Numerical Analysis – Spring 2024

Midterm Take-Home Examination Due on 3/19/24 by 1:30pm

- You can use any resource you want (including asking people, **except your instructor,** to explain a concept/topic).
- For the computations, you **cannot simply** use any standard software such as MATLAB, Mathematica or Maple. *However, you can write your own code.*

Show all your work. You will lose points if the information, justifications and explanations that you provide are insufficient.

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1. (20 points) Find the roots and their multiplicities of the following functions in the interval [0,2] using the Newton's method. Comment on your results and discuss any commonalities between the results of (i) and (ii).

(i) $f(x) = 21.12 - 32.4x + 12x^2$

(ii) $h(x) = 2.7951 - 8.954x + 10.56x^2 - 5.4x^3 + x^4$

2. (30 points) Solve the following systems of linear equations with the Jacobi iteration method using the initial guess as $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

In each case, will the Jacobi iteration converge to a solution? Give the justification for your answer? If yes, find the solutions.

(i) $\begin{bmatrix} 2 & 1 & 6 \\ 8 & 3 & 2 \\ 1 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 7 \end{bmatrix}$.

(ii) $\begin{bmatrix} 8 & 3 & 2 \\ 1 & 5 & 1 \\ 2 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \\ 9 \end{bmatrix}$.

3. (20 points) You would like to use the fixed-point iteration method to find the roots of $f(x) = x - x^2 = 0$. Consider the following two formulations.

   (i) $x = x + 2(x-x^2)$.

   (ii) $x = x - (x- x^2)/ (1 - 2x)$.

   For each formulation carry out the iterations first using the starting value 0.8 and then, using the starting value 0.2. Comment and justify your observations.

4. (30 points) (i) Use the Newton's method for finding the root of $f(x) = e^x - 2\cos(x) = 0$ in the interval [0,2] with the starting value 2. Carry out 6 iterations and comment on what you observe. If the observation needs any improvement, how will you go about it? Show that the Newton's method's convergence is of second order.

   (ii) Now, use the iteration method given below to find the root of $f(x) = e^x - 2\cos(x) = 0$ in the interval [0,2]. Choose your starting values as $x_0 = 0.6$ and $y_0 = 0.3388$.

   $y_{n+1} = y_n (2 - f'(x_n)\, y_n)$
   $x_{n+1} = x_n - y_{n+1}\, f(x_n)$

   Compare your methods and results in (i) and (ii) and discuss any connections between (i) and (ii).

Bonus Questions (25 points)

1. You would like to solve a linear system of equations Ax = b using the Jacobi iteration method. Here, the matrix A is the tridiagonal matrix [-1, 2, -1] of order 100. Find the spectral radius of the Jacobi iteration matrix and determine whether your iterations will converge or not. If converging, how fast will they converge. If not converging, why not?
2. Can you show that the Gauss-Seidel iteration method will converge twice as fast as the Jacobi iteration method for system of linear equations of order 2? Do you need to satisfy any conditions to obtain this result?