Quantitative Finance – Derivatives

Time value of money

- money has time value, because we can reinvest them
- always picture a timeline in thinking about this problem

Interest

- The easiest way to make money is by earning interest from the bank
- simple interest rate (return calculation): $a_t = a_0 * (1 + it)$
- compound interest rate (keep reinvest): $a_t = a_0(1 + i)^t$
- effective interest rate during a period: $i = \frac{a_n - a_{n-1}}{a_{n-1}}$
- nominal rate of interest: $i^{(m)}$, meaning that this interest i is paid m times a year
  an amount of $\frac{i^{(m)}}{m}$ will be paid at each time interval, therefore at the end $= (1 + \frac{i^{(m)}}{m})^m$
  let m goes to infinity, we have the continuous form $(1 + \frac{r}{m})^{mt} \longrightarrow e^{rt}$

Annuities and Perpetuities

- annuity: fixed payment at each time point
  annuity-immediate: payment at the end of interval
  annuity-due: payment at the beginning of interval
- perpetuities: fixed payment at each time point, until infinity
  perpetuity-immediate: payment at the end of interval
  perpetuity-due: payment at the beginning of interval

Bonds: a fixed income instrument

- market graph of bonds: what kinds of bonds are out there
- zero coupon bond: no coupon payment, only pay face value at maturity (treat as discount)
  zero rate: the annualized rate of return (annual compound) $\longrightarrow B(t,T)(1+R(t,T))^{T-t} = 1$
  continuous zero rate: $\longrightarrow B(t,T)e^{(T-t)R(t,T)} = 1$ [treat R(t,T) as an interest rate]
  k-times per year: $\longrightarrow B_{(t,T)}(1 + \frac{R_{(r,T)}}{m})^m(T-t) = 1$
- Bond with coupon: pay at specific time

  Yield-to-Maturity: the interest rate that makes present value of cash flow equals bond price
  or we can say that the YTM is the IRR at this point: internal rate of return
  it tells us the interest rate that this investment is representing, so that we can compare

  Annual Coupon Bond: $P = \sum \frac{CF_t}{(1+y)^t}$
  Semi annual coupon bond: $P = \sum \frac{CF_t}{(1+\frac{y}{2})^{2t}}$

  So what happened in the market $\longrightarrow$ we calculate the bond price, by the risk-free rate
  Therefore it is easy to imagine:
  if risk-free rate goes up, the current bond price should drop
  if risk-free rate goes down, the current bond price should rise
  In short, the bond price is negatively correlated to the YTM
  It depends on how we structure the dependent variable and independent variables:
  1. Calculate the bond price: the DV is bond price, IDV is the interest rate, negative correlated.
  2, Calculate YTM: the DV is YTM, IDV is bond price, if bond price is high, meaning the YTM must be low

- Relationship between zero coupon bond and coupon bond:
  When a coupon bond pays no coupon, it is the same as zero coupon bond $\longrightarrow$ same price and YTM
  With coupon bond introduce coupon, we have 2 dimension to interpret the results:
  1. how the price will change: with introduction of future cash flow, interest rate does not change, price will rise; with price rise, the YTM will drop
  2. how the YTM will change: not a good direction, stick to PRICE $\longrightarrow$ YTM path
- Since the zero coupon bond has a higher YTM than coupon bond, it is some how confusing because we may think the coupon bond gives us more return.
- We introduce the "Par Yield" to do standardization on the returns:

- Par Bond: the price = face value $\longrightarrow$ issue at par $\longrightarrow$ YTM = annual coupon rate
  Par Yield: when a bond is issue at par, the coupon rate (annualized return) is equal to YTM (calculated annualized return)
- Dollar Duration: the $\frac{dP}{dy}$ the derivatives of price w.r.t. YTM
  Modified Duration: $-\frac{1}{P}\frac{dp}{dy}$, Dur normalized by price
  Dollar Value of 1 basis point (DV01): $= Dur \cdot (-1bp)$

  At a specific price level, the Dollar Duration Dur, decrease as:
  1. maturity increase (Dur is a price sensitivity measure to yield, with T goes up, it is more sensitive)
  2. coupon increases (1. more money to discount, more sensitive. 2. more coupon means price up, then more sensitive)
  3. YTM decrease (look at the graph, when YTM is small, it is steeper)

Forward Rates

- Diagram and calculation: separate the time into different intervals, make investment using respective rates
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Futures

Options

* Memoryless:

Bonds

* Memoryless:

Financial Assets: Derivatives

Forwards / Futures

- Futures: Traded in the exchange, so basically standardized contract
- Forwards: customized
- Keywords: strike price, maturity day, long, short

Agencies

- Clearing house: ensure trader fulfill obligations
- Margin: Trader deposit margin to fulfill obligations:
  Initial margin ; Maintenance margin ; Variation margin ;
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Options

- Vanilla Option / European Option: fixed maturity date, strike price
- American Option: can exercise before maturity date
- Bermudan Option: can exercise on pre-determined date
- Barrier Option: can exercise once cross the barrier level:
  up and out, down and out, up and in, down and in
  1) Callable Bull: if always bull, then it is valid (never drop down barrier) [down and out]
  2) Callable Bear: if always bear, then it is valid (never go up barrier) [up and out]
- Asian Option: the payoff determined by average price of pre-set period of time

Option strategies: Notice difference between prices and payoff

- Straddle: long 1 call and 1 put with same K and T
- Strangle: long call and put with same K and T
- Bull Spread: make money when price up: Buy call with K_1, sell call with K_2
- Bear Spread: make money when price down: Buy call with K_1, sell call with K_2
- Butterfly: long 2 calls, short call * 2

Warrents

- equity warrants: issue by company. represent the right to subscribe to equity securities
- derivative warrants: issued by 3rd party.

Swap: 2 parties exchange financial instruments

- Interest rate swap
  A: expect Libor drops ; B: expect Libor ups
  ????
- Benefit: 1) hedge against interest rate exposure ; 2) leverage comparative advantage
  A: AAA rated company ; B: BBB rated company.
  Company A will definitely have less lending interest rate in the market, in all loans
  But that doesn't mean company A can't do better. We can calculate the spread and see what is company A's comparative advantage, and borrow only that kind of loan, other company can use their comparative advantage P38
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Financial Risk: Value at Risk

Return

- Arithmetic Return: $R_t = \frac{S_t - S_{t-1}}{S_{t-1}}$
- Log Return: $log(1 + R_t) = log(\frac{S_t}{S_{t-1}}) = log(S_t) - log(S_{t-1})$

Volatility Measure

- $\sigma_s = \sqrt{\frac{1}{n-1}\sum(\mu_i - \bar{\mu})^2}$
- Then the volatility over N periods: $\sqrt{N}\sigma_s$

VaR: Value-at-Risk

- if there is a distribution graph, starting from left to some point, this region is α
- the rest region to the right is $1 - \alpha$, stands for the confidence: 95%, 99% etc.
- We called α as the "tail probability", since it is the worst scenario
- In a graphical sense, VaR is the $p - th$ quantile of the PnL distribution
- "Relative VaR": $E[PnL] - VaR$

Modeling the PnL distribution

- The fundamental of finding VaR is to model the portfolio PnL distribution
- Parametric Approach:
  Keypoint: construct the mean μ and volatility σ
  Approach: convert to approach "standard normal distribution"
  Then we can calculate VaR ; or extend this to a multi-time period scenario
- Non-Parametric Approach: Historical Simulation
  1. we want to know the price at tomorrow
  2. we convert the a problem: how much will the portfolio value change
  3. based on past period-wise change data, we treat it as distribution
  4. select the change we need

Financial Risk: Hedging, Optimal Hedging

Idea

- Hedge against a instrument that is negatively correlated to your portfolio

Quantify

- we want to minimize variance of new portfolio : $S + NF$, with S original portfolio
- we first construct the variance: $\sigma_s + N^2\sigma_f + 2N\sigma_{s,f}$
- Then N can be solved by using the solution for a square equation

Coursera - Columbia - No Arbitrage

Example of using No-Arbitrage pricing:

- 1. A bond that pays A in 1 year:
  Pricing 1: portfolio that: buy 1 bond, borrow $\frac{A}{1+r}$
  Pricing 2: portfolio that: sell 1 bond, lend $\frac{A}{1+r}$
- 2. Use compounding interest rate for pricing:
- 3. Floating Rate Bond:
  The price of a floating rate bond, is equal to, the face value.
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Cousera - Columbia - No Arbitrage in various assets

Swaps

- What is it: a trade between fixed and float rates
- Why is it: leverage relative strength, capture trends in other markets
- How does it works: do your best, borrow your weakness
  1. A has relative strength in fixed rate
  2. B has relative strength in floating rate
  3. A will borrow fixed rate, B will borrow floating rate
  4. A borrow floating from B, pays fixed to B
- How to price it: What should be the exchange rate pay to each other
  1. Swap should be of 0 value to each party
  2. Write the Cash Flow PV for one side
  3. Set it to 0 and we calculate the payment rate

Futures

- What is it: An traded an exchange standardized contract
- Why need it: standardized, counter party risk, solved multiple price for same maturity
- How does it works: Margin call (go to initial margin), clearing house
- How to price it:
- Minimum variance hedging (since perfect hedge is impossible):

Options

- In the money, at the money, out of the money
- Put-Call Parity: call + cash = put + stock
  try to buy a stock, and use money to help me
  try to sell a stock, and hold a stock to help me
  $C + Ke^{-rt} = P + S$
  $C + Ke^{-rt} = P + S - D$ (with dividend)
- Bounds of option prices

Binomial Tree Pricing

- 1-period Binomial Tree: price goes up and down with probabilities, allow short sell
- Multi-period Binomial Tree: extend with same prob and up/down scale
- Replications: (buying shares, investing in cash) $\leftrightarrow$ (option)
  make the payoff equals
- European and American Pricing: use risk-neutral prob to price:
  $q = \frac{R-d}{u-d}$. $C_0 = \frac{1}{R}[qC_u + (1-q)C_d]$