

# First Order Logic of Sparse Random Graphs

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## First order language of graphs

- Variables  $x_1, \dots, x_n, \dots$
- Connectives  $\wedge, \vee$ , equality symbol  $=$ , and negation symbol  $\neg$ .
- Quantifiers  $\forall, \exists$ .
- A binary relation symbol  $R$
- Vertices.
- “And”, “or”, “equals”, “not”.
- “For all”, “there exists”.
- Edges  $x \sim y$ .

$$\forall x_1, x_2 R(x_1, x_2) \implies \exists x_3 \neg(x_3 = x_1) \wedge \neg(x_3 = x_1) \wedge R(x_1, x_3)$$

Binomial model of random graphs  $G(n, p)$ .

Start with vertex set  $[n] = \{1, \dots, n\}$  and each edge is added with probability  $p$  independently.

If  $G = ([n], E)$ :

$$\Pr(G) = p^{|E|} \cdot (1 - p)^{\binom{n}{2} - |E|}.$$

# The problem

For any F.O. sentence  $\phi$  we can talk about

$$Pr(G(n, p) \models \phi),$$

And if  $p = p(n)$  it also makes sense

$$\lim_{n \rightarrow \infty} Pr(G(n, p(n)) \models \phi).$$

What do we know about these limits?

# The problem

For  $p(n) = n^{-\alpha}$  and  $\alpha \in [0, 1]$  we have a nearly complete answer.

## Theorem

(Fagin 1976) If  $0 \leq p \leq 1$  is constant a Zero-One Law is satisfied

## Theorem

(Spencer, Shelah 1988)

- If  $\alpha \in (0, 1) \setminus \mathbb{Q}$  then a Zero-One law holds.
- If  $\alpha \in (0, 1) \cup \mathbb{Q}$  then a Zero-One law does not hold. Even more, there is a F.O. sentence  $\phi$  such that

$$\lim_{n \rightarrow \infty} \Pr(G(n, p(n)) \models \phi)$$

does not exist.

# The problem

## Theorem

(Lynch, 1992). If  $p(n) = c/n$  for some positive real  $c$  the limit

$$\lim_{n \rightarrow \infty} \Pr(G(n, p(n)) \models \phi)$$

*always exists and is analytic in  $c$ .*

# The landscape of $G(n, c/n)$

We are interested in last theorem.

- The number of cycles of length  $3, 4, \dots, r$  are asymptotically distributed like independent Poisson variables.
- Small cycles are a.a.s far away.
- Fixed vertices are a.a.s far away.
- The ball of a given radius centered in fixed vertex is a.a.s a tree. Any tree occurs with a positive probability.

# Outline of the proof

For any quantifier rank  $k$  we show (constructively) that there exist some classes of graphs  $C_1, \dots, C_m$  such that

- (1) A.a.s any two random graphs in the same class have the same rank  $k$  type.
- (2) A.a.s. a random graph belongs to some  $C_i$ .
- (3)  $\lim_{n \rightarrow \infty} \Pr(G \in C_i)$  exists for any  $i$  and is analytic in  $c$ .



# Outline of the proof

After this, for any F.O. sentence  $\phi$  we obtain

$$\begin{aligned}\lim_{n \rightarrow \infty} Pr(\phi) &= \sum_{i=1}^m \lim_{n \rightarrow \infty} Pr(\phi \wedge G \in C_i) + \lim_{n \rightarrow \infty} Pr(\phi \wedge G \notin C_1 \cup \dots \cup C_l) \\ &\stackrel{(2)}{=} \sum_{i=1}^m \lim_{n \rightarrow \infty} Pr(\phi \mid G \in C_i) Pr(G \in C_i) \\ &\stackrel{(1)}{=} \sum_{j=1}^l \lim_{n \rightarrow \infty} Pr(G \in C_{ij})\end{aligned}$$

And finally because of (3) this limit exists and is analytic in  $c$ .

## Outline of the proof

The classes  $C_1, \dots, C_m$  are obtained as the equivalence classes of a relation on graphs that **only** depends on the “small” neighborhoods of the small cycles.

$$x \in \text{Core}(G, r) \iff d(x, c) \leq r, \text{ for some cycle } c \text{ of length } \leq 2r + 1$$

## Outline of the proof

- The probability of any given “class” of core is given by a nested sequence of products of “Poisson expressions, analytic on  $c$ .
- To show that graphs with “equivalent cores share the same rank  $k$  type E.F games are used.

Can this be generalized to regular hypergraphs with edge size  $a$  y edge probability  $c/n^{a-1}$ ?

Yes it seems so. You can even consider arbitrary symmetry groups on the edges and multiple edge sets with different sizes.