Lecture 4: N-Player Games & Nash Theorem

Last Lecture Recap

- 2 Players zero sum finite game
- 2 Players non-zero sum finite game
- Computing all NE graphically in (2×2) games $\Rightarrow$ introduce (A, B)

Game Elements in This Lecture

Players $\mathcal{I} = \{1, \dots, i, \dots, N\}$ with $N \geq 2$ **(generalization of the $N = 2$ case).**
Strategy Mixed strategies $\mathbf{x}_i \in \Delta_i$, joint profile

$$\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \Delta_X := \prod_{i \in N} \Delta_i$$

Cost $\bar{J}_i : \Delta_X \to \mathbb{R}$ for each $i \in \mathcal{I}$

Why This Change? Many real applications involve > 2 players (markets, networks, etc.).

Central Question: Does NE exist for N-player games?

Key Theorems:

1. **Brouwer Fixed-Point Theorem:** Continuous $f : K \to K$ on compact convex K has fixed point
2. **Kakutani Fixed-Point Theorem:** Extends to set-valued maps (for BR correspondence)
3. **Nash Theorem:** Every N-player finite game has at least one mixed-strategy NE

4.1 N-Player Finite ("Matrix") Games (Ch. 3.3)

4.1.1 Setup and Nash equilibrium (NE) definition

Player Set. $\mathcal{I} = \{1, \dots, i, \dots, N\}$ with $N \geq 2$

Action Set.

- Ω_i = set of actions of P_i $\quad \forall\, i \in \mathcal{I}$
- $|\Omega_i| = m_i$ (cardinality)
- M_i = index set of actions = $\{1, \dots, j, \dots, m_i\}$

Mixed Strategy Space.

- Δ_i: set of mixed strategies of P_i.
- Mixed strategy of P_i:

$$\mathbf{x}_i = \begin{bmatrix} x_{i1} \\ \vdots \\ x_{ij} \\ \vdots \\ x_{im_i} \end{bmatrix}, \qquad x_{ij} = \Pr(P_i \text{ selects action } j).$$

- Definition (probability simplex):

$$\Delta_i := \underbrace{\left\{ \mathbf{x}_i \in \mathbb{R}^{m_i} \;\middle|\; \sum_{j=1}^{m_i} x_{ij} = 1,\ x_{ij} \geq 0\ \forall\, j \in M_i \right\}}_{\text{probability simplex in } \mathbb{R}^{m_i}}.$$

- Joint mixed-strategy profile:

$$\mathbf{x} := (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \underbrace{\Delta_1 \times \cdots \times \Delta_N}_{\Delta_X \text{ (Cartesian product)}}.$$

- Compact notation for "player i vs others $-i$":

$$\mathbf{x} = (\mathbf{x}_i, \mathbf{x}_{-i}), \qquad \underbrace{\mathbf{x}_{-i}}_{\text{all players except } P_i} := (\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_N).$$

Cost Functions.

- **Pure cost:** $J_i : \Omega_1 \times \cdots \times \Omega_N \to \mathbb{R}$
- **Expected cost:** $\bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}) : \Delta_1 \times \cdots \times \Delta_N \to \mathbb{R}$

$$\begin{aligned} \bar{J}_i(\mathbf{x}) &= \mathbb{E}[J_i(\mathbf{u})] \\ &= \underbrace{\sum_{u_1 \in \Omega_1} \cdots \sum_{u_N \in \Omega_N}}_{N\text{-fold sum}} J_i(u_1, \dots, u_N) \quad \underbrace{x(\mathbf{u})}_{\text{prob. of pure profile } \mathbf{u}} \end{aligned}$$

> Note. $\bar{J}_i$ is **multilinear** in $(\mathbf{x}_1, \dots, \mathbf{x}_N)$ and **continuous** on Δ_X.
> For fixed $\mathbf{x}_{-i}$, the map $\mathbf{x}_i \mapsto \bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is **linear** on Δ_i.

Def. 3.12 Nash Equilibrium. Nash Equilibrium (NE) is an N-tuple $\mathbf{x}^* = (\mathbf{x}_1^*, \dots, \mathbf{x}_N^*) \in \Delta_X$

Interpretation: No regret solution for each P_i (individually optimal)

$\boxed{1}$ **No regret (unilaterally) solution for each P_i**

$$\bar{J}_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}^*) \qquad \forall\, \mathbf{x}_i \in \Delta_i, \qquad \underbrace{\forall\, i \in \mathcal{I}}_{N \text{ inequalities to be satisfied}}$$

$\boxed{2}$ **Individually Optimal** Best Response of P_i: $\quad \mathrm{BR}_i : \Delta_{-i} \rightrightarrows \Delta_i \quad$ (set-valued map)

$$\mathbf{x}_i^* \in \underbrace{\arg\min_{\mathbf{x}_i \in \Delta_i} \bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}^*)}_{\text{usually a set}} =: \mathrm{BR}_i(\mathbf{x}_{-i}^*)$$

$\boxed{3}$ **Fixed-point**

$$\underbrace{\begin{cases} \mathbf{x}_1^* \in \mathrm{BR}_1(\mathbf{x}_{-1}^*) \\ \vdots \\ \mathbf{x}_i^* \in \mathrm{BR}_i(\mathbf{x}_{-i}^*) \\ \vdots \\ \mathbf{x}_N^* \in \mathrm{BR}_N(\mathbf{x}_{-N}^*) \end{cases}}_{\text{NE is the intersection of N individual BR maps}} \quad \Leftrightarrow \quad \underbrace{\mathbf{x}^* \in \mathrm{BR}(\mathbf{x}^*)}_{\text{NE is a fixed point of the overall BR map}}$$

where

$$\mathrm{BR}(\mathbf{x}) := \begin{bmatrix} \mathrm{BR}_1(\mathbf{x}_{-1}) \\ \vdots \\ \mathrm{BR}_N(\mathbf{x}_{-N}) \end{bmatrix} : \Delta_X \rightrightarrows \Delta_X$$

Note. Nash equilibrium is a mutual best response where no player can unilaterally improve by deviating. The fixed-point characterization connects NE existence to fixed-point theorems.

4.1.2 Best-response maps and NE

Def. Best Response Correspondence (pdf note 3.4.1). For player i and fixed strategies of others $\mathbf{x}_{-i} \in \Delta_{-i}$, the **best response set** is:

$$\mathrm{BR}_i(\mathbf{x}_{-i}) = \left\{\mathbf{x}_i \in \Delta_i \,\middle|\, \bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}) \leq \bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i}),\ \forall\, \mathbf{w}_i \in \Delta_i\right\}$$

This is also called the **optimal response set** or **rational action set**.

Overall BR map: $\mathrm{BR} : \Delta_X \rightrightarrows \Delta_X$

$$\mathrm{BR}(\mathbf{x}) = \mathrm{BR}_1(\mathbf{x}_{-1}) \times \cdots \times \mathrm{BR}_N(\mathbf{x}_{-N}) \subset \Delta_X$$

Properties of BR Correspondence For any $\mathbf{x}_{-i} \in \Delta_{-i}$:

1. $\mathrm{BR}_i(\mathbf{x}_{-i})$ is **non-empty** (by Weierstrass: continuous function on compact set)
2. $\mathrm{BR}_i(\mathbf{x}_{-i})$ is **convex** (by linearity of $\bar{J}_i$ in $\mathbf{x}_i$)
3. $\mathrm{BR}_i(\mathbf{x}_{-i})$ is **closed** (by continuity)
4. $\mathrm{BR}_i(\mathbf{x}_{-i}) \subseteq \Delta_i$ is a face of Δ_i (convex hull of some pure strategy vertices)
5. Can be set-valued: ranges from singleton to entire simplex Δ_i

4.2 Nash Theorem and Idea of Proof (Ch. 3.5)

Nash Theorem (existence of NE). Every N-player finite game has at least one mixed-strategy Nash equilibrium.

4.2.1 Two Fixed-Point Theorems (Appendix A)

Brouwer's Fixed Point Theorem.

- Let $S \subset \mathbb{R}^m$ be a **convex** and **compact** subset.
- Let $f : S \to S$ be a **continuous** function.
- Then $\exists\, x \in S$ s.t.

$$x = f(x), \quad \text{i.e. } x \text{ is a fixed point of } f$$

Kakutani's Fixed Point Theorem.

- Let $S \subset \mathbb{R}^m$ be a **convex** and **compact** subset.
- Let $\Phi : S \rightrightarrows S$ be a **set-valued map**, and denote the image of $x \in S$ by $\Phi(x) \subset S$.
- Assume that $\forall\, x \in S$:

 (1) $\Phi(x)$ is **non-empty** and **convex**;

 (2) Φ has a **closed graph**

$$\mathrm{Graph}(\Phi) := \{(x, y) \in S \times S \mid y \in \Phi(x)\}$$

- Then $\exists$ at least one $x \in S$ s.t.

$$x \in \Phi(x), \quad \text{i.e. } x \text{ is a fixed point of } \Phi$$

4.2.2 Applying Kakutani's Theorem to BR

Nash Theorem can be proved by Kakutani's Fixed-Point Theorem.

Step 1 — Figure out what is what.

- **Set:** $S \equiv \Delta_X := \Delta_1 \times \cdots \times \Delta_N$
- **Correspondence:** $\Phi \equiv \mathrm{BR} : \Delta_X \rightrightarrows \Delta_X$, $\quad \mathrm{BR}(\mathbf{x}) := \mathrm{BR}_1(\mathbf{x}_{-1}) \times \cdots \times \mathrm{BR}_N(\mathbf{x}_{-N})$ where, for each $i \in \mathcal{I}$,

$$\mathrm{BR}_i(\mathbf{x}_{-i}) := \arg\min_{\mathbf{x}_i \in \Delta_i} \bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}).$$

Step 2 — What needs to be shown?

(i) Δ_X is **convex** and **compact**;

(ii) For every $\mathbf{x} \in \Delta_X$, $\mathrm{BR}(\mathbf{x})$ is **non-empty** and **convex**;

(iii) $\mathrm{Graph}(\mathrm{BR})$ is **closed**, where

$$\mathrm{Graph}(\mathrm{BR}) := \{\, (\mathbf{x}, \mathbf{y}) \in \Delta_X \times \Delta_X \mid \mathbf{y} \in \mathrm{BR}(\mathbf{x}) \,\}.$$

Step 3 — Check all the conditions.

(i) Δ_X is convex and compact.

$$\Delta_i := \{\mathbf{x}_i \in \mathbb{R}^{m_i} \mid \mathbf{1}^\top \mathbf{x}_i = 1,\ x_{ij} \geq 0,\ \forall j \in M_i\}$$

Convexity of Δ_i. Take $\mathbf{x}_i, \mathbf{y}_i \in \Delta_i$ and $\alpha \in [0,1]$, and let $\mathbf{z}_i := \alpha \mathbf{x}_i + (1-\alpha)\mathbf{y}_i$.

Each component in $\mathbf{z}_i$ is non-negative

$$z_{ij} = \alpha x_{ij} + (1-\alpha) y_{ij} \geq 0, \quad \forall j \in M_i,$$

and the sum of the components in $\mathbf{z}_i$ is 1.

$$\mathbf{1}^\top \mathbf{z}_i = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \alpha x_{i1} + (1-\alpha) y_{i1} \\ \vdots \\ \alpha x_{im_i} + (1-\alpha) y_{im_i} \end{bmatrix} = \sum_{j=1}^{m_i} \big(\alpha x_{ij} + (1-\alpha) y_{ij}\big) = \alpha \sum_{j=1}^{m_i} x_{ij} + (1-\alpha) \sum_{j=1}^{m_i} y_{ij}$$
$$= \alpha \cdot 1 + (1-\alpha) \cdot 1 = 1.$$

$\mathbf{z}_i$ satisfies the defining constraints of Δ_i, and hence $\mathbf{z}_i \in \Delta_i \ \Rightarrow\ \Delta_i$ is convex.

Compactness of Δ_i.

- Bounded: $x_{ij} \geq 0$ and $\sum_j x_{ij} = 1$ imply $0 \leq x_{ij} \leq 1$, so $\Delta_i \subset [0,1]^{m_i}$.
- Closed: Let $\{\mathbf{x}_i^{(n)}\} \subset \Delta_i$ with $\mathbf{x}_i^{(n)} \to \mathbf{x}_i$. Then $x_{ij}^{(n)} \geq 0$ for all n implies $x_{ij} = \lim_n x_{ij}^{(n)} \geq 0$, and
$$\sum_j x_{ij} = \sum_j \lim_n x_{ij}^{(n)} = \lim_n \sum_j x_{ij}^{(n)} = \lim_n 1 = 1.$$
Thus $\mathbf{x}_i \in \Delta_i$, so Δ_i is closed.

Therefore Δ_i is closed and bounded, hence compact.

From Δ_i to Δ_X.

$$\Delta_X = \Delta_1 \times \cdots \times \Delta_N \subset \mathbb{R}^{m_1 + \cdots + m_N}.$$

- Cartesian product of convex sets is convex $\Rightarrow \Delta_X$ is convex.
- Cartesian product of compact sets in finite-dimensional Euclidean space is compact $\Rightarrow \Delta_X$ is compact.

Thus condition (i) holds.

(ii) $\mathrm{BR}(\mathbf{x})$ is non-empty and convex. Fix $i \in \mathcal{I}$ and $\mathbf{x}_{-i} \in \Delta_{-i}$.

Non-emptiness of $\mathrm{BR}_i(\mathbf{x}_{-i})$.

By Weierstrass theorem, a continuous function on a compact set attains its minimum. Since $\bar{J}_i(\cdot, \mathbf{x}_{-i})$ is continuous and Δ_i is compact, the minimizer set $\mathrm{BR}_i(\mathbf{x}_{-i})$ is non-empty.

Convexity of $\mathrm{BR}_i(\mathbf{x}_{-i})$.

Suppose $\mathbf{y}, \mathbf{z} \in \mathrm{BR}_i(\mathbf{x}_{-i})$ and let $\alpha \in [0, 1]$. Then for any $\mathbf{w} \in \Delta_i$:

$$\bar{J}_i(\alpha\mathbf{y} + (1-\alpha)\mathbf{z}, \mathbf{x}_{-i}) = \alpha\bar{J}_i(\mathbf{y}, \mathbf{x}_{-i}) + (1-\alpha)\bar{J}_i(\mathbf{z}, \mathbf{x}_{-i}) \leq \bar{J}_i(\mathbf{w}, \mathbf{x}_{-i})$$

by linearity of $\bar{J}_i$ in $\mathbf{x}_i$. Hence $\alpha\mathbf{y} + (1-\alpha)\mathbf{z} \in \mathrm{BR}_i(\mathbf{x}_{-i})$.

Since $\mathrm{BR}(\mathbf{x}) = \prod_i \mathrm{BR}_i(\mathbf{x}_{-i})$ is a product of non-empty convex sets, it is non-empty and convex.

(iii) $\mathrm{Graph}(\mathrm{BR})$ is closed (next lecture).

To show that Graph(BR) has a graph set that is closed, i.e. it contains all its limit points.

We prove by contradiction: assume that Graph(BR) is not closed, i.e., it does not contain all its limit points.

Real analysis & topology detour.

- **Compact set (in $\mathbb{R}^m$).** S is compact $\iff$ S is closed and bounded (Heine–Borel).
- **Convex set.** S is convex if $\forall x, y \in S$ and $\forall \alpha \in [0, 1]$,

$$\alpha x + (1-\alpha)y \in S.$$

- **Closed set.** S is closed iff for every sequence $\{x^{(n)}\} \subset S$ with $x^{(n)} \to x$, we have $x \in S$ (i.e., S contains all its limit points).
- **Graph of a correspondence.** For $\Phi : S \rightrightarrows S$,

$$\mathrm{Graph}(\Phi) := \{(x, y) \in S \times S \mid y \in \Phi(x)\}.$$