ECE1657 Problem Set 1 Solutions

Problem 1 Pure Security Strategies and Saddle-Points

Question

Consider the two-player zero-sum matrix game with the following cost matrix

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 5 & 2 & -1 & 2 \\ 2 & 0 & 3 & 1 \end{bmatrix}.$$

(i) Find the pure security strategies and the security levels (J_U, J_L) of the two players.

(ii) Does it admit a pure-strategy saddle-point equilibrium solution?

Solution

(i) Security strategies and security levels

Player 1 (row, minimizer)	security strategy: $j^* \in \arg\min_j \max_k a_{jk}$, security level: $J_U = \min_j \max_k a_{jk}$
Player 2 (column, maximizer)	security strategy: $k^* \in \arg\max_k \min_j a_{jk}$, security level: $J_L = \max_k \min_j a_{jk}$

Player 1.

$$\begin{aligned} J_U &= \min_j \max_k a_{jk} \\ &= \min\Big\{\underbrace{\max\{1,4,2,3\}}_{4=a_{12}}, \underbrace{\max\{5,2,-1,2\}}_{5=a_{21}}, \underbrace{\max\{2,0,3,1\}}_{3=a_{33}}\Big\} \\ &= \min\Big\{\underbrace{4}_{a_{12}}, \underbrace{5}_{a_{21}}, \underbrace{3}_{a_{33}}\Big\} = 3 = a_{33}. \end{aligned} \quad \Rightarrow \quad j^* = 3,\ J_U = 3.$$

Player 2.

$$\begin{aligned} J_L &= \max_k \min_j a_{jk} \\ &= \max\Big\{\underbrace{\min\{1,5,2\}}_{1=a_{11}}, \underbrace{\min\{4,2,0\}}_{0=a_{32}}, \underbrace{\min\{2,-1,3\}}_{-1=a_{23}}, \underbrace{\min\{3,2,1\}}_{1=a_{34}}\Big\} \\ &= \max\Big\{\underbrace{1}_{a_{11}}, \underbrace{0}_{a_{32}}, \underbrace{-1}_{a_{23}}, \underbrace{1}_{a_{34}}\Big\} = 1. \end{aligned} \quad \Rightarrow \quad k^* \in \{1,4\},\ J_L = 1.$$

(ii) Pure-strategy saddle point

By Proposition 2.3, a pure-strategy saddle point (j^*, k^*) must satisfy

$$a_{j^*k^*} = J_U = J_L$$

Since $J_U = 3 \neq J_L = 1$, the game admits no pure-strategy saddle-point equilibrium.

Problem 2 Pure Security Strategies and Saddle-Points

Question

Consider the two-player zero-sum matrix game with the following cost matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ -1 & 3 & 1 & 4 \end{bmatrix}.$$

(i) Find the pure security strategies and the security levels (J_U, J_L) of the two players.
(ii) Does it admit a pure-strategy saddle-point equilibrium solution?

Solution

(i) Security strategies and security levels

Player 1.

$$\begin{aligned} J_U &= \min_j \max_k a_{jk} \\ &= \min\Big\{\underbrace{\max\{2, 1, 0, -1\}}_{2=a_{11}}, \underbrace{\max\{-1, 3, 1, 4\}}_{4=a_{24}}\Big\} \\ &= \min\Big\{\underbrace{2}_{a_{11}}, \underbrace{4}_{a_{24}}\Big\} = 2 = a_{11}. \end{aligned} \qquad \Rightarrow \qquad j^* = 1,\ J_U = 2.$$

Player 2.

$$\begin{aligned} J_L &= \max_k \min_j a_{jk} \\ &= \max\Big\{\underbrace{\min\{2, -1\}}_{-1=a_{21}}, \underbrace{\min\{1, 3\}}_{1=a_{12}}, \underbrace{\min\{0, 1\}}_{0=a_{13}}, \underbrace{\min\{-1, 4\}}_{-1=a_{14}}\Big\} \\ &= \max\Big\{\underbrace{-1}_{a_{21}}, \underbrace{1}_{a_{12}}, \underbrace{0}_{a_{13}}, \underbrace{-1}_{a_{14}}\Big\} = 1 = a_{12}. \end{aligned} \qquad \Rightarrow \qquad k^* = 2,\ J_L = 1.$$

(ii) Pure-strategy saddle point (no-regret)

A pure action pair (j^*, k^*) is a saddle point (no-regret for both players) iff

$$\underbrace{a_{j^*k} \le a_{j^*k^*}}_{\text{no regret for } P_2} \quad \text{and} \quad \underbrace{a_{j^*k^*} \le a_{jk^*}}_{\text{no regret for } P_1}, \qquad \forall k, \forall j.$$

That is, a_{jk} is simultaneously the minimum in its column and the maximum in its row

- P1 cannot lower the cost by changing row
- P2 cannot increase the cost by changing column

Here the unique pair of security strategies is $(j^*, k^*) = (1, 2)$, with $a_{12} = 1$

- Column 2 is $(1, 3)^\top$, so
$$\min_j a_{j2} = 1 = a_{12},$$
hence player 1 has no regret at $(1, 2)$.
- Row 1 is $(2, 1, 0, -1)$, so
$$\max_k a_{1k} = 2 > 1 = a_{12},$$
thus player 2 regrets at $(1, 2)$ since it can switch to column 1 and increase the payoff from 1 to 2.

Therefore $(1, 2)$ is not a no-regret pair, so it is not a saddle point.

Problem 3 Strategic Equivalence under Constant Shifts

Question

Consider a two-player zero-sum matrix game with $m \times n$ cost matrix $A = [a_{jk}]$, $j \in \{1, \ldots, m\}$, $k \in \{1, \ldots, n\}$, and assume the pure (saddle-point) value of the game is J^*.

Consider the game with cost matrix $B = [b_{jk}]$, where $b_{jk} = a_{jk} + q$, for all $j \in \{1, \ldots, m\}$, $k \in \{1, \ldots, n\}$. Prove that this game has value $J^* + q$ and that the optimal strategies of the players are unchanged.

Solution

Let (j^*, k^*) be a saddle point of $A = [a_{jk}]$ with value J^*.

By Definition 2.2 (saddle-point equilibrium), this means

$$a_{j^*k} \leq a_{j^*k^*} \leq a_{jk^*}, \qquad \forall j, k.$$

Since $b_{jk} = a_{jk} + q$, adding q to each term yields

$$a_{j^*k} + q \leq a_{j^*k^*} + q \leq a_{jk^*} + q, \qquad \forall j, k.$$

Substituting $b_{jk} = a_{jk} + q$ in each term gives

$$b_{j^*k} \leq b_{j^*k^*} \leq b_{jk^*}, \qquad \forall j, k.$$

Thus, by the same definition, (j^*, k^*) is a saddle point of B.

The value of the game with matrix B is

$$b_{j^*k^*} = a_{j^*k^*} + q = J^* + q.$$

Hence the optimal pure strategies (j^*, k^*) are unchanged, and the game value is shifted by exactly q.

Problem 4 Ordered Interchangeable Saddle-point Strategies

Question

Consider a two-player zero-sum matrix game with $m \times n$ cost matrix $A = [a_{jk}]$ and mixed strategies $x \in \Delta_1$, $y \in \Delta_2$. Let (x_1, y_1) and (x_2, y_2) be two mixed-strategy saddle-point equilibria. Prove that (x_1, y_2) and (x_2, y_1) are also equilibria (saddle points), i.e., saddle points are interchangeable.

Proof

By Definition 2.8, for the two saddle points (x_1, y_1) and (x_2, y_2) we have

$$
\begin{aligned}
x_1^\top A y_1 &\le x^\top A y_1, && \forall x \in \Delta_1, && (1)\\
x_1^\top A y_1 &\ge x_1^\top A y, && \forall y \in \Delta_2, && (2)\\
x_2^\top A y_2 &\le x^\top A y_2, && \forall x \in \Delta_1, && (3)\\
x_2^\top A y_2 &\ge x_2^\top A y, && \forall y \in \Delta_2. && (4)
\end{aligned}
$$

Take special x, y in (1)–(4):

$$x_1^\top A y_1 \overset{(1),\, x=x_2}{\le} x_2^\top A y_1 \overset{(4),\, y=y_1}{\le} x_2^\top A y_2 \overset{(3),\, x=x_1}{\le} x_1^\top A y_2 \overset{(2),\, y=y_2}{\le} x_1^\top A y_1. \tag{5}$$

Hence all four terms in (5) are equal:

$$x_1^\top A y_1 = x_2^\top A y_1 = x_2^\top A y_2 = x_1^\top A y_2 =: J^*. \tag{6}$$

From (3) and (6),

$$x_1^\top A y_2 = x_2^\top A y_2 = J^* \le x^\top A y_2, \quad \forall x \in \Delta_1, \tag{7}$$

and from (2) and (6),

$$x_1^\top A y \le x_1^\top A y_1 = x_1^\top A y_2, \quad \forall y \in \Delta_2. \tag{8}$$

Thus (x_1, y_2) is a mixed-strategy saddle point with value J^*.

Similarly, using (1), (4) and (6),

$$x_2^\top A y_1 = J^* \le x^\top A y_1,\ \forall x \in \Delta_1, \qquad x_2^\top A y \le x_2^\top A y_1,\ \forall y \in \Delta_2, \tag{9}$$

so (x_2, y_1) is also a mixed-strategy saddle point with value J^*.

Therefore, (x_1, y_2) and (x_2, y_1) are mixed-strategy equilibria.

Problem 5 Minimax Theorem

Question

Consider the statement at the end of Step 1 in the proof of Theorem 2.10:

"Hence $x_0^\top Ay \le c, \ \forall y \in \Delta_2$ from which (b) follows."

Fill in and justify the missing steps of the logical argument to prove that indeed (b) follows from the inequality. The target statement (b) is

$$\min_{x\in\Delta_1}\max_{y\in\Delta_2} x^\top Ay \le c.$$

Solution

We are given

$$x_0^\top Ay \le c, \qquad \forall y \in \Delta_2.$$

Step 1: Take the maximum over y. Since the inequality holds for every $y \in \Delta_2$, it also holds for the maximizer in Δ_2:

$$\max_{y\in\Delta_2} x_0^\top Ay \le c.$$

Step 2: Take the minimum over x. Because $x_0 \in \Delta_1$ is one particular mixed strategy,

$$\min_{x\in\Delta_1}\max_{y\in\Delta_2} x^\top Ay \le \max_{y\in\Delta_2} x_0^\top Ay.$$

Step 3: Chain the inequalities. Combining the previous two inequalities,

$$\min_{x\in\Delta_1}\max_{y\in\Delta_2} x^\top Ay \le \max_{y\in\Delta_2} x_0^\top Ay \le c.$$

Thus,

$$\min_{x\in\Delta_1}\max_{y\in\Delta_2} x^\top Ay \le c,$$

which is exactly statement (b).

Takeaways

Problem 1 & 2

Proposition 2.3 (Saddle point and security levels). Consider an $m \times n$ matrix game $A = [a_{jk}]$ with security levels

$$J_L = \max_k \min_j a_{jk}, \qquad J_U = \min_j \max_k a_{jk},$$

and suppose $J_L = J_U$.

(i) The game has a saddle point in pure strategies.

(ii) A pair of pure strategies (j^*, k^*) is a saddle point iff j^* is a security strategy for player 1 and k^* is a security strategy for player 2.

(iii) The saddle-point value J^* is unique and equals

$$J^* = J_U = J_L.$$

Problem 3

Definition 2.2 (Saddle-point equilibrium). For a given $m \times n$ matrix game $A = [a_{jk}]$, let row j^* and column k^* (i.e. (j^*, k^*)) be a pair of (pure) strategies chosen by the two players. If

$$a_{j^*k} \;\le\; a_{j^*k^*} \;\le\; a_{jk^*}, \qquad \forall j = 1, \dots, m, \quad \forall k = 1, \dots, n,$$

then the pair (j^*, k^*) is a saddle-point equilibrium and the matrix game is said to have a saddle-point in pure strategies. The corresponding outcome

$$J^* = a_{j^*k^*}$$

is the saddle-point value of the game.

Problem 4

Definition 2.8 (Saddle-point equilibrium in mixed strategies). For a given two-player zero-sum matrix (2PZSM) game with $m \times n$ cost matrix $A = [a_{jk}]$, let $(x^*, y^*) \in \Delta_1 \times \Delta_2$ be a pair of mixed strategies chosen by the two players. Then (x^*, y^*) is a saddle-point game equilibrium in mixed strategies if

$$x^{*\top} A y^* \le x^\top A y^*, \qquad \forall x \in \Delta_1,$$

$$x^{*\top} A y \le x^{*\top} A y^*, \qquad \forall y \in \Delta_2.$$

The quantity

$$\bar{J}^* := x^{*\top} A y^*$$

is called the saddle-point value, or the value of the game in mixed strategies.

Definition 2.6 (Mixed security strategies). For a given two-player zero-sum matrix (2PZSM) game $\mathcal{G}(\mathcal{N}, \Delta_X, \bar{J})$ with $m \times n$ cost matrix A, let $x^* \in \Delta_1$ be a mixed strategy chosen by player 1. Then x^* is called a mixed security strategy for player 1 if the following holds for all $x \in \Delta_1$:

$$\bar{J}_U := \max_{y \in \Delta_2} x^{*\top} A y \;\le\; \max_{y \in \Delta_2} x^\top A y, \qquad \forall x \in \Delta_1,$$

hence

$$x^* = \arg\min_{x \in \Delta_1} \max_{y \in \Delta_2} x^\top A y, \qquad \bar{J}_U := \min_{x \in \Delta_1} \max_{y \in \Delta_2} x^\top A y.$$

Similarly, a mixed strategy $y^* \in \Delta_2$ chosen by player 2 is called a mixed security strategy for player 2 if the following holds for all $y \in \Delta_2$:

$$\bar{J}_L := \min_{x \in \Delta_1} x^\top A y^* \;\ge\; \min_{x \in \Delta_1} x^\top A y, \qquad \forall y \in \Delta_2,$$

hence

$$y^* = \arg\max_{y \in \Delta_2} \min_{x \in \Delta_1} x^\top A y, \qquad \bar{J}_L := \max_{y \in \Delta_2} \min_{x \in \Delta_1} x^\top A y.$$

The quantities $\bar{J}_U$ and $\bar{J}_L$ are the expected (average) security level of player 1 and player 2, respectively, or the expected (average) upper and lower value of the game.

Problem 5

Lemma 2.9. Let Q be any $m \times n$ matrix. Then at least one of the following two alternatives holds:

(i) There exists $y_0 \in \Delta_2$ such that
$$x^\top Q y_0 \le 0, \qquad \forall x \in \Delta_1;$$

(ii) There exists $x_0 \in \Delta_1$ such that
$$x_0^\top Q y \ge 0, \qquad \forall y \in \Delta_2.$$

Theorem 2.10 (Minimax Theorem). In any two-player zero-sum matrix (2PZSM) game with cost matrix $A \in \mathbb{R}^{m\times n}$, where the payoff is $\bar{J}(x, y) = x^\top A y$, we have
$$\min_{x\in\Delta_1} \max_{y\in\Delta_2} x^\top A y \;=\; \max_{y\in\Delta_2} \min_{x\in\Delta_1} x^\top A y.$$

Proof. Let Δ_1, Δ_2 be the simplices
$$\Delta_1 = \{x \in \mathbb{R}^m \mid \mathbf{1}_m^\top x = 1,\ x_j \ge 0\}, \qquad \Delta_2 = \{y \in \mathbb{R}^n \mid \mathbf{1}_n^\top y = 1,\ y_k \ge 0\},$$
so that $x^\top \mathbf{1}_{m\times n} y = 1$ for all $x \in \Delta_1$, $y \in \Delta_2$, where $\mathbf{1}_{m\times n}$ is the all-ones $m \times n$ matrix.

Step 1. Fix any constant $c \in \mathbb{R}$ and set
$$Q := -A + c\,\mathbf{1}_{m\times n}.$$
Apply Lemma 2.9 to Q.

Case (i) of Lemma 2.9. Then there exists $y_0 \in \Delta_2$ such that
$$0 \ge x^\top Q y_0 = x^\top(-A + c\mathbf{1}_{m\times n})y_0 = -x^\top A y_0 + c, \qquad \forall x \in \Delta_1,$$
so $x^\top A y_0 \ge c$ for all $x \in \Delta_1$. Hence
$$\max_{y\in\Delta_2} \min_{x\in\Delta_1} x^\top A y \;\ge\; \min_{x\in\Delta_1} x^\top A y_0 \;\ge\; c. \tag{a}$$

Case (ii) of Lemma 2.9. Then there exists $x_0 \in \Delta_1$ such that
$$0 \le x_0^\top Q y = x_0^\top(-A + c\mathbf{1}_{m\times n})y = -x_0^\top A y + c, \qquad \forall y \in \Delta_2,$$
so $x_0^\top A y \le c$ for all $y \in \Delta_2$. Hence
$$\min_{x\in\Delta_1} \max_{y\in\Delta_2} x^\top A y \;\le\; \max_{y\in\Delta_2} x_0^\top A y \;\le\; c. \tag{b}$$

Therefore, for every $c \in \mathbb{R}$ at least one of (a) or (b) holds.

Step 2. (Contradiction argument.) Let
$$\alpha := \max_{y\in\Delta_2} \min_{x\in\Delta_1} x^\top A y, \qquad \beta := \min_{x\in\Delta_1} \max_{y\in\Delta_2} x^\top A y.$$

Assume, for contradiction, that there is a gap $\beta > \alpha$. Then there exists $k > 0$ with
$$\beta = \alpha + k.$$

Choose
$$c := \alpha + \frac{k}{2},$$
so that
$$\alpha < c < \beta.$$

For this choice of c we have
$$\alpha = \max_y \min_x x^\top A y < c, \qquad \beta = \min_x \max_y x^\top A y > c.$$

Thus inequality (a) fails (its left-hand side is $< c$) and inequality (b) also fails (its left-hand side is $> c$). This contradicts Step 1, which says that for every c at least one of (a) or (b) must hold.

Hence our assumption $\beta > \alpha$ is false, so $\beta = \alpha$ and

$$\min_{x \in \Delta_1} \max_{y \in \Delta_2} x^\top A y = \max_{y \in \Delta_2} \min_{x \in \Delta_1} x^\top A y.$$