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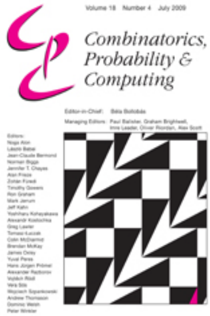

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Random Graphs from a Minor-Closed Class

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A minor-closed class of graphs is addable if each excluded minor is 2-connected. We see that such a class $\mathcal{A}$ of labelled graphs has smooth growth; and, for the random graph R_n sampled uniformly from the n-vertex graphs in $\mathcal{A}$, the fragment not in the giant component asymptotically has a simple 'Boltzmann Poisson distribution'. In particular, as $n \to \infty$ the probability that R_n is connected tends to $1/A(\rho)$, where $A(x)$ is the exponential generating function for $\mathcal{A}$ and ρ is its radius of convergence.

1. Introduction

We consider simple graphs. A class $\mathcal{A}$ of graphs is called *decomposable* (following Kolchin [16, 17]) if a graph is in $\mathcal{A}$ if and only if each component is. We say that $\mathcal{A}$ is *bridge-addable* if, for each graph in $\mathcal{A}$ and each pair u and v of vertices in different components, the graph obtained by adding an edge joining u and v must also be in $\mathcal{A}$; and it is *addable* if it is both decomposable and bridge-addable. (Note that that 'bridge-addable' was called 'weakly addable' in McDiarmid, Steger and Welsh [25, 26].) A useful result from the earlier of these papers is that, if $\mathcal{A}$ is bridge-addable, then

$$\mathbb{P}[R_n \text{ is connected}] \geqslant e^{-1}, \tag{1.1}$$

where $R_n \in_u \mathcal{A}_n$, that is, R_n is uniformly distributed over the set $\mathcal{A}_n$ of graphs in $\mathcal{A}$ on the vertices $1, \ldots, n$.

For most of the paper we consider classes of labelled graphs which are closed under isomorphism. We say that such a graph class $\mathcal{A}$ is *minor-closed* if any minor of a graph in $\mathcal{A}$ remains in $\mathcal{A}$. (We discard any loops and parallel edges.) A graph H is an *excluded minor* for a minor-closed class $\mathcal{A}$ if H is not in $\mathcal{A}$ but each proper minor of H is in $\mathcal{A}$. The theory of graph minors (see Robertson and Seymour [31] or, for example, Diestel [11]) shows that for any minor-closed class the set of excluded minors is finite.

Let the class $\mathcal{A}$ of graphs be minor-closed. It is easy to see that $\mathcal{A}$ is decomposable if and only if each excluded minor is connected, and $\mathcal{A}$ is addable if and only if each excluded minor is 2-connected. Bridge-addability is less well behaved, but note that $\mathcal{A}$ is bridge-addable if each excluded minor has no cut-point or isolated edge (and conversely, if $\mathcal{A}$ is bridge-addable then each excluded minor is bridgeless and so has minimum degree at least 2). For example, the class of planar graphs is addable, with excluded minors the complete graph K_5 and the complete bipartite graph $K_{3,3}$; and for any other surface S the class $\mathcal{G}^S$ of graphs embeddable on S is bridge-addable but it is not decomposable.

For any class $\mathcal{A}$ of graphs, let $A(x) = \sum_{n\geqslant 0} |\mathcal{A}_n| x^n/n!$ denote the corresponding exponential generating function, and let $\rho(\mathcal{A})$ denote its radius of convergence, that is,

$$\rho(\mathcal{A}) = \rho(A) = \left(\limsup_{n\to\infty} (|\mathcal{A}_n|/n!)^{1/n}\right)^{-1}.$$

We call $\mathcal{A}$ *small* if $\rho(\mathcal{A}) > 0$. We say that $\mathcal{A}$ has *growth constant* $\gamma = \gamma(\mathcal{A})$ if $0 < \gamma < \infty$ and $(|\mathcal{A}_n|/n!)^{1/n} \to \gamma$ as $n \to \infty$. If $\mathcal{A}$ has a growth constant then of course it must be $1/\rho(\mathcal{A})$. The result of Norine, Seymour, Thomas and Wollan [28] shows that every proper minor-closed class is small; and thus from Theorem 3.2 of McDiarmid, Steger and Welsh [26] we immediately obtain the following basic counting result for an addable such class (see also Bernardi, Noy and Welsh [6]).

Proposition 1.1. *Each addable proper minor-closed class $\mathcal{A}$ of graphs has a growth constant $\gamma = \gamma(\mathcal{A})$.*

For any class $\mathcal{A}$ of graphs let $r_n = n|\mathcal{A}_{n-1}|/|\mathcal{A}_n|$. Then

$$\liminf_{n\to\infty} r_n \leqslant \rho(\mathcal{A}) \leqslant \limsup_{n\to\infty} r_n.$$

If the ratio r_n tends to a finite positive limit as $n \to \infty$, then that limit must be $\rho(\mathcal{A})$, and we say that the class $\mathcal{A}$ is *smooth*. Such a condition crops up, for example, in Bell, Bender, Cameron and Richmond [3], Compton [10], McDiarmid, Steger and Welsh [25, 26], and McDiarmid [20].

Some classes of graphs known to be smooth include trees (by Cayley's counting formula), forests (Rényi [29], see also Moon [27]), outerplanar graphs and series-parallel graphs (Bodirsky, Giménez, Kang and Noy [7, 8]), planar graphs (Giménez and Noy [13]), apex graphs of planar graphs (McDiarmid [20]), and several classes of graphs specified by excluded minors: see Bernardi, Noy and Welsh [6], Bodirsky, Giménez, Kang and Noy [8], Gerke, Giménez, Noy and Weißl [12], and Giménez, Noy and Rué [14]. This is true also, for example, for 2-connected planar graphs (Bender, Gao and Wormald [5]) and cubic planar graphs (Bodirsky, Löffler, Kang and McDiarmid [9]) if we consider only even n. It was shown very recently in Bender, Canfield and Richmond [4] that graphs embeddable on any fixed surface S form a smooth class $\mathcal{G}^S$: we shall use the approach in that paper to prove the following result, which improves on Proposition 1.1.

Theorem 1.2. *Each addable proper minor-closed class of graphs is smooth.*

This result was conjectured in Bernardi, Noy and Welsh [6]. Smoothness of $\mathcal{A}$ will allow us to obtain detailed results about the asymptotic distribution of the fragments of $R_n \in_u \mathcal{A}_n$

not in the ‘giant’ component, as in the case of graphs on surfaces: see McDiarmid [20]. First we introduce a general distribution on the unlabelled graphs corresponding to a small class of labelled graphs; we return to minor-closed classes in Theorem 1.5 below. Given a graph G, let $|G|$ denote the number of vertices of G, and let $\mathrm{aut}(G)$ denote the number of automorphisms of G.

For any graph class $\mathcal{A}$ we let $\mathcal{UA}$ denote the corresponding unlabelled graph class, with members the equivalence classes of graphs in $\mathcal{A}$ under isomorphism. Now let $\mathcal{A}$ be any small decomposable class of (labelled) graphs which is closed under isomorphism, and let $A(x)$ be its exponential generating function. (This will be our general rule, so that a class $\mathcal{B}$ of graphs will have exponential generating function $B(x)$, *etc.*) If $\mathcal{C}$ denotes the class of connected graphs in $\mathcal{A}$ then the ‘exponential formula’ is that $A(x) = e^{C(x)}$. (By convention the empty graph $\emptyset$ is in $\mathcal{A}$ and not in $\mathcal{C}$.) As we shall observe at the beginning of Section 3, we may also write $A(x)$ in terms of the set $\mathcal{UA}$ of unlabelled graphs in $\mathcal{A}$ as

$$A(x) = \sum_{H \in \mathcal{UA}} \frac{x^{|H|}}{\mathrm{aut}(H)}. \tag{1.2}$$

If we choose $\rho > 0$ such that $A(\rho)$ is finite, then we may obtain a natural ‘Boltzmann Poisson distribution’ on $\mathcal{UA}$: see equation (1.3) below.

We need more notation to record some of the remarkable properties of this distribution. Let $\kappa(G)$ denote the number of components of a graph G; for a connected graph H let $\kappa(G, H)$ denote the number of components of G isomorphic to H; and for a class $\mathcal{D}$ of connected graphs let $\kappa(G, \mathcal{D})$ denote $\sum_{H \in \mathcal{UD}} \kappa(G, H)$, the number of components of G isomorphic to some graph in $\mathcal{D}$. The notation $X \sim \mathrm{Po}(\lambda)$ means that the random variable X has (precisely) the Poisson distribution with mean λ. Recall that a sum of independent Poisson random variables $\mathrm{Po}(\lambda_i)$ has distribution $\mathrm{Po}(\sum_i \lambda_i)$, as long as $\sum_i \lambda_i < \infty$.

Theorem 1.3. *Let $\mathcal{A}$ be any decomposable class of graphs which is closed under isomorphism, let $\rho > 0$ be such that $A(\rho)$ is finite, and let $\lambda(H) = \frac{\rho^{|H|}}{\mathrm{aut}(H)}$ for each graph H in $\mathcal{UA}$. Let the ‘Boltzmann Poisson random graph’ $R = R(\mathcal{A}, \rho)$ take values in $\mathcal{UA}$, with*

$$\mathbb{P}[R = H] = \frac{\lambda(H)}{A(\rho)} \quad \textit{for each} \quad H \in \mathcal{UA}. \tag{1.3}$$

Also, let $\mathcal{C}$ denote the class of connected graphs in $\mathcal{A}$. Then the random variables $\kappa(R, H)$ for $H \in \mathcal{UC}$ are independent, with $\kappa(R, H) \sim \mathrm{Po}(\lambda(H))$.

In particular, since $C(\rho) = \sum_{H \in \mathcal{UC}} \lambda(H)$ (by equation (1.2) applied to $\mathcal{C}$) we have $\kappa(R) \sim \mathrm{Po}(C(\rho))$. Let us record this as a corollary.

Corollary 1.4. *The random number $\kappa(R)$ of components of R satisfies $\kappa(R) \sim \mathrm{Po}(C(\rho))$.*

Recall that a random variable X is *discrete* if it takes values in a countable set B, where the distribution may be specified by the values $\mathbb{P}[X = b]$ for $b \in B$. For discrete random variables X and Y, the *total variation distance* $d_{TV}(X, Y)$ is $\frac{1}{2} \sum_{b \in B} |\mathbb{P}[X = b] - \mathbb{P}[Y = b]|$.

Given discrete random variables $X, X_1, X_2, \ldots$ we say that X_n tends to X *in total variation* as $n \to \infty$, and write $X_n \to_{TV} X$, if $d_{TV}(X_n, X) \to 0$ as $n \to \infty$. It is easy to check that this happens if and only if $\mathbb{P}[X_n = b] \to \mathbb{P}[X = b]$ as $n \to \infty$ for each $b \in B$; that is, if and only if we have pointwise convergence of probabilities.

The *big component* $\mathrm{Big}(G)$ of a graph G is the (lexicographically first) component with the most vertices, and $\mathrm{Frag}(G)$ is the *fragments* subgraph induced on the vertices not in the big component (which may be empty). (We have changed from the name $\mathrm{Miss}(G)$ used in McDiarmid [20].) We denote the numbers of vertices in $\mathrm{Big}(G)$ and $\mathrm{Frag}(G)$ by $\mathrm{big}(G)$ and $\mathrm{frag}(G)$ respectively, so $\mathrm{big}(G) + \mathrm{frag}(G)$ equals the number of vertices of G. Observe from the last theorem that $\mathbb{P}[\mathrm{big}(R) \leqslant n] = \exp\bigl(-\sum_{j>n} |\mathcal{C}_j| \rho^j / j!\bigr)$. (It is clear what $\mathrm{big}(H)$ means for an unlabelled graph H, though it is less obvious how to define $\mathrm{Big}(H)$.)

We focus on the limiting distribution of the random graph $\mathrm{Frag}(R_n)$. It is convenient to deal with the random unlabelled graph F_n corresponding to $\mathrm{Frag}(R_n)$. The following theorem and corollary, which are proved in Section 4 below, should be compared with the labelled part of Theorem 2 of Bell, Bender, Cameron and Richmond [3], and see also Compton [10].

Theorem 1.5. *Let $\mathcal{A}$ be an addable proper minor-closed class of graphs, and let $\rho = \rho(\mathcal{A})$. Then $0 < \rho < \infty$ and $A(\rho)$ is finite; and for $R_n \in_u \mathcal{A}_n$ the random unlabelled graph F_n corresponding to* $\mathrm{Frag}(R_n)$ *satisfies $F_n \to_{TV} R$, where R is the Boltzmann Poisson random graph for $\mathcal{A}$ and ρ defined in* (1.3) *above. Further,* $\mathbb{E}[|R|]$ *is finite.*

Corollary 1.6.

(a) *For any given distinct $H_1, \ldots, H_k$ in $\mathcal{UC}$, the k random variables $\kappa(F_n, H_i)$ are asymptotically independent with distribution* $\mathrm{Po}(\lambda(H_i))$.

(b) *For any class $\mathcal{D} \subseteq \mathcal{C}$ of connected graphs in $\mathcal{A}$, we have $\kappa(F_n, \mathcal{D}) \to_{TV}$* $\mathrm{Po}(D(\rho))$*, and each moment of $\kappa(F_n, \mathcal{D})$ tends to that of* $\mathrm{Po}(D(\rho))$.

(c) *As a special case of part* (b)*, $\kappa(R_n) \to_{TV} 1 +$* $\mathrm{Po}(C(\rho))$*, and as $n \to \infty$ we have $\mathbb{P}[R_n$ is connected$] \to e^{-C(\rho)} = A(\rho)^{-1}$, $\mathbb{E}[\kappa(R_n)] \to 1 + C(\rho)$, and the variance of $\kappa(R_n)$ tends to $C(\rho)$.*

(d) *The random numbers of vertices satisfy $|F_n| \to_{TV} |R|$; that is, for each non-negative integer k,*

$$\mathbb{P}[|F_n| = k] \to \frac{1}{A(\rho)} \frac{|\mathcal{A}_k| \rho^k}{k!} \quad \text{as} \quad n \to \infty.$$

Note that Theorem 1.2, together with the result from part (c) above that $\mathbb{P}[R_n$ is connected$]$ tends to a non-zero limit as $n \to \infty$, shows that $\mathcal{C}$ is smooth.

Let $\mathcal{F}$ denote the class of forests. Then for $R_n \in_u \mathcal{F}_n$ it is known that $\mathbb{P}[R_n$ is connected$] \to e^{-\frac{1}{2}}$ as $n \to \infty$ (Rényi [29]; see also Moon [27]). Now consider any bridge-addable class $\mathcal{A}$ of graphs, and let $R_n \in_u \mathcal{A}_n$. It was conjectured in McDiarmid, Steger and Welsh [26] that (if $\mathcal{A}$ is also closed under isomorphism) then the inequality (1.1) could be improved asymptotically to $\mathbb{P}[R_n$ is connected$] \geqslant (1 + o(1)) e^{-\frac{1}{2}}$. For progress on this conjecture see

Balister, Bollobás and Gerke [2] and Addario-Berry, McDiarmid and Reed [1]. Part (c) of the corollary above shows that, in the special case of an addable minor-closed graph class, $\mathbb{P}[R_n$ is connected] tends to a limit as $n \to \infty$, and identifies this limit as $e^{-C(\rho)} = 1/A(\rho)$: however, it is not clear if this helps to establish the conjecture for this case.

Consider a simple example which provides a contrast to Theorem 1.5 and Corollary 1.6. Let $\mathcal{C}$ be the class of paths, and let $\mathcal{A}$ be the class of *path forests* (forests in which every component is a path). Clearly $|\mathcal{C}_n| = n!/2$ so $\mathcal{C}$ has growth constant 1, and thus $\mathcal{A}$ also has growth constant 1 by the exponential formula. Observe that $\mathcal{A}$ is minor-closed and decomposable but not bridge-addable, and so Theorem 1.5 does not apply: indeed the conclusions of that theorem do not hold, and in particular $A(\rho)$ is clearly infinite. Further, for $R_n \in_u \mathcal{A}_n$ we have (McDiarmid [22]) that

$$\sqrt{2/n}\ \kappa(R_n) \to 1 \ \text{ in probability as } n \to \infty. \tag{1.4}$$

Similar results hold, for example, for caterpillar forests, which have growth constant $\xi \approx 1.76$: see the last paper and Bernardi, Noy and Welsh [6].

There is an 'appearances theorem'. Let H be a graph with vertex set $\{1,\ldots,h\}$, and let G be a graph on the vertex set $\{1,\ldots,n\}$ where $n > h$. Let $W \subset V(G)$ with $|W| = h$, and let the 'root' r_W denote the least element in W. We say that there is a *pendant appearance* of H at W in G if (a) the increasing bijection from $\{1,\ldots,h\}$ to W is an isomorphism from H to the induced subgraph $G[W]$ of G; and (b) there is exactly one edge in G between W and the rest of G, and this edge is incident with the root r_W. (The word 'pendant' was not used in McDiarmid, Steger and Welsh [25, 26], but is added here for clarity.) Immediately from Theorem 5.1 of [26] we have the following result.

Theorem 1.7. *Let $\mathcal{A}$ be an addable proper minor-closed class of graphs, let $R_n \in_u \mathcal{A}_n$, and let H be a fixed connected graph in $\mathcal{A}$. Then there is a constant $\alpha > 0$ such that with probability $1 - e^{-\Omega(n)}$, there are at least αn pairwise vertex-disjoint pendant appearances of H in R_n.*

Corollary 1.8. *Given an addable proper minor-closed graph class $\mathcal{A}$, there is a constant $\alpha > 1$ such that $\mathbb{P}[\mathrm{aut}(R_n) \geqslant \alpha^n] = 1 - e^{-\Omega(n)}$.*

To see why the corollary follows, take H as a 3-vertex path, rooted at the centre vertex. By the theorem, there is a constant $\alpha > 1$ such that, with probability $1 - e^{-\Omega(n)}$, R_n has at least αn pairwise disjoint pendant appearances of H and so has at least $2^{\alpha n}$ automorphisms. A corresponding upper bound is given in McDiarmid [21], where it is shown that, given $\mathcal{A}$ as here, there is a constant $\beta > 1$ such that $\mathbb{E}[\mathrm{aut}(R_n)]^{1/n} \to \beta$ as $n \to \infty$. Thus, for example, $\mathbb{P}[\mathrm{aut}(R_n) > (2\beta)^n] = e^{-\Omega(n)}$.

Let us briefly consider Hadwiger's conjecture, which proposes that, for each positive integer t, each graph with $\chi(G) \geqslant t$ has a minor K_t: see, for example, Diestel [11]. It is known (Robertson, Seymour and Thomas [30]) to hold for each $t \leqslant 6$. Now fix an integer $t \geqslant 7$, and let $\mathrm{Ex}(K_t)$ denote the addable class of graphs with no minor K_t. If Hadwiger's conjecture holds for t, then for each graph G in $\mathrm{Ex}(K_t)$ we of course have $\chi(G) \leqslant t - 1$.

But if Hadwiger's conjecture fails for t, then Theorem 1.7 shows that it should be 'easy' to find a counterexample: for if $R_n \in_u \mathrm{Ex}(K_t)_n$ then $\chi(R_n) \geqslant t$ with probability $1 - e^{-\Omega(n)}$.

Since by Theorem 1.2 the class $\mathcal{A}$ in Theorem 1.7 above is smooth, elementary counting yields a more precise new version of that result.

Proposition 1.9. *Let $\mathcal{A}$ be an addable proper minor-closed class of graphs, let $\rho = \rho(\mathcal{A})$, and let $R_n \in_u \mathcal{A}_n$. Let H be a fixed connected graph in $\mathcal{A}$, and let $X_n(H)$ be the number of pendant appearances of H in R_n. Then $X_n(H)/n \to \rho^{|H|}/|H|!$ in probability as $n \to \infty$.*

The same result holds if we count disjoint appearances. Indeed, if $\tilde{X}_n(H)$ denotes the number of pendant appearances of H in R_n that share a vertex or the root edge with some other pendant appearance of H, then $\mathbb{E}[\tilde{X}_n(H)] = O(1)$.

There are related results for unlabelled graphs, some of which are discussed in McDiarmid [21]. A major gap at present for unlabelled graphs is that there is no known 'smoothness' result corresponding to Theorem 1.2 above.

2. Proof of smoothness

In this section we shall show that, if $\mathcal{A}$ is an addable proper minor-closed class of graphs, then the class $\mathcal{C}$ of connected graphs in $\mathcal{A}$ is smooth, and later that $\mathcal{A}$ itself is smooth.

The *core* of a graph G, denoted by core(G), is the unique maximal subgraph (which may be the empty graph $\emptyset$) with each degree at least 2. We call a class $\mathcal{A}$ of graphs *trimmable* if, for each graph G and leaf v in G, G is in $\mathcal{A}$ if and only if $G - v$ is in $\mathcal{A}$. Thus, for a trimmable graph class, to tell if a graph is in the class it does not matter if we repeatedly 'trim' off leaves. Note that an equivalent definition of $\mathcal{A}$ being trimmable is that G is in $\mathcal{A}$ if and only if core(G) is in $\mathcal{A}$. (By convention the empty graph $\emptyset$ is in $\mathcal{A}$.) Also, a minor-closed class has this property if and only if each excluded minor has minimum degree at least 2.

We let core($\mathcal{A}$) be the set of all graphs of the form core(G) for some $G \in \mathcal{A}$. Observe that if $\mathcal{A}$ is trimmable then core($\mathcal{A}$) equals the set of all graphs in $\mathcal{A}$ with each degree at least 2.

Lemma 2.1. *Let the class $\mathcal{A}$ of graphs be trimmable, and let $\mathcal{C}$ be the class of connected graphs in $\mathcal{A}$. Let $\mathcal{B}$ be the class of connected graphs in* core($\mathcal{A}$)*, and assume that $\mathcal{B}$ has growth constant $\gamma(\mathcal{B})$ such that $1 < \gamma(\mathcal{B}) < \infty$. Then $\mathcal{C}$ is smooth.*

Proof. We shall use Theorem 1 of Bender, Canfield and Richmond [4]. Let $\mathcal{R}$ and $\mathcal{T}$ denote the classes of rooted and unrooted trees respectively, with exponential generating functions $R(x)$ and $T(x)$ respectively. Let $B(x)$ be the exponential generating function for $\mathcal{B}$ (with $B(0) = 0$). Then the exponential generating function $C(x)$ for $\mathcal{C}$ satisfies $C(x) = B(R(x)) + T(x)$. To see this, observe that each graph in $\mathcal{C}$ is either a tree or it may be obtained in a unique way from its core by substituting a rooted tree at each vertex.

Recall that $\rho(\mathcal{R}) = 1/e$ and $R(1/e) = 1$, and by assumption $0 < \rho(\mathcal{B}) < 1$. Hence $R(s) = \rho(\mathcal{B})$ for some $0 < s < \rho(\mathcal{R})$; that is, condition (c) of Theorem 1 in [4] holds. Hence, by that result, the coefficients of $B(R(x))$ are smooth; that is, if $g_n/n!$ is $[x^n]B(R(x))$ then ng_{n-1}/g_n tends to a limit (which is s) as $n \to \infty$. Hence, since $\rho(\mathcal{T}) = 1/e > s$ (or since $\mathcal{T}$ is smooth), it follows that $\mathcal{C}$ is smooth. □

We may base a direct combinatorial proof of this lemma on the ideas in Bender, Canfield and Richmond [4]; and indeed then we may learn more, in particular about the typical order of the core: see McDiarmid [23].

Given a graph G, let us say that a G-tree is a connected graph obtained by starting with some number n of disjoint copies of G and adding $n-1$ edges in a 'tree structure'. Thus a C_k-*tree* is a connected graph such that removing all bridges leaves a collection of disjoint k-cycles C_k.

Lemma 2.2. *Fix an integer $k \geqslant 3$. Let a_n be the number of C_k-trees on vertices $1, \ldots, n$. Then*

$$\left(\frac{a_n}{n!}\right)^{\frac{1}{n}} \to \left(\frac{ek}{2}\right)^{\frac{1}{k}} > 1$$

as $n \to \infty$ with $k|n$.

Proof. Note that a_n is the product of (i) the number of ways of partitioning $\{1, \ldots, n\}$ into k-cycles, (ii) the number of trees on n/k vertices, and (iii) the factor $(k^2)^{(n/k)-1)}$ to account for the k^2 choices of end points for an edge between two given k-cycles. Thus, for $k|n$ we have

$$a_n = \frac{n!}{(\frac{n}{k})!(2k)^{\frac{n}{k}}} \left(\frac{n}{k}\right)^{\frac{n}{k}-2} k^{2(\frac{n}{k}-1)}.$$

Hence

$$\begin{aligned}
\frac{a_n}{n!} &= \left(\frac{n}{k}\right)^{-2} k^{-2} \frac{(\frac{n}{k})^{\frac{n}{k}} k^{\frac{2n}{k}}}{(\frac{n}{k})!(2k)^{\frac{n}{k}}} \\
&= n^{-2} \frac{(\frac{n}{k})^{\frac{n}{k}}}{(\frac{n}{k})!} \left(\frac{k^2}{2k}\right)^{\frac{n}{k}} \\
&\sim n^{-2} \frac{1}{(2\pi)^{\frac{1}{2}} (\frac{n}{k})^{\frac{1}{2}} e^{-\frac{n}{k}}} \left(\frac{k}{2}\right)^{\frac{n}{k}} \\
&\sim n^{-\frac{5}{2}} \left(\frac{k}{2\pi}\right)^{\frac{1}{2}} \left(\frac{ek}{2}\right)^{\frac{n}{k}},
\end{aligned}$$

which yields the lemma. □

As may be seen from the proof, we could replace C_k above by any connected graph G with $\mathrm{aut}(G) < e|G|^2$ and still find that $\left(\frac{a_n}{n!}\right)^{\frac{1}{n}}$ tends to a limit > 1. We shall also need below that G has all degrees at least 2.

Lemma 2.3. *Let the class $\mathcal{A}$ of graphs be small, addable, trimmable and contain some cycle C_k. Then the class $\mathcal{C}$ of connected graphs in $\mathcal{A}$ is smooth.*

Proof. Let $\mathcal{B}$ be the set of connected graphs in core($\mathcal{A}$). Since core($\mathcal{A}$) is small and addable, it follows by Theorem 3.3 of McDiarmid, Steger and Welsh [25] that it has a growth constant γ, and by (1.1) that $|\mathcal{B}_n| \geqslant |\text{core}(\mathcal{A})_n|/e$; and so $\mathcal{B}$ also growth constant γ. But $\mathcal{B}$ contains all C_k-trees, so $\gamma > 1$ by the last lemma. Lemma 2.1 now shows that $\mathcal{C}$ is smooth. □

Every addable minor-closed class is trimmable, so we may use the last lemma to prove the following.

Lemma 2.4. *Let $\mathcal{A}$ be an addable proper minor-closed class of graphs, and let $\mathcal{C}$ be the class of connected graphs in $\mathcal{A}$. Then $\mathcal{C}$ is smooth.*

Proof. We know that the class $\mathcal{T}$ of trees is smooth. Since $\mathcal{A}$ contains the class $\mathcal{F}$ of forests, we may assume without loss of generality that this containment is strict and so the 3-cycle C_3 is in $\mathcal{A}$. The result now follows from the last lemma. □

The next two lemmas show that frag(R_n) is typically small. Both lemmas concern a bridge-addable class of graphs: in the second lemma we assume also that the class is minor-closed and we learn more.

Lemma 2.5. *Let $\mathcal{A}$ be a bridge-addable class of graphs, and let $R_n \in_u \mathcal{A}_n$. Then* frag(R_n) *is* tight; *that is, for any $\epsilon > 0$ there is a k such that*

$$\mathbb{P}[\text{frag}(R_n) > k] < \epsilon \quad \textit{for all} \quad n.$$

Proof. We use Theorems 2.2 and 2.5 of McDiarmid, Steger and Welsh [25]. By the latter result, for any positive integer s, the probability that R_n has more than one component of size $> s$ is at most

$$\mathbb{P}\left[\text{Po}\left(\frac{1}{s+1}\right) \geqslant 1\right] = 1 - e^{-\frac{1}{s+1}} < \epsilon/2$$

if s is sufficiently large. By the former result, for any positive integer k,

$$\mathbb{P}[\kappa(R_n) > k+1] \leqslant \mathbb{P}[\text{Po}(1) \geqslant k+1] \leqslant \frac{1}{(k+1)!} < \epsilon/2$$

if k is sufficiently large. But if a graph G has at most one component of size $> s$ and $\kappa(G) \leqslant k+1$, then Frag(G) consists of at most k components each of order at most s, and so frag(G) $\leqslant ks$. Thus $\mathbb{P}[\text{frag}(R_n) > ks] < \epsilon$ as required. □

Lemma 2.6. *Let $\mathcal{A}$ be a bridge-addable minor-closed class of graphs, and let $R_n \in_u \mathcal{A}_n$. Then there is a constant c such that $\mathbb{E}[\text{frag}(R_n)] \leqslant c$ for all n.*

Proof. We may assume that $\mathcal{A}$ is proper, since for the class of all graphs we have $\mathbb{E}[\text{frag}(R_n)] = o(1)$, as is well known and easy to see. By a result of Mader [19] (see also, for example, Diestel [11]), for any positive integer k there is a constant c such that every graph with average degree at least c has a minor K_k. Hence, since $\mathcal{A}$ is proper and minor-closed, there is a constant c such that each graph in $\mathcal{A}$ has average degree at most c. But then by Lemma 4.4 of [20], since $\mathcal{A}$ is bridge-addable,

$$\mathbb{E}[\text{frag}(R_n)] \leqslant (2/n)\ \mathbb{E}[|E(R_n)|] \leqslant (2/n)(cn/2) = c$$

for all n. □

Given a class $\mathcal{A}$ of graphs and a graph H in $\mathcal{A}$ we say that H is *freely addable* to $\mathcal{A}$ if, given any graph G disjoint from H, the union of G and H is in $\mathcal{A}$ if and only if G is in $\mathcal{A}$. Observe that $\mathcal{A}$ is decomposable if and only if each graph in $\mathcal{A}$ it is freely addable to $\mathcal{A}$.

Recall that $\mathcal{G}^S$ denotes the class of graphs embeddable on a given surface S; and note that $\mathcal{G}^S$ is bridge-addable, and the graphs freely addable to $\mathcal{G}^S$ are precisely the planar graphs. Note also from McDiarmid [20] that, for $R_n \in_u \mathcal{G}^S_n$, w.h.p. $\text{Frag}(R_n)$ is planar.

Lemma 2.7. *Let the class $\mathcal{A}$ of graphs be bridge-addable, and let $\mathcal{B}$ denote the class of all graphs freely addable to $\mathcal{A}$. Suppose that, for $R_n \in_u \mathcal{A}_n$, w.h.p.* $\text{Frag}(R_n) \in \mathcal{B}$. *Let $\mathcal{C}$ be the class of connected graphs in $\mathcal{A}$, let $\rho = \rho(\mathcal{C})$, and suppose that $\mathcal{C}$ is smooth. Then $B(\rho)$ is finite, and $\mathcal{A}$ is smooth.*

The proof in [4] that the class $\mathcal{G}^S$ of graphs embeddable on S is smooth shows that the class $\mathcal{C} = \mathcal{C}^S$ of connected graphs in $\mathcal{G}^S$ is smooth and then that $\mathcal{G}^S$ itself is smooth. The above lemma allows us to deduce directly that $\mathcal{G}^S$ is smooth from the fact that $\mathcal{C}$ is smooth.

Proof of Lemma 2.7. Let $a_n = |\mathcal{A}_n|$, $b_n = |\mathcal{B}_n|$ and $c_n = |\mathcal{C}_n|$. We shall show that $B(\rho)$ is finite and $a_n \sim B(\rho)c_n$, from which it will follow immediately that $\mathcal{A}$ is smooth. Let $0 < \eta < 1$. Then we are to show that $B(\rho)$ is finite, and for n sufficiently large,

$$(1-\eta)B(\rho)c_n \leqslant a_n \leqslant (1+\eta)B(\rho)c_n. \tag{2.1}$$

Let $0 < \epsilon < 1/(e+2)$ be such that $(1-2\epsilon)^{-1}(1+\epsilon) < 1+\eta$ and $(1-\epsilon)^2 > 1-\eta$. By our assumptions and Lemma 2.5, we may fix positive integers k and n_0 sufficiently large that $\sum_{j=0}^{k} b_j\rho^j/j!$ is at least $(1-\epsilon)B(\rho)$ if $B(\rho)$ is finite, and is at least $e+2$ otherwise; and $\mathbb{P}[\text{frag}(R_n) > k] < \epsilon$ and $\mathbb{P}[\text{Frag}(R_n) \notin \mathcal{B}] < \epsilon$ for all $n \geqslant n_0$. Let $n_1 \geqslant n_0$ be sufficiently large that for all $n \geqslant n_1$, the ratio $\tilde{r}_n = nc_{n-1}/c_n$ satisfies

$$(1-\epsilon)^{1/k}\rho < \tilde{r}_n < (1+\epsilon)^{1/k}\rho.$$

Then, for each $n \geqslant n_1 + k$ and each $j = 1, \ldots, k$, since $\frac{(n)_j c_{n-j}}{c_n} = \prod_{i=1}^{j} \tilde{r}_{n-i+1}$ we have

$$(1-\epsilon)\rho^j < \frac{(n)_j c_{n-j}}{c_n} < (1+\epsilon)\rho^j.$$

Denote $\sum_{j=0}^{k}\binom{n}{j}b_j c_{n-j}$ by $\tilde{a}_n$. Then $\tilde{a}_n = c_n \sum_{j=0}^{k} \frac{b_j}{j!}\frac{(n)_j c_{n-j}}{c_n}$; and so for each $n \geqslant n_1 + k$ we have $\tilde{a}_n \leqslant (1+\epsilon)c_n \sum_{j=0}^{k}\frac{b_j}{j!}\rho^j$ and $\tilde{a}_n \geqslant (1-\epsilon)c_n \sum_{j=0}^{k}\frac{b_j}{j!}\rho^j$.

We may now see that $B(\rho)$ is finite. For suppose not. Then, for each $n \geqslant n_1 + k$,

$$a_n \geqslant \tilde{a}_n \geqslant (1-\epsilon)(e+2)c_n \geqslant (e+1)c_n.$$

But since $\mathcal{A}$ is bridge-addable, by (1.1) the probability that R_n is connected is at least $1/e$, and we obtain the contradiction that $e \cdot c_n \geqslant a_n \geqslant (e+1)\cdot c_n$. Hence $B(\rho)$ must be finite.

From the above we have $\tilde{a}_n \leqslant (1+\epsilon)c_n B(\rho)$, and $\tilde{a}_n \geqslant (1-\epsilon)^2 c_n B(\rho)$. But

$$a_n = \tilde{a}_n + |\{G \in \mathcal{A}_n : \mathrm{frag}(G) > k \quad \text{or} \quad \mathrm{Frag}(G) \notin \mathcal{B}\}|.$$

Thus $a_n \leqslant \tilde{a}_n + 2\epsilon a_n$, and so

$$a_n \leqslant (1-2\epsilon)^{-1}\tilde{a}_n \leqslant (1+\eta)c_n B(\rho);$$

and

$$a_n \geqslant \tilde{a}_n \geqslant (1-\epsilon)^2 c_n B(\rho) \geqslant (1-\eta)c_n B(\rho).$$

So (2.1) holds and we are done. □

Proof of Theorem 1.2. We may now read the theorem off from Lemma 2.4 and Lemma 2.7 with $\mathcal{A}$ addable, since then $\mathrm{Frag}(R_n)$ is always in $\mathcal{A}$, and each graph in $\mathcal{A}$ is freely addable to $\mathcal{A}$. □

3. Distribution of the random graph F

Let us first check equation (1.2). Recall that the class $\mathcal{A}$ of graphs is closed under isomorphism. We identify an unlabelled graph on n vertices with an equivalence class of graphs on $\{1,\ldots,n\}$. Since each graph $H \in \mathcal{UA}$ consists of $\frac{|H|!}{\mathrm{aut}(H)}$ graphs $G \in \mathcal{A}$, we have

$$\frac{z^{|H|}}{\mathrm{aut}(H)} = \sum_{G\in H}\frac{\mathrm{aut}(H)}{|H|!}\frac{z^{|H|}}{\mathrm{aut}(H)} = \sum_{G\in H}\frac{z^{|G|}}{|G|!}.$$

Thus

$$A(z) = \sum_{H\in\mathcal{UA}}\sum_{G\in H}\frac{z^{|G|}}{|G|!} = \sum_{H\in\mathcal{UA}}\frac{z^{|H|}}{\mathrm{aut}(H)}.$$

Proof of Theorem 1.3. Each sum and product below is over all H in $\mathcal{UC}$. Let the unlabelled graph G consist of n_H components isomorphic to H for each $H \in \mathcal{UC}$, where $0 \leqslant \sum_H n_H < \infty$. Then

$$\rho^{|G|} = \prod_H \rho^{|H|n_H}$$

and

$$\mathrm{aut}(G) = \prod_H \mathrm{aut}(H)^{n_H} n_H!.$$

Also, since $\sum_H \lambda(H) = C(\rho)$ by (1.2) applied to $\mathcal{C}$,

$$\frac{1}{A(\rho)} = e^{-C(\rho)} = \prod_H e^{-\lambda(H)}.$$

Hence

$$\begin{aligned}\mathbb{P}[F = G] &= e^{-C(\rho)} \frac{\rho^{|G|}}{\operatorname{aut}(G)} \\ &= \prod_H e^{-\lambda(H)} \frac{\lambda(H)^{n_H}}{n_H\,!} \\ &= \prod_H \mathbb{P}[\operatorname{Po}(\lambda(H)) = n_H].\end{aligned}$$

Thus the probability factors appropriately, and the random variables $\kappa(F, H)$ for $H \in \mathcal{UC}$ satisfy

$$\mathbb{P}[\kappa(F, H) = n_H \ \forall H \in \mathcal{UC}] = \prod_H \mathbb{P}[\kappa(F, H) = n_H].$$

This holds for every choice of non-negative integers n_H with $\sum_{H \in \mathcal{UC}} n_H < \infty$, and thus also without this last restriction (since both sides are zero if the sum is infinite). This completes the proof of the theorem. □

4. Completing the proofs

It remains to prove Theorem 1.5 and Proposition 1.9. Recall that we use the notation $\lambda(H) = \rho^{|H|} / \operatorname{aut}(H)$ for each graph H, where $\rho > 0$ is given; and $A(z)$ for the exponential generating function of $\mathcal{A}$ and so on. Also, let $r_n = n|\mathcal{A}_{n-1}|/|\mathcal{A}_n|$, and assume that $|\mathcal{A}_n| \neq 0$ when necessary. Further, recall the notation $(n)_k = n(n-1)\cdots(n-k+1)$. The following lemma is essentially Lemma 5.1 of McDiarmid [20].

Lemma 4.1. *Let $\mathcal{A}$ be any class of graphs and let $\rho > 0$. Let $H_1, \ldots, H_m$ be pairwise non-isomorphic connected graphs, each freely addable to $\mathcal{A}$. Let $k_1, \ldots, k_m$ be non-negative integers, and let $K = \sum_{i=1}^m k_i|H_i|$. Then, for $R_n \in_u \mathcal{A}_n$,*

$$\mathbb{E}\left[\prod_{i=1}^m (\kappa(R_n, H_i))_{k_i}\right] = \prod_{i=1}^m \lambda(H_i)^{k_i} \prod_{j=1}^K (r_{n-j+1}/\rho).$$

When we add the assumption that $\mathcal{A}$ is smooth, we find convergence of distributions.

Lemma 4.2. *Let the graph class $\mathcal{A}$ be smooth, and let $\rho = \rho(\mathcal{A})$. Let $H_1, \ldots, H_m$ be a fixed family of pairwise non-isomorphic connected graphs, each freely addable to $\mathcal{A}$. Then, as $n \to \infty$ the joint distribution of the random variables $\kappa(R_n, H_1), \ldots, \kappa(R_n, H_m)$ converges in total variation to the product distribution* $\operatorname{Po}(\lambda(H_1)) \otimes \cdots \otimes \operatorname{Po}(\lambda(H_m))$.

Proof. Since $r_n \to \rho$ as $n \to \infty$, by the last lemma,

$$\mathbb{E}\left[\prod_{i=1}^{m} (\kappa(R_n, H_i))_{k_i}\right] \to \prod_{i=1}^{m} \lambda(H_i)^{k_i}$$

as $n \to \infty$, for all non-negative integers $k_1, \dots, k_m$. A standard result on the Poisson distribution now shows that the joint distribution of the random variables $\kappa(R_n, H_1), \dots, \kappa(R_n, H_m)$ tends to that of independent random variables $\mathrm{Po}(\lambda(H_1)), \dots, \mathrm{Po}(\lambda(H_m))$: see, for example, Lemma 5.4 of McDiarmid, Steger and Welsh [25], or see Janson, Łuczak and Ruciński [15]. Thus, for each m-tuple of non-negative integers $(t_1, \dots, t_m)$,

$$\mathbb{P}[\kappa(R_n, H_i) = t_i \ \forall i] \to \prod_i \mathbb{P}[\kappa(R_n, H_i) = t_i] \quad \text{as} \quad n \to \infty;$$

and so we have pointwise convergence of probabilities, which is equivalent to convergence in total variation. □

We now add further assumptions, and find that for $R_n \in_u \mathcal{A}_n$ the probability that R_n is connected tends to a limit. The following lemma parallels Lemma 2.7. Just as that lemma shows that, under suitable conditions, if the class $\mathcal{C}$ of connected graphs in $\mathcal{A}$ is smooth then the class $\mathcal{A}$ is smooth, the lemma below shows the converse result that if $\mathcal{A}$ is smooth then $\mathcal{C}$ is smooth.

Lemma 4.3. *Let the graph class $\mathcal{A}$ be bridge-addable; let $\rho = \rho(\mathcal{A})$; let $\mathcal{B}$ denote the class of all graphs freely addable to $\mathcal{A}$; and suppose that, for $R_n \in_u \mathcal{A}_n$, w.h.p.* $\mathrm{Frag}(R_n) \in \mathcal{B}$. *Let $\mathcal{C}$ and $\mathcal{D}$ be the classes of connected graphs in $\mathcal{A}$ and $\mathcal{B}$ respectively. Suppose that $\mathcal{A}$ is smooth. Then $B(\rho)$ and $D(\rho)$ are finite; $\kappa(R_n) \to_{TV} 1 + \mathrm{Po}(D(\rho))$; and in particular*

$$\Pr[R_n \text{ is connected}] \ \to \ e^{-D(\rho)} = 1/B(\rho) \quad \text{as} \quad n \to \infty,$$

and so $\mathcal{C}$ is smooth.

Proof. We follow the method of proof of Theorem 3.2 of McDiarmid [20]. We first show that $D(\rho)$ is finite. By Lemma 2.5 we may choose a (fixed) k sufficiently large that $\mathbb{P}[\mathrm{frag}(R_n) \geqslant k] \leqslant \frac{1}{3}$ for all n, and $\mathbb{P}[\mathrm{Po}(2k) \geqslant k] \geqslant \frac{2}{3}$. Suppose that $D(\rho) \geqslant 2k+1$. Then by (1.2) there are distinct $H_1, \dots, H_m$ in $\mathcal{UD}$ such that $\sum_{i=1}^{m} \lambda(H_i) = \lambda_0 \geqslant 2k$. It follows by Lemma 4.2 that, for $n > k \max_i v(H_i)$,

$$\frac{1}{3} \geqslant \mathbb{P}[\mathrm{frag}(R_n) \geqslant k] \geqslant \mathbb{P}\left[\sum_{i=1}^{m} \kappa(R_n, H_i) \geqslant k\right] \to \mathbb{P}[\mathrm{Po}(\lambda_0) \geqslant k] \geqslant \frac{2}{3}$$

as $n \to \infty$, a contradiction. Hence $D(\rho)$ is finite, and so $B(\rho)$ is finite too.

Now let $\lambda = D(\rho)$. Let k be a fixed positive integer and let $\epsilon > 0$. We want to show that for n sufficiently large we have

$$|\mathbb{P}[\kappa(\mathrm{Frag}(R_n)) = k] - \mathbb{P}[\mathrm{Po}(\lambda) = k]| < \epsilon. \tag{4.1}$$

By our assumptions, there is an n_0 such that, for each $n \geqslant n_0$,

$$\mathbb{P}[\mathrm{frag}(R_n) > n_0] + \mathbb{P}[\mathrm{Frag}(R_n) \notin \mathcal{B}] < \epsilon/3. \tag{4.2}$$

List the graphs in $\mathcal{UD}$ in non-decreasing order of the number of vertices as $H_1, H_2, \ldots$. For each positive integer m let $\lambda^{(m)} = \sum_{i=1}^{m} \lambda(H_i)$. Note that $D(\rho) = \sum_{H \in \mathcal{UD}} \lambda(H)$ by (1.2) applied to $\mathcal{D}$. Thus we may choose $n_1 \geqslant n_0$ such that, if m is the largest index such that $v(H_m) \leqslant n_1$, then

$$|\mathbb{P}[\mathrm{Po}(\lambda) = k] - \mathbb{P}[\mathrm{Po}(\lambda^{(m)}) = k]| < \epsilon/3. \tag{4.3}$$

Observe that for any graph G with more than $2n_1$ vertices, if $\mathrm{frag}(G) \leqslant n_0$ and $\mathrm{Frag}(G) \in \mathcal{B}$, then $\kappa(\mathrm{Frag}(G))$ is the number of components of G isomorphic to one of $H_1, \ldots, H_m$ (that is, with order at most n_1). Let X_n denote the number of components of R_n isomorphic to one of $H_1, \ldots, H_m$. Let $n > 2n_1$. Then

$$|\mathbb{P}[\kappa(\mathrm{Frag}(R_n)) = k] - \mathbb{P}[X_n = k]| \leqslant \mathbb{P}[\mathrm{frag}(R_n) > n_0] + \mathbb{P}[\mathrm{Frag}(R_n) \notin \mathcal{B}] < \epsilon/3. \tag{4.4}$$

But by Lemma 4.2, for n sufficiently large,

$$|\mathbb{P}[X_n = k] - \mathbb{P}[\mathrm{Po}(\lambda^{(m)}) = k]| < \epsilon/3,$$

and then by (4.3) and (4.4) the inequality (4.1) follows. Thus we have shown that $\kappa(R_n) \to_{TV} 1 + \mathrm{Po}(D(\rho))$, and in particular

$$|\mathcal{C}_n|/|\mathcal{A}_n| = \mathbb{P}(R_n \text{ is connected}) \to e^{-D(\rho)} \quad \text{as} \quad n \to \infty. \tag{4.5}$$

Finally observe that since $\mathcal{A}$ is smooth, and $|\mathcal{C}_n|/|\mathcal{A}_n|$ tends to a non-zero limit as $n \to \infty$ (namely $e^{-D(\rho)}$), it follows that $\mathcal{C}$ is smooth. □

The next lemma has similar premises to Lemmas 2.7 and 4.3, and obtains further conclusions.

Lemma 4.4. *Let $\mathcal{A}$ be bridge-addable; let $\rho = \rho(\mathcal{A})$; let $\mathcal{B}$ be the class of graphs freely addable to $\mathcal{A}$; and suppose that* $\mathrm{Frag}(R_n) \in \mathcal{B}$ *w.h.p. where $R_n \in_u \mathcal{A}_n$. Let $\mathcal{C}$ be the class of connected graphs in $\mathcal{A}$. Assume that either $\mathcal{A}$ or $\mathcal{C}$ is smooth. Then both $\mathcal{A}$ and $\mathcal{C}$ are smooth; $B(\rho)$ is finite; and the unlabelled graph F_n corresponding to* $\mathrm{Frag}(R_n)$ *satisfies $F_n \to_{TV} F$, where*

$$\mathbb{P}[F = H] = \frac{\lambda(H)}{B(\rho)} \quad \textit{for each} \quad H \in \mathcal{UB}.$$

Proof. Lemmas 2.7 and 4.3 show that both $\mathcal{A}$ and $\mathcal{C}$ are smooth, and that $B(\rho)$ is finite. Let $a_n = |\mathcal{A}_n|$ and $c_n = |\mathcal{C}_n|$. Let $\mathcal{D}$ be the class of connected graphs in $\mathcal{B}$. By Lemma 4.3,

$$c_n/a_n \to e^{-D(\rho)} = 1/B(\rho) \quad \text{as} \quad n \to \infty. \tag{4.6}$$

Given a graph G on a finite subset V of the positive integers let $\phi(G)$ be the natural copy of G moved down on to $\{1, \ldots, |V|\}$; that is, let $\phi(G)$ be the graph on $\{1, \ldots, |V|\}$ such that the increasing bijection between V and $\{1, \ldots, |V|\}$ is an isomorphism between G and $\phi(G)$.

Let H be any graph in $\mathcal{B}$ on $\{1,\ldots,h\}$. Then

$$\begin{aligned}\mathbb{P}[\phi(\mathrm{Frag}(R_n)) = H] &= \binom{n}{h}\,\frac{c_{n-h}}{a_n}\\ &= \frac{c_{n-h}}{a_{n-h}}\,\frac{1}{h!}\,\frac{(n)_h a_{n-h}}{a_n}\\ &= \frac{c_{n-h}}{a_{n-h}}\,\frac{1}{h!}\prod_{i=0}^{h-1} r_{n-i}\\ &\to e^{-D(\rho)}\frac{\rho^h}{h!},\end{aligned}$$

as $n \to \infty$ by (4.6) and the fact that $\mathcal{A}$ is smooth. Now, by symmetry,

$$\mathbb{P}[F_n \cong H] = \frac{h!}{\mathrm{aut}(H)}\,\mathbb{P}[\phi(\mathrm{Frag}(R_n)) = H],$$

and hence, as $n \to \infty$,

$$\mathbb{P}[F_n \cong H] \to e^{-D(\rho)}\frac{\rho^h}{\mathrm{aut}(H)} = \Pr(F \cong H).$$

Thus, for each $H \in \mathcal{UB}$, as $n \to \infty$ we have $\mathbb{P}[F_n = H] \to \mathbb{P}(F = H)$; that is, $F_n \to_{TV} U$. □

We need one last lemma to complete the proof of Theorem 1.5.

Lemma 4.5. *Let $\mathcal{A}$ be an addable proper minor-closed class of graphs, and let $\mathcal{C}$ be the class of connected graphs in $\mathcal{A}$. Let $\rho = \rho(\mathcal{A})$ $(= \rho(\mathcal{C}))$. Then $A'(\rho)$ and $C'(\rho)$ are finite.*

Proof. Note that $0 < \rho < \infty$. By the familiar 'exponential formula' $A(x) = e^{C(x)}$, it suffices to show that $C'(\rho)$ is finite.

By Lemma 2.6, there is a constant c such that $\mathbb{E}[\mathrm{frag}(R_n)] \leqslant c$ for all n. Suppose that $C'(\rho) \geqslant (c+3)/\rho$. Then by (1.2) applied to $\mathcal{C}$, there are distinct $H_1,\ldots,H_m$ in $\mathcal{UC}$ such that $\sum_{i=1}^m |H_i|\ \lambda(H_i) = \alpha \geqslant c+2$. Let $n_0 = \max_i |H_i|$. Then

$$\mathbb{E}[\mathrm{frag}(R_n)] \geqslant \mathbb{E}\left[\sum_{i=1}^m |H_i|\ \kappa(R_n, H_i)\right] - n_0\mathbb{P}[\mathrm{big}(R_n) \leqslant n_0].$$

Now $\mathcal{A}$ is smooth by Theorem 1.2. Thus, as $n \to \infty$,

$$\mathbb{E}\left[\sum_{i=1}^m |H_i|\kappa(R_n, H_i)\right] \to \alpha$$

by Lemma 4.2, and by Theorem 2.2 of McDiarmid, Steger and Welsh [25],

$$\mathbb{P}[\mathrm{big}(R_n) \leqslant n_0] \leqslant \mathbb{P}[\kappa(R_n) \geqslant n/n_0] \leqslant \mathbb{P}[\mathrm{Po}(1) \geqslant n/n_0 - 1] = o(1).$$

Hence $\mathbb{E}[\mathrm{frag}(R_n)] \geqslant \alpha - 1 \geqslant c+1$ for n sufficiently large, contradicting our choice of c. □

Proof of Theorem 1.5. We may deduce the theorem from Theorem 1.2 and Lemma 4.4 (with $\mathcal{B} = \mathcal{A}$), on noting that $|F|$ has mean $\rho\ C'(\rho)$, which is finite by the last lemma. □

Proof of Corollary 1.6. The only thing that does not follow directly from the fact that $F_n \to_{TV} F$ is the convergence of the moments in part (b) (which yields the results on moments in part (c)). But $0 \leqslant \kappa(F_n, \mathcal{D}) \leqslant \kappa(R_n) - 1 \leqslant \text{Po}(1)$ in distribution by Theorem 2.2 of McDiarmid, Steger and Welsh [25], and so convergence for the jth moment follows from convergence in total variation. □

Proof of Proposition 1.9. Observe first that

$$\mathbb{E}[X_n(H)] = \binom{n}{h}(n-h)\frac{|\mathcal{A}_{n-h}|}{|\mathcal{A}_n|} \sim n\rho^h/h!.$$

Now consider $\mathbb{E}[(X_n(H))_2]$. For each graph G on $\{1,\ldots,n\}$, let $Y_1(G,H)$ be the number of ordered pairs of appearances in G of H with disjoint vertex sets, and such that the roots are not adjacent; and let $Y_2(G,H)$ be the number of ordered pairs of appearances in G of H such that either the vertex sets meet or the roots are adjacent. Thus $\big(X_n(H)\big)_2 = Y_1(R_n,H) + Y_2(R_n,H)$. Now

$$\mathbb{E}[Y_1(R_n,H)] = \frac{(n)_{2h}}{(h!)^2}(n-2h)^2\frac{|\mathcal{A}_{n-2h}|}{|\mathcal{A}_n|} \sim (n\rho^h/h!)^2.$$

But a graph G of order at most $2h$ either consists of two appearances of H with adjacent roots (and then G has exactly two appearances of H), or the number of appearances of H is at most the number of bridges in G. Thus $Y_2(G,H)$ is at most $2h$ times the number of components of G of order at most $2h$, which is at most $2h\kappa(G)$; and so

$$\mathbb{E}[Y_2(R_n,H)] \leqslant 2h\mathbb{E}[\kappa(R_n)] \leqslant 4h$$

since $\mathbb{E}[\kappa(R_n)] \leqslant 2$ by Theorem 2.2 of McDiarmid, Steger and Welsh [25]. Hence

$$\mathbb{E}[(X_n(H))_2] = \mathbb{E}[Y_1(R_n,H)] + O(1) \sim (n\rho^h/h!)^2.$$

Thus the variance of $X_n(H)$ is $o(\mathbb{E}[(X_n(H))^2])$, and the result follows by Chebyshev's inequality. □

Finally, consider the remark following Proposition 1.9, concerning disjoint pendant appearances. By the above proof, it suffices to note that, in any graph G, the number of pendant appearances of H that share a vertex or edge with some other pendant appearance is at most $Y_2(G,H)$, and so $\mathbb{E}[\tilde{X}_n(H)] \leqslant \mathbb{E}[Y_2(R_n,H)] = O(1)$.

5. Concluding remarks

We have found that each addable proper minor-closed class $\mathcal{A}$ of graphs is smooth, and found various properties of the random graph $R_n \in_u \mathcal{A}_n$, but many questions are left open. For example, we know that the average degree of graphs in $\mathcal{A}$ is bounded: does the average degree of R_n tend in probability to a constant? Is the number of edges asymptotically normal, apart from the case when $\mathcal{A}$ is the class of forests? Does the giant component $\text{Big}(R_n)$ have diameter of order $\sqrt{n}$ w.h.p.? Is the maximum degree $O(\log n)$ w.h.p.? (This is true for the graphs embeddable on a fixed surface; McDiarmid and

Reed [24].) For which $\mathcal{A}$ is the maximum degree $O(\log n/\log\log n)$ w.h.p.? For which $\mathcal{A}$ is there w.h.p. a unique block of linear size?

What happens if we weaken the assumption of addability to, say, bridge-addability? A natural example to consider is the class of graphs embeddable on a fixed surface. This class is smooth (Bender, Canfield and Richmond [4]) as we noted earlier, and we may use Lemma 4.4 to show that, for random R_n from this class, $\mathrm{Frag}(R_n)$ behaves as for planar graphs: see McDiarmid [20]. Another natural example to consider is the class $\mathrm{Ex}(2C_3)$ of graphs with no two vertex-disjoint cycles. (Here $2C_3$ denotes the graph consisting of two disjoint triangles.) It is shown in Kurauskas and McDiarmid [18] that this class is smooth, and again we may use Lemma 4.4 to show that, for random R_n from this class, $\mathrm{Frag}(R_n)$ behaves as for forests. Now, however, w.h.p. the maximum degree $\Delta(R_n) \sim n/2$. (The last paper also considers $\mathrm{Ex}(kC_3)$ and $\mathrm{Ex}(kC_4)$, for example.)

Does every minor-closed class of graphs have a growth constant (as conjectured in Bernardi, Noy and Welsh [6]), and indeed is every such class smooth? What can we say about corresponding questions for unlabelled graphs? We noted earlier that some results are given in McDiarmid [21], but, in particular, is there a 'smoothness' result corresponding to Theorem 1.2 above?

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