Numerical Analysis - Final Exam

Spring 2024

Due on 4/30/24 in class by 1:30pm

- You can use any resource you want (**except communicating with your instructor**).
- For the computations, you **cannot simply** use any standard software such as MATLAB, Mathematica or Maple. *However, you can write your own code.*

Show all your work. You will lose points if the information, justifications and explanations that you provide are insufficient.

You can earn up to 25 points for each of the four problems and up to 20 points for the bonus questions.

1. (a) Use the Newton's method for finding a root of x^3 - $3x^2$ + x + 3 with the starting value 1. Comment on what you observe. If the observation needs any improvement, how will you go about it?

   (b) Find $(3)^{1/2}$ using the fixed-point iteration method.

2. For which values of b, the matrix $\begin{bmatrix} 2 & -1/2 & b \\ -1/2 & 2 & -1/2 \\ b & -1/2 & 2 \end{bmatrix}$ is positive definite? Find the LU decomposition of this matrix when b=0, using the Gaussian elimination method. Write down the matrices L and U. Explain and justify every step in the process.

   Now solve the system of equations given by
   $\begin{bmatrix} 2 & -1/2 & 0 \\ -1/2 & 2 & -1/2 \\ 0 & -1/2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 14 \end{bmatrix}$.

3. Construct a linear multi-step method to solve dy/dx = f(x,y) with y(0) = y_0 as follows. First, find a polynomial approximation in x for the function f(x,y) assuming that two points (x_m,f_m) and (x_{m+1},f_{m+1}) are known. Then, substitute this approximation for f(x,y) in the problem and integrate it from x_m to x_{m+1}.

   Find the order of the linear multi-step method that you constructed.

   Use this linear multistep method to solve the initial value problem

dy/dx = -5y with y(0) = 1. What is your choice of mesh width h? Justify your reason for choosing this h. Find the solution at the 4^{th} spatial step. How does this numerical solution compare with the exact solution? Are you satisfied with your numerical solution? If no, how can you improve it?

Now, if you would like to use your linear multistep method to solve the problem dy/dx = y^2 -10y +16 with y(0) =1, where y is non-negative and less or equal to 5, what will be your choice for h? Give your reason for choosing this h. In this problem, what is the maximum value y can achieve?

4. For the initial value problem given below, find the solution using the Picard iteration method.

   dy/dx = 2x - y with y (0) = 1, x ϵ [0,1].

Bonus Questions

1. You would like to solve the initial value problem dy/dx = 4y (1-y), y (0) =1/2. Write down a discrete numerical method whose exact solution will be the same as the exact solution of the given differential equation. Find both exact solutions (the one for the discrete method and the one for the differential equation) and compare to justify.

2. (a) If you are to describe yourself as a mathematical function or a mathematical concept, what will it be? Explain why?

   (b) What is your take away from this class?