Lecture 2: Two-Player Zero-Sum Finite Matrix Games with Mixed Strategies

Last Lecture Recap

Two-player zero-sum (2PZS) finite game with **pure strategies.**
2PZS: $J_1 + J_2 = 0 \Rightarrow$ use single cost function $J = J_1 = -J_2$

Player 1's objective: $\min_{\mathbf{u}_1} J(\mathbf{u}_1, \mathbf{u}_2)$

Action Set: $|\Omega_1| = m_1,\ M_1 = \{1, \ldots, m_1\}$

Action: $\mathbf{u}_1 = \mathbf{e}_{1j} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ (j-th component)

Player 2's objective: $\max_{\mathbf{u}_2} J(\mathbf{u}_1, \mathbf{u}_2)$

Action Set: $|\Omega_2| = m_2,\ M_2 = \{1, \ldots, m_2\}$

Action: $\mathbf{u}_2 = \mathbf{e}_{2k} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ (k-th component)

Cost for finite actions: $(j, k) \in M_1 \times M_2 \Rightarrow J(\mathbf{e}_{1j}, \mathbf{e}_{2k}) = a_{jk} = \mathbf{e}_{1j}^T A \mathbf{e}_{2k} \in \mathbb{R}$

- P_1's security strategy:

$$j^* \in \arg\min_j \max_k a_{jk} \Rightarrow J_U \neq J_L \text{in general}$$

- Minimax solution: both use security strategies $\Rightarrow (j^*, k^*)$ ("no-regret" not necessarily hold)
- Saddle point:

$$\underbrace{a_{j^*k} \leq a_{j^*k^*}}_{\text{no regret for } P_2} \qquad \underbrace{a_{j^*k^*} \leq a_{jk^*}}_{\text{no regret for } P_1} \qquad \forall k \in M_2 \qquad \forall j \in M_1$$

Game Elements in This Lecture

Players	Two players $\mathcal{N} = \{1, 2\}$ (denoted P_1, P_2)
Strategy	Mixed strategies $\mathbf{x}_i \in \Delta_i$ (probability simplex)
Cost	Expected cost: $\bar{J}(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A \mathbf{y}$

Why This Change? Pure strategy saddle-point may not exist $\Rightarrow$ need randomization.

Key Result: Minimax Theorem guarantees mixed strategy saddle-point *always* exists.

2.1 Mixed-Strategy Extension

When no pure-strategy equilibrium exists, we introduce **randomization** over actions by replacing one-hot pure action vectors with **probability distributions** over the action sets.

> **Example: Matching Pennies (MP).**
>
> $$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$
>
> P_1: both rows are security strategies
>
> $$\begin{aligned} j^* &\in \arg\min_j \max_k a_{jk} \\ &= \arg\min_j \max_k \{1, 1\} = \{1, 2\} \end{aligned}$$
>
> P_2: both columns are security strategies
>
> $$\begin{aligned} k^* &\in \arg\max_k \min_j a_{jk} \\ &= \arg\max_k \min_j \{-1, -1\} = \{1, 2\} \end{aligned}$$
>
> Security strategy solutions (minimax pairs): $(1, 1), (1, 2), (2, 1), (2, 2)$
>
> **Question:** Which one is saddle point (no regret property)?

Check $(j^*, k^*) = (1, 2)$:

- P_1 regrets: Given P_2 plays $k = 2$, P_1 checks column 2:

$$a_{12} = 1 > a_{22} = -1$$

P_1 could reduce cost by switching to row 2.
- P_2 no regret: Given P_1 plays $j = 1$, P_2 checks row 1:

$$a_{11} = -1 < a_{12} = 1$$

P_2 already maximizes P_1's cost.

$\Rightarrow (j^*, k^*) = (1, 2)$ is not a saddle point solution.

Problem: Checking all minimax pairs, we found that **no pure-strategy saddle point exists**.

Solution: Introduce randomization in selection of actions. Extend the game from

$$\mathcal{G}(\mathcal{N}, \underbrace{\Omega}_{\text{deterministic actions}}, J) \rightarrow \mathcal{G}(\mathcal{N}, \underbrace{\Delta_X}_{\text{probability distributions}}, \bar{J})$$

Notation 1: Mixed strategy.

- x_j = probability of P_1 selects j-th action

$$\Rightarrow \quad P_1\text{ 's mixed strategy : } \mathbf{x} = [x_1, \dots, x_j, \dots, x_{m_1}]^T \in \mathbb{R}^{m_1}$$

- y_k = probability of P_2 selects k-th action

$$\Rightarrow \quad P_2\text{ 's mixed strategy : } \mathbf{y} = [y_1, \dots, y_k, \dots, y_{m_2}]^T \in \mathbb{R}^{m_2}$$

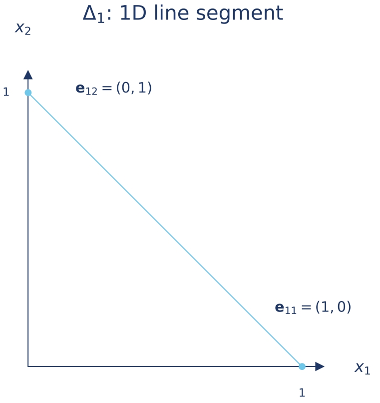
$$\text{Require} \quad \begin{cases} 0 \le x_j \le 1, & \sum_{j=1}^{m_1} x_j = 1 \\ 0 \le y_k \le 1, & \sum_{k=1}^{m_2} y_k = 1 \end{cases} \quad \text{for } \mathbf{x}, \mathbf{y} \text{ to be probability distributions.}$$

If P_1 selects j-th action with 100% probability, then $\mathbf{x} = \mathbf{e}_j = [0, \dots, \underbrace{1}_{j\text{-th}}, \dots, 0]^T$

2D example: 2 actions to choose from

$$\Delta_1 := \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid 0 \le x_1, x_2 \le 1, \mathbf{1}^T \mathbf{x} = 1 \right\}$$

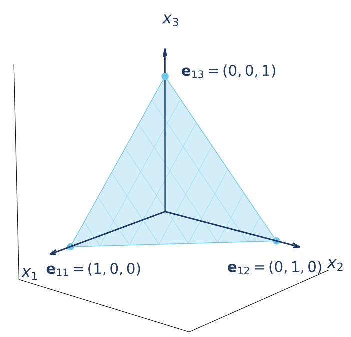
Δ_1: 1D line segment



3D example: 3 actions to choose from

$$\Delta_1 := \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid 0 \le x_j \le 1 \ \forall j, \mathbf{1}^T \mathbf{x} = 1 \right\}$$

Δ_1: 2D triangular region



Notation 2: Expected cost.

P_1 plays action j with probability x_j

$$\mathbb{P}(\chi_1 = \mathbf{e}_{1j}) = \begin{bmatrix} x_1 & \cdots & x_{m_1} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = x_j$$

P_2 plays action k with probability y_k

$$\mathbb{P}(\chi_2 = \mathbf{e}_{2k}) = \begin{bmatrix} y_1 & \cdots & y_{m_2} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = y_k$$

Assume players randomize independently

$$\begin{aligned} \bar{J}(\mathbf{x}, \mathbf{y}) &= \mathbb{E}[J] \\ &= \sum_{j=1}^{m_1} \sum_{k=1}^{m_2} \underbrace{a_{jk}}_{\text{cost}} \cdot \underbrace{x_j \cdot y_k}_{\Pr(\text{outcome } (j,k))} \\ &= \sum_{j=1}^{m_1} x_j \left(\sum_{k=1}^{m_2} a_{jk} \cdot y_k \right) \\ &= \mathbf{x}^T A \mathbf{y} \end{aligned}$$

P_1 aims to minimize the expected cost:

$$\min_{\mathbf{x} \in \Delta_1} \bar{J}(\mathbf{x}, \mathbf{y})$$

P_2 aims to maximize the expected cost:

$$\max_{\mathbf{y} \in \Delta_2} \bar{J}(\mathbf{x}, \mathbf{y})$$

2.2 Mixed Security Strategies

Def. 2.6 Security Mixed Strategy.

P_1 has security mixed strategy $\mathbf{x}^* \in \Delta_1$ if

$$\mathbf{x}^* = \arg\min_{\mathbf{x} \in \Delta_1} \max_{\mathbf{y} \in \Delta_2} \mathbf{x}^\top A \mathbf{y}$$

$$\Rightarrow \bar{J}_U = \min_{\mathbf{x} \in \Delta_1} \max_{\mathbf{y} \in \Delta_2} \mathbf{x}^\top A \mathbf{y}$$

The expected cost $\bar{J}_U$ is called the **upper value**.

P_2 has security mixed strategy $\mathbf{y}^* \in \Delta_2$ if

$$\mathbf{y}^* = \arg\max_{\mathbf{y} \in \Delta_2} \min_{\mathbf{x} \in \Delta_1} \mathbf{x}^\top A \mathbf{y}$$

$$\Rightarrow \bar{J}_L = \max_{\mathbf{y} \in \Delta_2} \min_{\mathbf{x} \in \Delta_1} \mathbf{x}^\top A \mathbf{y}$$

The expected cost $\bar{J}_L$ is called the **lower value**

Note. Allowing mixed strategies enlarges each player's feasible set:

$$J_L \le \bar{J}_L \le \bar{J}_U \le J_U$$

Minimax Theorem shows that the interval $[J_L, J_U]$ collapses to: $\bar{J}_L = \bar{J}_U$ in finite zero-sum games,

2.3 Mixed-Strategy Saddle-Point Equilibrium

Def 2.8 Mixed-Strategy Saddle-Point Equilibrium. saddle-point equilibrium is $(\mathbf{x}^*, \mathbf{y}^*) \in \Delta_1 \times \Delta_2$ such that

$$\mathbf{x}^{*\top} A \mathbf{y} \;\le\; \mathbf{x}^{*\top} A \mathbf{y}^* \;\le\; \mathbf{x}^\top A \mathbf{y}^* \qquad \forall\, \mathbf{x} \in \Delta_1,\ \forall\, \mathbf{y} \in \Delta_2$$

The common value $\bar{J} := \mathbf{x}^{*\top} A \mathbf{y}^*$ is called the **value of the game**.

Note. At saddle-point equilibrium, neither player can unilaterally improve outcome (no-regret)

$$\bar{J} = \bar{J}_U = \bar{J}_L \Leftrightarrow (\mathbf{x}^*, \mathbf{y}^*)$$

2.4 Minimax Theorem (von Neumann)

Thm 2.10 Minimax Theorem. In any two-player zero-sum matrix (2PZSM) game with cost matrix A, where A is an $m \times n$ matrix, or $\mathcal{G}(\mathcal{N}, \Delta_X, \bar{J})$ with $\bar{J}(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top A\mathbf{y}$, we have

$$\min_{\mathbf{x}\in\Delta_1} \max_{\mathbf{y}\in\Delta_2} \bar{J}(\mathbf{x}, \mathbf{y}) = \max_{\mathbf{y}\in\Delta_2} \min_{\mathbf{x}\in\Delta_1} \bar{J}(\mathbf{x}, \mathbf{y}),$$

or equivalently

$$\min_{\mathbf{x}\in\Delta_1} \max_{\mathbf{y}\in\Delta_2} \mathbf{x}^\top A\mathbf{y} = \max_{\mathbf{y}\in\Delta_2} \min_{\mathbf{x}\in\Delta_1} \mathbf{x}^\top A\mathbf{y}.$$

where Δ_1 and Δ_2 denote the simplices of appropriate dimensions:

$$\Delta_1 = \left\{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{1}_m^\top \mathbf{x} - 1 = 0,\ x_j \geq 0,\ \forall j\right\}, \qquad \Delta_2 = \left\{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{1}_n^\top \mathbf{y} - 1 = 0,\ y_k \geq 0,\ \forall k\right\}.$$

Proof. **Goal.** Show the mixed lower and upper values coincide:

$$\bar{J}_L := \max_{\mathbf{y}\in\Delta_2} \min_{\mathbf{x}\in\Delta_1} \mathbf{x}^\top A\mathbf{y}, \qquad \bar{J}_U := \min_{\mathbf{x}\in\Delta_1} \max_{\mathbf{y}\in\Delta_2} \mathbf{x}^\top A\mathbf{y}, \qquad \Rightarrow \qquad \bar{J}_L = \bar{J}_U.$$

Lemma 2.9 Separating Hyperplane. Let $Q \in \mathbb{R}^{m\times n}$ be arbitrary. Exactly one of the following holds:

$$\text{(i) } \exists\, \mathbf{y}_0 \in \Delta_2 \text{ such that } \mathbf{x}^\top Q\mathbf{y}_0 \leq 0, \quad \forall \mathbf{x} \in \Delta_1,$$
$$\text{(ii) } \exists\, \mathbf{x}_0 \in \Delta_1 \text{ such that } \mathbf{x}_0^\top Q\mathbf{y} \geq 0, \quad \forall \mathbf{y} \in \Delta_2.$$

Interpretation: either P_2 has a mixed strategy $\mathbf{y}_0$ making P_1's payoff non-positive for all $\mathbf{x}$, or P_1 has a mixed strategy $\mathbf{x}_0$ making the payoff non-negative for all $\mathbf{y}$, but not both.

Step 1: Relate lemma to the game. Fix a scalar $c \in \mathbb{R}$ and define a shifted matrix

$$Q := -A + c\,\mathbf{1}_{m\times n}.$$

Case (i). Suppose there exists $\mathbf{y}_0 \in \Delta_2$ such that $\mathbf{x}^\top Q\mathbf{y}_0 \leq 0$ for all $\mathbf{x} \in \Delta_1$. Then

$$0 \geq \mathbf{x}^\top Q\mathbf{y}_0 = \mathbf{x}^\top(-A + c\mathbf{1})\mathbf{y}_0 = -\mathbf{x}^\top A\mathbf{y}_0 + c \quad \Rightarrow \quad \mathbf{x}^\top A\mathbf{y}_0 \geq c,\ \forall \mathbf{x}.$$

Taking $\min_{\mathbf{x}}$ and then $\max_{\mathbf{y}}$,

$$\bar{J}_L = \max_{\mathbf{y}\in\Delta_2} \min_{\mathbf{x}\in\Delta_1} \mathbf{x}^\top A\mathbf{y} \ \geq\ c. \tag{1}$$

Case (ii). Suppose instead there exists $\mathbf{x}_0 \in \Delta_1$ such that $\mathbf{x}_0^\top Q\mathbf{y} \geq 0$ for all $\mathbf{y} \in \Delta_2$. Then

$$0 \leq \mathbf{x}_0^\top Q\mathbf{y} = \mathbf{x}_0^\top(-A + c\mathbf{1})\mathbf{y} = -\mathbf{x}_0^\top A\mathbf{y} + c \quad \Rightarrow \quad \mathbf{x}_0^\top A\mathbf{y} \leq c,\ \forall \mathbf{y}.$$

Taking $\max_{\mathbf{y}}$ and then $\min_{\mathbf{x}}$,

$$\bar{J}_U = \min_{\mathbf{x}\in\Delta_1} \max_{\mathbf{y}\in\Delta_2} \mathbf{x}^\top A\mathbf{y} \ \leq\ c. \tag{2}$$

Step 2: Eliminate the gap by contradiction. Assume for contradiction, that there is a positive gap:

$$\bar{J}_U > \bar{J}_L \quad \Rightarrow \quad k := \bar{J}_U - \bar{J}_L > 0.$$

Choose the midpoint

$$c := \frac{\bar{J}_L + \bar{J}_U}{2} = \bar{J}_L + \frac{k}{2} = \bar{J}_U - \frac{k}{2}.$$

Apply Lemma 2.9 to $Q = -A + c\mathbf{1}_{m\times n}$:

- If case (i) holds, then by (1)

$$\bar{J}_L \geq c = \bar{J}_L + \frac{k}{2},$$

  which implies $0 \geq \frac{k}{2} > 0$, a contradiction.
- If case (ii) holds, then by (2)

$$\bar{J}_U \leq c = \bar{J}_U - \frac{k}{2},$$

  which implies $0 \leq -\frac{k}{2} < 0$, also a contradiction.

In both cases we obtain a contradiction, so the assumption $\bar{J}_U > \bar{J}_L$ is false. Therefore

$$\bar{J}_L = \bar{J}_U$$

and hence

$$\min_{\mathbf{x}\in\Delta_1} \max_{\mathbf{y}\in\Delta_2} \mathbf{x}^\top A\mathbf{y} = \max_{\mathbf{y}\in\Delta_2} \min_{\mathbf{x}\in\Delta_1} \mathbf{x}^\top A\mathbf{y}.$$

2.5 Computing Mixed Saddle-Point

2.5.1 Dominated Strategies

Def. Dominated Strategies.

Row j dominates row r if

$$\begin{aligned} a_{jk} &\leq a_{rk}, && \forall k, \\ a_{jk} &< a_{rk}, && \text{for at least one } k. \end{aligned}$$

Column k dominates column c if

$$\begin{aligned} a_{jk} &\geq a_{jc}, && \forall j, \\ a_{jk} &> a_{jc}, && \text{for at least one } j. \end{aligned}$$

Dominated strategies can be eliminated since they will never be chosen in any optimal strategy.

Prop. 2.13 Elimination of Dominated Strategies. If rows $j_1, \ldots, j_l$ are dominated, then P_1 has an optimal strategy with $x_{j_1} = \cdots = x_{j_l} = 0$. Any optimal strategy for the reduced game (after removing dominated strategies) is optimal for the original game.

Example 2.14 Iterative Elimination.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 1 \\ 1 & 5 & 3 \\ 4 & 3 & 2 \end{bmatrix}$$

Step 1: Remove Row 4, which is dominated by Row 2 (compare: $0 \leq 4$, $2 \leq 3$, $1 < 2$).

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 1 \\ 1 & 5 & 3 \end{bmatrix}$$

Step 2: Remove Column 1, which is dominated by Column 3 (compare: $4 \geq 2$, $1 \geq 0$, $3 > 1$).

$$\begin{bmatrix} 1 & 4 \\ 2 & 1 \\ 5 & 3 \end{bmatrix}$$

Step 3: Remove Row 3, which is dominated by Row 2 (compare: $2 \leq 5$, $1 < 3$).

$$A' = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$$

2.5.2 Graphical Solution

Graphical Solution for $2 \times n$ Games. Suppose P_1 has two actions ($m_1 = 2$), and P_2 has m_2 actions.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m_2} \\ a_{21} & a_{22} & \cdots & a_{2m_2} \end{bmatrix}.$$

Mixed strategies:

$$\Delta_1 = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; 0 \leq x_j \leq 1,\ x_1 + x_2 = 1 \right\}, \quad \Delta_2 = \left\{ y = \begin{bmatrix} y_1 \\ \vdots \\ y_{m_2} \end{bmatrix} \in \mathbb{R}^{m_2} \;\middle|\; 0 \leq y_k \leq 1,\ \sum_{k=1}^{m_2} y_k = 1 \right\}.$$

P_1's security strategy. By definition,

$$\mathbf{x}^* \in \arg\min_{\mathbf{x} \in \Delta_1} \max_{\mathbf{y} \in \Delta_2} \mathbf{x}^\top A\, \mathbf{y}.$$

For fixed $\mathbf{x}$, $\mathbf{x}^\top A \mathbf{y}$ is linear in $\mathbf{y}$, hence its maximum is attained at a vertex $\mathbf{e}_{2k}$:

$$\begin{aligned}
\mathbf{x}^* &= \arg\min_{\mathbf{x} \in \Delta_1} \max_{k \in M_2} \mathbf{x}^\top A\, \mathbf{e}_{2k} \\
&= \arg\min_{\mathbf{x} \in \Delta_1} \max_{k \in M_2} \left\{ \mathbf{x}^\top \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \ldots, \mathbf{x}^\top \begin{bmatrix} a_{1k} \\ a_{2k} \end{bmatrix}, \ldots, \mathbf{x}^\top \begin{bmatrix} a_{1m_2} \\ a_{2m_2} \end{bmatrix} \right\}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 1 - x_1 \end{bmatrix}, \qquad x_1 \in [0,1]. \\
&= \arg\min_{x_1 \in [0,1]} \underbrace{\max_{k \in M_2} \{ \underbrace{(a_{11} - a_{21})x_1 + a_{21}}_{\text{best response function } R_1(x_1)} \ \ldots, \underbrace{(a_{1k} - a_{2k})x_1 + a_{2k}}_{R_k(x_1)} \ \ldots, \underbrace{(a_{1m_2} - a_{2m_2})x_1 + a_{2m_2}}_{R_{m_2}(x_1)} \}}_{\text{worst-case expected loss } V_1(x_1)}
\end{aligned}$$

Example: Matching Pennies (MP).

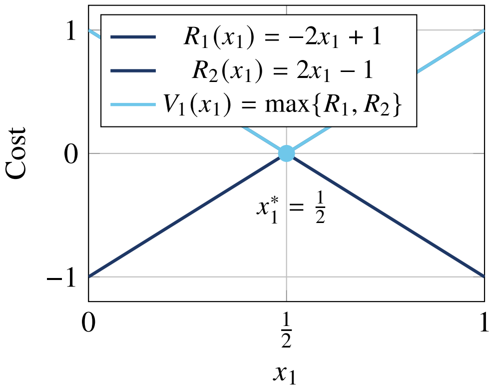
$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

from graphical solution, we obtain

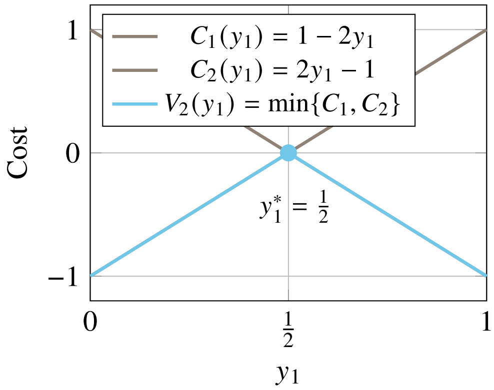
$$R_k(x_1) = (a_{1k} - a_{2k})x_1 + a_{2k} \Rightarrow \begin{cases} R_1(x_1) = (-1-1)x_1 + 1 = -2x_1 + 1, \\ R_2(x_1) = (1-(-1))x_1 - 1 = 2x_1 - 1. \end{cases}$$

Player 1 minimizes

$$V_1(x_1) = \max\{R_1(x_1),\ R_2(x_1)\}, \qquad x_1 \in [0,1].$$



Graphical method: P_1's mixed security strategy



Graphical method: P_2's mixed security strategy

The optimal x_1^* occurs at the intersection of R_1 and R_2:

$$R_1(x_1) = R_2(x_1) \;\Rightarrow\; -2x_1^* + 1 = 2x_1^* - 1 \;\Rightarrow\; \begin{cases} x_1^* = \frac{1}{2} \\ x_2^* = 1 - x_1^* = \frac{1}{2}. \end{cases}$$

Thus

$$x^* = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \qquad V_1(x_1^*) = R_1\left(\tfrac{1}{2}\right) = R_2\left(\tfrac{1}{2}\right) = 0.$$

By symmetry, Player 2's optimal mixed strategy is

$$y^* = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix},$$

and the saddle-point value (expected cost) is $\bar{J}^* = 0$.