe?

Intuitionistically speaking, yes.

Under classical assumptions, not so much.

Wait, are we safe?

The Kock-Lawvere axiom is not consistent with the Principle of the Excluded Middle:

$$P \vee \neg P$$

We must use "intuitionistic" logic.

Derivatives and Taylor series.

Derivatives are defined in a natural way.

Definition

Let $f \in R^R$. The derivative of f at the point $x \in R$ is the unique constant $f'(x) \in R$ such that

$$\forall d \in D : \quad f(x+d) = f(x) + f'(x)d.$$

And functions "locally" coincide with their linear approximations.

Derivatives and Taylor series.

D is not an ideal.

What can we say about

$$D_2 := \{ d \in R \mid d^3 = 0 \}?$$

Derivatives and Taylor series.

Not much. We need an additional axiom:

Axiom

For any $f \in R^{D_2}$ there exist unique $c_1, c_2 \in R$ such that

$$\forall d \in D_2 : \quad f(d) = f(0) + c_1 d + c_2 d^2$$

Derivatives and Taylor series.

This way we have:

Theorem

For any $f \in R^R$

$$\forall d \in D_2, \forall x \in R: \quad f(x+d) = f(0) + f'(x)d + \frac{f''(x)}{2}d^2.$$

Derivatives and Taylor series.

Similarly, we would need an additional axiom for any of

$$D_k := \{\, d \in R \,|\, d^{k+1} = 0 \,\} \text{ for } k = 1, 2, \ldots$$

We can state them all together:

Axiom

Let $f \in R^{D_k}$ for some $k \in \mathbb{N}$. Then there exists a unique k-tuple of constants $c_1, \ldots, c_k \in R$ such that

$$\forall d \in D_k : \quad f(d) = f(0) + \sum_{i=1}^{k} c_k d^k$$

Derivatives and Taylor series.

If we define $D_\infty = \bigcup_{i=0}^{\infty} D_i$ it follows

Theorem (Taylor series)

For all $f \in R^R$ and $x \in R$ there exists a unique formal power series $\Phi(X)$ such that

$$\forall d \in D_\infty : \quad f(x+d) = \Phi(d).$$

Namely,

$$\Phi(X) = \sum_{i=0}^{\infty} \frac{f^{(k)}}{k!} X^k.$$

Don't let the axiom-party stop.

There are still families of infinitesimals we cannot deal with.

$$D(2) := \{\, (d_1, d_2) \in D^2 \mid d_1 d_2 = 0 \,\}$$

Our current axioms state that

$$W_k := R[X]/\langle X^{k+1} \simeq R^{D_k}, \text{ where "} D_k = Spec_R(W_k) \text{"}$$

Don't let the axiom-party stop.

In general, if $W = R[X_1, \ldots, X_n]$ satisfies some technical condition (it is a **Weil algebra**) we can state:

Axiom

$$W \simeq R^{Spec_R(W)}$$

Tangent Vectors

Definition

A **tangent vector** to M at the point $p \in M$ is a map $t \in M^D$ such that $t(0) = p$.

Thus, M^D is the tangent bundle of M.

To give $(M^D)_p$ a tangent space structure we need M to be "infinitesimally linear".

Tangent Vectors

Definition

An object M is said to be **infinitesimally linear** if for any $p \in M$ and any n-tuple of maps $t_1, \ldots, t_n \in (M^D)_p$ there is a unique map $l \in M^{D(n)}$ satisfying $l \circ incl_i = t_i$ for all $i = 1, \ldots, n$.

Differentials

Theorem

Let M and N be infinitesimally linear, and $f \in N^M$. Then, for any $p \in M$ the map $f^D(t) = f \circ t$ restricts to a linear map from $(M^D)_p$ to $(N^D)_{f(p)}$.

Vector Fields

Definition

A vector field X over M is any of the following:

A "section of the tangent bundle", $\hat{X} : M \to M^D$.

An "infinitesimal flow of the aditive group R" $X : M \times D \to M$

An "infinitesimal deformation of the identity map" $\check{X} : D \to M^M$.

Directional Derivatives

Definition

The directional derivative of f in the direction of X is the unique map $X(f) \in M^R$ such that, for any $p \in M$

$$\forall d \in D : \quad f(X(p,d)) = f(p) + dX(f)(p).$$

Under some additional hypotheses on M we can define:

Definition

Let $X, Y \in Vect(M)$. The **Lie bracket** $[X, Y]$ is the unique vector field such that

$$\forall d_1, d_2 \in D : \quad [X, Y]^{\vee}(d_1 d_2) = \check{Y}(-d_2) \circ \check{X}(-d_1) \circ \check{Y}(d_2) \circ \check{X}(d_1)$$

And it is satisfied

$$[X, Y](f) = X(f) - Y(f)$$

Questions?

Let them be easy please