A Logical Approach to Asymptotic Combinatorics II: Monadic Second-Order Properties

KEVIN J. COMPTON*

Wesleyan University, Middletown, Connecticut 06457

Communicated by Gian-Carlo Rota

Received June 29, 1987

INTRODUCTION

In Compton [3] we began an investigation of a general approach to asymptotic problems in combinatorics. These problems involve determining probabilities of certain properties holding in large, finite structures randomly chosen from certain classes. The properties we considered were those expressible in first order logic $L_{\omega\omega}$, and the classes were those closed under disjoint unions and components. We now consider, for the same classes, a general approach to problems about properties expressible in monadic second order logic (here denoted L_{2M}).

The main results in [3] characterized classes for which the asymptotic probabilities of a set of first-order sentences called *component-bounded sentences* exist, and classes for which every first-order sentence has asymptotic probability either 0 or 1 (such classes are said to have a first order 0–1 *law*). The latter characterization—which, strictly speaking, holds only for non-fast growing classes—yielded a precise description of the set of first-order sentences having probability 1. In a later paper (Compton [2]) we showed that this set need not be recursive, even if the class of structures is finitely first-order axiomatizable, and thus that the problem of determining asymptotic probabilities is undecidable.

There are good reasons for considering monadic second-order properties. Within monadic second-order logic we may express combinatorially interesting properties, such as connectivity, not expressible within first-order logic; and although monadic second-order logic lacks completeness and compactness theorems, tools instrumental in our work on first-order properties, e.g. Ehrenfeucht game techniques, do pertain (Lynch [9, 10]

110

has used Ehrenfeucht games to obtain results on asymptotic probabilities of first order properties).

Our approach will be as follows. In Section 2 we prove by means of Ehrenfeucht games some fundamental results about classes satisfying monadic second-order sentences. In Section 3 we define an extended notion of asymptotic probability. Using the results of Section 2, we give, in Section 4, sufficient conditions for the existence of extended asymptotic probabilities of monadic second-order sentences. We must then be able to infer the existence of asymptotic probabilities from the existence of extended asymptotic probabilities. They key for doing this is a pair of Tauberian theorems discussed in Section 5. The main results of the paper are in Section 6. Theorem 6.1 shows that whenever the extended asymptotic of a monadic second-order sentence is 0 or 1 then the sentence has an asymptotic probability (necessarily the same value). From this follows Corollary 6.2, a result showing that in rather general circumstances the asymptotic probability of connectivity in a random structure is 0. Theorems 6.3 and 6.4 extend our earlier results to a characterization of non-fast growing classes with monadic second-order 0–1 laws. Unfortunately, our methods do not yield, as they did in the first-order case, a precise description of the set of sentences having asymptotic probability 1. Theorems 6.6 and 6.8 give sufficient conditions for the existence of asymptotic probabilities for all monadic second-order sentences.

The author would like to thank Ward Henson for helpful discussions.

1. PRELIMINARIES

We will assume that the reader is familiar with the notation and definitions of Compton [3].

As we noted in the Introduction, our chief concern is with the properties expressible in monadic second-order logic. For a language L of relation and constant symbols (for technical reasons we exclude function symbols), we denote by L_{2M} the set of monadic second-order sentences over these symbols. Besides the symbols in L and the usual element variables, connectives, quantifiers, and delimiters of first-order logic, monadic second-order sentences may include set variables $X, Y, Z, ...$ which may be quantified, and instances of the member relation symbol ε. The syntax and semantics of this language are natural extensions of those for first-order logic. We may also define the *quantifier rank* of a monadic second-order sentence φ, denoted $\mathrm{qr}(\varphi)$, by extension of the first-order notion: $\mathrm{qr}(\varphi)$ is the maximal number of quantifier nestings (counting quantifications of both element and set variables) in φ. If structures $\mathfrak{A}$ and $\mathfrak{B}$ satisfy precisely the same

monadic second-order sentences we write $\mathfrak{A} \equiv'_n \mathfrak{B}$. It is straightforward to show that if L is finite, there are only finitely many $\equiv'_n$-equivalence classes.

Let L_i, $i < k$, be languages with the same relation symbols but such that no two have constant symbols in common. Suppose that for each $i < k$, $\mathfrak{A}_i$ is an L_i-structure, and let $L = \bigcup_{i<k} L_i$. Define $\amalg_{i<k} \mathfrak{A}_i$ to be the L-structure with the disjoint union of the sets A_i as universe, with each relation symbol interpreted by the union of the interpretations in structures $\mathfrak{A}_i$, $i < k$, and with each constant symbol in L_i interpreted by the element which interpreted it in $\mathfrak{A}_i$.

We recall the definition of an *admissible function* in Hayman [8]. This was used in Compton [3].

DEFINITION. Suppose that $a(z) = \sum_{n=0}^\infty a_n z^n$ has radius of convergence $R > 0$, where $a_n \in \mathscr{C}$, and for some $R_0 < R$, $a(r) > 0$ when $R_0 < r < R$. Let

$$f(z) = za'(z)/a(z)$$

$$g(z) = zf'(z).$$

$a(z)$ is admissible if

$$\lim_{r \to R} g(r) = \infty$$

and there is a function $\delta(r)$ defined for $R_0 < r < R$ with $0 < \delta(r) < \pi$ such that

$$a(re^{i\theta}) \sim a(r) \exp(i\theta f(r) - \tfrac{1}{2}\theta^2 g(r))$$

as $r \to R$, uniformly for $|\theta| \leqq \delta(r)$, and

$$a(re^{i\theta}) = o(a(r) \theta g(r)^{-1/2})$$

as $r \to R$, uniformly for $\delta(r) \leqq |\theta| \leqq \pi$ (here r ranges over the reals).

The main theorem of Hayman [8] gives an asymptotic formula for the coefficients of admissible functions:

THEOREM 1.1. *If* $a(z) = \sum_{n=0}^\infty a_n z^n$ *is admissible, then, letting* $r_n = f^{-1}(n)$,

$$a_n \sim \frac{1}{2\pi r_n^n} \int_{-\delta(r_n)}^{\delta(r_n)} a(r_n e^{i\theta}) e^{-in\theta} d\theta$$

$$\sim \frac{a(r_n)}{r_n^n \sqrt{2\pi b(r_n)}}$$

as $n \to \infty$. (*We also remark, for later reference, that we may always assume that* $\delta(r) \to 0$ *as* $r \to R$.)

2. Ehrenfeucht Games and L_{2M} Sentences

L_{2M} is a more powerful logic than $L_{\omega\omega}$; in L_{2M} we can express interesting combinatorial notions such as connectivity. Unfortunately, we have fewer logical tools for L_{2M}. It has no completeness or compactness theorem so we cannot prove an 0–1 law for L_{2M} by showing, as was done for $L_{\omega\omega}$ in Fagin [5] and Compton [3], that axioms for some complete theory have asymptotic probability 1.

One of the few logical techniques that does carry over from $L_{\omega\omega}$ to L_{2M} is the method of Ehrenfeucht games, introduced in Ehrenfeucht [4]. It will be the primary technique of our investigation. In this section, we describe Ehrenfeucht games for L_{2M} and prove some basic facts about them.

Let L be a language containing relation symbols R_i, $i \in I$, and constant symbols c_j, $j \in J$. Unless otherwise noted, I and J may be either finite or infinite in this section.

An Ehrenfeucht game is played by two players I and II on L-structures $\mathfrak{A}$ and $\mathfrak{B}$. Player I attempts to show that $\mathfrak{A}$ and $\mathfrak{B}$ do not look alike and player II attempts to show the contrary. The game lasts for n rounds (or has *length* n). During each round player I chooses either an element or subset from the universe of either one of the L-structures and player II responds by choosing the same kind of object (element or subset) from the universe of the other L-structure. If the following conditions hold when play terminates then player II wins; otherwise player I wins:

(i)   If $a_0, a_1 \in A$, $b_0, b_1 \in B$, and for $i = 0, 1$, either a_i and b_i interpret the same constant symbol or are elements chosen in the same round, then

$$\mathfrak{A} \models a_0 = a_1 \qquad \text{iff} \qquad \mathfrak{B} \models b_0 = b_1.$$

(ii)   If R is a relation symbol in L of arity α, $a_0, a_1, ..., a_{\alpha-1} \in A$, $b_0, b_1, ..., b_{\alpha-1} \in B$, and for each $i < \alpha$ either a_i and b_i interpret the same constant symbol in L or are elements chosen in the same round, then

$$\mathfrak{A} \models R(a_0, a_1, ..., a_{\alpha-1}) \qquad \text{iff} \qquad \mathfrak{B} \models R(b_0, b_1, ..., b_{\alpha-1})$$

(iii)   If $a \in A$ and $b \in A$ interpret the same constant symbol in L or are elements chosen in the same round, and $A' \subseteq A$ and $B' \subseteq B$ are subsets chosen in the same round, then

$$\mathfrak{A} \models a \in A' \qquad \text{iff} \qquad \mathfrak{B} \models b \in B'.$$

In other words, player II wins if the substructures of $\mathfrak{A}$ and $\mathfrak{B}$ generated by elements chosen during the game, with new relations formed from subsets chosen during the game, are isomorphic in the obvious way.

The following theorem shows the utility of Ehrenfeucht games.

THEOREM 2.1.   *Player* II *has a winning strategy for the Ehrenfeucht game of length n played on L-structures* $\mathfrak{A}$ *and* $\mathfrak{B}$ *iff* $\mathfrak{A} \equiv_n' \mathfrak{B}$.

We remark that in the original theorem for $L_{\omega\omega}$ due to Fraïsse [6] and Ehrenfeucht [4], the players choose only elements during the game; the proof here is nearly identical.

THEOREM 2.2.   *Let* L_i, $i < k$, *be as in the definition of disjoint union in Section* 1, *and let* $\mathfrak{A}_i, \mathfrak{B}_i$ *be* L_i-*structures with* $\mathfrak{A}_i \equiv_n' \mathfrak{B}_i$ *for* $i < k$. *Then* $\amalg_{i \leqslant k} \mathfrak{A}_i \equiv_n' \amalg_{i < k} \mathfrak{B}_i$.

Proof. This is an easy consequence of Theorem 2.1. Since $\mathfrak{A}_i \equiv_n' \mathfrak{B}_i$, $i < k$, player II has a winning strategy for the Ehrenfeucht game of length n on $\mathfrak{A}_i$ and $\mathfrak{B}_i$. She has a winning strategy for the game on $\amalg_{i<k} \mathfrak{A}_i$ and $\amalg_{i<k} \mathfrak{B}_i$ by combining strategies in the obvious way. ∎

THEOREM 2.3.   *Assume that* L *has no constant symbols. For every* $n \in \omega$ *there is an* $N \in \omega$ *such that whenever* $\kappa \geq \lambda \geq N$ *and* $\mathfrak{A}_i$, $i < \kappa$, *is a sequence of L-structures from some* $\equiv_n'$-*equivalence class, then*

$$\underset{i < \kappa}{\amalg} \mathfrak{A}_i \equiv_n' \underset{i < \lambda}{\amalg} \mathfrak{A}_i.$$

Proof. Suppose L consists of relation symbols R_i of arity α_i, $i \in I$, and constant symbols c_j, $j \in J$, with I and J finite. Let $\mathfrak{A}$ be an L-structure, $\mathbf{a} \in A^p$ an arbitrary sequence of elements, and $\mathbf{A} \in (2^A)^q$ an arbitrary sequence of subsets. We define $\tau_n^{\mathfrak{A}}(\mathbf{a}, \mathbf{A})$, the n-quantifier rank type of $\mathbf{a}, \mathbf{A}$ in $\mathfrak{A}$ inductively:

$$\tau_0^{\mathfrak{A}}(\mathbf{a}, \mathbf{A}) = \langle f, \mathbf{g}, h \rangle,$$

where $f: (p \cup J)^2 \to 2$, $\mathbf{g} = \langle g_i \rangle_{i \in I}$, $g_i: (p \cup J)^{\alpha_i} \to 2$, and $h: (p \cup J) \times q \to 2$ (assume that $p \cap J = \varnothing$). For $j \in J$ let a_j be interpretation of c_j in $\mathfrak{A}$. These functions are defined by

$$f(i, j) = 1 \quad \text{iff} \quad \mathfrak{A} \models a_i = a_j, \quad \cdot$$

$$g_i(j_0, j_1, ..., j_{\alpha-1}) = 1 \quad \text{iff} \quad \mathfrak{A} \models R_i(a_{j_0}, a_{j_1}, ..., a_{j_{\alpha-1}}), \; \alpha = \alpha_i,$$

$$h(i, j) = 1 \quad \text{iff} \quad \mathfrak{A} \models a_i \in A_j.$$

Clearly, there are only finitely many such f, g_i, and h for given p and q. Now define

$$\tau_{k+1}^{\mathfrak{A}}(\mathbf{a}, \mathbf{A}) = \{ \tau_k^{\mathfrak{A}}(\langle \mathbf{a}, a_p \rangle, \mathbf{A}) : a_p \in A \} \cup \{ \tau_k^{\mathfrak{A}}(a, \langle \mathbf{A}, A_q \rangle) : A_q \subseteq A \}.$$

It follows easily by induction that for given n, p, and q that $\tau_n^{\mathfrak{A}}(\mathbf{a}, \mathbf{A}) = \tau_n^{\mathfrak{B}}(\mathbf{b}, \mathbf{B})$ iff $\langle \mathfrak{A}, \mathbf{a}, \mathbf{A} \rangle \equiv_n' \langle \mathfrak{B}, \mathbf{b}, \mathbf{B} \rangle$. Let

$$\rho_0^{\mathfrak{A}}(\mathbf{A}) = 1,$$

$$\rho_{k+1}^{\mathfrak{A}}(\mathbf{A}) = \max\left(1 + \rho_k^{\mathfrak{A}}(\mathbf{A}), \sum \rho_k^{\mathfrak{A}}(\langle \mathbf{A}, A_q \rangle)\right),$$

where the summation in the second expression is taken over a finite set of subsets A_q such that $\tau_k^{\mathfrak{A}}(\phi, \langle \mathbf{A}, A_q \rangle)$ ranges over all possible values. It is easily verified that if $\tau_n^{\mathfrak{A}}(\phi, \mathbf{A}) = \tau_n^{\mathfrak{B}}(\phi, \mathbf{B})$ then $\rho_n^{\mathfrak{A}}(\mathbf{A}) = \rho_n^{\mathfrak{B}}(\mathbf{B})$.

We now show that if $\langle \mathfrak{A}, \mathbf{A} \rangle \equiv_n' \langle \mathfrak{A}_i, \mathbf{A}_i \rangle$, $i < \kappa$ and $\kappa > \lambda \geqq \rho_n^{\mathfrak{A}}(\mathbf{A})$ then $\amalg_{i < \kappa} \langle \mathfrak{A}_i, \mathbf{A}_i \rangle \equiv_n' \amalg_{i < \lambda} \langle \mathfrak{A}_i, \mathbf{A}_i \rangle$. This is easily seen by induction on n.

Observe I and II playing an Ehrenfeucht game of length n on these two disjoint unions.

Suppose player I begins by choosing an element from (say) the first of these disjoint unions. He has, in effect, picked a structure $\langle \mathfrak{A}_k, \mathbf{A}_k \rangle$ for some $k < \kappa$ and then picked an element a in $\mathfrak{A}_k$. Player II responds by picking any structure $\langle \mathfrak{A}_l, \mathbf{A}_l \rangle$, $l < \lambda$, and then using her winning strategy to pick an element a' in $\mathfrak{A}_l$. Thus,

$$\langle \mathfrak{A}_k, \mathbf{A}_k, a \rangle \equiv_{n-1}' \langle \mathfrak{A}_l, \mathbf{A}_l, a' \rangle.$$

But also, by the induction hypothesis and the definition of $\rho_n^{\mathfrak{A}}(\mathbf{A})$,

$$\coprod_{\substack{i < \kappa \\ i \neq k}} \langle \mathfrak{A}_i, \mathbf{A}_i \rangle \equiv_{n-1}' \coprod_{\substack{i < \lambda \\ i \neq l}} \langle \mathfrak{A}_i, \mathbf{A}_i \rangle.$$

Combining these two facts, we see from Theorem 2.2 that player II has a winning strategy for the remaining $n - 1$ rounds of the game.

Suppose player I begins by choosing a subset from one of the disjoint unions (again, say the first). He has, in effect, chosen subsets $A_{iq} \subseteq \mathfrak{A}_i$, $i < \kappa$. Player II responds by choosing a subset from the second disjoint union—i.e., subsets $A_{iq} \subseteq \mathfrak{A}_i$, $i < \lambda$—in the following way. For any structure $\langle \mathfrak{A}, \mathbf{A}, A_q \rangle$, if there are less than $\rho = \rho_{n-1}^{\mathfrak{A}}(\langle \mathbf{A}, A_q \rangle)$ structures of type $\tau_{n-1}^{\mathfrak{A}}(\phi, \langle \mathbf{A}, A_q \rangle)$ among the structures $\langle \mathfrak{A}_i, \mathbf{A}_i, A_{iq} \rangle$, $i < \kappa$, then she chooses so that there are precisely the same number of this type among the structures $\langle \mathfrak{A}_i, \mathbf{A}_i, A_{iq} \rangle$, $i < \lambda$; if there are not less than ρ structures of this type among $\langle \mathfrak{A}_i, \mathbf{A}_i, A_{iq} \rangle$, $i < \kappa$, then she chooses so that there are not less than ρ structures of this type (but possibly not the same number) among $\langle \mathfrak{A}_i, \mathbf{A}_i, A_{iq} \rangle$, $i < \lambda$. Notice that $\rho_n^{\mathfrak{A}}(\mathbf{A})$ was chosen large enough so that this can always be done. Therefore, by Theorem 2.2 and the induction hypothesis, the disjoint union of the structures among $\langle \mathfrak{A}_i, \mathbf{A}_i, A_{iq} \rangle$, $i < \kappa$, in a given $\equiv_{n-1}'$-equivalence class is $\equiv_{n-1}'$ to the disjoint union of

structures among $\langle \mathfrak{A}_i, \mathbf{A}_i, A_{iq} \rangle$ in the same equivalence class. Again invoking Theorem 2.2 we have

$$\coprod_{i < \kappa} \langle \mathfrak{A}_i, \mathbf{A}_i, A_{iq} \rangle \equiv'_{n-1} \coprod_{i < \lambda} \langle \mathfrak{A}_i, \mathbf{A}_i, A_{iq} \rangle$$

and again player II has a winning strategy for the remaining $n - 1$ moves in the game.

The result now follows since there are finitely many $\equiv'_n$-classes. For $n \in \omega$ take N to be the maximum of the values $\rho_n^{\mathfrak{A}}(\varnothing)$. ∎

3. Extended Asymptotic Probabilities

In this section, we extend the notion of asymptotic probability given in Section 1. This will serve two purposes. First, we will sometimes be able to infer the existence of asymptotic probabilities for monadic second-order sentences from the existence of extended asymptotic probabilities. The means for doing this are Tauberian theorems discussed in Section 5. Second, the asymptotic probability of a sentence may fail to exist but we may nonetheless wish to assign some kind of probability to it.

Consider the following class $\mathscr{C}$ which is closed under disjoint unions and components. The language of $\mathscr{C}$ contains two symbols—R, a binary relation symbol, and S, a unary relation symbol. $\mathscr{C}$ consists of all structures such that R interprets a partial order for which the elements lying below any given element are linearly ordered and each element has no more than two immediate successors, and such that S holds of an element if and only if the number elements strictly above it is even. $\mathscr{C}$ may be identified with the class of unary–binary forests with odd subtree relation (this is similar to Example 7.10 of [3]). $\mathscr{C}$ is finitely first-order axiomatizable and has an exponential generating series

$$a(x) = \sum_{}^{\infty} \frac{a_n}{n!} x^n = \exp(k(x)),$$

where

$$k(x) = (1 - x - \sqrt{1 - 2x - x^2})/x.$$

(See [3].) Let φ be a first-order sentence which says "S holds at no roots" (i.e., "all trees in the forest have even cardinality"). The exponential generating series for the subclass satisfying φ is

$$c(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n = \exp \frac{k(x) + k(-x)}{2}.$$

Now by Darboux's theorem (see Szegö [11] or Bender [1]), one can show that

$$\frac{a_n}{n!} \sim \left(\frac{1+R^2}{4\pi n^3}\right)^{1/2} \frac{a(R)}{R^{n+1}},$$

$$\frac{c_n}{n!} \sim \left(\frac{1+R^2}{4\pi n^3}\right)^{1/2} \frac{c(R)}{R^{n+1}}, \qquad n \text{ even},$$

where $R = -1 + \sqrt{2}$. Clearly $c_n = 0$ when n is odd. Thus $\mu(\varphi)$ does not exist. However, φ does have labeled asymptotic probability $c(R)/a(R)$ when the class is restricted to even structures and 0 when the class is restricted to odd structures. An extended asymptotic probability should somehow average these two values. Also, it should agree with the value of the asymptotic probability whenever the asymptotic probability exists. The following proposition will allow us to formulate such a definition.

PROPOSITION 3.1. *Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence $R > 0$ with $a_n \geqq 0$ and $\lim_{x \to R} a(x) = \infty$. If b_n is a sequence such that $\lim_{n \to \infty} b_n/a_n = P$ and $b(x) = \sum_{n=0}^{\infty} b_n x^n$ then $\lim_{x \to R} b(x)/a(x) = P$.*

The proof is straightforward and will not be included here. See Titchmarsh [12, Section 7.5], or Hardy's exposition [7], in which he investigates "summing" divergent series. It is used to show that some of the summability methods he uses are *regular*, i.e., that they coincide with ordinary summation for series that do converge.

Our concern is not with "summing" divergent series, but with finding the "limit" of a divergent sequence. This is really what Hardy did—he worked with an extended notion of a limit for the partial sums of his series.

Now suppose that $a(x)$ is the exponential generating series for a class $\mathscr{C}$ of structures and that $a(x)$ has radius of convergence $R > 0$, with $\lim_{x \to R} a(x) = \infty$. For a sentence φ let c_n be the number of structures in $\mathscr{C}$ of size n which satisfy φ and $c(x) = \sum_{n=0}^{\infty} (c_n/n!) x^n$. If we define

$$\bar{\mu}(\varphi) = \lim_{x \to R} c(x)/a(x)$$

then Proposition 3.1 shows that $\bar{\mu}(\varphi) = \mu(\varphi)$ whenever $\mu(\varphi)$ exists. In fact, we can easily show that $\mu^*(\varphi) = \mu(\varphi)$ whenever $\mu^*(\varphi)$ exists. ($\mu^*(\varphi)$ was defined in Section 6 of Compton [3].)

These results are contingent on $a(x)$ approaching ∞ as $x \to R$. Suppose that this is not the case, as in our example of unary–binary forests. If $\lim_{x \to R} a'(x) = \infty$ we could define

$$\mu(\varphi) = \lim_{x \to R} c'(x)/a'(x).$$

By l'Hopital's rule, this has the same value as $\lim_{x \to R} c(x)/a(x)$ when $\lim_{x \to R} a(x) = \infty$. In general, if $\lim_{x \to R} a^{(j)}(x) = \infty$ we could define

$$\mu(\varphi) = \lim_{x \to R} c^{(j)}(x)/a^{(j)}(x).$$

Again, this extends the notion of labeled asymptotic probability. For if $\lim_{n \to \infty} c_n/a_n = P$ and $\lim_{x \to R} a^{(j)}(x) = \infty$ then

$$\lim_{x \to R} \frac{c^{(j)}(x)}{a^{(j)}(x)} = \lim_{x \to R} \frac{\sum_{n=0}^{\infty} (c_{n+j}/n!) \, x^n}{\sum_{n=0}^{\infty} (a_{n+j}/n!) \, x^n}$$

$$= \lim_{n \to \infty} \frac{c_{n+j}/n!}{a_{n+j}/n!}$$

$$= P$$

by Proposition 3.1. A similar argument shows that this extends the defintion of $\mu^*(\varphi)$, the generalized asymptotic probability of φ.

Let us see how this definition works in our example. Since $\lim_{x \to \infty} a'(x) = \infty$ we have by a simple calculation

$$\mu(\varphi) = \lim_{x \to R} c'(x)/a'(x)$$

$$= (c(R)/a(R))/2,$$

which is the value we sought.

But what if $\lim_{x \to R} a^{(j)}(x) < \infty$ for all j? We use the following definition for this case.

DEFINITION. Let $\mathscr{C}$ be a class of structures with exponential generating series $a(x)$ and let $c(x)$ be the exponential generating series for the subclass of structures that satisfy φ. The *extended labeled asymptotic probability* of φ is defined as

$$\bar{\mu}(\varphi) = \lim_{j \to \infty} \lim_{x \to R} c^{(j)}(x)/a^{(j)}(x).$$

Similarly, if $\mathscr{C}$ has an ordinary generating series with a radius of convergence S and the subclass of structures satisfying φ has an ordinary generating series $d(x)$, then the *extended unlabeled asymptotic probability* of φ is defined as

$$\bar{v}(\varphi) = \lim_{j \to \infty} \lim_{x \to S} d^{(j)}(x)/b^{(j)}(x).$$

The following proposition justifies these definitions.

PROPOSITION 3.2.   *Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence $R > 0$ and $a_n \geq 0$. If b_n is a sequence such that $\lim_{n \to \infty} b_n/a_n = P$ and $b(x) = \sum_{n=0}^{\infty} b_n x^n$, then*

$$\lim_{j \to \infty} \lim_{x \to R} b^{(j)}(x)/a^{(j)}(x) = P.$$

Proof.   For $\varepsilon > 0$, choose N so that $|b_k/a_k - P| < \varepsilon$ when $k > N$. Then

$$\left| \frac{b^{(j)}(x)}{a^{(j)}(x)} - P \right| \left| \frac{\sum_{n=0}^{\infty} (n+1)(n+2) \cdots (n+j)(b_{n+j} - P a_{n+j}) x^n}{\sum_{n=0}^{\infty} (n+1)(n+2) \cdots (n+j) a_{n+j} x^n} \right| < \varepsilon$$

when $j > N$, since $|b_{n+j} - P a_{n+j}| < \varepsilon a_{n+j}$. We obtain the desired result by making ε small.   ∎

This proposition shows that the notion of extended asymptotic probability coincides with the usual notion whenever the usual notion is defined.

For the next two results, we assume that $\mathscr{C}$ is closed under disjoint unions and components, has exponential generating series $a(x)$ with radius of convergence R, and has ordinary generating series $b(x)$ with radius of convergence S. Component-bounded sentences $\theta_{\mathfrak{R}, j}$ were defined in Section 5 of Compton [3].

PROPOSITION 3.3.   *Let $\mathfrak{R}_0, \mathfrak{R}_1, ..., \mathfrak{R}_{q-1}$ be non-isomorphic finite connected structures in $\mathscr{C}$ with $|\mathfrak{R}_i| = m_i$, $i < q$, and $j_0, ..., j_{q-1} \in \omega$:*

(i)   *If $\lim_{x \to R} a^{(j)}(x) = \infty$ for some j then*

$$\bar{\mu} \left(\bigwedge_{i < q} \theta_{\mathfrak{R}_i, j_i} \right) = \prod_{i < q} \frac{\lambda_i^{j_i}}{j_i!},$$

where $\lambda_i = R^{m_i}/\sigma(\mathfrak{R}_i)$.

(ii)   *If $\lim_{v \to S} b^{(j)}(x) = \infty$ for some j then*

$$\bar{v} \left(\bigwedge_{i < q} \theta_{\mathfrak{R}_i, j_i} \right) = \prod_{i < q} S^{m_i j_i} (1 - S^{m_i}).$$

Proof.   (i)   From Lemma 2.1(ii) in Compton [3] the exponential generating series for $\varphi = \bigwedge_{i < q} \theta_{\mathfrak{R}_i, j_i}$ is

$$c(x) = a(x) \prod_{i < q} \frac{1}{j_i!} \left(\frac{x^{m_i}}{\sigma(\mathfrak{R}_i)} \right)^{j_i} \exp \left(-\frac{x^{m_i}}{\sigma(\mathfrak{R}_i)} \right).$$

Take j to be the least integer such that $\lim_{x \to R} a^{(j)}(x) = \infty$. Then

$$\bar{\mu}(\varphi) = \lim_{x \to R} \frac{c^{(j)}(x)}{a^{(j)}(x)},$$

which is equal to

$$\lim_{x \to R} \left(a^{(j)}(x) \prod_{i < q} \frac{1}{j_i!} \left(\frac{x^{m_i}}{\sigma(\Re_i)} \right)^{j_i} \exp\left(-\frac{x^{m_i}}{\sigma(\Re_i)} \right) \right.$$
$$\left. + \sum_{k < j} a^{(k)}(x) 0(1) \bigg/ a^{(j)}(x) \right)$$

This establishes (i). We prove (ii) similarly. ∎

PROPOSITION 3.4. *Let φ be a component-bounded sentence:*

(i) *If $\lim_{x \to R} a^{(j)}(x) = \infty$ for some j then $\bar{\mu}(\varphi)$ exists.*

(ii) *If $\lim_{x \to S} b^{(j)}(x) = \infty$ for some j then $\bar{v}(\varphi)$ exists.*

Proof. This follows from the previous proposition and Lemmas 5.2 and 5.5 in Compton [3]. ∎

4. EXTENDED ASYMPTOTIC PROBABILITIES OF MONADIC SECOND-ORDER SENTENCES

The main result of the section is Theorem 4.3 which states general conditions under which extended asymptotic probabilities of monadic second-order sentences will exist.

LEMMA 4.1. *Let φ be an L_{2M} sentence with $\mathrm{qr}(\varphi) = n$ and N the integer associated with n in Theorem 2.3. Let $\Re_i$, $i < q$, be representative structures from each of the $\equiv'_n$-equivalence classes of finite connected structures in $\mathscr{C}$, and $k_i(x)$ the exponential generating series for the $\equiv'_n$-equivalence class of connected structures containing $\Re_i$ for each $i < q$. Define $\hat{k}_i^m(x)$, $m \leq N$, by*

$$\hat{k}_i^m(x) = k_i(x)^m / m!, \qquad m < N,$$
$$\hat{k}_i^N(x) = \exp(k_i(x)) - \sum_{m < N} \hat{k}_i^m(x).$$

Then the exponential generating series $c(x)$ for the subclass of structures in $\mathscr{C}$ satisfying φ is

$$\sum \hat{k}_0^{m_0}(x) \hat{k}_1^{m_1}(x) \cdots \hat{k}_{q-1}^{m_{q-1}}(x),$$

where the summation is taken over all $m_0, m_1, ..., m_{q-1} \leqq N$ *such that*

$$\prod_{i < q} m_i \cdot \mathfrak{K}_i \models \varphi.$$

Proof. Define an equivalence relation $\approx$ on the class of finite structures in $\mathscr{C}$ as follows. $\mathfrak{A} = \mathfrak{B}$ iff for each $i < q$, $\mathfrak{A}$ and $\mathfrak{B}$ have either the same number of components in the $\equiv'_n$-equivalence class that contains $\mathfrak{K}_i$ or both have at least N components in this equivalence class.

We claim $\approx$ is a refinement of $\equiv'_n$ on the class of finite structures in $\mathscr{C}$. This follows easily from Theorems 2.2 and 2.3. The former implies that any component of a structure may be replaced by any other component in the same $\equiv'_n$-equivalence class without affecting the $\equiv'_n$-equivalence class to which the structure belongs; the latter implies that if there are at least N components of a structure in a particular $\equiv'_n$-equivalence, this number may be increased or decreased without affecting the equivalence class to which the structure belongs as long as the number remains at least N. Thus, if $\mathfrak{A} \approx \mathfrak{B}$, $\mathfrak{A}$ may be transformed into $\mathfrak{B}$ by a series of such operations, so $\mathfrak{A} \equiv'_n \mathfrak{B}$.

The subclass of structures in $\mathscr{C}$ satisfying φ is a union of $\equiv'_n$-equivalence classes and hence a union of $\approx$-equivalence classes. Therefore $c(x)$ is a sum of exponential generating series for the $\approx$-equivalences that compose the union. The exponential generating series for the subclass of structures in $\mathscr{C}$ with exactly m components, each belonging to the $\equiv'_n$-equivalence class containing $\mathfrak{K}_i$, is $k_i(x)^m/m! = \hat{k}_i^m(x)$. From this we see that the exponential generating series for the subclass of structures in $\mathscr{C}$ with at least N components, each belonging to the $\equiv'_n$-equivalence class containing $\mathfrak{K}_i$, is $\hat{k}_i^N(x)$ (see Compton [3] for details). Consequently, the exponential generating series for a typical $\approx$-equivalence class is of the form

$$\hat{k}_0^{m_0}(x) \, \hat{k}_1^{m_1}(x) \cdots \hat{k}_{q-1}^{m_{q-1}}(x).$$

Summing over $\approx$-equivalence classes contained in the subclass of structures in $\mathscr{C}$ that satisfy φ, we have the expression in the statement of the theorem. ∎

As always, there is a version for unlabeled structures. This one is slightly more complicated than the version for labeled structures.

LEMMA 4.2. *Let* φ *be an* L_{2M} *sentence with* $\mathrm{qr}(\varphi) = n$ *and* N *the integer associated with* n *in Theorem 2.3. Let* $\mathfrak{K}_i$, $i < q$, *be representative structures from each of the* $\equiv'_n$-*equivalence classes of finite connected structures in* $\mathscr{C}$, *and* $l_i(x)$ *the ordinary generating series for the* $\equiv'_n$-*equivalence class of*

connected structures containing $\mathfrak{R}_i$ *for each* $i < q$. *Define* $\hat{l}_i^m(x)$, $i < q$, $m < N$, *by*

$$\hat{l}_i^m(x) = \sum \frac{l_i(x)^{j_1} l_i(x^2)^{j_2} \cdots l_i(x^m)^{j_m}}{1^{j_1} 2^{j_2} \cdots m^{j_m} j_1! j_2! \cdots j_m!},$$

where the summation is taken over non-negative integers $j_1, j_2, ..., j_m$ *such that* $\sum_{i \leq m} i j_i = m$, *and*

$$\hat{l}_i^N(x) = \exp\left(\sum_{m \geq 1} \frac{1}{m} l(x^m)\right) - \sum_{m < N} \hat{l}_i^m(x).$$

Then the ordinary generating series $d(x)$ *for the subclass of structures in* C *satisfying* φ *is*

$$\sum \hat{l}_0^{m_0}(x) \hat{l}_1^{m_1}(x) \cdots \hat{l}_{q-1}^{m_{q-1}}(x),$$

where the summation is taken over all $m_0, m_1, ..., m_{q-1} \leq N$ *such that*

$$\coprod_{i < q} m_i \cdot \mathfrak{R}_i \models \varphi.$$

Proof. We show that when $m < N$, $\hat{l}_1^m(x)$ is the ordinary generating series for the subclass of structures in $\mathscr{C}$ with exactly m components, each belonging to the $\equiv'_n$-equivalence class containing $\mathfrak{R}_i$. The result then follows by Theorem 2.1(iv) in Compton [3] and in the previous theorem.

If $l_i(x) = l(x) = \sum_{n \geq 1} l_n x^n$ is the ordinary generating series for the subclass of connected structures within a class $\mathscr{C}$ closed under disjoint unions and components, then

$$\prod_{n \geq 1} (1 - x^n)^{-l_n}$$

is the ordinary generating series for the entire class. This is shown by observing that $(1 - x^n)^{-1}$ is the ordinary generating series for the subclass of structures with all components isomorphic to a particular connected structure of cardinality n and then taking the product over all connected structures in $\mathscr{C}$. Now if we modify this by taking the product of series $(1 - x^n y)^{-1}$ instead of $(1 - x^n)^{-1}$, the y exponent will count the number of components—i.e., the coefficient of the $x^n y^m$ term in the product

$$\exp\left(\sum_{m \geq 1} \frac{1}{m} l(x^m) y^m\right) = \prod_{m \geq 1} \exp\left(\frac{1}{m} l(x^m) y^m\right)$$

$$= \prod_{m \geq 1} \sum_{j \geq 0} \frac{1}{j!} \left(\frac{1}{m} l(x^m) y^m\right)^j.$$

If we compute, for a fixed $m < N$, the sum of the $x^n y^m$ terms and factor out y^m, we have the expression for $\hat{l}_i^m(x)$ given above. This is the ordinary generating series for the subclass of structures in $\mathscr{C}$ with m components, each belonging to the $\equiv_n'$-equivalence class containing $\Re_i$. ∎

Using Lemmas 4.1 and 4.2 we can prove that extended asymptotic probabilities of L_{2M} sentences exist in many commonly occurring cases.

THEOREM 4.3. *Let φ be an L_{2M} sentence:*

(i) *If $\lim_{x \to R} a(x) = \infty$ then $\bar{\mu}(\varphi)$ exists.*

(ii) *If $\lim_{x \to S} b(x) = \infty$ then $\bar{v}(\varphi)$ exists.*

Proof. We prove (i); the proof of (ii) is only slightly different.

Let $c(x)$ be the exponential generating series for the subclass of structures in $\mathscr{C}$ satisfying φ and suppose $qr(\varphi) = n$. We need to show that $c(x)/a(x)$ approaches a limit as $x \to R$. Since $c(x)$ may be written in the form given in Theorem 4.1, it suffices to show (adopting the notation of Theorem 4.1) that

$$\hat{k}_0^{m_0}(x)\, \hat{k}_1^{m_1}(x) \cdots \hat{k}_{q-1}^{m_{q-1}}(x)/a(x)$$

approaches a limit as $x \to R$. But $a(x) = \exp(\sum_{i < q} k_i(x))$, so we may write the above expression as a product of factors $\hat{k}_i^{m_i}(x)/\exp(k_i(x))$; if we can show that each of these approaches a limit as $x \to R$ we will be done. When $m_i < N$ this is clear, because then $\hat{k}_i^{m_i}(x) = k_i(x)^{m_i}/m_i!$ and $k_i(x)$ is increasing on the interval $[0, R)$. If $m_i = N$, $\hat{k}_i^{m_i}(x)/\exp(k_i(x))$ is

$$1 - \sum_{m < N} \hat{k}_i^m(x)/\exp(k_i(x))$$

and again the limit as $x \to R$. ∎

EXAMPLE 4.4. Theorem 4.3 demonstrates that extended asymptotic probabilities of L_{2M} sentences exist for the following examples in Compton [3]: unary functions (Example 7.5), permutations (Example 7.6), oriented forests of height 1 (Example 7.12), linear forests (Example 7.14), classes with finitely many connected structures (Example 7.15), equivalence relations (Example 7.16), and partitions with selected subsets (Example 7.17).

5. TAUBERIAN THEOREMS

If we set $a_n = 1$, $n = 0, 1, 2, ...$, in Proposition 3.1, we have Abel's theorem. A theorem of this type—which draws a conclusion about the limiting value

of an analytic function at its radius of convergence from information about the limiting value of the coefficients of its power series expansion—is thus called an *Abelian* theorem. Within our framework, we use Abelian theorems to show that extended asymptotic probabilities agree with asymptotic probabilities wherever the latter are defined. Compton [3] uses partial converses to Abelian theorems to show that under certain circumstances the existence of extended asymptotic probabilities implies the existence of asymptotic probabilities. A partial converse to an Abelian theorem is call a *Tauberian* theorem, after A. Tauber, one of the first mathematicians to prove such a theorem (see Hardy [7] for a discussion and history of Tauberian theorems). Theorems 5.1 and 5.3 are examples of Tauberian theorems. For later reference, we reformulate, in Corollaries 5.2 and 5.4, these theorems in terms relating asymptotic probabilities and extended aymptotic probabilities.

THEOREM 5.1. *Suppose that* $a(z) = \sum_{n=0}^{\infty} a_n z_n$, $a_n \in \mathbb{C}$, *and* $\lim_{n \to \infty} ((a_{n-k})/a_n) = R^k$ *for some* $k > 0$ *and* $R \in \mathbb{R}$, $R \geq 0$. *Suppose also that* $b(z)/a(z) = c(z^k)$ *for some series* $b(z) = \sum_{n=0}^{\infty} b_n z_n$ *and* $c(z) = \sum_{n=0}^{\infty} c_n z^n$:

(i)   *If* $c(z^k)$ *has radius of convergence* $S > R$ *and* $\lim_{z \to R} c(z^k) = P$ *then* $\lim_{n \to \infty} b_n/a_n = P$.

(ii)   *If* $c(z^k)$ *has radius of convergence* R, $c(R^k)$ *is absolutely convergent,* $|a_{n-i}|/|a_n| \leq AR^i$ *for some* A *and all sufficiently large* n *and all* $i \leq n$, *and* $\lim_{z \to R} c(z^k) = P$, *then* $\lim_{n \to \infty} b_n/a_n = P$.

Proof. (i) is an easy extension of Theorem 2 in Bender [1]. (ii) follows by a similar argument. ∎

For the next theorem recall the definitions of the period of a class, and of $\mathscr{A} \xrightarrow{*} R$, $\mathscr{B} \xrightarrow{*} S$, $\mu^*(\varphi)$, and $v^*(\varphi)$ defined in Section 6 of Compton [3].

COROLLARY 5.2. *Let* $\mathscr{C}$ *be a class of L-structures closed under disjoint unions and components and suppose* d *is the period of* $\mathscr{C}$. *Let* $a(x)$ *be the exponential generating series and* $b(x)$ *the ordinary generating series for* $\mathscr{C}$. *For a sentence* φ *let* $c(x)$ *be the exponential generating series, and* $d(x)$ *the ordinary generating series, for the subclass of structures in* $\mathscr{C}$ *that satisfy* φ. *Set* $t(x) = c(x)/a(x)$ *and* $u(x) = d(x)/b(x)$; ($t(x)$ *and* $u(x)$ *should be considered power series resulting from the formal division of generating series*):

(i)   *If* $\mathscr{A} \xrightarrow{*} R$ *and* $t(x)$ *has a radius of convergence greater than* R, *then* $\mu^*(\varphi) = \bar{\mu}(\varphi) = t(R)$.

(ii)   *If* $\mathscr{B} \xrightarrow{*} S$ *and* $u(x)$ *has a radius of convergence greater than* S, *then* $v^*(\varphi) = \bar{v}(\varphi) = u(S)$.

(iii)   If $\mathscr{A} \overset{*}{\to} R$, $t(R)$ is absolutely convergent, and for some A,

$$\left| \frac{a(n-i)d/((n-i)\,d)!}{a_{nd}/(nd)!} \right| \le AR^{id}$$

for all sufficiently large n and $i \le n$, then $\mu^*(\varphi) = \bar{\mu}(\varphi) = t(R)$.

(iv)   If $\mathscr{B} \overset{*}{\to} S$, $u(S)$ is absolutely convergent, and for some B, $|b_{(n-1)d}/b_{nd}| \le BS^{id}$ for all sufficiently large n and $i \le n$, then $v^*(\varphi) = \bar{v}(\varphi) = u(S)$.

We remark that this corollary does not really depend on $\mathscr{C}$ being closed under disjoint unions and components. It is true for any class $\mathscr{C}$ of L-structures when the period d is defined in the obvious way.

THEOREM 5.3.   *Suppose that $a(z) = \sum_{n=0}^{\infty} b_n z^n$ is admissible with a radius of convergence R, $b(z) = \sum_{n=0}^{\infty} b_n z^n$, $c(z) = \sum_{n=0}^{\infty} c_n z^n = b(z)/a(z)$, and $c(R)$ is absolutely convergent. Then $\lim_{n \to \infty} b_n/a_n = c(R)$.*

Proof.   Choose $\delta(r)$ for $a(z)$ as in the definition of admissibility. We may assume, by the remark in Theorem 1.1 that $\delta(r) \to 0$ as $r \to R$. Now $b(z) = c(z)a(z)$, so by Cauchy's theorem,

$$b_n = \frac{1}{2\pi r_n^n} \int_{-\pi}^{\pi} c(r_n e^{i\theta}) a(r_n e^{i\theta}) e^{-n\theta} d\theta,$$

where r_n is as in the statement of Theorem 1.1. We will break the interval of integration here into two smaller regions, the first given by $\delta(r_n) < |\theta| \le \pi$ and the second by $|\theta| \le \delta(r_n)$. Since $a(re^{i\theta}) = o(a(r)\,g(r)^{-1/2})$ uniformly and $c(R)$ is absolutely convergent, the first integral is $o(a(r)\,g(r)^{-1/2})$. The second integral

$$\int_{-\delta(r_n)}^{\delta(r_n)} c(r_n e^{i\theta})\, a(r_n e^{i\theta})\, e^{-in\theta}\, d\theta,$$

is asymptotic to

$$c(R) \int_{-\delta(r_n)}^{\delta(r_n)} a(r_n e^{i\theta})\, e^{-in\theta}\, d\theta$$

as $n \to \infty$. This follows because $c(R)$ is absolutely convergent and because, from the definition of admissibility, $a(r_n e^{i\theta})\, e^{-in\theta} \sim a(r) \exp(-\frac{1}{2}\theta^2 g(r))$ (which is positive) uniformly for $|\theta| \le \delta(r_n)$. Combining this with Theorem 1.1, we have $b_n/a_n \sim c(R)$. We have shown, by the way, that $b(z)$ is admissible.   ∎

COROLLARY 5.4. *Let $\mathscr{C}$ be a class of L-structures with exponential generating series $a(x)$ and suppose $a(x)$ has radius of convergence R, $0 < R < \infty$. For a sentence φ let $c(x)$ be the exponential generating series for the subclass of structures in $\mathscr{C}$ that satisfy φ. Set $t(x) = c(x)/a(x)$. If $a(z)$ is admissible and $t(R)$ is absolutely convergent then $\mu(\varphi) = \bar{\mu}(\varphi) = t(R)$.*

Of course an analogous theorem holds in the unlabeled case, but it is not as useful.

6. ASYMPTOTIC PROBABILITIES OF MONADIC SECOND-ORDER SENTENCES

The theorems of this section combine results of previous sections to describe different circumstances under which asymptotic probabilities of monadic second-order sentences will exist. The idea is to establish the existence of extended asymptotic probabilities using, for example, results of Section 3 or 4, and from this conclude, by the Tauberian theorems of Section 5 or other means, that asymptotic probabilities exist.

The first theorem shows that very little is needed to demonstrate that a monadic second-order sentence has asymptotic probability 0 or 1 if its extended asymptotic probability is one of these values.

THEOREM 6.1. *Let φ be an L_{2M} sentence:*

(i) *If $\lim_{x \to R} a(x) = \infty$ and $\mathscr{A} \to R$, then $\mu(\varphi)$ exists whenever $\bar{\mu}(\varphi) = 0$ or 1.*

(ii) *If $\lim b(x) = \infty$ and $\mathscr{B} \to S$, then $\nu(\varphi)$ exists whenever $\bar{\nu}(\varphi) = 0$ or 1.*

Proof. We prove (i). Suppose $\mu(\varphi) = 0$ (otherwise substitute $\neg \varphi$ for φ). Let $c(x)$ be the exponential generating for the subclass of structures in $\mathscr{C}$ satisfying φ. Write $c(x)$ in the form given in Theorem 4.1,

$$c(x) = \sum \hat{k}_0^{m_0}(x) \, \hat{k}_1^{m_1}(x) \cdots \hat{k}_{q-1}^{m_{q-1}}(x),$$

where the summation is taken over certain of the values $m_0, m_1, ..., m_{q-1} \leq N$. A typical term

$$\prod_{i < q} \hat{k}_i^{m_i}(x) \tag{6.1}$$

of this sum is the exponential generating series for one of the $\approx$-equivalence classes defined in the proof of Theorem 4.1. Since $\bar{\mu}(\varphi) = 0$,

$\lim_{x \to R} c(x)/a(x) = 0$, so if we divide (6.1) by $a(x) = \prod_{i < q} \exp(k_i(x))$ and let $x \to R$, we have

$$\lim_{x \to R} \prod_{i < q} k_i^{m_i}(x)/\exp(k_i(x)) = 0.$$

Now each factor approaches a limit as $x \to R$, so for some $j < q$, the jth factor approaches 0 and $x \to R$. Consequently, $m_j < N$ and $\lim_{x \to R} k_j(x) = \infty$. Let $\bar{k}(x) = \sum_{i \neq j} k_i(x)$. Then

$$c^*(x) = \left(\sum_{m < N} k_j(x)^m/m! \right) \exp(\bar{k}(x)) \tag{6.2}$$

is the exponential generating series for the subclass of structures in $\mathscr{C}$ with fewer than N components in the $\equiv'_n$-equivalence class containing $\mathfrak{K}_j$. This subclass clearly contains the subclass whose exponential generating series was given in (6.1) and $\lim_{x \to R} c^*(x)/a(x) = 0$. Put

$$k_j(x) = \sum_{n \geq 0} \frac{k_{j,n}}{n!} x^n,$$

$$\tilde{k}^r(x) = \sum_{n \geq r} \frac{k_{j,n}}{n!} x^n,$$

$$\tilde{k}_r(x) = \sum_{n < r} \frac{k_{j,n}}{n!} x^n.$$

Then

$$c^*(x) = \left(\sum_{m < N} \tilde{k}_r(x)^m/m! \right) \exp(\tilde{k}^r(x) + \bar{k}(x))$$

is the exponential generating series for the subclass of structures in $\mathscr{C}$ with fewer than N components of cardinality less than r in the same $\equiv'_n$-equivalence class as $\mathfrak{K}_j$. This subclass clearly contains the one with an exponential generating series given by (6.2). It may be described by a component-bounded sentence ψ_r, since it restricts only components of cardinality less than a fixed bound r. By Theorem 5.8, in Compton [3] $\mu(\psi_r)$ exists and can be found by dividing $c^*(x)$ by $a(x) = \exp(\tilde{k}_r(x) + \tilde{k}^r(x) + \bar{k}(x))$ and letting $x \to R$. The result is

$$\mu(\psi_r) = \left(\sum_{m < N} \tilde{k}_r(R)^m/m! \right) \exp(-\tilde{k}_r(R)).$$

As $r \to \infty$, $\tilde{k}_r(R) \to \infty$, so this value approaches 0. Therefore, $\mu(\varphi) = 0$.

The proof of (ii) is essentially the same. ∎

COROLLARY 6.2. *Let φ be the sentence asserting that an L-structure is connected:*

(i) *If $\lim_{x \to R} a(x) = \infty$ and $\mathscr{A} \to R$, then $\mu(\varphi) = 0$.*

(ii) *If $\lim_{x \to S} b(x) = \infty$ and $\mathscr{B} \to S$, then $\nu(\varphi) = 0$.*

Proof. Let $k(x)$ be the exponential generating series for the subclass of connected structures in $\mathscr{C}$. Then $a(x) = \exp(k(x))$, $\bar{\mu}(\varphi) = \lim_{x \to R} k(x)/\exp(k(x)) = 0$. The result follows from Theorem 6.1. The unlabeled case is similar. ∎

In Theorem 6.1 and Corollary 6.2 we may replace every occurrence of $\to$ with $\overset{*}{\to}$, of μ with μ^*, and of ν with ν^*.

We are ready to characterize non-fast growing classes with L_{2M} 0–1 laws.

THEOREM 6.3. *Let $\mathscr{C}$ be closed under disjoint unions and components, and suppose that the radius of convergence of $a(x)$, the exponential generating series for $\mathscr{C}$, is greater than 0. Then the following are equivalent:*

(i) $\mathscr{A} \to \infty$.

(ii) *C has an $L_{\omega\omega}$ labeled 0–1 law.*

(iii) *C has an L_{2M} labeled 0–1 law.*

Proof. The equivalence of (i) and (ii) was shown in Theorem 5.9 in Compton [3]. It is obvious that (iii) implies (ii). We will show that (i) implies (iii). By Theorem 6.1 we need only show that when $\mathscr{A} \to \infty$, $\bar{\mu}(\varphi) = 0$ or 1 for each L_{2M} sentence φ. We saw in Theorem 4.1 that the exponential generating series for the subclass of structures in C satisfying φ is a finite sum of the form

$$\sum \hat{k}_0^{m_0}(x) \, \hat{k}_1^{m_1}(x) \cdots \hat{k}_{q-1}^{m_{q-1}},$$

where $\hat{k}_i^{m_i}(x)$ is $k_i(x)^{m_i}/m_i!$ when $m_i < N$ and $\exp(k_i(x)) - \sum_{m < N} \hat{k}_i^m(x)$ when $m_i = N$. Note that $\lim_{x \to \infty} k_i(x) = \infty$, so, dividing the above summation by $a(x) = \prod \exp(k_i(x))$ and letting $x \to \infty$, we have $\bar{\mu}(\varphi) = 0$ or 1. ∎

THEOREM 6.4. *Let $\mathscr{C}$ be closed under disjoint unions and components, and suppose that the radius of convergence of $b(x)$, the ordinary generating series of C, is greater than 0. Then the following are equivalent:*

(i) $\mathscr{B} \to 1$.

(ii) *C has an $L_{\omega\omega}$ unlabeled 0–1 law.*

(iii) *C has an L_{2M} unlabeled 0–1 law.*

Proof. Proceed as in the previous theorem. Although $\lim_{x \to 1} l_i(x)$ may be finite, $\lim_{x \to 1} \exp(\sum 1/m \, l_i(x^m)) = \infty$ so the same argument applies. ∎

In Theorems 6.3 and 6.4 we may replace $\to$ with $\overset{*}{\to}$ and add the qualifier "generalized" before the words "0-1 law" (see Section 6 of Compton [3]).

The last two theorems of the section illustrate the use of the Tauberian theorems in Section 5 to show that L_{2M} sentences has asymptotic probabilities. Both theorems require the following lemma.

LEMMA 6.5. *Let φ be an L_{2M} sentence, and $c(x) = \sum_{n \geq 0} (c_n/n!) \, x^n$ the exponential generating series and $d(x) = \sum_{n \geq 0} d_n x^n$ the ordinary generating series for the subclass of structures in $\mathscr{C}$ that satisfy φ:*

(i) *If $\lim_{x \to R} a(x) = \infty$ and $\mathscr{A} \to R$ then $c(x)$ may be written as a sum $c^s(x) + c^a(x)$, where $c^s(x) = \sum_{n \geq 0} (c\gamma_n^s/n!) \, x^n$, $\lim_{n \to \infty} c_n^s/a_n = 0$, and the series $t(x) = c^a(x)/a(x)$ is absolutely convergent at R.*

(ii) *If $\lim_{x \to S} b(x) = \infty$ and $\mathscr{B} \to S$ then $d(x)$ may be written as a sum $d^s(x) + d^a(x)$, where $d^s(x) = \sum_{n \geq 0} d_n^s x^n$, $\lim_{n \to \infty} d_n^s/b_n = 0$, and the series $u(x) = d^a(x)/a(x)$ is absolutely convergent at S.*

Proof. By Theorem 4.1, $c(x)$ may be written as a finite sum

$$\sum \hat{k}_0^{m_0}(x) \, \hat{k}_1^{m_1}(x) \cdots \hat{k}_{q-1}^{m_{q-1}}(x),$$

where $\hat{k}_i^{m_i}(x)$ is $k(x)^{m_i}/m!$ if $m_i < N$ and $\exp(k_i(x)) - \sum_{m < N} \hat{k}_i^m(x)$ if $m_i = N$. Expand this into a sum of products in which each factor is either of the form $k_i(x)^m/m!$ or $\exp(k_i(x))$ (for the moment we ignore the minus signs that precede some of terms in this expression). A typical term in this expression will be of the form

$$\prod_{i < p} k_i(x)^{m_i}/m_i! \prod_{p \leq i < q} \exp(k_i(x)) \tag{6.3}$$

(this may require permuting indices). This term is the exponential generating series for the subclass of structures satisfying an L_{2M} sentence ψ which says, "for each $i < p$ there are exactly m_i components in the same $\equiv_n'$-equivalence class as $\mathfrak{R}_i$." (We will not show that there is such a sentence; the details are easily worked out, and the argument below depends only on the form of the generating series of (6.3). We use ψ only to avoid repeating the argument in the proof of Theorem 6.1.) Dividing (6.3) by $a(x)$ and letting x approach R we have

$$\bar{\mu}(\psi) = \lim_{x \to R} \prod_{i < p} (k_i(x)^{m_i}/m_i!) \exp(-k_i(x)).$$

Now there are two possibilities: if $\lim_{x \to R} k_i(x) = \infty$ for some $i < p$ then $\bar{\mu}(\psi) = 0$ and by Theorem 6.1 $\mu(\psi) = 0$, so the ratio of coefficients in the series expansion of (6.3) to corresponding coefficients in $a(x)$ approaches 0; if $\lim_{x \to R} k_i(x) < \infty$ for all $i < p$ then each series $k_i(R)$ converges and the terms in the series expansion of

$$\bar{\mu}(\psi) = \prod_{i < p} (k_i(R)^{m_i}/m_i!) \exp(-k_i(R))$$

are dominated in absolute value by those of

$$\prod_{i < p} (k_i(x)^{m_i}/m_i!) \exp(k_i(R)),$$

which is convergent and has only positive terms.

The result now follows by grouping terms of the former type together to form $c^s(x)$, and those of the latter type together to form $c^a(x)$.

The proof of (ii) is analogous. ∎

THEOREM 6.6. *Let* $\mathscr{C}$, $a(x)$, $b(x)$, R, *and* S *be as described in Theorems* 6.3 *and* 6.4:

(i) *If* $\lim_{x \to R} a(x) = \infty$, $\mathscr{A} \to R$, *and* $|a_{n-i}/(n-i)!|/|a_n/n!| \leq AR^i$ *for some* A, *all sufficiently large* n *and all* $i \leq n$, *then* $\mu(\varphi)$ *exists for all* L_{2M} *sentences* φ.

(ii) *If* $\lim_{x \to S} b(x) = \infty$, $\mathscr{B} \to S$, *and* $|b_{n-i}/b_n| \leq AS^i$ *for some* A, *all sufficiently large* N *and all* $i \leq n$, *then* $\nu(\varphi)$ *exists for all* L_{2M} *sentences*.

Proof. (i) $\bar{\mu}(\varphi)$ exists by Theorem 4.3. By Lemma 6.5 and Theorem 5.2, (ii) $\mu(\varphi)$ exists.

(ii) Proved in the same way. ∎

EXAMPLE 6.7. By Theorem 6.6 every L_{2M} sentence about permutations (Example 7.6 of Compton [3]) has a labeled asymptotic probability (this was first noticed by Ward Henson). If we add a unary relation symbol to L then we have (still considering permutations) $a_n = 2^n n!$, $a(x) = (1 - 2x)^{-1}$. Again Theorem 6.6 applies and each L_{2M} sentence has a labeled asymptotic probability.

THEOREM 6.8. *If* $a(x)$ *is admissible then* $\mu(\varphi)$ *exists for all* L_{2M} *sentences* φ.

Proof. It is implicit in the statement of Theorem 1.1 (and shown in Hayman [8]) that $\lim_{x \to R} a(x) = \infty$. Corollary 4.2 of Compton [3] shows that $\mathscr{A} \to R$. Thus $\bar{\mu}(\varphi)$ exists by Theorem 4.3. By Lemma 6.5 and Theorem 5.3, $\mu(\varphi)$ exists. ∎

We will not state the corresponding theorem for the unlabeled case because admissibility is more useful in the labeled case.

EXAMPLE 6.9. Every L_{2M} sentence about linear forests (Example 7.14 of Compton [3]) has a labeled asymptotic probability. As in Example 6.7, if we add a unary predicate to L it is still true that every L_{2M} sentence will have an asymptotic probability.

7. OPEN PROBLEMS

We close with a list of questions suggested by our results. Assume that C is closed under disjoint unions and components.

Question 7.1. Is it true for every sentence φ in L_{2M} that $\mu(\varphi)$ exists iff $v(\varphi)$ exists?

Question 7.2. Does $\tilde{\mu}(\varphi)$ $(\tilde{v}(\varphi))$ exist for all sentences φ in L_{2M} when $\lim_{x \to R} a(x) < \infty$ $(\lim_{x \to S} b(x) < \infty)$ (cf. Theorem 4.3)?

Question 7.3. If $\mathscr{A} \to R > 0$ and $\lim_{x \to R} a(x) = \infty$, does $\mu(\varphi)$ exist for all φ in L_{2M}? If $\mathscr{B} \to S > 0$ and $\lim_{x \to S} b(x) = \infty$, does $v(\varphi)$ exist for all φ in L_{2M}?

Question 7.4. If $\mathscr{A} \to R > 0$ and $\lim_{x \to R} a(x) < \infty$, is it true that for every L_{2M} sentence φ there is a p such that $\lim_{n \to \infty} c_{np+r}/a_{np+r}$ exists for every r, $0 \leqq r < p$? (Here c_n is the number of labeled structures of cardinality n satisfying φ.) We ask the analogous question for the unlabeled case.

REFERENCES

1. E. A. BENDER, Asymptotic methods in enumeration, *SIAM Rev.* **16** (1974), 485–515.
2. K. COMPTON, An undecidable problem in finite combinatorics, *J. Symbolic Logic* **49** (1984), 842–850.
3. K. COMPTON, A logical approach to asymptotic combinatorics I: First-order properties, *Adv. in Math.* **65** (1987), 65–96.
4. A. EHRENFEUCHT, An application of games to the completeness problem for formalized theories, *Fund. Math.* **49** (1961), 129–141.
5. R. FAGIN, Probabilities on finite models, *J. Symbolic Logic* **41** (1976), 50–58.
6. R. FRAÏSSE, Sur les classifications des systèmes de relations, *Publ. Sci. Univ. Alger* **1** (1954).
7. G. H. HARDY, "Divergent Series," Oxford Univ. Press (Clarendon), Oxford, 1949.
8. W. K. HAYMAN, A generalisation of Stirling's formula, *J. Reine Angew Math.* **196** (1956), 67–95.
9. J. F. LYNCH, Almost sure theories, *Ann. Math. Logic* **18** (1980), 91–135.
10. J. F. LYNCH, Probabilities of first-order sentences about unary functions, *Trans. Amer. Math. Soc.* **287** (1985), 543–568.
11. G. SZEGÖ, "Orthogonal Polynomials," Amer. Math. Soc., Providence, RI, 1959.
12. E. C. TITCHMARSH, "Theory of Functions," Oxford Univ. Press, London, 1952.