Time Series Analysis

What is time series

A collection of stochastic random variables **indexed by time**, their stochastic **distributions are similar but not the same**, the data are **correlated across time**
In plain words, time series refers to **a series of data points**, for which they follow some kind of distribution.
The goal for studying time series, is really to investigate in those data points, and try to figure out its **underlying distribution, and internal correlation**

Important Characteristics

1. Trend
2. Seasonality
3. Periodicity
4. Cyclical Trend
5. Heteroskedasticity
6. Dependence

Presentation & Basic Decomposition

Y_t, in which t indexes time: $Y_t = m_t + s_t + x_t$ $, with:$
m_t — trend component, trend means that over the time scope, the values are going up or going down,
s_t — seasonal component, seasonality means that it is a periodicity movement, it has a circle of appearance, within the circle, the overall trend should be zero .
x_t — stationary component, stationary means still, it pictures the characteristic that the component is not changing its value

Estimate the trend

1. Moving Average:
   estimate the trend with a moving window
2. Parametric Regression
   fit in a polynomial regression to estimate
3. Non-Parametric Approach
   - Kernel Regression
   - Local Polynomial Regression
   - Other Approaches

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Example: Temperature Data
1. Read the data file in R
2. Visualize the data (Avg Temp)
3. Estimate the trend (Moving Average)
4. Estimate the trend (Parametric)
5. Estimate the trend (Non-Parametric)
6. Comparison
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Estimate the Seaonality

General Approach: estimate and subtract m_t and s_t

1. Seasonal Average
   $\hat{s}_k = w_k - \frac{1}{d}\sum_{j=1}^{d} w_j$

   k: the seasonal group, let's say it has d groups. If the seasonality is monthly, then d = 12
   w_k: the average of all seasonal group value
2. Parametric Regression

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Example: Temperature Data
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Estimate the Stationary

Important Concepts in Time Series

1. **Auto Correlation Function (ACF)**
Like correlation or covariance, the auto correlation between 2 variables can be expressed as:
$\gamma(s,t) = Cov(X_s, X_t) = E[(X_s - \mu_s)(X_t - \mu_t)]$
$\gamma(t,t) = Var[X_t] = \sigma^2$
then we have the auto correlation defined as:
$\gamma_k = \gamma(t, t+k) \approx c_k \rightarrow \frac{\sum(x_t - \bar{x})(x_{t+k} - \bar{x})}{N}$
The underlying assumption is that the time series is stationary, therefore, across all k time horizon, this auto correlation relationship exist

2. **Auto Correlation Coefficient**
Following the auto correlation, the auto correlation coefficient can be defined as: $\rho_K = \frac{\gamma_k}{\gamma_0} \approx r_k \rightarrow \frac{c_k}{c_0}$

3. **Random Walk**
$X_t = X_{t-1} + Z_t$
Z as the white noise, X as stock price
$X_t = \sum_{i=1}^t Z_i$
we can interpret X as accumulated white noise
$E[X_t] = \mu t \ , \ Var[X_t] = \sigma^2 t$

4. **Moving Average Process: q**
How long back need to trace, to have weighted average:
The longer we trace back, distribution be smoother:
For instance, the MA(2) process is correlated back in 2 steps:
$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Zt - 2$
$Z_t \sim N(0,1)$
Moving Average is Weakly Stationary:
$cov(X_t, X_{t+k}) = \sigma^2 \sum_{i=0}^{q-k} \beta_i \beta_{i+k}$

5. **Autoregressive Process: p**
How long back need to trace, to have weighted average:
The longer we trace back, distribution be smoother:
Example: random walk: build on historical value
$X_t = Z_t + \phi_1 X_{t-1} + ... + +\phi_p X_{t-p}$
$Z_t = (1 - \phi_1 B + ... + +\phi_p B^p) X_t = \Phi(B) X_t$
$X_t = \frac{1}{1-(1-\phi_1 B + ... + +\phi_p B^p)} Z_t = (1 + \theta_1 B + \theta_2 B^2 + ...) Z_t$
Then we have: $E[X_t] = 0$; $Var[X_t] = \sigma_z^2 \sum_{i=0}^{\infty} \theta_i^2$
$\gamma(k) = \sigma_z^2 \sum_{i=0}^{\infty} \theta_i \theta_{i+k}$; $\rho(k) = \frac{\sum_{i=0}^{\infty} \theta_i \theta_{i+k}}{\sum_{i=0}^{\infty} \theta_i \theta_i}$

6. **a**

7. **Strict Stationary**
Definition: shifted distribution is the same:
$P\{X(t_1), X(t_2) ... X(t_k)\} = P\{X(t_{1+\tau}), X(t_{2+\tau}) ... X(t_{k+\tau})\}$
$\mu(t) = \mu$; $\sigma^2(t) = \sigma^2$; $\gamma(t_1, t_2) = \gamma(t_2 - t_1) = \gamma(\tau)$
Which leads to the properties:
1. random variables are **independently** distributed
2. the mean and variance function is identical
3. the covariance (ACF) depends only on **lag spaces**

8. **Weak Stationary**
Definition: constant mean and ACF

9. **Backward Shift Operator**
$BX_t = X_{t-1}$; $B^2 X_t = X_{t-2}$; $B^k X_t = X_{t-k}$
Further decompose the equation, we can have relationship between X and Z:
For $MA(q)$:
$X_t - \mu = \beta(B) Z_t$
$\beta(B) = \phi_0 + \phi_1 B + ... + \phi_q B^q$
For $AR(p)$:
$\phi(B) X_t = Z_t$
$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p$

10. **Invertibility**
Definition:
Stochastic process: $\{X_t\}$,
Innovation (random process): $\{Z_t\}$
if $Z_t = \sum_{k=0}^{\infty} \pi_k X_{t-k}$ (AR), then $\{X_t\}$ invertible

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We try to see if we can make the invert:
Invert a MA process to AR process:
1. First try to express Z_t using X_t, X_{t-1}
2. Change the Z_t in MA definition using X_{t-1}
3. Therefore we change MA to a AR

11. **Duality**
a). Invertibility Condition for MA(q)
$\beta(B) = \beta_0 + \beta_1 B + ... + \beta_q B^q$
its roots all lies **outside** in unit circle, $\rightarrow$, invertible
b). Stationarity Condition for AR(p)
$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - ... - -\phi_p B^p$
its roots all lies **outside** in unit circle, $\rightarrow$, invertible
c). How MA and AR are related
if Invertible: $MA(q) \longrightarrow AR(\infty)$
if Stationary: $AR(p) \longrightarrow MA(\infty)$

12. **Mean-Square Convergence**

13. **Difference Equation**
Recall the way we solve recursive series for general form
Difference equation is dealing with the same problems.

14. **Yule-Walker Equations**
A set of Difference Equations, that governs AR's ACF

15. **a**

16. **a**