First Order Logic of Sparse Random Graphs

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Preliminaries: The first order logic (FO) of graphs

Variables $x_1, \ldots, x_n, \ldots \rightarrow$ Vertices
Boolean connectives $\wedge, \vee, \neg$ and equality symbol $=$.
Quantifiers $\forall, \exists$.
A binary relation symbol E.

It can express properties like:

- The existence of a given subgraph
- The existence of a covering set of size k

It cannot express connectivity, k-colorability, existence of a Hamiltonian path ...

Preliminaries: The binomial model of random graphs $G(n, p)$

Start with vertex set $[n] = \{1, \ldots, n\}$ and each edge is added with probability p independently.

If $G = ([n], E)$:

$$Pr(G) = p^{|E|} \cdot (1 - p)^{\binom{n}{2} - |E|}.$$

The problem

For any F.O. sentence ϕ we can talk about

$$Pr(G(n,p) \models \phi),$$

And if $p = p(n)$ it also makes sense

$$\lim_{n \to \infty} Pr(G(n,p(n)) \models \phi).$$

What do we know about these limits?

The problem

For $p(n) = n^{-\alpha}$ and $\alpha \in [0, 1]$ we have a nearly complete answer.

Theorem

(Fagin 1976) If $0 \leq p \leq 1$ *is constant a Zero-One Law is satisfied*

Theorem

(Spencer, Shelah 1988)

- *If* $\alpha \in (0, 1) \setminus \mathbb{Q}$ *then a Zero-One law holds.*
- *If* $\alpha \in (0, 1) \cup \mathbb{Q}$ *then a Zero-One lay does not hold. Even more, there is a F.O. sentence* ϕ *such that*

  $$\lim_{n \to \infty} Pr(G(n, p(n)) \models \phi)$$

  does not exist.

The problem

Theorem

(Lynch, 1992). If $p(n) = c/n$ for some positive real c the limit

$$\lim_{n \to \infty} Pr(G(n, p(n)) \models \phi)$$

always exists and is analytic in c.

The landscape of $G(n, c/n)$

We are interested in last theorem.

- The number of cycles of length $3, 4 \ldots, r$ are asymptotically distributed like independent Poisson variables.
- Small cycles are a.a.s far away.
- Fixed vertices are a.a.s far away.
- The ball of a given radius centered in fixed vertex is a.a.s a tree. Any tree occurs with a positive probability.

Outline of the proof

For any quantifier rank k we show (constructively) that there exist some classes of graphs $C_1, .., C_m$ such that

(1) A.a.s any two random graphs in the same class have the same rank k type.

(2) A.a.s. a random graph belongs to some C_i.

(3) $\lim_{n \to \infty} Pr(G \in C_i)$ exists for any i and is analytic in c.

Outline of the proof

After this, for any F.O. sentence ϕ we obtain

$$\begin{aligned}
\lim_{n\to\infty} Pr(\phi) &= \sum_{i=1}^{m} \lim_{n\to\infty} Pr(\,\phi \wedge G \in C_i\,) + \lim_{n\to\infty} Pr(\,\phi \wedge G \notin C_1 \cup \cdots \cup C_l\,) \\
&\stackrel{(2)}{=} \sum_{i=1}^{m} \lim_{n\to\infty} Pr(\,\phi \,|\, G \in C_i\,) Pr(\,G \in C_i\,) \\
&\stackrel{(1)}{=} \sum_{j=1}^{l} \lim_{n\to\infty} Pr(\,G \in C_{i_j}\,)
\end{aligned}$$

And finally because of (3) this limit exists and is analytic in c.

Outline of the proof

The classes $C_1, \dots, C_m$ are obtained as the equivalence classes of a relation on graphs that **only** depends on the "small" neighborhoods of the small cycles.

$$x \in Core(G, r) \iff d(x, c) \leq r, \text{ for some cycle } c \text{ of length } \leq 2r + 1$$

Outline of the proof

- The probability of any given "class" of core is given by a nested sequence of products of "Poisson expressions, analytic on c.

- To show that graphs with "equivalent cores share the same rank k type E.F games are used.

Can this be generalized to regular hypergraphs with edge size a y edge probability c/n^{a-1}?

Yes it seems so. You can even consider arbitrary symmetry groups on the edges and multiple edge sets with different sizes.