## Lecture 5: Nash Theorem Proof & Repeated Games

**Last Lecture Recap**

- N-player finite (matrix) game
  - 1.1 Setup & NE definition
  - 1.2 BR map & NE characterization
- Two fixed point theorems (Brouwer, Kakutani)
- Nash theorem & idea of proof
  - Verified: $\Delta_X$ is compact & convex
  - Verified: $\mathrm{BR}(\mathbf{x})$ is non-empty & convex
  - **Remaining: Graph(BR) is closed (3rd condition)**

### 5.1 Finishing Nash's Existence Theorem

**Game Elements in This Lecture**

**Players** $\mathcal{I} = \{1, \ldots, i, \ldots, N\}$ with $N \geq 2$ **(generalization of the $N = 2$ case)**.
**Strategy** Mixed strategies $\mathbf{x}_i \in \Delta_i$, joint profile $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \Delta_X := \prod_{i \in N} \Delta_i$
**Cost** $\bar{J}_i : \Delta_X \to \mathbb{R}$ for each $i \in \mathcal{I}$

**Recall: Nash's theorem states that any finite $N$-player game admits at least one mixed NE.**

$$\mathbf{x}^* = \text{NE} \quad \Leftrightarrow \quad \mathbf{x}_i^* \in \mathrm{BR}_i(\mathbf{x}_{-i}^*), \ \forall\, i \in \mathcal{I}$$

$$\underbrace{(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \quad \Leftrightarrow \quad \mathbf{x}^* \in \mathrm{BR}(\mathbf{x}^*)}_{\text{fixed point of the overall BR map (set-valued)}} \qquad \text{where } \mathrm{BR}(\mathbf{x}^*) = \begin{bmatrix} \mathrm{BR}_1(\mathbf{x}_{-1}^*) \\ \vdots \\ \mathrm{BR}_N(\mathbf{x}_{-N}^*) \end{bmatrix}$$

where, for a given $\mathbf{x}_{-i}$,

$$\mathrm{BR}_i(\mathbf{x}_{-i}) = \arg\min_{\mathbf{x}_i \in \Delta_i} \bar{J}_i(\mathbf{x}_i, \mathbf{x}_{-i}).$$

Overall: $\mathrm{BR}_i : \Delta_{-i} \rightrightarrows \Delta_i$ and $\mathrm{BR} : \Delta \rightrightarrows \Delta$.

**Three Conditions for Kakutani's Theorem.**

To apply Kakutani's fixed-point theorem to prove Nash's theorem, we need:

(i) The set $\Delta_X$ is **convex** and **compact**. ✓ (verified in Lecture 4)
(ii) $\mathrm{BR}(\mathbf{x})$ is a **non-empty convex set** for all $\mathbf{x} \in \Delta$. ✓ (verified in Lecture 4)
(iii) $\mathrm{Graph}(\mathrm{BR})$ is a **closed set** (it contains all its limit points).
$\Rightarrow$ We show this by contradiction: a set $S$ is closed if for every sequence $\{x^{(n)}\} \subset S$ with $\lim_{n\to\infty} x^{(n)} = x$, we have $x \in S$.

**Graph of a correspondence.**

$\mathrm{Graph}(\Phi) = \{(y, x) \in S \times S \mid y \in \Phi(x) \text{ for some } x \in S\}$

In our case, $S = \Delta_X$ and $\Phi = \mathrm{BR}$.

$\Rightarrow \mathrm{Graph}(\mathrm{BR}_i) = \{(\bar{\mathbf{z}}_i, \mathbf{x}_{-i}) \in \Delta_i \times \Delta_{-i} \mid \bar{\mathbf{z}}_i \in \mathrm{BR}_i(\mathbf{x}_{-i}) \text{ for some } \mathbf{x}_{-i} \in \Delta_{-i}\}$

**Proof that $\mathrm{Graph}(\mathrm{BR}_i)$ is Closed.**

**Assume (by contradiction) that $\mathrm{Graph}(\mathbf{BR}_i)$ not closed**

**i.e.** for some $i \in \mathcal{I}$, the graph of the $i$-th component is not closed set.

$\Leftrightarrow$ Graph$(\mathrm{BR}_i)$ doesn't contain all its limit points.

$\Leftrightarrow \exists$ a point $(\bar{\mathbf{z}}_i, \bar{\mathbf{x}}_{-i})$ that is a limit point of a sequence $\{(\mathbf{z}_i^{(n)}, \mathbf{x}_{-i}^{(n)})\} \subset G(\mathrm{BR}_i), \forall\, n \in \mathbb{N}$

s.t. $\begin{cases} \lim\limits_{n\to\infty} \mathbf{z}_i^{(n)} = \bar{\mathbf{z}}_i \\ \lim\limits_{n\to\infty} \mathbf{x}_{-i}^{(n)} = \bar{\mathbf{x}}_{-i} \end{cases}$

**but** $(\bar{\mathbf{z}}_i, \bar{\mathbf{x}}_{-i})$ is not in $G(\mathrm{BR}_i)$ itself **i.e.** $(\bar{\mathbf{z}}_i, \bar{\mathbf{x}}_{-i}) \notin G(\mathrm{BR}_i)$

**Given** any sequence

$$\{(\mathbf{z}_i^{(n)}, \mathbf{x}_{-i}^{(n)})\}_{n\in\mathbb{N}} \in \mathrm{Graph}(\mathrm{BR}_i),$$

i.e. each $\mathbf{z}_i^{(n)}$ is a best response to $\mathbf{x}_{-i}^{(n)}$, we have

$$\bar{J}_i(\mathbf{z}_i^{(n)}, \mathbf{x}_{-i}^{(n)}) \le \bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i}^{(n)}) \quad \forall\, \mathbf{w}_i \in \Delta_i,\ \forall\, n \in \mathbb{N}. \qquad (*)$$

**Assume** this sequence converges to a limit point $(\bar{\mathbf{z}}_i, \bar{\mathbf{x}}_{-i})$, i.e.

$$\begin{cases} \lim\limits_{n\to\infty} \mathbf{z}_i^{(n)} = \bar{\mathbf{z}}_i \\ \lim\limits_{n\to\infty} \mathbf{x}_{-i}^{(n)} = \bar{\mathbf{x}}_{-i} \end{cases}$$

and the limit point is not in the graph i.e. it's not a best response to every possible $\bar{\mathbf{x}}_{-i}$

$$(\bar{\mathbf{z}}_i, \bar{\mathbf{x}}_{-i}) \notin \mathrm{Graph}(\mathrm{BR}_i) \quad \Leftrightarrow \quad \exists\, \mathbf{w}_i \in \Delta_i \text{ s.t. } \bar{J}_i(\mathbf{w}_i, \bar{\mathbf{x}}_{-i}) < \bar{J}_i(\bar{\mathbf{z}}_i, \bar{\mathbf{x}}_{-i}).$$

Express the limit point using the limits of the sequence:

$$\Rightarrow \bar{J}_i(\mathbf{w}_i, \lim_{n\to\infty} \mathbf{x}_{-i}^{(n)}) < \bar{J}_i(\lim_{n\to\infty} \mathbf{z}_i^{(n)}, \lim_{n\to\infty} \mathbf{x}_{-i}^{(n)}).$$

Since $\bar{J}_i$ is continuous, we can move the limit outside the function:

$$\Rightarrow \lim_{n\to\infty} \bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i}^{(n)}) < \lim_{n\to\infty} \bar{J}_i(\mathbf{z}_i^{(n)}, \mathbf{x}_{-i}^{(n)}).$$

**Lemma.** If two real sequences satisfy

$$\lim_{n\to\infty} a^{(n)} < \lim_{n\to\infty} b^{(n)},$$

then there exists an index $k$ such that

$$a^{(n)} < b^{(n)} \quad \forall\, n \ge k.$$

Applying this to $a^{(n)} = \bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i}^{(n)})$ and $b^{(n)} = \bar{J}_i(\mathbf{z}_i^{(n)}, \mathbf{x}_{-i}^{(n)})$, we obtain

$$\Rightarrow \exists\, k \quad \text{s.t. } \bar{J}_i(\mathbf{w}_i, \mathbf{x}_{-i}^{(n)}) < \bar{J}_i(\mathbf{z}_i^{(n)}, \mathbf{x}_{-i}^{(n)}) \quad \forall\, n \ge k, \quad \textbf{which contradicts (*)}.$$

Therefore, the assumption that Graph$(\mathrm{BR}_i)$ is not closed is false. $\Rightarrow$ Graph$(\mathrm{BR}_i)$ is closed.

### 5.2 Repeated N-Player Finite Games and Finding an NE

### 5.2.1 Setup(Ch 9.1)

**Game Elements in This Lecture**

| | |
|---|---|
| **Players** | $N$ players playing repeatedly at $k = 0, 1, 2, \dots$ |
| **Strategy** | Use action $j$-th or $e_{ij}$ with $\mathbb{P}(e_i^k = e_{ij}) = x_{ij}^k$ where $\mathbf{x}_i^k \in \Delta_i$ at iteration $k$ |
| **Cost** | $\bar{J}_i(\mathbf{x}^k)$ evaluated at each iteration |

**Difference** Players update their strategies over time based on information/observations.

**Consider an $N$-player finite game $G(\mathcal{N}, \Omega_i, J_i)$ played repeatedly.**

**At each iteration** $k = 0, 1, 2, \dots$

- $\mathbf{x}_i^k \in \Delta_i$ mixed strategy of $P_i$ at iteration $k$, and use an action $e_i^k$ according to this mixed strategy.
  $\Rightarrow$ $\text{Prob}\big(e_i^k = e_{ij}\big) = x_{ij}^k$ the probability that $P_i$ selects action $e_{ij}$ is $x_{ij}^k$.
- $J(\mathbf{x}_i^k, \mathbf{x}_{-i}^k) = J(\mathbf{x}^k)$ where $\mathbf{x}_{-i}^k \in \Delta_{-i}$.

**The goal is to**

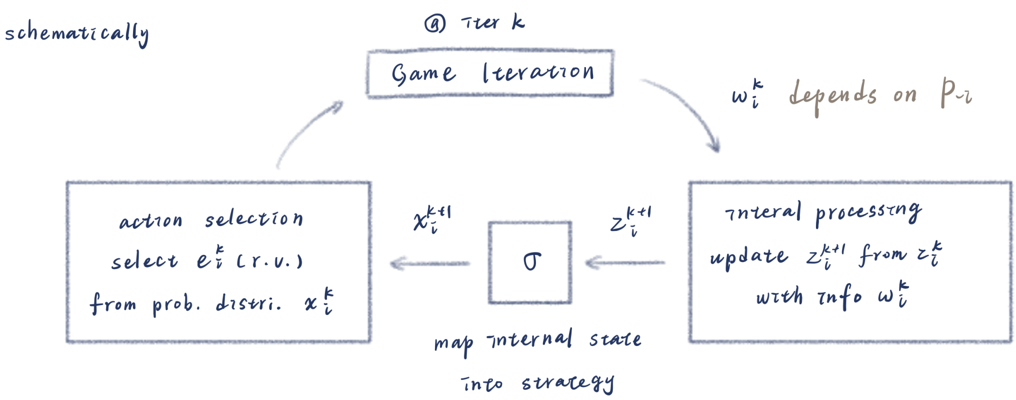
$$\min_{\mathbf{x}_i^k \in \Delta_i} \bar{J}_i \quad \Longleftrightarrow \quad \max_{\mathbf{x}_i^k \in \Delta_i} \bar{U}_i \quad \text{where} \quad \bar{J}_i = -\bar{U}_i$$

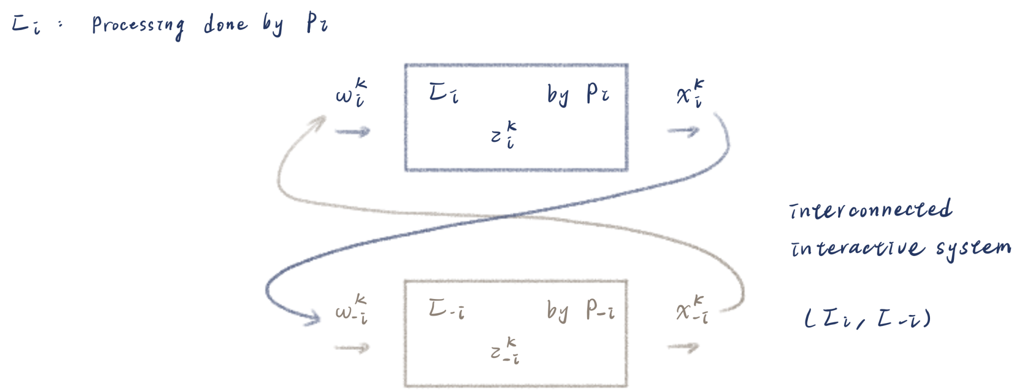
Player $i$ adjusts its strategy $\mathbf{x}_i^k$ based on:

- $w_i^k$: information it has (observations, feedback)
- $z_i^k$: processing it does (internal state variable)

**HOPE:** In the long run ($k \to \infty$), this adjustment makes $\mathbf{x}_i^k$ converge to a NE (unknown prior).

$$\begin{aligned} &\lim_{k\to\infty} \mathbf{x}_i^k = \mathbf{x}_i^*, \quad \forall\, i \in \mathcal{I} \\ \Leftrightarrow\ &\lim_{k\to\infty} \mathbf{x}^k = \mathbf{x}^*, \quad \forall\, \underbrace{\mathbf{x}^0 \in \Delta_X}_{\text{initial state}} \end{aligned}$$



### 5.2.2 Generic learning process as a DT stochastic process

**Learning Process(Algorithm).**

iteratively updates the internal state $z_i^k$ and the mixed strategy $\mathbf{x}_i^k$ so that the process converges to a stationary solution (e.g. a NE).

- **DT – stochastic process**

$$\begin{cases} z_i^{k+1} = z_i^k + f_i^k\left(w_i^k, z_i^k\right), \\ \mathbf{x}_i^k = \sigma_i(z_i^k), \end{cases}$$

  where $w_i^k$ is the information/observation at iteration $k$
- **CT – ODE**

$$\begin{cases} \dot{z}_i = f_i(z_i), \\ \mathbf{x}_i = \sigma_i(z_i). \end{cases}$$

Different learning algorithms correspond to different choices of:

- Information structure $w_i^k$
- Update rule $f_i$
- Strategy mapping $\sigma_i$