暨大資工系　線性代數　期末考 112. 6. 14

Theorem 1

4.6 Change of Basis

Change of Basis Problem: If we change the basis for a vector space ∇ from some old basis B to some new basis B', how is the old coordinate matrix $[v]_B$ of a vector v related to the new coordinate matrix $[v]_{B'}$?

sol: Let $B=\{u_1,u_2\}$ & $B'=\{u_1',u_2'\}$

If $[u_1']_B=\begin{bmatrix}a\\b\end{bmatrix}$ & $[u_2']_B=\begin{bmatrix}c\\d\end{bmatrix}$

i.e. $u_1'=au_1+bu_2$; $u_2'=cu_1+du_2$

Let $v\in\nabla$ & $[v]_{B'}=\begin{bmatrix}k_1\\k_2\end{bmatrix}$

$$\begin{aligned}\therefore v&=k_1u_1'+k_2u_2'\\&=k_1(au_1+bu_2)+k_2(cu_1+du_2)\\&=(k_1a+k_2c)u_1+(k_1b+k_2d)u_2\end{aligned}$$

i.e. $[v]_B=\begin{bmatrix}k_1a+k_2c\\k_1b+k_2d\end{bmatrix}=\begin{bmatrix}a&c\\b&d\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix}$

$=P[v]_{B'}$

Solution of the change of Basis Problem: If we change the basis for a vector space ∇ from some old basis $B=\{u_1, u_2, \ldots, u_n\}$ to some new basis $B'=\{u_1', u_2', \ldots, u_n'\}$, then the old coordinate matrix $[v]_B$ of a vector v related to the new coordinate matrix $[v]_{B'}$ of the same vector v by the equation $[v]_B=P[v]_{B'}$,where the column of P are the coordinate matrices of the new basis vectors relative to the old basis.

i.e. $P=\left[[u_1']_B\middle|[u_2']_B\middle|\cdots\middle|[u_n']_B\right]$ P is called the transition matrix from B' to B

Theorem 2 If $S=\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for an inner product space ∇ , and u is any vector in ∇, then $u=\langle u, v_1\rangle v_1+\langle u, v_2\rangle v_2+\ldots+\langle u, v_n\rangle v_n$

Theorem 3. Let ϖ be a finite-dimensional subspace of an inner product space ∇.

(a) If $\{v_1, v_2, \ldots, v_r\}$ is an orthonormal basis for ϖ, and u is any vector in ∇, then $proj_{\varpi}u=\langle u,v_1\rangle v_1+\langle u,v_2\rangle v_2+\cdots+\langle u,v_r\rangle v_r$

(b) If $\{v_1, v_2, \ldots, v_n\}$ is an orthogonal basis for ϖ , and u is any vector in ∇,

then $$proj_{\varpi}u=\frac{\langle u,v_1\rangle}{\|v_1\|^2}v_1+\frac{\langle u,v_2\rangle}{\|v_2\|^2}v_2+\cdots+\frac{\langle u,v_n\rangle}{\|v_n\|^2}v_n$$

***Theorem* 4: A least squares solution of Ax=b must satisfy the equality $A^TAx=A^Tb$ (i.e. $x=(A^TA)^{-1}A^Tb$) [it is called the *normal system* associated with Ax=b].**

If A is an $m\times n$ matrix with linearly independent column vectors, then for every $n\times 1$ matrix b, the linear system Ax=b has a unique least squares solution. This solution is given by $x=(A^TA)^{-1}A^Tb$. Moreover, if ϖ is the column space of A, then the orthogonal projection of b on ϖ is $proj_{\varpi}b=Ax=A(A^TA)^{-1}A^Tb$

Theorem 5.

If $T:\mathbb{R}^n\to\mathbb{R}^n$ is a linear transformation, and $e_1,e_2,\cdots,e_n$ are the standard basic vectors for $\mathbb{R}^n$, then the standard matrix for T is $[T]=\left[T(e_1)\middle|T(e_2)\middle|\cdots\middle|T(e_n)\right]$

Theorem 6.

Section 8.4
Matrices for General Linear Transformations

$$T : V \to W$$

V: $\dim = n$, $B = \{u_1, u_2, \cdots, u_n\}$

W: $\dim = m$, $B' = \{v_1, v_2, \cdots, v_m\}$

找 Matrix A s.t.

$$\boxed{[T(x)]_{B'} = A[x]_B \quad \forall x \in V}$$

Pf:

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$ $\because [u_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1} \cdots [u_n]_B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}$

$$\therefore [T(u_1)]_{B'} = A[u_1]_B = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

同理 $[T(u_2)]_{B'} = A[u_2]_B = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$

$\Downarrow$ 推得

$$\therefore A = \Big[[T(u_1)]_{B'}, [T(u_2)]_{B'}, \cdots, [T(u_n)]_{B'} \Big] \triangleq [T]_{B',B}$$

Theorem 6-1

基底轉換可視作一 identity linear operator on V

$$I : V \to V$$

basis B basis B'

$I(x) = x$

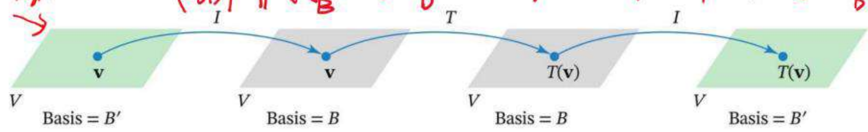
$$I([x]_B) = [x]_{B'} = Q[x]_B$$

$$[I]_{B',B} = Q_{B \to B'}$$

Theorem 7.

Section 8.5 Similarity

input $\quad I([x]_{B'}) = P[x]_{B'} = [x]_B \quad x \in V \quad I([x]_B) = Q[x]_B = [x]_{B'}$



THEOREM 8.5.1

If B and B' are bases for a finite-dimensional vector space V, and if $I : V \to V$ is the identity operator on V, then

$$Q_{B\to B'} = [I]_{B',B} \quad \text{and} \quad P_{B'\to B} = [I]_{B,B'} \qquad Q = P^{-1}$$

THEOREM 8.5.2

Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V, and let B and B' be bases for V. Then

$$[T]_{B'} = P^{-1}[T]_B P \tag{11}$$

Theorem 8.

THEOREM 2.3.6 Inverse of a Matrix Using Its Adjoint

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$$

Theorem 9. Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

$$\|1\|^2 = \int_0^{2\pi} 1\, dx = 2\pi$$

$$\|\cos kx\|^2 = \int_0^{2\pi} \cos^2 kx\, dx = \int_0^{2\pi} \frac{1+\cos 2kx}{2}\, dx = \pi$$

$$\|\sin kx\|^2 = \int_0^{2\pi} \sin^2 kx\, dx = \int_0^{2\pi} \frac{1-\cos 2kx}{2}\, dx = \pi$$

$$\frac{a_0}{2} = \frac{\langle f(x), 1\rangle}{\|1\|^2} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$$

$$a_k = \frac{\langle f(x), \cos kx\rangle}{\|\cos kx\|^2} = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\, dx$$

$$b_k = \frac{\langle f(x), \sin kx\rangle}{\|\sin kx\|^2} = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\, dx$$

1. (10%) Find a 3×3 matrix A that has eigenvalues 1, 2, and 3, and for which (0,1,0), (-1,2,2), and (-1,1,1) are their corresponding eigenvectors.

2. (30%) Let W be the plane with equation $5x - 3y + z = 0$.
   (a) (10%) Find an orthonormal basis for W.
   (b) (10%) Find the standard matrix for the orthogonal projection onto W.
   (c) (10%) Find all of the points in R^3, such that all of them are orthogonally projected to the same vector (1, 1, -2) which is on the plane W.

3. (10%) Let W be the line in R^3 with parametric equations $x = 2t, y=-t, z=4t$ $(-\infty<t<\infty)$. Find an equation for $W^{\perp}$.

4. (15%) Let T: $M_{22} \rightarrow M_{22}$ be a linear operator and defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)=\begin{bmatrix} 2c & a+c \\ b-2c & d \end{bmatrix}$$

   (a) (10%) Find a basis B of M_{22}, then the standard matrix $[T]_B$ of T with respect to the basis B is a diagonal matrix.
   (b) (5%) Find $[T]_B^{100}$.

5. (10%) Find a curve of the form $a+(b/x)$ that best fits the data points (1, 7), (3, 3), (6, 1) by making the substitution $X = 1/x$.

6. (10%) Find the Fourier series of $f(x) = 1$, $0 \le x < \pi$ and $f(x) = 0$, $\pi \le x \le 2\pi$ over the interval $[0, 2\pi]$.

7. (15%)

EXAMPLE 3 Matrix for a Linear Transformation

Let $T: R^2 \rightarrow R^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the matrix for the transformation T with respect to the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2 and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}; \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

1. (10%) Find a 3×3 matrix $\boldsymbol{A}$ that has eigenvalues 1, 2, and 3, and for which (0,1,0), (-1,2,2), and (-1,1,1) are their corresponding eigenvectors.

solution:

$\because A\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$ $A\begin{bmatrix}-1\\2\\2\end{bmatrix}=\begin{bmatrix}-2\\4\\4\end{bmatrix}$ $A\begin{bmatrix}-1\\1\\1\end{bmatrix}=\begin{bmatrix}-3\\3\\3\end{bmatrix}$ $Ax=\lambda x$

$$\therefore A\underbrace{\begin{bmatrix}0&-1&-1\\1&2&1\\0&2&1\end{bmatrix}}_{B}=\underbrace{\begin{bmatrix}0&-2&-3\\1&4&3\\0&4&3\end{bmatrix}}_{C}$$

$$A=CB^{-1}$$

$\because Av = \lambda v$

$A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$A\begin{bmatrix}-1\\2\\2\end{bmatrix} = \begin{bmatrix}-2\\4\\4\end{bmatrix} \quad \Rightarrow A\begin{bmatrix}0 & -1 & -1\\1 & 2 & 1\\0 & 2 & 1\end{bmatrix} = \begin{bmatrix}0 & -2 & -3\\1 & 4 & 3\\0 & 4 & 3\end{bmatrix}$

$A\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}-3\\3\\3\end{bmatrix}$

$\therefore A\begin{bmatrix}0 & -1 & -1\\1 & 2 & 1\\0 & 2 & 1\end{bmatrix} = \begin{bmatrix}0 & -2 & -3\\1 & 4 & 3\\0 & 4 & 3\end{bmatrix}$

$A = \begin{bmatrix}0 & -2 & -3\\1 & 4 & 3\\0 & 4 & 3\end{bmatrix}\begin{bmatrix}0 & -1 & -1\\1 & 2 & 1\\0 & 2 & 1\end{bmatrix}^{-1}$

$A = \begin{bmatrix}0 & -2 & -3\\1 & 4 & 3\\0 & 4 & 3\end{bmatrix}\begin{bmatrix}0 & 1 & -1\\1 & 0 & 1\\-2 & 0 & -1\end{bmatrix}$

$= \begin{bmatrix}4 & 0 & 1\\-2 & 1 & 0\\-2 & 0 & 1\end{bmatrix}$

$\therefore A = \begin{bmatrix}4 & 0 & 1\\-2 & 1 & 0\\-2 & 0 & 1\end{bmatrix}$ #

$\left[\begin{array}{ccc|ccc}0 & -1 & -1 & 1 & 0 & 0\\1 & 2 & 1 & 0 & 1 & 0\\0 & 2 & 1 & 0 & 0 & 1\end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|ccc}0 & 1 & 1 & -1 & 0 & 0\\1 & 0 & -1 & 2 & 1 & 0\\0 & 0 & -1 & 2 & 0 & 1\end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|ccc}0 & 1 & 1 & -1 & 0 & 0\\1 & 0 & -1 & 2 & 1 & 0\\0 & 0 & 1 & -2 & 0 & -1\end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|ccc}0 & 1 & 0 & 1 & 0 & 1\\1 & 0 & 0 & 0 & 1 & -1\\0 & 0 & 1 & -2 & 0 & -1\end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 0 & 1 & -1\\0 & 1 & 0 & 1 & 0 & 1\\0 & 0 & 1 & -2 & 0 & -1\end{array}\right]$

solution 2

$$P = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \qquad \operatorname{adj}(P)^T = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \qquad \det(P) = -2+1 = -1$$

$$P^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix}$$

$$\because\ P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \therefore\ A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 & -3 \\ 1 & 4 & 3 \\ 0 & 4 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \ \#$$

2. (30%) Let W be the plane with equation $5x - 3y + z = 0$.

   (a) (10%) Find an orthonormal basis for W.

   (b) (10%) Find the standard matrix for the orthogonal projection onto W.

   (c) (10%) Find all of the points in R^3, such that all of them are orthogonally projected to the same vector (1, 1, -2) which is on the plane W.

2. $5x-3y+z=0$

(a) $x=t,\ y=s,\ z=-5t+3s$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=t\begin{pmatrix}1\\0\\-5\end{pmatrix}+s\begin{pmatrix}0\\1\\3\end{pmatrix}$$

~~用 Theorem 3~~

Solution 1:

$u_1=v_1=(1,0,-5)$ $\quad \|v_1\|^2=1+25=26$, $\langle u_2,v_1\rangle=-15$

$v_2=u_2-\frac{\langle u_2,v_1\rangle}{\|v_1\|^2}v_1=(0,1,3)-\frac{-15}{26}(1,0,-5)=\left(\frac{15}{26},1,\frac{3}{26}\right)$

orthonormal basis: $\quad \|v_2\|=\sqrt{\frac{225}{676}+\frac{676}{676}+\frac{9}{676}}=\frac{\sqrt{910}}{26}$

$\frac{v_1}{\|v_1\|}=\frac{1}{\sqrt{26}}(1,0,-5)=\left(\frac{1}{\sqrt{26}},0,\frac{-5}{\sqrt{26}}\right)$

$\frac{v_2}{\|v_2\|}=\frac{26}{\sqrt{910}}\left(\frac{15}{26},1,\frac{3}{26}\right)=\left(\frac{15}{\sqrt{910}},\frac{26}{\sqrt{910}},\frac{3}{\sqrt{910}}\right)$ #

Theorem 3 & 用 Theorem 5

(b) $T=[T(e_1)\,|\,T(e_2)\,|\,T(e_3)]$

$T(e_1)=\langle(1,0,0),v_1\rangle v_1+\langle(1,0,0),v_2\rangle v_2=\left(\frac{2}{7},\frac{3}{7},\frac{-1}{7}\right)$

$T(e_2)=\langle(0,1,0),v_1\rangle v_1+\langle(0,1,0),v_2\rangle v_2=\left(\frac{3}{7},\frac{26}{35},\frac{3}{35}\right)$

$T(e_3)=\langle(0,0,1),v_1\rangle v_1+\langle(0,0,1),v_2\rangle v_2=\left(\frac{-1}{7},\frac{3}{35},\frac{34}{35}\right)$

$$\Rightarrow\begin{bmatrix}\frac{2}{7}&\frac{3}{7}&\frac{-1}{7}\\\frac{3}{7}&\frac{26}{35}&\frac{3}{35}\\\frac{-1}{7}&\frac{3}{35}&\frac{34}{35}\end{bmatrix}$$

~~(c)~~

$$\begin{bmatrix}\frac{2}{7}&\frac{3}{7}&\frac{-1}{7}\\\frac{3}{7}&\frac{26}{35}&\frac{3}{35}\\\frac{-1}{7}&\frac{3}{35}&\frac{34}{35}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\-2\end{bmatrix}$$ #

求(x,y,z)解

(a) If $x=s$ and $y=t$, then a point on the plane is $(s,t,-5s+3t)=s(1,0,-5)+t(0,1,3)$.

Solution2 $\mathbf{w}_1=(1,0,-5)$ and $\mathbf{w}_2=(0,1,3)$ form a basis for W (they are linearly independent since neither of them is a scalar multiple of the other).

繼續求得 orthonormal basis 如上頁

(b) Letting $A=\begin{bmatrix}1&0\\0&1\\-5&3\end{bmatrix}$, Formula (11) yields

用 Theorem 4

$$P=A\left(A^TA\right)^{-1}A^T=\begin{bmatrix}1&0\\0&1\\-5&3\end{bmatrix}\left(\underbrace{\begin{bmatrix}1&0&-5\\0&1&3\end{bmatrix}\begin{bmatrix}1&0\\0&1\\-5&3\end{bmatrix}}_{A^TA\text{ 不會是 }I_{2\times2}}\right)^{-1}\begin{bmatrix}1&0&-5\\0&1&3\end{bmatrix}$$

$$=\begin{bmatrix}1&0\\0&1\\-5&3\end{bmatrix}\left(\begin{bmatrix}26&-15\\-15&10\end{bmatrix}\right)^{-1}\begin{bmatrix}1&0&-5\\0&1&3\end{bmatrix}=\begin{bmatrix}1&0\\0&1\\-5&3\end{bmatrix}\left(\frac{1}{(26)(10)-(-15)(-15)}\begin{bmatrix}10&15\\15&26\end{bmatrix}\right)\begin{bmatrix}1&0&-5\\0&1&3\end{bmatrix}$$

$$=\tfrac{1}{35}\begin{bmatrix}1&0\\0&1\\-5&3\end{bmatrix}\begin{bmatrix}10&15\\15&26\end{bmatrix}\begin{bmatrix}1&0&-5\\0&1&3\end{bmatrix}=\tfrac{1}{35}\begin{bmatrix}10&15&-5\\15&26&3\\-5&3&34\end{bmatrix}.=\begin{bmatrix}\frac{2}{7}&\frac{3}{7}&\frac{-1}{7}\\\frac{3}{7}&\frac{26}{35}&\frac{3}{35}\\\frac{-1}{7}&\frac{3}{35}&\frac{34}{35}\end{bmatrix}$$

Solution3

#

(a) $5x - 3y + z = 0$

$z = -5x + 3y$

$x = s$ $\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$y = t$

let $v_1 = (0,1,3)$ $\qquad \|v_1\| = \sqrt{10}$

$v_2 = (1,0,-5) - \dfrac{\langle (0,1,3),(1,0,-5)\rangle}{10}(0,1,3)$

$= (1,0,-5) - \dfrac{\cancel{-15}\ {-3}}{\cancel{10}\ 2}(0,1,3)$ $\qquad 4+9+1=14$

$= \left(1, \frac{3}{2}, \frac{-1}{2}\right) \Leftrightarrow (2,3,-1)$ $\qquad \|v_2\| = \sqrt{14}$

orthonormal basis $= \left\{ \left(0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right), \left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right) \right\}$ #

(b) $A = \begin{bmatrix} 0 & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{14}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{14}} \end{bmatrix}$

$$\begin{aligned} \mathrm{Proj}_W &= A(A^TA)^{-1}A^T \\ &= A I A^T \\ &= \begin{bmatrix} 0 & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{14}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{14}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} & \frac{-1}{\sqrt{14}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{14} & \frac{6}{14} & \frac{-2}{14} \\ \frac{6}{14} & \frac{1}{10}+\frac{9}{14} & \frac{3}{10}+\frac{-3}{14} \\ \frac{-2}{14} & \frac{3}{10}+\frac{-3}{14} & \frac{9}{10}+\frac{1}{14} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{-1}{7} \\ \frac{3}{7} & \frac{26}{35} & \frac{3}{35} \\ \frac{-1}{7} & \frac{3}{35} & \frac{34}{35} \end{bmatrix} \quad \# \end{aligned}$$

用 Theorem4

$\frac{14+90}{140} = \frac{104}{140} = \frac{26}{35}$

$\frac{42-30}{140} = \frac{12}{140} = \frac{3}{35}$

$\frac{126+10}{140} = \frac{136}{140} = \frac{34}{35}$

$\frac{15}{2} + \frac{6}{2} = \frac{21}{2}$

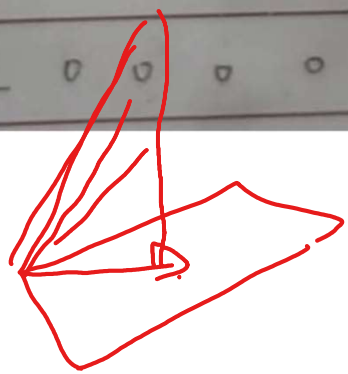
(c) $\left[\begin{array}{ccc|c} \frac{2}{7} & \frac{3}{7} & \frac{-1}{7} & 1 \\ \frac{3}{7} & \frac{26}{35} & \frac{3}{35} & 1 \\ \frac{-1}{7} & \frac{3}{35} & \frac{34}{35} & -2 \end{array}\right] \Leftrightarrow \begin{bmatrix} 2 & 3 & -1 & 7 \\ 15 & 26 & 3 & 35 \\ -5 & 3 & 34 & -70 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & 3 & -1 & 7 \\ 0 & 35 & 105 & -175 \\ -5 & 3 & 34 & -70 \end{bmatrix}$

A

$$\Leftrightarrow \begin{bmatrix} 2 & 3 & -1 & 7 \\ 0 & 1 & 3 & 5 \\ -5 & 3 & 34 & -70 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{-1}{2} & \frac{7}{2} \\ 0 & 1 & 3 & 5 \\ -5 & 3 & 34 & -70 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{-1}{2} & \frac{7}{2} \\ 0 & 1 & 3 & 5 \\ 0 & \frac{21}{2} & \frac{63}{2} & \frac{-105}{2} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{-1}{2} & \frac{7}{2} \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & -5 & -4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x = -4 + 5t \\ y = 5 - 3t \\ z = t \end{cases} \quad t \in \mathbb{R} \quad \#$$



$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

3. (10%) Let W be the line in R^3 with parametric equations x = 2t, y=-t, z=4t ($-\infty<t<\infty$). Find an equation for $W^{\perp}$.

$(x, y, z) = t(2, -1, 4)$

sol: $(x, y, z) \cdot (2, -1, 4) = 0$

$\Rightarrow 2x - y + 4z = 0$

4. (15%) Let $T: M_{22} \to M_{22}$ be a linear operator and defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2c & a+c \\ b-2c & d \end{bmatrix}$$

(a) (10%) Find a basis B of M_{22}, then the standard matrix $[T]_B$ of T with respect to the basis B is a diagonal matrix.

(b) (5%) Find $[T]_B^{100}$.

用 Theorem 6 & Theorem 6-1

Theorem 7

(a)

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} 2c \\ a+c \\ b-2c \\ d \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B_0 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

標準基底

$$\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \triangleq [T]_{B_0} \Rightarrow \begin{bmatrix} \lambda & 0 & -2 & 0 \\ -1 & \lambda & -1 & 0 \\ 0 & -1 & \lambda+2 & 0 \\ 0 & 0 & 0 & \lambda-1 \end{bmatrix}$$

$$(\lambda-1)(\lambda^2(\lambda+2)-2-\lambda)$$
$$= (\lambda-1)(\lambda^3+2\lambda^2-\lambda-2)$$
$$= (\lambda-1)^2(\lambda^2+3\lambda+2)$$
$$= (\lambda-1)^2(\lambda+1)(\lambda+2)$$

$\therefore \lambda = 1, -1, -2$

$\lambda = 1 \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = -1 \Rightarrow \begin{bmatrix} -1 & 0 & -2 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = -2 \Rightarrow \begin{bmatrix} -2 & 0 & -2 & 0 \\ -1 & -2 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$P = \begin{bmatrix} 0 & 2 & -2 & -1 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad [T]_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

$\text{basis}_B = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ ✓

注意基底是4个 2x2 矩陣

(b) $[T]_B^{100}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2^{100} \end{bmatrix}$

P 為基底 B 至 B_0 之 transition matrix

$P^{-1}[T]_{B_0}P = [T]_B$

5. (10%) Find a curve of the form $a+(b/x)$ that best fits the data points (1, 7), (3, 3), (6, 1) by making the substitution $X = 1/x$.

用 Theorem 4

Sol:

With the substitution $X=\frac{1}{x}$, the problem becomes to find a line of the form $y=a+b\cdot X$ that best fits the data points $(1, 7)$, $(\frac{1}{3}, 3)$, $(\frac{1}{6}, 1)$.

$$\begin{bmatrix} 1 & 1 \\ 1 & \frac{1}{3} \\ 1 & \frac{1}{6} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

$M \quad v \quad \neq \quad y$

$M^T M v = M^T y$

$v^* = (M^T M)^{-1} M^T y$

We have $M=\begin{bmatrix} 1 & 1 \\ 1 & \frac{1}{3} \\ 1 & \frac{1}{6} \end{bmatrix}$, $M^T M=\begin{bmatrix} 3 & \frac{3}{2} \\ \frac{3}{2} & \frac{41}{36} \end{bmatrix}$, $\left(M^T M\right)^{-1}=\frac{1}{42}\begin{bmatrix} 41 & -54 \\ -54 & 108 \end{bmatrix}$, and

$\mathbf{v}^*=\left(M^T M\right)^{-1} M^T \mathbf{y}=\frac{1}{42}\begin{bmatrix} 41 & -54 \\ -54 & 108 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{3} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}=\begin{bmatrix} \frac{5}{21} \\ \frac{48}{7} \end{bmatrix}$. The line in terms of X is $y=\frac{5}{21}+\frac{48}{7}X$, so the required curve is $y=\frac{5}{21}+\frac{48}{7x}$.

6. (10%) Find the Fourier series of $f(x) = 1$, $0 \le x < \pi$ and $f(x) = 0$, $\pi \le x \le 2\pi$ over the interval $[0, 2\pi]$.

Sol:

用 Theorem 9

Let $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$.

$$a_0 = \tfrac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \tfrac{1}{\pi}\int_0^{\pi} dx = 1$$

$$a_k = \tfrac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx = \tfrac{1}{\pi}\int_0^{\pi} \cos kx\,dx = 0$$

$$b_k = \tfrac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx = \tfrac{1}{\pi}\int_0^{\pi} \sin kx\,dx = \tfrac{1}{k\pi}(1-(-1)^k)$$

$$\left. \frac{-1}{k}\cos kx \right|_0^{\pi} \qquad -(-1)^k + 1$$

So the Fourier series is $\frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{k\pi}(1-(-1)^k)\sin kx$.

7. (15%)

▶ **EXAMPLE 3** **Matrix for a Linear Transformation**

Let $T : R^2 \to R^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the matrix for the transformation T with respect to the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2 and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}; \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Solution

From the formula for T,

$$T(\mathbf{u}_1) = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad T(\mathbf{u}_2) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

用 Theorem 6

Expressing these vectors as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, we obtain (verify)

$$T(\mathbf{u}_1) = \mathbf{v}_1 - 2\mathbf{v}_3, \quad T(\mathbf{u}_2) = 3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$$

Thus,

$$[T(\mathbf{u}_1)]_{B'} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad [T(\mathbf{u}_2)]_{B'} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

so

$$[T]_{B',B} = [[T(\mathbf{u}_1)]_{B'} \mid [T(\mathbf{u}_2)]_{B'}] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$