Improving the accuracy of the mini-element approximation to Navier-Stokes equations

Article *in* IMA Journal of Numerical Analysis · May 2006

DOI: 10.1093/imanum/drl010

3 authors:

Blanca Ayuso de Dios
King Abdullah University of Science and Technology

Javier De Frutos
Universidad de Valladolid

Julia Novo
Universidad Autónoma de Madrid

IMA Journal of Numerical Analysis (2005) Page 1 of 20
doi: 10.1093/imanum/dri017

Improving the Accuracy of the Mini–Element Approximation to Navier–Stokes equations

BLANCA AYUSO ∗
IMATI-CNR, Pavia, Italy

JAVIER DE FRUTOS †
Departamento de Matemática Aplicada, Universidad de Valladolid
Valladolid, Spain

JULIA NOVO ‡
Departamento de Matemáticas, Universidad Autónoma de Madrid
Madrid, Spain

A technique to improve the accuracy of the mini–element approximation to incompressible Navier–Stokes equations is introduced. Once the mini–element approximation has been computed at a fixed time, the linear part of this approximation is post-processed by solving a discrete Stokes problem. The bubble functions needed to stabilize the approximation to the Navier–Stokes equations are not used at the postprocessing step. This postprocessing procedure allows the H^1 norm rate of convergence of the velocity and corresponding L^2 rate of the pressure to be increased by one unit (up to a logarithmic term). An error analysis of the algorithm is performed.

Keywords: Navier–Stokes equations, mini–element

1. Introduction

The postprocessing technique we consider has been developed as a way to enhance, at low cost, Galerkin approximations to nonlinear evolutionary dissipative equations. It was first introduced in García-Archilla *et al.* (1998) for Fourier approximation and later extended to polynomial spectral methods in de Frutos *et al.* (2000), de Frutos & Novo (2000a) and to the spectral element method (p-version of finite element method) in de Frutos & Novo (2000b), de Frutos & Novo (2000c). In García-Archilla & Titi (2000), de Frutos & Novo (2002) the finite element (h-version) was considered. More recently, in Ayuso (2003), the postprocessing procedure has been considered for the mixed finite element approximation of the incompressible Navier–Stokes equations. The results for the high order case are collected in Ayuso *et al.* (2005), where the authors focus on the Hood-Taylor element approximation.

In this paper we shall concentrate on the low order case. Very often, due to complexity or to computational requirements, a first–order approximation based on continuous polynomials is preferred for the Navier–Stokes problem. However, it is well known that a naive combination of continuous linear polynomials for the approximations to both the velocity and the pressure (the so-called P^1P^1 element) leads to an unstable discretization because the resulting mixed-element fails to satisfy the Ladyzhenskaya-Babuška-Brezzi (LBB) condition. To overcome this difficulty Arnold *et al.* (1984) proposed a stable

∗e-mail: blanca@imati.cnr.it
†e-mail: frutos@mac.cie.uva.es
‡Corresponding author. e-mail: julia.novo@uam.es

mixed-element by enriching the discrete velocity space with bubble functions. The resulting stable mixed finite element is called the mini–element. In Pierre (1988), Pierre (1989) and Bank & Welfert (1990) the authors showed the equivalence between the Galerkin method employing the mini–element of Arnold *et al.* and the stabilized methods of Brezzi & Pitkäranta (1984), Hughes *et al.* (1986) for the Stokes flow. These authors prove that the linear part of the mini–element approximation satisfies the equations of a stabilized method with a well determined stability parameter that emanates from the static-condensation of the bubble functions.

In this work we propose a procedure to enhance the accuracy of the mini–element approximation for the Navier–Stokes problem. The accuracy is improved by increasing the rate of convergence of the velocity and pressure approximations in the H^1 and L^2-norms, respectively.

Let us consider the incompressible Navier–Stokes equations in a bounded smooth domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$,

$$\begin{aligned} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, &\quad \text{in } \Omega \\ \operatorname{div} u = 0, &\quad \text{in } \Omega \end{aligned} \tag{1.1}$$

subject to homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$. In (1.1) u denotes the velocity field, p the pressure and f represents the external forces. We next describe the postprocessing technique we apply, which can be seen as a two-level method. Suppose that we are interested in the solution (u, p) at a certain time $T > 0$, corresponding to a given initial condition

$$u(\cdot,0) = u_0. \tag{1.2}$$

We first compute the mini–element approximation u_h and p_h to the velocity and pressure, respectively, by integrating in time the corresponding discretization of (1.1), (1.2) from $t = 0$ to $t = T$. Then, we can postprocess the previously computed approximation by means of a single Stokes problem. The first idea is to consider the problem

$$\begin{cases} -\nu\Delta\tilde u + \nabla\tilde p = f - \dfrac{d}{dt}u_h(T) - (u_h(T)\cdot\nabla)u_h(T), & \text{in } \Omega, \\ \operatorname{div}\tilde u = 0, & \text{in } \Omega, \\ \tilde u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

To improve the accuracy of the method, the approximation to the above Stokes problem is carried out by using a more accurate mixed finite element pair than in the first level. However, it has been observed and reported in the literature (see for instance Verfürth (1989), Verfürth (1998), Bank & Welfert (1991), Kim & Lee (2000), Pierre (1988) and Pierre (1989)) that the linear part of the approximation to the velocity, u_h^l, is a better approximation to the solution u than u_h itself. The bubble part of the approximation is only introduced for stability reasons and does not improve the approximation to the velocity and pressure terms. This suggest the possibility that the correct Stokes problem to consider for the postprocessed method, rather than (1.3), is the one obtained by retaining on the right hand side only the linear part of the computed approximation at time T, that is,

$$\begin{cases} -\nu\Delta\tilde u + \nabla\tilde p = f - \dfrac{d}{dt}u_h^l(T) - (u_h^l(T)\cdot\nabla)u_h^l(T), & \text{in } \Omega, \\ \operatorname{div}\tilde u = 0, & \text{in } \Omega, \\ \tilde u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In this paper we shall show that the rate of convergence is improved for the postprocessed velocity and pressure, with respect to that dictated by the Mini–element approximation. To improve the accuracy of the method, the approximation to the above Stokes problem (1.4) is carried out by using either the Hood-Taylor (P^2P^1) element over the same grid or the mini–element over a finer grid. The postprocessed method has in both cases better accuracy than the standard approximation. Let us remark that although it is possible to postprocess the full approximation to the velocity, this procedure has the drawback of being difficult to implement. This is due to the fact that the bubble functions which are essential for the stability of the mini–element, are not local when passing from a coarse to a fine grid, see Verfürth (1998).

We observe that the computational cost of the postprocessed method is nearly the same as that of the mini–element approximation since the most time–consuming part is the time integration of the Navier–Stokes problem whereas the postprocessing step requires only the numerical solution of a stationary linear Stokes problem. As a result the postprocessed mini–element method is more efficient than the mini–element approximation itself.

The rest of the paper is as follows. In Section 2 we state some preliminaries and notations. In Section 3 we describe the postprocessed method. Finally, in Section 4 the convergence analysis is presented.

2. Preliminaries and notations

Let H and V be the Hilbert spaces

$$H = \{u \in \left(L^2(\Omega)\right)^d \mid \operatorname{div} \tilde{u} = 0, \quad u\cdot n_{|_{\partial\Omega}} = 0\},$$
$$V = \{u \in \left(H_0^1(\Omega)\right)^d \mid \operatorname{div} \tilde{u} = 0\},$$

endowed with the inner product of $L^2(\Omega)^d$ and $H_0^1(\Omega)^d$ respectively. For $1 \leqslant q \leqslant \infty$ and $k \geqslant 0$, we consider the standard Sobolev spaces $W^{k,q}(\Omega)^d$, of functions with derivatives up to order k in $L^q(\Omega)$, and $H^k(\Omega)^d = W^{k,2}(\Omega)^d$. The norm in $H^k(\Omega)^d$ will be denoted by $\|\cdot\|_k$ while $\|\cdot\|_{-k}$ will represent the norm of its dual space. $H^m(\Omega)/\mathbb{R}$ is the quotient space consisting of equivalence classes of elements of $H^m(\Omega)$ differing by constants; it is denoted by $L^2(\Omega)/\mathbb{R}$ if $m = 0$. Its norm is

$$\|p\|_{H^m(\Omega)/\mathbb{R}} = \inf_{c\in\mathbb{R}} \|p+c\|_m.$$

For the sake of brevity, when no confusion can arise, we shall write $\|p\|_m = \|p\|_{H^m(\Omega)/\mathbb{R}}$.

We shall frequently use the following Sobolev imbeddings Adams (1975). There exists a constant $C = C(\Omega, q)$ such that for $q \in [1,\infty)$, $q' < \infty$, it holds

$$\|v\|_{L^{q'}(\Omega)^d} \leqslant C\|v\|_{W^{s,q}(\Omega)^d}, \qquad \frac{1}{q} \geqslant \frac{1}{q'} \geqslant \frac{1}{q} - \frac{s}{d} > 0, \quad v \in W^{s,q}(\Omega)^d. \tag{2.1}$$

For $q' = \infty$, (2.1) holds with $\frac{1}{q} < \frac{s}{d}$.

The Leray projection $\Pi : L^2(\Omega)^d \to H$ is the orthogonal projection that maps each function in $L^2(\Omega)^d$ onto its divergence-free part. The Stokes operator A is defined by

$$A : D(A) \subset H \to H, \quad A = -\Pi\Delta, \quad D(A) = H^2(\Omega)^d \cap V.$$

The pressure free formulation of the Navier–Stokes equations (1.1) is the following

$$u_t + \nu Au + B(u,u) = \Pi f \quad \text{ in } \Omega,$$

where $B(u,v)=\Pi((u\cdot\nabla)v)$, $u,v\in H_0^1(\Omega)^d$.

We shall use the trilinear form $b(\cdot,\cdot,\cdot)$ defined by

$$b(u,v,w)=(F(u,v),w)\quad \forall u,v,w\in H_0^1(\Omega)^d, \tag{2.2}$$

where

$$F(u,v)=(u\cdot\nabla)v+\frac{1}{2}(\nabla\cdot u)v\quad \forall u,v\in H_0^1(\Omega)^d.$$

It is straightforward to verify that b enjoys the skew-symmetry property

$$b(u,v,w)=-b(u,w,v)\quad \forall u,v,w\in H_0^1(\Omega)^d. \tag{2.3}$$

Let us observe that $B(u,v)=\Pi F(u,v)$, if $u\in V$.

In what follows we will assume that the solution (u,p) of (1.1), (1.2) satisfies, for a given $T>0$,

$$\max_{0\leqslant t\leqslant T}(\|u(t)\|_2+\|p(t)\|_1)<\infty, \tag{2.4}$$

$$\max_{0\leqslant t\leqslant T}(\|u_t(t)\|_2+\|p_t(t)\|_1)<\infty, \tag{2.5}$$

$$\max_{0\leqslant t\leqslant T}(\|u_{tt}(t)\|_2+\|p_{tt}(t)\|_1)<\infty. \tag{2.6}$$

We refer the reader to Heywood & Rannacher (1982), Temam (1983) for a study of the regularity of the solutions to the Navier–Stokes equations. The regularity assumed requires some nonlocal compatibility assumptions at $t=0$ to be satisfied. From a practical point of view, the initial time $t=0$ can be replaced by an appropriate $t_0>0$ provided that $u(t_0)$ is accurately approximated.

In order to simplify the description of the method, in the following we shall suppose that Ω is a convex polygonal or polyhedral domain, although the results we present are applicable to the general case in which Ω is a bounded domain with smooth boundary, see Ayuso (2003), Ayuso & García-Archilla (2005).

Let $T_h=(\tau_i^h,\phi_i^h)_{i\in I_h}$, $h>0$, be a family of partitions of Ω, where the parameter h is the maximum diameter of the elements $\tau_i^h\in T_h$ and ϕ_i^h are affine mappings of the reference simplex τ_0 onto τ_i^h. We restrict ourselves to regular quasi-uniform meshes T_h.

Let $r\geqslant 2$, we consider the finite-element spaces

$$S_{h,r}=\left\{\chi_h\in C^0(\overline{\Omega}):\chi_h|_{\tau_i^h}\circ\phi_i^h\in P^{r-1}(\tau_0)\right\}\subset H^1(\Omega),$$

where $P^{r-1}(\tau_0)$ denotes the space of polynomials of degree at most $r-1$ on τ_0.

As a consequence of restricting our study to quasi-uniform partitions the following inverse inequality holds, for all $\tau\in T_h$ and $v_h\in(S_{h,r})^d$

$$\|v_h\|_{W^{m,q}(\tau)^d}\leqslant Ch^{l-m-d(\frac{1}{q'}-\frac{1}{q})}\|v_h\|_{W^{l,q'}(\tau)^d},\ 0\leqslant l\leqslant m\leqslant 2, 1\leqslant q'\leqslant q\leqslant\infty. \tag{2.7}$$

For a proof of (2.7) see (Ciarlet, 1991, Theorem 17.2).

In order to guarantee convergence of the mixed finite element (MFE) approximation we shall need to choose a stable combination of two finite element spaces. We shall denote by (X_h,Q_h) the so-called mini–element (see Arnold *et al.* (1984), Brezzi & Fortin (1991)) where

$$X_h=X_h^l\oplus\mathbb{B}_h,\quad X_h^l=S_{h,2}^d\cap H_0^1(\Omega)^d,\quad Q_h=S_{h,2}\cap L^2(\Omega)/\mathbb{R},$$

and $\mathbb{B}_h$ is the bubble space defined as,

$$\mathbb{B}_h = \Big\{ \sum_{\tau \in T_h} \alpha_\tau b_\tau : \alpha_\tau \in \mathbb{R},\ b_\tau(x) = (d+1)^{d+1} \prod_{j=1}^{d+1} \lambda_j(x) \in H_0^1(\tau) \Big\}.$$

Here $\lambda_j(x),\ j = 1,\dots,d+1$, denote the barycentric coordinates of each $x \in \tau$.

For $u_h \in X_h$, we will use the notation

$$u_h = u_h^l + u_h^b, \quad u_h^l \in X_h^l, \quad u_h^b \in \mathbb{B}_h.$$

Let us observe that the above decomposition is orthogonal with respect to the $H_0^1(\Omega)^d$-inner product. It is easy to prove that the bubble functions satisfy the following properties for all $x \in \tau$, $\tau \in T_h$:

$$\|b_\tau\|_{0,\tau} = Ch^{d/2}, \ \begin{cases} C_1 \leqslant \|\nabla b_\tau\|_{0,\tau} \leqslant C_2, & d = 2, \\ \\ \|b_\tau\|_{0,\tau} \leqslant Ch \|\nabla b_\tau\|_{0,\tau}, & d = 3. \end{cases} \tag{2.8}$$

For this mixed finite element method a uniform inf–sup condition is satisfied Arnold *et al.* (1984), see also Brezzi & Fortin (1991). There exists a constant $\beta > 0$ independent of the mesh grid size h such that

$$\inf_{q_h \in Q_h} \sup_{v_h \in X_h} \frac{(q_h, \operatorname{div} v_h)}{\|v_h\|_1 \|q_h\|_0} \geqslant \beta.$$

The discrete divergence free velocity space is defined as

$$V_h = \{\chi_h \in X_h : \int_\Omega q_h \operatorname{div} \chi_h \,\mathrm{d}x = 0 \quad \forall q_h \in Q_h\}.$$

Notice that $V_h \not\subset V$.

Let $\Pi_h : L^2(\Omega)^d \to V_h$ be the discrete Leray projection defined by

$$(\Pi_h(u), \chi_h) = (u, \chi_h) \quad \forall \chi_h \in V_h.$$

By definition, the projection is stable in the L^2 norm.

We will denote by A_h the discrete Stokes operator defined by

$$(\nabla v_h, \nabla \phi_h) = (A_h v_h, \phi_h) = (A_h^{1/2} v_h, A_h^{1/2} \phi_h) \quad \forall v_h, \phi_h \in V_h.$$

Due to the fact that A_h is a discrete self-adjoint operator, it is easy to show that, for each α, $0 \leqslant \alpha < 1$, there exists a positive constant C_α, which is independent of h, such that,

$$\|A_h^\alpha e^{-tA_h}\|_0 \leqslant C_\alpha t^{-\alpha}. \tag{2.9}$$

For $f \in L^2(\Omega)^d$, since for $v_h \in V_h$, we have $(A_h^{-1/2} \Pi_h f, v_h) = (f, A_h^{-1/2} v_h)$, it follows that

$$\|A_h^{-1/2} \Pi_h f\|_0 \leqslant C \|f\|_{-1}. \tag{2.10}$$

There exists a piecewise linear (quasi) interpolant such that (see Bernardi (1989))

$$\|v - I_h(v)\|_0 + h \|v - I_h(v)\|_1 \leqslant Ch^2 \|v\|_2 \quad \forall v \in H^2(\Omega)^d \cap H_0^1(\Omega)^d. \tag{2.11}$$

Following Girault & Raviart (1986), we consider the projection $J_h : L^2(\Omega)^d \to Q_h$ defined by

$$J_h q = P_h q - \frac{1}{\text{meas}(\Omega)} \int_\Omega P_h q \, \mathrm{d}x,$$

where P_h is the standard $L^2(\Omega)$-projection onto $S_{h,2}$. For $q \in H^1(\Omega)/\mathbb{R}$ the following bound holds:

$$\|q - J_h q\|_{L^2(\Omega)/\mathbb{R}} \leqslant Ch\|q\|_{H^1(\Omega)/\mathbb{R}}. \tag{2.12}$$

Moreover, the projection is stable in the H^1-norm (see Bramble *et al.* (2002)):

$$\|J_h q\|_{H^1(\Omega)/\mathbb{R}} \leqslant C\|q\|_{H^1(\Omega)/\mathbb{R}}. \tag{2.13}$$

We next introduce a projection operator that will play a key role in our analysis, the so-called discrete Stokes projection, see Heywood & Rannacher (1982). Let $(u,p) \in \left(H^2(\Omega)^d \cap V\right) \times \left(H^1(\Omega)/\mathbb{R}\right)$ be the solution of a Stokes problem with right–hand side g, its discrete Stokes projection $S_h(u) \in V_h$ is defined by

$$\nu(\nabla S_h(u), \nabla \chi_h) = \nu(\nabla u, \nabla \chi_h) - (p, \text{div } \chi_h) = (g, \chi_h) \quad \forall \chi_h \in V_h.$$

We will also consider the above definition of $S_h(u)$ written in mixed form: find $(s_h, q_h) \in (X_h, Q_h)$, $s_h = S_h(u)$, such that

$$\nu(\nabla s_h, \nabla \phi_h) + (\nabla q_h, \phi_h) = (g, \phi_h) \quad \forall \phi_h \in X_h, \tag{2.14}$$

$$(\text{div } s_h, \psi_h) = 0 \qquad \forall \psi_h \in Q_h. \tag{2.15}$$

The following bounds hold Arnold *et al.* (1984),

$$\|u - s_h\|_0 + h\|u - s_h\|_1 \leqslant Ch^2\left(\|u\|_2 + \|p\|_1\right), \tag{2.16}$$

$$\|p - q_h\|_0 \leqslant Ch\left(\|u\|_2 + \|p\|_1\right). \tag{2.17}$$

In the sequel we will apply the above estimates to the particular case in which (u,p) is the solution of the Navier–Stokes problem (1.1), (1.2). In that case $S_h(u)$ is the discrete velocity in problem (2.14), (2.15) with $g = f - u_t - (u \cdot \nabla u)$. Note that the temporal variable t appears here merely as a parameter and then, taking the time derivative and given the regularity assumptions (2.5), (2.6), the error bounds (2.16), (2.17) can also be applied to the first and second time derivatives of $S_h(u)$ replacing u, p by u_t, p_t or u_{tt}, p_{tt}, respectively.

We finish this section with a lemma that establishes the rate of convergence of the linear part of the discrete approximation to the velocity in a general Stokes problem.

LEMMA 2.1 Let $(u,p) \in \left(H^2(\Omega)^d \cap V\right) \times \left(H^1(\Omega)/\mathbb{R}\right)$ be the solution of a Stokes problem with right–hand side g, and let (s_h, q_h) be its mini–element approximation. Then, the following error bound holds

$$\|u - s_h^l\|_0 + h\|u - s_h^l\|_1 \leqslant Ch^2\left(\|u\|_2 + \|p\|_1\right), \tag{2.18}$$

where $s_h = s_h^l + s_h^b$, $s_h^l \in X_h^l$, $s_h^b \in \mathbb{B}_h$.

Proof. We first observe that (s_h, q_h) satisfies (2.14)-(2.15). Let us write $s_h^b = \sum_{\tau\in T_h} c_\tau b_\tau$, with $b_\tau \in \mathbb{B}_h$. Taking in (2.14) $\phi_h = b_\tau$, as test function, we get

$$\nu(\nabla s_h, \nabla b_\tau) + (\nabla q_h, b_\tau) = \nu(\nabla(s_h^l + \sum_{\tau\in T_h} c_\tau b_\tau), \nabla b_\tau) + (\nabla q_h, b_\tau) = (g, b_\tau),$$

so that

$$c_\tau \nu(\nabla b_\tau, \nabla b_\tau) + (\nabla q_h, b_\tau) = (g, b_\tau) \quad \forall \tau \in T_h.$$

Then, the coefficients c_τ satisfy

$$c_\tau = \frac{(g, b_\tau)}{\nu(\nabla b_\tau, \nabla b_\tau)} - \frac{(\nabla q_h, b_\tau)}{\nu(\nabla b_\tau, \nabla b_\tau)} \quad \forall \tau \in T_h.$$

Let us bound $\|s_h^b\|_0$. Using (2.8) we get

$$\begin{aligned}
\|s_h^b\|_0^2 &= \sum_{\tau\in T_h} c_\tau^2 \|b_\tau\|_0^2 \leqslant Ch^d \sum_{\tau\in T_h} c_\tau^2 \\
&\leqslant Ch^d \sum_{\tau\in T_h} \left(\frac{|(g,b_\tau)|^2}{\nu^2\|\nabla b_\tau\|_0^4} + \frac{|(q_h - p, \operatorname{div} b_\tau)|^2}{\nu^2\|\nabla b_\tau\|_0^4} + \frac{|(\nabla p, b_\tau)|^2}{\nu^2\|\nabla b_\tau\|_0^4} \right) \\
&\leqslant Ch^d \sum_{\tau\in T_h} \left(\frac{\|g\|_{0,\tau}^2 \|b_\tau\|_{0,\tau}^2}{\nu^2\|\nabla b_\tau\|_0^4} + \frac{\|p - q_h\|_{0,\tau}^2}{\nu^2\|\nabla b_\tau\|_0^2} + \frac{\|\nabla p\|_{0,\tau}^2 \|b_\tau\|_{0,\tau}^2}{\nu^2\|\nabla b_\tau\|_0^4} \right).
\end{aligned}$$

We observe that in view of (2.8) it follows that, for $\tau \in T_h$, $d = 2, 3$,

$$\begin{aligned}
\|\nabla b_\tau\|_{0,\tau} &\geqslant Ch^{d/2-1}, \\
\frac{\|b_\tau\|_{0,\tau}}{\|\nabla b_\tau\|_{0,\tau}} &\leqslant Ch.
\end{aligned}$$

Using the above inequalities together with (2.17) we get

$$\|s_h^b\|_0^2 \leqslant Ch^4 \left(\|g\|_0^2 + \|u\|_2^2 + \|p\|_1^2 \right) \leqslant Ch^4 \left(\|u\|_2^2 + \|p\|_1^2 \right),$$

and so

$$\|s_h^b\|_0 \leqslant Ch^2(\|u\|_2 + \|p\|_1). \tag{2.19}$$

Using now the inverse inequality (2.7) we obtain

$$\|\nabla s_h^b\|_0 \leqslant Ch(\|u\|_2 + \|p\|_1). \tag{2.20}$$

To estimate $\|u - s_h^l\|_1$ we observe that

$$\begin{aligned}
\|\nabla(u - s_h^l)\|_0^2 &= \|\nabla(u - s_h)\|_0^2 + \|\nabla s_h^b\|_0^2 + 2(\nabla(u - s_h), \nabla s_h^b) \\
&\leqslant 2(\|\nabla(u - s_h)\|_0^2 + \|\nabla s_h^b\|_0^2) \\
&\leqslant Ch^2(\|u\|_2 + \|p\|_1)^2,
\end{aligned} \tag{2.21}$$

where we have used (2.16) and (2.20) in the last inequality.

Next, we estimate $\|u-s_h^l\|_0$ reasoning by duality.

$$\begin{aligned}\|u-s_h^l\|_0 &= \sup_{\|\phi\|_0=1} |(u-s_h^l,\phi)| \\ &\leqslant \sup_{\|\phi\|_0=1} (|(\nabla(u-s_h^l),\nabla w)| + |(\text{div }(u-s_h^l),z)|),\end{aligned}$$

where (w,z) are given by the dual Stokes problem

$$-\Delta w + \nabla z = \phi, \quad \text{in } \Omega, \quad w=0 \quad \text{at } \partial\Omega, \quad \text{div } w = 0 \quad \text{in } \Omega.$$

Let us denote by $w_h = w_h^l + w_h^b \in X_h^l \oplus \mathbb{B}_h$ the finite element approximation to w. We observe that taking $\phi_h = v_h \in X_h^l$ as test function in (2.14) we get

$$(\nabla s_h^l, \nabla v_h) + (\nabla q_h, v_h) = (g, v_h) \quad \forall v_h \in X_h^l,$$

then

$$\begin{aligned}(\nabla(u-s_h^l),\nabla w) &= (\nabla(u-s_h^l),\nabla(w-w_h^l)) + (\nabla(q_h-p),w_h^l) \\ &\leqslant \|u-s_h^l\|_1\|w-w_h^l\|_1 + (q_h-p,\nabla\cdot(w-w_h^l)) \\ &\leqslant \|u-s_h^l\|_1\|w-w_h^l\|_1 + \|q_h-p\|_0\|w-w_h^l\|_1 \\ &\leqslant Ch^2(\|u\|_2+\|p\|_1)(\|w\|_2+\|z\|_1) \\ &\leqslant Ch^2(\|u\|_2+\|p\|_1)\|\phi\|_0,\end{aligned}$$

after using (2.21) and the regularity of the Stokes problem (see Kellog & Osborn (1975), Dauge (1989)).

For the second term we have

$$\begin{aligned}(\text{div }(u-s_h^l),z) &= (\text{div }(u-s_h^l),z-J_hz) + (\text{div }(u-s_h^l),J_hz) \\ &\leqslant \|u-s_h^l u\|_1\|z-J_hz\|_0 - (\text{div } s_h^l, J_hz).\end{aligned}$$

Using (2.15) we get

$$-(\text{div } s_h^l, J_hz) = (\text{div } s_h^b, J_hz).$$

Finally, applying the above identity together with (2.21) and (2.12) we obtain

$$\begin{aligned}(\text{div }(u-s_h^l),z) &\leqslant Ch(\|u\|_2+\|p\|_1)Ch\|z\|_1 - (s_h^b,\nabla J_hz) \\ &\leqslant Ch^2(\|u\|_2+\|p\|_1)\|\phi\|_0 + \|s_h^b\|_0\|J_hz\|_1 \\ &\leqslant 2Ch^2(\|u\|_2+\|p\|_1)\|\phi\|_0,\end{aligned}$$

after using (2.13), (2.19) and the regularity of the Stokes problem. The proof is now complete. □

3. The postprocessed method

Let us suppose that we want to approximate the solution of (1.1), (1.2) at a given time $T>0$. The postprocessing technique can be seen as a two-level method.

1. Compute the mixed finite element approximation to (1.1), (1.2) at time T. Given $u^h(0)$ an initial approximation to $u(0)$ we find $u_h : [0,T] \to X_h$, and $p_h : [0,T] \to Q_h$ satisfying,

$$(\dot{u}_h, \phi_h) + \nu(\nabla u_h, \nabla \phi_h) + b(u_h, u_h, \phi_h) + (\nabla p_h, \phi_h) = (f, \phi_h) \qquad \forall \phi_h \in X_h \tag{3.1}$$

$$(\mathrm{div}\ u_h, \psi_h) = 0 \qquad \forall \psi_h \in Q_h \tag{3.2}$$

where $b(\cdot,\cdot,\cdot)$ has been defined in (2.2). As initial condition we take $u_h(0) = S_h(u_0)$ although other choices are possible.

2. Retain only the linear part of $u_h(T)$ and compute $(\tilde{u}_h, \tilde{p}_h) \in (\widetilde{X}, \widetilde{Q})$ satisfying,

$$\nu(\nabla \tilde{u}_h, \nabla \tilde{\phi}) + (\nabla \tilde{p}_h, \tilde{\phi}) = (f, \tilde{\phi}) - b(u_h^l(T), u_h^l(T), \tilde{\phi}) - (\dot{u}_h^l(T), \tilde{\phi}) \qquad \forall \tilde{\phi} \in \widetilde{X} \tag{3.3}$$

$$(\mathrm{div}\ \tilde{u}_h, \tilde{\psi}) = 0 \qquad \forall \tilde{\psi} \in \widetilde{Q} \tag{3.4}$$

where $(\widetilde{X}, \widetilde{Q})$ is a given stable mixed finite element space with better approximation capabilities than (X_h, Q_h).

Note that the second step requires only the solution of a single discrete Stokes problem with right–hand side depending on the previously computed approximation at the final time T. The computational cost of this step is usually very small (depending of the particular choice of $(\widetilde{X}, \widetilde{Q})$) compared with the computational cost of the first step.

Several choices of $(\widetilde{X}, \widetilde{Q})$ are possible. In this paper we consider two options that lead to optimal order (up to a logarithmic term) for the errors in velocity $\|u(T) - \tilde{u}(T)\|_1$ and pressure $\|p(T) - \tilde{p}(T)\|_0$:

- The mini–element over a finer grid

$$(\widetilde{X}, \widetilde{Q}) = (X_{h'}, Q_{h'}),\ h' < h.$$

- The Hood-Taylor P^2P^1 element over the same grid

$$(\widetilde{X}, \widetilde{Q}) = \left(S_{h,3}^d \cap H_0^1(\Omega)^d, Q_h\right).$$

In next section we show that $(\tilde{u}_h, \tilde{p}_h)$ is an approximation to the solution of (1.1), (1.2) at time T that has essentially the order of approximation of the space $(\widetilde{X}, \widetilde{Q})$. More precisely, we will prove that the rate of convergence of the approximate velocity can be improved by one unit, up to a logarithmic term, in the H^1 norm, by using the postprocessing technique. The same improvement is obtained for the L^2 norm of the discrete pressure, see Theorems 4.4 and 4.5 in Section 4. The rate of convergence for the postprocessed velocity in the L^2 norm is not increased, see Ayuso (2003). This was first observed for reaction-convection-diffusion equations in de Frutos & Novo (2002).

4. Analysis of the postprocessed method

This section is devoted to the analysis of convergence of the postprocessed MFE method. Our first goal will be to show a superconvergence result for the error between the MFE approximation to the velocity, u_h and the Stokes projection of the velocity field u, s_h. This superconvergence behaviour occurs only in the H^1 norm and it will be shown in Theorem 4.3 in which we shall concentrate our efforts on the first part of the section. The argument is based on a stability plus consistency argument (Theorems 4.1

and 4.2, respectively). For the purpose of analysis, we shall mainly be concerned with the pressure-free formulation associated with (3.1), (3.2). If (u_h, p_h) is the mixed finite element approximation to the solution (u, p) of (1.1), (1.2) then $u_h \in V_h$ is the solution of

$$(\dot{u}_h, \chi_h) + \nu(\nabla u_h, \nabla \chi_h) + b(u_h, u_h, \chi_h) = (f, \chi_h) \quad \forall \chi_h \in V_h,$$

which can also be expressed in abstract operator form as,

$$\dot{u}_h + \nu A_h u_h + \Pi_h F(u_h, u_h) = \Pi_h f.$$

The Stokes projection s_h satisfies the abstract equation,

$$\dot{s}_h + \nu A_h s_h + \Pi_h F(s_h, s_h) = \Pi_h f + T_h, \tag{4.1}$$

where $T_h(t)$ is the truncation error, defined by

$$T_h(t) = \dot{s}_h - \Pi_h(u_t) + \Pi_h F(s_h, s_h) - \Pi_h F(u, u). \tag{4.2}$$

Let us now consider mappings $v_h : [0, T] \to V_h$ satisfying the following threshold condition:

$$\|s_h(t) - v_h(t)\|_1 \leqslant ch \qquad \forall\, t \in [0, t_1], \quad 0 < t_1 \leqslant T. \tag{4.3}$$

We define their truncation error as

$$\widehat{T}_h = \dot{v}_h + \nu A_h v_h + \Pi_h F(v_h, v_h) - \Pi_h f. \tag{4.4}$$

Prior to establishing the stability restricted to the threshold (4.3) (Theorem 4.1) we state and prove a lemma that can also be found in Ayuso *et al.* (2005) (see also de Frutos & Novo (2000c)). The proof of the lemma is included for the reader's convenience.

LEMMA 4.1 Let (u, p) be the solution of the Navier–Stokes problem (1.1), (1.2) satisfying (2.4). Let $s_h = S_h(u)$ be the discrete Stokes projection of the velocity field u and let $v_h : [0, T] \to V_h$ satisfying the threshold condition (4.3). Then, there exist a constant $K > 0$, independent of t_1 in (4.3), such that, for all $t \in [0, t_1]$,

$$\|F(s_h(t), s_h(t)) - F(v_h(t), v_h(t))\|_0 \leqslant K\|s_h(t) - v_h(t)\|_1, \tag{4.5}$$

$$\|F(s_h(t), s_h(t)) - F(v_h(t), v_h(t))]\|_{-1} \leqslant K\|s_h(t) - v_h(t)\|_0, \tag{4.6}$$

where the constant K depends on c and $\max_{0 \leqslant t \leqslant T}(\|u(t)\|_2 + \|p(t)\|_1)$.

Proof. In order to simplify the notation, we shall omit the dependence on t in the proof. We define $e_h = v_h - s_h$. We proceed by standard duality arguments, using the splitting

$$F(v_h, v_h) - F(s_h, s_h) = F(v_h, e_h) + F(e_h, s_h). \tag{4.7}$$

We start by showing (4.5). Observe that

$$\begin{aligned} \left\|F(e_h, s_h)\right\|_0 &= \sup_{\|\phi\|_0 = 1} \left|(e_h \cdot \nabla s_h, \phi) + \frac{1}{2}((\mathrm{div}\, e_h) s_h, \phi)\right| \\ &\leqslant C\|e_h\|_{L^{2d/(d-1)}(\Omega)^d} \|\nabla s_h\|_{L^{2d}(\Omega)^d} + C\|e_h\|_1 \|s_h\|_\infty. \end{aligned}$$

Let us show that both $\|s_h\|_\infty$, $\|\nabla s_h\|_{L^{2d}(\Omega)^d}$ are bounded. Since, by virtue of the Sobolev imbeddings (2.1) we have $\|s_h\|_\infty \leqslant C\|\nabla s_h\|_{L^{2d}(\Omega)^d}$ we only need to bound the second term. Application of the inverse inequality (2.7) and the error estimates (2.16) and (2.11) together with (2.1) give

$$\begin{aligned}\|\nabla s_h\|_{L^{2d}(\Omega)^d} &\leqslant Ch^{-(1+d)/2}(\|s_h-u\|_0+\|u-I_hu\|_0)+\|\nabla I_hu\|_{L^{2d}(\Omega)^d}\\ &\leqslant Ch^{(3-d)/2}(\|u\|_2+\|p\|_1)+C\|u\|_{W^{1,2d}(\Omega)^d}\leqslant K.\end{aligned} \tag{4.8}$$

Applying again (2.1) we obtain $\|e_h\|_{L^{2d/(d-1)}(\Omega)^d} \leqslant C\|e_h\|_1$ and so

$$\big\|F(e_h,s_h)\big\|_0 \leqslant K\|e_h\|_1.$$

To estimate the other term in (4.7), we note that the same arguments lead to

$$\begin{aligned}\big\|F(v_h,e_h)\big\|_0 &= \sup_{\|\phi\|_0=1}\big|(v_h\cdot\nabla e_h,\phi)+\frac{1}{2}((\nabla\cdot v_h)e_h,\phi)\big|\\ &\leqslant C\|v_h\|_\infty\|e_h\|_1+C\|\nabla\cdot v_h\|_{L^{2d}(\Omega)^d}\|e_h\|_{L^{2d/(d-1)}(\Omega)^d}.\end{aligned}$$

As before, to conclude we must show that the above norms of v_h are bounded. We only need to estimate $\|\nabla v_h\|_{L^{2d}(\Omega)^d}$, using the inverse inequality (2.7), the threshold condition (4.3) and (4.8) we find

$$\|\nabla v_h\|_{L^{2d}(\Omega)^d} \leqslant h^{-(d-1)/2}\|v_h-s_h\|_1+\|\nabla s_h\|_{L^{2d}(\Omega)^d}\leqslant c_kh^{(3-d)/2}+K\leqslant K.$$

Therefore, (4.5) follows. We now show (4.6). Applying (4.7) we find

$$\|F(v_h,v_h)-F(s_h,s_h)\|_{-1}\leqslant \|F(v_h,e_h)\big\|_{-1}+\|F(e_h,s_h)\big\|_{-1},$$

so that the proof is reduced to estimating each of the above negative norms on the right-hand-side. Using the skew-symmetry property (2.3), one gets for the first term

$$\begin{aligned}\big\|F(v_h,e_h)\big\|_{-1} &= \sup_{\|\phi\|_1=1}\big|((v_h\cdot\nabla)\phi,e_h)+\frac{1}{2}((\nabla\cdot v_h)\phi,e_h)\big|\\ &\leqslant \sup_{\|\phi\|_1=1}\Big(\|e_h\|_0\|v_h\|_\infty\|\phi\|_1+\|e_h\|_0\|\nabla\cdot v_h\|_{L^{2d/(d-1)}(\Omega)^d}\|\phi\|_{L^{2d}(\Omega)^d}\Big)\\ &\leqslant K\|e_h\|_0.\end{aligned}$$

To deal with the other term in (4.7) we integrate by parts and obtain

$$\begin{aligned}\big\|F(e_h,s_h)\big\|_{-1} &= \sup_{\|\phi\|_1=1}\big|\frac{1}{2}\big((e_h\cdot\nabla)s_h,\phi\big)-\frac{1}{2}\big((e_h\cdot\nabla)\phi,s_h\big)\big|\\ &\leqslant \sup_{\|\phi\|_1=1}\Big(\|e_h\|_0\|\nabla s_h\|_{L^{2d/(d-1)}(\Omega)^d}\|\phi\|_{L^{2d}(\Omega)^d}+\|e_h\|_0\|\phi\|_1\|s_h\|_\infty\Big)\\ &\leqslant K\|e_h\|_0.\end{aligned}$$

This completes the proof of (4.6). □

THEOREM 4.1 (Stability). Fix $T > 0$, let $s_h = S_h(u)$ be the discrete Stokes projection of the velocity field u solution of (1.1), (1.2) satisfy (2.4) and let $v_h : [0,T] \to V_h$ satisfying the threshold condition (4.3). Then, there exists a positive constant $K_s > 0$ such that the following estimate holds

$$\max_{0\leqslant t\leqslant t_1} \|s_h(t) - v_h(t)\|_1 \leqslant e^{K_s t_1}\Big(\|s_h(0) - v_h(0)\|_1 + \max_{0\leqslant t\leqslant t_1} \Big\|\int_0^t e^{-\nu(t-s)A_h} A_h^{1/2}[T_h(s) - \widehat{T}_h(s)]\,\mathrm{d}s\Big\|_0\Big),$$

where $T_h(s)$ and $\widehat{T}_h(s)$ are the truncation errors given in (4.2) and (4.4) respectively.

Proof. We define $e_h = s_h - v_h$. Subtracting (4.4) from (4.1) it follows that e_h satisfies the error equation

$$\dot{e}_h(t) + \nu A_h e_h(t) = \Pi_h F(v_h(t), v_h(t)) - \Pi_h F(s_h(t), s_h(t)) + T_h(t) - \widehat{T}_h(t).$$

Then, by integrating from time 0 up to time t the above error equation we find that

$$e_h(t) = e^{-\nu t A_h} e_h(0) + \int_0^t e^{-\nu(t-s)A_h} \Pi_h\big[F(v_h, v_h) - F(s_h, s_h)\big]\,\mathrm{d}s + \int_0^t e^{-\nu(t-s)A_h}[T_h(s) - \widehat{T}_h(s)]\,\mathrm{d}s.$$

Applying $A_h^{1/2}$ and taking norms we get

$$\|A_h^{1/2} e_h(t)\|_0 \leqslant \|e^{-\nu t A_h} A_h^{1/2} e_h(0)\|_0 + \Big\|\int_0^t e^{-\nu(t-s)A_h} A_h^{1/2} \Pi_h\big[F(v_h, v_h) - F(s_h, s_h)\big]\,\mathrm{d}s\Big\|_0 + \Big\|\int_0^t e^{-\nu(t-s)A_h} A_h^{1/2}[T_h(s) - \widehat{T}_h(s)]\,\mathrm{d}s\Big\|_0.$$

Since $\{e^{-\nu t A_h}\}_{t>0}$ is a contraction

$$\|e^{-\nu t A_h} A_h^{1/2} e_h(0)\|_0 \leqslant \|e_h(0)\|_1.$$

As regards the second term, estimates (2.9) and (4.5) from Lemma 4.1, lead to

$$\Big\|\int_0^t e^{-\nu(t-s)A_h} A_h^{1/2} \Pi_h(F(s_h, s_h) - \Pi_h F(v_h, v_h)) ds\Big\|_0 \leqslant \frac{C}{\sqrt{\nu}} \int_0^t \frac{\big\|F(s_h, s_h) - F(v_h, v_h)\big\|_0}{\sqrt{t-s}}\,\mathrm{d}s \leqslant \frac{KC}{\sqrt{\nu}} \int_0^t \frac{\|e_h(s)\|_1}{\sqrt{t-s}}\,\mathrm{d}s.$$

Then,

$$\|e_h(t)\|_1 \leqslant \|e_h(0)\|_1 + \frac{KC}{\sqrt{\nu}} \int_0^t \frac{\|e_h(s)\|_1}{\sqrt{t-s}}\,\mathrm{d}s + \Big\|\int_0^t e^{-\nu(t-s)A_h} A_h^{1/2}[T_h(s) - \widehat{T}_h(s)]\,\mathrm{d}s\Big\|.$$

And now, a standard application of the generalized Gronwall Lemma (see, for example, (Henry, 1991, pp. 188-189)) allow us to conclude the proof. □

The following lemma will be required in the proof of Theorem 4.2

LEMMA 4.2 Let (u,p) be the solution of (1.1–1.2) satisfying (2.4-2.5-2.6). Then, there exists a positive constant $K=K(u,p)$ such that for all $t\in[0,T]$ the truncation error defined in (4.2) and its time derivative satisfy the following bound

$$\|A_h^{-1/2}T_h(t)\|_0 \leqslant Kh^2, \quad \|A_h^{-1/2}\dot{T}_h(t)\|_0 \leqslant Kh^2.$$

Proof. In view of definition (4.2) we observe that

$$\|A_h^{-1/2}T_h(t)\|_0 \leqslant \|A_h^{-1/2}\Pi_h(\dot{s}_h-u_t)\|_0 + \|A_h^{-1/2}\Pi_h(F(s_h,s_h)-F(u,u))\|_0.$$

Applying (2.10) it follows

$$\|A_h^{-1/2}T_h(t)\|_0 \leqslant \|\dot{s}_h-u_t\|_{-1} + \|F(s_h,s_h)-F(u,u)\|_{-1} \leqslant \|\dot{s}_h-u_t\|_0 + K\|s_h-u\|_0,$$

where we have used (4.6) from Lemma 4.1 in the last inequality. We observe that Lemma 4.1 can also be applied by taking u instead of v_h. Now applying (2.16) to u and its time derivative u_t we obtain

$$\|A_h^{-1/2}T_h(t)\|_{-1} \leqslant Kh^2.$$

The same reasoning applies to the time derivative of the truncation error

$$\begin{aligned}\|A_h^{-1/2}\dot{T}_h(t)\|_0 \leqslant{} & \|(s_h-u)_{tt}\|_{-1} + \|F(\dot{s}_h-u_t,s_h)\|_{-1} + \|F(u_t,s_h-u)\|_{-1} \\ & + \|F(s_h-u,\dot{s}_h)\|_{-1} + \|F(u,\dot{s}_h-u_t)\|_{-1}.\end{aligned} \tag{4.9}$$

Applying (2.16) to the second time derivative of u we get for the first term on the right-hand side of (4.9)

$$\|(s_h-u)_{tt}\|_{-1} \leqslant \|(s_h-u)_{tt}\|_0 \leqslant Ch^2(\|u_{tt}\|_2 + \|p_{tt}\|_1).$$

For the remaining terms reasoning similarly as in the proof of Lemma 4.1 (using that $\|\nabla s_h\|_{L^{2d}}$, $\|\nabla \dot{s}_h\|_{L^{2d}}$ are bounded) and applying again (2.16) we obtain

$$\begin{aligned}\|F(\dot{s}_h-u_t,s_h)\|_{-1}, \|F(u,\dot{s}_h-u_t)\|_{-1} &\leqslant K\|\dot{s}_h-u_t\|_0 \leqslant Kh^2, \\ \|F(u_t,s_h-u)\|_{-1}, \|F(s_h-u,\dot{s}_h)\|_{-1} &\leqslant K\|s_h-u\|_0 \leqslant Kh^2,\end{aligned}$$

which completes the proof. □

THEOREM 4.2 (Consistency). Let (u,p) be the solution of (1.1), (1.2) satisfying (2.4), (2.5), (2.6). Then, there exists a positive constant $K=K(u,p)$ such that

$$\max_{0\leqslant t\leqslant T}\Big\|\int_0^t e^{-\nu(t-s)A_h}A_h^{1/2}T_h(s)\,\mathrm{d}s\Big\|_0 \leqslant \frac{K}{\nu}h^2. \tag{4.10}$$

Proof. We first integrate by parts to get

$$\begin{aligned}\int_0^t e^{-\nu(t-s)A_h}A_h^{1/2}T_h(s)\,\mathrm{d}s = {} & (\nu A_h)^{-1}(A_h^{1/2}T_h(t) - e^{-\nu t A_h}A_h^{1/2}T_h(0)) \\ & - \frac{1}{\nu}\int_0^t e^{-\nu(t-s)A_h}A_h^{-1/2}\dot{T}_h(s)\,\mathrm{d}s.\end{aligned}$$

Taking into account that $e^{-\nu A_h(t-s)}$ is a contraction, the left–hand side of (4.10) can be bounded by means of

$$\max_{0\leqslant s\leqslant T}\|A_h^{-1/2}T_h(s)\|_0, \quad \max_{0\leqslant s\leqslant T}\|A_h^{-1/2}\dot{T}_h(s)\|_0$$

and then, since Lemma 4.2 provides the required estimates for the truncation error, we obtain (4.10). □

THEOREM 4.3 (Superconvergence for the velocity). Let (u,p) be the solution of (1.1), (1.2) satisfying (2.4), (2.5), (2.6), let s_h be the Stokes projection of u and let u_h be the mini–element approximation to u. Then, there exists a positive constant $K(u,p,\nu)$ such that for h small enough the following bound holds

$$\max_{0\leqslant t\leqslant T} \|s_h(t)-u_h(t)\|_1 \leqslant K(u,p,\nu)h^2. \tag{4.11}$$

Proof. Since $u_h(0)=s_h(0)$, the proof follows from Theorem 4.1 (applied to $v_h=u_h$) and Theorem 4.2. The threshold condition (4.3) needed for Theorem 4.1 to be valid is proved by a standard bootstrap argument as follows. Let us define

$$t_1 = \max\left\{t \mid \ v_h(\tilde t)=u_h(\tilde t) \text{ satisfies } (4.3) \text{ for } \tilde t\in[0,t]\right\}.$$

By continuity we have that $t_1>0$. We now show that $t_1=T$ for h small enough. Since u_h satisfies (4.3) for $t\in[0,t_1]$ applying Theorems 4.1 and 4.2 it follows that for h small enough

$$\|s_h(t_1)-u_h(t_1)\|_1 \leqslant e^{K_sT}\frac{K}{\nu}h^2 \leqslant \frac{c_k h}{2}.$$

Then, there exists $t_2>t_1$ such that (4.3) holds for $v_h=u_h$ and $t\in[0,t_2]$. But this contradicts the definition of t_1. Therefore t_1 must be equal to T. □

COROLLARY 4.1 Let (u,p) be the solution of (1.1), (1.2) satisfying (2.4), (2.5), (2.6), let s_h^l be the linear part of the Stokes projection of u and let u_h^l be the linear part of the mini–element approximation to u. Then, there exist positive constants $K(u,p,\nu)$ and h_0 such that $\forall h\leqslant h_0$ the following bound holds

$$\max_{0\leqslant t\leqslant T} \|s_h^l(t)-u_h^l(t)\|_1 \leqslant K(u,p,\nu)h^2. \tag{4.12}$$

Proof. By virtue of the H_0^1-orthogonal decomposition of the velocity space X_h we have

$$\|s_h-u_h\|_1^2 = \|s_h^l-u_h^l\|_1^2+\|s_h^b-u_h^b\|_1^2,$$

so that (4.12) is deduced by applying (4.11). □

As a consequence of Theorem 4.3 and Corollary 4.1 the rates of convergence for u_h and u_h^l are obtained.

COROLLARY 4.2 Let (u,p) be the solution of (1.1), (1.2) satisfying (2.4), (2.5), (2.6) and (u_h,p_h) its mini–element approximation. Then, there exist positive constants h_0 and $K(u,p,\nu)$ such that the following bounds hold for all $h\leqslant h_0$:

$$\max_{0\leqslant t\leqslant T} \|u(t)-u_h(t)\|_0 \leqslant Kh^2, \tag{4.13}$$
$$\max_{0\leqslant t\leqslant T} \|u(t)-u_h(t)\|_1 \leqslant Kh,$$
$$\max_{0\leqslant t\leqslant T} \|u(t)-u_h^l(t)\|_0 \leqslant Kh^2, \tag{4.14}$$
$$\max_{0\leqslant t\leqslant T} \|u(t)-u_h^l(t)\|_1 \leqslant Kh.$$

Proof. By decomposing $u-u_h=(u-s_h)+(s_h-u_h)$ and $u-u_h^l=(u-s_h^l)+(s_h^l-u_h^l)$ and appealing to Theorem 4.3 and Corollary 4.1 together with estimates (2.16) and (2.18) from Lemma 2.1 one easily reaches the conclusion of the corollary. □

Lemma 4.4 provides an estimate for the time derivative of the error in the linear part of the MFE approximation to the velocity. The following lemma will be required in its proof.

LEMMA 4.3 Let $f \in C([0,T];L^2(\Omega)^d)$; then following estimate holds for all $t \in [0,T]$:

$$\int_0^t \left\|A_h e^{-\nu(t-s)A_h}\Pi_h f(s)\right\|_0 \,\mathrm{d}s \leqslant \frac{C}{\nu}|\log(h)| \max_{0\leqslant t\leqslant T} \|f(t)\|_0.$$

Proof. The proof follows essentially the same steps as those given in de Frutos & Novo (2000c) and García-Archilla & Titi (2000). □

LEMMA 4.4 Let (u,p) be the solution of (1.1), (1.2) satisfying (2.4), (2.5), (2.6) and let $u_h : [0,T] \to V_h$ be the mini–element approximation to the velocity. Then, the following estimate holds

$$\max_{0\leqslant t\leqslant T} \left\|u_t(t) - \dot{u}_h^l(t)\right\|_0 \leqslant K(u,p,\nu)h^2|\log(h)|. \tag{4.15}$$

Proof. For simplicity, we shall drop the explicit dependence on the time t in the proof. We consider the splitting

$$u_t - \dot{u}_h^l = (u_t - \dot{s}_h^l) + (\dot{s}_h^l - \dot{u}_h^l).$$

Using (2.18) from Lemma 2.1 we obtain for the first term

$$\|u_t - \dot{s}_h^l\|_0 \leqslant Ch^2(\|u_t\|_2 + \|p_t\|_1).$$

For the second one, applying the H_0^1-orthogonality of the decomposition of the velocity space $X_h = X_h^l \oplus \mathbb{B}_h$, we get

$$\|\dot{s}_h^l - \dot{u}_h^l\|_0 \leqslant \|\dot{s}_h^l - \dot{u}_h^l\|_1 \leqslant \|\dot{s}_h - \dot{u}_h\|_1.$$

Then, we are left with the task of getting a bound for the error $\|\dot{s}_h - \dot{u}_h\|_1$. We reason as in the proof of Theorem 4.1. Denoting $e_h = s_h - u_h$, it follows easily that

$$\dot{e}_h = -\nu A_h e_h + \Pi_h F(u_h,u_h) - \Pi_h F(u,u) + \Pi_h(\dot{s}_h - \dot{u}_t).$$

Setting $r_h = \dot{e}_h = \dot{s}_h - \dot{u}_h$ and differentiating with respect to t we obtain

$$\dot{r}_h = -\nu A_h r_h + \Pi_h F(-r_h,u_h) + \Pi_h F(u_h,-r_h) + \Pi_h R_h,$$

where

$$\begin{aligned} R_h = {}& \Pi_h((s_h-u)_{tt}) + \Pi_h F(\dot{s}_h - u_t, u) + \Pi_h F(\dot{s}_h, u_h - u) \\ & + \Pi_h F(u, \dot{s}_h - u_t) + \Pi_h F(u_h - u, \dot{s}_h). \end{aligned}$$

Then, integrating from time 0 up to time t

$$\begin{aligned} r_h(t) = {}& e^{-\nu t A_h} r_h(0) + \int_0^t e^{-\nu(t-s)A_h}\left(\Pi_h F(-r_h,u_h) + \Pi_h F(u_h,-r_h)\right) \,\mathrm{d}s \\ & + \int_0^t e^{-\nu(t-s)A_h} R_h \,\mathrm{d}s. \end{aligned}$$

Applying $A_h^{1/2}$ and taking into account that $r_h(0) = 0$ we get

$$\begin{aligned} \|A_h^{1/2} r_h(t)\|_0 \leqslant {}& \left\|\int_0^t e^{-\nu(t-s)A_h} A_h^{1/2}\left(\Pi_h F(r_h,u_h) + \Pi_h F(u_h,r_h)\right) \,\mathrm{d}s\right\|_0 \\ & + \left\|\int_0^t e^{-\nu(t-s)A_h} A_h^{1/2} R_h \,\mathrm{d}s\right\|_0. \end{aligned} \tag{4.16}$$

Let us bound the first term on the right-hand-side of (4.16), using (2.9) we get

$$\begin{aligned}\Big\|\int_0^t e^{-\nu(t-s)A_h}A_h^{1/2}\left(\Pi_h F(r_h,u_h)+\Pi_h F(u_h,r_h)\right)\,\mathrm{d}s\Big\|_0 &\leqslant \frac{C}{\sqrt{\nu}}\int_0^t \frac{\big\|F(r_h,u_h)+F(u_h,r_h)\big\|_0}{\sqrt{t-s}}\,\mathrm{d}s\\ &\leqslant \frac{C}{\sqrt{\nu}}\int_0^t \frac{\|F(r_h,u_h)\|_0+\|F(u_h,r_h)\|_0}{\sqrt{t-s}}\,\mathrm{d}s\\ &\leqslant \frac{KC}{\sqrt{\nu}}\int_0^t \frac{\|r_h(s)\|_1}{\sqrt{t-s}}\,\mathrm{d}s,\end{aligned}$$

where in the last inequality we have used that

$$\|F(r_h,u_h)\|_0 \leqslant K\|r_h\|_1, \quad \|F(u_h,r_h)\|_0 \leqslant K\|r_h\|_1. \tag{4.17}$$

These inequalities are obtained reasoning as in the proof of (4.6) from Lemma 4.1. The constant K in (4.17) depends on $\|\nabla u_h\|_{L^{2d}(\Omega)^d}$ which is easily proved to be bounded. For the second term on the right-hand-side of (4.16), Lemma 4.3 gives

$$\begin{aligned}\Big\|\int_0^t e^{-\nu(t-s)A_h}A_h^{1/2}R_h(s)\,\mathrm{d}s\Big\|_0 &\leqslant \int_0^t \big\|A_h e^{-\nu(t-s)A_h}A_h^{-1/2}R_h(s)\big\|_0\,\mathrm{d}s\\ &\leqslant \frac{C}{\nu}|\log(h)|\max_{0\leqslant t\leqslant T}\|A_h^{-1/2}R_h(t)\|_0.\end{aligned}$$

Applying (2.10) we get

$$\begin{aligned}\|A_h^{-1/2}R_h\|_0 &\leqslant \|(s_h-u)_{tt}\|_{-1}+\|F(\dot s_h-u_t,u)\|_{-1}+\|F(\dot s_h,u_h-u)\|_{-1}\\ &\quad+\|F(u,\dot s_h-u_t)\|_{-1}+\|F(u_h-u,\dot s_h)\|_{-1}\\ &\leqslant Ch^2(\|u_{tt}\|_2+\|p_{tt}\|_1)+2K\|\dot s_h-u_t\|_0+2K\|u_h-u\|_0.\end{aligned}$$

In the last inequality we have use (2.16) and we have reasoned as in the proof of (4.5) from Lemma 4.1 to bound the nonlinear terms. The constant K depends on $\|\nabla \dot s_h\|_{L^{2d}(\Omega)^d}$ which is also bounded independently of h. Applying now (2.16) we obtain

$$\|\dot s_h-u_t\|_0 \leqslant Ch^2(\|u_t\|_2+\|p_t\|_1),$$

which together with (4.13) gives

$$\|A_h^{-1/2}R_h\|_0 \leqslant Kh^2,$$

and then

$$\int_0^t \big\|A_h e^{-\nu(t-s)A_h}A_h^{-1/2}R_h(s)\big\|_0\,\mathrm{d}s \leqslant \frac{K}{\nu}|\log(h)|h^2.$$

Finally, applying Gronwall inequality the proof is complete. □

4.1 *Rate of Convergence of the postprocessed method*

The next two theorems state the rate of convergence of the postprocessed MFE approximation $(\tilde u_h,\tilde p_h)\in(\widetilde X,\widetilde Q)$ that solves (3.3), (3.4). We will denote by $\widetilde V$ the corresponding discretely divergence–free space which may vary depending on the selection of the postprocessed space. The discrete Leray projection into $\widetilde{V}$ will be denoted by $\widetilde{\Pi}_h$ and we will represent by $\widetilde{A}_h$ the discrete Stokes operator acting on functions in $\widetilde{V}$.

The postprocessed approximation to the velocity, $\tilde{u}_h$, is the solution of the pressure-free formulation

$$\nu(\nabla\tilde{u}_h, \nabla\tilde{\chi}) = (f, \tilde{\chi}) - b(u_h^l(T), u_h^l(T), \tilde{\chi}) - (\dot{u}_h^l(T), \tilde{\chi}) \quad \forall\tilde{\chi} \in \widetilde{V}. \tag{4.18}$$

We shall make use of the Stokes projection of the solution of (1.1), (1.2) at time T relative to the postprocessing space $(\widetilde{X}, \widetilde{Q})$. To this end we define $(\tilde{s}_h, \tilde{q}_h) \in (\widetilde{X}, \widetilde{Q})$ satisfying

$$\nu(\nabla\tilde{s}_h, \nabla\tilde{\phi}) - (\tilde{q}_h, \nabla\cdot\tilde{\phi}) = (f - u_t - (u\cdot\nabla u), \tilde{\phi}) \quad \forall\tilde{\phi} \in \widetilde{X}, \tag{4.19}$$

$$(\nabla\cdot\tilde{s}_h, \tilde{\psi}) = 0 \quad \forall\tilde{\psi} \in \widetilde{Q}, \tag{4.20}$$

where all the time–dependent functions appearing in (4.19) are evaluated at time $t = T$. The Stokes projection $\tilde{s}_h \in \widetilde{V}$ satisfies the equation

$$\nu(\nabla\tilde{s}_h, \nabla\tilde{\chi}) = (f - u_t - (u\cdot\nabla)u, \tilde{\chi}) \quad \forall\chi \in \widetilde{V}. \tag{4.21}$$

Let us observe that for the the Hood-Taylor element, $(\widetilde{X}, \widetilde{Q}) = (S_{h,3}(\Omega)^d \cap H_0^1(\Omega)^d, Q_h)$, which is known to be LBB stable (see Hood & Taylor (1973)), the following estimates hold for $(u, p) \in (H^3(\Omega)^d \cap V) \times H^2(\Omega)/\mathbb{R}$ (see Heywood & Rannacher (1988)):

$$\|u(T) - \tilde{s}_h\|_0 + h\|u(T) - \tilde{s}_h\|_1 \leqslant Ch^3\left(\|u(T)\|_3 + \|p(T)\|_2\right), \tag{4.22}$$

$$\|p(T) - \tilde{q}_h\|_0 \leqslant Ch^2\left(\|u(T)\|_3 + \|p(T)\|_2\right). \tag{4.23}$$

THEOREM 4.4 Fix $T > 0$, let (u_h, p_h) be the mini–element approximation to the solution (u, p) of (1.1), (1.2) satisfying (2.4), (2.5), (2.6) and let $(\tilde{u}_h, \tilde{p}_h)$ be the postprocessed MFE approximation at time T. Then, there exist positive constants $K(u, p, \nu)$ and h_0 such that $\forall h \leqslant h_0$ the following bounds hold:
(i) If the postprocessing element is $(\widetilde{X}, \widetilde{Q}) = (X_{h'}, Q_{h'})$, $h' < h$, then

$$\|u(T) - \tilde{u}_h\|_1 \leqslant Ch'(\|u(T)\|_2 + \|p(T)\|_1) + K(u, p, \nu)h^2|\log(h)|.$$

(ii) If at time T the solution enjoys more regularity, say $(u(T), p(T))$ belongs to $(H^3(\Omega)^d \cap V) \times H^2(\Omega)/\mathbb{R}$ and the postprocessing element is $(\widetilde{X}, \widetilde{Q}) = (S_{h,3})^d \cap H_0^1(\Omega)^d, Q_h)$, then

$$\|u(T) - \tilde{u}_h\|_1 \leqslant Ch^2(\|u(T)\|_3 + \|p(T)\|_2) + K(u, p, \nu)h^2|\log(h)|.$$

Proof. Let $\tilde{s}_h$ be the Stokes projection defined in (4.19–4.20). We consider the splitting

$$\|u(T) - \tilde{u}_h\|_1 \leqslant \|u(T) - \tilde{s}_h\|_1 + \|\tilde{s}_h - \tilde{u}_h\|_1.$$

The first term can be readily estimated by using (2.16), (4.22) so that

$$\|u(T) - \tilde{s}_h\|_1 \leqslant \begin{cases} Ch'(\|u(T)\|_2 + \|p(T)\|_1), & \text{if } (\widetilde{X}, \widetilde{Q}) = (X_{h'}, Q_{h'}), \\ Ch^2(\|u(T)\|_3 + \|p(T)\|_2), & \text{if } (\widetilde{X}, \widetilde{Q}) = (S_{h,3}^d \cap H_0^1(\Omega)^d, Q_h). \end{cases}$$

We will concentrate now on the second term. Subtracting (4.21) from (4.18) one finds

$$\begin{aligned} \nu(\nabla(\tilde{u}_h - \tilde{s}_h), \nabla\tilde{\chi}) &= b(u(T), u(T), \tilde{\chi}) - b(u_h^l(T), u_h^l(T), \tilde{\chi}) \\ &\quad + \left(u_t(T) - \dot{u}_h^l(T), \tilde{\chi}\right) \quad \forall\tilde{\chi} \in \widetilde{V}. \end{aligned}$$

Then, by setting $\tilde{\chi} = \tilde{u}_h - \tilde{s}_h \in \widetilde{V}$ we find

$$\nu \|\nabla(\tilde{u}_h - \tilde{s}_h)\|_0 \leqslant \left\| F(u(T), u(T)) - F(u_h^l(T), u_h^l(T)) \right\|_{-1} + \|u_t(T) - \dot{u}_h^l(T)\|_{-1}.$$

For the first term above, applying (4.6) from Lemma 4.1 and (4.14) we obtain

$$\left\| F(u(T), u(T)) - F(u_h^l(T), u_h^l(T)) \right\|_{-1} \leqslant K\|u(T) - u_h^l(T)\|_0 \leqslant Kh^2.$$

As regards the second term we apply (4.15) from Lemma 4.4 to get

$$\|u_t - \dot{u}_h^l\|_{-1} \leqslant \|u_t - \dot{u}_h^l\|_0 \leqslant Kh^2|\log(h)|.$$

Then

$$\|\tilde{u}_h - \tilde{s}_h\|_1 \leqslant K(u, p, \nu)h^2|\log(h)|, \tag{4.24}$$

and the proof is complete. □

THEOREM 4.5 Fix $T > 0$, let (u_h, p_h) be the mini–element approximation to the solution (u, p) of (1.1), (1.2) satisfying (2.4), (2.5), (2.6) and let $(\tilde{u}_h, \tilde{p}_h)$ be the postprocessed MFE approximation at time T. Then, there exist positive constants $K(u, p, \nu)$ and h_0 such that $\forall h \leqslant h_0$ the following bounds hold:
(i) If the postprocessing element is $(\widetilde{X}, \widetilde{Q}) = (X_{h'}, Q_{h'})$, then

$$\|p(T) - \tilde{p}_h\|_0 \leqslant Ch'(\|u(T)\|_2 + \|p(T)\|_1) + K(u, p, \nu)h^2|\log(h)|.$$

(ii) If at time T the solution possesses more regularity, say $(u(T), p(T))$ belongs to $(H^3(\Omega)^d \cap V) \times H^2(\Omega)/\mathbb{R}$ and the postprocessing element is $(\widetilde{X}, \widetilde{Q}) = (S_{h,3})^d \cap H_0^1(\Omega)^d, Q_h)$, then

$$\|p(T) - \tilde{p}_h\|_0 \leqslant Ch^2(\|u(T)\|_3 + \|p(T)\|_2) + K(u, p, \nu)h^2|\log(h)|.$$

Proof. Let $\tilde{q}_h$ be defined in (4.19). Then, the standard triangle inequality gives

$$\|p(T) - \tilde{p}_h\|_0 \leqslant \|p(T) - \tilde{q}_h\|_0 + \|\tilde{q}_h - \tilde{p}_h\|_0.$$

The first term can be easily estimated applying (2.17) and (4.23)

$$\|p(T) - \tilde{q}_h\|_0 \leqslant \begin{cases} Ch'(\|u(T)\|_2 + \|p(T)\|_1) & \text{if } (\widetilde{X}, \widetilde{Q}) = (X_{h'}, Q_{h'}), \\ Ch^2(\|u(T)\|_3 + \|p(T)\|_2) & \text{if } (\widetilde{X}, \widetilde{Q}) = (S_{h,3}^d \cap H_0^1(\Omega)^d, Q_h). \end{cases}$$

Let us now deal with the second term. From the equations that satisfy $\tilde{p}_{\tilde{h}}$ and $\tilde{q}_{\tilde{h}}$ ((3.3), (4.19) respectively) we deduce that for all $\tilde{\phi} \in \widetilde{X}$,

$$(\tilde{p}_h - \tilde{q}_h, \nabla \cdot \tilde{\phi}) = \nu(\nabla \tilde{u}_h - \tilde{s}_h, \nabla \tilde{\phi}) + (F(u_h^l, u_h^l) - F(u, u), \tilde{\phi}) + (\dot{u}_h^l - u_t, \tilde{\phi}).$$

Taking into account the LBB condition we obtain

$$\beta\|\tilde{p}_h - \tilde{q}_h\|_0 \leqslant \nu\|\tilde{u}_h - \tilde{s}_h\|_1 + \|F(u_h^l, u_h^l) - F(u, u)\|_{-1} + \|\dot{u}_h^l - u_t\|_{-1}.$$

By applying (4.24), (4.6) from Lemma 4.1 and (4.15) from Lemma 4.4 we obtain

$$\|\tilde{p}_h - \tilde{q}_h\|_0 \leqslant K\left(h^2 + \|u - u_h^l\|_0 + h^2|\log(h)|\right)$$

and so using estimate (4.14) the proof is concluded. □

REMARK 4.1 In view of Theorems 4.4 and 4.5 we observe that the postprocessed method achieves a gain of one order in the rate of convergence in the H^1 norm, up to a logarithmic term, whenever the solution is smooth enough and the P^2P^1 Hood-Taylor element is used at the postprocessed step or a, sufficiently refined, new mesh is used with the same mini–element. In the later case one should take $h' = h^2$ in the postprocessed step. Although this is a very demanding requirement, for practical computations a slightly refined mesh is usually sufficient in order to observe a considerable reduction of the error in the H^1 norm.

Acknowledgements The first author was supported by project HPRN-CT-2002-00284 and HPRN-CT-2002-00286. This research has been partially supported by MEC under project MTM2004-02847 (cofinanced by FEDER funds).

REFERENCES

ADAMS R. (1975) *Sobolev Spaces*, Academic Press: New York.

ARNOLD, D. N., BREZZI, F. & FORTIN, M. (1984) A stable finite element for the Stokes equations, *Calcolo*, **23**, 337–344.

AYUSO, B. (2003), *The postprocessed mixed finite element method for the Navier–Stokes equations*, Ph.D. Thesis, Universidad Autónoma de Madrid: Madrid.

AYUSO, B. & GARCÍA-ARCHILLA, B. (2005) Regularity constants of the Stokes problem. Application to finite-element methods on curved domains, *Math. Models Methods Appl. Sci.*, **15**, 437-470.

AYUSO, B., GARCÍA-ARCHILLA, B. & NOVO, J. (2005) Postprocessing mixed finite element method for the Navier–Stokes equations, *SIAM J. Numer. Anal.*, **43**, 1091-1111.

BANK, R. E. & WELFERT, B. D. (1990) A comparison between the mini–element and the Petrov-Galerkin Formulations for the generalized Stokes problem, *Comput. Methods. Appl. Mech. Engrg.*, **83**, 61–68.

BANK, R. E. & WELFERT, B. D. (1991) A posteriori error estimates for the Stokes problem, *SIAM J. Numer. Anal.*, **28**, 591-623.

BERNARDI, C. (1989) Optimal finite-element interpolation on curved domains, *SIAM J. Numer. Anal.*, **26**, 1212–1240.

BRAMBLE, J. H., PASCIAK, J. E. & STEINBACH, O. (2002) On the stability of the L^2 projection in $H^1(\Omega)$, *Math. Comp.*, **71**, 147–156.

BREZZI, F. & FORTIN, M. (1991) *Mixed and hybrid finite element methods*, Springer-Verlag: New-York.

BREZZI F. & PITKÄRANTA, J. (1984) On the stabilization of finite element approximations of the Stokes problem, in W. Hackbusch, ed., *Efficient Solutions of Elliptic Systems, Notes on Numerical Fluids Mechanics*, **10**, Vieweg, Wiesbaden, 11–19.

CIARLET, P.G. (1991) *Basic error estimates for elliptic problems, Handbook of Numerical Analysis, Vol. II, Finite Element Methods (Part 1)*, P.G. Ciarlet and J. Lions, eds., North-Holland: New York, 17–352.

DAUGE, M. (1989) Stationary Stokes and Navier–Stokes equations on two or three dimensional domains with corners, *SIAM J. Numer. Anal.*, **20**, 74–97.

DE FRUTOS, J., GARCÍA-ARCHILLA, B. & NOVO, J. (2000) A postprocessed Galerkin method with Chebyshev or Legendre polynomials, *Numer. Math.*, **86**, 419–442.

DE FRUTOS, J. & NOVO, J. (2000a) An enhanced pseudospectral Chebyshev method for dissipative partial differential equations, *J. Comput. Appl. Math.*, **115**, 137–150.

DE FRUTOS, J. & NOVO, J. (2000b) A postprocess based improvement of the spectral element method, *Appl. Numer. Math.*, **33**, 799–819

DE FRUTOS, J. & NOVO, J. (2000c) A spectral element method for the Navier–Stokes equations with improved accuracy, *SIAM J. Numer. Anal.*, **38**, 799–819.

DE FRUTOS, J. & NOVO, J. (2002) Postprocessing the linear finite element method, *SIAM J. Numer. Anal.,* **40**, 805–819.

GARCÍA-ARCHILLA, B.,NOVO, J. & TITI, E. S. (1998) Postprocessing the Galerkin method: a novel approach to approximate inertial manifolds, *SIAM J. Numer. Anal.,* **35**, 941-972.

GARCÍA-ARCHILLA, B. & TITI, E. S. (2000) Postprocessing the Galerkin method: the finite-element case, *SIAM J. Numer. Anal.,* **37**, 470-499.

GIRAULT, V. & RAVIART, P.A. (1986) *Finite element methods for Navier–Stokes equations*, Springer-Verlag: Berlin.

HENRY, D. (1991) *Geometric Theory of Semilinear Parabolic Equations* Springer-Verlag: Berlin.

HEYWOOD, J.G. & RANNACHER, R. (1982) Finite element approximation of the nonstationary Navier–Stokes problem, Part I: Regularity of solutions and second-order error estimates for spatial discretization, *SIAM J. Numer. Anal.,* **19**, 275-311.

HEYWOOD, J.G. & RANNACHER, R. (1988) Finite element approximation of the nonstationary Navier–Stokes problem, Part III: Smoothing property and higher order error estimates for spatial discretization, *SIAM J. Numer. Anal.,* **25**, 489-512.

HUGHES, T. J. R., FRANCA, L. P. & BALESTRA, M. (1986) A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuška-Brezzi condition: A stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations, *Comput. Methods Appl. Mech. Engrg.,* **59**, 85–89.

HOOD, P. & TAYLOR, C. (1973) A numerical solution of the Navier–Stokes equations using the finite element technique, *Comp. and Fluids,* **1**, 73-100.

KELLOG, R.B. & OSBORN, J.P. (1975) A regularity result for the Stokes problem in a convex polygon, *J. Funct. Anal.,* **21**, 397–431.

KIM, Y. & LEE, S. (2000) Modified mini finite element for the Stokes problem in $\mathbb{R}^2$ or $\mathbb{R}^3$, *Advances in Computational Mathematics,* **12**, 261-272.

PIERRE, R. (1988) Simple C^0 Approximations for the computation of incompressible flows, *Comput. Methods Appl. Mech. Engrg.,* **68**, 205-227.

PIERRE, R. (1989) Regularization procedures of mixed finite element approximations of the Stokes problem, *Numer. Methods Partial Diff. Eq.,* **5**, 241-258.

TEMAN, R. (1983) *Navier–Stokes equations and nonlinear functional analysis*, in CBMS-NFS Regional Conference Series in Applied Mathematics, Philadelphia: Pennsylvania.

VERFÜRTH, R. (1989) A posteriori error estimators for the Stokes equations, *Numer. Math.,* **55**, 309-325.

VERFÜRTH, R. (1998) Multilevel algorithms for mixed problems II. Treatment of the mini–element, *SIAM J. Numer. Anal.,* **25**, 285-293.