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Two-loop QED and QCD corrections to massless fermion-boson scattering*

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Abstract: We present the NNLO QCD virtual corrections for $q\bar{q} \to g\gamma$, $q\bar{q} \to \gamma\gamma$ and the NNLO QED virtual corrections for $e^-e^+ \to \gamma\gamma$, and all processes related by crossing symmetry. We perform an explicit evaluation of the two-loop diagrams in conventional dimensional regularisation, and our results are renormalised in the $\overline{\rm MS}$ scheme. The infrared pole structure of the amplitudes is in agreement with the prediction of Catani's general formalism for the singularities of two-loop amplitudes, while expressions for the finite remainder are given for all processes in terms of logarithms and polylogarithms that are real in the physical region.

Keywords: QCD, Jets, LEP HERA and SLC Physics, NLO and NNLO Computations.

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1. Introduction

Scattering processes involving either initial or final state photons are common in electron-positron annihilation, electron-proton collisions and hadron-hadron collisions. In the initial state, we may be interested in direct processes where the photon behaves in a pointlike way, or in processes where the internal structure of the photon is probed. Similarly, prompt final state photons may be produced directly at large transverse momentum in the hard scattering or from the fragmentation of a large transverse momentum jet.[1]

To illustrate the wide range of interesting processes involving photons, let us select a few examples relevant to e^-e^+, e^-p, and hadron-hadron collider experiments in the next decade or so. In hadron-hadron collisions, attention is mainly focussed on single direct photon production,

$$q\bar{q} \to \gamma g, \qquad gg \to \gamma q$$

which is sensitive to the gluonic content of the proton and is an important test of perturbative QCD and the pair production of high mass prompt photon pairs,

$$q\bar{q} \to \gamma\gamma, \qquad gg \to \gamma\gamma$$

which is a background to the discovery of a light Higgs boson via its decay into photons at the TEVATRON or LHC,

$$gg \to H \to \gamma\gamma.$$

Important processes in electron-positron collisions include jet photoproduction,

$$\gamma q \to qg, \qquad \gamma g \to q\bar{q},$$

which is sensitive to the value of the strong coupling constant, as well as the photoproduction of prompt photons from both the direct

$$\gamma q \to \gamma q,$$

and resolved processes such as

$$gq \to \gamma q,$$

which is sensitive to the gluonic content of the photon. Finally, the proposed gamma-gamma option of future linear colliders will, amongst other things, measure dijet production,

$$\gamma\gamma \to q\bar{q}$$

[1] Non-prompt photons may also be produced from the decay of a single large transverse momentum hadron such as the π^0.

which is once again a background to the Higgs boson detection via its decay into bottom quarks,

$$\gamma\gamma \to H \to b\bar{b}.$$

State-of-the-art theoretical predictions for prompt photon production and the vast majority of QED and QCD scattering processes, incorporate corrections at next-to-leading order (NLO) in perturbation theory. However, these NLO calculations generally exhibit a large sensitivity to the variation of unphysical renormalisation and factorization scales as a consequence of truncating the perturbative expansion. The inclusion of higher order terms is therefore desirable in order to stabilise the theoretical predictions and to reduce the inherent theoretical uncertainties.

Until recently, a major stumbling block toward a next-to-next-to-leading order (NNLO) numerical prediction for scattering processes has been the evaluation of the two-loop matrix elements. The first calculation of a two-loop four-point scattering amplitude was performed by Bern, Dixon, and Kosower for the maximal-helicity-violating gluon-gluon scattering [1]. Subsequently, generic $2 \to 2$ scattering matrix elements at two-loops have now become tractable for massless particle exchanges in the loops and where all of the external particles are on-shell. Smirnov [2] and Tausk [3] have provided analytic expansions in $\epsilon = \frac{4-D}{2}$, where D is the dimension, in terms of Nielsen polylogarithms for the two most challenging integrals emerging; the double-box [2] and the cross-box [3]. At the same time, algorithms based on integration by parts [4] and Lorentz invariance [5] recursion relations were also developed for the tensor reduction to master integrals of all relevant two-loop topologies [6, 7, 8, 9, 10].

Already, this technology has been applied to a wide range of physically interesting processes. The interference of tree and two-loop graphs (together with the simpler self interference of one-loop diagrams) for various processes have now been computed, including Bhabha scattering ($e^+e^- \to e^+e^-$) in the massless electron limit [11] and all the QCD $2 \to 2$ parton-parton scattering processes ($gg \to gg$, $gg \to q\bar{q}$ and $q\bar{q} \to q\bar{q}$) [12, 13, 14, 15, 16, 17]. Two-loop helicity amplitudes have also been derived for gluon fusion into photons ($gg \to \gamma\gamma$) [18], light-by-light scattering ($\gamma\gamma \to \gamma\gamma$) [19] and gluon-gluon scattering ($gg \to gg$) [20].

The case where the internal propagators are massless but one external leg is off-shell has also been intensively studied, leading to the evaluation of all associated planar and non-planar master integrals [21, 22] in terms of two-dimensional harmonic polylogarithms [23]. These integrals arise in the decay of an off-shell photon to three partons ($\gamma^* \to q\bar{q}g$) relevant for three jet production in electron-positron annihilation and the NNLO matrix elements have been evaluated in Ref. [24]. Even more recently, Mellin Barnes integral techniques have been applied to the planar double box diagram with four massive and three massless lines, all four legs on the mass shell [25].

In this paper, we present the NNLO virtual corrections for the processes

$$\begin{aligned} q + \bar{q} &\to g + \gamma, \\ q + \bar{q} &\to \gamma + \gamma, \\ e^- + e^+ &\to \gamma + \gamma, \end{aligned}$$

and the ones related by crossing symmetry or time-reversal. For clarity and calculational convenience, we decompose the NNLO virtual corrections into the self-interference of the one-loop amplitude and the interference of the tree and two-loop amplitude. Our results are valid in the limit where the masses of the quarks and electrons can be ignored. The Feynman diagrams for the processes in consideration are only a subset of those for quark-gluon scattering ($q\bar{q} \to gg$), which we presented in [15]. However, due to the complicated colour structure of the quark-gluon two-loop amplitude and the different flavour content of the processes with photons we choose to present them independently. As in [15], we work in dimensional regularisation treating all external particles in D dimensions, and renormalise with the $\overline{\mathrm{MS}}$ scheme.

Our paper is organised as follows. In Section 2 we introduce our notation and define the perturbative expansion of the matrix elements summed over colours and spins. In Section 3 we study the singular behavior of the NNLO contributions, and verify that it agrees with the general formalism developed by Catani for the infrared structure of two-loop amplitudes [26]. For the simpler case of one-loop amplitudes, their poles in ϵ can be expressed in terms of the tree amplitude and a colour-charge operator $\mathrm{I}^{(1)}(\epsilon)$, constructed in a universal manner [27]. In the same fashion, the divergences of the two-loop amplitude can be written as a sum of two terms: the action of the $\mathrm{I}^{(1)}(\epsilon)$ operator on the one-loop amplitude and the action of a new operator $\mathrm{I}^{(2)}(\epsilon)$ on the tree amplitude. The $\mathrm{I}^{(2)}(\epsilon)$ operator includes a renormalisation scheme dependent term $H^{(2)}$ multiplied by a $1/\epsilon$ pole. Although in [26] it is anticipated that $H^{(2)}$ can be constructed for any given process in a universal manner, such a prescription is not yet available. We give explicit expressions for $\mathrm{I}^{(1)}(\epsilon)$ and $\mathrm{I}^{(2)}(\epsilon)$ relevant for each of the processes in Eq. (1.1) valid in the $\overline{\mathrm{MS}}$ scheme. In Section 4 we present the finite remainders of the interference of the tree and the two-loop amplitude after subtraction of the singular poles of Section 3 from the explicit result of the two-loop Feynman diagrams. We organize the finite part according to the colour and flavour content of the two-loop amplitude. Similarly, in Section 5 we give the finite contributions of the self-interference of the one-loop amplitude. The finite remainders are expressed in terms of logarithms and polylogarithms which are real in the physical domain. Finally, we conclude in Section 6.

2. Notation

We consider the processes

$$q(p_1) + \bar{q}(p_2) + g(p_3) + \gamma(p_4) \to 0, \tag{2.1}$$
$$q(p_1) + \bar{q}(p_2) + \gamma(p_3) + \gamma(p_4) \to 0, \tag{2.2}$$
$$e^-(p_1) + e^+(p_2) + \gamma(p_3) + \gamma(p_4) \to 0, \tag{2.3}$$

where all particles are incoming with light-like momenta satisfying

$$p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu = 0, \quad p_i^2 = 0. \tag{2.4}$$

The amplitudes exhibit infrared and ultraviolet divergences so we work in conventional dimensional regularisation and treat all external particle states in D dimensions. The ultraviolet divergences are renormalised with the $\overline{\rm MS}$ scheme where the bare strong coupling α_s^0 is related to the running coupling $\alpha_s \equiv \alpha_s(\mu^2)$ at renormalisation scale μ via

$$\alpha_s^0\, S_\epsilon = \alpha_s \left[1 - \frac{\beta_0}{\epsilon}\left(\frac{\alpha_s}{2\pi}\right) + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right)\left(\frac{\alpha_s}{2\pi}\right)^2 + \mathcal{O}\left(\alpha_s^3\right)\right]. \tag{2.5}$$

In this expression

$$S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma}, \qquad \gamma = 0.5772\ldots = \text{Euler constant} \tag{2.6}$$

is the typical phase-space volume factor in $D = 4 - 2\epsilon$ dimensions and β_0, β_1 are the first two coefficients of the QCD beta function for N_F (massless) quark flavours

$$\beta_0 = \frac{11C_A - 4T_R N_F}{6}\quad , \qquad \beta_1 = \frac{17C_A^2 - 10C_A T_R N_F - 6C_F T_R N_F}{6}\quad . \tag{2.7}$$

For an $SU(N)$ gauge theory

$$C_F = \frac{N^2-1}{2N}, \qquad C_A = N, \qquad T_R = \frac{1}{2}. \tag{2.8}$$

In a similar way, the bare QED coupling α^0 is related to the running coupling $\alpha \equiv \alpha(\mu'^2)$ at renormalisation scale μ' via

$$\alpha^0\, S_\epsilon = \alpha \left[1 - \frac{\beta_0'}{\epsilon}\left(\frac{\alpha}{2\pi}\right) + \left(\frac{{\beta_0'}^2}{\epsilon^2} - \frac{\beta_1'}{2\epsilon}\right)\left(\frac{\alpha}{2\pi}\right)^2 + \mathcal{O}\left(\alpha^3\right)\right]. \tag{2.9}$$

where β_0', β_1' are now the first two coefficients of the QED beta function,

$$\beta_0' = \frac{-2N_F'}{3}\quad , \qquad \beta_1' = -N_F'\quad , \tag{2.10}$$

and

$$N'_F = \sum_f Q_f^2, \tag{2.11}$$

with the sum running over the active (massless) fermion flavours.

The renormalised amplitudes may be expanded as

$$|\mathcal{M}_{q\bar{q}g\gamma}\rangle = 4\pi\sqrt{\alpha\alpha_s}\left[|\mathcal{M}^{(0)}_{q\bar{q}g\gamma}\rangle + \left(\frac{\alpha_s}{2\pi}\right)|\mathcal{M}^{(1)}_{q\bar{q}g\gamma}\rangle + \left(\frac{\alpha_s}{2\pi}\right)^2|\mathcal{M}^{(2)}_{q\bar{q}g\gamma}\rangle + \mathcal{O}\left(\alpha_s^3\alpha^0\right)\right], \tag{2.12}$$

$$|\mathcal{M}_{q\bar{q}\gamma\gamma}\rangle = 4\pi\alpha\left[|\mathcal{M}^{(0)}_{q\bar{q}\gamma\gamma}\rangle + \left(\frac{\alpha_s}{2\pi}\right)|\mathcal{M}^{(1)}_{q\bar{q}\gamma\gamma}\rangle + \left(\frac{\alpha_s}{2\pi}\right)^2|\mathcal{M}^{(2)}_{q\bar{q}\gamma\gamma}\rangle + \mathcal{O}\left(\alpha_s^3\alpha^0\right)\right], \tag{2.13}$$

$$|\mathcal{M}_{e^-e^+\gamma\gamma}\rangle = 4\pi\alpha\left[|\mathcal{M}^{(0)}_{e^-e^+\gamma\gamma}\rangle + \left(\frac{\alpha}{2\pi}\right)|\mathcal{M}^{(1)}_{e^-e^+\gamma\gamma}\rangle + \left(\frac{\alpha}{2\pi}\right)^2|\mathcal{M}^{(2)}_{e^-e^+\gamma\gamma}\rangle + \mathcal{O}\left(\alpha^3\right)\right], \tag{2.14}$$

where $|\mathcal{M}^{(i)}_{\mathcal{P}}\rangle$ represents the colour-space vector describing the renormalised $i-$loop amplitude for the $\mathcal{P} = q\bar{q}g\gamma$, $q\bar{q}\gamma\gamma$, $e^-e^+\gamma\gamma$ processes of Eqs. (2.1)- (2.3). The dependence on the renormalisation scale μ and renormalisation scheme is implicit.

We denote the squared amplitudes summed over spins and colours by

$$\langle\mathcal{M}_{q\bar{q}g\gamma}|\mathcal{M}_{q\bar{q}g\gamma}\rangle = \sum|\mathcal{M}\left(q+\bar{q}\to\gamma+g\right)|^2 = \mathcal{A}_{q\bar{q}g\gamma}\left(s,t,u\right), \tag{2.15}$$

$$\langle\mathcal{M}_{q\bar{q}\gamma\gamma}|\mathcal{M}_{q\bar{q}\gamma\gamma}\rangle = \sum|\mathcal{M}\left(q+\bar{q}\to\gamma+\gamma\right)|^2 = \mathcal{A}_{q\bar{q}\gamma\gamma}\left(s,t,u\right), \tag{2.16}$$

$$\langle\mathcal{M}_{e^-e^+\gamma\gamma}|\mathcal{M}_{e^-e^+\gamma\gamma}\rangle = \sum\left|\mathcal{M}\left(e^-+e^+\to\gamma+\gamma\right)\right|^2 = \mathcal{A}_{e^-e^+\gamma\gamma}\left(s,t,u\right), \tag{2.17}$$

where the Mandelstam variables are

$$s = (p_1+p_2)^2, \quad t = (p_2+p_3)^2, \quad u = (p_1+p_3)^2, \quad s+t+u = 0. \tag{2.18}$$

The squared matrix elements for the crossed processes are obtained by permuting the Mandelstam variables and introducing a minus sign for each fermion exchange between initial and final states,

$$\sum|\mathcal{M}(g+\gamma\to q+\bar{q})|^2 = \mathcal{A}_{q\bar{q}g\gamma}(s,t,u), \tag{2.19}$$

$$\sum|\mathcal{M}(q+\gamma\to q+g)|^2 = -\mathcal{A}_{q\bar{q}g\gamma}(u,t,s), \tag{2.20}$$

$$\sum|\mathcal{M}(q+g\to\gamma+q)|^2 = -\mathcal{A}_{q\bar{q}g\gamma}(t,u,s), \tag{2.21}$$

$$\sum|\mathcal{M}(g+\bar{q}\to\gamma+\bar{q})|^2 = -\mathcal{A}_{q\bar{q}g\gamma}(u,t,s), \tag{2.22}$$

$$\sum|\mathcal{M}(\gamma+\bar{q}\to\bar{q}+g)|^2 = -\mathcal{A}_{q\bar{q}g\gamma}(t,u,s). \tag{2.23}$$

Similarly,

$$\sum|\mathcal{M}(\gamma+\gamma\to q+\bar{q})|^2 = \mathcal{A}_{q\bar{q}\gamma\gamma}(s,t,u), \tag{2.24}$$

$$\sum|\mathcal{M}(q+\gamma\to q+\gamma)|^2 = -\mathcal{A}_{q\bar{q}\gamma\gamma}(u,t,s), \tag{2.25}$$

$$\sum|\mathcal{M}(\gamma+\bar{q}\to\gamma+\bar{q})|^2 = -\mathcal{A}_{q\bar{q}\gamma\gamma}(u,t,s), \tag{2.26}$$

and

$$\sum |\mathcal{M}(\gamma+\gamma\to e^- + e^+)|^2 = \mathcal{A}_{e^-e^+\gamma\gamma}(s,t,u), \tag{2.27}$$
$$\sum |\mathcal{M}(e^- +\gamma\to e^- + \gamma)|^2 = -\mathcal{A}_{e^-e^+\gamma\gamma}(u,t,s), \tag{2.28}$$
$$\sum |\mathcal{M}(\gamma+e^+\to \gamma + e^+)|^2 = -\mathcal{A}_{e^-e^+\gamma\gamma}(u,t,s). \tag{2.29}$$

The functions $\mathcal{A}_{\mathcal{P}}(s,t,u)$ are symmetric under the exchange of t and u and can be expanded perturbatively to yield,

$$\mathcal{A}_{q\bar{q}g\gamma}(s,t,u) = 16\pi^2\alpha\alpha_s\left[\mathcal{A}^{\rm LO}_{q\bar{q}g\gamma}(s,t,u) + \left(\frac{\alpha_s}{2\pi}\right)\mathcal{A}^{\rm NLO}_{q\bar{q}g\gamma}(s,t,u)\right.$$
$$\left. + \left(\frac{\alpha_s}{2\pi}\right)^2\mathcal{A}^{\rm NNLO}_{q\bar{q}g\gamma}(s,t,u) + \mathcal{O}\left(\alpha_s^3\alpha^0\right)\right], \tag{2.30}$$

$$\mathcal{A}_{q\bar{q}\gamma\gamma}(s,t,u) = 16\pi^2\alpha^2\left[\mathcal{A}^{\rm LO}_{q\bar{q}\gamma\gamma}(s,t,u) + \left(\frac{\alpha_s}{2\pi}\right)\mathcal{A}^{\rm NLO}_{q\bar{q}\gamma\gamma}(s,t,u)\right.$$
$$\left. + \left(\frac{\alpha_s}{2\pi}\right)^2\mathcal{A}^{\rm NNLO}_{q\bar{q}\gamma\gamma}(s,t,u) + \mathcal{O}\left(\alpha_s^3\alpha^0\right)\right], \tag{2.31}$$

and

$$\mathcal{A}_{e^-e^+\gamma\gamma}(s,t,u) = 16\pi^2\alpha^2\left[\mathcal{A}^{\rm LO}_{e^-e^+\gamma\gamma}(s,t,u) + \left(\frac{\alpha}{2\pi}\right)\mathcal{A}^{\rm NLO}_{e^-e^+\gamma\gamma}(s,t,u)\right.$$
$$\left. + \left(\frac{\alpha}{2\pi}\right)^2\mathcal{A}^{\rm NNLO}_{e^-e^+\gamma\gamma}(s,t,u) + \mathcal{O}\left(\alpha^3\right)\right]. \tag{2.32}$$

At leading-order (LO) the self-interference of the tree amplitudes is

$$\mathcal{A}^{\rm LO}_{\mathcal{P}}(s,t,u) = \langle\mathcal{M}^{(0)}_{\mathcal{P}}|\mathcal{M}^{(0)}_{\mathcal{P}}\rangle, \tag{2.33}$$

so that for each process we have

$$\mathcal{A}^{\rm LO}_{q\bar{q}g\gamma}(s,t,u) = NC_F\mathcal{T}(s,t,u), \tag{2.34}$$
$$\mathcal{A}^{\rm LO}_{q\bar{q}\gamma\gamma}(s,t,u) = N\mathcal{T}(s,t,u), \tag{2.35}$$
$$\mathcal{A}^{\rm LO}_{e^-e^+\gamma\gamma}(s,t,u) = \mathcal{T}(s,t,u), \tag{2.36}$$

where we have defined the tree-type structure

$$\mathcal{T}(s,t,u) = 8(1-\epsilon)\left(\frac{u}{t}+\frac{t}{u}-\epsilon\frac{s^2}{tu}\right). \tag{2.37}$$

The next-to-leading order (NLO) term consists of the interference of the tree and the one-loop amplitudes

$$\mathcal{A}^{\rm NLO}_{\mathcal{P}}(s,t,u) = \langle\mathcal{M}^{(0)}_{\mathcal{P}}|\mathcal{M}^{(1)}_{\mathcal{P}}\rangle + \langle\mathcal{M}^{(1)}_{\mathcal{P}}|\mathcal{M}^{(0)}_{\mathcal{P}}\rangle, \tag{2.38}$$

and their expansion in ϵ up to $\mathcal{O}(\epsilon)$ was given in Ref. [28]. As will be discussed in Section 3, the singular structure of the two-loop amplitude is expressed in terms of the tree amplitude multiplied by a singular operator of $\mathcal{O}(1/\epsilon^4)$ and the one-loop amplitude multiplied by another operator diverging as $1/\epsilon^2$. Therefore, one needs to know the expansion of the one-loop amplitude up to and including the $\mathcal{O}(\epsilon^2)$ term. In Section 3 we give expressions for $\langle\mathcal{M}^{(0)}_{\mathcal{P}}|\mathcal{M}^{(1)}_{\mathcal{P}}\rangle$ valid in all kinematic regions and to all orders in ϵ, in terms of two one-loop master integrals. The ϵ-expansions of these master integrals to the appropriate order are given in the Appendix C.

At next-to-next-to-leading-order (NNLO) contributions from the self-interference of the one-loop amplitude and the interference of the tree and the two-loop amplitude must be taken into account, so that

$$\mathcal{A}^{\rm NNLO}_{\mathcal{P}}(s,t,u) = \mathcal{A}^{\rm NNLO(1\times1)}_{\mathcal{P}}(s,t,u) + \mathcal{A}^{\rm NNLO(0\times2)}_{\mathcal{P}}(s,t,u), \tag{2.39}$$

with

$$\mathcal{A}^{\rm NNLO(1\times1)}_{\mathcal{P}}(s,t,u) = \langle\mathcal{M}^{(1)}_{\mathcal{P}}|\mathcal{M}^{(1)}_{\mathcal{P}}\rangle, \tag{2.40}$$

and

$$\mathcal{A}^{\rm NNLO(0\times2)}_{\mathcal{P}}(s,t,u) = \langle\mathcal{M}^{(0)}_{\mathcal{P}}|\mathcal{M}^{(2)}_{\mathcal{P}}\rangle + \langle\mathcal{M}^{(2)}_{\mathcal{P}}|\mathcal{M}^{(0)}_{\mathcal{P}}\rangle. \tag{2.41}$$

In the following sections, we present expressions for the infrared singular and finite contributions to $\mathcal{A}^{\rm NNLO}_{\mathcal{P}}$. In Section 3 we write down the singular parts according to the formalism of Catani [26]. The pure two-loop finite contributions of $\mathcal{A}^{\rm NNLO\,(2\times0)}_{\mathcal{P}}(s,t,u)$ are described in Sec. 4 and the finite remainders of the self-interference of the one-loop amplitude $\mathcal{A}^{\rm NNLO\,(1\times1)}_{\mathcal{P}}(s,t,u)$ are described in Sec. 5.

As in Refs. [12, 13, 14, 15, 16, 17], we use `QGRAF` [29] to produce the tree, one and two-loop Feynman diagrams to construct $|\mathcal{M}^{(i)}_{\mathcal{P}}\rangle$. We then project by $\langle\mathcal{M}^{(0)}_{\mathcal{P}}|$ or $\langle\mathcal{M}^{(1)}_{\mathcal{P}}|$ and perform the summation over colours and spins. It should be noted that when summing over the gluon polarisation, we ensure that the polarisation states are transversal (i.e. physical) by using an axial gauge

$$\sum_{\rm spins} \epsilon_3^\mu \epsilon_3^{\nu *} = -g^{\mu\nu} + \frac{n^\mu p_3^\nu + n^\nu p_3^\mu}{n\cdot p_3}, \tag{2.42}$$

where n is an arbitrary light-like 4-vector. For simplicity, we choose $n^\mu = p_4^\mu$. Finally, the trace over the Dirac matrices is carried out in D dimensions using conventional dimensional regularisation. It is then straightforward to identify the scalar and tensor integrals present and replace them with combinations of the basis set of master integrals using the tensor reduction of two-loop integrals described in [6, 7, 8], based on integration-by-parts [4] and Lorentz invariance [5] identities. The final result is a combination of master integrals in $D = 4-2\epsilon$ for which the expansions around $\epsilon = 0$ are given in [2, 3, 6, 7, 8, 9, 10, 30, 31].

3. Infrared Pole Structure

We further decompose the one-loop self-interference and the two-loop contributions as a sum of singular and finite terms,

$$\mathcal{A}_{\mathcal{P}}^{\mathrm{NNLO}\,(1\times1)}(s,t,u) = \mathcal{P}^{1\times1}_{oles,\,\mathcal{P}}(s,t,u) + \mathcal{F}^{1\times1}_{inite,\mathcal{P}}(s,t,u) \tag{3.1}$$

and

$$\mathcal{A}_{\mathcal{P}}^{\mathrm{NNLO}\,(2\times0)}(s,t,u) = \mathcal{P}^{0\times2}_{oles,\,\mathcal{P}}(s,t,u) + \mathcal{F}^{0\times2}_{inite,\mathcal{P}}(s,t,u), \tag{3.2}$$

for each of the processes $\mathcal{P} = q\bar{q}g\gamma,\ q\bar{q}\gamma\gamma,\ e^-e^+\gamma\gamma$. $\mathcal{P}^{1\times1}_{oles,\,\mathcal{P}}$ and $\mathcal{P}^{0\times2}_{oles,\,\mathcal{P}}$ contain infrared singularities that will be analytically canceled by the infrared singularities occurring in radiative processes of the same order (ultraviolet divergences having already being removed by renormalisation). $\mathcal{F}^{1\times1}_{inite,\mathcal{P}}$ and $\mathcal{F}^{0\times2}_{inite,\mathcal{P}}$ are the remainders which are finite as $\epsilon \to 0$.

The poles of the one-loop amplitude self-interference can be written in terms of a universal operator $\mathrm{I}^{(1)}(\epsilon)$ acting on the colour-space of the amplitude. Due to the simple colour structure of the processes we study, the action of $\mathrm{I}^{(1)}(\epsilon)$ factorises yielding,

$$\mathcal{P}^{1\times1}_{oles,\,\mathcal{P}}(s,t,u) = -\left|\mathrm{I}^{(1)}_{\mathcal{P}}(\epsilon)\right|^2 \langle\mathcal{M}^{(0)}_{\mathcal{P}}|\mathcal{M}^{(0)}_{\mathcal{P}}\rangle + 2\mathrm{Re}\left\{\mathrm{I}^{(1)}_{\mathcal{P}}(\epsilon)^\dagger\langle\mathcal{M}^{(0)}_{\mathcal{P}}|\mathcal{M}^{(1)}_{\mathcal{P}}\rangle\right\} \tag{3.3}$$

where

$$\begin{aligned}\mathrm{I}^{(1)}_{q\bar{q}g\gamma}(\epsilon) = \frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)} \quad &\Bigg[-N\left(\frac{1}{\epsilon^2}+\frac{3}{4\epsilon}+\frac{\beta_0}{2N\epsilon}\right)\left\{\left(-\frac{\mu^2}{t}\right)^{\epsilon}+\left(-\frac{\mu^2}{u}\right)^{\epsilon}\right\}\\ &+\frac{1}{N}\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)\left(-\frac{\mu^2}{s}\right)^{\epsilon}\Bigg],\end{aligned} \tag{3.4}$$

$$\mathrm{I}^{(1)}_{q\bar{q}\gamma\gamma}(\epsilon) = -C_F\frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)\left(-\frac{\mu^2}{s}\right)^{\epsilon} \tag{3.5}$$

and

$$\mathrm{I}^{(1)}_{e^-e^+\gamma\gamma}(\epsilon) = -\frac{e^{\epsilon\gamma}}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}+\frac{\beta_0'}{\epsilon}\right)\left(-\frac{\mu^2}{s}\right)^{\epsilon}. \tag{3.6}$$

The generic form of the $\mathrm{I}^{(1)}(\epsilon)$ operator was found by Catani and Seymour [27] and it was derived for the general one-loop QCD amplitude by integrating the real radiation graphs of the same order in perturbation series in the one-particle unresolved limit. For the processes $q\bar{q}g\gamma$ and $q\bar{q}\gamma\gamma$ we consider only NNLO QCD corrections, therefore in Eq. (3.5) there are no contributions from soft or collinear photon emission at the same order in α_s. Note that on expanding the $\mathrm{I}^{(1)}(\epsilon)$ operator, imaginary parts are generated, the sign of which are fixed by the small imaginary part $+i0$ assigned to each Mandelstam variable. Terms that involve the hermitian conjugate of this operator are to be modified accordingly.

The renormalised interference of the tree and the one-loop amplitudes can be expressed in terms of the one-loop bubble graph (Bub) and the one-loop box integral in $D = 6 - 2\epsilon$ dimensions (Box^6), as follows

$$\langle \mathcal{M}^{(0)}_{q\bar{q}g\gamma} | \mathcal{M}^{(1)}_{q\bar{q}g\gamma} \rangle = N C_F \left[C_F f_1(s,t,u) - \frac{C_A}{2} \left\{ f_1(s,t,u) + f_2(s,t,u) \right\} \right. \\ \left. + (t \leftrightarrow u) \right] - \frac{\beta_0}{2\epsilon} \langle \mathcal{M}^{(0)}_{q\bar{q}g\gamma} | \mathcal{M}^{(0)}_{q\bar{q}g\gamma} \rangle, \tag{3.7}$$

$$\langle \mathcal{M}^{(0)}_{q\bar{q}\gamma\gamma} | \mathcal{M}^{(1)}_{q\bar{q}\gamma\gamma} \rangle = N C_F f_1(s,t,u) + (t \leftrightarrow u) \tag{3.8}$$

and

$$\langle \mathcal{M}^{(0)}_{e^-e^+\gamma\gamma} | \mathcal{M}^{(1)}_{e^-e^+\gamma\gamma} \rangle = f_1(s,t,u) + f_1(s,u,t) - \frac{\beta_0'}{\epsilon} \langle \mathcal{M}^{(0)}_{e^-e^+\gamma\gamma} | \mathcal{M}^{(0)}_{e^-e^+\gamma\gamma} \rangle, \tag{3.9}$$

where we have used

$$f_1(s,t,u) = -\frac{8(1-2\epsilon)\left[s^2+t^2-\epsilon(s^2+t^2+u^2)+\epsilon^2(t^2+s^2-st)-\epsilon^3 st\right]}{t}\mathrm{Box}^6(s,t) \\ +\frac{4(1-2\epsilon)\left[t^2+u^2+\epsilon(\epsilon-2)s^2\right]}{\epsilon u t}\mathrm{Bub}(s) \\ -\frac{4(1-\epsilon)\left[3s+t+\epsilon(5t+2u)-3\epsilon^2 s\right]}{t}\mathrm{Bub}(t) \tag{3.10}$$

and

$$f_2(s,t,u) = \frac{4s(1-2\epsilon)\left[t^2+u^2+\epsilon(ut-2t^2-2u^2)+\epsilon^2(s^2+tu)\right]}{ut}\mathrm{Box}^6(t,u) \\ -\frac{4(1-\epsilon)^2\left[2(t^2+u^2)-\epsilon(t-s)^2-\epsilon^2 us\right]}{\epsilon t u}\mathrm{Bub}(t). \tag{3.11}$$

These expressions are valid in all kinematic regions and for all orders in ϵ. However, to evaluate them in a particular region, the one-loop master integrals Bub and Box^6, must be expanded as a series in ϵ (see Appendix C).

The infrared poles of the interference of the tree and the two-loop amplitudes follow a generic formula developed by Catani [26], such that

$$\mathcal{P}^{0\times 2}_{oles,\,\mathcal{P}}(s,t,u) = 2\mathrm{Re}\left\{ \mathrm{I}^{(1)}_{\mathcal{P}}(\epsilon)\langle M^{(0)}_{\mathcal{P}} | M^{(1)}_{\mathcal{P}} \rangle + \mathrm{I}^{(2)}_{\mathcal{P}}(\epsilon)\langle M^{(0)}_{\mathcal{P}} | M^{(0)}_{\mathcal{P}} \rangle \right\} \tag{3.12}$$

where once again the simple colour structure of the processes under consideration allows the action of $\mathrm{I}^{(1)}(\epsilon)$ and $\mathrm{I}^{(2)}(\epsilon)$ to be factrorised. For the QCD processes $\mathcal{P} = q\bar{q}g\gamma,\ q\bar{q}\gamma\gamma$ we have

$$\mathrm{I}^{(2)}_{\mathcal{P}}(\epsilon) = -\frac{1}{2}\mathrm{I}^{(1)}_{\mathcal{P}}(\epsilon)\left(\mathrm{I}^{(1)}_{\mathcal{P}}(\epsilon) + \frac{2\beta_0}{\epsilon}\right) + \frac{e^{-\epsilon\gamma}\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{\beta_0}{\epsilon}+K\right)\mathrm{I}^{(1)}_{\mathcal{P}}(2\epsilon) \\ +\frac{e^{\epsilon\gamma}}{4\epsilon\Gamma(1-\epsilon)}H^{(2)}_{\mathcal{P}} \tag{3.13}$$

with

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R N_F. \tag{3.14}$$

and for the QED process $\mathcal{P} = e^- e^+ \gamma\gamma$ we have

$$\begin{aligned} \mathrm{I}^{(2)}_{e^-e^+\gamma\gamma}(\epsilon) = & -\frac{1}{2}\mathrm{I}^{(1)}_{e^-e^+\gamma\gamma}(\epsilon)\left(\mathrm{I}^{(1)}_{e^-e^+\gamma\gamma}(\epsilon) + \frac{2\beta_0'}{\epsilon}\right) + \frac{e^{-\epsilon\gamma}\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{\beta_0'}{\epsilon} + K'\right)\mathrm{I}^{(1)}_{e^-e^+\gamma\gamma}(2\epsilon) \\ & + \frac{e^{\epsilon\gamma}}{4\epsilon\Gamma(1-\epsilon)} H^{(2)}_{e^-e^+\gamma\gamma} \end{aligned} \tag{3.15}$$

with

$$K' = -\frac{10}{9} N_F'. \tag{3.16}$$

The process and renormalisation scheme dependent $H^{(2)}$ constants are related to the colour space operator $\mathrm{H}^{(2)}$. The full structure of $\mathrm{H}^{(2)}$ is not known at present, although it certainly contains non-trivial colour structures [20] that have been investigated in the case of gluon-gluon scattering. However, because of the projection by the tree-level amplitude and the summation over colours we are only sensitive to the trivial colour part of $\mathrm{H}^{(2)}$, and we find that the constants $H^{(2)}$ can be written as a sum of factors associated with each of the external legs. We derive them directly from the $1/\epsilon$ term of the explicit expansion in ϵ of the two-loop amplitudes. In conventional dimensional regularisation and the $\overline{\mathrm{MS}}$ scheme, we find that for the processes under consideration

$$H^{(2)}_{q\bar{q}g\gamma} = H^{(2)}_q + H^{(2)}_{\bar{q}} + H^{(2)}_g, \tag{3.17}$$

$$H^{(2)}_{q\bar{q}\gamma\gamma} = H^{(2)}_q + H^{(2)}_{\bar{q}}, \tag{3.18}$$

$$H^{(2)}_{e^-e^+\gamma\gamma} = H^{(2)}_{e-} + H^{(2)}_{e^+} + 2H^{(2)}_\gamma, \tag{3.19}$$

where for an external quark or an antiquark we find

$$\begin{aligned} H^{(2)}_q = H^{(2)}_{\bar{q}} = & \left(\frac{\pi^2}{2} - 6\,\zeta_3 - \frac{3}{8}\right) C_F^2 + \left(\frac{13}{2}\zeta_3 + \frac{245}{216} - \frac{23}{48}\pi^2\right) C_A C_F \\ & + \left(-\frac{25}{54} + \frac{\pi^2}{12}\right) T_R N_F C_F \end{aligned} \tag{3.20}$$

and for each external gluon

$$\begin{aligned} H^{(2)}_g = & \frac{20}{27} T_R^2 N_F^2 + T_R C_F N_F - \left(\frac{\pi^2}{36} + \frac{58}{27}\right) T_R N_F C_A \\ & + \left(\frac{\zeta_3}{2} + \frac{5}{12} + \frac{11}{144}\pi^2\right) C_A^2. \end{aligned} \tag{3.21}$$

These are the same factors as found in [26, 12, 13, 15, 16, 24]. Similarly, for each external electron or positron we find

$$H^{(2)}_{e^-} = H_{e^+} = \left(\frac{\pi^2}{2} - 6\zeta_3 - \frac{3}{8}\right) + \left(-\frac{25}{54} + \frac{\pi^2}{12}\right) N_F' \tag{3.22}$$

while for the external photon

$$H_{\gamma}^{(2)} = \frac{20}{27} {N'_F}^2 + N'_F. \tag{3.23}$$

It should be noted that $H_{e^-}^{(2)}$, $H_{e^+}^{(2)}$ and $H_{\gamma}^{(2)}$ can be obtained from the expressions for $H_q^{(2)}$,$H_{\bar{q}}^{(2)}$ and $H_g^{(2)}$ accordingly, by taking the limit $T_R \to 1$, $C_F \to 1$ and $C_A \to 0$. It should also be emphasized that contributions of $\mathcal{O}(\epsilon)$ to $H^{(2)}$ are undetermined at present.

4. Finite two-loop contributions

In this section, we give explicit expressions for the finite remainder of the two-loop contributions $\mathcal{F}_{inite,\mathcal{P}}^{0\times2}$ defined as,

$$\mathcal{F}_{inite,\mathcal{P}}^{0\times2}(s,t,u) = \mathcal{A}_{\mathcal{P}}^{\mathrm{NNLO}\,(2\times0)}(s,t,u) - \mathcal{P}_{oles,\,\mathcal{P}}^{0\times2}(s,t,u). \tag{4.1}$$

Note that the $\mathcal{P}_{oles,\,\mathcal{P}}^{0\times2}(s,t,u)$ is expanded through to $\mathcal{O}(1)$ and therefore contains finite as well as singular contributions. Using the standard polylogarithm identities [38], we express our results in terms of a basis set of logarithms and polylogarithms[2] with arguments x, $1-x$ and $(x-1)/x$, where

$$x = -\frac{t}{s}, \qquad y = -\frac{u}{s} = 1-x, \qquad z = -\frac{u}{t} = \frac{x-1}{x}. \tag{4.3}$$

In the physical region $s > 0$ and $t, u < 0$, our basis set of functions are all real. For convenience, we also introduce the following logarithms

$$X = \log\left(\frac{-t}{s}\right), \qquad Y = \log\left(\frac{-u}{s}\right), \qquad S = \log\left(\frac{s}{\mu^2}\right), \qquad U = \log\left(\frac{-u}{\mu^2}\right), \tag{4.4}$$

where μ is the renormalisation scale.

From Eqs. (2.15)-(2.29) we see that we need to evaluate the function $\mathcal{F}_{inite,\mathcal{P}}^{0\times2}$ with arguments (s,t,u), (u,t,s) and (t,u,s). Since t and u are both negative they can be interchanged leaving the polylogarithms and logarithms of our basis well defined and real. Therefore the crossing symmetry $(t \leftrightarrow u)$ can be performed trivially. It is a more involved operation obtain expressions valid under the exchange of s with either t or u, since the logarithms and polylogarithms may acquire imaginary parts and

[2]As usual, the polylogarithms $\mathrm{Li}_n(w)$ are defined by

$$\begin{aligned} \mathrm{Li}_n(w) &= \int_0^w \frac{dt}{t}\mathrm{Li}_{n-1}(t) \qquad \text{for } n = 2,3,4 \\ \mathrm{Li}_2(w) &= -\int_0^w \frac{dt}{t}\log(1-t). \end{aligned} \tag{4.2}$$

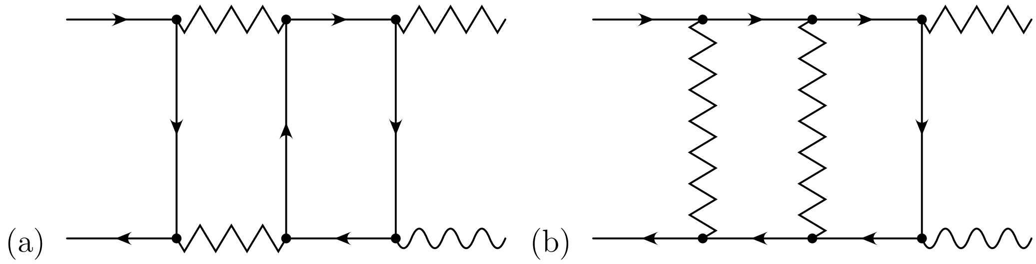

Figure 1: Feynman diagrams representative of (a) "light-by-light" scattering and (b) "abelian gluon" exchange. These graphs contribute to A_c and B_c respectively. Depending on the process, the zigzag lines represent either photons or gluons.

appropriate analytic continuations need to be considered. We therefore give explicit expressions for the "s-channel" function $\mathcal{F}^{0\times 2}_{inite,\mathcal{P}}(s,t,u)$ and the "u-channel" function $\mathcal{F}^{0\times 2}_{inite,\mathcal{P}}(u,t,s)$ that are directly valid in the physical region $s > 0$ and $u,t < 0$. The "t-channel", $\mathcal{F}^{0\times 2}_{inite,\mathcal{P}}(t,u,s)$ can be obtained from the $u-$channel expression by exchanging t and u. Note that here we define a "channel" according to the first argument of the function $\mathcal{F}^{0\times 2}_{inite,\mathcal{P}}$.

We organize our results according to the colour and flavour structure of the two-loop amplitude, so that for the three different processes, we have

$$\begin{aligned}\mathcal{F}^{0\times 2}_{inite,q\bar{q}g\gamma;\,c} &= 2NC_F\Bigg[\left(\sum_q Q_q\right)\left(2T_RC_F - \frac{3T_RC_A}{4}\right)A_c \\ &\quad + C_F^2B_c + C_A^2C_c + C_FC_AD_{1;c} + N_FC_FE_{1;c} + N_FC_AE_{2;c} + N_F^2F_{1;c}\Bigg],\end{aligned} \tag{4.5}$$

$$\mathcal{F}^{0\times 2}_{inite,q\bar{q}\gamma\gamma;\,c} = 2N\left[\left(\sum_q Q_q^2\right)T_RC_FA_c + C_F^2B_c + C_FC_AD_{2;c} + N_FC_FE_{3;c}\right], \tag{4.6}$$

and

$$\mathcal{F}^{0\times 2}_{inite,e^-e^+\gamma\gamma c} = 2\left[\left(\sum_f Q_f^4\right)A_c + B_c + N_F'E_{4;c} + {N_F'}^2F_{2;c}\right], \tag{4.7}$$

where the subscript $c = s,u$ denotes the channel. Q_q refers to the charge of each of the q active massless quark flavours for the QCD processes and equivalently Q_f denotes the charge of each of the f active massless fermion flavours participating in the $e^-e^+\gamma\gamma$ process. We see that the "light-by-light" contribution, A_c, due to internal fermion box graphs and the "abelian gluon" contribution, B_c, illustrated in Fig. 1, occur in all three processes.

The values of A_c, B_c, C_c, $D_{i;c}$, $E_{i;c}$ and $F_{i;c}$ are presented in Appendix A.1 for the s-channel and in Appendix A.2 for the u-channel.

5. Finite one-loop contributions

In this section, we give explicit expressions for the finite remainder of the one-loop self-interference contributions $\mathcal{F}^{1\times1}_{inite,\mathcal{P}}$ defined as,

$$\mathcal{F}^{1\times1}_{inite,\mathcal{P}}(s,t,u) = \mathcal{A}^{\mathrm{NNLO}\,(1\times1)}_{\mathcal{P}}(s,t,u) - \mathcal{P}^{1\times1}_{oles,\,\mathcal{P}}(s,t,u). \tag{5.1}$$

Note again that the $\mathcal{P}^{1\times1}_{oles,\,\mathcal{P}}(s,t,u)$ is expanded through to $\mathcal{O}(1)$ and therefore contains finite as well as singular contributions. In fact, as in [15, 14, 17], all polylogarithms are collected in the singular part ($\mathcal{P}^{1\times1}_{oles,\,\mathcal{P}}$), leaving finite remainders with only logarithms and constants such as π^2 and ζ_3. This is because in the expansion of the box integral in $D=6$ dimensions, (see Appendix C) the polylogarithms appear in the $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ terms. In order for these polylogarithms to contribute at $\mathcal{O}(1)$, they must be multiplied by an infrared singular term and are therefore contained in $\mathcal{P}^{1\times1}_{oles,\,\mathcal{P}}$. The finite part of the one-loop self-interference contribution due to the interference of one box graph with another only collects the purely logarithmic $\mathcal{O}(1)$ terms in each.

Specifically we find,

$$\begin{aligned}\mathcal{F}^{1\times1}_{inite,q\bar{q}g\gamma;\,c} &= NC_F\Big[C_F^2 G_{1;c} + C_A C_F G_{2;c} + C_A^2 G_{3;c} \\ &\qquad + N_F C_F X_{1;c} + N_F C_A X_{2;c} + N_F^2 X_{3;c}\Big], \end{aligned}\tag{5.2}$$

$$\mathcal{F}^{1\times1}_{inite,q\bar{q}\gamma\gamma;\,c} = NC_F^2 G_{1;c} \tag{5.3}$$

and

$$\mathcal{F}^{1\times1}_{inite,e^-e^+\gamma\gamma_c} = G_{1;c} + N_F' X_{4;c} + {N_F'}^2 X_{5;c}, \tag{5.4}$$

where the subscript $c = s, u$ denotes the channel.

The values of $G_{i;c}$ and $X_{i;c}$ are presented in Appendix B.1 for the s-channel and in Appendix B.2 for the u-channel.

6. Conclusions

In this paper we calculated the NNLO QED and QCD virtual corrections for a range of $2 \to 2$ massless scattering processes with external state containing photons, namely $q\bar{q}g\gamma$, $q\bar{q}\gamma\gamma$, $e^-e^+\gamma\gamma$ and the processes related by crossing symmetry and time reversal, in the high energy limit where the fermion masses can safely be ignored. We used conventional dimensional regularisation and to remove the UV divergences we renormalised using the $\overline{\rm MS}$ scheme. The renormalised amplitudes are infrared divergent and contain poles to $\mathcal{O}\left(1/\epsilon^4\right)$. In fact we were able to check our results by comparing the infrared structure of the calculated amplitudes with the prediction of Catani's formalism [26] for the infrared structure of generic one- and two-loop QCD amplitudes. Catani's method does not determine the $1/\epsilon$ poles exactly, but suggests that the undetermined non-logarithmic contribution $H^{(2)}$ can be extracted from a few basic two-loop amplitudes. From the amplitudes calculated in this paper together with Refs. [15, 16, 24] we find that, when summing over colours, each external-leg contributes independently to the leading order term in ϵ of $H^{(2)}$. The contributions of quarks, anti-quarks and gluons and their QED analogues are given in Section 3 for the $\overline{\rm MS}$ scheme.

The main result of our paper is the finite remainders of the NNLO virtual corrections after we subtract the pole structures of Catani's formalism expanded through to $\mathcal{O}\left(1\right)$. In Section 4 we gave the finite remainders of the interference of the tree and the two-loop amplitude and in Section 5 we presented the finite remainders of the self-interference of the one-loop amplitude for each of the processes under consideration. The results are expressed in terms of polylogarithms and logarithms that are real in the physical region. The renormalised amplitudes for the three processes $q\bar{q}g\gamma$, $q\bar{q}\gamma\gamma$ and $e^-e^+\gamma\gamma$, differ with respect to group invariants, the renormalisation procedure, and their flavour content.

The aim, of course, are more precise predictions for the basic scattering processes. Initial studies suggest that at NNLO an accuracy of a few percent is attainable for strong interaction processes. However, much work is needed to accomplish this goal. An important ingredient is a systematic procedure for analytically carrying through the cancellation of the infrared singularities present in the virtual contributions against the contributions from the one-loop $2 \to 3$ processes when one particle is unresolved and the tree-level $2 \to 4$ processes when two particles are unresolved. Such a method has not yet been established, although the single and double unresolved limits of the relevant matrix elements are well known [32, 33, 34, 35]. In fact, many of the analytic phase space integrations for the double unresolved and single unresolved loop contributions have already been studied in the context of $e^+e^- \to$ photon + jet at $\mathcal{O}(\alpha\alpha_s)$ [36] and Higgs production in hadron colliders [37]. A further complication may be due to initial state radiation where the three loop splitting functions [39, 40, 41, 42, 43] are needed to extract parton density functions [44] at an accuracy matching that of the hard scattering matrix element. Nevertheless, it is an important task to achieve more reliable theoretical calculations that can take advantage of the improving experimental data and make a better test of the underlying physics at short distances.

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A. Finite two-loop contributions

A.1 s-channel

In this section we list the finite contributions for one-and two-loop contributions in the s-channel as defined in Eqs. (4.5), (4.6), (4.7), (5.2), (5.3) and (5.4).

$$
\begin{aligned}
A_s = & \Bigg[128\,\mathrm{Li}_4(z) - 128\,\mathrm{Li}_4(x) + 128\,\mathrm{Li}_4(y) + \left(-\frac{64}{3} + 128\,Y\right)\mathrm{Li}_3(x) \\
& + \left(\frac{64}{3}\,X - \frac{64}{3}\,\pi^2\right)\mathrm{Li}_2(x) + \frac{16}{3}\,X^4 - \frac{64}{3}\,X^3\,Y + \left(-16 + \frac{32}{3}\,\pi^2 + 32\,Y^2\right)X^2 \\
& + \left(-\frac{64}{3}\,\pi^2\,Y + 48 + \frac{160}{9}\,\pi^2\right)X + \frac{64}{3}\,\zeta_3 + \frac{224}{45}\,\pi^4 - 128\,Y\,\zeta_3\Bigg]\frac{t}{u} \\
& + \Bigg[\frac{32}{3}\,\mathrm{Li}_3(x) - \frac{32}{3}\,\mathrm{Li}_3(y) + \left(-\frac{32}{3}\,X - \frac{32}{3}\,Y\right)\mathrm{Li}_2(x) \\
& + \left(-\frac{32}{3}\,Y^2 - \frac{80}{9}\,\pi^2 - \frac{64}{3}\right)X + \left(\frac{64}{3} + \frac{32}{3}\,\pi^2\right)Y\Bigg]\frac{t^2}{s^2} + 24\,X^2\,\frac{t^2}{u^2} \\
& + \Bigg[\frac{416}{3}\,\mathrm{Li}_3(x) + 64\,\mathrm{Li}_3(y)\,X - \frac{416}{3}\,\mathrm{Li}_2(x)\,X + \left(8\,Y^2 + 16\right)X^2 \\
& + \left(-\frac{8}{3}\,Y + \frac{80}{3} + \frac{112}{9}\,\pi^2 - 64\,\zeta_3 - 64\,Y^2\right)X - \frac{416}{3}\,\zeta_3 - \frac{148}{9}\,\pi^2 + \frac{44}{45}\,\pi^4\Bigg] \\
& + \Big\{t \leftrightarrow u\Big\}
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
B_s = & \Bigg(-112\,\mathrm{Li}_4(z) - 88\,\mathrm{Li}_4(y) + \left(-128\,Y + 48\,X - 64\right)\mathrm{Li}_3(x) \\
& + \left(-16\,Y - 16\,X + 12\right)\mathrm{Li}_3(y) + \left(12\,Y - 4\,Y^2 + 8\,X^2 - 8\,\pi^2 + 64\,X\right)\mathrm{Li}_2(x) \\
& + \frac{2}{3}\,X^4 + \frac{56}{3}\,X^3\,Y + \left(44\,Y - 4\,\pi^2 + 2 - 32\,Y^2\right)X^2 \\
& + \left(-4\,Y^3 - 8 - 32\,\zeta_3 - \frac{80}{3}\,\pi^2 + 6\,Y^2 + \frac{56}{3}\,\pi^2\,Y\right)X \\
& + Y^4 + 6\,Y^3 + \left(-\frac{10}{3}\,\pi^2 - 5\right)Y^2 + \left(-39 - 18\,\pi^2 + 144\,\zeta_3\right)Y \\
& + 3\,S + \frac{187}{4} - 4\,\pi^2\,S + \frac{4}{45}\,\pi^4 - 5\,\pi^2 - 20\,\zeta_3 + 48\,\zeta_3\,S\Bigg)\frac{t}{u} \\
& + \left(-12\,X^2 + \left(24\,Y + 24\right)X - 12\,Y^2 - 24\,Y - 12\,\pi^2\right)\frac{t^2}{s^2} + 8\,X^2\,\frac{t^2}{u^2} \\
& + \Bigg(-80\,\mathrm{Li}_4(y) + 32\,X\,\mathrm{Li}_3(x) + \left(-128\,X - 152\right)\mathrm{Li}_3(y) + 152\,\mathrm{Li}_2(x)\,X
\end{aligned}
$$

$$
\begin{aligned}
&+8\,Y^2\,\mathrm{Li}_2(y)+\Big(-16\,Y^2-24\Big)\,X^2+\Big(60\,Y^2+\Big(28+\frac{32}{3}\,\pi^2\Big)\,Y-58\Big)\,X\\
&+\frac{14}{3}\,Y^4+\frac{44}{3}\,Y^3+\frac{8}{3}\,Y^2\,\pi^2+\Big(96\,\zeta_3-\frac{32}{3}\,\pi^2\Big)\,Y+\frac{32}{45}\,\pi^4+16\,\zeta_3-\frac{86}{3}\,\pi^2-2\Big)\\
&+\Big\{t\leftrightarrow u\Big\} \qquad (A.2)
\end{aligned}
$$

$$
\begin{aligned}
C_s=&\Big[-20\,\mathrm{Li}_4(z)+28\,\mathrm{Li}_4(x)+\Big(-28\,Y-10\,X+\frac{1}{3}\Big)\,\mathrm{Li}_3(x)\\
&+\Big(6\,X^2+\Big(-\frac{1}{3}-4\,Y\Big)\,X-2\,Y^2+\frac{58}{3}\,Y+\frac{4}{3}\,\pi^2\Big)\,\mathrm{Li}_2(x)\\
&+\Big(\frac{58}{3}-12\,X-12\,Y\Big)\,\mathrm{Li}_3(y)-\frac{1}{6}\,X^4+\Big(\frac{10}{3}\,Y+\frac{13}{9}\Big)\,X^3\\
&+\Big(-9\,Y^2-\frac{1}{3}\,\pi^2+\frac{4}{9}+\frac{11}{2}\,S+13\,Y\Big)\,X^2\\
&+\Big(-\frac{8}{3}\,Y^3+\frac{50}{3}\,Y^2+\Big(\frac{17}{3}\,\pi^2-\frac{28}{3}+11\,S\Big)\,Y-\frac{563}{27}-\frac{233}{36}\,\pi^2+\frac{55}{3}\,S\Big)\,X\\
&+\Big(-\frac{2}{3}\,\pi^2+\frac{80}{9}\Big)\,Y^2+\Big(-\frac{284}{27}-\frac{299}{36}\,\pi^2+26\,\zeta_3+\frac{55}{3}\,S\Big)\,Y-\frac{209}{36}\,\pi^2\,S\\
&-2\,\zeta_3\,S+\frac{121}{12}\,S^2-13\,S-\frac{1142}{81}-\frac{197}{360}\,\pi^4+\frac{461}{36}\,\pi^2-\frac{55}{18}\,\zeta_3\Big]\,\frac{t}{u}\\
&+\Big[-\frac{5}{4}\,X^2+\Big(\frac{5}{2}+\frac{5}{2}\,Y\Big)\,X-\frac{5}{4}\,Y^2-\frac{5}{2}\,Y-\frac{5}{4}\,\pi^2\Big]\,\frac{t^2}{s^2}+\frac{1}{2}\,X^2\,\frac{t^2}{u^2}\\
&+\Big[24\,\mathrm{Li}_4(y)-20\,X\,\mathrm{Li}_3(x)+\Big(-40\,X-22\Big)\,\mathrm{Li}_3(y)+22\,\mathrm{Li}_2(x)\,X+8\,Y^2\,\mathrm{Li}_2(y)\\
&+\frac{4}{3}\,X^3\,Y+\Big(-6\,Y^2-\frac{575}{36}\Big)\,X^2+\Big(\frac{46}{3}\,Y^2+\Big(\frac{73}{12}+4\,\pi^2\Big)\,Y-\frac{637}{18}\Big)\,X\\
&+\frac{1}{3}\,Y^4+\frac{59}{9}\,Y^3+\Big(\frac{2}{3}\,\pi^2+11\,S\Big)\,Y^2+\Big(44\,\zeta_3-\frac{4}{9}\,\pi^2+11\,S\Big)\,Y\\
&-\frac{38}{45}\,\pi^4+\frac{77}{72}\,\pi^2+2\,\zeta_3\Big]+\Big\{t\leftrightarrow u\Big\} \qquad (A.3)
\end{aligned}
$$

$$
\begin{aligned}
D_{1;s}=&\Big[96\,\mathrm{Li}_4(z)-48\,\mathrm{Li}_4(x)+52\,\mathrm{Li}_4(y)+\Big(124\,Y-8\,X+46\Big)\,\mathrm{Li}_3(x)\\
&+\Big(-16\,X^2+\Big(-46+8\,Y\Big)\,X+6\,Y^2-30\,Y+\frac{4}{3}\,\pi^2\Big)\,\mathrm{Li}_2(x)\\
&+\Big(-30+28\,Y+36\,X\Big)\,\mathrm{Li}_3(y)+\frac{1}{2}\,X^4+\Big(-\frac{56}{3}\,Y-\frac{100}{9}\Big)\,X^3\\
&+\Big(-\frac{125}{3}\,Y+39\,Y^2+\frac{214}{9}+3\,\pi^2-22\,S\Big)\,X^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{14}{3}\,Y^3-\frac{73}{3}\,Y^2+\left(-24\,\pi^2+4\right)Y+\frac{155}{9}\,\pi^2+\frac{148}{3}\right)X\\
&-\frac{74}{9}\,Y^3+\left(\frac{10}{3}\,\pi^2-\frac{55}{9}-11\,S\right)Y^2+\left(-33\,S+\frac{136}{9}\,\pi^2-140\,\zeta_3+\frac{241}{3}\right)Y\\
&-\frac{43417}{324}+\frac{23}{6}\,\pi^2\,S-52\,\zeta_3\,S-\frac{173}{18}\,\pi^2+\frac{227}{180}\,\pi^4+\frac{1834}{27}\,S+\frac{515}{9}\,\zeta_3\Bigg]\,\frac{t}{u}\\
&+\left[14\,X^2+\left(-28-28\,Y\right)X+14\,Y^2+28\,Y+14\,\pi^2\right]\frac{t^2}{s^2}-5\,X^2\,\frac{t^2}{u^2}\\
&+\Bigg[-8\,\mathrm{Li}_4(y)+24\,X\,\mathrm{Li}_3(x)+\left(144\,X+120\right)\mathrm{Li}_3(y)-120\,\mathrm{Li}_2(x)\,X-20\,Y^2\,\mathrm{Li}_2(y)\\
&-\frac{8}{3}\,X^3\,Y+\left(20\,Y^2+\frac{472}{9}\right)X^2+\left(-\frac{182}{3}\,Y^2+\left(-\frac{40}{3}\,\pi^2-\frac{104}{3}\right)Y+\frac{898}{9}\right)X\\
&-3\,Y^4-\frac{184}{9}\,Y^3+\left(-22\,S-\frac{8}{3}\,\pi^2\right)Y^2+\left(-22\,S+\frac{56}{9}\,\pi^2-136\,\zeta_3\right)Y\\
&+\frac{4}{3}\,\pi^4+\frac{148}{9}\,\pi^2+2-12\,\zeta_3\Bigg]+\Big\{t\leftrightarrow u\Big\}
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
E_{1;s}=&\left[\frac{22}{9}\,X^3+\left(-\frac{76}{9}+4\,S+\frac{2}{3}\,Y\right)X^2+\left(\frac{1}{3}\,Y^2+Y+\frac{16}{9}\,\pi^2-\frac{7}{3}\right)X\right.\\
&+\frac{11}{9}\,Y^3+\left(\frac{7}{9}+2\,S\right)Y^2+\left(6\,S+\frac{8}{9}\,\pi^2-\frac{37}{3}\right)Y\\
&\left.+\frac{19}{9}\,\pi^2-\frac{328}{27}\,S+\frac{3401}{162}-\frac{2}{9}\,\zeta_3-\frac{1}{3}\,\pi^2\,S\right]\frac{t}{u}\\
&+\left[-\frac{46}{9}\,X^2+\left(\frac{2}{3}\,Y+\frac{2}{3}\,Y^2-\frac{76}{9}\right)X\right.\\
&\left.+\frac{22}{9}\,Y^3+4\,Y^2\,S+\left(\frac{16}{9}\,\pi^2+4\,S\right)Y+\frac{8}{9}\,\pi^2\right]+\Big\{t\leftrightarrow u\Big\}
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
E_{2;s}=&\left[-\frac{4}{3}\,\mathrm{Li}_3(x)-\frac{4}{3}\,\mathrm{Li}_3(y)+\left(-\frac{4}{3}\,Y+\frac{4}{3}\,X\right)\mathrm{Li}_2(x)\right.\\
&-\frac{11}{18}\,X^3+\left(-\frac{1}{2}\,Y+\frac{5}{6}-S\right)X^2\\
&+\left(-\frac{5}{3}\,Y^2+\left(\frac{10}{9}-2\,S\right)Y-\frac{1}{9}\,\pi^2+\frac{37}{9}-\frac{31}{6}\,S\right)X\\
&-\frac{41}{18}\,Y^2+\left(\frac{5}{9}\,\pi^2+\frac{43}{9}-\frac{31}{6}\,S\right)Y+\frac{65}{81}+\frac{19}{18}\,\pi^2\,S-\frac{11}{3}\,S^2\\
&\left.-\frac{13}{9}\,\zeta_3+\frac{206}{27}\,S-\frac{275}{108}\,\pi^2\right]\frac{t}{u}\\
&+\left[-X^2+\left(2\,Y+2\right)X-Y^2-2\,Y-\pi^2\right]\frac{t^2}{s^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{14}{9}\,X^2+\left(\frac{2}{3}\,Y-\frac{1}{3}\,Y^2+\frac{38}{9}\right)X-\frac{11}{9}\,Y^3-2\,Y^2\,S\right.\\
&\left.-\frac{17}{18}\,\pi^2+\left(-\frac{8}{9}\,\pi^2-2\,S\right)Y\right]+\Big\{t\leftrightarrow u\Big\} \qquad \text{(A.6)}
\end{aligned}
$$

$$
\begin{aligned}
F_{1;s}=&\left[\frac{5}{36}\,X^2+\left(-\frac{10}{27}+\frac{1}{3}\,S+\frac{1}{18}\,Y\right)X+\frac{5}{36}\,Y^2+\left(-\frac{10}{27}+\frac{1}{3}\,S\right)Y\right.\\
&\left.+\frac{1}{3}\,S^2+\frac{1}{54}\,\pi^2-\frac{20}{27}\,S\right]\frac{t}{u}+\Big\{t\leftrightarrow u\Big\} \qquad \text{(A.7)}
\end{aligned}
$$

$$
\begin{aligned}
D_{2;s}=&\Bigg[48\,\mathrm{Li}_4(z)-16\,\mathrm{Li}_4(x)+24\,\mathrm{Li}_4(y)+\Big(56\,Y-8\,X+20\Big)\,\mathrm{Li}_3(x)\\
&+\Big(8\,X-12+16\,Y\Big)\,\mathrm{Li}_3(y)+\left(\frac{16}{3}\,\pi^2-20\,X-12\,Y-8\,X^2+4\,Y^2\right)\mathrm{Li}_2(x)\\
&+\frac{1}{3}\,X^4+\left(-8\,Y-\frac{70}{9}\right)X^3+\left(-4\,\pi^2+\frac{286}{9}-16\,Y+14\,Y^2-\frac{44}{3}\,S\right)X^2\\
&+\left(-\frac{22}{9}\,\pi^2+4\,Y^3-8\,\pi^2\,Y-6\,Y^2\right)X-\frac{44}{9}\,Y^3+\left(-\frac{4}{3}\,\pi^2+\frac{35}{9}-\frac{22}{3}\,S\right)Y^2\\
&+\left(57-\frac{26}{9}\,\pi^2-72\,\zeta_3-22\,S\right)Y+\frac{479}{9}\,\zeta_3+\frac{19}{60}\,\pi^4-52\,\zeta_3\,S+\frac{1141}{27}\,S-\frac{215}{18}\,\pi^2\\
&-\frac{43417}{324}+\frac{23}{6}\,\pi^2\,S\Bigg]\frac{t}{u}\\
&+\left[6\,X^2+\Big(-12-12\,Y\Big)X+6\,Y^2+12\,Y+6\,\pi^2\right]\frac{t^2}{s^2}-6\,X^2\,\frac{t^2}{u^2}\\
&+\Bigg[16\,\mathrm{Li}_4(y)+48\,\mathrm{Li}_3(x)\,Y+64\,\mathrm{Li}_3(y)-8\,Y^2\,\mathrm{Li}_2(y)-64\,\mathrm{Li}_2(x)\,X\\
&-\frac{4}{3}\,X^4+\left(-\frac{20}{3}\,\pi^2+6\,Y^2\right)X^2+\left(-24\,Y^2+\left(-\frac{16}{3}\,\pi^2-14\right)Y-\frac{148}{9}\,\pi^2\right)X\\
&-\frac{112}{9}\,Y^3+\left(-\frac{44}{3}\,S+\frac{298}{9}\right)Y^2+\left(\frac{538}{9}-48\,\zeta_3-\frac{44}{3}\,S\right)Y-8\,\zeta_3-\frac{1}{3}\,\pi^4+\frac{61}{9}\,\pi^2\Bigg]\\
&+\Big\{t\leftrightarrow u\Big\} \qquad \text{(A.8)}
\end{aligned}
$$

$$
\begin{aligned}
E_{3;s}=&\left[\frac{16}{9}\,X^3+\left(-\frac{76}{9}+\frac{8}{3}\,S\right)X^2+\frac{16}{9}\,\pi^2\,X\right.\\
&+\frac{8}{9}\,Y^3+\left(\frac{4}{3}\,S-\frac{2}{9}\right)Y^2+\left(\frac{8}{9}\,\pi^2+4\,S-10\right)Y\\
&\left.-\frac{1}{3}\,\pi^2\,S-\frac{202}{27}\,S+\frac{19}{9}\,\pi^2-\frac{2}{9}\,\zeta_3+\frac{3401}{162}\right]\frac{t}{u}\\
&+\left[\frac{16}{9}\,\pi^2\,X+\frac{16}{9}\,Y^3+\left(\frac{8}{3}\,S-\frac{52}{9}\right)Y^2\right.
\end{aligned}
$$

$$+\left(-\frac{76}{9}+\frac{8}{3}\,S\right)Y+\frac{8}{9}\,\pi^2\Bigg]+\Big\{t\leftrightarrow u\Big\} \tag{A.9}$$

$$\begin{aligned}
E_{4;s} = &\Bigg[\frac{16}{9}\,Y^3-\frac{4}{9}\,Y^2+\frac{16}{3}\,Y^2\,S-20\,Y+\frac{64}{9}\,\pi^2\,Y+16\,Y\,S\\
&+\frac{32}{9}\,X^3-\frac{152}{9}\,X^2+\frac{32}{3}\,X^2\,S+\frac{128}{9}\,\pi^2\,X\\
&-\frac{2}{3}\,\pi^2\,S+\frac{110}{9}\,\pi^2+\frac{3401}{81}-\frac{908}{27}\,S-\frac{4}{9}\,\zeta_3\Bigg]\,\frac{t}{u}\\
&+\Bigg[\frac{32}{9}\,X^3-\frac{104}{9}\,X^2+\frac{32}{3}\,X^2\,S-\frac{152}{9}\,X+\frac{32}{3}\,X\,S\\
&+\frac{128}{9}\,\pi^2\,X+\frac{64}{9}\,\pi^2\Bigg]+\Big\{t\leftrightarrow u\Big\}
\end{aligned} \tag{A.10}$$

$$F_{2;s} = \Bigg[-\frac{92}{27}\,\pi^2+\frac{32}{9}\,S^2-\frac{160}{27}\,S\Bigg]\,\frac{t}{u}+\Big\{t\leftrightarrow u\Big\} \tag{A.11}$$

A.2 u-channel

In this section we list the finite contributions for one-and two-loop contributions in the u-channel as defined in Eqs. (4.5), (4.6), (4.7), (5.2), (5.3) and (5.4).

$$\begin{aligned}
A_u = &\Bigg[24\,\pi^2-48\,X\,Y+24\,Y^2+24\,X^2\Bigg]\,\frac{t^2}{s^2}+24\,Y^2\,\frac{s^2}{t^2}\\
&+\Bigg[\left(64\,Y+\frac{32}{3}\right)\mathrm{Li}_3(x)+64\,\mathrm{Li}_3(y)\,X-\frac{32}{3}\,\mathrm{Li}_2(x)\,X+\left(-8+16\,Y^2\right)X^2\\
&+\left(\left(-\frac{64}{3}\,\pi^2+16\right)Y-\frac{16}{9}\,\pi^2+24-64\,\zeta_3+\frac{32}{3}\,Y^3-\frac{32}{3}\,Y^2\right)X+\frac{64}{9}\,Y^3\\
&+\left(-48+64\,\zeta_3+\frac{32}{9}\,\pi^2\right)Y+\left(-\frac{16}{3}\,\pi^2-16\right)Y^2-\frac{16}{3}\,Y^4-8\,\pi^2+\frac{32}{3}\,\zeta_3\\
&+\frac{88}{45}\,\pi^4\Bigg]\,\frac{t^2+s^2}{st}\\
&+\Bigg[-128\,\mathrm{Li}_4(x)-128\,\mathrm{Li}_4(y)+128\,\mathrm{Li}_4(z)+\left(64\,Y+\frac{32}{3}\right)\mathrm{Li}_3(x)\\
&+\left(128\,Y+\frac{64}{3}-64\,X\right)\mathrm{Li}_3(y)+\left(-\frac{32}{3}\,X+\frac{64}{3}\,Y+\frac{64}{3}\,\pi^2\right)\mathrm{Li}_2(x)\\
&+\frac{16}{3}\,X^4-\frac{64}{3}\,X^3\,Y+\left(16\,Y^2-8+\frac{32}{3}\,\pi^2\right)X^2\\
&+\left(\frac{32}{3}\,Y^2+\frac{32}{3}\,Y^3+64\,\zeta_3-\frac{16}{9}\,\pi^2+24+16\,Y\right)X+\frac{88}{15}\,\pi^4-\frac{32}{3}\,\zeta_3\\
&+\left(-\frac{32}{9}\,\pi^2-64\,\zeta_3\right)Y+16\,Y^2\,\pi^2-8\,\pi^2\Bigg]\,\frac{t^2-s^2}{st}
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg[-\frac{32}{3}\,\mathrm{Li}_3(x)-\frac{64}{3}\,\mathrm{Li}_3(y)+\left(\frac{32}{3}\,X-\frac{64}{3}\,Y\right)\mathrm{Li}_2(x)\\
&+\left(-\frac{32}{3}\,Y^2-\frac{64}{3}+\frac{16}{9}\,\pi^2\right)X+\frac{32}{9}\,\pi^2\,Y+\frac{32}{3}\,\zeta_3\Bigg]\,\frac{t^2-s^2}{u^2}\\
&+\Bigg[\left(64\,Y-\frac{416}{3}\right)\mathrm{Li}_3(x)+64\,\mathrm{Li}_3(y)\,X+\frac{416}{3}\,\mathrm{Li}_2(x)\,X+\left(64\,Y+16+16\,Y^2\right)X^2\\
&+\left(-\frac{160}{3}\,Y^2+\frac{32}{3}\,Y^3+\left(-\frac{80}{3}-\frac{64}{3}\,\pi^2\right)Y+\frac{208}{9}\,\pi^2-64\,\zeta_3+\frac{80}{3}\right)X+\frac{320}{9}\,Y^3\\
&+\left(-\frac{160}{3}+\frac{160}{9}\,\pi^2+64\,\zeta_3\right)Y+\left(-\frac{16}{3}\,\pi^2+\frac{80}{3}\right)Y^2-\frac{16}{3}\,Y^4+\frac{88}{45}\,\pi^4-\frac{200}{9}\,\pi^2\\
&-\frac{416}{3}\,\zeta_3\Bigg] \qquad \text{(A.12)}
\end{aligned}
$$

$$
\begin{aligned}
B_u =\; & -12\,X^2\,\frac{t^2+s^2}{u^2}+24\,X\,\frac{t^2-s^2}{u^2}\\
&+8\,Y^2\,\frac{s^2}{t^2}+\left[-16\,X\,Y+8\,Y^2+8\,\pi^2+8\,X^2\right]\frac{t^2}{s^2}\\
&+\Bigg[44\,\mathrm{Li}_4(z)+44\,\mathrm{Li}_4(y)+\left(-16\,X-56\,Y+26\right)\mathrm{Li}_3(x)-88\,\mathrm{Li}_3(y)\,X\\
&+\left(-6\,X^2+\left(12\,Y-26\right)X-6\,\pi^2\right)\mathrm{Li}_2(x)+5\,X^4+\left(-20\,Y+3\right)X^3\\
&+\left(-\frac{3}{2}+\frac{10}{3}\,\pi^2-28\,Y+Y^2\right)X^2+\Bigg(-\frac{28}{3}\,Y^3+40\,Y^2+\left(\frac{20}{3}\,\pi^2+3\right)Y\\
&-\frac{1}{3}\,\pi^2-\frac{47}{2}+72\,\zeta_3\Bigg)X+\frac{14}{3}\,Y^4-\frac{80}{3}\,Y^3+\left(-3-\frac{10}{3}\,\pi^2\right)Y^2\\
&+\left(47-56\,\zeta_3-\frac{28}{3}\,\pi^2\right)Y-\frac{13}{2}\,\pi^2-46\,\zeta_3-4\,\pi^2\,U-\frac{28}{9}\,\pi^4\\
&+3\,U+48\,\zeta_3\,U+\frac{187}{4}\Bigg]\,\frac{t^2+s^2}{st}\\
&+\Bigg[-44\,\mathrm{Li}_4(z)+44\,\mathrm{Li}_4(y)+112\,\mathrm{Li}_4(x)\\
&+\left(-32\,X+38-24\,Y\right)\mathrm{Li}_3(x)+\left(24\,X-48\,Y+76\right)\mathrm{Li}_3(y)\\
&+\left(-2\,X^2+\left(-38+4\,Y\right)X+76\,Y-4\,Y^2+6\,\pi^2\right)\mathrm{Li}_2(x)\\
&+\frac{1}{3}\,X^4+\left(-\frac{4}{3}\,Y-3\right)X^3+\left(-6\,\pi^2-Y^2+\frac{7}{2}-16\,Y\right)X^2\\
&+\left(-8\,Y^3+54\,Y^2+\left(12\,\pi^2-7\right)Y-\frac{7}{3}\,\pi^2-56\,\zeta_3+\frac{31}{2}\right)X
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{4}{3}\,Y^2\,\pi^2 + \left(24\,\zeta_3 - \frac{86}{3}\,\pi^2\right) Y - \frac{206}{45}\,\pi^4 - 38\,\zeta_3 + \frac{7}{2}\,\pi^2\Bigg]\,\frac{t^2-s^2}{st} \\
& +\Bigg[80\,\mathrm{Li}_4(z) + 80\,\mathrm{Li}_4(y) + \left(-96\,Y - 32\,X + 152\right)\mathrm{Li}_3(x) - 160\,\mathrm{Li}_3(y)\,X \\
& +\left(-8\,X^2 + \left(-152 + 16\,Y\right) X - 8\,\pi^2\right)\mathrm{Li}_2(x) + 8\,X^4 + \left(\frac{44}{3} - 32\,Y\right) X^3 \\
& +\left(\frac{16}{3}\,\pi^2 - 4\,Y^2 - 104\,Y - 24\right) X^2 + \Bigg(-16\,Y^3 + 72\,Y^2 + \left(16\,\pi^2 - 8\right) Y \\
& +\frac{4}{3}\,\pi^2 + 128\,\zeta_3 - 58\Bigg) X + 8\,Y^4 - 48\,Y^3 + \left(8 - \frac{8}{3}\,\pi^2\right) Y^2 \\
& +\left(116 - \frac{56}{3}\,\pi^2 - 96\,\zeta_3\right) Y - 4 - \frac{76}{3}\,\pi^2 - 120\,\zeta_3 - \frac{24}{5}\,\pi^4\Bigg]
\end{aligned}
\tag{A.13}
$$

$$
\begin{aligned}
C_u = {} & \left[-X\,Y + \frac{1}{2}\,\pi^2 + \frac{1}{2}\,Y^2 + \frac{1}{2}\,X^2\right]\frac{t^2}{s^2} + \frac{1}{2}\,Y^2\,\frac{s^2}{t^2} \\
& -\frac{5}{4}\,X^2\,\frac{t^2+s^2}{u^2} + \frac{5}{2}\,X\,\frac{t^2-s^2}{u^2} \\
& +\Bigg[-14\,\mathrm{Li}_4(y) - 14\,\mathrm{Li}_4(z) + \left(-\frac{59}{6} + 11\,X - 31\,Y\right)\mathrm{Li}_3(x) - 9\,\mathrm{Li}_3(y)\,X \\
& +\left(-4\,X^2 + \left(\frac{59}{6} + 8\,Y\right) X - 4\,\pi^2\right)\mathrm{Li}_2(x) - \frac{1}{4}\,X^4 + \left(\frac{4}{3}\,Y + \frac{13}{18}\right) X^3 \\
& +\left(-\frac{7}{2}\,Y^2 - \frac{22}{3}\,Y - \frac{5}{2}\,\pi^2 + \frac{11}{4}\,U + \frac{14}{3}\right) X^2 \\
& +\left(-\frac{4}{3}\,Y^3 + \frac{55}{2}\,Y^2 + \left(\frac{35}{6}\,\pi^2 - \frac{33}{2}\,U\right) Y + 2\,\zeta_3 + \frac{55}{3}\,U - \frac{847}{54} + \frac{1}{18}\,\pi^2\right) X \\
& +\frac{2}{3}\,Y^4 - \frac{55}{3}\,Y^3 + \left(-\frac{1}{2}\,\pi^2 + \frac{33}{2}\,U\right) Y^2 + \left(\frac{847}{27} - \frac{28}{9}\,\pi^2 + 5\,\zeta_3 - \frac{110}{3}\,U\right) Y \\
& -\frac{1142}{81} + \frac{61}{9}\,\zeta_3 - 13\,U - \frac{166}{9}\,\pi^2 + \frac{47}{360}\,\pi^4 - 2\,\zeta_3\,U + \frac{143}{18}\,\pi^2\,U + \frac{121}{12}\,U^2\Bigg]\,\frac{t^2+s^2}{st} \\
& +\Bigg[20\,\mathrm{Li}_4(x) + 14\,\mathrm{Li}_4(y) - 14\,\mathrm{Li}_4(z) + \left(-7\,Y + \frac{19}{2} - X\right)\mathrm{Li}_3(x) \\
& +\left(-14\,Y + 7\,X + 19\right)\mathrm{Li}_3(y) + \left(-2\,X^2 - \frac{19}{2}\,X + \frac{2}{3}\,\pi^2 + 19\,Y\right)\mathrm{Li}_2(x) \\
& -\frac{1}{4}\,X^4 + \left(\frac{2}{3}\,Y + \frac{13}{18}\right) X^3 + \left(-\frac{3}{2}\,Y^2 - \frac{38}{9} - \frac{3}{2}\,\pi^2 - 10\,Y + \frac{11}{4}\,U\right) X^2 \\
& +\left(-\frac{4}{3}\,Y^3 + \frac{39}{2}\,Y^2 + \left(\frac{76}{9} + \frac{13}{6}\,\pi^2 - \frac{11}{2}\,U\right) Y + \frac{77}{36}\,\pi^2 - \frac{31}{6} - 12\,\zeta_3\right) X \\
& -\frac{3}{2}\,Y^2\,\pi^2 + \left(7\,\zeta_3 - \frac{79}{6}\,\pi^2\right) Y - \frac{19}{2}\,\zeta_3 - \frac{38}{9}\,\pi^2 - \frac{5}{6}\,\pi^4 + \frac{11}{4}\,\pi^2\,U\Bigg]\,\frac{t^2-s^2}{st}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[-24\,\mathrm{Li}_4(y)-24\,\mathrm{Li}_4(z)+\Bigg(22-60\,Y+20\,X\Bigg)\,\mathrm{Li}_3(x)-20\,\mathrm{Li}_3(y)\,X\\
&+\Bigg(-8\,X^2+\Bigg(-22+16\,Y\Bigg)\,X-8\,\pi^2\Bigg)\,\mathrm{Li}_2(x)-\frac{2}{3}\,X^4+\Bigg(\frac{4}{3}\,Y+\frac{59}{9}\Bigg)\,X^3\\
&+\Bigg(-35\,Y-\frac{4}{3}\,\pi^2-\frac{575}{36}-6\,Y^2+11\,U\Bigg)\,X^2\\
&+\Bigg(-\frac{8}{3}\,Y^3+\frac{131}{3}\,Y^2+\Bigg(\frac{26}{3}\,\pi^2-22\,U+\frac{178}{9}\Bigg)\,Y+11\,U-\frac{637}{18}+24\,\zeta_3+\frac{53}{9}\,\pi^2\Bigg)\,X\\
&+\frac{4}{3}\,Y^4-\frac{262}{9}\,Y^3+\Bigg(-\frac{178}{9}+\frac{2}{3}\,\pi^2+22\,U\Bigg)\,Y^2+\Bigg(\frac{637}{9}-28\,\zeta_3-\frac{67}{9}\,\pi^2-22\,U\Bigg)\,Y\\
&-18\,\zeta_3+\frac{2}{45}\,\pi^4+11\,\pi^2\,U-\frac{71}{3}\,\pi^2\Bigg] \qquad\qquad (\mathrm{A.14})
\end{aligned}
$$

$$
\begin{aligned}
D_{1;u}=&\Bigg[10\,X\,Y-5\,\pi^2-5\,Y^2-5\,X^2\Bigg]\,\frac{t^2}{s^2}-5\,Y^2\,\frac{s^2}{t^2}\\
&+14\,X^2\,\frac{t^2+s^2}{u^2}-28\,X\,\frac{t^2-s^2}{u^2}\\
&+\Bigg[-2\,\mathrm{Li}_4(y)-2\,\mathrm{Li}_4(z)+\Bigg(-8-10\,X+90\,Y\Bigg)\,\mathrm{Li}_3(x)+70\,\mathrm{Li}_3(y)\,X\\
&+\Bigg(11\,X^2+\Bigg(8-22\,Y\Bigg)\,X+11\,\pi^2\Bigg)\,\mathrm{Li}_2(x)-\frac{11}{6}\,X^4+\Bigg(\frac{28}{3}\,Y-\frac{29}{3}\Bigg)\,X^3\\
&+\Bigg(-\frac{33}{2}\,U+\frac{53}{6}+47\,Y+\frac{13}{2}\,Y^2\Bigg)\,X^2+\Bigg(\frac{22}{3}\,Y^3-75\,Y^2+\Bigg(-\frac{65}{3}-15\,\pi^2+33\,U\Bigg)\,Y\\
&+\frac{389}{6}-60\,\zeta_3-\frac{31}{3}\,\pi^2-\frac{33}{2}\,U\Bigg)\,X-\frac{11}{3}\,Y^4+50\,Y^3+\Bigg(\frac{8}{3}\,\pi^2+\frac{65}{3}-33\,U\Bigg)\,Y^2\\
&+\Bigg(50\,\zeta_3+\frac{29}{3}\,\pi^2-\frac{389}{3}+33\,U\Bigg)\,Y+\frac{587}{9}\,\zeta_3+\frac{326}{9}\,\pi^2-52\,\zeta_3\,U+\frac{61}{36}\,\pi^4\\
&+\frac{1834}{27}\,U-\frac{43417}{324}-\frac{38}{3}\,\pi^2\,U\Bigg]\,\frac{t^2+s^2}{st}\\
&+\Bigg[-96\,\mathrm{Li}_4(x)-50\,\mathrm{Li}_4(y)+50\,\mathrm{Li}_4(z)+\Bigg(18\,X+26\,Y-38\Bigg)\,\mathrm{Li}_3(x)\\
&+\Bigg(52\,Y-26\,X-76\Bigg)\,\mathrm{Li}_3(y)+\Bigg(5\,X^2+\Bigg(-2\,Y+38\Bigg)\,X-76\,Y+2\,Y^2-\frac{13}{3}\,\pi^2\Bigg)\,\mathrm{Li}_2(x)\\
&+\frac{1}{3}\,X^4+\Bigg(-\frac{2}{3}\,Y-\frac{13}{9}\Bigg)\,X^3+\Bigg(\frac{269}{18}+6\,\pi^2-\frac{11}{2}\,U+28\,Y+\frac{7}{2}\,Y^2\Bigg)\,X^2\\
&+\Bigg(\frac{20}{3}\,Y^3-66\,Y^2+\Bigg(-\frac{269}{9}-\frac{31}{3}\,\pi^2+11\,U\Bigg)\,Y+\frac{8}{9}\,\pi^2+52\,\zeta_3+\frac{33}{2}\,U-\frac{31}{2}\Bigg)\,X\\
&+\frac{11}{3}\,Y^2\,\pi^2+\Bigg(-26\,\zeta_3+\frac{122}{3}\,\pi^2\Bigg)\,Y+38\,\zeta_3+\frac{178}{45}\,\pi^4+\frac{269}{18}\,\pi^2-\frac{11}{2}\,\pi^2\,U\Bigg]\,\frac{t^2-s^2}{st}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[8\,\mathrm{Li}_4(y)+8\,\mathrm{Li}_4(z)+\Bigg(-120+168\,Y-24\,X\Bigg)\,\mathrm{Li}_3(x)+120\,\mathrm{Li}_3(y)\,X\\
&+\Bigg(20\,X^2+\Bigg(120-40\,Y\Bigg)\,X+20\,\pi^2\Bigg)\,\mathrm{Li}_2(x)-\frac{8}{3}\,X^4+\Bigg(\frac{40}{3}\,Y-\frac{184}{9}\Bigg)\,X^3\\
&+\Bigg(\frac{472}{9}+14\,Y^2+122\,Y-22\,U\Bigg)\,X^2+\Bigg(\frac{40}{3}\,Y^3-\frac{370}{3}\,Y^2+\Bigg(-\frac{76}{3}\,\pi^2+44\,U-\frac{320}{9}\Bigg)\,Y\\
&-\frac{112}{9}\,\pi^2+\frac{898}{9}-112\,\zeta_3-22\,U\Bigg)\,X-\frac{20}{3}\,Y^4+\frac{740}{9}\,Y^3+\Bigg(\frac{320}{9}-44\,U\Bigg)\,Y^2\\
&+\Bigg(44\,U+\frac{218}{9}\,\pi^2+104\,\zeta_3-\frac{1796}{9}\Bigg)\,Y+96\,\zeta_3+\frac{104}{45}\,\pi^4+4-22\,\pi^2\,U+60\,\pi^2\Bigg]
\end{aligned}
\tag{A.15}
$$

$$
\begin{aligned}
E_{1;u}=&\Bigg[\frac{11}{6}\,X^3+\Bigg(3\,U-\frac{23}{6}-6\,Y\Bigg)\,X^2+\Bigg(7\,Y^2+\Bigg(-6\,U+\frac{20}{3}\Bigg)\,Y+\frac{11}{6}\,\pi^2-\frac{22}{3}+3\,U\Bigg)\,X\\
&-\frac{14}{3}\,Y^3+\Bigg(-\frac{20}{3}+6\,U\Bigg)\,Y^2+\Bigg(-6\,U+\frac{44}{3}+\frac{4}{3}\,\pi^2\Bigg)\,Y\\
&-\frac{2}{9}\,\zeta_3+\frac{8}{3}\,\pi^2\,U-\frac{121}{18}\,\pi^2+\frac{3401}{162}-\frac{328}{27}\,U\Bigg]\,\frac{t^2+s^2}{st}\\
&+\Bigg[\frac{11}{18}\,X^3+\Bigg(-\frac{83}{18}+U-2\,Y\Bigg)\,X^2+\Bigg(2\,Y^2+\Bigg(\frac{83}{9}-2\,U\Bigg)\,Y-3\,U+5+\frac{11}{18}\,\pi^2\Bigg)\,X\\
&-2\,\pi^2\,Y+\frac{1}{18}\,\pi^2\,\Bigg(-83+18\,U\Bigg)\Bigg]\,\frac{t^2-s^2}{st}\\
&+\Bigg[\frac{22}{9}\,X^3+\Bigg(-\frac{46}{9}+4\,U-8\,Y\Bigg)\,X^2+\Bigg(\frac{28}{3}\,Y^2+\Bigg(\frac{80}{9}-8\,U\Bigg)\,Y+\frac{22}{9}\,\pi^2-\frac{76}{9}+4\,U\Bigg)\,X\\
&-\frac{56}{9}\,Y^3+\Bigg(-\frac{80}{9}+8\,U\Bigg)\,Y^2+\Bigg(\frac{152}{9}-8\,U+\frac{16}{9}\,\pi^2\Bigg)\,Y+2\,\pi^2\,\Bigg(-5+2\,U\Bigg)\Bigg]
\end{aligned}
\tag{A.16}
$$

$$
\begin{aligned}
E_{2;u}=&\Bigg[\frac{4}{3}\,\mathrm{Li}_3(x)-\frac{4}{3}\,\mathrm{Li}_2(x)\,X-\frac{11}{36}\,X^3+\Bigg(\frac{4}{3}\,Y-\frac{13}{18}-\frac{1}{2}\,U\Bigg)\,X^2\\
&+\Bigg(-\frac{7}{2}\,Y^2+\Bigg(\frac{1}{3}+3\,U\Bigg)\,Y+\frac{1}{36}\,\pi^2-\frac{31}{6}\,U+\frac{40}{9}\Bigg)\,X\\
&+\frac{7}{3}\,Y^3+\Bigg(-\frac{1}{3}-3\,U\Bigg)\,Y^2+\Bigg(\frac{31}{3}\,U-\frac{5}{9}\,\pi^2-\frac{80}{9}\Bigg)\,Y\\
&-\frac{25}{9}\,\zeta_3-\frac{13}{9}\,\pi^2\,U-\frac{11}{3}\,U^2+\frac{206}{27}\,U+\frac{65}{81}+\frac{487}{108}\,\pi^2\Bigg]\,\frac{t^2+s^2}{st}\\
&+\Bigg[-\frac{11}{36}\,X^3+\Bigg(\frac{14}{9}-\frac{1}{2}\,U+Y\Bigg)\,X^2+\Bigg(-Y^2+\Bigg(-\frac{28}{9}+U\Bigg)\,Y-\frac{11}{36}\,\pi^2-\frac{1}{3}\Bigg)\,X\\
&+\pi^2\,Y-\frac{1}{18}\,\pi^2\,\Bigg(-28+9\,U\Bigg)\Bigg]\,\frac{t^2-s^2}{st}\\
&-X^2\,\frac{t^2+s^2}{u^2}+2\,X\,\frac{t^2-s^2}{u^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[-\frac{11}{9}X^3+\left(\frac{14}{9}-2U+4Y\right)X^2\\
&+\left(-\frac{14}{3}Y^2+\left(-\frac{40}{9}+4U\right)Y+\frac{38}{9}-2U-\frac{11}{9}\pi^2\right)X+\left(-\frac{76}{9}+4U-\frac{8}{9}\pi^2\right)Y\\
&+\frac{28}{9}Y^3+\left(\frac{40}{9}-4U\right)Y^2-\pi^2\left(-5+2U\right)\Bigg] \qquad \text{(A.17)}\\
F_{1;u} =& \Bigg[\frac{5}{36}X^2+\left(\frac{1}{3}U-\frac{10}{27}-\frac{1}{3}Y\right)X+\frac{1}{3}Y^2+\left(\frac{20}{27}-\frac{2}{3}U\right)Y\\
&-\frac{20}{27}U-\frac{13}{108}\pi^2+\frac{1}{3}U^2\Bigg]\frac{t^2+s^2}{st} \qquad \text{(A.18)}\\
D_{2;u} =& \Bigg[-6\pi^2-6Y^2+12XY-6X^2\Bigg]\frac{t^2}{s^2}-6Y^2\frac{s^2}{t^2}\\
&+6X^2\frac{t^2+s^2}{u^2}-12X\frac{t^2-s^2}{u^2}\\
&+\Bigg[-4\,\mathrm{Li}_4(y)-4\,\mathrm{Li}_4(z)+\left(-4-4X+36Y\right)\mathrm{Li}_3(x)+28\,\mathrm{Li}_3(y)X\\
&+\left(6X^2-12XY+6\pi^2+4X\right)\mathrm{Li}_2(x)+20Y^3+\left(\frac{107}{3}+3X^2-22U-30X\right)Y^2\\
&+\left(4X^3+24X^2+\left(-4\pi^2+22U-\frac{107}{3}\right)X-57+36\zeta_3+22U-\frac{2}{3}\pi^2\right)Y\\
&-X^4-\frac{19}{3}X^3+\left(-11U+\frac{107}{6}+\frac{1}{3}\pi^2\right)X^2+\left(-11U-32\zeta_3+\frac{57}{2}-\frac{20}{3}\pi^2\right)X\\
&-\frac{43417}{324}-\frac{43}{6}\pi^2U-52\zeta_3U+\frac{515}{9}\zeta_3+\frac{251}{9}\pi^2+\frac{1141}{27}U+\frac{65}{36}\pi^4\Bigg]\frac{t^2+s^2}{st}\\
&+\Bigg[-20\,\mathrm{Li}_4(y)-48\,\mathrm{Li}_4(x)+20\,\mathrm{Li}_4(z)+\left(12Y-16+12X\right)\mathrm{Li}_3(x)\\
&+\left(4Y^2+\left(-4X-32\right)Y+2X^2+16X-\frac{10}{3}\pi^2\right)\mathrm{Li}_2(x)\\
&+\left(-12X+24Y-32\right)\mathrm{Li}_3(y)+4Y^3X+\left(-\frac{94}{3}X+X^2+\frac{4}{3}\pi^2\right)Y^2\\
&+\left(\frac{46}{3}X^2+\left(-\frac{20}{3}\pi^2+\frac{22}{3}U-\frac{251}{9}\right)X-12\zeta_3+\frac{62}{3}\pi^2\right)Y\\
&-\frac{13}{9}X^3+\left(\frac{251}{18}+3\pi^2-\frac{11}{3}U\right)X^2+\left(-\frac{16}{9}\pi^2+24\zeta_3-\frac{57}{2}+11U\right)X\\
&+16\zeta_3+\frac{251}{18}\pi^2-\frac{11}{3}\pi^2U+\frac{94}{45}\pi^4\Bigg]\frac{t^2-s^2}{st}\\
&+\Bigg[-16\,\mathrm{Li}_4(y)-16\,\mathrm{Li}_4(z)+\left(-64+48Y\right)\mathrm{Li}_3(x)+48\,\mathrm{Li}_3(y)X
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-16\,X\,Y+64\,X+8\,X^2+8\,\pi^2\right)\mathrm{Li}_2(x)+\frac{272}{9}\,Y^3\\
&+\left(4\,X^2+\frac{344}{9}-\frac{136}{3}\,X-\frac{88}{3}\,U\right)Y^2+\left(8\,X^3+\frac{184}{3}\,X^2+\left(\frac{88}{3}\,U-\frac{344}{9}-\frac{16}{3}\,\pi^2\right)X\right.\\
&\left.+\frac{88}{3}\,U+\frac{32}{9}\,\pi^2+48\,\zeta_3-\frac{1076}{9}\right)Y-2\,X^4-\frac{112}{9}\,X^3+\left(-\frac{44}{3}\,U+\frac{298}{9}\right)X^2\\
&\left.+\left(-\frac{76}{9}\,\pi^2-\frac{44}{3}\,U+\frac{538}{9}-48\,\zeta_3\right)X+48\,\pi^2+48\,\zeta_3-\frac{44}{3}\,\pi^2\,U+\frac{92}{45}\,\pi^4\right] \qquad \text{(A.19)}
\end{aligned}
$$

$$
\begin{aligned}
E_{3;u} = &\left[-\frac{8}{3}\,Y^3+\left(4\,U-\frac{26}{3}+4\,X\right)Y^2+\left(-4\,X^2+\left(\frac{26}{3}-4\,U\right)X+\frac{4}{3}\,\pi^2+10-4\,U\right)Y\right.\\
&+\frac{4}{3}\,X^3+\left(2\,U-\frac{13}{3}\right)X^2+\left(\frac{4}{3}\,\pi^2+2\,U-5\right)X\\
&\left.-\frac{2}{9}\,\zeta_3+\frac{5}{3}\,\pi^2\,U+\frac{3401}{162}-\frac{56}{9}\,\pi^2-\frac{202}{27}\,U\right]\frac{t^2+s^2}{st}\\
&+\left[\frac{4}{3}\,Y^2\,X+\left(-\frac{4}{3}\,X^2+\left(-\frac{4}{3}\,U+\frac{74}{9}\right)X-\frac{4}{3}\,\pi^2\right)Y+\frac{4}{9}\,X^3+\left(-\frac{37}{9}+\frac{2}{3}\,U\right)X^2\right.\\
&\left.+\left(-2\,U+5+\frac{4}{9}\,\pi^2\right)X+\frac{1}{9}\,\pi^2\left(-37+6\,U\right)\right]\frac{t^2-s^2}{st}\\
&+\left[-\frac{32}{9}\,Y^3+\left(\frac{16}{3}\,U+\frac{16}{3}\,X-\frac{104}{9}\right)Y^2+\left(-\frac{16}{3}\,X^2+\left(-\frac{16}{3}\,U+\frac{104}{9}\right)X\right.\right.\\
&\left.+\frac{152}{9}-\frac{16}{3}\,U+\frac{16}{9}\,\pi^2\right)Y+\frac{16}{9}\,X^3+\left(\frac{8}{3}\,U-\frac{52}{9}\right)X^2+\left(\frac{8}{3}\,U-\frac{76}{9}+\frac{16}{9}\,\pi^2\right)X\\
&\left.+\frac{4}{3}\,\pi^2\left(2\,U-7\right)\right] \qquad \text{(A.20)}
\end{aligned}
$$

$$
\begin{aligned}
E_{4;u} = &\left[\frac{8}{3}\,X^3+\left(8\,U-\frac{26}{3}-8\,Y\right)X^2+\left(8\,Y^2+\left(\frac{52}{3}-16\,U\right)Y+8\,U+\frac{8}{3}\,\pi^2-10\right)X\right.\\
&-\frac{16}{3}\,Y^3+\left(-\frac{52}{3}+16\,U\right)Y^2+\left(\frac{8}{3}\,\pi^2-16\,U+20\right)Y\\
&\left.-\frac{112}{9}\,\pi^2-\frac{908}{27}\,U-\frac{4}{9}\,\zeta_3+\frac{22}{3}\,\pi^2\,U+\frac{3401}{81}\right]\frac{t^2+s^2}{st}\\
&+\left[\frac{8}{9}\,X^3+\left(\frac{8}{3}\,U-\frac{74}{9}-\frac{8}{3}\,Y\right)X^2\right.\\
&+\left(\frac{8}{3}\,Y^2+\left(-\frac{16}{3}\,U+\frac{148}{9}\right)Y+\frac{8}{9}\,\pi^2+10-8\,U\right)X\\
&\left.-\frac{8}{3}\,\pi^2\,Y+\frac{2}{9}\,\pi^2\left(12\,U-37\right)\right]\frac{t^2-s^2}{st}\\
&+\left[\frac{32}{9}\,X^3+\left(-\frac{104}{9}-\frac{32}{3}\,Y+\frac{32}{3}\,U\right)X^2\right.
\end{aligned}
$$

$$+\left(\frac{32}{3}\,Y^2+\left(-\frac{64}{3}\,U+\frac{208}{9}\right)Y-\frac{152}{9}+\frac{32}{3}\,U+\frac{32}{9}\,\pi^2\right)X$$
$$-\frac{64}{9}\,Y^3+\left(-\frac{208}{9}+\frac{64}{3}\,U\right)Y^2+\left(-\frac{64}{3}\,U+\frac{304}{9}+\frac{32}{9}\,\pi^2\right)Y$$
$$+\frac{8}{3}\,\pi^2\left(-7+4\,U\right)\Bigg] \tag{A.21}$$

$$F_{2;u}=\left[\frac{4}{27}\,\pi^2+\frac{32}{9}\,U^2-\frac{160}{27}\,U\right]\frac{t^2+s^2}{st} \tag{A.22}$$

B. Finite one-loop contributions

B.1 s-channel

$$G_{1;s}=\Bigg[14\,X^4+28\,X^3+8\,X^2\,Y^2+56\,X^2\,\pi^2-48\,X^2+12\,X^2\,Y+32\,X\,Y\,\pi^2+80\,\pi^2\,X$$
$$+2\,Y^4+12\,Y^3-10\,Y^2+8\,Y^2\,\pi^2+26\,\pi^2+24\,\pi^2\,Y-84\,Y+102\Bigg]\frac{t}{u}$$
$$+8\,X\left[X^3+X^2+4\,\pi^2\,X+2\,\pi^2\right]\frac{t^2}{u^2}+2\,X^2\left[X^2+4\,\pi^2\right]\frac{t^3}{u^3}$$
$$+\Bigg[32\,X^3+8\,X^2\,Y^2+80\,\pi^2\,X+32\,X\,Y\,\pi^2+8\,Y^2\,X$$
$$+8\,Y^4+32\,Y^2\,\pi^2-32\,Y^2-4-56\,Y+24\,\pi^2\Bigg]+\Big\{t\leftrightarrow u\Big\} \tag{B.1}$$

$$G_{2;s}=\Bigg[-10\,X^4-\frac{106}{3}\,X^3-8\,X^3\,Y-2\,X^2\,Y^2-52\,X^2\,\pi^2-\frac{44}{3}\,X^2\,S+6\,X^2-\frac{40}{3}\,X^2\,Y$$
$$-4\,Y^3\,X-\frac{56}{3}\,Y^2\,X-32\,X\,Y\,\pi^2+8\,X\,Y-80\,\pi^2\,X+\frac{140}{3}\,X-\frac{20}{3}\,Y^3-\frac{22}{3}\,Y^2\,S$$
$$-20\,Y^2-6\,Y^2\,\pi^2-18\,\pi^2\,Y-22\,Y\,S+\frac{140}{3}\,Y-4+\frac{154}{3}\,S-40\,\pi^2\Bigg]\frac{t}{u}$$
$$-8\,X\left[X^3+X^2+4\,\pi^2\,X+2\,\pi^2\right]\frac{t^2}{u^2}-2\,X^2\left[X^2+4\,\pi^2\right]\frac{t^3}{u^3}$$
$$+\Bigg[-8\,X^3\,Y-32\,X^3+4\,X^2\,\pi^2-4\,X^2\,Y^2-\frac{52}{3}\,X^2\,Y-8\,Y^2\,X-\frac{40}{3}\,X\,Y$$
$$-32\,X\,Y\,\pi^2-76\,\pi^2\,X-\frac{44}{3}\,X\,S-4\,Y^4+\frac{8}{3}\,Y^3+\frac{8}{3}\,Y^2-\frac{44}{3}\,Y^2\,S-32\,Y^2\,\pi^2$$
$$+28\,Y+4-24\,\pi^2\Bigg]+\Big\{t\leftrightarrow u\Big\} \tag{B.2}$$

$$G_{3;s}=\Bigg[2\,X^4+2\,X^3\,Y+\frac{22}{3}\,X^3+\frac{11}{3}\,X^2\,S+2\,X^2\,Y^2+10\,X^2\,Y+\frac{68}{9}\,X^2+13\,X^2\,\pi^2$$
$$+\frac{20}{3}\,Y^2\,X+6\,X\,Y\,\pi^2+\frac{100}{9}\,X\,Y+\frac{22}{3}\,X\,Y\,S+\frac{110}{9}\,X\,S+\frac{50}{3}\,\pi^2\,X+\frac{50}{9}\,Y^2$$

$$+2\,Y^2\,\pi^2+\frac{8}{3}\,\pi^2\,Y+\frac{110}{9}\,Y\,S+2+\frac{121}{18}\,S^2-\frac{11}{3}\,\pi^2\,S+\frac{1}{2}\,\pi^4+\frac{13}{2}\,\pi^2\Bigg]\,\frac{t}{u}$$
$$+2\,X\left[X^3+X^2+4\,\pi^2\,X+2\,\pi^2\right]\frac{t^2}{u^2}+\frac{1}{2}\,X^2\left[X^2+4\,\pi^2\right]\frac{t^3}{u^3}$$
$$+\Bigg[4\,X^3\,Y+8\,X^3-2\,X^2\,\pi^2+\frac{26}{3}\,X^2\,Y+2\,Y^2\,X+8\,X\,Y\,\pi^2+\frac{20}{3}\,X\,Y+\frac{22}{3}\,X\,S$$
$$+18\,\pi^2\,X-\frac{4}{3}\,Y^3+\frac{20}{3}\,Y^2+\frac{22}{3}\,Y^2\,S+8\,Y^2\,\pi^2+6\,\pi^2\Bigg]+\Big\{t\leftrightarrow u\Big\} \tag{B.3}$$

$$X_{1;s}=\frac{2}{3}\left(3\,Y+Y^2-7+2\,X^2\right)\left(2\,S+Y+X\right)\frac{t}{u}$$
$$+\left(\frac{4}{3}\,X^2\,Y+\frac{8}{3}\,X\,S+\frac{4}{3}\,X\,Y+\frac{4}{3}\,Y^3+\frac{4}{3}\,Y^2+\frac{8}{3}\,Y^2\,S\right)+\Big\{t\leftrightarrow u\Big\} \tag{B.4}$$

$$X_{2;s}=-\frac{1}{9}\left(2\,S+Y+X\right)\left(11\,S+3\,X^2+6\,X\,Y+10\,Y+10\,X-3\,\pi^2\right)\frac{t}{u}$$
$$+\left(-\frac{2}{3}\,X\,Y-\frac{2}{3}\,Y^2-\frac{2}{3}\,X^2\,Y-\frac{4}{3}\,Y^2\,S-\frac{4}{3}\,X\,S-\frac{2}{3}\,Y^3\right)+\Big\{t\leftrightarrow u\Big\} \tag{B.5}$$

$$X_{3;s}=\frac{1}{18}\left(2\,S+Y+X\right)^2\frac{t}{u}+\Big\{t\leftrightarrow u\Big\} \tag{B.6}$$

$$X_{4;s}=\left(-\frac{32}{3}\,Y\,\pi^2+\frac{32}{3}\,S\,X^2+16\,S\,Y-\frac{64}{3}\,X\,\pi^2-16\,\pi^2+\frac{16}{3}\,S\,Y^2-\frac{112}{3}\,S\right)\frac{t}{u}$$
$$+\left(\frac{32}{3}\,S\,X^2+\frac{32}{3}\,S\,Y-\frac{64}{3}\,X\,\pi^2-\frac{32}{3}\,\pi^2\right)+\Big\{t\leftrightarrow u\Big\} \tag{B.7}$$

$$X_{5;s}=\left(\frac{32}{9}\,S^2+\frac{32}{9}\,\pi^2\right)\frac{t}{u}+\Big\{t\leftrightarrow u\Big\} \tag{B.8}$$

B.2 u-channel

$$G_{1;u}=\Bigg[8\,X^4+\left(-32\,Y+8\right)X^3+\left(48\,Y^2-24\,Y+16\,\pi^2\right)X^2$$
$$+\left(-32\,Y^3+24\,Y^2-32\,\pi^2\,Y+8\,\pi^2\right)X$$
$$+8\,Y^4-8\,Y^3+16\,Y^2\,\pi^2-8\,\pi^2\,Y+8\,\pi^4\Bigg]\frac{t^2}{s^2}$$
$$+\left[8\,Y^4-8\,Y^3+32\,\pi^2\,Y^2-16\,\pi^2\,Y\right]\frac{s^2}{t^2}$$
$$+\Bigg[2\,X^4-8\,X^3\,Y+\left(4\,\pi^2+12\,Y^2\right)X^2+\left(-8\,Y^3-8\,\pi^2\,Y\right)X$$
$$+2\,Y^4+4\,Y^2\,\pi^2+2\,\pi^4\Bigg]\frac{t^3}{s^3}+\left[2\,Y^4+8\,\pi^2\,Y^2\right]\frac{s^3}{t^3}$$

$$
\begin{aligned}
&+\Bigg[8\,X^4+\Big(-32\,Y+20\Big)\,X^3+\Big(16\,\pi^2+56\,Y^2-66\,Y-29\Big)\,X^2\\
&+\Big(-48\,Y^3+78\,Y^2+\Big(58-32\,\pi^2\Big)\,Y-42+20\,\pi^2\Big)\,X\\
&+24\,Y^4-52\,Y^3+\Big(56\,\pi^2-58\Big)\,Y^2+\Big(-66\,\pi^2+84\Big)\,Y\\
&+102+8\,\pi^4-29\,\pi^2\Bigg]\,\frac{t^2+s^2}{st}\\
&+\Bigg[6\,X^4+\Big(-24\,Y+8\Big)\,X^3+\Big(36\,Y^2-19-30\,Y+12\,\pi^2\Big)\,X^2\\
&+\Big(-24\,Y^3+30\,Y^2+\Big(38-24\,\pi^2\Big)\,Y+8\,\pi^2+42\Big)\,X\\
&-12\,Y^2\,\pi^2+2\,\pi^2\,Y+3\,\pi^2\Big(2\,\pi^2-3\Big)\Bigg]\,\frac{t^2-s^2}{st}\\
&+\Bigg[8\,X^4+\Big(32-32\,Y\Big)\,X^3+\Big(-104\,Y+64\,Y^2-32+16\,\pi^2\Big)\,X^2\\
&+\Big(-64\,Y^3+120\,Y^2+\Big(-32\,\pi^2+64\Big)\,Y-56+32\,\pi^2\Big)\,X\\
&+32\,Y^4-80\,Y^3+64\Big(\pi-1\Big)\Big(\pi+1\Big)\,Y^2\\
&+\Big(-104\,\pi^2+112\Big)\,Y-8+8\,\pi^4-32\,\pi^2\Bigg]
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
G_{2;u}=&\Bigg[-8\,X^4+\Big(32\,Y-8\Big)\,X^3+\Big(-48\,Y^2+24\,Y-16\,\pi^2\Big)\,X^2\\
&+\Big(32\,Y^3-24\,Y^2+32\,\pi^2\,Y-8\,\pi^2\Big)\,X\\
&-8\,Y^4+8\,Y^3-16\,Y^2\,\pi^2+8\,\pi^2\,Y-8\,\pi^4\Bigg]\,\frac{t^2}{s^2}\\
&-\Bigg[8\,Y^4-8\,Y^3+32\,\pi^2\,Y^2-16\,\pi^2\,Y\Bigg]\,\frac{s^2}{t^2}\\
&+\Bigg[-2\,X^4+8\,X^3\,Y+\Big(-4\,\pi^2-12\,Y^2\Big)\,X^2+\Big(8\,Y^3+8\,\pi^2\,Y\Big)\,X\\
&-2\,Y^4-4\,Y^2\,\pi^2-2\,\pi^4\Bigg]\,\frac{t^3}{s^3}-\Bigg[2\,Y^4+8\,\pi^2\,Y^2\Bigg]\,\frac{s^3}{t^3}\\
&+\Bigg[-5\,X^4+\Big(-21+26\,Y\Big)\,X^3+\Big(-11\,U-7-50\,Y^2-13\,\pi^2+79\,Y\Big)\,X^2\\
&+\Big(48\,Y^3-111\,Y^2+\Big(6+44\,\pi^2+22\,U\Big)\,Y+\frac{140}{3}-30\,\pi^2-11\,U\Big)\,X
\end{aligned}
$$

$$
\begin{aligned}
&-24\,Y^4+74\,Y^3+\Big(-6-56\,\pi^2-22\,U\Big)\,Y^2+\Big(-\frac{280}{3}+85\,\pi^2+22\,U\Big)\,Y\\
&+\frac{154}{3}\,U-4+7\,\pi^2-11\,\pi^2\,U-8\,\pi^4\Big]\,\frac{t^2+s^2}{st}\\
&+\Big[-5\,X^4+\Big(22\,Y-\frac{43}{3}\Big)\,X^3+\Big(\frac{121}{3}\,Y-11\,\pi^2-36\,Y^2+13-\frac{11}{3}\,U\Big)\,X^2\\
&+\Big(24\,Y^3-\frac{121}{3}\,Y^2+\Big(-26+\frac{22}{3}\,U+20\,\pi^2\Big)\,Y-\frac{52}{3}\,\pi^2+11\,U\Big)\,X\\
&+12\,Y^2\,\pi^2-5\,\pi^2\,Y-\frac{1}{3}\,\pi^2\,\Big(18\,\pi^2+11\,U-3\Big)\Big]\,\frac{t^2-s^2}{st}\\
&+\Big[-4\,X^4+\Big(-\frac{88}{3}+24\,Y\Big)\,X^3+\Big(-\frac{44}{3}\,U+\frac{8}{3}-56\,Y^2-12\,\pi^2+\frac{340}{3}\,Y\Big)\,X^2\\
&+\Big(64\,Y^3-164\,Y^2+\Big(\frac{64}{3}+48\,\pi^2+\frac{88}{3}\,U\Big)\,Y+28-\frac{124}{3}\,\pi^2-\frac{44}{3}\,U\Big)\,X\\
&-32\,Y^4+\frac{328}{3}\,Y^3+\Big(-\frac{64}{3}-64\,\pi^2-\frac{88}{3}\,U\Big)\,Y^2+\Big(-56+\frac{364}{3}\,\pi^2+\frac{88}{3}\,U\Big)\,Y\\
&+8+\frac{8}{3}\,\pi^2-\frac{44}{3}\,\pi^2\,U-8\,\pi^4\Big]
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
G_{3;u}=\;&\Big[2\,X^4+\Big(-8\,Y+2\Big)\,X^3+\Big(12\,Y^2-6\,Y+4\,\pi^2\Big)\,X^2\\
&+\Big(-8\,Y^3+6\,Y^2-8\,\pi^2\,Y+2\,\pi^2\Big)\,X\\
&+2\,Y^4-2\,Y^3+4\,Y^2\,\pi^2-2\,\pi^2\,Y+2\,\pi^4\Big]\,\frac{t^2}{s^2}\\
&+\Big[2\,Y^4-2\,Y^3+8\,\pi^2\,Y^2-4\,\pi^2\,Y\Big]\,\frac{s^2}{t^2}\\
&+\Big[\frac{1}{2}\,X^4-2\,X^3\,Y+\Big(\pi^2+3\,Y^2\Big)\,X^2+\Big(-2\,Y^3-2\,\pi^2\,Y\Big)\,X\\
&+\frac{1}{2}\,Y^4+Y^2\,\pi^2+\frac{1}{2}\,\pi^4\Big]\,\frac{t^3}{s^3}+\Big[\frac{1}{2}\,Y^4+2\,\pi^2\,Y^2\Big]\,\frac{s^3}{t^3}\\
&+\Big[X^4+\Big(\frac{11}{3}-5\,Y\Big)\,X^3+\Big(\frac{11}{6}\,U+\frac{59}{9}+11\,Y^2+\frac{9}{2}\,\pi^2-\frac{58}{3}\,Y\Big)\,X^2\\
&+\Big(-12\,Y^3+36\,Y^2+\Big(-14\,\pi^2-\frac{218}{9}-11\,U\Big)\,Y+\frac{110}{9}\,U+\frac{41}{3}\,\pi^2\Big)\,X\\
&+6\,Y^4-24\,Y^3+\Big(11\,U+\frac{218}{9}+14\,\pi^2\Big)\,Y^2+\Big(-26\,\pi^2-\frac{220}{9}\,U\Big)\,Y\\
&+\frac{121}{18}\,U^2+2\,\pi^4+\frac{11}{2}\,\pi^2\,U+\frac{59}{9}\,\pi^2+2\Big]\,\frac{t^2+s^2}{st}
\end{aligned}
$$

$$+\Bigg[X^4+\left(\frac{11}{3}-5\,Y\right)X^3+\left(1-\frac{38}{3}\,Y+\frac{5}{2}\,\pi^2+9\,Y^2+\frac{11}{6}\,U\right)X^2$$
$$+\left(-6\,Y^3+\frac{38}{3}\,Y^2+\left(-4\,\pi^2-\frac{11}{3}\,U-2\right)Y+\frac{11}{3}\,\pi^2\right)X$$
$$-3\,Y^2\,\pi^2+2\,\pi^2\,Y+\frac{1}{6}\,\pi^2\left(9\,\pi^2+11\,U-6\right)\Bigg]\frac{t^2-s^2}{st}$$
$$+\Bigg[\left(\frac{20}{3}-4\,Y\right)X^3+\left(12\,Y^2-\frac{92}{3}\,Y+\frac{20}{3}+\frac{22}{3}\,U+2\,\pi^2\right)X^2$$
$$+\left(-16\,Y^3+52\,Y^2+\left(-\frac{44}{3}\,U-\frac{80}{3}-16\,\pi^2\right)Y+\frac{22}{3}\,U+\frac{38}{3}\,\pi^2\right)X$$
$$+8\,Y^4-\frac{104}{3}\,Y^3+\left(16\,\pi^2+\frac{80}{3}+\frac{44}{3}\,U\right)Y^2+\left(-\frac{104}{3}\,\pi^2-\frac{44}{3}\,U\right)Y$$
$$+\frac{2}{3}\,\pi^2\left(3\,\pi^2+10+11\,U\right)\Bigg] \tag{B.11}$$

$$X_{1;u}=\Bigg[X^3+\left(2\,U-4\,Y+1\right)X^2+\left(6\,Y^2+\left(-4-4\,U\right)Y+\pi^2+2\,U-\frac{14}{3}\right)X$$
$$-4\,Y^3+\left(4+4\,U\right)Y^2+\left(-4\,U+\frac{28}{3}-4\,\pi^2\right)Y+2\,\pi^2\,U-\frac{28}{3}\,U+\pi^2\Bigg]\frac{t^2+s^2}{st}$$
$$+\Bigg[\frac{1}{3}\,X^3+\left(-\frac{4}{3}\,Y-1+\frac{2}{3}\,U\right)X^2+\left(\frac{4}{3}\,Y^2+\left(-\frac{4}{3}\,U+2\right)Y+\frac{1}{3}\,\pi^2-2\,U\right)X$$
$$+\frac{1}{3}\,\pi^2\left(2\,U+3\right)\Bigg]\frac{t^2-s^2}{st}$$
$$+\Bigg[\frac{4}{3}\,X^3+\left(-\frac{16}{3}\,Y+\frac{8}{3}\,U+\frac{4}{3}\right)X^2+\left(8\,Y^2+\left(-\frac{16}{3}\,U-\frac{16}{3}\right)Y+\frac{4}{3}\,\pi^2+\frac{8}{3}\,U\right)X$$
$$-\frac{16}{3}\,Y^3+\left(\frac{16}{3}+\frac{16}{3}\,U\right)Y^2+\left(-\frac{16}{3}\,U-\frac{16}{3}\,\pi^2\right)Y+\frac{4}{3}\,\pi^2\left(2\,U+1\right)\Bigg] \tag{B.12}$$

$$X_{2;u}=\Bigg[-\frac{1}{6}\,X^3+\left(-\frac{1}{3}\,U+\frac{4}{3}\,Y-\frac{10}{9}\right)X^2+\left(-3\,Y^2+\left(2\,U+\frac{40}{9}\right)Y-\frac{7}{6}\,\pi^2-\frac{31}{9}\,U\right)X$$
$$+2\,Y^3+\left(-2\,U-\frac{40}{9}\right)Y^2+\left(2\,\pi^2+\frac{62}{9}\,U\right)Y-\frac{22}{9}\,U^2-\pi^2\,U-\frac{10}{9}\,\pi^2\Bigg]\frac{t^2+s^2}{st}$$
$$+\Bigg[-\frac{1}{6}\,X^3+\left(\frac{2}{3}\,Y-\frac{1}{3}\,U\right)X^2+\left(-\frac{2}{3}\,Y^2-\frac{1}{6}\,\pi^2+\frac{2}{3}\,Y\,U\right)X-\frac{1}{3}\,\pi^2\,U\Bigg]\frac{t^2-s^2}{st}$$
$$+\Bigg[-\frac{2}{3}\,X^3+\left(\frac{8}{3}\,Y-\frac{4}{3}\,U-\frac{2}{3}\right)X^2+\left(-4\,Y^2+\left(\frac{8}{3}\,U+\frac{8}{3}\right)Y-\frac{2}{3}\,\pi^2-\frac{4}{3}\,U\right)X$$
$$+\frac{8}{3}\,Y^3+\left(-\frac{8}{3}-\frac{8}{3}\,U\right)Y^2+\left(\frac{8}{3}\,U+\frac{8}{3}\,\pi^2\right)Y-\frac{2}{3}\,\pi^2\left(2\,U+1\right)\Bigg] \tag{B.13}$$

$$X_{3;u}=\left[\frac{1}{18}\,X^2+\left(-\frac{2}{9}\,Y+\frac{2}{9}\,U\right)X-\frac{4}{9}\,Y\,U+\frac{1}{18}\,\pi^2+\frac{2}{9}\,U^2+\frac{2}{9}\,Y^2\right]\frac{t^2+s^2}{st} \tag{B.14}$$

$$
\begin{aligned}
X_{4;u} &= \frac{32}{3}\, U \left[X + X^2 - 2\,Y - 2\,Y\,X + 2\,Y^2 + \pi^2 \right] \\
&\quad + \frac{8}{3}\, U \left[3\,\pi^2 - 6\,Y - 6\,Y\,X - 14 + 3\,X^2 + 6\,Y^2 + 3\,X \right] \frac{t^2 + s^2}{st} \\
&\quad - \frac{8}{3}\, U \left[-\pi^2 + 2\,Y\,X - X^2 + 3\,X \right] \frac{t^2 - s^2}{st} \qquad \text{(B.15)} \\
X_{5;u} &= \frac{32}{9}\, U^2\, \frac{t^2 + s^2}{st} \qquad \text{(B.16)}
\end{aligned}
$$

C. One-loop master integrals

In this appendix, we list the expansions for the one-loop box integrals in $D = 6-2\epsilon$. We remain in the physical region $s > 0$, $u, t < 0$, and write coefficients in terms of logarithms and polylogarithms that are real in this domain. More precisely, we use the notation of Eqs. (4.3) and (4.4) to define the arguments of the logarithms and polylogarithms. The polylogarithms are defined as in Eq. (4.2).

We find that the box integrals have the expansion,

$$\begin{aligned}
\mathrm{Box}^6(u,t) = & \frac{e^{\epsilon\gamma}\Gamma\left(1+\epsilon\right)\Gamma\left(1-\epsilon\right)^2}{2s\Gamma\left(1-2\epsilon\right)\left(1-2\epsilon\right)}\left(\frac{\mu^2}{s}\right)^{\epsilon}\Bigg\{\frac{1}{2}\left[(X-Y)^2+\pi^2\right] \\
& +2\epsilon\left[\mathrm{Li}_3(x)-X\mathrm{Li}_2(x)-\frac{1}{3}X^3-\frac{\pi^2}{2}X\right] \\
& -2\epsilon^2\left[\mathrm{Li}_4(x)+Y\mathrm{Li}_3(x)-\frac{1}{2}X^2\mathrm{Li}_2(x)-\frac{1}{8}X^4-\frac{1}{6}X^3Y+\frac{1}{4}X^2Y^2\right. \\
& \left.-\frac{\pi^2}{4}X^2-\frac{\pi^2}{3}XY-\frac{\pi^4}{45}\right]+(u\leftrightarrow t)\Bigg\}+\mathcal{O}\left(\epsilon^3\right),
\end{aligned} \tag{C.1}$$

and

$$\begin{aligned}
\mathrm{Box}^6(s,t) = & \frac{e^{\epsilon\gamma}\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{2u\Gamma(1-2\epsilon)(1-2\epsilon)}\left(-\frac{\mu^2}{u}\right)^{\epsilon}\Bigg\{\left(X^2+2i\pi X\right) \\
& +\epsilon\Bigg[\left(-2\mathrm{Li}_3(x)+2X\mathrm{Li}_2(x)-\frac{2}{3}X^3+2YX^2-\pi^2X+2\zeta_3\right) \\
& +i\pi\left(2\mathrm{Li}_2(x)+4YX-X^2-\frac{\pi^2}{3}\right)\Bigg] \\
& +\epsilon^2\Bigg[\Bigg(2\mathrm{Li}_4(z)+2\mathrm{Li}_4(y)-2Y\mathrm{Li}_3(x)-2X\mathrm{Li}_3(y)+(2XY-X^2-\pi^2)\mathrm{Li}_2(x) \\
& +\frac{1}{3}X^4-\frac{5}{3}X^3Y+\frac{3}{2}X^2Y^2+\frac{2}{3}\pi^2X^2-2\pi^2XY+2Y\zeta_3+\frac{1}{6}\pi^4\Bigg) \\
& +i\pi\Bigg(-2\mathrm{Li}_3(x)-2\mathrm{Li}_3(y)+2Y\mathrm{Li}_2(x)+\frac{1}{3}X^3-2X^2Y+3XY^2 \\
& -\frac{\pi^2}{3}Y+2\zeta_3\Bigg)\Bigg]\Bigg\}+\mathcal{O}\left(\epsilon^3\right).
\end{aligned} \tag{C.2}$$

$\mathrm{Box}^6(s,u)$ is obtained from Eq. (C.2) by exchanging u and t.

Finally, the one-loop bubble integral in $D = 4-2\epsilon$ dimensions is given by

$$\mathrm{Bub}(s) = \frac{e^{\epsilon\gamma}\Gamma\left(1+\epsilon\right)\Gamma\left(1-\epsilon\right)^2}{\Gamma\left(2-2\epsilon\right)\epsilon}\left(-\frac{\mu^2}{s}\right)^{\epsilon}. \tag{C.3}$$

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