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Optimal Numberings and Isoperimetric Problems on Graphs

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I. At the end of [1], the following unsolved problem was noted: How may the numbers 1, 2, ..., 2^n be assigned to the 2^n vertices of the n-cube so that the maximum absolute difference of numbers assigned to opposite ends of the same edge is minimized? As was noted in [1], A. W. Hales had pointed out that any numbering which, assigned to the $\binom{n}{w}$ vertices of weight w the $\binom{n}{w}$ numbers, l, satisfying

$$\sum_{i=0}^{w-1} \binom{n}{i} < l \leq \sum_{i=0}^{w} \binom{n}{i},$$

gives a lower maximum than the natural numbering if n is sufficiently large. By the "natural numbering" we mean that the n-tuple of zeros and ones $(v_0, v_1, ..., v_{n-1})$ is assigned $\sum_{i=0}^{n-1} v_i 2^i$. Hales' observation was significant because the natural numbering is the canonical numbering which minimizes the *average* absolute difference of the numbers assigned to neighboring vertices. In this paper we shall show that a subclass of Hales' numberings actually does solve the problem of minimizing the max.

The methods used are sufficiently general to solve related problems. Each minimization problem seems to have a dual maximization problem. In this case it is to find the numberings which maximize the minimum absolute difference. As it turns out the solution set for this problem is a subset of the numberings which maximize the average absolute difference and so both functionals are maximized simultaneously.

All of these problems may be stated as purely graph-theoretical problems, and we shall do so in the next section. The max min problem is especially interesting in that context as a little reflection shows it to be intimately related to the general problem of graph-coloring, which may be stated as follows: given a graph, what is the minimal integer ϱ such that 1, 2, ..., ϱ may be assigned to all vertices (numbers may be used more than once) so that the minimum absolute difference between numbers assigned to neighboring vertices will be at least one.

II. DEFINITIONS. A *graph* is an ordered pair (V, E), where V is the set of *vertices* and E is a set of (unordered) pairs of vertices called *edges*. If a graph has N vertices, a *numbering* of the graph is any function

$$\varphi : V \rightarrow \{1, 2, ..., N\},$$

which is 1-1 onto. A numbering gives a functional Δ_e on E:

$$\text{if } e = \{\mathrm{v}, w\},\ e \in E,$$

then

$$\Delta_e = | \varphi(\mathrm{v}) - \varphi(w) | .$$

Given a numbering φ, Sl, *the beginning segment of length* l of φ is the set

$$\{\mathrm{v} \in V : \varphi(\mathrm{v}) \leq l\}.$$

The *n-cube*, considered as a graph, has

$$V = \{(\mathrm{v}_0, \mathrm{v}_1, ..., \mathrm{v}_{n-1}) : \mathrm{v}_i = 0 \text{ or } 1\},$$

the set of all n-tuples of zeros and ones, and two n-tuples are in E if they differ in exactly one entry.

Suppose we are given a graph having N vertices and asked to construct a numbering φ which minimizes

$$\max_{e \in E} \Delta_e .$$

Immediately below we present a program for constructing such a numbering provided that the graph fulfils certain conditions. The program does not work for all graphs, but we shall see later that it does work for the n-cube.

PROGRAM. If there exists a numbering all of whose beginning segments obey the following two conditions, then $\max_e \Delta_e$ for this numbering will be the minimum $\max_e \Delta_e$ over all numberings of the graph:

(i) For a set of l vertices, let Φ_l be the number of vertices in the set having neighbors not in the set. Φ_l must be minimized for all beginning segments S_l.

(ii) The $\Phi_l' = l - \Phi_l$ "interior vertices" of S_l must be numbered 1, 2, ..., Φ_l', i.e., have the lowest possible numbers on them. If they do exist, call such numberings *Hales numberings.*

PROOF. Let φ be any numbering and let Φ_l^c be the number of vertices not in S_l but having nearest neighbors in S_l. It is easy to see that, if S_l is contained in the interior of $S_{l'}$, then $l' \geq l + \Phi_l^c$. Therefore

$$\max_e \Delta_e \geq \max_l \Phi_l^c.$$

But Φ_l^c is just Φ_{N-l} for the numbering φ^c derived from φ by reversing the order. Since $\max_e \Delta_e$ for φ is the same as $\max_e \Delta_e$ for φ^c, we have

$$\max_e \Delta_e \geq \max_l \Phi_l.$$

Conditions (i) and (ii) make it clear then that Hales numberings minimize $\max_e \Delta_e$ if they exist and that for them

$$\max_e \Delta_e = \max_l \Phi_l\,.$$

In some ways there is a great deal of similarity between the program presented here and the one given in [1] to minimize $\sum_{e \in E} \Delta_e$. In both cases we reduce the problem of minimizing a functional on the numberings to one of solving a series of isoperimetric problems. Of course for different functionals the definition of the boundary of a set of vertices must be changed. In the previous paper Θ_l was the same as Φ_l here except each vertex in the set was counted as many times as it had neighbors not in the set. There, corresponding to the above identity, we had

$$\sum_e \Delta_e = \sum_{l=1}^{2^n} \Theta_l\,.$$

In [1] all numberings which minimize $\sum_e \Delta_e$ on the n-cube were found. This has not been possible for $\max_e \Delta_e$, but some uniqueness for our numberings will be shown later.

THEOREM 1. *On the n-cube the following numberings have the smallest possible value of* $\max_e \Delta_e$: *Number any vertex with* 1; *having assigned* 1, 2, ..., *l, assign* $l + 1$ *to any open nearest neighbor of the lowest numbered vertex having open neighbors.*

A gross feature of the structure of these numberings that is obvious (and originally in Hales' crude class of numberings) is that, if $l = \sum_{i=0}^{w} \binom{n}{i}$, then S_l will be just the set of vertices within Hamming distance w of the first, i.e., the vertex numbered 1. Before proceeding with the proof of Theorem 1 however, we need some definitions and a lemma.

DEFINITION [2]. Consider a set, S, of l vertices on the n-cube. The *origin*, A, of such a set is that vertex whose i-th coordinate, a_i, is 0 if at least half the vertices in S have 0 in that coordinate, and 1 otherwise. Let the i-th coordinate divide the n-cube into two $(n - 1)$-cubes. Then the set S is said to be *stable in i* if for each vertex v in S such that $\mathrm{v}_i = \bar{a}_i$ ($\bar{a}_i$ is 1 if $a_i = 0$ and 0 if $a_i = 1$), the corresponding vertex with $\mathrm{v}_i = a_i$ is also in S. If S is stable in i for all $i = 0, 1, ..., n - 1$, then it is said to be *one-dimensionally stable*.

Let coordinates i and j divide the n-cube into four $(n - 2)$-cubes, and let S be a one-dimensionally stable set of vertices. Also let S' be the subset of S consisting of those vertices in the subcube $\mathrm{v}_i = a_i$, $\mathrm{v}_j = \bar{a}_j$ and let S'' be the subset of S consisting of those vertices in the subcube $\mathrm{v}_i = \bar{a}_i$, $\mathrm{v}_j = a_j$. Then S is *stable in coordinates i and j* if either the set obtained by complimenting the i-th and j-th coordinates of each vertex in S' is contained in S'' or vice versa. S is *two-dimensionally stable* if it is stable in every pair of coordinates.

DEFINITION. A set of vertices which is two-dimensionally stable (and by definition then one-dimensionally stable) is said to be in *normal form* if its origin, A, is the all zero vector, and if v in S implies that every vector gotten by shifting ones to the left in v is also in S. Now given a two-dimensionally stable set of vertices, the first requirement for normal form may be fulfilled by complimenting the non-zero coordinates of A. If we then reorder the coordinates from right to left so that the number of members of S having zero in that coordinate is increasing, then the second condition is fulfilled. Thus we have

LEMMA. *Every two-dimensionally stable set of vertices may be put into normal form by reflections and rotations of the cube.*

PROOF OF THEOREM 1. Referring back to the definition of Hales numberings now, we see that it only remains to prove that the initial segments, S_l, of the numberings described fulfil Condition (i), i.e., that they maximize $\Phi_l' = l - \Phi_l$ for each l. We shall actually show that the S_l's are essentially all of the sets which maximize Φ_l' and thus that they are the unique solutions of the isoperimetric problem.

First we note that all sets of vertices maximizing Φ_l' are essentially two-dimensionally stable. To demonstrate this we will now describe a series of transformations which will change S to a two-dimensionally

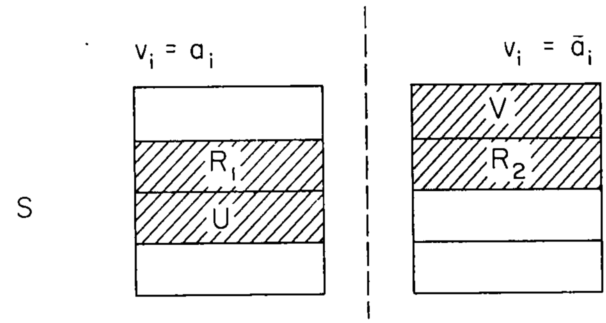

FIGURE 1

stable set without decreasing Φ_l'. Divide the n-cube into two $(n-1)$-cubes, $v_i = a_i$ and $v_i = \bar{a}_i$. Then S is also divided into two parts. Each of these can, in turn, be divided into two parts, R_1, U and R_2, V as shown in Figure 1. The subsets R_1 and R_2 are equinumerous and correspond to pairs of vertices differing only in the i-th coordinate. U and V are

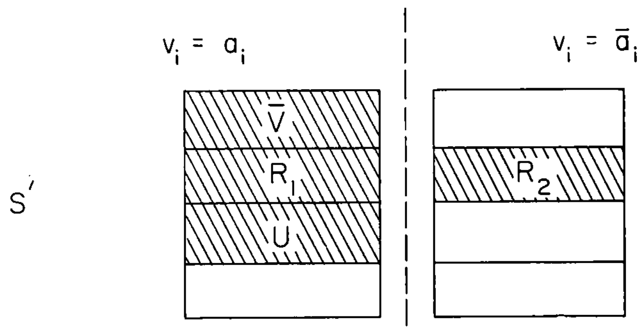

FIGURE 2

the sets of vertices whose corresponding vertices are not present in S. Define the transformazion T_i then to compliment the i-th coordinate of V and leave everything else fixed. (Figure 2.) Note that the distances between vertices within $\bar{V}$ are unchanged from those in V. Also every edge between a vertex in V and one in R_2 is replaced by a corresponding edge between $\bar{V}$ and R_1. Then only difference between S and S' where $T_i : S \to S'$ is that the distances between vertices in U and $\bar{V}$ are less than the corresponding distances between U and V. Thus $\Phi_l{}'$ of S' cannot be less than $\Phi_l{}'$ of S.

Let S be a one-dimensionally stable set with origin A. Divide the n-cube into quadrants by restricting the i-th and j-th coordinates as in Figure 3. Then define the transformation T_{ij} to compliment U_3 in

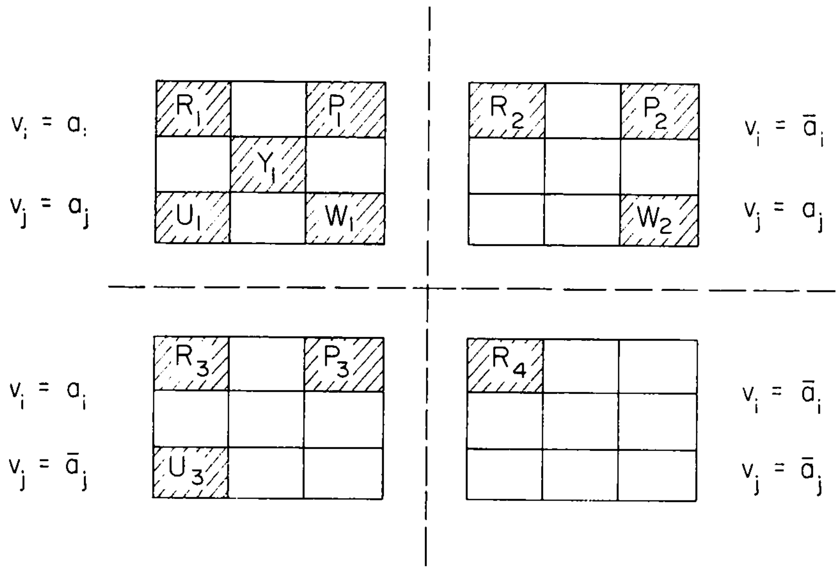

FIGURE 3

the i-th and j-th coordinates. This changes S to S' as shown in Figure 4. Again the effect of T_{ij} is to decrease distance between vertices in S' and thereby increase $\Phi_l{}'$. This same arguement appears in [2].

If we apply T_i to S for all i, then S will become one-dimensionally stable. If we then apply T_{ij} for all i and j it will become two-dimensionally stable. Conversely, if the set S maximizes $\Phi_l{}'$, then the inverse of any of these transformations will strictly decrease $\Phi_l{}'$ unless it just moves those vertices in S having no interior vertices as nearest neighbors. Thus, except for this small extra set, S will be two-dimensionally stable.

If $l = 1, 2, ..., n + 1$, then it is trivial that any set of l vertices which maximizes Φ_l' is essentially S_l, the beginning segment of some numbering

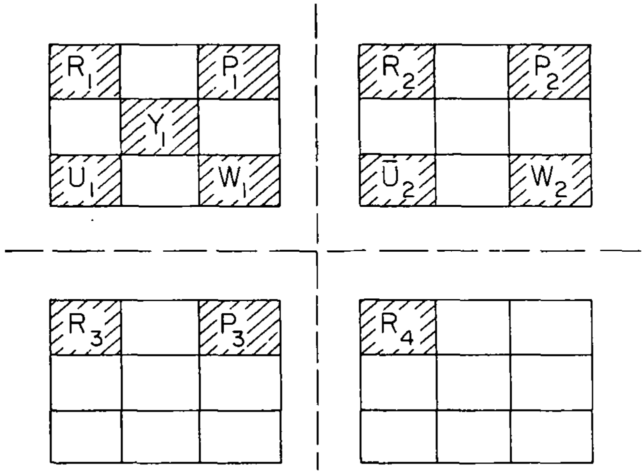

FIGURE 4

as described in Theorem 1. Assume be way of induction then that this is true for $l - 1 \geq n + 1$ and that S is a set of l vertices maximizing Φ_l'. If S contains a vertex none of whose nearest neighbors are interior vertices and which is not an interior vertex itself, then we could remove it from S without lowering the number of interior vertices and $\Phi_l' = \Phi_{l-1}'$. By induction, then, S is an S_{l-1} with an extra vertex placed at random. If this is not the case, then every vertex in S is an interior vertex or the nearest neighbor of one. Since this means that S is two-dimensionally stable, we may assume that it is in normal form.

Now, dividing the n-cube into two $(n - 1)$-cubes by the rightmost coordinate also divides S into two parts having a and b vertices in each part $a \geq b$ and $a + b = l$. Obviously a and b are non-zero since $l \geq n + 1$ implies $\Phi_l' > 0$. The quantity

$$\max \Phi_b'(n - 1) + \min\{\max \Phi_a'(n - 1), b\}$$

sets an upper bound on Φ_l'. But by induction on n we can maximize $\Phi_a'(n - 1)$ and $\Phi_b'(n - 1)$ in each subcube with beginning segments

S_a and S_b of numberings of the $(n-1)$-cube, and this can obviously be done so that the upper bound is achieved.

We desire to demonstrate the existence of a numbering φ such that S is the beginning segment S_l for φ. So far we have only shown that the a configuration in the subcube $v_{n-1} = 0$ is an S_a for some numbering φ_0 of the $(n-1)$-cube; that the b configuration in the subcube $v_{n-1} = 1$ is an S_b for some numbering φ_1 of the $(n-1)$-cube; and that S_b is either a subset of the vertices in that subcube which correspond (by being the unique neighbors in the subcube) to the interior vertices of S_a, or it contains them. If we can show that

$$\Phi'_{a-1}(n-1) < b \leq \Phi_a'(n-1)$$

then we can say that S_b is actually contained in the set corresponding to the interior vertices of S_a but not in any smaller such set. Also these inequalities make it clear that the beginning segment of length a for φ_1 is just the set corresponding to S_a. Thus we may assume that $\varphi_0 = \varphi_1$ modulo the natural isomorphism between the two subcubes. From φ_0 we construct a numbering φ as follows: the vertices of the subcube $v_{n-1} = 0$ will be numbered before vertices in $v_{n-1} = 1$ whenever there is an arbitrary choice in the algorithm of Theorem 1. Also the vertices in $v_{n-1} = 0$ will be numbered in the order given by φ_0. It is easily seen that φ is a numbering as described in Theorem 1 and that $S_l = S_a \, US_b = S$ for φ.

To demonstrate that the inequalities

$$\Phi'_{a-1}(n-1) < b \leq \Phi_a'(n-1)$$

hold we must eliminate two possibilities:

(i) $b \leq \Phi'_{a-1}(n-1)$. But then

$$\Phi_l' = \Phi'_{a-1}(n-1) + \Phi_b'(n-1) = \Phi'_{l-1},$$

contrary to hypothesis.

(ii) $b > \Phi_a'(n-1)$. In this case some vertex in the b configuration must be the nearest neighbor of a vertex in the subcube $v_{n-1} = 0$, which is either not in the a configuration and thus not in S, or which is in the a configuration but is not an interior vertex of it. These two possibilities both contradict the assumption that S is a two-dimensionally stable set.

III. Many variants and corollaries of the preceding analysis have been worked out but we shall only list the results here:

a) Any set of 2^n real numbers may bes assigned. Order them, and (the same analysis shows that the Hale numberings will achieve $\min_\varphi \max_e \Delta_e$ for all sets of 2^n real numbers and so the numberings of Theorem 1 are unique in this respect.

(b) The numberings of Theorem 1 will also minimzie $\max_e \Delta_e$ if the set of edges, E, is extended to E_d, the set of all pairs of vertices within Hamming distance d, $0 \leq d \leq n$.

(c) $\max \min_e \Delta_e$ may be achieved by taking any Hales numbering, first numbering the even weight vertices and them the odd weight vertices in the order of the Hales numbering.

(d) By induction

$$\min_\varphi \max_e \Delta_e = \sum_{m=0}^{n-1} \binom{m}{[m/2]}$$

on the n-cube.

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