ECE1657 Problem Set 2 Solutions

Problem 1: Mixed Saddle-Point Equilibrium (2×2 Zero-Sum Game)

Consider the two-player zero-sum matrix game with the following cost matrix:

$$A = \begin{bmatrix} 1 & -3 \\ -3 & -2 \end{bmatrix}$$

Find the mixed saddle-point equilibrium solution $(\mathbf{x}^*, \mathbf{y}^*)$ by using the graphical method of mixed security strategies for P_1 and P_2.

Solution

Player 1 chooses $\mathbf{x}$ to minimize his worst-case loss against every possible pure choice of player 2,

$$V_1(\mathbf{x}) = \min_{\mathbf{x}\in\Delta_1} \max_{k\in\{1,2\}} \mathbf{x}^\top A e_k = \min_{\mathbf{x}\in\Delta_1} \max \left\{ \mathbf{x}^\top \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \mathbf{x}^\top \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \right\},$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Delta_1, \qquad x_2 = 1 - x_1, \quad x_1 \in [0, 1].$$

$$\begin{aligned} V_1(x_1) &= \min_{x_1\in[0,1]} \max\Big\{x_1(a_{11} - a_{21}) + a_{21},\ x_1(a_{12} - a_{22}) + a_{22}\Big\} \\ &= \min_{x_1\in[0,1]} \max\Big\{x_1(1 - (-3)) + (-3),\ x_1(-3 - (-2)) + (-2)\Big\} \\ &= \min_{x_1\in[0,1]} \max\{\underbrace{4x_1 - 3}_{R_1(x_1)},\ \underbrace{-x_1 - 2}_{R_2(x_1)}\}. \end{aligned}$$

The **upper envelope** $V_1(x_1)$ is the pointwise maximum of R_1, R_2; the security strategy is at $R_1 = R_2$:

$$\begin{aligned} 4x_1 - 3 &= -x_1 - 2, \\ 5x_1 &= 1, \\ x_1^* &= \tfrac{1}{5}, \end{aligned} \qquad \mathbf{x}^* = \begin{bmatrix} x_1^* \\ 1 - x_1^* \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix}.$$

Player 2 chooses $\mathbf{y}$ to maximize his worst-case payoff against every possible pure choice of player 1,

$$V_2(\mathbf{y}) = \max_{\mathbf{y}\in\Delta_2} \min_{k\in\{1,2\}} e_k^\top A\mathbf{y} = \max_{\mathbf{y}\in\Delta_2} \min\Big\{[\,a_{11}\ a_{12}\,]\mathbf{y},\ [\,a_{21}\ a_{22}\,]\mathbf{y}\Big\},$$

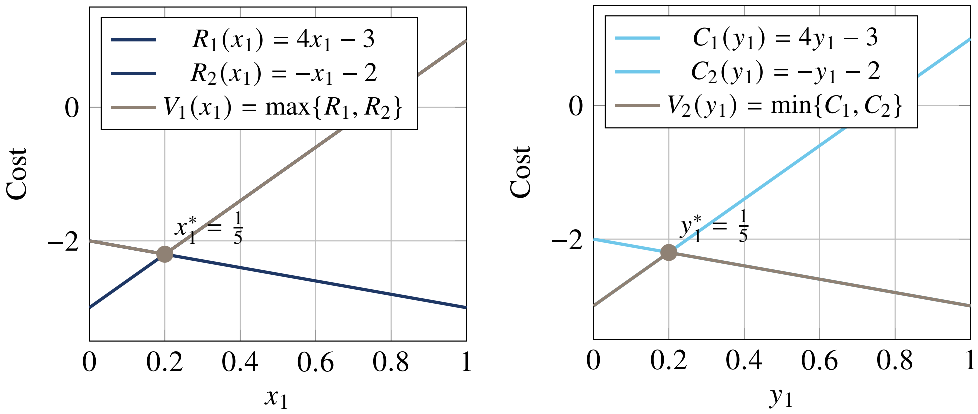
where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \Delta_2, \qquad y_2 = 1 - y_1, \quad y_1 \in [0, 1].$$

$$\begin{aligned} V_2(y_1) &= \max_{y_1\in[0,1]} \min\Big\{y_1(a_{11} - a_{12}) + a_{12},\ y_1(a_{21} - a_{22}) + a_{22}\Big\} \\ &= \max_{y_1\in[0,1]} \min\Big\{\underbrace{4y_1 - 3}_{C_1(y_1)},\ \underbrace{-y_1 - 2}_{C_2(y_1)}\Big\}. \end{aligned}$$

The **lower envelope** $V_2(y_1)$ is the pointwise minimum of C_1, C_2; the security strategy is at $C_1 = C_2$:

$$\begin{aligned} 4y_1 - 3 &= -y_1 - 2, \\ 5y_1 &= 1, \\ y_1^* &= \tfrac{1}{5}, \end{aligned} \qquad \mathbf{y}^* = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix}.$$



(a) P_1 mixed security strategy

(b) P_2 mixed security strategy

The game value is

$$\bar{J}^* = V_1(x_1^*) = 4 \cdot \frac{1}{5} - 3 = -\frac{11}{5} = V_2(y_1^*).$$

Since $\bar{J}_U = \bar{J}_L = -\frac{11}{5}$, by the Minimax Theorem the game admits a unique mixed saddle point

$$(\mathbf{x}^*, \mathbf{y}^*) = \left(\begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix}, \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix} \right), \qquad \bar{J}^* = -\frac{11}{5}.$$

Problem 2: Mixed Equilibrium for 2×3 Zero-Sum Game

Consider the two-player zero-sum matrix game with the following cost matrix:

$$A = \begin{bmatrix} -1 & 1 & -3 \\ -3 & -5 & 3 \end{bmatrix}$$

(i) Use the graphical method to find the mixed security strategy $\mathbf{x}^*$ for P_1 and the averaged (expected) security value $\bar{J}_U$.

(ii) Find the mixed saddle-point $(\mathbf{x}^*, \mathbf{y}^*)$ using reduction: assume that P_2 knows that P_1 plays this optimal $\mathbf{x}^*$ and evaluates the expected outcome $\bar{J}(\mathbf{x}^*, \mathbf{e}_k^2)$ for every pure strategy $\mathbf{e}_k^2$, $k \in M_2$. Based on this evaluation, P_2 eliminates strategies that give lower expected outcome. Show that this leads to a reduced 2×2 game A_r. Find $\mathbf{y}_r^*$ in the reduced game and map back to the original game.

(iii) Relate the reduction procedure in (ii) to how $\mathbf{x}^*$ was obtained in (i) using the graphical plots.

Solution

(i): Finding P_1's Mixed Security Strategy

$$\begin{aligned}
V_1(x) &= \min_{x\in\Delta_1}\max_{y\in\Delta_2} x^\top A y = \min_{x\in\Delta_1}\max_{k\in\{1,2,3\}} x^\top A e_k \\
&= \min_{x\in\Delta_1}\max\left\{x^\top\begin{bmatrix}a_{11}\\a_{21}\end{bmatrix}, x^\top\begin{bmatrix}a_{12}\\a_{22}\end{bmatrix}, x^\top\begin{bmatrix}a_{13}\\a_{23}\end{bmatrix}\right\} \\
&= \min_{x\in\Delta_1}\max\{x_1(a_{11}-a_{21})+a_{21},\ x_1(a_{12}-a_{22})+a_{22},\ x_1(a_{13}-a_{23})+a_{23}\} \\
&= \min_{x\in\Delta_1}\max\{x_1(-1-(-3))+(-3),\ x_1(1-(-5))+(-5),\ x_1(-3-3)+3\} \\
&= \min_{x\in\Delta_1}\max\{\underbrace{2x_1-3}_{R_1(x_1)},\ \underbrace{6x_1-5}_{R_2(x_1)},\ \underbrace{-6x_1+3}_{R_3(x_1)}\}.
\end{aligned}$$

- P_1's security strategy: min of the upper envelope is at $R_2(x_1) = R_3(x_1) \Rightarrow x_1^* = \frac{2}{3} \Rightarrow \mathbf{x}^* = \begin{bmatrix}2/3\\1/3\end{bmatrix}$.
- P_1's security level: $\bar{J}_U = V_1(x_1^*) = \max\left\{2\cdot\frac{2}{3}-3,\ 6\cdot\frac{2}{3}-5,\ -6\cdot\frac{2}{3}+3\right\} = \max\left\{-\frac{5}{3},\ -1,\ -1\right\} = -1.$

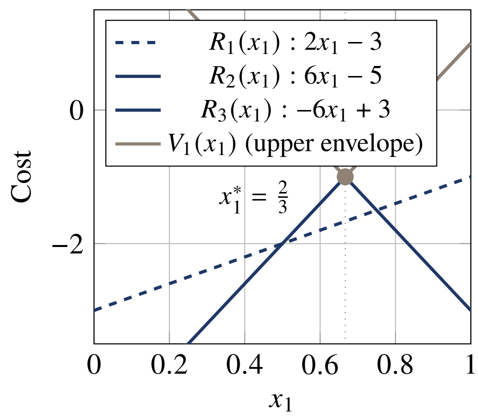


Figure 2: P_1's security strategy

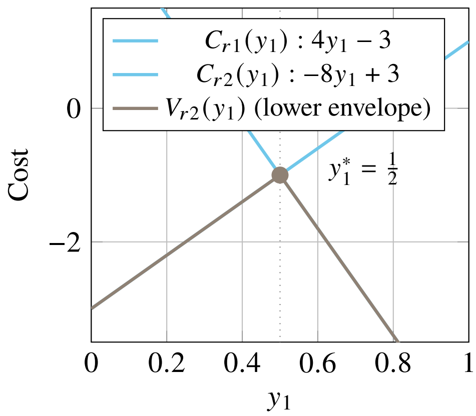


Figure 3: P_2's security strategy (reduced)

(ii): Finding Mixed Saddle-Point via Reduction

Step 1: P_2's expected cost $\bar{J}(\mathbf{x}^*, \mathbf{e}_k^2)$ given $\mathbf{x}^*$ for every pure strategies

$$\begin{aligned}\bar{J}(\mathbf{x}^*, \mathbf{e}_k^2) &= \mathbf{x}^{*\top} A \mathbf{e}_k^2, \qquad k \in \{1,2,3\} \\ &= \left\{ \mathbf{x}^{*\top} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \ \mathbf{x}^{*\top} \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}, \ \mathbf{x}^{*\top} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ -3 \end{bmatrix}, \ \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \ \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -3 \\ 3 \end{bmatrix} \right\} \\ &= \left\{ -\tfrac{5}{3}, -1, -1 \right\}.\end{aligned}$$

Step 2: Reduced game A_r

P_2 eliminates column 1 since $\bar{J}(\mathbf{x}^*, \mathbf{e}_1^2) = -\frac{5}{3} < -1 = \bar{J}(\mathbf{x}^*, \mathbf{e}_2^2) = \bar{J}(\mathbf{x}^*, \mathbf{e}_3^2)$

$$A_r = \begin{bmatrix} 1 & -3 \\ -5 & 3 \end{bmatrix}$$

Step 3: P_2's security strategy in reduced game A_r

$$\begin{aligned}V_{r2}(\mathbf{y_r}) &= \max_{\mathbf{y_r} \in \Delta_{r2}} \min_{k \in \{1,2\}} e_k^{1\top} A_r \mathbf{y_r} \\ &= \max_{\mathbf{y_r} \in \Delta_{r2}} \min \left\{ \begin{bmatrix} 1 & -3 \end{bmatrix} \mathbf{y_r}, \ \begin{bmatrix} -5 & 3 \end{bmatrix} \mathbf{y_r} \right\} \\ &= \max_{\mathbf{y_r} \in \Delta_{r2}} \min\{ \underbrace{4y_1 - 3}_{C_{r1}(y_1)}, \ \underbrace{-8y_1 + 3}_{C_{r2}(y_1)} \}.\end{aligned}$$

- P_2's security strategy: max of the lower envelope is at $C_{r1}(y_1) = C_{r2}(y_1) \Rightarrow y_1^* = \frac{1}{2} \Rightarrow \mathbf{y}^* = \begin{bmatrix} 0 \\ \mathbf{y}_r^* \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$
- P_2's security level: $\bar{J}_L = V_{r2}(y_1^*) = \min\left\{ 4 \cdot \frac{1}{2} - 3, \ -8 \cdot \frac{1}{2} + 3 \right\} = \min\{-1, -1\} = -1$

Saddle-point equilibrium:

$$(\mathbf{x}^*, \mathbf{y}^*) = \left(\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix} \right)$$

(iii): Envelope view of the reduction

From Figure 1, at $x_1^* = \frac{2}{3}$ the upper envelope

$$V_1(x_1) = \max\{R_1(x_1), R_2(x_1), R_3(x_1)\}$$

takes the value $v = -1$ on R_2 and R_3, while $R_1(x_1^*) = -\frac{5}{3} < v$.

- **On–envelope lines at x_1^*** (here columns 2 and 3) are exactly the pure strategies that can be mixed by P_2 in equilibrium and therefore remain in the reduced game.
- **Off–envelope lines at x_1^*** (here column 1) are strictly worse for P_2, so they are eliminated in the reduction and get zero probability in $\mathbf{y}^*$.

In Figure 2, restricting to the surviving columns $\{2, 3\}$, the two remaining lines form the lower envelope $V_{r2}(y_1)$ and are equalized at $y_1^* = \frac{1}{2}$ with the same value $v = -1$, giving the reduced-game security strategy of P_2.

Problem 3: Pure NE in Bimatrix Game

Consider the two-player bimatrix game with the following cost matrices:

$$A = \begin{bmatrix} 8 & -4 & 8 \\ 6 & 1 & 6 \\ 12 & 0 & 0 \\ 0 & 0 & 12 \\ 0 & 4 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Find, if there exist, the set of pure Nash equilibrium solutions. Is there one pure NE better than another?

Solution

According to **Definition 3.1**, pure NE are at the intersections of the two players' best responses. Among the pure NE, we use **Definition 3.2** to compare outcomes componentwise and select the better ones.

P_1**'s Best Responses**: For each column k, find minimum of A:

$$\begin{aligned} k = 1: &\quad \min\{8, 6, 12, 0, 0\} = 0 && \Rightarrow j \in \{4, 5\} \\ k = 2: &\quad \min\{-4, 1, 0, 0, 4\} = -4 && \Rightarrow j = 1 \\ k = 3: &\quad \min\{8, 6, 0, 12, 0\} = 0 && \Rightarrow j \in \{3, 5\} \end{aligned}$$

P_1's BR candidates: $\{(4, 1), (5, 1), (1, 2), (3, 3), (5, 3)\}$

P_2**'s Best Responses**: For each row j, find minimum of B:

$$\begin{aligned} j = 1: &\quad \min\{4, 0, 0\} = 0 && \Rightarrow k \in \{2, 3\} \\ j = 2: &\quad \min\{0, 0, 4\} = 0 && \Rightarrow k \in \{1, 2\} \\ j = 3: &\quad \min\{0, 4, 0\} = 0 && \Rightarrow k \in \{1, 3\} \\ j = 4: &\quad \min\{0, 4, 0\} = 0 && \Rightarrow k \in \{1, 3\} \\ j = 5: &\quad \min\{0, 1, 0\} = 0 && \Rightarrow k \in \{1, 3\} \end{aligned}$$

P_2's BR candidates: $\{(1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 3), (5, 1), (5, 3)\}$

Pure Nash Equilibria: Results & Comparison The pure NE are the intersections of best responses.

$NE(\mathcal{G})(j^*, k^*)$	P_1**'s cost** a_{jk}	P_2**'s cost** b_{jk}	**note**
$(4, 1)$	0	0	dominated by $(1, 2)$
$(5, 1)$	0	0	dominated by $(1, 2)$
$(1, 2)$	-4	0	**Pareto-best**
$(3, 3)$	0	0	dominated by $(1, 2)$
$(5, 3)$	0	0	dominated by $(1, 2)$

Problem 4: NE in 2×2 Bimatrix Game: Best Response (BR) Correspondences

Find the set of all (pure or mixed) Nash equilibrium (NE) solutions, $NE(\mathcal{G})$, for the bimatrix 2×2 game with the following cost matrices:

$$A = \begin{bmatrix} 10 & -2 \\ -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & 2 \\ 1 & -1 \end{bmatrix}$$

Use the graphical method of best response mappings (correspondences).

1. Does it admit a pure-strategy NE?
2. What about a mixed-strategy NE?
3. What is the NE of the bimatrix game $(-A, -B)$?

Solution

Best-response (BR) correspondences in bimatrix games.

$$\begin{aligned} BR_1(\mathbf{y}) &= \left\{\boldsymbol{\xi} \in \Delta_1 \;\middle|\; \boldsymbol{\xi}^\top A\mathbf{y} \le \mathbf{x}^\top A\mathbf{y},\ \forall\, \mathbf{x} \in \Delta_1\right\}, \\ BR_2(\mathbf{x}) &= \left\{\boldsymbol{\eta} \in \Delta_2 \;\middle|\; \boldsymbol{\eta}^\top B\mathbf{x} \le \mathbf{y}^\top B\mathbf{x},\ \forall\, \mathbf{y} \in \Delta_2\right\}. \end{aligned}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ 1-x_1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ 1-y_1 \end{bmatrix}, \quad \boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ 1-\xi_1 \end{bmatrix}, \quad \boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ 1-\eta_1 \end{bmatrix}, \quad x_1, y_1, \xi_1, \eta_1 \in [0,1].$$

Player 1's best response given $\mathbf{y} = [y_1, 1-y_1]^\top$

$$\begin{aligned} BR_1(y_1) &= \left\{\xi_1 \in [0,1] \;\middle|\; \begin{bmatrix}\xi_1 & 1-\xi_1\end{bmatrix} A \begin{bmatrix} y_1 \\ 1-y_1 \end{bmatrix} \le \begin{bmatrix}x_1 & 1-x_1\end{bmatrix} A \begin{bmatrix} y_1 \\ 1-y_1 \end{bmatrix},\ \forall x_1 \in [0,1]\right\} \\ &= \{\xi_1 \in [0,1] \mid \xi_1(\tilde{a}y_1 - \tilde{c}_1) \le x_1(\tilde{a}y_1 - \tilde{c}_1),\ \forall x_1 \in [0,1]\} = \arg\min_{x_1 \in [0,1]} (\tilde{a}y_1 - \tilde{c}_1)x_1 \\ &= \begin{cases} \{0\}, & \tilde{a}y_1 - \tilde{c}_1 > 0, \\ [0,1], & \tilde{a}y_1 - \tilde{c}_1 = 0, \\ \{1\}, & \tilde{a}y_1 - \tilde{c}_1 < 0, \end{cases} \end{aligned}$$

Player 2's best response given $\mathbf{x} = [x_1, 1-x_1]^\top$

$$\begin{aligned} BR_2(x_1) &= \left\{\eta_1 \in [0,1] \;\middle|\; \begin{bmatrix}\eta_1 & 1-\eta_1\end{bmatrix} B \begin{bmatrix} x_1 \\ 1-x_1 \end{bmatrix} \le \begin{bmatrix}y_1 & 1-y_1\end{bmatrix} B \begin{bmatrix} x_1 \\ 1-x_1 \end{bmatrix},\ \forall y_1 \in [0,1]\right\} \\ &= \left\{\eta_1 \in [0,1] \;\middle|\; \eta_1(\tilde{b}x_1 - \tilde{d}_2) \le y_1(\tilde{b}x_1 - \tilde{d}_2),\ \forall y_1 \in [0,1]\right\} = \arg\min_{y_1 \in [0,1]} (\tilde{b}x_1 - \tilde{d}_2)y_1 \\ &= \begin{cases} \{0\}, & \tilde{b}x_1 - \tilde{d}_2 > 0, \\ [0,1], & \tilde{b}x_1 - \tilde{d}_2 = 0, \\ \{1\}, & \tilde{b}x_1 - \tilde{d}_2 < 0, \end{cases} \end{aligned}$$

where

$$\begin{cases} \tilde{a} & = a_{11} - a_{12} - a_{21} + a_{22} \\ \tilde{c}_1 & = a_{22} - a_{12} \\ \tilde{b} & = b_{11} - b_{12} - b_{21} + b_{22} \\ \tilde{d}_2 & = b_{22} - b_{21} \end{cases}$$

The mixed Nash equilibrium assuming $\frac{\tilde{d}_2}{\tilde{b}}, \frac{\tilde{c}_1}{\tilde{a}} \in (0,1)$:

$$(\mathbf{x}^*, \mathbf{y}^*) = \left(\begin{bmatrix} \frac{\tilde{d}_2}{\tilde{b}} \\ 1 - \frac{\tilde{d}_2}{\tilde{b}} \end{bmatrix}, \begin{bmatrix} \frac{\tilde{c}_1}{\tilde{a}} \\ 1 - \frac{\tilde{c}_1}{\tilde{a}} \end{bmatrix}\right).$$

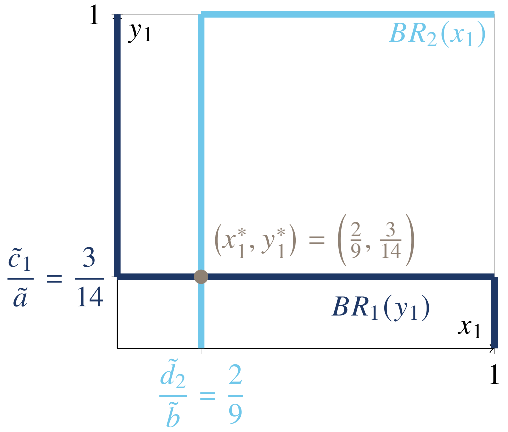
For this game,

$$\begin{aligned} & \tilde{a} = 14, \quad \tilde{c}_1 = 3, \quad \tilde{b} = -9, \quad \tilde{d}_2 = -2, \\ \Rightarrow \quad & \tilde{a}y_1 - \tilde{c}_1 = 14y_1 - 3, \quad \tilde{b}x_1 - \tilde{d}_2 = -9x_1 + 2. \end{aligned}$$

The BR sets are the piecewise (segment) functions

$$BR_1(y_1) = \begin{cases} \{0\}, & y_1 > \frac{3}{14}, \\ [0,1], & y_1 = \frac{3}{14}, \\ \{1\}, & y_1 < \frac{3}{14}, \end{cases} \qquad BR_2(x_1) = \begin{cases} \{0\}, & x_1 < \frac{2}{9}, \\ [0,1], & x_1 = \frac{2}{9}, \\ \{1\}, & x_1 > \frac{2}{9}. \end{cases}$$

The intersection of two BR maps gives the NE:



$$(\mathbf{x}^*, \mathbf{y}^*) = \left(\begin{bmatrix} \frac{2}{9} \\ \frac{7}{9} \end{bmatrix}, \begin{bmatrix} \frac{3}{14} \\ \frac{11}{14} \end{bmatrix} \right)$$

There is no pure-strategy NE.

NE of the transformed game $(-A, -B)$

For bimatrix game $(-A, -B)$, the best responses for player 1 and player 2 are:

$$BR_1(y_1) = \begin{cases} 0, & \text{if } y_1 < \frac{3}{14}, \\ [0,1], & \text{if } y_1 = \frac{3}{14}, \\ 1, & \text{if } y_1 > \frac{3}{14}, \end{cases} \qquad BR_2(x_1) = \begin{cases} 0, & \text{if } x_2 > \frac{2}{9}, \\ [0,1], & \text{if } x_2 = \frac{2}{9}, \\ 1, & \text{if } x_2 < \frac{2}{9}. \end{cases}$$

This will flip the plots of $BR_1(y_1), BR_2(x_1)$ in against axes y_1, x_1 respectively, but will not change the mixed-strategy NE. Thus, for game $(-A, -B)$, there is still only one mixed-strategy:

$$(x^*, y^*) = \left(\begin{bmatrix} \frac{2}{9} \\ \frac{7}{9} \end{bmatrix}, \begin{bmatrix} \frac{3}{14} \\ \frac{11}{14} \end{bmatrix} \right).$$

And no pure-strategy NE.

Takeaways

Problem 1

Player 1 chooses $\mathbf{x}$ to minimize his worst-case loss against every possible pure choice of player 2,

$$V_1(\mathbf{x}) = \min_{\mathbf{x}\in\Delta_1} \max_{k\in\{1,2\}} \mathbf{x}^\top A e_k = \min_{\mathbf{x}\in\Delta_1} \max \left\{ \mathbf{x}^\top \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \mathbf{x}^\top \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \right\},$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Delta_1, \qquad x_2 = 1 - x_1, \quad x_1 \in [0,1].$$

Problem 3

Definition 3.1 (Pure-strategy Nash equilibrium). Consider a bimatrix game (A, B) with m pure strategies for player 1 (rows) and n pure strategies for player 2 (columns). Let (j^*, k^*) be a pair of pure strategies, one for each player. If

$$a_{j^*k^*} \le a_{jk^*}, \quad \forall j = 1, \dots, m, \qquad b_{j^*k^*} \le b_{j^*k}, \quad \forall k = 1, \dots, n,$$

then (j^*, k^*) is called a Nash equilibrium in pure strategies of the bimatrix game, and the pair of payoffs

$$(a_{j^*k^*}, b_{j^*k^*})$$

is the corresponding NE outcome.

Definition 3.2 (Better pure-strategy pair). A pair of pure strategies (j^*, k^*) is said to be better than another pair $(\tilde{j}, \tilde{k})$ if

$$a_{j^*k^*} \le a_{\tilde{j}\tilde{k}} \quad \text{and} \quad b_{j^*k^*} \le b_{\tilde{j}\tilde{k}},$$

and at least one of these inequalities is strict.

Problem 4

Definition 3.3 (Strategic equivalence of bimatrix games). Two bimatrix games (A, B) and $(\tilde{A}, \tilde{B})$ are said to be strategically equivalent if there exist positive constants $\alpha_1, \alpha_2 > 0$ and scalars β_1, β_2 such that

$$\tilde{a}_{jk} = \alpha_1 a_{jk} + \beta_1, \qquad \tilde{b}_{jk} = \alpha_2 b_{jk} + \beta_2, \quad \forall j = 1, \dots, m, \ \forall k = 1, \dots, n.$$

(That is, an affine transformation on each player's payoffs: column-wise in A, row-wise in B.)