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Abstract

This is a survey of logical results concerning random structures. A class of relational structures on which a (finitely additive) probability measure has been defined has a 0–1 law for a particular logic if every sentence of that logic has probability either 0 or 1. The measure may be an asymptotic probability on finite structures or generated on a class of infinite structures by assigning fixed probabilities to independently occurring properties. Conditions under which all sentences of a logic have a probability, and under which 0–1 laws occur, are examined. Also, the complexity of computing probabilities of sentences is considered.

1 Introduction

The study of random structures has become one of the main branches of combinatorics. Logicians, always eager to undertake a systematic investigation of an established field of mathematics, have recently turned their attention to this area. In this survey, we will give an account of recent logical results concerning three general problems:

1. For a given space of random structures, will all the properties expressible in a particular logic be "measurable"? (I.e., will the notion of probability be defined for all sentences in the logic?)

2. Can the probabilities of sentences be computed, and, if so, what is the complexity of the computation?

3. Under what circumstances will all the sentences of a particular logic have probability either 0 or 1?

In the situation described in the third problem, we say that a *0-1 law* holds. Establishing a 0–1 law makes precise the intuition that random structures in a given class look alike: from the point of view of a particular logic, they share the same properties almost surely. When a 0–1 law holds, the second question becomes a decision problem; we ask if it can be decided whether or not a property has probability 1.

Let us make these ideas precise. By a *structure* we always mean a *relational structure*; this is a tuple $\langle A, R_0, R_1, \ldots, R_{k-1}\rangle$ where A (the *universe* of the structure) is a set and $R_0, R_1, \ldots, R_{k-1}$ are relations on A. (Functions and constants are treated as restricted relations. For example, we regard a unary

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functions as binary relations.) The classes of structures we consider will consist of structures of the same *similarity type*. That is, they are all of the form $\langle A, R_0, R_1, \ldots, R_{k-1}\rangle$, where k is fixed and the arity of each relation R_i, for $i < k$, is also fixed.

Our notion of a probability measure on a class C of structures is much less stringent than the usual definition from probability theory. A *probability measure* is a finitely additive function $\mu : \mathcal{F} \longrightarrow [0, 1]$, where $\mathcal{F}$ is a collection of subsets of C with the following properties: $\mathcal{F}$ contains $\emptyset$ and C with $\mu(C) = 1$; $\mathcal{F}$ is closed under finite disjoint unions; and if $A, B \in \mathcal{F}$ with $A \subseteq B$, then $B - A \in \mathcal{F}$.

There are two general methods in combinatorics for defining a probability measure μ on a class C.

The first is to take a limit of uniform measures on sets of structures of fixed finite cardinalities. This method is closely related to the traditional enumeration problems of combinatorics (see Goulden and Jackson [32] for a comprehensive account). For example, suppose C is closed under isomorphism, C_n is the set of structures in C with universe $\{0, 1, \ldots, n-1\}$, and C_n is nonempty for all large n. If we define $\mu_n(S) = |C_n \cap S|/|C_n|$, then $\mu(S) = \lim_{n\to\infty} \mu_n(S)$ is the *labeled asymptotic probability* of S (with respect to C). Notice that taking $\mathcal{F}$ to be the collection of sets S for which $\mu(S)$ is defined satisfies our definition of probability measure. Take D_n to be a set of representative structures from each of the isomorphism classes in C_n. Then $\nu_n(S) = |D_n \cap S|/|D_n|$ and $\nu(S) = \lim_{n\to\infty} \nu_n(S)$ is the *unlabeled asymptotic probability* of S.

In certain cases we may use a more general type of limit. For example, if $\mu_n(S)$ is 0 when n is even and 1 when n is odd, then $\lim_{n\to\infty} \mu_n(S)$ is not defined, but the *Ceŝaro limit*

$$\lim_{n\to\infty} (\mu_0(S) + \mu_1(S) + \cdots + \mu_{n-1}(S))/n$$

is defined and is, quite reasonably, equal to 1/2. The Ceŝaro limit is just one of many summability methods (so called because they are used to "sum" divergent series) that can be used to define probability measures. We will present others in §4.

The second method for defining a probability measure is to specify probabilities of independently occurring properties of structures and then define a product measure. The most well known example occurs in the theory of random graphs, where a probability p is assigned the event of an edge occurrence between each pair of distinct vertices from a given set. There are two ways that *independent probability measures* (as we will call them) can be assigned on the class of graphs.

The first is to let C_n be the class of graphs on $n = \{0, 1, \ldots, n-1\}$, and $C = \bigcup_{n\geq 0} C_n$. We take p to be a function of n and identify C_n with the product space $\{0, 1\}^{[n]^2}$. Here $[n]^2$ is the set of all subsets of $\{0, 1, \ldots, n-1\}$ of cardinality 2 and each element of $\{0, 1\}^{[n]^2}$ represents a graph with edges between those pairs where the coordinate values are 1. An *edge occurrence event* is obtained by fixing a coordinate position and taking the set of graphs in $\{0, 1\}^{[n]^2}$ whose coordinate value is 1 at that position. Let μ_n be the usual product measure on C_n, with $p = p(n)$ the probability of each edge occurrence event, and $\mu_n(S) = \mu_n(C_n \cap S)$. As we did with uniform distributions, take $\mu(S) = \lim_{n\to\infty} \mu_n(S)$. This, of course, is the approach to random graphs initiated by Erdős and Rényi [24]. (See Palmer [58] for an up to date introduction and Bollobás [8] for a comprehensive account of the work in this area.)

The second way to generate an independent probability measure on a class of graphs is to let C be the class of graphs on some infinite set A and p be constant. As in the preceding paragraph, C can be regarded as a product space. Let μ be the product measure, and $\mathcal{F}$ the Borel sets, on C. In this case we can also take the usual topology on C and consider the topological analogues of the measure 0 and measure 1 sets, viz., the meager and comeager sets.

We must still say how logic figures in the study of random structures. Fix a class C of structures of the same similarity type. A logic L for C consists of a set objects called *sentences* and a relation $\models$ holding between structures in C and sentences from L. When $\mathcal{M} \models \varphi$ holds we say "$\mathcal{M}$ satisfies φ" or "$\mathcal{M}$ is a model of φ." We require that isomorphic structures in C satisfy precisely the same sentences. We can now speak of probabilities of sentences, rather than probabilities of subsets of C, by defining

$$\mu(\varphi) = \mu(\{\mathcal{M} \in C \mid \mathcal{M} \models \varphi\}).$$

In the next section we will describe a number of logics that are appropriate to the study of random structures. In §3 we make the basic distinction between fast and slow growing classes of structures, and review a few basic facts about generating series. In §4 we describe techniques used to prove convergence and 0–1 laws for slow growing structures. In §5 we survey 0–1 laws for fast growing classes and their associated complexity problems. In §6 we survey basic results about classes of structures with underlying relations, such as successor relations and linear orders. In §7 we briefly describe results on independent probability measures, including measures on infinite structures and recent work on 0–1 laws for random graphs and random partial orders. In §8 will conclude with some open problems and suggestions for new directions.

We have attempted to include statements of all of the major results in the area. Omissions are inadvertent and the author would appreciate having them brought to his attention. We will follow the approach taken by Oberschelp [54] in an earlier survey by emphasizing the intuitive ideas behind the formal results; detailed proofs of all the theorems cited here would run to many pages. We also note that Taylor [67] has reviewed several of the major papers in the area.

2 Logics

Much of the material in this section, particularly that related to first-order and second-order logic, will be familiar to readers with a logical background. We have included just enough material to give a feel for the area and to understand statements of results. For more details, consult Chang and Keisler [11].

To consider the computational complexity of probabilities, we usually regard sentences as consisting of finite strings of symbols from a finite alphabet. (The only exception considered here will be sentences of the logic $L_{\omega_1\omega}$.) In all the logics we describe, we inductively define *formulas* and at the same time define the set of *free variables* in a formula. Sentences are formulas whose set of free variables is empty. We omit the definition of the satisfaction relation $\models$ holding between relational structures and formulas as it is always straightforward.

The symbols that make up formulas are of two types. There are *extralogical* symbols $P_0, P_1, \ldots, P_{k-1}$ that are interpreted by the relations $R_0, R_1, \ldots, R_{k-1}$ in the structures $\langle A, R_0, R_1, \ldots, R_{k-1}\rangle$ in C. There are also *logical* symbols that are particular to the type of logic.

The most well known and most widely investigated logic is *first-order logic*, denoted FO here. The logical symbols of FO include logical connectives $\wedge$ (and), $\vee$ (or), $\neg$ (not); the equality symbol $=$; quantifiers $\forall$, $\exists$; delimiters such as parentheses, comma, and space; and element variables $x, y, z, x_0, x_1, \ldots$. To be precise, we should point out that subscripts on variables are really strings of symbols—this is important for complexity considerations. Formulas are either *atomic formulas*, of the form $x = y$ or $P_i(x_0, x_1, \ldots, x_{j-1})$ where j is the arity of R_i, or built up from simpler formulas by application of connectives and quantifiers. The set of free variables in a formula φ and its *quantifier rank* are defined inductively. If φ is atomic, then $free(\varphi)$ is the set of variables occurring in φ and $qr(\varphi) = 0$. Also,

$$\begin{aligned}
free(\neg\psi) &= free(\psi) \\
free(\psi \wedge \theta) &= free(\psi \vee \theta) = free(\psi) \cup free(\theta) \\
free(\forall x\, \psi) &= free(\exists x\, \psi) = free(\psi) - \{x\}
\end{aligned}$$

and

$$\begin{aligned}
qr(\neg\psi) &= qr(\psi) \\
qr(\psi \wedge \theta) &= qr(\psi \vee \theta) = \max(qr(\psi), qr(\theta)) \\
qr(\forall x\, \psi) &= qr(\exists x\, \psi) = qr(\psi) + 1
\end{aligned}$$

The other logics we consider extend FO in some way. They fall into three general categories. First, there is second-order logic, in which relations as well as elements may be quantified, and various restrictions of second-order logic. Second, there are logics with relational operators which are applied to defined relations. These logics arise in characterizations of complexity classes and elsewhere in computer science. Third, there are infinitary logics which have infinite connectives. We will consider only one infinitary logic, $L_{\omega_1\omega}$.

Second-order logic contains all the logical symbols of FO as well as relation variables X_j^i for $i, j \in \omega$. The intention is that X_j^i ranges over relations of arity i on the universe of a structure. (Whenever the superscript is clear from context, we will omit it.) Besides the atomic formulas from FO, there are atomic formulas of the form $X_j^i(x_0, \ldots, x_{i-1})$ whose set of free variables is $\{X_j^i, x_0, \ldots, x_{i-1}\}$. We must also add a formula formation rule for quantification of relation variables and accordingly extend the definition of the set of free variables and the quantifier rank of a formula. Second-order logic is too powerful for our purposes because every reasonable notion of asymptotic probability will be undefined for some second order sentence. Therefore, we need to consider various restrictions.

One obvious way to restrict second-order logic is to include just the relation variables of arity 1 (the *monadic variables*). This logic, called *monadic second-order logic* and denoted *MSO*, has been widely investigated (particularly its decision problems).

Another obvious way to restrict second-order logic is by restricting the quantifier prefixes of sentences. For example, the Σ_1^1 formulas are those of the form

$$\exists X_0 \, \exists X_1 \cdots \exists X_{i-1} \, \psi \tag{1}$$

and the Π_1^1 formulas are those of the form

$$\forall X_0 \, \forall X_1 \cdots \forall X_{i-1} \, \psi, \tag{2}$$

where $X_0, X_1, \ldots, X_{i-1}$ are relation variables and ψ is an *FO* formula (in the logic extended by adding relation symbols $X_0, X_1, \ldots, X_{i-1}$). Σ_{k+1}^1 is the set of second-order formulas of the form (1), where ψ is a Π_k^1 formula, and Π_{k+1}^1 is the set of second-order formulas of the form (2), where ψ is a Σ_k^1 formula.

Σ_1^1 is an important fragment of second order logic. Fagin [26] showed that a set of finite structures (of the same similarity type) is the set satisfying a Σ_1^1 sentence if and only if it is in the complexity class *NP*. Stockmeyer [65] extended this result by showing that the classes Σ_k^1 and Π_k^1 correspond to the levels of the polynomial time hierarchy.

Kolaitis and Vardi [46] investigated a subset of the Σ_1^1 sentences that they call *strict*-Σ_1^1 sentences. It consists of sentences of the form (1) where ψ is an *FO* sentence $\exists x_0, \ldots, x_{j-1} \forall y_0, \ldots, y_{k-1} \, \theta$, with θ a quantifier-free formula. The dual set of strict-Π_1^1 sentences consists of sentences of the form (2) where ψ is an *FO* sentence $\forall x_0, \ldots, x_{j-1} \exists y_0, \ldots, y_{k-1} \, \theta$, with θ a quantifier-free formula.

The theorem of Fagin showing that Σ_1^1 "captures" *NP* was the first of several results relating logics and complexity classes. Immerman [39] and Vardi [69] showed that *least fixed point logic* (denoted *LFP*), a logic that incorporates inductive definitions, captures the complexity class *P* when structures have an underlying linear order (see also Gurevich [34] and Gurevich and Shelah [35]). The first investigation of inductive definitions for *FO* was made by Moschovakis [53]. Aho and Ullman [1] added least fixed point operators to *FO*. Gurevich and Shelah [35] showed that this logic has the same expressive power on finite structures as a more general logic they call *inductive logic*; inductive logic is more expressive than least fixed point logic on infinite structures.

Immerman [40] later showed that other familiar complexity classes were captured by logics. We will describe one of these logics, *transitive closure logic* (denoted *TC*), as a prelude to our definition of least fixed point logic. On structures with an underlying linear order *TC* captures the complexity class formed by taking a union of the levels of the logspace hierarchy (see Immerman [40]).

The logical symbols of transitive closure logic include those of first-order logic and the relation variables X_j^i of second-order logic. (In *TC* we will require only the relation variables of even arity.) We do not have unlimited quantification of relation variables. Instead, we have a transitive closure operator that allows us to say that a relation variable should be interpreted as the least transitive relation extending some defined relation. More specifically, we have, besides the formula formation rules of first-order logic, a rule that says if ψ and θ are formulas, X_j^{2i} is a relation variable not free in θ, and $x_0, \ldots, x_{i-1}, y_0, \ldots, y_{i-1}$ is a sequence of distinct variables, then φ given by

$$\left[X_j^{2i}(x_0, \ldots, x_{i-1}, y_0, \ldots, y_{i-1}) \equiv \mathrm{TC}(\theta)\right] \psi$$

is also a formula with

$$\begin{aligned} \mathit{free}(\varphi) \;=\; & \Big((\mathit{free}(\theta) - \{x_0, \ldots, x_{i-1}, y_0, \ldots, y_{i-1}\}) \\ & \qquad \cup \mathit{free}(\psi) \Big) - \{X_j^{2i}\} \end{aligned}$$

Let $\vec{x} = x_0, \ldots, x_{i-1}$ and $\vec{y} = y_0, \ldots, y_{i-1}$. For each universe A of a structure, we can regard $\theta = \theta(\vec{x}, \vec{y})$ as defining a binary relation on A^i. Formula φ says that X_j^{2i} should be interpreted in ψ as the least transitive relation extending the relation θ. For example, consider the logic whose only extralogical symbol is a binary relation symbol E which we take to denote the edge relation on the class of graphs. The TC sentence

$$[X(x,y) \equiv \mathrm{TC}(E(x,y))]\ \forall x, y(x \neq y \rightarrow X(x,y))$$

asserts that a graph is connected. X interprets the path relation in each graph: it holds between two points precisely when there is a path between them.

We now describe how to build formulas of least fixed point logic. The logical symbols include those of first-order and the relation variables X^i_j. Besides the formula formation rules of first-order logic, we have a rule that says if ψ and θ are formulas, X is a relation variable, and $x_0, \ldots, x_{i-1}$ is a sequence of distinct variables, then φ given by

$$[X(x_0, \ldots, x_{i-1}) \equiv \theta]\ \psi$$

is also a formula, where we require that X occurs only positively in θ. (This means, roughly, that X occurs only within the scope of an even number of negations; to be rigorous we would have to define positive and negative occurrences by an induction on formulas.) The part of φ within brackets *implicitly defines* the interpretation of X in ψ. Let us make this idea precise.

Fix a structure M with universe A and assign values in A to the symbols in $\mathit{free}(\varphi)$; i.e., assign elements of A to element variables and relations on A of the appropriate arity to relation variables. For every such assignment, $\theta = \theta(X, x_0, \ldots, x_{i-1})$ defines an operator F on the set of i-ary relations on A:

$$F(R) = \{(a_0, \ldots, a_{i-1}) \mid \mathcal{M} \models \theta(R, a_0, \ldots, a_{i-1})\}$$

It is easy to show that if X occurs only positively in θ, F is *monotone*, i.e., $F(R) \subseteq F(R')$ whenever $R \subseteq R'$. Let $F^0(R) = R$, $F^{\alpha+1}(R) = F(F^\alpha(R))$ and if α is a limit ordinal, $F^\alpha(R) = \bigcup_{\beta<\alpha} F^\beta(R)$. By induction $F^\alpha(\emptyset) \subseteq F^\beta(\emptyset)$ whenever $\alpha < \beta$. There must be an ordinal κ such that $F^\alpha(\emptyset) = F^\kappa(\emptyset)$ whenever $\alpha \geq \kappa$. Thus, $F^\kappa(\emptyset)$ is a fixed point for F, in fact, by a well known theorem of Tarski [66], it is the least least fixed point. Then φ is true in M (at the given assignment) just in case ψ is true in M when $F^\kappa(\emptyset)$ interprets X (all other free symbols interpreted as in the assignment). This describes the semantics for φ in least fixed point logic.

We also use the notation

$$[X(x_0, \ldots, x_{i-1}) \equiv \theta]_m\ \psi,$$

where m is a non-negative integer, to indicate that $F^m(\emptyset)$ interprets X in ψ. As m increases we obtain better approximations to the fixed-point interpretation. Notice that a sentence which is formed using only connectives, quantifiers, and these approximations is equivalent to a first-order sentence since $F^m(\emptyset)$ is first-order definable for finite m.

It is easy to see that LFP is at least as powerful as TC. For example, the transitive closure sentence above asserting connectivity in graph can be rewritten

$$\begin{aligned}&[X(x,y) \equiv E(x,y) \vee \exists z(X(x,z) \wedge X(z,y))]\\ &\qquad\qquad\forall x, y(x \neq y \rightarrow X(x,y))\end{aligned}$$

in LFP.

Kolaitis and Vardi [46] define another logic which they call *iterative logic*, denoted here as IT. We will not give their definition; instead we note that one can obtain an equivalent logic by dropping the restriction in LFP that X occur only positively in θ in order that $[X(x_0, \ldots, x_{i-1}) \equiv \theta]\ \psi$ be a formula. Of course, the operator F defined from θ may no longer be monotone, so that the definition makes sense only on finite structures and even then there may not be a κ such that $F^\alpha(\emptyset) = F^\kappa(\emptyset)$ whenever $\alpha \geq \kappa$. If there is no convergence in the evaluation of the truth value of a sentence φ on a structure M, then we will say that it is not the case that $\mathcal{M} \models \varphi$. (This convention has the unfortunate consequence that for some sentences φ neither $\mathcal{M} \models \varphi$ nor $\mathcal{M} \models \neg\varphi$ holds.) One can show that on structures with an underlying linear order, IT captures $PSPACE$.

The infinitary logic $L_{\omega_1\omega}$ contains all the symbols of FO and two infinitary connectives, countable conjunction $\bigwedge$, and countable disjunction $\bigvee$. The rules for forming formulas of $L_{\omega_1\omega}$ include all those for FO and a rule that says if ψ_i is a formula of $L_{\omega_1\omega}$ for each $i \in \omega$, then so are $\bigwedge_{i\in\omega} \psi_i$ and $\bigvee_{i\in\omega} \psi_i$. The set of free variables for both of these formulas is $\bigcup_{i\in\omega} \mathit{free}(\psi_i)$. Clearly $L_{\omega_1\omega}$ is too powerful for classes of finite structures since every class of finite structures is the class of finite structures satisfying some $L_{\omega_1\omega}$ sentence. We will see, however, that $L_{\omega_1\omega}$ is a natural logic for independent probabilities on infinite structures.

This concludes our list of logics. One would naturally like to know the expressive powers of these logics. Write $L \leq L'$ if for every sentence in the logic L there is an equivalent sentence (a sentence having precisely the same models) in L'; if $L \leq L'$ and there is a sentence of L' not equivalent to any sentence in L, write $L < L'$. It is not difficult to show that $FO < MSO$, $FO <$ strict-$\Sigma_1^1 < \Sigma_1^1$, $FO <$ strict-$\Pi_1^1 < \Pi_1^1$, $\Sigma_k^1 \cup \Pi_k^1 < \Sigma_{k+1}^1 \cap \Pi_{k+1}^1$, $FO < TC \leq LFP$, and $TC < L_{\omega_1\omega}$. On classes of finite structures $LFP \leq IT$ and $LFP \leq \Sigma_1^1 \cup \Pi_1^1$. This list of comparisons is incomplete. The missing comparisons (and incomparisons) can probably be resolved without too much difficulty.

The only result about first-order logic we use in the paper that is not found in Chang and Keisler [11] concerns Ehrenfeucht games, which are used to determine when structures satisfy precisely the same first-order sentences. These games are particularly useful in the study of finite structures, and one of the few techniques that generalize to second-order logic. We describe Ehrenfeucht games for first-order and monadic second-order logic.

Given two structures $\mathcal{M}$ and $\mathcal{N}$, write $\mathcal{M} \equiv_r \mathcal{N}$ to indicate that $\mathcal{M}$ and $\mathcal{N}$ satisfy precisely the same first-order sentences of quantifier rank at most r. The game used to characterize $\equiv_r$ is played for r moves (numbered $1, 2, \ldots, r$) between players I and II on a pair of structures $\mathcal{M}$ and $\mathcal{N}$. On each move player I chooses an element of $\mathcal{M}$ (or $\mathcal{N}$) and player II responds by choosing a corresponding element of $\mathcal{N}$ (respectively, $\mathcal{M}$). Player I is not constrained to choose all elements from the same structure; he may alternate between them as often as he likes. Let a_i be the element chosen from $\mathcal{M}$ and b_i be the element chosen from $\mathcal{N}$ on move i. Player II wins if the set $\{(a_i, b_i) \mid i \leq r\}$ is an isomorphism between substructures of $\mathcal{M}$ and $\mathcal{N}$; otherwise, player I wins.

The basic result concerning this game is due to Ehrenfeucht [23]: player II has a winning strategy if and only if $\mathcal{M} \equiv_r \mathcal{N}$.

Write $\mathcal{M} \approxeq_r \mathcal{N}$ to indicate that $\mathcal{M}$ and $\mathcal{N}$ satisfy the same monadic second-order sentences of quantifier rank at most r. In the game characterizing $\approxeq_r$, players I and II again play for r moves on structures $\mathcal{M}$ and $\mathcal{N}$. On each move player I picks a subset of $\mathcal{M}$ (or $\mathcal{N}$) and player II responds picking subset of $\mathcal{N}$ (respectively $\mathcal{M}$). Whenever player I picks a singleton set, player II must respond with a singleton set. (Singleton set moves correspond to element quantifiers.) Let R_i be the subset chosen from $\mathcal{M}$ and S_i be the subset chosen from $\mathcal{N}$ on move i. If the set

$$\{(x, y) \mid \text{for some } i,\ R_i = \{x\} \text{ and } S_i = \{y\}\}$$

is an isomorphism between substructures of $\langle \mathcal{M}, R_1, \ldots, R_r \rangle$ and $\langle \mathcal{N}, S_1, \ldots, S_r \rangle$, then player II wins; otherwise, player I wins.

The basic result concerning this game is completely analogous to the first-order case: player II has a winning strategy if and only if $\mathcal{M} \approxeq_r \mathcal{N}$. (For a nice treatment of monadic second-order Ehrenfeucht games and some interesting applications, see Ladner [48]).

3 Generating Series and Growth Rates

Labeled and unlabeled probabilities are closely related to enumeration problems in combinatorics. The most important tool for enumeration is generating series (see, for example, Goulden and Jackson [32] which approaches enumeration from this point of view), so we begin with definitions of the two kinds of generating series that arise most frequently in asymptotic combinatorics.

Let C be a class of structures and C_n and D_n be, respectively, the classes of labeled and unlabeled structures in C on sets of size n, as defined in §1. Put $a_n = |C_n|$ and $b_n = |D_n|$. Then C has as its *exponential generating series* the formal power series $a(x) = \sum_{n=0}^{\infty}(a_n/n!)x^n$, and as its *ordinary generating series* the formal power series $b(x) = \sum_{n=0}^{\infty} b_n x^n$. Let $S \subseteq C$ and put $c_n = |C_n \cap S|$, $d_n = |D_n \cap S|$. Then $c(x) = \sum_{n=0}^{\infty}(c_n/n!)x^n$ and $d(x) = \sum_{n=0}^{\infty} d_n x^n$ are the exponential and ordinary generating series for S.

Generating series are useful because combinatorial operations on classes of structures correspond to algebraic operations on their generating series. Let us describe several instances of this correspondence.

(We will omit proofs; see Goulden and Jackson [32] for details). Generating series should be regarded as formal power series: convergence does not matter. (We will see, however, that convergence does determine asymptotic techniques that may be used and the kinds of 0–1 laws that may hold.)

For a structure $\mathcal{M} = \langle A, R_0, R_1, \ldots, R_{k-1}\rangle$, we say that two elements a and b are *directly connected* if there is a tuple in one of the relations R_i containing both a and b (so, in particular, the direct connection relation is reflexive and symmetric). Elements a and b are in the same *component* if (a, b) belongs to the transitive closure of the direct connection relation. $\mathcal{M}$ is *connected* if it has just one component.

If C_1 and C_2 are classes of structures of the same similarity type, with exponential generating series $a_1(x)$ and $a_2(x)$, respectively, and no component of a structure in C_1 is isomorphic to a component of structure in C_2, then the exponential generating series for $C_1 \cup C_2$ is $a_1(x) + a_2(x)$, and for $\{\mathcal{M} \dot\cup \mathcal{N} \mid \mathcal{M} \in C_1, \mathcal{N} \in C_2\}$ is $a_1(x)a_2(x)$. (Here $\dot\cup$ denotes disjoint union.) If C consists of connected structures, then the exponential generating series for the class consisting of structures with k components, all taken from C, is $a(x)^k/k!$, so the exponential generating series for the class obtained by closing C under disjoint unions is

$$\sum_{k=0}^{\infty} a(x)^k/k! = \exp(a(x))$$

Similarly, if C_1 and C_2 have ordinary generating series $b_1(x)$ and $b_2(x)$, respectively, and no component of a structure in C_1 is isomorphic to a component of structure in C_2, then the ordinary generating series for $C_1 \cup C_2$ is $b_1(x) + b_2(x)$, and for $\{\mathcal{M} \dot\cup \mathcal{N} \mid \mathcal{M} \in C_1, \mathcal{N} \in C_2\}$ is $b_1(x)b_2(x)$. Define the series $Z_k(x_1, x_2, \ldots, x_k)$ as follows.

$$\exp(y \sum_{k=1}^{\infty} x_k/k) = \sum_{k=0}^{\infty} Z_k(x_1, x_2, \ldots, x_k) y^k$$

Z_k is the Pólya cycle indicator polynomial for the group S_k (see Pólya [59] or Harary and Palmer [37]). Let $Z_k(b(x)) = Z_k(b(x), b(x^2), \ldots, b(x^k))$. If C consists of connected structures, then the ordinary generating series for the class consisting of structures with k components, all taken from C, is $Z_k(b(x))$, so the ordinary generating series for the class obtained by closing C under disjoint unions is

$$\sum_{k=0}^{\infty} Z_k(b(x)) = \exp(\sum_{k=1}^{\infty} b(x^k)/k)$$

For labeled and unlabeled probabilities there seems to be a fundamental distinction between *fast growing* and *slow growing* classes of structures. We will say that the set of labeled structures in C is fast growing if C has an exponential generating series with radius of convergence 0; otherwise, it is slow growing. Similarly, the set of unlabeled structures in C is fast growing if C has an ordinary generating series with radius of convergence 0, and slow growing otherwise. We will consider the slow growing and fast growing classes in separate sections. Moreover, for the fast growing classes Lynch [50, 51] has developed techniques for structures with functions and underlying relations such as linear orders. These techniques will be considered in another section.

4 Slow Growing Classes

In this section we will investigate 0–1 laws and convergence theorems for slow growing classes closed under disjoint unions and components. That is, the disjoint union of two structures in the class is again in the class, and each component of a structure in the class is also in the class. Many familiar classes satisfy these conditions: among them are the classes of unary functions, of one-to-one unary functions, of equivalence relations, of forests, and of binary forests.

We saw in the last section that if a class C closed under disjoint unions and components has exponential generating series $a(x)$ and the subclass of connected structures in C has exponential generating series $c(x)$, then $a(x) = \exp(c(x))$. If C has ordinary generating series $b(x)$ and the the subclass of connected structures in C has exponential generating series $l(x)$, then $b(x) = \exp(\sum_{k \geq 1} l(x^k)/k)$. These fundamental relationships have been used extensively in the enumeration of labeled and unlabeled structures (see Goulden and Jackson [32]).

We begin by extending the definition of labeled asymptotic probability when $a(x)$, the exponential generating series for C, has radius of convergence $R > 0$ and $\lim_{x\to R} a(x) = \infty$. Suppose that the subclass of structures in C satisfying some sentence φ has exponential generating series $c(x)$. It is not difficult to show that if $\mu(\varphi)$ is defined, it is equal to $\bar{\mu}(\varphi) = \lim_{x\to R} c(x)/a(x)$. This a simple example of an Abelian theorem—"... roughly, one which asserts that if a sequence or function behaves regularly, then some average of the sequence or function behaves regularly" [38, p. 148]. Now $\bar{\mu}(\varphi)$ may be defined in some cases where $\mu(\varphi)$ is not. For example, if C is the class of one-to-one unary functions, then the number of labeled structures with universe n is $n!$ so $a(x) = \sum_{k\geq 0} x^k = 1/(1-x)$. Now if φ asserts that the universe has an even number of elements (we can say this in second order logic, for example), then the subclass of structures satisfying φ is $\sum_{k\geq 0} x^{2k} = 1/(1-x^2)$ so

$$\bar{\mu}(\varphi) = \lim_{x\to 1} \frac{1-x}{1-x^2} = \frac{1}{2}$$

but $\mu(\varphi)$ is not defined.

Using ordinary generating series rather than exponential generating series, we can define $\bar{\nu}$ extending the unlabeled asymptotic probability ν.

Theorem 4.1 (Compton [17]) *If C has exponential generating series $a(x)$ with radius of convergence $R > 0$ and $\lim_{x\to R} a(x) = \infty$, then $\bar{\mu}(\varphi)$ is defined for every* MSO *sentence φ. Similarly, if C has ordinary generating series with radius of convergence $S > 0$ and $\lim_{x\to S} b(x) = \infty$, then $\bar{\nu}(\varphi)$ is defined for every* MSO *sentence φ.*

Proof: We use Ehrenfeucht games to establish combinatorial properties of classes satisfying *MSO* sentences. From these properties we can show that the corresponding generating series will have a form that insures extended asymptotic probabilities exist.

Two useful facts follow by Ehrenfeucht game arguments.

First, if $\mathcal{M}_0 \approxeq_r \mathcal{N}_0$ and $\mathcal{M}_1 \approxeq_r \mathcal{N}_1$, then $\mathcal{M}_0 \dot{\cup} \mathcal{M}_1 \approxeq_r \mathcal{N}_0 \dot{\cup} \mathcal{N}_1$. This is obvious because player II can combine the winning strategies for the Ehrenfeucht games of length r on the pair $\mathcal{M}_0, \mathcal{N}_0$ and on the pair $\mathcal{M}_1, \mathcal{N}_1$ to obtain a winning strategy on the disjoint unions.

Second, for every r there is an integer s depending only on the similarity type and r, such that whenever $i, j \geq s$ and $\mathcal{M}$ is a structure, $i \cdot \mathcal{M} \approxeq_r j \cdot \mathcal{M}$. (Here $i \cdot \mathcal{M}$ is the disjoint union of i copies of $\mathcal{M}$.) We show by induction on r that player II has a winning strategy on the Ehrenfeucht game of length r played on these two structures. When player I picks a subset, say of $i \cdot \mathcal{M}$, on the first move, we can view this as adding a unary relation to the similarity type and expanding each of the copies of $\mathcal{M}$ in the $i \cdot \mathcal{M}$ to structures in this new similarity type. It is easy to see that the number t of $\approxeq_{r-1}$-classes in this new similarity type is finite. The induction hypothesis tells us that there is an s' such that whenever $i, j \geq s'$ and $\mathcal{M}'$ is a structure in this new similarity type, $i \cdot \mathcal{M}' \approxeq_{r-1} j \cdot \mathcal{M}'$. Let $s = s't$. It is clear that player II can choose a subset that expands the j copies of $\mathcal{M}$ in the second structure so that in each $\approxeq_{r-1}$-class, either the same number of expanded copies of $\mathcal{M}$ in the class are present in each structure, or at least s' are present in each structure. It follows that the expansions of $i \cdot \mathcal{M}$ and $j \cdot \mathcal{M}$ satisfy the same *MSO* sentences of quantifier rank $r-1$, so player II has a winning strategy for the remaining $r-1$ moves.

Let $D_0, \ldots, D_{l-1}$ be the $\approxeq_r$-classes of connected structures in C and $c_0(x), \ldots, c_{l-1}(x)$ be their exponential generating series so that $c(x)$, the exponential generating series for the class of connected structures in C, is $\sum_{i<l} c_i(x)$. The first fact shows that the *MSO* sentences of quantifier rank r holding in a structure are determined by the number of components in each class D_i. The two facts together show that it only necessary to know the precise number of components in each class D_i up to s components; for more than s, the *MSO* sentences of quantifier rank r that hold are the same as if there were exactly s. For each $i < l$ there are $s+1$ possibilities: the number of components in D_i may be $0, 1, \ldots, s-1$, or $\geq s$. We can represent this information in each structure by a sequence of l integers in the interval from 0 to s. The exponential generating series for the class of structures in C associated with a particular sequence $(j_0, \ldots, j_{l-1})$ is a product of l series. The i-th series is $c_i(x)^{j_i}/j_i!$ when $0 \leq j_i \leq s-1$ and $\exp(c_i(x)) - \sum_{j<s} c_i(x)^j/j!$ when $j_i = s$. Multiplying out this product, we obtain a sum in which each term is a product of l factors, the i-th factor being either of the form $c_i(x)^j/j!$ or $\exp(c_i(x))$. Now divide each term by $a(x) = \prod_{i<l} \exp(c_i(x))$ and let x approach R. Clearly, the limit always exists.□

We would like to be able to show now that if $\bar{\mu}(\varphi)$ is defined, then so is $\mu(\varphi)$. Unfortunately, this is not true in general. Our task is to find conditions under which we may make this inference. Results of this kind are Tauberian theorems, or "... corrected forms of false converses of Abelian theorems" [38, p. 149].

Let us define a condition that figures in some of our Tauberian theorems. We will say that the number of labeled structures in a class C with exponential generating series $a(x) = \sum_{n\geq 0}(a_n/n!)x^n$ *grows smoothly* if

$$\lim_{n\to\infty} \frac{a_{n-m}/(n-m)!}{a_n/n!} = R^m$$

for each $m \geq 0$. In other words, the radius of convergence R of $a(x)$ is a limit of a ratio of coefficients. When $0 < R < \infty$ this condition can be simplified to

$$\lim_{n\to\infty} \frac{a_{n-1}/(n-1)!}{a_n/n!} = R$$

The following result appears in Compton [16].

Theorem 4.2 *Let C be a class of structures whose exponential generating series $a(x)$ has radius of convergence $R > 0$. Suppose that the number of labeled structures in C grows smoothly and $\lim_{x\to R} a(x) = \infty$. If φ is an* MSO *sentence with $\bar{\mu}(\varphi)$ equal to 0 or 1, then $\mu(\varphi) = \bar{\mu}(\varphi)$.*

The proof uses the analysis of the exponential generating series for *MSO* sentences given in the previous theorem. We can extract a great deal of information from this analysis when $\bar{\mu}(\varphi) = 0$ or 1. This result is used in the proof of the following theorem, which characterizes slow growing classes with label *MSO* 0–1 laws.

Theorem 4.3 *Let C be a class of structures closed under disjoint unions and components and suppose that $a(x)$, the exponential generating series for C, has radius of convergence $R > 0$. Then C has a labeled* MSO *0-1 law if and only if $R = \infty$ and the number of labeled structures in C grows smoothly.*

Proof: We know that $a(x) = \exp(c(x))$ where $c(x)$ is the exponential generating series for subclass of connected structures in C.

First we show that if $0 < R < \infty$, then C does not have a labeled *MSO* 0–1 law. Let $\mathcal{M}$ be a connected structure in C of size m and with σ automorphisms. Let φ be a sentence that says there is precisely one component isomorphic to $\mathcal{M}$. The exponential generating series for the class of structures satisfying φ is

$$(x^m/\sigma)\exp(c(x) - x^m/\sigma)$$

If $\lim_{x\to R} a(x) = \infty$, then we compute $\bar{\mu}(\varphi)$ by dividing by $a(x)$ and letting x approach R. We obtain

$$\bar{\mu}(\varphi) = (R^m/\sigma)\exp(-R^m/\sigma)$$

which is strictly between 0 and 1. If $\lim_{x\to R} a(x) < \infty$ we can give a separate argument to obtain the same value for $\bar{\mu}(\varphi)$.

To prove the other direction, observe that when $R = \infty$, then $\bar{\mu}(\varphi) = 0$. That is, the extended asymptotic probability that precisely one component is isomorphic to $\mathcal{M}$ is 0. Similarly, we can show for every nonnegative integer j that the extended asymptotic probability that precisely j components are isomorphic to $\mathcal{M}$ is 0. Thus, for each j there are almost surely at least j components isomorphic to $\mathcal{M}$.

Using facts from the proof of Theorem 4.1, we see that there is an s for each r such that structures with at least s components from each $\approxeq_r$-class satisfy the same *MSO* sentences of quantifier rank r. When $R = \infty$ structures in C will almost surely have s components from each $\approxeq_r$ class, so they almost surely satisfy the same *MSO* sentences of quantifier rank r. That is, $\bar{\mu}(\varphi)$ is either 0 or 1 for each *MSO* sentence φ. But by the previous theorem, $\mu(\varphi) = \bar{\mu}(\varphi)$.□

This theorem provides easily verifiable criteria for 0–1 laws. Compton [12, 16] gives a detailed analysis of the logical properties of the theory of the almost sure first-order sentences in cases where the theorem applies. This theory may be very complex. Compton [14] gives an example of a finitely axiomatizable class where this theory is undecidable.

Let us examine some familiar classes to see what this theorem tells us. (For details, see [16, 17].)

The class of equivalence relations has a labeled *MSO* 0–1 law. Since there is precisely one connected equivalence relation of each finite cardinality, this class has exponential generating series

$$a(x) = \exp(\sum_{k\geq 1} \frac{x^k}{k!}) = \exp(e^x - 1)$$

which clearly has radius of convergence ∞. Well known asymptotic results for this series show that the number of labeled structures in C grows smoothly.

If a class C closed under disjoint unions and components contains only finitely many finite connected structures and the sizes of their universes have greatest common divisor 1, then the class has a labeled *MSO* 0–1 law. The exponential generating series for such a class will be of the form $\exp(p(x))$, where $p(x)$ is a polynomial, so it is easy to see that $R = \infty$. Again, well known techniques show that the number of labeled structures in C grows smoothly.

The class of one-to-one unary functions has exponential generating series $1/(1-x)$ since there are $n!$ structures in C_n. Hence $R = 1$ and the class does not have a labeled *MSO* 0–1 law.

Let C be the class of cycle-free directed graphs in which every vertex has in-degree at most 1 and out-degree at most 1. Thus, connected structures in C are chains. There are $n!$ possible chains on n elements, so C has exponential generating series $\exp(x/(1-x))$. Again, $R = 1$ so the class does not have a labeled *MSO* 0–1 law.

Let C be the class of unary functions. The exponential generating series is

$$\sum_{k\geq 0} \frac{k^k}{k!} x^k$$

which is easily seen to have radius of convergence $1/e$, and once more the class does not have a labeled *MSO* 0–1 law.

There are many other examples where a labeled *MSO* 0–1 law fails. In all theses cases, we would like to know if $\mu_n(\varphi)$ converges for all *MSO* sentences φ. This is where Tauberian theorems play a significant role.

In [17], we prove a Tauberian theorem stating that if the ratios of coefficients mentioned in the definition of smooth growing classes are uniformly bounded, then $\mu(\varphi) = \bar{\mu}(\varphi)$ for every *MSO* sentence φ. Clearly, the class of one-to-one unary functions satisfies the hypotheses of this theorem, for every *MSO* sentence has a labeled asymptotic probability for this class.

In the same paper, we prove a Tauberian theorem stating that if $a(x)$ is *admissible*, then $\mu(\varphi) = \bar{\mu}(\varphi)$ for every *MSO* sentence φ. We will not define admissible functions here, since the definition is somewhat technical (although well known in the study of asymptotics). One can show that $\exp(x/(1-x))$ is admissible, so every *MSO* sentence has a labeled asymptotic probability for the class of directed graphs whose components are chains.

We prove a Tauberian theorem in [15] stating that if the coefficients of $a(x)$ are asymptotic to Kn^α for some $K > 0$ and $\alpha > -1$, then

$$\lim_{n\to\infty} (\mu_0(\varphi) + \mu_1(\varphi) + \cdots + \mu_{n-1}(\varphi))/n$$

is equal to $\mu(\varphi)$. We see by Stirling's formula that the class of unary functions satisfies the hypotheses, so every *MSO* sentence has a Ceśaro probability for this class.

There has been very little work on 0–1 laws and convergence theorems for the slow growing classes in logics other than *FO* and *MSO*. The only example we know is Kolaitis [43] where it is shown that the class of equivalence relations has a labeled *LFP* 0–1 law since *LFP* is no more expressive than *FO* on this class, but that there are classes having an unlabeled *FO* 0–1 law but no unlabeled *LFP* 0–1 law. It is not known for these classes if $\mu_n(\varphi)$ converges for each *LFP* sentence φ.

Not much is known about the computational complexity of probabilities for the slow growing classes. Compton [20] gives bounds for computing probabilities of *FO* and *MSO* sentences for the class of one-to-one functions. Kolaitis [43] states that the problem of deciding whether an *FO* sentence has probability 1 for the class of equivalence relations is *PSPACE*-complete. For *LFP* sentences the problem is *EXPTIME*-complete. These results are proved by techniques similar to those introduced in the next section for fast growing classes.

For simplicity we have stated all the theorems in this section for labeled structures, but they also hold for unlabeled structures. Use the ordinary generating series $b(x) = \sum_{k \geq 0} b_k x^k$ rather than the exponential generating series and define smoothness of growth as $\lim_{n \to \infty} b_{n-1}/b_n = S$. The analogue of Theorem 4.3 has $S = 1$ rather than $R = \infty$. We must use properties of the Pòlya cycle indicator generating series described in §3 so the arguments are somewhat more complicated, but the proofs follow along the same lines.

5 Fast Growing Classes

In this section we present 0–1 laws for fast growing classes. The techniques are quite different than those for slow growing classes because we can no longer use analytic properties of generating series. The first-order 0–1 laws discussed in this section could be proved using Ehrenfeucht games, but we will see that we can get 0–1 laws for other logics by another technique. We begin with one of the first 0–1 laws to be proved.

Theorem 5.1 (Glebskiĭ, et. al. [31]) *Let C be the class of all structures for a given relational similarity type. Then a labeled* FO *0–1 law holds.*

Proof: The proof of Glebskiĭ, et. al. is by induction on formula complexity, but rather than building formulas with the usual quantifiers $\exists x$ and $\forall x$, we consider formulas with the *excluding quantifiers* $(\exists x \neq y_0, \ldots, y_{k-1})$ and $(\forall x \neq y_0, \ldots, y_{k-1})$. If the set of free variables in a formula ψ is a subset of $\{x, y_0, \ldots, y_{k-1}\}$, then $(\exists x \neq y_0, \ldots, y_{k-1})\psi$ and $(\forall x \neq y_0, \ldots, y_{k-1})\psi$ are also formulas. The intended meaning of these quantifiers is evident, and it is easy to show that any *FO* formula is equivalent to a formula constructed from atomic formulas using only the usual connectives and excluding quantifiers. In fact, only the existential excluding quantifier is required, since the universal excluding quantifier may be obtained from it using negations.

Proceeding with the proof, we claim that if $\varphi(x_0, \ldots, x_{j-1})$ is such a formula, $n_0, \ldots, n_{j-1}$ are distinct nonnegative integers, and every atomic subformula of φ contains a quantified variable, then $\mu_n(\varphi(n_0, \ldots, n_{j-1}))$ approaches either 0 or 1 exponentially fast. (Note that $\mu_n(\varphi(n_0, \ldots, n_{j-1}))$ is defined when n is at least $\max(n_0, \ldots, n_{j-1})$.)

The claim is proved by induction on formula complexity. It is true by default for atomic formulas, and the induction steps for connectives are easy. Consider a formula φ of the form $(\exists x \neq n_0, \ldots, n_{j-1})\psi$. It is not difficult to show that ψ is equivalent to a formula $\bigvee_{i<k}(\sigma_i \wedge \tau_i)$ where σ_i is quantifier free and every atomic subformula of each formula τ_i contains a quantified variable. Thus, φ is equivalent to

$$\bigvee_{i<k} (\exists x \neq n_0, \ldots, n_{j-1})(\sigma_i \wedge \tau_i)$$

so it suffices to prove the claim for each of the formulas θ of the form

$$(\exists x \neq n_0, \ldots, n_{j-1})(\sigma_i \wedge \tau_i)$$

Now $\tau_i = \tau_i(x, x_0, \ldots, x_{j-1})$ has smaller quantifier rank than φ so by the induction hypothesis, if $m \neq n_0, \ldots, n_{j-1}$, then $\mu_n(\tau_i) = \mu_n(\tau_i(m, n_0, \ldots, n_{j-1}))$ approaches 0 or 1 exponentially fast.

Now there are two cases to consider. In the first case either $\mu_n(\tau_i)$ approaches 0 or there is no assignment of distinct nonnegative integers that makes $\sigma_i = \sigma_i(x, y_0, \ldots, y_{j-1})$ true. Then it is easy to show that $\mu_n(\theta)$ approaches 0 exponentially fast. In the second case, $\mu_n(\tau_i)$ approaches 1 and there is an assignment of distinct nonnegative integers that makes σ_i true. Then it can be shown that $\mu_n(\theta)$ approaches 1 exponentially fast. This proves the claim, and the labeled *FO* 0–1 law is an immediate consequence.□

We have left a number of technical details to be verified in this proof, but it is surprisingly elementary. It involves no sophisticated techniques or difficult results from logic. It uses just the definition of satisfaction of *FO* formulas and a few simple transformations for obtaining equivalent formulas. Unfortunately, this proof did not lead to the discovery of other 0–1 laws, even though some 0–1 laws described in this section might be proved in a similar manner, because it hides the model theoretic characteristics of fast growing classes with of 0–1 laws. Fagin [28] later gave another proof of this 0–1 law which serves as a paradigm for fast growing classes.

Alternate Proof: Consider the theory T consisting of all sentences of the form

$$(\forall \text{ distinct } x_0, \ldots, x_{j-1})(\exists x_j \neq x_0, \ldots, x_{j-1})(\delta \rightarrow \delta')$$

where $\delta = \delta(x_0, \ldots, x_{j-1})$ is a conjunction of formulas in $\Delta(x_0, \ldots, x_{j-1})$, a maximal consistent set of of atomic and negated atomic formulas in the variables $x_0, \ldots, x_{j-1}$; $\delta' = \delta'(x_0, \ldots, x_j)$ is defined similarly for $\Delta'(x_0, \ldots, x_j)$, a maximal consistent set of atomic and negated atomic formulas in the variables $x_0, \ldots, x_j$; and $\Delta(x_0, \ldots, x_{j-1}) \subseteq \Delta'(x_0, \ldots, x_j)$. (If j is 0, then δ is a tautology.) The sentences in T assert that every finite substructure can be extended by one element in all possible ways. We informally refer to sentences asserting that finite submodels can be extended as *extension axioms*. Extension axioms figure in all known 0–1 laws for fast growing classes.

An easy computation shows that $\mu(\varphi) = 1$ for each $\varphi \in T$. T is complete since it is $\aleph_0$-categorical and has no finite models. The the Compactness Theorem implies, then, that for each *FO* sentence ψ, either ψ is a consequence of finitely many sentences in T, whence $\mu(\psi) = 1$; or else $\neg\psi$ is a consequence of finitely many sentences in T, whence $\mu(\psi) = 0$.□

The theory T described above has a long history. Lynch [50] attributes the discovery of this theory to Jaškowski, who proposed it as an example of an $\aleph_0$-categorical theory not finitely axiomatizable over the axiom schema of infinity. (By $\aleph_0$-categorical we mean that the theory has precisely one countable model, up to isomorphism.) The $\aleph_0$-categoricity of T is easily demonstrated by a common model theoretic technique known as a back-and-forth argument. One simply builds an isomorphism between an arbitrary pair of countable models of T by successively extending finite partial isomorphisms between the two models according to the extension axioms. Since the models are countable, one can arrange that the union of the partial isomorphisms is an isomorphism between the two structures. Erdős used essentially the same argument to show that almost all countable graphs are isomorphic (see [25]) as did Gaifman [30] to show that almost all countable structures satisfy T (see §7). Rado [61] studied the properties of the almost sure graph, but his motivation was not probabilistic. Blass, Exoo, and Harary [4] give an explicit construction for graphs satisfying finite subtheories of T. (For model theorists, we note that T is the model completion of the empty theory.)

A proof virtually identical to the one above establishes labeled *FO* 0–1 laws for graphs, directed graphs, and tournaments. Oberschelp generalized this observation by defining a *parametric condition* to be a finite set of sentences of the form

$$(\forall \text{ distinct } x_0, \ldots, x_{j-1})\psi$$

where ψ is a Boolean combination of formulas of the form $R(y_0, \ldots, y_{k-1})$ with $\{y_0, \ldots, y_{k-1}\} = \{x_0, \ldots, x_{j-1}\}$. (The qualifier "distinct" can be omitted when $j = 1$.) The usual axiomatizations for graphs, directed graphs, and tournaments can easily be expressed as parametric conditions.

Theorem 5.2 (Oberschelp [54]) *A class of structures given by a parametric condition has a labeled* FO *0–1 law.*

Oberschelp gives only the statement of the theorem, not the proof. However, with some effort we can give a proof along the same lines as Fagin's proof by formulating the appropriate extension axioms for parametric classes. For example, models of the parametric condition consisting of the two sentences

$$\forall x\, \neg E(x, x)$$
$$(\forall \text{ distinct } x, y) E(x, y) \rightarrow E(y, x)$$

are irreflexive, symmetric relations, i.e., graphs. Let T consist of these two sentences and the extension axioms asserting that for all distinct elements $x_0, \ldots, x_{j-1}, y_0, \ldots, y_{k-1}$ there is a z not equal to any of these elements such that

$$\bigwedge_{i<j} E(x_i, z) \wedge \bigwedge_{i<k} \neg E(y_i, z)$$

for each $j, k > 0$. It is easy to see that T is $\aleph_0$-categorical and that $\mu(\varphi) = 1$ for each $\varphi \in T$. The argument proceeds as before. For arbitrary parametric conditions the proof is more complicated, but along the same lines.

To state the next result we need to define a *d-complex*. This is a set A together with a collection $\mathcal{S}$ of nonempty subsets of A of cardinality at most $d+1$ such that every nonempty subset of a set in $\mathcal{S}$ is also in $\mathcal{S}$. For each i, with $1 \leq i \leq d$, we have an $(i+1)$-ary relation symbol R_i. $R_i(x_0, \ldots, x_i)$ holds only when $x_0, \ldots, x_i$ are distinct and $\{x_0, \ldots, x_i\} \in \mathcal{S}$.

Theorem 5.3 (Blass and Harary [6]) *For each $d > 0$, the class of d-complexes has a labeled* FO *0–1 law.*

The proof is similar to previous proofs. Formulate extension axioms and carry out the argument outlined earlier.

By Kleitman and Rothschild's asymptotic estimate for the number of labeled partial orders [42] we can prove a labeled *FO* 0–1 law for partial orders. (See also Remlinger [62]).

Theorem 5.4 (Compton [19]) *The class of partial orders has a labeled* FO *0–1 law.*

Proof: Kleitman and Rothschild obtain a detailed description of the structure of random finite partial orders. With labeled asymptotic probability 1 a partial order will have no chains of length greater than 3. Thus, almost every partial order can be partitioned into 3 levels: L_0, the set of minimal elements, L_1, the set of elements immediately succeeding elements in L_0, and L_2, the set of elements immediately succeeding elements in L_1. Moreover, in partial orders of size n, $|L_0| = n/4 + o(n)$, $|L_1| = n/2 + o(n)$, and $|L_2| = n/4 + o(n)$ almost surely. Now it is easy to formulate extension axioms for random partial orders. First, for all distinct $x_0, \ldots, x_{j-1}$ and $y_0, \ldots, y_{k-1}$ in L_1 and all distinct $z_0, \ldots, z_{l-1}$ in L_0, there is an element z in L_0 not equal to $z_0, \ldots, z_{l-1}$, such that

$$\bigwedge_{i<j} z \leq x_i \wedge \bigwedge_{i<k} z \not\leq y_i$$

Second, for all distinct $x_0, \ldots, x_{j-1}$ and $y_0, \ldots, y_{k-1}$ in L_1 and all distinct $z_0, \ldots, z_{l-1}$ in L_2, there is an element z in L_2 not equal to $z_0, \ldots, z_{l-1}$, such that

$$\bigwedge_{i<j} x_i \leq z \wedge \bigwedge_{i<k} y_i \not\leq z$$

Finally, for all distinct $x_0, \ldots, x_{j-1}$ and $y_0, \ldots, y_{k-1}$ in L_0, all distinct $x'_0, \ldots, x'_{j'-1}$ and $y'_0, \ldots, y'_{k'-1}$ in L_2, and all distinct $z_0, \ldots, z_{l-1}$ in L_1, there is an element z in L_1 not equal to $z_0, \ldots, z_{l-1}$, such that

$$\bigwedge_{i<j} x_i \leq z \wedge \bigwedge_{i<k} y_i \not\leq z \wedge \bigwedge_{i<j'} z \leq x'_i \wedge \bigwedge_{i<k'} z \not\leq y'_i$$

These axioms can be written as *FO* sentences. Taken together with the axioms for partial orders and a sentence asserting that there is a chain of length 3 but no chain has length greater than three, they form a an $\aleph_0$-categorical theory. This is easily shown by a back-and-forth argument, but one must take care. Before attempting to extend partial isomorphisms between two countable models of the theory, one should first add three unary relations to the models for the levels L_0, L_1, and L_2 so that elements are mapped to elements at the same level. The labeled *FO* 0–1 law now follows by Fagin's argument.□

The K_{m+1}-free graphs are those graphs having no subgraph isomorphic to K_{m+1}, the complete graph on $m+1$ vertices. These classes are all fast growing.

Theorem 5.5 (Kolaitis, Prömel, and Rothschild [44, 45]) *The class of K_{m+1}-free graphs has a labeled* FO *0–1 law for each $m \geq 2$.*

Proof: Kolaitis, Prömel, and Rothschild give an argument similar to previous arguments although the details are more difficult. First, they derive an asymptotic estimate for the number of labeled K_{m+1}-free graphs, and in so doing analyze the structure of random K_{m+1}-free graphs. They show that with labeled asymptotic probability 1 a K_{m+1}-free graph will be uniquely m-colorable, i.e., there is a unique partition of its vertices into m sets so that no two vertices in the same set are adjacent. The property of m-colorability is not *FO* expressible for arbitrary graphs; random K_{m+1}-free graphs, however, possess an *FO* property that implies m-colorability. A *spindle* connecting two vertices x and y in a K_{m+1}-free graph is a subgraph isomorphic to K_{m-1} such that all its vertices are adjacent to both x and y. Let φ be an *FO* sentence asserting that the relation of being connected by a spindle is an equivalence relation of index m and no two vertices in an equivalence class of this relation are adjacent. Kolaitis, Prömel, and Rothschild show that φ holds almost surely in a K_{m+1}-free graph.

Suppose that the edge relation is denoted by E and that $L_0, \ldots, L_{m-1}$ are the equivalence classes for the spindle connection relation in a random K_{m+1}-free graph. For each $i < m$ there are extension axioms saying that for all distinct $x_0, \ldots, x_{j-1}$ and $y_0, \ldots, y_{k-1}$ not in L_i and all distinct $z_0, \ldots, z_{l-1}$ in L_i, there is an element z in L_i not equal to $z_0, \ldots, z_{l-1}$, such that

$$\bigwedge_{i<j} E(x_i, z) \wedge \bigwedge_{i<k} \neg E(y_i, z)$$

These axioms together with φ form an $\aleph_0$-categorical theory. Again this is proved by a back-and-forth argument (this time with unary relations added for $L_0, \ldots, L_{m-1}$), and again a labeled 0–1 law follows by showing that each extension axiom has labeled asymptotic probability 1.□

The proofs we have sketched provide the ideas needed to determine the complexity of labeled asymptotic probability computations and to extend labeled 0–1 laws to more general logics. Let T be the the theory of almost all finite relational structures (i.e., the set of extension axioms we listed for this class) and let $\mathcal{M}$ be the unique countable model of T. We have seen that an *FO* sentence has labeled asymptotic probability 1 if and only if it is a consequence of T, or equivalently, is true in $\mathcal{M}$. Let us therefore consider the problem of determining the truth of formulas in $\mathcal{M}$. The proof that T is $\aleph_0$-categorical shows that $\mathcal{M}$ has the following homogeneity property. An isomorphism between finite substructures of $\mathcal{M}$ can always be extended to an automorphism of $\mathcal{M}$. Hence, the truth of a formula $\varphi(x_0, \ldots, x_{j-1})$ in $\mathcal{M}$, where $x_0, \ldots, x_{j-1}$ is a sequence of elements from $\mathcal{M}$, is completely determined by the isomorphism type of the substructure formed by restricting $\mathcal{M}$ to $\{x_0, \ldots, x_{j-1}\}$, and this in turn is completely determined by $\delta(x_0, \ldots, x_{j-1})$, the conjunction of the atomic and negated atomic formulas true of elements $x_0, \ldots, x_{j-1}$ in $\mathcal{M}$. Grandjean uses this idea to obtain the following result.

Theorem 5.6 (Grandjean [33]) *Let C be the class of all relational structures. The problem of whether $\mu(\varphi) = 1$ for an* FO *sentence φ is* PSPACE*-complete.*

Proof: Let us say that $\delta(x_0, \ldots, x_{j-1})$ is a j-description if it is the conjunction of the atomic and negated atomic formulas true of a sequence of distinct elements $x_0, \ldots, x_{j-1}$ in $\mathcal{M}$. By convention, the only 0-description will be a tautology τ. Now if $\delta = \delta(x_0, \ldots, x_{j-1})$ is a j-description and $\varphi = \varphi(x_0, \ldots, x_{j-1})$ is an *FO* formula, we would like to determine when $\mathcal{M} \models \delta \rightarrow \varphi$. The following equivalences allow us to do this.

(i) If φ is atomic, then $\mathcal{M} \models \delta \rightarrow \varphi$ if and only if φ is a conjunct of δ.

(ii) If φ is of the form $\neg\psi$, then $\mathcal{M} \models \delta \rightarrow \varphi$ if and only if $\mathcal{M} \not\models \delta \rightarrow \psi$.

(iii) If φ is of the form $\psi_0 \vee \psi_1$, then $\mathcal{M} \models \delta \rightarrow \varphi$ if and only if $\mathcal{M} \models \delta \rightarrow \psi_0$ or $\mathcal{M} \models \delta \rightarrow \psi_1$.

(iv) If φ is of the form $\exists y\,\psi(x_0,\ldots,x_{j-1},y)$, then $\mathcal{M} \models \delta \to \varphi$ if and only if either for some $i < j$, $\mathcal{M} \models \delta \to \psi(x_0,\ldots,x_{j-1},x_i)$; or for some $j+1$-description $\delta' = \delta'(x_0,\ldots,x_j)$ extending $\delta(x_0,\ldots,x_{j-1})$, $\mathcal{M} \models \delta' \to \psi(x_0,\ldots,x_j)$.

Now it is easy to translate this list of equivalences, as Grandjean does, into a polynomial time program for an alternating Turing machine. (Chandra, Kozen, and Stockmeyer [10] is the standard reference on alternating Turing machines.) This program takes as input a j-description δ and formula φ, and returns a value of *true* or *false* depending on whether or not $\delta \to \varphi$ is true in $\mathcal{M}$. We may regard the program as consisting of a recursive procedure with formal parameters δ and φ. We assume that conjunction and universal quantification are defined in terms of negation, disjunction, and existential quantification, so φ will be in one of the forms given above. If φ is atomic, the procedure computes the return value directly according to (i). In all other cases it computes the return value according to recursive procedure calls according to the equivalences listed.

To see that this program operates in alternating polynomial time note that all the $j+1$-descriptions extending a particular j-description can be generated in alternating time uniformly polynomial in the length of the j-description. Now a for an *FO* sentence φ, $\mu(\varphi) = 1$ if and only if $\mathcal{M} \models \tau \to \varphi$, which can be determined by applying the program to τ and φ. Chandra, Kozen, and Stockmeyer [10] show that the set of problems solved by alternating Turing machines in polynomial time is precisely the set of problems solved by ordinary Turing machines in *PSPACE*. Therefore, the problem of computing labeled asymptotic probabilities of *FO* sentences is in *PSPACE*.

Stockmeyer [65] shows that for any *FO* theory T having a model with at least two elements, the problem of determining whether an *FO* sentence φ is a consequence of T is *PSPACE*-hard. (The reductions used here are either polynomial time or log space reductions.) Thus, the problem of computing labeled asymptotic probabilities of *FO* sentences is *PSPACE*-complete.□

In view of Stockmeyer's result, the problem of computing labeled asymptotic probabilities for relational structures has the lowest possible complexity. It is interesting to compare this problem to the problem of determining whether an *FO* sentence is true in all relational structures (rather than almost all relational structures). Trakhtenbrot [68] showed that for some similarity type this problem is undecidable, and Vaught [70, 71] extended the result to every similarity type with at least one non-unary relation (these are precisely the similarity types that give rise to fast growing classes). Consideration of almost all structures drastically reduces complexity.

Kolaitis and Vardi [46] note that Grandjean's algorithm requires only a linear number of alternations on an alternating Turing machine. This would seem to give a slightly better bound since many believe that *TA(poly,lin)*, the set of problems solved by alternating Turing machines in polynomial time with a linear number of alternations, is properly contained in *PSPACE*.

Grandjean's argument shows that the problem of whether a sentence has labeled asymptotic probability 1 is *PSPACE*-complete for parametric classes, the class of d-complexes, the class of partial orders, and the classes of K_{m+1}-free graphs.

Let us now consider complexities of 0–1 laws for other logics.

Theorem 5.7 (Blass, Gurevich, and Kozen [5]) *If a class $\mathcal{C}$ of structures has an* FO *0–1 law and the set of* FO *sentences with probability 1 is an $\aleph_0$-categorical theory, then that class has an* LFP *0–1 law.*

Proof: The Ryll-Nardzewski Theorem (Theorem 2.3.12(e) of Chang and Keisler [11]) states that a necessary and sufficient condition for $\aleph_0$-categoricity of a complete theory T is that for each $j > 0$, there are only a finite number of formulas $\varphi(x_0,\ldots,x_{j-1})$, logically inequivalent with respect to T. This is obvious for the $\aleph_0$-categorical theories we have seen because the truth value of each formula in j variables is determined by conjunctions of atomic and negated atomic formulas in those variables (for partial orders and K_{m+1}-free graphs, this is true only after the addition of extra relations), and there are just a finite number of atomic formulas.

Consider the *LFP* formula φ of the form

$$\left[X(x_0,\ldots,x_{j-1}) \equiv \theta\right] \psi$$

where θ is equivalent to an *FO* formula. Recall that $\theta = \theta(X, x_0, \ldots, x_{j-1})$ gives rise to an operator F on the set of j-ary relations on each relational structure and the least least fixed point of F is $F^\kappa(\emptyset)$, where κ is the least ordinal such that $F^\alpha(\emptyset) = F^\kappa(\emptyset)$ whenever $\alpha \geq \kappa$. If α is finite, $F^\alpha(\emptyset)$ is defined by a *FO* formula. But there are only finitely many inequivalent formulas with respect to T so κ must be finite. Thus, in models of T, $F^\kappa(\emptyset)$, the least fixed point of F, is defined by an *FO* formula $\lambda(x_0, \ldots, x_{j-1})$, and there is an *FO* formula asserting that $\lambda(x_0, \ldots, x_{j-1})$ is the least least fixed point. By the Compactness Theorem, the formula asserting that λ is the least least fixed point of F is a consequence of some finite subtheory T' of T. But the sentences in T' hold with labeled asymptotic probability 1, so φ is equivalent to an *FO* formula φ' on almost all finite relational structures. (From our remarks it is not difficult to show that if the number of variables in φ is k, we can take φ' to be equivalent to a formula formed by replacing every implicit definition $[X(x_0, \ldots, x_{j-1}) \equiv \theta]$ in φ with an approximation $[X(x_0, \ldots, x_{j-1}) \equiv \theta]_m$, where m is the number of inequivalent k-descriptions.) From this it follows that every *LFP* sentence has labeled asymptotic probability either 0 or 1.□

Blass, Gurevich, and Kozen also give complexity bounds for computing the labeled asymptotic probabilities of *LFP* sentences.

Theorem 5.8 ([5]) *Let C be the class of all relational structures. The problem of computing labeled asymptotic probabilities of* LFP *sentences is* EXPTIME-*complete.*

The idea is that in an *LFP* sentence φ we can replace the implicit definitions with approximations, as in the proof of Theorem 5.7, and then apply a recursive procedure of the sort given for *FO* sentences in Theorem 5.6 to determine if it holds in the countable model $\mathcal{M}$. (Some care must be taken to insure that the algorithm takes just exponential time; see Compton [19].)

Kolaitis and Vardi use similar ideas to prove the following.

Theorem 5.9 (Kolaitis and Vardi [46]) *Let C be the class of all relational structures. A labeled* IT *0–1 law holds and its associated decision problem is* EXPSPACE-*complete. Also, the decision problem for* TC *is* PSPACE-*complete.*

The proof carries over for all of the other classes discussed in this section. In the same paper, Kolaitis and Vardi give a simple model theoretic argument showing the following.

Theorem 5.10 (Kolaitis and Vardi) *Let C be the class of all relational structures. The logic strict-$\Sigma_1^1 \cup$ strict-Π_1^1 has a labeled 0–1 law and the problem of deciding whether a strict-Σ_1^1 sentence has labeled asymptotic probability 1 is* NEXPTIME-*complete.*

The proof that this problem is in *NEXPTIME* is much more difficult than the complexity proofs for the other logics we have mentioned. We do not know if we can modify the *NEXPTIME* decision procedure in the paper of Kolaitis and Vardi for the other fast growing classes we have examined, but we can modify the 0–1 law for strict-$\Sigma_1^1 \cup$ strict-Π_1^1 for these classes.

Liogon'kiĭ [49] showed that the class of all relational structures has an unlabeled *FO* 0–1 law. In fact, all the fast growing classes presented in this section have unlabeled 0–1 laws for the logics *FO*, *TC*, *LFP*, and *IT*, and the complexity bounds for the related decision problems are the same.

Let C be any of the classes discussed in this section. If a_n is the number of labeled structures and b_n is the number of unlabeled structures in C_n, then $b_n \sim a_n/n!$. Whenever this is the case, the labeled asymptotic probability of a sentence is the same as its unlabeled asymptotic probability. To see this, let $\mu_n(\varphi)$ be the fraction of labeled structures in C_n satisfying φ and $\nu_n(\varphi)$ be the fraction of unlabeled structures in C_n satisfying φ. If $\mu(\varphi) = 1$, then

$$n!\, b_n\, \nu_n(\varphi) \geq a_n\, \mu_n(\varphi)$$

so

$$1 \geq \nu_n(\varphi) \geq \frac{a_n/n!}{b_n}\, \mu_n(\varphi)$$

and therefore $\nu_n(\varphi)$ approaches 1.

It is necessary to consider the various fast growing classes separately to show that $b_n \sim a_n/n!$. The proofs rely on methods, pioneered by Pólya [59], for enumerating unlabeled structures. Let the symmetric group S_n act on the universes of the labeled structures in C_n. The Frobenius-Burnside Lemma tells us that the number of unlabeled structures in C_n is

$$b_n = \frac{1}{n!} \sum_{\alpha \in S_n} \mathrm{Fix}(\alpha)$$

where $\mathrm{Fix}(\alpha)$ is the number of structures fixed by α (i.e., for which α is an automorphism). The identity element ι of S_n fixes all structures, so $\mathrm{Fix}(\iota) = a_n$. Thus, $b_n - a_n/n!$ is

$$\frac{1}{n!} \sum_{\alpha \neq \iota} \mathrm{Fix}(\alpha)$$

All of the proofs proceed by estimating the number of labeled structures fixed by each $\alpha \neq \iota$ to show that this sum is $o(a_n/n!)$.

There is a close relationship between this method and the the probability of rigidity in a random structure. (A structure is *rigid* if it has no automorphisms other than the trivial automorphism.) It is easy to show that $b_n \sim a_n/n!$ if and only if the unlabeled asymptotic probability of rigidity is 1. Furthermore, if $b_n \sim a_n/n!$, then the unlabeled asymptotic probability of rigidity is 1 (but the converse may fail).

Pólya showed that unlabeled graphs are almost surely rigid. Wright [73] extended this result to classes of graphs with a given number of edges. Oberschelp [56], and Fagin [27] showed that unlabeled relational structures are almost surely rigid (assuming that there is a relation of arity at least 2), and Oberschelp [55] extended this result to all classes given by a parametric condition. Prömel [60] proved that unlabeled structures from a class C, with the similarity type of one binary relation, closed under substructures, and with $\log a_n = c_0 n^2 + c_1 n + o(n)$, are almost surely rigid. The asymptotic estimate of Kleitman and Rothschild [42] for the number of labeled partial orders, and of Kolaitis, Prömel, and Rothschild [44] for the number of K_{m+1}-free graphs, shows that both kinds of classes satisfy $\log a_n = c_0 n^2 + c_1 n + o(n)$, so they are covered by Prömel's theorem. Bollobás and Palmer [9] show that unlabeled d-complexes are almost surely rigid. Hence, for all of the fast growing classes in this section we have unlabeled 0–1 laws for FO, TC, LFP, and IT, and strict-Σ^1_1 $\cup$ strict-Π^1_1.

This leads us to ask whether rigidity can be expressed in any of these logics. Blass and Harary [6], show that the answer for the class of graphs is no. The reason is that the proofs of 0–1 laws for these logics all depend on showing that if φ, a sentence in the logic, is true in $\mathcal{M}$, the unique countable model of theory T consisting of extension axioms, then φ is a consequence of finitely many sentences in T on finite structures. Thus, the sentences holding in $\mathcal{M}$ are precisely the sentences with labeled asymptotic probability 1. Blass and Harary show that every finite subset of T has a finite model with nontrivial automorphisms, so rigidity cannot be expressed in any of these logics. They also show that every finite subset of T has a finite model with a Hamilton cycle. One of the jewels of random graph theory is the result that a labeled or unlabeled graph is almost surely Hamiltonian. (See Bollobás [8] for the history of this problem and an account of its proof.) It follows that Hamiltonicity cannot be expressed in any of these logics.

This points out the main difficulty in our present state of knowledge of 0–1 laws for fast growing classes. The logics for which we have 0–1 laws can express only those almost sure properties that follow from extension axioms. Bollobás [7, p. 131] remarks that the first-order labeled 0–1 law for graphs looks sophisticated, but follows from shallow computations. This observation does not hold for all the 0–1 laws we have seen since determining the asymptotic growth of the class, say for partial orders or K_{m+1}-free graphs, may be quite difficult. However, another of his criticisms does pertain: most combinatorially interesting properties cannot be expressed in the logics for which we have 0–1 laws. (Blass and Harary [6] point out that a few interesting combinatorial properties do follow from the extension axioms for graphs; they include nonplanarity, having diameter 2, k-connectivity for each k, and the property of not being a line graph.)

A 0–1 law for graphs in a logic in which Hamiltonicity or rigidity could be expressed would be significant. A natural direction to take in the search for such a 0–1 law would be to consider fragments of second-order logic, but work so far in this direction holds little hope. Kaufmann and Shelah [41], for example, show that there is not a labeled *MSO* 0–1 law for the class of relational structures. In the example they give of an *MSO* sentence φ without an asymptotic probability, $\mu_n(\varphi)$ is so badly behaved that no reasonable modification of the notion of asymptotic probability will serve.

6 Structures with Functions and Underlying Relations

In the last section we established 0–1 laws by showing that the *FO* sentences in an $\aleph_0$-categorical theory T have asymptotic probability 1. Unfortunately, in cases where a 0–1 law does not hold but we would like to show that $\mu_n(\varphi)$ converges for every sentence φ in some logic, this approach offers no guidance. Even in cases where a 0–1 law holds, but the set of sentences with probability 1 is not $\aleph_0$-categorical, this approach fails. In this section we will discuss a technique due to Lynch [50, 51] for dealing with some of these cases. The technique seems most applicable to fast growing classes of structures with underlying relations, but has also been used for slow growing classes, particularly classes of functions. Unlike the method of the last section, Lynch's method does not generalize to extensions of *FO*, in fact, cannot generalize because, as we will see, asymptotic probabilities are not defined in general for the the extensions of first-order logic we have introduced.

To illustrate the ideas in this approach, let us derive the labeled *FO* 0–1 law for the class C of relational structures supposing that we know nothing about $\aleph_0$-categoricity. Instead, we use the *FO* Ehrenfeucht games described in §2. It is enough to show for a fixed $r > 0$ that two sufficiently large, randomly chosen labeled structures $\mathcal{M}$ and $\mathcal{N}$ (not necessarily the same size) will almost surely satisfy $\mathcal{M} \equiv_r \mathcal{N}$. A sentence φ of quantifier rank r is then either true in almost all structures or false in almost all structures. Hence, we must determine a winning strategy for player II in the Ehrenfeucht game of length r on $\mathcal{M}$ and $\mathcal{N}$. Since the extension axioms each have probability 1, we may assume that in $\mathcal{M}$ and $\mathcal{N}$ substructures on sets of size less than r may be extended by one element in all possible ways. This shows that the naive strategy is a winning strategy for player II. Whenever player I picks an element from one model, player II is guaranteed an element in the other structure that will extend the partial isomorphism between elements previously chosen.

Now consider what happens if we have an underlying successor relation on structures. That is, we add a binary relation symbol S which interprets the successor relation on the labeled structures of C. (Recall that the universe of a labeled structure is always of the form $\{0, \ldots, n-1\}$; $S(x, y)$ holds precisely when $y = x + 1$.) The *FO* 0–1 law no longer holds. Suppose, for example, that R is a binary relation symbol in the similarity type. The probability that $R(0, 0)$ holds is 1/2. We can, however, modify our Ehrenfeucht game argument to show the following.

Theorem 6.1 (Lynch [50]) *Let C be the class of all relational structures with an underlying successor relation. Then $\mu_n(\varphi)$ converges for each* FO *sentence φ.*

Proof: For a sequence of elements $x_0, \ldots, x_{j-1}$ from $\mathcal{M}$ (empty sequence permitted) and a nonnegative integer r, form $\mathcal{B}^r(\mathcal{M}, x_0, \ldots, x_{j-1})$, the substructure of $\mathcal{M}$ whose universe consists of elements at distance no more than 3^r from some x_i or one of the two end points, with constants added for each of the elements $x_0, \ldots, x_{j-1}$.

We claim that the probability of $\mathcal{M} \equiv_r \mathcal{N}$, given that $\mathcal{B}^r(\mathcal{M})$ and $\mathcal{B}^r(\mathcal{N})$ are isomorphic, can be made as close to 1 as we like by taking $\mathcal{M}$ and $\mathcal{N}$ large enough. Consequently, the sentences of quantifier rank r satisfied by a structure are almost surely determined by the intervals of length 3^r at the beginning and end of the structure. There are only finitely many ways to specify relations on the beginning and ending 3^r elements of a structure, and all such specifications have equal probability. For φ of quantifier rank r, obtain $\mu(\varphi)$ by summing the probabilities of the beginning-ending combinations for which φ holds.

To prove the claim, we show that player II almost surely has a winning strategy on $\mathcal{M}$ and $\mathcal{N}$ when $\mathcal{B}^r(\mathcal{M})$ and $\mathcal{B}^r(\mathcal{N})$ are isomorphic. Suppose that x_i and y_i are the elements chosen on move i from $\mathcal{M}$ and $\mathcal{N}$ respectively. Player II responds so that

$$\mathcal{B}^{r-j}(\mathcal{M}, x_1, \ldots, x_j) \text{ and } \mathcal{B}^{r-j}(\mathcal{N}, y_1, \ldots, y_j)$$

are isomorphic for $0 \le j \le r$. To see that player II can effect such a strategy, let us examine her options on move $j+1$ supposing that she has managed to satisfy this condition on move j.

Player I picks an element from one of the two structures, say x_{j+1} from $\mathcal{M}$. Now if x_{j+1} differs from some x_i already chosen or one of the end points by at most $2 \cdot 3^{r-j-1}$, then the universe of $\mathcal{B}^{r-j-1}(\mathcal{M}, x_1, \ldots, x_{j+1})$ is entirely contained in the universe of $\mathcal{B}^{r-j}(\mathcal{M}, x_1, \ldots, x_j)$ so player II has only to pick the corresponding element in $\mathcal{B}^{r-j}(\mathcal{N}, y_1, \ldots, y_j)$ On the other hand, if x_j differs from each x_i and both end points by more than $2 \cdot 3^{r-j-1}$, then consider a maximal set of elements y that are at least distance $2 \cdot 3^{r-j-1}$ apart and at least the same distance from each y_i. Almost surely for one such value y_{j+1}, $\mathcal{B}^{r-j-1}(\mathcal{N}, y_1, \ldots, y_{j+1})$ is isomorphic to $\mathcal{B}^{r-j-1}(\mathcal{M}, x_1, \ldots, x_{j+1})$. (Notice that although the mapping from $x_1, \ldots, x_{j+1}$ to $y_1, \ldots, y_{j+1}$ preserves the successor relation, it may not preserve order.)$\Box$

Let us describe the condition assuring a winning strategy for player II as an extension axiom. We define an *(r, j)-description* $\delta = \delta(x_1, \ldots, x_j)$ for $x_0, \ldots, x_{j-1}$ to be the conjunction of formulas giving the relations and negations of relations holding between the elements in $\mathcal{B}^{r-j+1}(\mathcal{M}, x_1, \ldots, x_j)$. Let $\delta = \delta'(x_1, \ldots, x_{j+1})$ be the $(r, j+1)$-description for $x_1, \ldots, x_{j+1}$. Then the extension axiom

$$\forall x_1, \ldots, x_j \exists x_{j+1}(\delta \rightarrow \delta')$$

has labeled asymptotic probability 1.

Lynch [50] formalizes this technique and applies it to other examples. It is not difficult to see that if, rather than the successor relation, we take the modular successor relation as an underlying relation (i.e., let $S(n-1, 0)$ hold on labeled structures with universe n), we then have an *FO* 0–1 law because we no longer need to consider intervals at the ends of structures.

A much more difficult example considered by Lynch is relational structures with the successor relation and modular addition as underlying relations. (For the latter, we have a ternary relation symbol P with $P(x, y, z)$ holding precisely when $x + y = z \pmod n$ in structures with universe n.) Here it is no longer the case that $\mu_n(\varphi)$ converges for all *FO* sentences φ. For example, the sentence $\forall x \exists y\, P(y, y, x)$ holds only in structures with universes of odd cardinality. However, Lynch shows that $\mu_n(\varphi)$ is "almost convergent" in the following sense.

Theorem 6.2 (Lynch) *Let C be the class of relational structures with underlying successor and modular addition relations. For each* FO *sentence φ there is a positive integer p such that for each $i < p$, $\mu_{i+np}(\varphi)$ converges as n approaches ∞.*

The same analysis shows that if we have underlying modular successor and modular addition relations, then for each *FO* sentence φ there is a positive integer p such that for each $i < p$, $\mu_{i+np}(\varphi)$ approaches either 0 or 1 as n approaches ∞. In both of these examples, the Cesàro limit of $\mu_n(\varphi)$ (defined in the introduction) exists. The analysis of this example is similar in spirit to the analysis for relational structures with an underlying successor relation, but is too involved to present here.

Lynch's technique is strikingly similar to Ferrante and Rackoff's technique for obtaining upper complexity bounds for the satisfiability problem for *FO* theories [29]. They use Ehrenfeucht games to show for certain theories T that it is possible to determine whether whether φ is satisfied in some model of T by checking the truth of φ in just a finite number of models; moreover, to verify if φ holds in one of these models it is possible to restrict the search to finitely many elements when searching for witnesses to existentially quantified formulas. By bounding the number of models and number of elements considered, one obtains an upper bound. Grandjean's *PSPACE* upper bound for the theory of almost all relational structures in Theorem 5.6 be cast in this form.

Close scrutiny of Lynch's arguments [50] reveals that they also give upper bounds. Let T be the set of extension axioms described above for the class of relational structures with an underlying successor relation. To determine that an *FO* sentence φ of quantifier rank r has labeled asymptotic probability 1, we must verify that φ holds in models of T with every combination of relations for the beginning and ending intervals of length 3^r. That is, we must show that for every $(r, 0)$-description δ, $\delta \rightarrow \varphi$ is true in models of T.

A straightforward modification of Grandjean's algorithm allows us to determine for a formula $\varphi(x_1, \ldots, x_j)$ of quantifier rank r and an (r, j)-description $\delta(x_1, \ldots, x_j)$ whether $\delta \rightarrow \varphi$ is true in models of T. Notice that to justify the negation clause in this algorithm we must show that when $\delta(x_1, \ldots, x_j)$ is an (r, j)-description, $T \cup \{\delta\}$ completely determines the formulas of quantifier rank r holding in any model. This follows from our Ehrenfeucht game analysis. Notice also in the existential quantifier clause that generating an $(r-1, j+1)$-description δ' extending an (r, j)-description δ requires exponential rather than polynomial time. Consequently, the algorithm gives an upper bound of $TA(2^{cn}, n)$ (i.e., time 2^{cn} on an alternating Turing machine making n alternations), for some $c > 0$, to determine whether an *FO* sentence φ of length n has labeled asymptotic probability 1. The same bound holds for relational structures with an underlying modular successor relation. For relational structures with underlying successor and modular addition relations Lynch's analysis together with these observation show that the upper bound is $TA(2^{2^{cn}}, n)$ for some $c > 0$.

These upper bounds are the best possible. Compton and Henson [21] give a general method for proving lower bounds for theories. They show, roughly, that if it is possible, using formulas of length $O(n)$, to interpret in models of a theory T' all binary relations on sets of size at most $f(n)$ together with the power sets of their universes, then the problem of whether a sentence φ is true in some model of T' has a lower bound of $TA(f(dn), dn)$ for some $d > 0$.

Let T' be the set of sentences having labeled asymptotic probability 1 in the class of relational structures with an underlying successor relation. Using standard techniques we can specify formulas $\alpha_n(x, u, u')$ of length $O(n)$ that hold when $u \leq x \leq u'$ and $u' - u \leq 2^n$. Thus, by varying the parameters u and u', we can define all intervals of length at most 2^n. Now if the similarity type contains a non-unary relation symbol, say a binary relation symbol R, the interpretation of R restricted to these intervals almost surely will range over all binary relations on sets of size at most 2^n. Moreover, the formulas $\alpha_n(x, u, u') \wedge R(x, v)$ almost surely define all subsets of the interval $u \leq x \leq u'$ as v ranges over elements in the structure. (See [21] for a detailed discussion of these methods.) It follows that the problem of determining whether a sentence has labeled asymptotic probability 1 in the class of relational structures with an underlying successor relation has a lower bound of $TA(2^{dn}, dn)$ for some $d > 0$. A similar argument shows that for the class of structures with underlying successor and modular addition relations the lower bound is $TA(2^{2^{dn}}, dn)$ for some $d > 0$.

Of course, in those cases where a 0–1 law does not hold we would like to know the labeled asymptotic probabilities of sentences. Lynch's technique gives this directly. For the class of relational structures with an underlying successor relation, $\mu(\varphi)$ is always a dyadic rational number that can be computed in $SPACE(2^{dn})$. Similarly, for the class of relational structures with underlying successor and modular addition relations, there is, for each *FO* sentence φ, an integer p and dyadic rational numbers l_i, $i < p$, such that $\mu_{i+np}(\varphi)$ approaches l_i as n approaches ∞; p and l_i, $i < p$ can be computed in $SPACE(2^{2^{dn}})$.

Can we obtain similar results for relational structures with an underlying linear order? Compton, Henson, and Shelah [22] show that the answer is no. There is an *FO* sentence φ for this class such that $\mu_n(\varphi)$ is badly behaved: it does not converge and there is no periodic subsequence that converges. The proof is similar to the proof of Kaufmann and Shelah [41] showing that there is an *MSO* sentence φ for the class of relational structures such that $\mu(\varphi)$ does not converge. With an underlying successor relation we can quantify over intervals in *FO*, and this is enough set quantification to define, in almost all structures, the largest interval on which a binary relation R codes addition and multiplication. An interesting corollary of this result is that for the class of relational structures with an underlying successor relation, the labeled asymptotic probability of sentences from *TC*, *LFP*, or *IT*. The reason is that with a transitive closure operator we can define a linear order from a successor relation.

Theorem 6.3 (Lynch [51]) *Let C be the class of structures with k unary functions. Then every* FO *sentence has a labeled asymptotic probability.*

Proof: The approach is like the one for the class of relational structures with an underlying successor relation but the details are significantly harder.

The distance between elements is determined not by a successor relation now, but rather by the shortest path between elements in the graph formed by taking an edge between two elements precisely when one is mapped to the other by one of the k functions. Fix $r > 0$. We saw that for relational structures with a successor relation, restrictions to the beginning and ending intervals of length 3^r almost surely determine which *FO* sentences of quantifier rank r will hold. With a similar view we define a *critical substructure* of one of our k-function structures to be the restriction to the set of elements at distance no more than 3^r from a given cycle of length at most $2 \cdot 3^r + 1$. (*Cycle* refers to a cycle in the graph.) Lynch shows that critical substructures are almost surely disjoint for each r (a condition he calls r-simplicity). He also shows that structures are almost surely *r-rich*; this condition can be expressed by extension axioms which insure a winning strategy for player II in the Ehrenfeucht game of length r. These axioms are precisely the same as for relational structures with an underlying successor relation except that $\mathcal{B}^r(\mathcal{M}, x_1, \ldots, x_j)$ is the substructure of $\mathcal{M}$ whose universe consists of elements at distance no more than 3^r from some x_i or a cycle of length at most $2 \cdot 3^r$, with constants added for each of the elements $x_1, \ldots, x_j$.

Computing the probabilities of relations holding on beginning and ending intervals is straightforward, but computing probabilities of occurrences of critical substructures is not easy: for a fixed r the number of isomorphism types of critical structures may be infinite, and there is no bound on the number of critical substructures of a structure. Lynch overcomes these problems by defining for each r an equivalence relation he calls r-morphism on classes of critical structures. The r-morphism relation respects isomorphism classes and has finite index l. Lynch shows that the *FO* sentences holding in a structure are almost surely determined by the number of critical substructures in each r-morphism class. An Ehrenfeucht game argument shows it is only necessary to know the precise number of critical substructures in an r-morphism class up to r; for more than r, the *FO* sentences of quantifier rank r that hold are no different than if there were exactly r. For each of the l r-morphism classes there are $r+1$ possibilities: the number of critical structures in the class may be $0, 1, \ldots, r-1$, or $\geq r$. Thus, we need to consider $(r+1)^l$ cases. If we can compute the labeled asymptotic probability of each case, the labeled asymptotic probability of any *FO* sentence of quantifier rank r can be expressed as a sum of these values.

The computations proceed by induction, since r-morphism is defined by induction on r. We omit details since they are lengthy and several combinatorial and analytic results must first be proved.□

The proof is effective in the sense that it provides a procedure which, for a given first-order sentence φ, yields an expression for $\mu(\varphi)$ in terms of integers, e, the usual arithmetic operations, and exponentiation. There can be up to r nestings of exponentiation in these expressions for sentences of quantifier rank r. The proof implicitly gives a decision procedure for the set of *FO* sentences with labeled asymptotic probability 1. In fact we can give an algorithm similar to the one for deciding whether an *FO* sentence about relational structures with an underlying successor relation has probability 1. However, the best bound we can obtain for number of r-morphism classes is $\exp_\infty(cr)$ for some $c > 0$, where

$$\exp_\infty(n) = \left.2^{2^{\cdot^{\cdot^{2}}}}\right\} n \text{ times}$$

We thereby obtain an upper time bound of $\exp_\infty(cn)$ for determining whether an *FO* sentence of length n has labeled asymptotic probability 1. (For time bounds growing this fast it makes no difference whether the model of computation is a Turing machine or an alternating Turing machine.)

Compton, Henson, and Shelah [22] show that this bound is the best possible. Using the lower bound techniques from Compton and Henson [21] they show an $\exp_\infty(dn)$ lower time bound for the problem of determining whether an *FO* sentence has labeled asymptotic probability 1 in the class of structures consisting of a single unary function. In the same paper they prove that for the class of structures with two or more unary functions there are *MSO* sentences without a labeled asymptotic probability. They also prove that for the class of structures with a binary function, there are *FO* sentences without a labeled asymptotic probability. The proofs are similar to the other nonconvergence proofs we have cited: addition and multiplication can be coded on certain sets and the largest such set can be picked out almost surely. Nothing is known about labeled asymptotic probabilities of sentences from *TC*, *LFP*, or *IT* for the class of structures containing two or more unary functions.

Lynch's results for functions remain true if some of the unary functions are required to be one-to-one. When the structures have just a one-to-one function, the upper bound for determining whether an *FO* sentence has labeled asymptotic probability *less than* 1 can be improved to $NTIME(2^{dn^2})$. The best lower bound known for this problem is $NTIME(2^{cn})$ (shown in Compton [20]). However, in the case of monadic second-order logic on structures with two or more one-to-one functions, Compton, Henson, and Shelah show that there are sentences without an asymptotic probability, and that the problem of determining probability 1 sentences is undecidable.

7 Independent Probability Measures

Here we relate various findings concerning independent probability measures, including measures on classes of infinite structures.

The most well known example of an independent probability measure is found in the study of random graphs initiated by Erdős and Rényi [24]. As we noted in the introduction, this measure is defined by assigning a probability $p = p(n)$ to the independent edge occurrence events in a random graph on n vertices. Palmer [58] calls this Model A. This measure is closely related to the labeled asymptotic probability on graphs with $\lfloor p \cdot \binom{n}{2} \rfloor$ edges on n vertices – Model B in [58]. A result for Model A often holds for Model B as well, but we will confine our remarks to Model A.

Erdős and Rényi showed that many graph properties have associated *threshold functions.* We say that $t(n)$ is a threshold function for a property S if $\mu(S) = 0$ when $p(n) \ll t(n)$ and $\mu(S) = 1$ when $t(n) \ll p(n)$. (Here $f(n) \ll g(n)$ means $f(n) = o(g(n))$.) When $p(n) \sim ct(n)$ for some $0 < c < \infty$, we usually have $0 < \mu(S) < 1$, in which case we do not have a 0–1 law for logics in which S is can be expressed.

We would naturally like to know which functions are threshold functions for first-order sentences, and whether first-order 0–1 laws hold between the threshold functions. A survey of the literature on random graphs (see Bollobás [8] or Palmer [58]) reveals that n^{-2} is a threshold function for the sentence that says "there is a pair of vertices joined by an edge"; $n^{-1-1/k}$ is a threshold function for the sentence that says "there is a component isomorphic to $\mathcal{T}$," where $\mathcal{T}$ is any tree with k vertices; n^{-1} is a threshold function for the sentence that says "there is an m-cycle," where m is any integer greater than 2; and $n^{-1} \log n$ is a threshold function for the sentence that says "there is no vertex of degree m," where m is any nonnegative integer. Shelah and Spencer [64] give a fairly complete picture (the gaps are discussed below) of where first-order 0–1 laws occur between these threshold functions.

Theorem 7.1 (Shelah and Spencer) *Let $p(n)$ be the edge occurrence probability for a class of random graphs. An* FO *0–1 law holds when $p(n)$ satisfies any of the following conditions.*

(i) $p(n) \ll n^{-2}$.

(ii) $n^{-1-1/k} \ll p(n) \ll n^{-1-1/(k+1)}$ *for some positive integer k.*

(iii) $n^{-1-\varepsilon} \ll p(n) \ll n^{-1}$ *for all $\varepsilon > 0$.*

(iv) $n^{-1} \ll p(n) \ll n^{-1} \log n$.

(v) $n^{-1} \log n \ll p(n) \ll n^{-1+\varepsilon}$ *for all $\varepsilon > 0$.*

(vi) $n^{-\varepsilon} \ll p(n)$ *for all $\varepsilon > 0$.*

The proof of this theorem (actually six theorems) employs techniques similar to those used in sections 4 and 5 to prove 0–1 laws.

When condition (i), (ii), or (vi) holds, the set of *FO* sentences with probability 1 turns out to be $\aleph_0$-categorical, so by the results in §5 we also have 0–1 laws for the logics *TC*, *LFP*, and *IT*. (Shelah and Spencer point out that the alternate proof we sketched for Theorem 5.1 gives case (vi) immediately.) It is not difficult to show that the complexity of the set of probability 1 sentences in these cases is the same as it was in §5: for *FO* and *TC* the set is *PSPACE*-complete, for *LFP* it is *EXPTIME*-complete, and for *IT* it is *EXPSPACE*-complete.

When condition (i), (ii), (iii), or (iv) holds, we can use Ehrenfeucht games as described in §4 to show that an *MSO* 0–1 law holds. Rather than invoking generating series methods, we can show directly that for every r, any two random graphs will almost surely satisfy the same *MSO* sentences of quantifier rank r. In cases (iii) and (iv) the set of *MSO* sentences with probability 1 is decidable, but even the set of *FO* sentences with probability 1 cannot be decided in time $\exp_\infty(dn)$ for some $d > 0$. The lower bound follows by essentially the same argument used by Compton, Henson, and Shelah [22] to obtain a lower bound for random unary functions.

Theorem 7.1 above does not tell us what happens when $n^{-1+\varepsilon} \ll p(n) \ll n^{-\varepsilon}$ for all $\varepsilon > 0$. Here Shelah and Spencer provide a partial solution.

Theorem 7.2 (Shelah and Spencer) *If $p(n) = n^{-\alpha}$, where α is an irrational number between 0 and 1, then an* FO *0–1 law holds.*

This is the most difficult result in [64]. We will not attempt to summarize the proof. The paper gives an example where the *FO* 0–1 law fails for a $p(n)$ near $n^{-1/7}$ and it appears that similar examples can be given near $n^{-\alpha}$ for every rational α strictly between 0 and 1. The probability 1 theories that result from different irrational α are distinct, so almost all of them will be undecidable since there are only countably many decidable theories.

Recall that the probability measure on the class of graphs with an infinite vertex set A is defined by setting the probability of each edge occurrence event to be a fixed real number p strictly between 0 and 1. The space of graphs is identified with the product space $\{0,1\}^{[A]^2}$. Now it is easy to see that the extension axioms for the theory of graphs (given after Theorem 5.2) each have probability 1. Since we are now dealing with a countably additive measure, the set S of graphs satisfying all of the extension axioms has probability 1. Since there is only one countable graph satisfying this theory, the set of graphs satisfying a sentence of *any* logic will either contain S or be disjoint from S when A is countable. Thus, for countable graphs, we have a 0–1 law for every logic. Erdős and Rényi make this observation in [24], although without reference to the logical framework.

Gaifman [30] gives essentially the same argument, using the extension axioms for relational structures rather than graphs, to prove an *FO* 0–1 law for the infinite random relational structures of a finite similarity type. As with graphs, if the universe is countable, the argument shows that every logic has a 0–1 law; if the universe is uncountable, we can only conclude that there is a 0–1 law for *FO*, and by $\aleph_0$-categoricity, for *TC*, *LFP*, and *IT*. The complexity bounds for the decision problems are the same as in §5.

Reyes [63] observed that the set of relational structures satisfying the extension axioms is comeager, so we have a categorical analogue of a 0–1 law – a *meager-comeager law* – corresponding to each 0–1 law. (See Oxtoby [57] for a general discussion of the parallels between measure and category.) Let us summarize these observations.

Theorem 7.3 *The class of structures on an infinite set A has a 0–1 law and a meager-comeager law for* FO, TC, LFP, *and* IT. *The probability 1 sentences coincide with the comeager sentences in each case, and the complexities of the sets of probability 1 sentences are as in §5. If A is countable, a 0–1 law and meager-comeager law holds for every logic.*

We see that the techniques of §5 apply to classes of infinite structures. Lynch [50] applies the techniques of §6 to infinite structures with underlying relations. For example, Theorem 6.1 holds for the class of relational structures on $\omega = \{0,1,2,\ldots\}$ with the usual successor relation. The proof is the same except that now we have only one end point to worry about when we play the Ehrenfeucht game. Results similar to those in §6 can be obtained for structures with ω or $\mathbf{Z} = \{\ldots,-1,0,1,\ldots\}$ as the universe and underlying successor and addition relations. Moreover, in all of Lynch's examples 0–1 laws hold precisely when meager-comeager laws hold, and the probability 1 sentences coincide with the comeager sentences.

One might be tempted to conclude from the discussion above that in the matter of logical probabilities, infinite structures behave identically to their finite counterparts. This is not so for structures with underlying linear orders. In §6 we saw that we do not in general have convergence for *FO* sentences about relational structures with underlying linear orders. But for the set of relational structures on $\mathbf{Z}$ with the usual order we have not only convergence, but also an $L_{\omega_1\omega}$ 0–1 law. In fact, Mycielski showed that if the automorphism group of the structure formed by restricting to the underlying relations (in this case $\langle \mathbf{Z}, \leq \rangle$) has no finite orbits, then the set of structures satisfying an $L_{\omega_1\omega}$ sentence will have measure 0 or 1, and be meager or comeager. (The proof is given in Lynch [50].)

For the special case of structures on $\mathbf{Z}$ with the usual order and the similarity type containing only unary relations, this result follows from the classical 0–1 law of Kolmogorov and its topological analogue (see Oxtoby [57]): if $\mathbf{X}$ is a Baire space with a probability measure, then every shift invariant Borel set $S \subseteq \mathbf{X}^{\mathbf{Z}}$ has measure 0 or 1, and is meager or comeager. By *shift invariant* we mean that if $a = (\ldots, a_{-1}, a_0, a_1, \ldots)$ is in S, then for each integer j, $b = (\ldots, b_{-1}, b_0, b_1, \ldots)$ is also in S, where $b_i = a_{i+j}$; the mapping that takes a to b is a *shift* by j coordinate places. If there are k unary relations in the similarity type, each element of $\mathbf{Z}$ satisfies one of 2^k possible subsets of this set of relations, and the set of random structures may be identified with $(2^k)^{\mathbf{Z}}$. A shift on this space clearly preserves isomorphism types, so the set of structures satisfying any sentence is shift invariant. Makkai and Mycielski [52] point out that by the Lopez-Escobar Interpolation Theorem, every shift invariant Borel set is the set of structures satisfying some sentence of $L_{\omega_1\omega}$.

Now measure and category may not agree here. The standard example of a meager set with probability 1 is the set of sequences in $\{0,1\}^{\mathbf{Z}}$ where the asymptotic proportion of 0's (and hence 1's) is 1/2. This set is Borel, so there is a meager $L_{\omega_1\omega}$ sentence with probability 1. Benda [3] showed that measure and category do agree for *FO* sentences and Compton [13] extended this to *MSO* sentences. (Both results are for similarity types having only unary relations.) Notice that it is not obvious that the set of structures satisfying an *MSO* sentence should be Borel; this is shown in [13].

Another example of independent probability measures are defined on random partial orders of fixed dimension. They were studied by Winkler in [72]. Fix a dimension k and randomly choose n points from the (solid) k-cube $[0,1]^k$. (The probability measure is obtained by extending the uniform measure on $[0,1]$ to the product.) Let $a = (a_i \mid i < k)$ and $b = (b_i \mid i < k)$ be points in $[0,1]^k$. Write $a \leq b$ if $a_i \leq b_i$ for each $i < k$. Now $\mu_n(\varphi)$ is the probability that a randomly chosen order satisfies φ. Winkler displays an *FO* sentence φ for $k = 2$ such that $\lim_{n\to\infty} \mu_n(\varphi) = 1/e$. It is not known whether convergence occurs for every *FO* sentence when $k \geq 2$. It is evident that the measure we have defined is equivalent to taking the labeled asymptotic probability on the class of structures with k linear orders and letting $\leq$ be the intersection of these linear orders in each structure.

For $k = 1$ an *FO* 0–1 law does hold. If one allows random unary predicates in addition to the random linear order, we no longer have a 0–1 law, but Ehrenfeucht has shown that we still have convergence for every *FO* sentence φ. (The proof is given in Lynch [50].) This result relies on simple properties of Markov chains. For unary relations with an underlying linear order there is a well known correspondence between sets satisfying *MSO* sentences and sets accepted by finite automata (see Ladner [48] for a nice treatment). From the finite automata one can construct Markov chains to compute probabilities of *MSO* sentences and show that their Ceŝaro limit exists.

Winkler also examines random countable orders of dimension k. Here we pick a sequence of points from $[0,1]^k$ (or, more formally, take the product measure on $([0,1]^k)^\omega$). In the case $k = 1$ we have a random sequence from the unit interval, which forms, almost surely, a dense linear order without end points. Cantor, in the first use of a back-and-forth argument, showed that any two such structures are isomorphic; i.e., we have $\aleph_0$-categoricity and therefore a 0–1 law. Winkler extends this argument to higher dimensions.

Theorem 7.4 (Winkler [72]) *The class of countable partial orders of dimension k has an* FO *0–1 law for each finite k. The set of* FO *sentences with probability 1 is an $\aleph_0$-categorical theory.*

Again, $\aleph_0$-categoricity yields a 0–1 law for every logic. It is not difficult to see that we also have a meager-comeager law for every logic, and that measure and category agree.

We note that Bankston and Ruitenburg [2] discuss a general method for assigning probability measures, metrics, and game strategies on classes of countably infinite structures. Their results are concerned mostly with the model theoretic properties of sets of structures satisfying various notions of ubiquity, such as having probability 1 and being comeager.

8 Problems and New Directions

We close with a list of problems and suggestions for further research directions.

Problem 8.1 *Develop techniques for showing the existence of asymptotic probabilities of* FO *and* MSO *sentences in classes whose generating series converge at the radius of convergence.*

In §4 we described techniques used with slow growing classes whose generating series diverge at the radius of convergence, but there are interesting examples not covered by this condition.

The most well known is the class of labeled rooted trees (and forests). A famous theorem of Cayley, probably the first difficult enumeration result, says there are n^{n-1} labeled rooted trees on n vertices, so the exponential generating series for the class is $c(x) = \sum_{n>0}(n^{n-1}/n!)x^n$ (see Harary and Palmer [37]). It is not difficult to show that $c(x)$ has radius of convergence e^{-1} and $c(e^{-1}) = 1$. The exponential generating series for forests is $a(x) = \exp(c(x))$ so it also converges at e^{-1}. Does every *MSO* sentence have a labeled asymptotic probability in the class of rooted trees (or equivalently, in the class of rooted forests)?

Another example is the class of unit interval graphs. Hanlon [36] explicitly derives the ordinary generating series (enumerating unlabeled structures) for this class. It converges at its radius of convergence. Does every *FO* sentence have an unlabeled asymptotic probability in this class? What about in the class of interval graphs?

Compton [18] gives techniques for computing probabilities of certain properties when convergence occurs at the radius of convergence. For example, if the exponential generating series $a(x)$ of a class closed under disjoint unions and components satisfies certain general conditions, then the labeled asymptotic probability of connectivity is $1/a(R)$, where R is the radius of convergence of $a(x)$. This result indicates the difficulty when $a(R)$ is finite: we must take into account not only the number of components that occur in a structure, but also what is true within components.

Problem 8.2 *Develop techniques for showing the existence of asymptotic probabilities of* LFP *sentences in slow growing classes.*

As we noted in §4, nothing is known in this area other than a few results of Kolaitis [43].

Problem 8.3 *Prove a 0–1 law for graphs in a logic powerful enough to express Hamiltonicity.*

One approach to this problem would be to see what kinds of combinatorial operators must be added to *FO* to quantify over the long paths that occur in the proof of almost sure Hamiltonicity (see [8]). An easier problem might be to prove a 0–1 law for graphs in a logic powerful enough to express rigidity.

Problem 8.4 *Investigate asymptotic probabilities for naturally occurring fast growing classes not covered in sections 5 and 6.*

Classess where the techniques of Compton, Henson, and Shelah [22] might show that labeled asymptotic probabilities of *FO* sentences fail to exist include structures consisting of a binary relation plus unary function, and of a unary function with an underlying linear order. A class where the techniques of Lynch [51] might show that asymptotic probabilities of *FO* sentences do exist is pairs of equivalence relations. After all, a unary function f induces an equivalence relation $\{(x, y) \mid f(x) = f(y)\}$. This gives a somewhat different distribution than the labeled asymptotic probability, however. The techniques of [22] might show that labeled asymptotic probabilities of *MSO* sentences fail to exist for pairs of equivalence relations.

Problem 8.5 *We use the notation in the discussion of random graphs in §7. If $t(n)$ is one of the functions n^{-2}, $n^{-1-1/k}$, n^{-1}, or $n^{-1}\log n$, then do all* FO *sentences have a probability when $p(n) = ct(n)$?*

These are all threshold functions for *FO* sentences, so we will not have an *FO* 0–1 law in any of these cases.

Problem 8.6 (Winkler [72]) *Show that all* FO *sentences have asymptotic probabilities on the class of random partial orders of dimension k, where k is finite.*

Winkler notes that this would follow if *FO* sentences have labeled asymptotic probabilities on the class of structures consisting of k linear orders. In the case of k successor relations, the methods of Lynch [51] can probably be applied to show that *FO* sentences have asymptotic probabilities.

Problem 8.7 *Develop a theory of asymptotic probabilities for structures (such as groups) where direct product is an appropriate operation.*

We saw in §4 that for structures such as graphs where disjoint union is an appropriate operation, extended asymptotic probabilities can be defined using ratios of power series. When direct products occur we often use Dirichlet series for enumeration (see Goulden and Jackson [32]), so it is natural to define asymptotic probabilities in these cases using ratios of Dirichlet series. Let C be a class of structures and a_n be the number of unlabeled structures in C of size n. For a sentence φ let c_n be the number of unlabeled structures of size n in C satisfying φ. The *Dirichlet series* for C and φ are $a(s) = \sum_{n\geq 1} a_n n^{-s}$ and $c(s) = \sum_{n\geq 1} c_n n^{-s}$, respectively. Dirichlet series have a right half-plane of convergence rather than a circle of convergence. Suppose the half-plane of convergence for $a(s)$ consists of those complex numbers with real part greater than R and $\lim_{s\to R} a(s) = \infty$. Define $\hat{\mu}(\varphi) = \lim_{s\to R} c(s)/a(s)$. Does $\hat{\mu}(\varphi)$ exist for every *FO* sentence φ when C is the class of Abelian groups? What about other classes of algebraic structures?

Problem 8.8 *Investigate asymptotic probabilities and 0–1 laws for classes of regular graphs.*

Bollobás [8] has a short discussion on known asymptotic results for regular graphs.

Problem 8.9 *Show that all* FO *sentences have labeled asymptotic probabilities for classes of directed graphs having the amalgamation property and closed under substructures.*

Kolaitis, Prömel, and Rothschild [45] proved that classes of (undirected) graphs with these properties have labeled *FO* 0–1 laws by invoking a result of Lachlan and Woodrow [47] which characterizes such classes and then observing that each class has a labeled 0–1 law. Is this an accident? The question is how amalgamation figures in the study of asymptotic probabilities. Bankston and Ruitenburg [2] use some of the well known model theoretic properties of classes with amalgamation in their work, but it does not seem to shed light on the Kolaitis-Prömel-Rothschild result. It would be instructive to have a proof that used the amalgamation property directly.

A class of directed graphs with the desired properties, but without a labeled 0–1, is given in [45], but it is easily shown that labeled asymptotic probabilities of *FO* sentences do exist in this example.

References

[1] A. Aho and J. Ullman. 'Universality of data retrieval languages,' *Proc. 6th ACM Symp. on Principles of Programming Languages* (1979), 110–120.

[2] P. Bankston and W. Ruitenburg, 'Notions of relative ubiquity for invariant sets of relational structures,' preprint.

[3] M. Benda. 'Infinite words as universes,' unpublished.

[4] A. Blass, G. Exoo, and F. Harary. 'Paley graphs satisfy all first-order adjacency axioms,' *J. Graph Theory* **5**(1981), 435–439.

[5] A. Blass, Y. Gurevich, and D. Kozen. 'A zero-one law for logic with a fixed-point operator,' *Inform. and Control* **67**(1985), 70–90.

[6] A. Blass and F. Harary. 'Properties of almost all graphs and complexes,' *J. Graph Theory* **3**(1979), 225–240.

[7] B. Bollobás. *Graph Theory—An Introductory Course*, Springer-Verlag, New York, 1979.

[8] B. Bollobás. *Random Graphs*, Academic Press, Orlando, Florida, 1985.

[9] B. Bollobás and E. Palmer. 'Almost all simplicial complexes are pure,' *Utilitas Math.* **12**(1977), 101-111.

[10] A. Chandra, D. Kozen, and L. Stockmeyer. 'Alternation,' *J. Assoc. Comp. Mach.* **28**(1981), 99–128.

[11] C.C. Chang and H. J. Keisler. *Model Theory*, North-Holland, Amsterdam, 1973.

[12] K.J. Compton. 'Some useful preservation theorems,' *J. Symbolic Logic* **48**(1983), 427–440.

[13] K.J. Compton. 'On rich words,' in *Progress in Combinatorics on Words* (L. Cummings, ed.), Academic Press, New York, 1983, 39–61.

[14] K.J. Compton. 'An undecidable problem in finite combinatorics,' *J. Symbolic Logic* **49**(1984), 842–850.

[15] K.J. Compton. 'Application of a Tauberian theorem to finite model theory,' *Arch. Math. Logic* **25**(1985), 91–98.

[16] K.J. Compton. 'A logical approach to asymptotic combinatorics I: first-order properties,' *Adv. Math.* **65**(1987), 65–96.

[17] K.J. Compton. 'A logical approach to asymptotic combinatorics II: monadic second-order properties,' to appear.

[18] K.J. Compton. 'Some methods for computing component distribution probabilities in relational structures,' *Discrete Math.*, to appear.

[19] K.J. Compton. 'The computational complexity of asymptotic problems I: partial orders,' *Inform. and Computation*, to appear.

[20] K.J. Compton. 'The computational complexity of asymptotic problems II: permutations,' in preparation.

[21] K.J. Compton and C.W. Henson. 'A uniform method for proving lower bounds on the complexity of logical theories,' *Annals Pure Appl. Logic*, to appear.

[22] K.J. Compton, C.W. Henson, and S. Shelah. 'Nonconvergence, undecidability, and intractability in asymptotic problems,' *Annals Pure Appl. Logic*, to appear.

[23] A. Ehrenfeucht. 'An application of games to the completeness problem for formalized theories,' *Fund. Math.* **49**(1961), 129–141.

[24] P. Erdős and A. Rényi. 'On the evolution of random graphs,' *Magyar Tud. Akad. Mat. Kutató Int. Kőzl.* **5**(1960), 17–61.

[25] P. Erdős and A. Rényi. 'Asymmetric graphs,' *Acta Math. Sci. Hungar.* **14**(1963), 295–315.

[26] R. Fagin. 'Generalized first-order spectra and polynomial time recognizable sets,' in *Complexity of Computations* (R. Karp, ed.), SIAM-AMS Proc. **7**, American Mathematical Society, New York, 43–73.

[27] R. Fagin. 'The number of finite relational structures,' *Discrete Math.* **19**(1977), 17–21.

[28] R. Fagin. 'Probabilities on finite models,' *J. Symbolic Logic* **41**(1976), 50–58.

[29] J. Ferrante and C. Rackoff. *The Computational Complexity of Logical Theories*, Lecture Notes in Math. 718, Springer-Verlag, Berlin, 1979.

[30] H. Gaifman. 'Concerning measures in first-order calculi,' *Israel J. Math.* **2**(1964), 1–18.

[31] Y.V. Glebskiĭ, D.I. Kogan, M.I. Liogon'kiĭ, and V.A. Talanov. 'Range and degree of realizability of formulas in the restricted predicate calculus,' *Kibernetika (Kiev)* **1969, no. 2**, 17–28; English translation, *Cybernetics* **5**(1969), 142–154.

[32] I. Goulden and D. Jackson. *Combinatorial Enumeration*, Wiley, New York, 1983.

[33] E. Grandjean. 'Complexity of the first-order theory of almost all finite structures,' *Inform. and Control* **57**(1983), 180–204.

[34] Y. Gurevich. 'Toward logic tailored for computational complexity,' in *Computation and Proof Theory* (M.M. Richter, et. al., eds.), Lecture Notes in Math. 1104, Springer-Verlag, New York, 1984, 175–216.

[35] Y. Gurevich and S. Shelah. 'Fixed-point extensions of first-order logic,' *Proc. 26th IEEE Symp. on Foundations of Comp. Sci.* (1985), 346–353.

[36] P. Hanlon. 'Counting interval graphs,' *Trans. Amer. Math. Soc.* **272**(1982), 383–426.

[37] F. Harary and E. Palmer. *Graphical Enumeration*, Academic Press, New York, 1973.

[38] G.H. Hardy. *Divergent Series*, Clarendon Press, Oxford, 1949.

[39] N. Immerman. 'Relational queries computable in polynomial time,' *Inform. and Control* **68**(1986), 86–104.

[40] N. Immerman. 'Languages which capture complexity classes,' *SIAM J. Comput.*, to appear.

[41] M. Kaufmann and S. Shelah. 'On random models of finite power and monadic logic,' *Discrete Math.* **54**(1983), 285–293.

[42] D.J. Kleitman and B.L. Rothschild. 'Asymptotic enumeration of partial orders on a finite set,' *Trans. Amer. Math. Soc.* **205**(1975), 205–220.

[43] Ph. Kolaitis. 'On asymptotic probabilities of inductive queries and their decision problem,' in *Logics of Programs, Brooklyn, June 1985*, Lecture Notes in Comp. Sci. 193, Springer-Verlag, 1985, 153–166.

[44] Ph. Kolaitis, H. Prömel, and B. Rothschild. 'Asymptotic enumeration and a 0–1 law for m-clique free graphs,' *Bull. Amer. Math. Soc. (N.S.)***13**(1985), 160–162.

[45] Ph. Kolaitis, H. Prömel, and B. Rothschild. 'K_{l+1}-free graphs: asymptotic structure and a 0–1 law,' preprint.

[46] Ph. Kolaitis and M. Vardi. 'The decision problem for the probabilities of higher-order properties,' *Proc. 19th ACM Symp. on Theory of Computing* (1987).

[47] A. Lachlan and R. Woodrow, 'Countable ultrahomogeneous undirected graphs,' *Trans. Amer. Math. Soc.* **262**(1980), 51–94.

[48] R. Ladner. 'Application of model theoretic games to discrete linear orders and finite automata,' *Inform. and Control* **33**(1977), 281–303.

[49] M.I. Liogon'kiĭ. 'On the question of quantitative characteristics of logical formulas,' *Kibernetika (Kiev)* **1970, no. 3**, 16–22; English translation, *Cybernetics* **6**(1970), 205–211.

[50] J.F. Lynch. 'Almost sure theories,' *Annals Math. Logic* **18**(1980), 91–135.

[51] J.F. Lynch. 'Probabilities of first-order sentences about unary functions,' *Trans. Amer. Math. Soc.* **287**(1985), 543–568.

[52] M. Makkai and J. Mycielski. 'An $L_{\omega_1\omega}$ complete and consistent theory without models,' *Proc. Amer. Math. Soc.* **62**(1977), 131–133.

[53] Y.N. Moschovakis. *Elementary Induction on Abstract Structures*, North-Holland, Amsterdam, 1974.

[54] W. Oberschelp. 'Asymptotic 0–1 laws in combinatorics,' in *Combinatorial Theory, Proc. Schloss Rauischholzhausen*, Lecture Notes in Math. 969, Springer-Verlag New York, 1982, 276–292.

[55] W. Oberschelp. 'Monotonicity for structure numbers in theories without identity,' in *Combinatoire et Représentation du Groupe Symétrique* (D. Foata, ed.), Lecture Notes in Math. 579, Springer-Verlag, Berlin, 1977, 297-308.

[56] W. Oberschelp. 'Strukturzahlen in endlichen relationssystemen,' in *Logic Colloquium 1966* (H.A. Schmidt, et. al., eds.), North-Holland, Amsterdam, 1968.

[57] J.C. Oxtoby. *Measure and Category*, Springer-Verlag, New York, 1971.

[58] E. Palmer. *Graphical Evolution*, Wiley, New York, 1985.

[59] G. Pólya. 'Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen,' *Acta Math.* **68**(1937), 145–254.

[60] H.J. Prömel. 'Counting unlabeled structures,' *J. Comb. Theory(A)*, to appear.

[61] R. Rado. 'Universal graphs and universal functions,' *Acta Arith.* **9**(1964), 331–340.

[62] J. Remlinger. '0–1-Gesetze für partielle Ordnungen und spezielle Teilklassen,' RWTH Aachen Schriften zur Informatik und angewandten Mathematik Bericht 117, 1986.

[63] G.E. Reyes. 'Local definability theory,' *Annals Math. Logic* **1**(1970), 95–137.

[64] S. Shelah and J. Spencer. 'Zero-one laws for sparse random graphs,' *Proc. 19th ACM Symp. on Theory of Computing* (1987).

[65] L. Stockmeyer. 'The polynomial-time hierarchy,' *Theoret. Comp. Sci.* **3**(1977), 1–22.

[66] A. Tarski. 'A lattice-theoretical fixpoint theorem and its applications,' *Pacific J. Math.* **5**(1955), 285-309.

[67] W. Taylor. Review, *J. Symbolic Logic* **50**(1985), 1073–1076.

[68] B.A. Trakhtenbrot. 'The impossibility of an algorithm for the decision problem for finite models,' *Dokl. Akad. Nauk SSSR* **70**(1950), 569–572.

[69] M. Vardi. 'Complexity of relational query languages,' *Proc. 14th ACM Symp. on Theory of Computing* (1982), 137–146.

[70] R. Vaught. 'Sentences true in all constructive models,' *J. Symbolic Logic* **25**(1960), 39–58.

[71] R. Vaught. 'On a theorem of Cobham concerning undecidable theories,' in *Proc. 1960 Intl. Cong. Logic, Phil. and Methodology of Sci.*, Stanford University Press, Stanford, Calif., 1962, 14–25.

[72] P. Winkler. 'Random orders,' *Order* **1**(1985), 317–331.

[73] E.M. Wright, 'Graphs on unlabelled nodes with a given number of edges,' *Acta Math.* **168**(1971), 1-9.