Lecture 3: Non-Zero-Sum Games & Nash Equilibrium

Last Lecture Recap

Two-player zero-sum (2PZS) finite game with **mixed strategies.**

1. Mixed strategy extension

$$\mathbf{x} \in \Delta_1, \quad \mathbf{y} \in \Delta_2, \quad \Delta_1 = \left\{ \mathbf{x} \,\middle|\, 0 \le x_j \le 1, \ \sum_j x_j = 1 \right\},$$

$$\mathbf{x} = \begin{bmatrix} \vdots \\ x_j \\ \vdots \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \vdots \\ y_k \\ \vdots \end{bmatrix} \longrightarrow \text{probability of action } j, k,$$

$$\Rightarrow \quad \bar{J}(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top A \mathbf{y} \quad \text{(expected cost)}.$$

$$P_1 : \min_{\mathbf{x} \in \Delta_1} \bar{J}(\mathbf{x}, \mathbf{y}), \qquad P_2 : \max_{\mathbf{y} \in \Delta_2} \bar{J}(\mathbf{x}, \mathbf{y}).$$

2. Mixed strategy saddle point $(\mathbf{x}^*, \mathbf{y}^*)$

$$\underbrace{\mathbf{x}^{*\top} A \mathbf{y} \ \le \ \mathbf{x}^{*\top} A \mathbf{y}^*}_{\text{no-regret for } P_2} \qquad \underbrace{\mathbf{x}^{*\top} A \mathbf{y}^* \ \le \ \mathbf{x}^\top A \mathbf{y}^*}_{\text{no-regret for } P_1} \qquad \forall\, \mathbf{x} \in \Delta_1, \ \forall\, \mathbf{y} \in \Delta_2.$$

3. Computing (x^*, y^*) using graphical method in $2 \times m_2$ games

Game Elements in This Lecture

Players $N = 2$

Strategy Mixed strategies $\mathbf{x}_i \in \Delta_i$

Cost **Non-zero-sum:** $J_1 + J_2 \neq 0$ with $\bar{J}_1(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A \mathbf{y}, \bar{J}_2(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T B \mathbf{y}$

Why This Change? Real-world games have cooperation/conflict beyond pure opposition.

Key Concept: Nash Equilibrium (NE) generalizes saddle-point equilibrium to non-zero-sum games.

$$J_i(\mathbf{x}^*, \mathbf{y}^*) \le J_i(\mathbf{x}^*, \mathbf{y}^*) \quad \forall \mathbf{x}^* \in \Delta_1, \ \forall \mathbf{y}^* \in \Delta_2$$

3.1 Extension of all concepts from 2PZSG

Corresponds to Chapter 3.2 in pdf notes

3.1.1 From single matrix to bimatrix

In non-zero-sum games, $A + B \neq 0$, and each player has their own cost function:

P_1's cost matrix A

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m_2} \\ \vdots & & \vdots \\ a_{m_1 1} & \cdots & a_{m_1 m_2} \end{bmatrix}$$

$$P_1 : \min_{\mathbf{x} \in \Delta_1} \bar{J}_1(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{x} \in \Delta_1} \mathbf{x}^\top A \ \mathbf{y}$$

P_2's cost matrix B

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1m_2} \\ \vdots & & \vdots \\ b_{m_1 1} & \cdots & b_{m_1 m_2} \end{bmatrix}$$

$$P_2 : \min_{\mathbf{y} \in \Delta_2} \bar{J}_2(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{y} \in \Delta_2} \mathbf{x}^\top B \ \mathbf{y}$$

Special Case. Pure strategy saddle point $(j^*, k^*) \in M_1 \times M_2$ satisfies:

$$\underbrace{e_{j^*}^\top A e_k}_{a_{j^*k}} \le \underbrace{e_{j^*}^\top A e_{k^*}}_{a_{j^*k^*}} \le \underbrace{e_j^\top A e_{k^*}}_{a_{jk^*}}, \qquad \forall j \in M_1,\ \forall k \in M_2.$$

3.1.2 From saddle point to Nash equilibrium (no regret)

Def. 3.7 Mixed-Strategy NE in Bimatrix Game.

A pair of **mixed strategies** $(\mathbf{x}^*, \mathbf{y}^*) \in \Delta_1 \times \Delta_2$ is a **no-regret (Nash) pair for both** P_1 **and** P_2 in a 2-player non-zero-sum game if

$$\underbrace{\mathbf{x}^{*\top} A \mathbf{y}^* \le \mathbf{x}^\top A \mathbf{y}^*}_{\text{no-regret for } P_1} \qquad \forall \mathbf{x} \in \Delta_1, \tag{1}$$

$$\underbrace{\mathbf{x}^{*\top} B \mathbf{y}^* \le \mathbf{x}^{*\top} B \mathbf{y}}_{\text{no-regret for } P_2} \qquad \forall \mathbf{y} \in \Delta_2 \tag{2}$$

Def. 3.1 Pure-Strategy NE in 2P Non-Zero-Sum Game.

A pair of **pure strategies** $(j^*, k^*) \in M_1 \times M_2$ is a **no-regret (Nash) pair** if

$$\underbrace{a_{j^*k^*} \le a_{jk^*}}_{\text{no-regret for } P_1} \quad \underbrace{b_{j^*k^*} \le b_{j^*k},}_{\text{no-regret for } P_2} \qquad \forall j \in M_1, \forall k \in M_2,$$

where $A = [a_{jk}]$ and $B = [b_{jk}]$ are the cost matrices of P_1 and P_2 respectively.

Example: Prisoner's Dilemma (PD).

Two prisoners can either **confess** or **not confess**.

P_1's cost matrix A

$$A = \begin{bmatrix} 5 & 0 \\ 15 & 1 \end{bmatrix}$$

P_2's cost matrix B

$$B = \begin{bmatrix} 5 & 15 \\ 0 & 1 \end{bmatrix}$$

- P_1 looks at worst case in A: $\min_j \max_k \{a_{jk}\} = \min_j \{\underbrace{\max\{5,0\}}_{k=1}, \underbrace{\max\{15,1\}}_{k=2}\} = 5 \Rightarrow j^* = 1$
- P_2 looks at worst case in B: $\min_k \max_j \{b_{jk}\} = \min_k \{\underbrace{\max\{5,15\}}_{j=1}, \underbrace{\max\{0,1\}}_{j=2}\} = 5 \Rightarrow k^* = 1$

$\Rightarrow (j^*, k^*) = (1, 1)$ both confess.

Example: Chicken Game.

Two drivers head toward each other and can either **swerve** or **go straight**.

P_1's cost matrix A

$$A = \begin{bmatrix} 0 & 1 \\ \underbrace{-1}_{\text{temptation cost}} & 10 \end{bmatrix}$$

P_2's cost matrix B

$$B = \begin{bmatrix} 0 & -1 \\ 1 & \underbrace{10}_{\text{crash cost}} \end{bmatrix}$$

- P_1 looks at worst case in A: $\min_j \max_k \{a_{jk}\} = \min_j \{\underbrace{\max\{0,1\}}_{k=1}, \underbrace{\max\{-1,10\}}_{k=2}\} = 1 \Rightarrow j^* = 1.$

- P_2 looks at worst case in B: $\min_k \max_j \{b_{jk}\} = \min_k \{\underbrace{\max\{0,1\}}_{j=1}, \underbrace{\max\{-1,10\}}_{j=2}\} = 1 \Rightarrow k^* = 1$.

$\Rightarrow (j^*, k^*) = (1,1)$ both swerve.

3.2 Best response strategy/maps and NE equilibrium

Def. Best-Response Strategy.

P_1 **always responds optimally** to any mixed strategy $\mathbf{y} \in \Delta_2$ of P_2, and so does P_2:

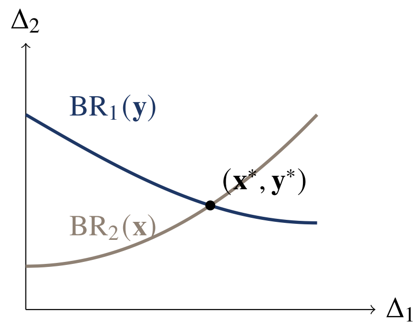
$$\underbrace{\mathbf{x} = \mathrm{BR}_1(\mathbf{y}) := \arg\min_{\mathbf{x}\in\Delta_1} \mathbf{x}^\top A\mathbf{y}}_{\text{set-valued mapping } \Delta_2 \rightrightarrows \Delta_1} \qquad \forall \text{ given } \mathbf{y} \in \Delta_2 \tag{3}$$

$$\underbrace{\mathbf{y} = \mathrm{BR}_2(\mathbf{x}) := \arg\min_{\mathbf{y}\in\Delta_2} \mathbf{x}^\top B\mathbf{y}}_{\text{(not a single value!)}} \qquad \forall \text{ given } \mathbf{x} \in \Delta_1 \tag{4}$$

Interpretation. NE $(\mathbf{x}^*, \mathbf{y}^*)$ is the best response to each other.

$$(1) \quad \underbrace{\mathbf{x}^{*\top} A\mathbf{y}^* \leq \mathbf{x}^\top A\mathbf{y}^*}_{\text{no-regret for } P_1} \quad \Rightarrow \quad \underbrace{\mathbf{x}^* \in \mathrm{BR}_1(\mathbf{y}^*)}_{\mathbf{x}^* \text{ is the best response to } \mathbf{y}^*} \quad (3)$$

$$(2) \quad \underbrace{\mathbf{x}^{*\top} B\mathbf{y}^* \leq \mathbf{x}^{*\top} B\mathbf{y}}_{\text{no-regret for } P_2} \quad \Rightarrow \quad \underbrace{\mathbf{y}^* \in \mathrm{BR}_2(\mathbf{x}^*)}_{\mathbf{y}^* \text{ is the best response to } \mathbf{x}^*} \quad (4)$$



$\Rightarrow$ $(\mathbf{x}^*, \mathbf{y}^*)$ NE are at the $\cap$ intersection of individual BR maps. We'll use this to compute NEs.

Computing all NEs graphically in 2x2 games

Given the bimatrix game

$$(A, B) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right),$$

with mixed strategies

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top \in \Delta_1 \quad \Rightarrow \quad x_1 \in [0,1] \quad \text{and} \quad x_2 = 1 - x_1,$$
$$\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^\top \in \Delta_2 \quad \Rightarrow \quad y_1 \in [0,1] \quad \text{and} \quad y_2 = 1 - y_1,$$

the expected costs are

$$
\begin{aligned}
\bar{J}_1(x_1, y_1) &= \mathbf{x}^\top A\, \mathbf{y} \\
&= \begin{bmatrix} x_1 & 1 - x_1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ 1 - y_1 \end{bmatrix} \\
&= a_{22} + (a_{21} - a_{22}) y_1 + x_1 \Big[\underbrace{(a_{11} - a_{12} - a_{21} + a_{22})}_{\tilde{a}}\, y_1 - \underbrace{(a_{22} - a_{12})}_{\tilde{c}_1} \Big] \\
&= \big(a_{22} + (a_{21} - a_{22}) y_1 \big) + x_1 (\tilde{a} y_1 - \tilde{c}_1), \\
\bar{J}_2(x_1, y_1) &= \mathbf{x}^\top B\, \mathbf{y} \\
&= \begin{bmatrix} x_1 & 1 - x_1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ 1 - y_1 \end{bmatrix} \\
&= b_{22} + (b_{12} - b_{22}) x_1 + y_1 \Big[\underbrace{(b_{11} - b_{12} - b_{21} + b_{22})}_{\tilde{b}}\, x_1 - \underbrace{(b_{22} - b_{21})}_{\tilde{d}_2} \Big] \\
&= \big(b_{22} + (b_{12} - b_{22}) x_1 \big) + y_1 (\tilde{b} x_1 - \tilde{d}_2),
\end{aligned}
$$

where

$$
\begin{cases}
\tilde{a} = a_{11} - a_{12} - a_{21} + a_{22} \\
\tilde{c}_1 = a_{22} - a_{12} \\
\tilde{b} = b_{11} - b_{12} - b_{21} + b_{22} \\
\tilde{d}_2 = b_{22} - b_{21}
\end{cases}
$$

Given y_1, P_1's best response map.

$$
\begin{aligned}
\xi_1 = \underbrace{\mathrm{BR}_1(y_1)}_{\text{set-valued map}} &= \arg\min_{x_1 \in [0,1]} \underbrace{\bar{J}_1(x_1, y_1)}_{\text{linear in } x_1 \,\in\, [0,1]} && \forall\, y_1 \in [0, 1] \\
&= \arg\min_{x_1 \in [0,1]} x_1 \underbrace{(\tilde{a} y_1 - \tilde{c}_1)}_{\text{slope}} && \forall\, y_1 \in [0, 1].
\end{aligned}
$$

$$
\text{slope} = \begin{cases} \nearrow \\ \leftrightarrow \\ \searrow \end{cases} \Rightarrow \quad \xi_1 = \begin{cases} 0, & \text{if } (\tilde{a} y_1 - \tilde{c}_1) > 0 \quad \Rightarrow \quad y_1 > \frac{\tilde{c}_1}{\tilde{a}} \\ [0, 1], & \text{if } (\tilde{a} y_1 - \tilde{c}_1) = 0 \quad \Rightarrow \quad y_1 = \frac{\tilde{c}_1}{\tilde{a}} \longrightarrow \text{might be not unique!} \\ 1, & \text{if } (\tilde{a} y_1 - \tilde{c}_1) < 0 \quad \Rightarrow \quad y_1 < \frac{\tilde{c}_1}{\tilde{a}}, \end{cases}
$$

Given x_1, P_2's best response map.

$$
\begin{aligned}
\eta_1 = \underbrace{\mathrm{BR}_2(x_1)}_{\text{set-valued map}} &= \arg\min_{y_1 \in [0,1]} \underbrace{\bar{J}_2(x_1, y_1)}_{\text{linear in } y_1 \in [0,1]} && \forall\, x_1 \in [0, 1] \\
&= \arg\min_{y_1 \in [0,1]} y_1 \underbrace{(\tilde{b} x_1 - \tilde{d}_2)}_{\text{slope}} && \forall\, x_1 \in [0, 1].
\end{aligned}
$$

$$
\text{slope} = \begin{cases} \nearrow \\ \leftrightarrow \\ \searrow \end{cases} \Rightarrow \quad \eta_1 = \begin{cases} 0, & \text{if } (\tilde{b} x_1 - \tilde{d}_2) > 0 \quad \Rightarrow \quad x_1 > \dfrac{\tilde{d}_2}{\tilde{b}}, \\ [0, 1], & \text{if } (\tilde{b} x_1 - \tilde{d}_2) = 0 \quad \Rightarrow \quad x_1 = \dfrac{\tilde{d}_2}{\tilde{b}} \longrightarrow \text{might be not unique!} \\ 1, & \text{if } (\tilde{b} x_1 - \tilde{d}_2) < 0 \quad \Rightarrow \quad x_1 < \dfrac{\tilde{d}_2}{\tilde{b}}. \end{cases}
$$

$\Rightarrow$ $(\mathbf{x}^*, \mathbf{y}^*)$ NE are at the intersection of BR maps. $\Rightarrow$ Plot them on the same graph.

$$(\mathbf{x}^*, \mathbf{y}^*) = \left(\begin{bmatrix} \frac{\tilde{d}_2}{\tilde{b}} \\ 1 - \frac{\tilde{d}_2}{\tilde{b}} \end{bmatrix}, \begin{bmatrix} \frac{\tilde{c}_1}{\tilde{a}} \\ 1 - \frac{\tilde{c}_1}{\tilde{a}} \end{bmatrix} \right).$$

Example: Chicken Game.

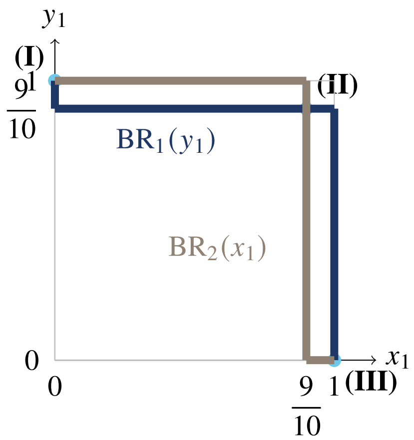
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 10 \end{bmatrix}$$

From the general formulas

$$\begin{cases} \tilde{a} = a_{11} - a_{12} - a_{21} + a_{22} = 10 \\ \tilde{c}_1 = a_{22} - a_{12} = 9 \\ \tilde{b} = b_{11} - b_{12} - b_{21} + b_{22} = 10 \\ \tilde{d}_2 = b_{22} - b_{21} = 9 \end{cases}$$

We obtain the best response maps:

$$\xi_1(y_1) = \begin{cases} 0, & y_1 > \frac{9}{10}, \\ [0,1], & y_1 = \frac{9}{10}, \\ 1, & y_1 < \frac{9}{10}, \end{cases} \qquad \eta_1(x_1) = \begin{cases} 0, & x_1 > \frac{9}{10}, \\ [0,1], & x_1 = \frac{9}{10}, \\ 1, & x_1 < \frac{9}{10}. \end{cases}$$



The three intersections of the best–response graphs yield three Nash equilibria:

$$\text{(I)} \quad x_1^* = 0,\ y_1^* = 1 \quad \Longrightarrow \quad \mathbf{x}^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \mathbf{y}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\text{(II)} \quad x_1^* = \tfrac{9}{10},\ y_1^* = \tfrac{9}{10} \quad \Longrightarrow \quad \mathbf{x}^* = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix},\ \mathbf{y}^* = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix},$$

$$\text{(III)} \quad x_1^* = 1,\ y_1^* = 0 \quad \Longrightarrow \quad \mathbf{x}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{y}^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In the mixed equilibrium (II), each player swerves with probability 0.9.

Next time: Given 3 NEs, how to decide which one is better? For whom?