ECE1657 Problem Set 4 Solutions

Debreu–Fan–Glicksberg (DFG) NE Existence Theorem.

Any N player continuous-kernel game admits at least one (pure) Nash equilibrium if the following conditions hold.

A1 (Action sets). For every player i, $\Omega_i \subset \mathbb{R}^{n_i}$ is nonempty, convex, and compact.

A2 (Cost functions). For every player i, cost function J_i is

- continuous in $\mathbf{u} = (\mathbf{u}_i, \mathbf{u}_{-1})$
- convex in own action $\mathbf{u}_i$ for every fixed $\mathbf{u}_{-i}$.

Problem 1

Consider the two-player zero-sum continuous-kernel game where each of the two players can choose pure actions $\mathbf{u}_1 \in \Omega_1,\ \mathbf{u}_2 \in \Omega_2$, where $\Omega_1 = \Omega_2 = [0, 1]$. The cost function for player 1 is

$$J(\mathbf{u}_1, \mathbf{u}_2) = -\frac{1}{2}\mathbf{u}_2^2 + 2\mathbf{u}_1^2 + 2\mathbf{u}_1\mathbf{u}_2 - \frac{7}{2}\mathbf{u}_1 - \frac{5}{4}\mathbf{u}_2.$$

Does this game have a pure Nash equilibrium? If yes, find it.

Solution

Check NE existence by DFG Theorem

Let $J_1 = J,\ J_2 = -J$, where player 1 minimizes J_1 and player 2 minimizes J_2.

A1 holds. Action sets $\Omega_1 = \Omega_2 = [0, 1]$ are convex, compact, and non-empty.

A2 holds. Both cost functions (polynomials) are continuous in $u = (u_1, u_2)$.

For convexity in each player's own action,

$$\begin{aligned}
\frac{\partial J_1}{\partial u_1} &= \frac{\partial J}{\partial u_1} = 4u_1 + 2u_2 - \frac{7}{2} & \frac{\partial^2 J_1}{\partial u_1^2} &= \frac{\partial^2 J}{\partial u_1^2} = 4 > 0, & &J_1 \text{ strictly convex in } \mathbf{u}_1 \\
\frac{\partial J_2}{\partial u_2} &= \frac{\partial(-J)}{\partial u_2} = u_2 - 2u_1 + \frac{5}{4} & \frac{\partial^2 J_2}{\partial u_2^2} &= \frac{\partial^2 (-J)}{\partial u_2^2} = 1 > 0, & &J_2 \text{ strictly convex in } \mathbf{u}_2
\end{aligned}$$

Thus, by DFG Theorem, there exists at least one pure NE.

Compute NE by pseudo-gradient characterization

At NE, the pseudo-gradient vanishes:

$$\nabla_p J(u^*) = \begin{bmatrix} \nabla_{u_1} J_1(u^*) \\ \nabla_{u_2} J_2(u^*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving for the first-order conditions,

$$\begin{cases} \dfrac{\partial J}{\partial u_1} = 0 \implies 4u_1 + 2u_2 = \dfrac{7}{2} \\ \dfrac{\partial J}{\partial u_2} = 0 \implies 2u_1 - u_2 = \dfrac{5}{4} \end{cases}$$

yields NE

$$\mathbf{u}^* = \left(\frac{3}{4}, \frac{1}{4}\right)$$

Problem 2

Consider the two-player zero-sum continuous-kernel game where the cost for player 1 is

$$J(\mathbf{u}_1, \mathbf{u}_2) = \mathbf{u}_1^3 - 3\mathbf{u}_1\mathbf{u}_2 + \mathbf{u}_2^3,$$

where each of the two players can choose pure actions $\mathbf{u}_1 \in \Omega_1,\ \mathbf{u}_2 \in \Omega_2$, where $\Omega_1 = \Omega_2 = [0, 1]$. Find the pure Nash equilibrium strategies, if they exist.

Hint. Note that J is strictly convex in $\mathbf{u}_1$ and that, for any given $\mathbf{u}_1$, the maximum value of J with respect to $\mathbf{u}_2$ is attained at either $\mathbf{u}_2 = 0$ or $\mathbf{u}_2 = 1$.

Solution

Check NE existence by DFG Theorem and Best Response Maps

Let $J_1 = J,\ J_2 = -J$, where player 1 minimizes J_1 and player 2 minimizes J_2.

Cost convexity in each player's own action:

$$\frac{\partial J_1}{\partial \mathbf{u}_1} = \frac{\partial J}{\partial \mathbf{u}_1} = 3\mathbf{u}_1^2 - 3\mathbf{u}_2 \qquad \frac{\partial^2 J_1}{\partial \mathbf{u}_1^2} = 6\mathbf{u}_1 > 0 \text{ for } \mathbf{u}_1 > 0 \qquad J_1 \text{ strictly convex in } \mathbf{u}_1$$

$$\frac{\partial J_2}{\partial \mathbf{u}_2} = \frac{\partial(-J)}{\partial \mathbf{u}_2} = -3\mathbf{u}_2^2 + 3\mathbf{u}_1 \qquad \frac{\partial^2 J_2}{\partial \mathbf{u}_2^2} = -6\mathbf{u}_2 < 0 \text{ for } \mathbf{u}_2 > 0 \qquad J_2 \text{ strictly concave in } \mathbf{u}_2$$

$\Rightarrow$ **DFG doesn't apply.** Check if best response maps intersect.

Player 1's best response. For fixed $\mathbf{u}_2$,

$$\mathrm{BR}_1(\mathbf{u}_2) = \arg\min_{\mathbf{u}_1} J_1(\mathbf{u}_1, \mathbf{u}_2) = \arg\min_{\mathbf{u}_1} \left[\mathbf{u}_1^3 - 3\mathbf{u}_1\mathbf{u}_2 + \mathbf{u}_2^3\right].$$

FOC:

$$\frac{\partial J_1}{\partial \mathbf{u}_1} = 3\mathbf{u}_1^2 - 3\mathbf{u}_2 = 0 \quad \Rightarrow \mathbf{u}_1^2 = \mathbf{u}_2 \quad \Rightarrow \mathrm{BR}_1(\mathbf{u}_2) = \sqrt{\mathbf{u}_2}.$$

Player 2's best response.

$$\mathrm{BR}_2(\mathbf{u}_1) = \arg\min_{\mathbf{u}_2} J_2(\mathbf{u}_1, \mathbf{u}_2) = \arg\min_{\mathbf{u}_2} \left[-\mathbf{u}_1^3 + 3\mathbf{u}_1\mathbf{u}_2 - \mathbf{u}_2^3\right].$$

Since J_2 is strictly concave in $\mathbf{u}_2$, so the minimum on $[0, 1]$ is at a boundary:

$$J_2(\mathbf{u}_1, 0) = -\mathbf{u}_1^3, \qquad J_2(\mathbf{u}_1, 1) = -\mathbf{u}_1^3 + 3\mathbf{u}_1 - 1,$$

$$J_2(\mathbf{u}_1, 1) - J_2(\mathbf{u}_1, 0) = 3\mathbf{u}_1 - 1. \Rightarrow \mathrm{BR}_2(\mathbf{u}_1) = \begin{cases} 1, & \mathbf{u}_1 < \frac{1}{3}, \\ \{0, 1\}, & \mathbf{u}_1 = \frac{1}{3}, \\ 0, & \mathbf{u}_1 > \frac{1}{3}. \end{cases}$$

If there's an intersection of the best-response maps, then there exists $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ such that

$$u_1^* = \sqrt{u_2^*}, \qquad u_2^* \in \begin{cases} \{1\}, & u_1^* < \frac{1}{3}, \\ \{0, 1\}, & u_1^* = \frac{1}{3}, \\ \{0\}, & u_1^* > \frac{1}{3}. \end{cases}$$

(i) $u_1^* < \frac{1}{3} \Rightarrow u_2^* = 1 \Rightarrow u_1^* = \sqrt{1} = 1$ (contradiction);
(ii) $u_1^* = \frac{1}{3} \Rightarrow u_2^* = (u_1^*)^2 = \frac{1}{9} \notin \{0, 1\}$;
(iii) $u_1^* > \frac{1}{3} \Rightarrow u_2^* = 0 \Rightarrow u_1^* = \sqrt{0} = 0$ (contradiction).

There is no pair (u_1^*, u_2^*) satisfying both best-response conditions

The game admits no pure Nash equilibrium.

Problem 3

Consider the zero-sum continuous-kernel game with

$$J(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1 - \mathbf{u}_2)^2 - \alpha \mathbf{u}_2^2,$$

where α is a scalar and each of the two players can choose pure actions $\mathbf{u}_1 \in \Omega_1$, $\mathbf{u}_2 \in \Omega_2$, where $\Omega_1 = \Omega_2 = [0, 1]$.

Player 1 is minimizing J while player 2 is maximizing J.

Find the pure Nash equilibrium solutions (if they exist) in each of the following cases:

$$\text{(i) } \alpha \in (1, 2] \quad \text{(ii) } \alpha = 0 \quad \text{(iii) } \alpha \in (0, 1]$$

Solution

Player 1 (minimizer). For fixed $\mathbf{u}_2$,

$$\frac{\partial J}{\partial \mathbf{u}_1} = 2(\mathbf{u}_1 - \mathbf{u}_2), \qquad \frac{\partial^2 J}{\partial \mathbf{u}_1^2} = 2 > 0.$$

Thus, J is strictly convex in $\mathbf{u}_1$, and P_1's BR_1 (min) is at:

$$\frac{\partial J}{\partial \mathbf{u}_1} = 0 \;\Rightarrow\; \mathbf{u}_1 = \mathbf{u}_2. \;\Rightarrow\; \boxed{\text{BR}_1(\mathbf{u}_2) = \mathbf{u}_2, \quad \forall \mathbf{u}_2 \in [0, 1].}$$

Player 2 (maximizer). For fixed $\mathbf{u}_1$,

$$\frac{\partial J}{\partial \mathbf{u}_2} = -2\mathbf{u}_1 + 2(1 - \alpha)\mathbf{u}_2, \qquad \frac{\partial^2 J}{\partial \mathbf{u}_2^2} = 2(1 - \alpha).$$

The curvature in $\mathbf{u}_2$ and P_2's maximization problem depend on α:

- $\alpha > 1$: $\frac{\partial^2 J}{\partial \mathbf{u}_2^2} < 0 \Rightarrow$ J is strictly concave in $\mathbf{u}_2 \Rightarrow P_2$'s BR_2 (max) solves $\frac{\partial J}{\partial \mathbf{u}_2} = 0$
- $\alpha = 1$: $\frac{\partial^2 J}{\partial \mathbf{u}_2^2} = 0 \Rightarrow$ J is linear in $\mathbf{u}_2$. $\Rightarrow P_2$'s BR_2 (max) over $[0, 1]$ occurs at boundary ($\mathbf{u}_2 = 0$ or 1)
- $\alpha < 1$: $\frac{\partial^2 J}{\partial \mathbf{u}_2^2} > 0 \Rightarrow$ J is strictly convex in $\mathbf{u}_2$. $\Rightarrow P_2$'s BR_2 (max) occurs at boundary ($\mathbf{u}_2 = 0$ or 1)

Derive BR_2 for each parameter range and intersect with BR_1.

(i) $\alpha \in (1, 2]$ $\quad \Rightarrow \alpha > 1 \quad \Rightarrow P_2$'s BR (max) solves $\frac{\partial J}{\partial \mathbf{u}_2} = 0$

$$J(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1 - \mathbf{u}_2)^2 - \alpha \mathbf{u}_2^2 \qquad \Rightarrow \frac{\partial J}{\partial \mathbf{u}_2} = 0 \;\Rightarrow\; \mathbf{u}_2 = \frac{\mathbf{u}_1}{1 - \alpha} \le 0,$$

which lies outside $[0, 1]$ for all $\mathbf{u}_1 > 0$ (and equals 0 when $\mathbf{u}_1 = 0$). Thus the maximum over $[0, 1]$ is attained at the boundary; comparing

$$\underbrace{J(\mathbf{u}_1, 1)}_{=(\mathbf{u}_1 - 1)^2 - \alpha} - \underbrace{J(\mathbf{u}_1, 0)}_{=\mathbf{u}_1^2} = 1 - \alpha - 2\mathbf{u}_1$$

$$\boxed{\text{BR}_2(\mathbf{u}_1) = 0, \quad \forall \mathbf{u}_1 \in [0, 1], \; \alpha > 1.}$$

A pure Nash equilibrium might occur at the intersection of BR_1 and BR_2.

$$\begin{cases} \mathbf{u}_1^* = \text{BR}_1(\mathbf{u}_2^*) = \mathbf{u}_2^*, \\ \mathbf{u}_2^* = \text{BR}_2(\mathbf{u}_1^*) = 0, \end{cases} \qquad \Rightarrow \qquad (\mathbf{u}_1^*, \mathbf{u}_2^*) = (0, 0).$$

Thus for all $\alpha \in (1, 2]$,

$$\boxed{\text{unique pure Nash equilibrium: } (\mathbf{u}_1^*, \mathbf{u}_2^*) = (0, 0).}$$

(ii) $\alpha = 0 \quad \Rightarrow \alpha < 1 \quad \Rightarrow P_2$**'s** BR_2 **(max) occurs at boundary (**$\mathbf{u}_2 = 0$ **or** 1**)**

$$J(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1 - \mathbf{u}_2)^2. \qquad \Rightarrow \underbrace{J(\mathbf{u}_1, 1)}_{=(1-\mathbf{u}_1)^2} - \underbrace{J(\mathbf{u}_1, 0)}_{=\mathbf{u}_1^2} = -2\mathbf{u}_1 + 1$$

$$\mathrm{BR}_2(\mathbf{u}_1) = \begin{cases} \{1\}, & \mathbf{u}_1 < \frac{1}{2}, \\ \{0,1\}, & \mathbf{u}_1 = \frac{1}{2}, \\ \{0\}, & \mathbf{u}_1 > \frac{1}{2}. \end{cases}$$

There's no intersection between BR_1 and BR_2.

$$\boxed{\text{no pure Nash equilibrium when } \alpha = 0.}$$

(iii) $\alpha \in (0, 1] \quad \Rightarrow \alpha < 1$ **or** $\alpha = 1 \quad \Rightarrow P_2$**'s** BR_2 **(max) occurs at boundary (**$\mathbf{u}_2 = 0$ **or** 1**)**

Case $\alpha \in (0, 1)$

$$\underbrace{J(\mathbf{u}_1, 1)}_{=\mathbf{u}_1^2 - 2\mathbf{u}_1 + 1 - \alpha} - \underbrace{J(\mathbf{u}_1, 0)}_{=\mathbf{u}_1^2} = 1 - \alpha - 2\mathbf{u}_1.$$

$$\mathrm{BR}_2(\mathbf{u}_1) = \begin{cases} \{1\}, & \mathbf{u}_1 < \frac{1-\alpha}{2}, \\ \{0,1\}, & \mathbf{u}_1 = \frac{1-\alpha}{2}, \\ \{0\}, & \mathbf{u}_1 > \frac{1-\alpha}{2}. \end{cases}$$

There's no intersection between BR_1 and BR_2.

$$\boxed{\text{no pure Nash equilibrium for } \alpha \in (0, 1).}$$

Case $\alpha = 1$

$$J(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1 - \mathbf{u}_2)^2 - \mathbf{u}_2^2 = \mathbf{u}_1^2 - 2\mathbf{u}_1\mathbf{u}_2. \qquad \Rightarrow \frac{\partial J}{\partial \mathbf{u}_2} = -2\mathbf{u}_1$$

so J is linear in $\mathbf{u}_2$:

- If $\mathbf{u}_1 = 0$, then $J(0, \mathbf{u}_2) \equiv 0$ for all $\mathbf{u}_2 \in [0, 1]$, so every $\mathbf{u}_2$ is a best response, i.e. $\mathrm{BR}_2(0) = [0, 1]$.
- If $\mathbf{u}_1 > 0$, then $\dfrac{\partial J}{\partial \mathbf{u}_2} = -2\mathbf{u}_1 < 0$, so $J(\mathbf{u}_1, \mathbf{u}_2)$ is strictly decreasing in $\mathbf{u}_2$ on $[0, 1]$ and the maximizer is $\mathrm{BR}_2(\mathbf{u}_1) = \{0\}$.

$$\Rightarrow \mathrm{BR}_2(\mathbf{u}_1) = \begin{cases} [0,1], & \mathbf{u}_1 = 0, \\ \{0\}, & \mathbf{u}_1 > 0. \end{cases}$$

If $\mathbf{u}_1^* > 0$, then $\mathbf{u}_2^* = 0$ by BR_2, contradicting $\mathbf{u}_1^* = \mathbf{u}_2^* > 0$. Thus $\mathbf{u}_1^* = 0$. Then $\mathrm{BR}_2(0) = [0, 1]$ and $\mathrm{BR}_1(\mathbf{u}_2) = \mathbf{u}_2$ imply $\mathbf{u}_2^* = 0$.

$$\boxed{\text{for } \alpha = 1 \text{ the unique pure Nash equilibrium is } (\mathbf{u}_1^*, \mathbf{u}_2^*) = (0, 0).}$$

Summarizing case (iii):

$$\boxed{\begin{cases} \text{no pure NE}, & \alpha \in (0, 1), \\ (\mathbf{u}_1^*, \mathbf{u}_2^*) = (0, 0), & \alpha = 1. \end{cases}}$$

Problem 4

(i) Consider the cost function $J : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$J(\mathbf{u}_1, \mathbf{u}_2) = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} Q \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} + p^\top \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} + s,$$

where Q is a 2×2 matrix, p is a 2×1 vector, and s is a scalar. Assume J is strictly convex on $\mathbb{R}^2$.

- Find the team optimization solution $\mathbf{u}^{\mathrm{opt}}$ that jointly minimizes J with respect to $\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$.
- Consider a two-player game with $J_1 = J_2 = J$. Find the best-response functions $R_1(\mathbf{u}_2)$ and $R_2(\mathbf{u}_1)$. Prove that these can have at most one point in common; hence the Nash equilibrium solution $\mathbf{u}^* = \begin{bmatrix} \mathbf{u}_1^* \\ \mathbf{u}_2^* \end{bmatrix}$ is unique. Show that the NE solution $\mathbf{u}^*$ and the team optimization solution $\mathbf{u}^{\mathrm{opt}}$ are identical.

(ii) Consider the same setup as in (i), where

$$Q = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}.$$

Does J have the same properties as in (i)? Comment on the team optimization problem. Show that the NE solution in the two-player game is still unique.

Solution

Let $\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$ and write

$$J(\mathbf{u}_1, \mathbf{u}_2) = \mathbf{u}^\top Q \mathbf{u} + p^\top \mathbf{u} + s, \qquad Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

For a quadratic cost,

$$\nabla J(\mathbf{u}) = (Q + Q^\top)\mathbf{u} + p, \qquad \nabla^2 J(\mathbf{u}) = Q + Q^\top =: S.$$

(i) Team optimization and Nash equilibrium

Team optimization. The assumption that J is strictly convex on $\mathbb{R}^2$ is equivalent to $S \succ 0$ (positive definite Hessian). Hence there is a unique global minimizer $\mathbf{u}^{\mathrm{opt}}$ characterized by the first-order condition

$$\nabla J(\mathbf{u}^{\mathrm{opt}}) = 0 \iff S\mathbf{u}^{\mathrm{opt}} + p = 0,$$

so

$$\boxed{\mathbf{u}^{\mathrm{opt}} = -S^{-1}p = -(Q + Q^\top)^{-1}p.}$$

Best-response functions. Expanding the cost function,

$$J(\mathbf{u}_1, \mathbf{u}_2) = q_{11}\mathbf{u}_1^2 + (q_{12} + q_{21})\mathbf{u}_1\mathbf{u}_2 + q_{22}\mathbf{u}_2^2 + p_1\mathbf{u}_1 + p_2\mathbf{u}_2 + s,$$

the partial derivatives are

$$\frac{\partial J}{\partial \mathbf{u}_1} = 2q_{11}\mathbf{u}_1 + (q_{12} + q_{21})\mathbf{u}_2 + p_1, \qquad \frac{\partial J}{\partial \mathbf{u}_2} = (q_{12} + q_{21})\mathbf{u}_1 + 2q_{22}\mathbf{u}_2 + p_2.$$

Since $S \succ 0$, we have $2q_{11} > 0$ and $2q_{22} > 0$ (positive diagonal entries from positive definiteness), so for any fixed opponent action, the minimizer in each variable is unique. The best-response functions are

$$R_1(\mathbf{u}_2) := \arg\min_{v_1 \in \mathbb{R}} J(v_1, \mathbf{u}_2) = -\frac{(q_{12} + q_{21})\mathbf{u}_2 + p_1}{2q_{11}},$$

$$R_2(\mathbf{u}_1) := \arg\min_{v_2 \in \mathbb{R}} J(\mathbf{u}_1, v_2) = -\frac{(q_{12} + q_{21})\mathbf{u}_1 + p_2}{2q_{22}}.$$

Each R_i is an affine function, hence a line in the $(\mathbf{u}_1, \mathbf{u}_2)$-plane.

Uniqueness of NE and coincidence with team optimum. A Nash equilibrium $\mathbf{u}^* = (\mathbf{u}_1^*, \mathbf{u}_2^*)$ satisfies

$$\mathbf{u}_1^* = R_1(\mathbf{u}_2^*), \qquad \mathbf{u}_2^* = R_2(\mathbf{u}_1^*),$$

which is equivalent to the first-order conditions

$$\frac{\partial J}{\partial \mathbf{u}_1}(\mathbf{u}^*) = 0, \qquad \frac{\partial J}{\partial \mathbf{u}_2}(\mathbf{u}^*) = 0.$$

In vector form, using the pseudo-gradient notation,

$$\nabla_p J(\mathbf{u}^*) = \begin{bmatrix} \nabla_{\mathbf{u}_1} J_1(\mathbf{u}^*) \\ \nabla_{\mathbf{u}_2} J_2(\mathbf{u}^*) \end{bmatrix} = \begin{bmatrix} \frac{\partial J}{\partial \mathbf{u}_1}(\mathbf{u}^*) \\ \frac{\partial J}{\partial \mathbf{u}_2}(\mathbf{u}^*) \end{bmatrix} = 0 \iff S\mathbf{u}^* + p = 0.$$

Because $S \succ 0$, it is invertible, so there is exactly one solution

$$\mathbf{u}^* = -S^{-1} p.$$

Hence the two best-response lines can intersect at most one point; this unique intersection is the unique Nash equilibrium. Since it satisfies the same equation as $\mathbf{u}^{\mathrm{opt}}$, we conclude

$$\boxed{\mathbf{u}^* = \mathbf{u}^{\mathrm{opt}}.}$$

(ii) Case $Q = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

Here $Q = Q^\top$ (symmetric) and

$$Q = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}, \qquad S = Q + Q^\top = 2Q = \begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}.$$

Properties of J. To check convexity, compute the eigenvalues of Q:

$$\det(Q - \lambda I) = (2-\lambda)^2 - 9 = \lambda^2 - 4\lambda - 5 = 0 \implies \lambda_1 = 5 > 0, \quad \lambda_2 = -1 < 0.$$

Thus Q (and $S = 2Q$) is indefinite, so J is **not strictly convex** on $\mathbb{R}^2$ and does not have the property assumed in (i).

Along an eigenvector v_- associated with $\lambda_2 = -1$,

$$J(tv_-) = t^2 v_-^\top Q v_- + t\, p^\top v_- + s = -\|v_-\|^2 t^2 + O(t) \xrightarrow{t \to \pm\infty} -\infty,$$

so the team optimization problem $\min_{\mathbf{u} \in \mathbb{R}^2} J(\mathbf{u})$ is nonconvex and unbounded below. **There is no team optimum.**

NE remains unique. For fixed $\mathbf{u}_2$, $J(\cdot, \mathbf{u}_2)$ is a quadratic in $\mathbf{u}_1$ with

$$\frac{\partial^2 J}{\partial \mathbf{u}_1^2} = 2q_{11} = 4 > 0,$$

and similarly, for fixed $\mathbf{u}_1$,

$$\frac{\partial^2 J}{\partial \mathbf{u}_2^2} = 2q_{22} = 4 > 0.$$

Hence each player's problem is strictly convex in their own action, so the first-order conditions uniquely determine the best responses. Therefore, the NE is characterized by

$$S\mathbf{u}^* + p = 0, \qquad S = \begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}.$$

Since $\det S = 4 \cdot 4 - 6 \cdot 6 = -20 \neq 0$, S is invertible and there is a unique solution

$$\boxed{\mathbf{u}^* = -S^{-1} p.}$$

Thus the two-player game has a unique Nash equilibrium even though J is not strictly convex.