ECE1657 Problem Set 3 Solutions

Problem 1: NE at the intersection of best-response correspondences

Consider the two-player bimatrix game with the following cost matrices

$$A = \begin{bmatrix} -2 & \alpha \\ \alpha & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & \alpha \\ \alpha & -2 \end{bmatrix}$$

where $\alpha \geq 0$. This is a modified "battle of the sexes" game. For what value of α is the unique mixed-strategy Nash equilibrium (NE) solution such that the couple will spend the evening together:

(a) with highest probability
(b) with lowest probability

Solution

Spending the evening together means both players choose the same pure strategy.

- Let P_1's mixed strategy be $\mathbf{x} = [x_1, x_2]^\top$ where $x_1 + x_2 = 1$, so $x_2 = 1 - x_1$.
- Let P_2's mixed strategy be $\mathbf{y} = [y_1, y_2]^\top$ where $y_1 + y_2 = 1$, so $y_2 = 1 - y_1$.

$$\begin{aligned} P(\text{together}) &= P(\text{both choose 1}) + P(\text{both choose 2}) \\ &= x_1 y_1 + x_2 y_2 \\ &= x_1 y_1 + (1 - x_1)(1 - y_1) \end{aligned} \tag{1}$$

Step 1: Find NE using BR map

P_1's Best-Response Correspondence given $y_1 \in [0, 1]$

$$\begin{aligned} \mathrm{BR}_1(y_1) &= \arg\min_{x_1 \in [0,1]} \bar{J}_1(x_1, y_1) \\ &= \arg\min_{x_1 \in [0,1]} \left[\tilde{a} x_1 y_1 - \tilde{c}_1 x_1 - \tilde{c}_2 y_1 + a_{22}\right] \\ &= \arg\min_{x_1 \in [0,1]} (\tilde{a} y_1 - \tilde{c}_1) x_1 \end{aligned} \qquad \begin{aligned} \bar{J}_1(x_1, y_1) &= \begin{bmatrix} x_1 & 1 - x_1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ 1 - y_1 \end{bmatrix} \\ &= \tilde{a} x_1 y_1 - \tilde{c}_1 x_1 - \tilde{c}_2 y_1 + a_{22} \end{aligned}$$

Since $\bar{J}_1$ is linear in x_1 on $[0, 1]$, the BR depends on the slope $(\tilde{a} y_1 - \tilde{c}_1)$:

$$\mathrm{BR}_1(y_1) = \begin{cases} 0, & \tilde{a} y_1 - \tilde{c}_1 > 0, \\ [0,1], & \tilde{a} y_1 - \tilde{c}_1 = 0, \\ 1, & \tilde{a} y_1 - \tilde{c}_1 < 0, \end{cases} \quad \Rightarrow \quad \mathrm{BR}_1(y_1) = \begin{cases} 0, & y_1 > \dfrac{\tilde{c}_1}{\tilde{a}}, \\ [0,1], & y_1 = \dfrac{\tilde{c}_1}{\tilde{a}}, \\ 1, & y_1 < \dfrac{\tilde{c}_1}{\tilde{a}}, \end{cases} \quad \Rightarrow \quad y_1^* = \frac{\tilde{c}_1}{\tilde{a}} = \frac{1 + \alpha}{3 + 2\alpha}.$$

P_2's Best-Response Correspondence given $x_1 \in [0, 1]$

$$\begin{aligned} \mathrm{BR}_2(x_1) &= \arg\min_{y_1 \in [0,1]} \bar{J}_2(x_1, y_1) \\ &= \arg\min_{y_1 \in [0,1]} \left[\tilde{b} x_1 y_1 - \tilde{d}_1 x_1 - \tilde{d}_2 y_1 + b_{22}\right] \\ &= \arg\min_{y_1 \in [0,1]} (\tilde{b} x_1 - \tilde{d}_2)\, y_1 \end{aligned} \qquad \begin{aligned} \bar{J}_2(x_1, y_1) &= \begin{bmatrix} y_1 & 1 - y_1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ 1 - x_1 \end{bmatrix} \\ &= \tilde{b} x_1 y_1 - \tilde{d}_1 x_1 - \tilde{d}_2 y_1 + b_{22} \end{aligned}$$

Since $\bar{J}_2$ is linear in y_1 on $[0, 1]$, the BR depends on the slope $(\tilde{b} x_1 - \tilde{d}_2)$:

$$\mathrm{BR}_2(x_1) = \begin{cases} 0, & \tilde{b} x_1 - \tilde{d}_2 > 0, \\ [0,1], & \tilde{b} x_1 - \tilde{d}_2 = 0, \\ 1, & \tilde{b} x_1 - \tilde{d}_2 < 0, \end{cases} \quad \Rightarrow \quad \mathrm{BR}_2(x_1) = \begin{cases} 0, & x_1 < \dfrac{\tilde{d}_2}{\tilde{b}}, \\ [0,1], & x_1 = \dfrac{\tilde{d}_2}{\tilde{b}}, \\ 1, & x_1 > \dfrac{\tilde{d}_2}{\tilde{b}}, \end{cases} \quad \Rightarrow \quad x_1^* = \frac{\tilde{d}_2}{\tilde{b}} = \frac{2 + \alpha}{3 + 2\alpha}.$$

The unique interior mixed-strategy Nash equilibrium is

$$(x_1^*, y_1^*) = \left(\frac{2+\alpha}{3+2\alpha}, \frac{1+\alpha}{3+2\alpha}\right), \quad \text{and we have the symmetry}$$

$$\begin{cases} 1 - x_1^* = 1 - \dfrac{2+\alpha}{3+2\alpha} = \dfrac{1+\alpha}{3+2\alpha} = y_1^*, \\ 1 - y_1^* = 1 - \dfrac{1+\alpha}{3+2\alpha} = \dfrac{2+\alpha}{3+2\alpha} = x_1^*. \end{cases}$$

Step 2: Finding Critical Points

Substituting the NE values into equation (1):

$$\begin{aligned} P(\alpha) &= x_1^* y_1^* + (1-x_1^*)(1-y_1^*) \\ &= 2x_1^* y_1^* \\ &= 2 \cdot \frac{2+\alpha}{3+2\alpha} \cdot \frac{1+\alpha}{3+2\alpha} \\ &= \frac{2(2+\alpha)(1+\alpha)}{(3+2\alpha)^2} \\ &= \frac{4+6\alpha+2\alpha^2}{(3+2\alpha)^2} \end{aligned}$$

$$\begin{aligned} \frac{dP}{d\alpha} &= \frac{d}{d\alpha}\left[\frac{4+6\alpha+2\alpha^2}{(3+2\alpha)^2}\right] \\ &= \frac{(6+4\alpha)(3+2\alpha)^2 - (4+6\alpha+2\alpha^2)\cdot 4(3+2\alpha)}{(3+2\alpha)^4} \\ &= \frac{(6+4\alpha)(3+2\alpha) - 4(4+6\alpha+2\alpha^2)}{(3+2\alpha)^3} \\ &= \frac{18+24\alpha+8\alpha^2 - (16+24\alpha+8\alpha^2)}{(3+2\alpha)^3} \\ &= \frac{2}{(3+2\alpha)^3} \end{aligned}$$

Since $\dfrac{dP}{d\alpha} > 0$ for all $\alpha \geq 0$, $P(\alpha)$ is strictly increasing on $[0, \infty)$.

$$\text{At } \alpha = 0: \quad \begin{aligned} P(0) &= \frac{4+0+0}{9+0+0} \\ &= \frac{4}{9} \end{aligned}$$

$$\text{As } \alpha \to \infty: \quad \begin{aligned} P(\infty) &= \lim_{\alpha\to\infty} \frac{4+6\alpha+2\alpha^2}{9+12\alpha+4\alpha^2} \\ &= \lim_{\alpha\to\infty} \frac{\alpha^2(4/\alpha^2+6/\alpha+2)}{\alpha^2(9/\alpha^2+12/\alpha+4)} \\ &= \lim_{\alpha\to\infty} \frac{4/\alpha^2+6/\alpha+2}{9/\alpha^2+12/\alpha+4} \\ &= \frac{1}{2} \end{aligned}$$

Summary

The unique mixed-strategy Nash equilibrium is

$$(\mathbf{x}^*, \mathbf{y}^*) = \left(\begin{bmatrix} \dfrac{2+\alpha}{3+2\alpha} \\ \dfrac{1+\alpha}{3+2\alpha} \end{bmatrix}, \begin{bmatrix} \dfrac{1+\alpha}{3+2\alpha} \\ \dfrac{2+\alpha}{3+2\alpha} \end{bmatrix}\right).$$

The probability of spending the evening together is

$$P(\alpha) = \frac{4+6\alpha+2\alpha^2}{9+12\alpha+4\alpha^2}.$$

Since $P(\alpha)$ is strictly increasing on $[0, \infty)$,

(a) Highest probability: as $\alpha \to \infty$, $P(\alpha) \to \dfrac{1}{2}$.

(b) Lowest probability: at $\alpha = 0$, with $P(0) = \dfrac{4}{9}$.

Problem 2: Convexity of Best-Reply Correspondence

Problem

Consider a N-player finite matrix game. For player i and, for any given $\mathbf{x}_{-i} \in \Delta_{-i}$, let $\Phi_i(\mathbf{x}_{-i}) \subset \Delta_i$ denote its mixed-strategy best-reply correspondence set.

Prove that $\Phi_i(\mathbf{x}_{-i})$ is a convex set. Then show that for every $\mathbf{x} \in \Delta_X$ the overall set $\Phi(\mathbf{x})$ is a convex set.

What property of the cost function $\bar{J}_i$ was instrumental in your proof? Can you give a weaker property of $\bar{J}_i$ under which the same claim can be proved? Justify your answer.

Solution

Convexity of Individual Best Response $\phi_i(\mathbf{x}_{-i})$

Let $\phi_i(\mathbf{x}_{-i})$ be player i's best–response set

$$\phi_i(\mathbf{x}_{-i}) := \{\mathbf{x}_i \in \Delta_i \mid \bar{J}_i(\mathbf{x}_i,\ \mathbf{x}_{-i}) \le \bar{J}_i(\mathbf{w}_i,\ \mathbf{x}_{-i}),\ \forall \mathbf{w}_i \in \Delta_i\}.$$

Let $\mathbf{a}, \mathbf{b} \in \phi_i(\mathbf{x}_{-i})$ be two best responses, and $\mathbf{z}_i = \lambda \mathbf{a} + (1-\lambda)\mathbf{b}$ with $\lambda \in [0,1]$. By definition of $\phi_i(\mathbf{x}_{-i})$,

$$\begin{aligned}
\bar{J}_i(\mathbf{a},\ \mathbf{x}_{-i}) &\le \bar{J}_i(\mathbf{w}_i,\ \mathbf{x}_{-i}), && \forall \mathbf{w}_i \in \Delta_i,\\
\bar{J}_i(\mathbf{b},\ \mathbf{x}_{-i}) &\le \bar{J}_i(\mathbf{w}_i,\ \mathbf{x}_{-i}), && \forall \mathbf{w}_i \in \Delta_i.
\end{aligned}$$

Using linearity of $\bar{J}_i(\mathbf{x}_i,\ \mathbf{x}_{-i})$ in its first argument,

$$\begin{aligned}
\bar{J}_i(\mathbf{z}_i,\ \mathbf{x}_{-i}) &= \bar{J}_i\big(\lambda \mathbf{a} + (1-\lambda)\mathbf{b},\ \mathbf{x}_{-i}\big)\\
&= \lambda\, \bar{J}_i(\mathbf{a},\ \mathbf{x}_{-i}) + (1-\lambda)\, \bar{J}_i(\mathbf{b},\ \mathbf{x}_{-i}) && \text{(linearity in } \mathbf{x}_i)\\
&\le \lambda\, \bar{J}_i(\mathbf{w}_i,\ \mathbf{x}_{-i}) + (1-\lambda)\, \bar{J}_i(\mathbf{w}_i,\ \mathbf{x}_{-i}) && \text{(since } \mathbf{a}, \mathbf{b} \in \phi_i(\mathbf{x}_{-i}))\\
&= \bar{J}_i(\mathbf{w}_i,\ \mathbf{x}_{-i}), \qquad \forall \mathbf{w}_i \in \Delta_i.
\end{aligned}$$

Hence $\bar{J}_i(\mathbf{z}_i,\ \mathbf{x}_{-i}) \le \bar{J}_i(\mathbf{w}_i,\ \mathbf{x}_{-i})$ for all $\mathbf{w}_i \in \Delta_i$, so

$$\mathbf{z}_i = \lambda \mathbf{a} + (1-\lambda)\mathbf{b} \in \phi_i(\mathbf{x}_{-i}).$$

Therefore each individual best–response set $\phi_i(\mathbf{x}_{-i})$ is convex.

Convexity of Overall Best Response $\Phi(\mathbf{x})$

Define the joint best–response correspondence

$$\Phi(\mathbf{x}) := \phi_1(\mathbf{x}_{-1}) \times \cdots \times \phi_N(\mathbf{x}_{-N}) \subset \Delta_X.$$

Let $\mathbf{a}, \mathbf{b} \in \Phi(\mathbf{x})$ with

$$\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_N), \qquad \mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_N), \qquad \mathbf{a}_i, \mathbf{b}_i \in \phi_i(\mathbf{x}_{-i})\ \ \forall i.$$

For any $\lambda \in [0,1]$ set

$$\mathbf{z} := \lambda \mathbf{a} + (1-\lambda)\mathbf{b} = \big(\lambda \mathbf{a}_1 + (1-\lambda)\mathbf{b}_1, \dots, \lambda \mathbf{a}_N + (1-\lambda)\mathbf{b}_N\big).$$

Since each $\phi_i(\mathbf{x}_{-i})$ is convex,

$$\mathbf{z}_i := \lambda \mathbf{a}_i + (1-\lambda)\mathbf{b}_i \in \phi_i(\mathbf{x}_{-i}), \quad \forall i,$$

hence

$$\mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_N) \in \phi_1(\mathbf{x}_{-1}) \times \cdots \times \phi_N(\mathbf{x}_{-N}) = \Phi(\mathbf{x}).$$

Therefore $\Phi(\mathbf{x})$ is convex.

Discussion of Cost Function Properties

Q1: What property of $\bar{J}_i$ was instrumental in your proof?

The key property is **linearity** of $\bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i})$ in the $\mathbf{w}_i$ argument (Section 3.3.2, lecture notes).

Any mixed strategy $\mathbf{w}_i \in \Delta_i$ can be written as a convex combination of pure strategies:

$$\mathbf{w}_i = \sum_{j \in M_i} \mathbf{e}_{ij} \lambda_{ij}$$

where $\lambda_{ij} \geq 0$ and $\sum_{j \in M_i} \lambda_{ij} = 1$.

Since the expected cost $\bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i})$ is linear in $\mathbf{w}_i$ for any given $\mathbf{x}_{-i}$, it follows that:

$$\bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i}) = \bar{J}_i \left(\sum_{j \in M_i} \mathbf{e}_{ij} \lambda_{ij}, \mathbf{x}_{-i} \right) = \sum_{j \in M_i} \bar{J}_i(\mathbf{e}_{ij}, \mathbf{x}_{-i}) \lambda_{ij} \tag{3.10}$$

This linearity guarantees that any convex combination of best responses achieves the same minimal cost, ensuring $\Phi_i(\mathbf{x}_{-i})$ is convex.

Q2: Can you give a weaker property under which the same claim holds?

Yes. A weaker sufficient condition is that $\bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i})$ is **quasi-convex** in $\mathbf{w}_i$.

Justification. A function $f : S \to \mathbb{R}$ is quasi-convex if all sublevel sets $\{z \in S : f(z) \leq c\}$ are convex.

The best-response correspondence is defined as:

$$\Phi_i(\mathbf{x}_{-i}) = \left\{ \mathbf{x}_i \in \Delta_i : \bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \min_{\mathbf{w}_i \in \Delta_i} \bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i}) \right\}$$

This is equivalently the minimum sublevel set:

$$\Phi_i(\mathbf{x}_{-i}) = \left\{ \mathbf{z}_i \in \Delta_i : \bar{J}_i(\mathbf{z}_i, \mathbf{x}_{-i}) \leq \bar{J}_i^* \right\}$$

where $\bar{J}_i^* = \min_{\mathbf{w}_i \in \Delta_i} \bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i})$.

If $\bar{J}_i$ is quasi-convex in $\mathbf{w}_i$, then all sublevel sets are convex, ensuring $\Phi_i(\mathbf{x}_{-i})$ is convex.

Linearity of the expected cost in the mixed strategy. For player i, any mixed strategy $w_i \in \Delta_i$ can be written as a convex combination of pure strategies

$$w_i = \sum_{j \in M_i} \alpha_{ij} e_{ij}, \qquad \alpha_{ij} \geq 0, \sum_{j \in M_i} \alpha_{ij} = 1.$$

Since the expected cost $\bar{J}_i(w_i, x_{-i})$ is linear in w_i, for any fixed x_{-i} we have

$$\bar{J}_i(w_i, x_{-i}) = \bar{J}_i \left(\sum_{j \in M_i} \alpha_{ij} e_{ij}, x_{-i} \right) = \sum_{j \in M_i} \alpha_{ij} \bar{J}_i(e_{ij}, x_{-i}).$$

That is, the cost of a mixed strategy is the weighted average of the costs of pure strategies, with weights given by the mixing probabilities α_{ij}.

Problem 3: Nash Equilibrium Theorem via Brouwer's Fixed-Point Theorem

This question has three parts, related to proving Nash equilibrium theorem (Theorem 3.19 in the PDF notes) by using Brouwer's fixed-point theorem.

(a) Consider Lemma 3.17 and its proof. The main steps for getting to the last inequality in the proof (which is a contradiction) are only described in words. Show mathematically, justifying each step, that indeed you obtain that inequality.

(b) Prove Lemma 3.18.

(c) The proof of Theorem 3.19 on page 49 involves a specially constructed, continuous function $\eta : \Delta_X \to \Delta_X$. The last part of the proof shows that every fixed point of η is a Nash equilibrium (NE). Prove the reverse, namely that every NE $\mathbf{x}^*$ is a fixed point of η.

Solution

Part (a): Mathematical Justification of Lemma 3.17

Lemma 3.17: Consider any mixed-strategy $\mathbf{x} = (\mathbf{x}_i, \mathbf{x}_{-i}) \in \Delta_X$, $\mathbf{x}_i \in \Delta_i$. Then for every player $i \in \mathcal{N}$, there exists a $k \in \sup(\mathbf{x}_i)$ such that:

$$\bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}) \leq \bar{J}_i(\mathbf{e}_{ik}, \mathbf{x}_{-i})$$

Write a mixed strategy $\mathbf{x}_i \in \Delta_i$ as a convex combination of pure strategies:

$$\mathbf{x}_i = \sum_{j \in M_i} x_{ij}\mathbf{e}_{ij} = \sum_{k \in \sup(\mathbf{x}_i)} x_{ik}\mathbf{e}_{ik}, \qquad x_{ik} \geq 0, \quad \sum_{k \in \sup(\mathbf{x}_i)} x_{ik} = 1.$$

Assume by contradiction that

$$\bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}) > \bar{J}_i(\mathbf{e}_{ik}, \mathbf{x}_{-i}), \qquad \forall k \in \sup(\mathbf{x}_i).$$

Multiplying by $x_{ik} > 0$ and summing over $k \in \sup(\mathbf{x}_i)$ gives

$$\underbrace{\sum_k x_{ik}\bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i})}_{LHS} > \underbrace{\sum_k x_{ik}\bar{J}_i(\mathbf{e}_{ik}, \mathbf{x}_{-i})}_{RHS}.$$

$$\begin{aligned}\sum_k x_{ik}\,\bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}) &= \bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}) \sum_k x_{ik} \\ &= \bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i})\end{aligned} \qquad \begin{aligned}\sum_k x_{ik}\,\bar{J}_i(\mathbf{e}_{ik}, \mathbf{x}_{-i}) &= \bar{J}_i\left(\sum_k x_{ik}\mathbf{e}_{ik}, \mathbf{x}_{-i}\right) \\ &= \bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i})\end{aligned}$$

Which leads to a contradiction.

$$\bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}) > \bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}),$$

Therefore there exists some $k \in \sup(\mathbf{x}_i)$ such that $\bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}) \leq \bar{J}_i(\mathbf{e}_{ik}, \mathbf{x}_{-i})$

Part (b): Proof of Lemma 3.18

Lemma 3.18. Let $\mathbf{x}^* = (\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \in \Delta_X$. Then $\mathbf{x}^* \in NE(\mathcal{G})$ iff for every $i \in \mathcal{N}$ and $j \in M_i$,

$$\bar{J}_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \bar{J}_i(\mathbf{e}_{ij}, \mathbf{x}_{-i}^*). \tag{3.13}$$

$$\begin{aligned}&\mathbf{A}: \ \mathbf{x}^* \in NE(\mathcal{G}), \\ &\mathbf{B}: \ \bar{J}_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \bar{J}_i(\mathbf{e}_{ij}, \mathbf{x}_{-i}^*), \ \forall i \in \mathcal{N}, \ \forall j \in M_i.\end{aligned}$$

(A $\Rightarrow$ B) Necessity. Assume **A** holds. By the NE definition (Eq. 3.6), for every i and every $\mathbf{w}_i \in \Delta_i$,

$$\bar{J}_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i}^*).$$

Since each pure strategy $\mathbf{e}_{ij} \in \Delta_i$ is a special case of $\mathbf{w}_i$, choosing $\mathbf{w}_i = \mathbf{e}_{ij}$ gives

$$\bar{J}_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \bar{J}_i(\mathbf{e}_{ij}, \mathbf{x}_{-i}^*), \quad \forall i, \ \forall j,$$

which is exactly **B**.

(B ⇒ A) Sufficiency. Assume **B** holds. Fix a player i and an arbitrary mixed strategy $\mathbf{w}_i \in \Delta_i$. Write it as a convex combination of pure strategies:

$$\mathbf{w}_i = \sum_{j \in M_i} \alpha_{ij} \mathbf{e}_{ij}, \qquad \alpha_{ij} \geq 0, \sum_{j \in M_i} \alpha_{ij} = 1.$$

Multiply (3.13) by α_{ij} and sum over j:

$$\sum_{j \in M_i} \alpha_{ij} \bar{J}_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \sum_{j \in M_i} \alpha_{ij} \bar{J}_i(\mathbf{e}_{ij}, \mathbf{x}_{-i}^*).$$

$$\begin{aligned} \sum_{j \in M_i} \alpha_{ij} \bar{J}_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) &= \bar{J}_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \sum_{j \in M_i} \alpha_{ij} \\ &= \bar{J}_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \end{aligned} \qquad \begin{aligned} \sum_{j \in M_i} \alpha_{ij} \bar{J}_i(\mathbf{e}_{ij}, \mathbf{x}_{-i}^*) &= \bar{J}_i\left(\sum_{j \in M_i} \alpha_{ij} \mathbf{e}_{ij}, \mathbf{x}_{-i}^* \right) \\ &= \bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i}^*) \end{aligned}$$

Since $\mathbf{w}_i \in \Delta_i$ was arbitrary, this holds for all players i and all mixed deviations $\mathbf{w}_i$, i.e.

$$\bar{J}_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i}^*), \quad \forall i, \ \forall \mathbf{w}_i \in \Delta_i,$$

which is the NE condition, so **A** holds.

Part (c): If $\mathbf{x}^* \in NE(\mathcal{G})$, then $\mathbf{x}^* = \boldsymbol{\eta}(\mathbf{x}^*)$.

Brouwer's Fixed-Point Theorem.

Let $S \subset \mathbb{R}^m$ be nonempty, compact, and convex. If $f : S \to S$ is continuous, then

$$\exists x^\star \in S \quad \text{s.t.} \quad f(x^\star) = x^\star.$$

Intuition (1D case). If $S = [0, 1]$ and $f : [0, 1] \to [0, 1]$ is continuous, set $g(x) = f(x) - x$. Then $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$; by IVT $\exists x^\star$ with $g(x^\star) = 0$, so $f(x^\star) = x^\star$.

Graph of a map.

$$\text{Graph}(f) = \{(x, y) \in S \times S : \ y = f(x)\}.$$

For $S = [0, 1]$, $\text{Graph}(f) \subset [0, 1] \times [0, 1]$ (a curve inside the unit square).

Supose $\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_N^*) \in NE(\mathcal{G})$. By Lemma 3.18, for all $i \in \mathcal{N}$ and $j \in M_i$:

$$\bar{J}_i(\mathbf{x}^*) \leq \bar{J}_i(\mathbf{e}_{ij}, \mathbf{x}_{-i}^*)$$

This implies:

$$\bar{J}_i(\mathbf{x}^*) - \bar{J}_i(\mathbf{e}_{ij}, \mathbf{x}_{-i}^*) \leq 0, \quad \forall i \in \mathcal{N}, \forall j \in M_i$$

Therefore, by definition of $C_{i,j}$:

$$C_{i,j}(\mathbf{x}^*) = \max\{\bar{J}_i(\mathbf{x}^*) - \bar{J}_i(\mathbf{e}_{ij}, \mathbf{x}_{-i}^*), 0\} = 0, \quad \forall i \in \mathcal{N}, \forall j \in M_i$$

For each $i \in \mathcal{N}$ and $j \in M_i$, compute $\eta_{i,j}(\mathbf{x}^*)$ using equation (3.14):

$$\eta_{i,j}(\mathbf{x}^*) = \frac{x_i^*(\mathbf{e}_{ij}) + C_{i,j}(\mathbf{x}^*)}{1 + \sum_{j \in M_i} C_{i,j}(\mathbf{x}^*)}$$

Substituting $C_{i,j}(\mathbf{x}^*) = 0$ for all j:

$$\eta_{i,j}(\mathbf{x}^*) = \frac{x_i^*(\mathbf{e}_{ij}) + 0}{1 + 0} = x_i^*(\mathbf{e}_{ij}) = x_{i,j}^*$$

Therefore, for each player i:

$$\boldsymbol{\eta}_i(\mathbf{x}^*) = (x_{i,1}^*, x_{i,2}^*, \ldots, x_{i,M_i}^*) = \mathbf{x}_i^*$$

Combining all players:

$$\boldsymbol{\eta}(\mathbf{x}^*) = (\boldsymbol{\eta}_1(\mathbf{x}^*), \ldots, \boldsymbol{\eta}_N(\mathbf{x}^*)) = (\mathbf{x}_1^*, \ldots, \mathbf{x}_N^*) = \mathbf{x}^*$$

Thus, every NE $\mathbf{x}^*$ is a fixed point of $\boldsymbol{\eta}$.