几何处理

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目录

1 微分几何基础

向量空间是一种代数结构, 元素集合 V, 加法操作 $+$, 标量乘法 $\cdot$, 和 additive identity 0, 满足如下八条定理:

流形空间

Permutation 定义为 $\sigma \in S_n$, $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \downarrow & \downarrow & \dots & \downarrow \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix} \Leftrightarrow \begin{pmatrix} \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$$

permutation 的符号定义为 $sgn : S_n \to \{+1,-1\}$

如果一个 permutation 只有两个相交换, 则可记为 $\tau_{i,j} \in S_n$, 称为一个 transposition .

$$\begin{pmatrix} 1 & \dots & i & \dots & j & \dots & n \\ \downarrow & \dots & \downarrow & \dots & \downarrow & \dots & \downarrow \\ 1 & \dots & j & \dots & i & \dots & n \end{pmatrix} \Leftrightarrow \tau_{i,j}$$

determinant, 记为 D, 定义为

$$D(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{\sigma(i)i}$$

性质, 若 $A \in \mathcal{R}^{n\times n}$

- $D(I) = 1$
- 如果 A 有两列一样, 则 $D(A) = 0$
- 如果固定 A 中任意 $n-1$ 列, 则对于剩余列是线性函数

对于函数 $f : \mathcal{R}^n \to \mathcal{R}^m$, 则有

$$f(x) \approx Df(x_0)(x - x_0) + f(x_0)$$

D 是一个线性映射, 称为 f 的 Jacobian 矩阵

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} = \frac{\partial f_i}{\partial x_j}$$

在 $\mathcal{R}^n$ 中的点 p 的向量 v 分别记为

$$p = (x_1,\dots,x_n), \quad v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

流形 $\mathcal{R}^n$ 由点集合 $p = (x_1,\dots,x_n), x_i \in \mathcal{R}$ 构成. 向量空间 $\mathcal{R}^n$ 由向量集合 $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, x_i \in \mathcal{R}$ 构成.

对于函数 $f : \mathcal{R}^n \to \mathcal{R}$, 如果函数定义在流形 $\mathcal{R}^n$ 上称为 functions, 如果定义在向量空间 $\mathcal{R}^n$ 上, 则称为 functionals. 对于函数 $f : \mathcal{R}^n \to \mathcal{R}^m$, 若 $\mathcal{R}^n, \mathcal{R}^m$ 都是向量空间, 则称为 transformations.

coordinate function

$$\begin{aligned} x :& \mathcal{R}^2 \to R & y :& \mathcal{R}^2 \to R & z :& \mathcal{R}^2 \to R \\ & p \mapsto x(p) & & p \mapsto y(p) & & p \mapsto z(p) \end{aligned}$$

T_pM 指代流形 M 在点 p 处的切向量空间 (tangent space), 流形 M 和其所有点上相关的点的切空间 T_pM 一起被称作 tangent bundle, 记为 TM. 若 M 是 n 维的则 TM 为 $2n$ 维的流形.

流形上的 vector field(向量场) 在流形上每一点指定了一个向量, 点 p 向量为 $v_p \in T_pM$. 所以指定一个 vector field 等价于给 M 上每个切空间一个元素, 称为 section of tangent bundle TM.

假设一个 $f : \mathcal{R}^3 \to \mathcal{R}$, 设单位向量 $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$,$p =$

(x_0,y_0,z_0) 处方向导数为

$$D_uf(x_0,y_0,z_0)=\lim_{t\to 0}\frac{f(x_0+ta,y_0+tb,z_0+tc)-f(x_0,y_0,z_0)}{t}$$
$$=\frac{d}{dt}(f(p+tu))|_{t=0}$$

显然 $D_{e_0}f=\frac{\partial f}{\partial x}, D_{e_1}f=\frac{\partial f}{\partial y}$

$$D_uf(x,y,z)=\frac{\partial f}{\partial x}a+\frac{\partial f}{\partial y}b+\frac{\partial f}{\partial z}c$$

记

$$v_p[f]=D_{v_p}f=\frac{d}{dt}(f(p+tv_p))|_{t=0}$$

设 $v_p=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$

$$v_p[f]=\sum_{i=1}^{3}v_i\frac{\partial f}{\partial x_i}|_p$$

$v_p\in T_pR^n$ 等价于一个微分符号

$$v_p=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}_p=v_1\frac{\partial f}{\partial x_1}|_p+\cdots+v_n\frac{\partial f}{\partial x_n}|_p$$

$T_p(R^n)$ 是一个向量空间, $T_p^*(R^n)$ 是其 dual space, 在流形 M 上的 differential one form, $\alpha: T_pR^n\to R\in T_p^*(R^n)$, 满足

$$\alpha(v_p+w_p)=\alpha(v_p)+\alpha(w_p)$$

$$\alpha(av_p)=a\alpha(v_p)$$

其中 $v_p,w_p\in T_pR^n$ 且 $a\in R$

可以将 $T_p(R^3)$ 的基向量写作

$$\{\frac{\partial}{\partial x_1}|_p,\frac{\partial}{\partial x_2}|_p,\frac{\partial}{\partial x_3}|_p\}$$

则 $T_p^*(R^3)$ 的基向量写作

$$\{dx_{1p},dx_{2p},dx_{3p}\}$$

假设 funtion $f:\mathcal{R}^n\to R$, f 的 differential df 是定义在 R^n 上的 one form, 即 $df\in T_p^*(R^n)$.

$$df(v_p)=v_p[f]=v_1\frac{\partial f}{\partial x_1}|_p+\cdots+v_n\frac{\partial f}{\partial x_n}|_p$$
$$\frac{\partial f}{\partial x_1}|_pdx_1(v_p)+\cdots+\frac{\partial f}{\partial x_n}|_pdx_n(v_p)$$
$$=(\frac{\partial f}{\partial x_1}|_pdx_1+\cdots+\frac{\partial f}{\partial x_n}|_pdx_n)(v_p)$$

即

$$df=\frac{\partial f}{\partial x_1}|_pdx_1+\cdots+\frac{\partial f}{\partial x_n}|_pdx_n$$

对于向量空间 V,W, 如果存在映射 $\phi: V\to W$, 满足

$$\phi(v_1+v_2)=\phi(v_1)+\phi(v_2),\phi(cv)=c\phi(v)$$

其中 $v_1,v_2,v\in V$, 则两向量空间 V,W 是 isomorphics, 记为 $V\simeq W$,

$$manifold\ R^3\simeq T_pR^3\simeq T_p^*R^3\simeq vector\ space\ R^3$$

通过 isomorphic 将向量空间中使用的向量微积分概念推广到流形空间.

wedgeproduct 定义,

$$dx_i\wedge dx_j(v_p,w_p)=\begin{vmatrix}dx_i(v_p)&dx_i(w_p)\\dx_j(v_p)&dx_j(w_p)\end{vmatrix}$$

具体几何意义是, v_p,w_p 投影到 dx_i,dx_j 平面上形成平行四边形的面积.

$$dx_{i_1}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_n}(v_1,v_2,\ldots,v_n)=$$
$$\begin{vmatrix}dx_{i_1}(v_1)&dx_{i_1}(v_2)&\cdots&dx_{i_1}(v_n)\\dx_{i_2}(v_1)&dx_{i_2}(v_2)&\cdots&dx_{i_2}(v_n)\\\vdots&\vdots&\ddots&\vdots\\dx_{i_n}(v_1)&dx_{i_n}(v_2)&\cdots&dx_{i_n}(v_n)\end{vmatrix}$$

$\wedge_p^2(R^3)$ 的基向量为

$$\{dx_1\wedge dx_2,dx_2\wedge dx_3,dx_3\wedge dx_1\}$$

$\alpha\in\wedge^k(R^n)$, 则

$$\alpha=\sum_I a_Idx^I$$

$$I\in J_{k,n}=\{(i_1i_2\ldots i_k|1\le i_1<i_2<\cdots<i_k\le n,i_i\in Z\}$$

举例若 $k=2,n=4$

$$J_{2,4}=\{12,13,14,23,24,34\}$$

$$dx^{i_1i_2\ldots i_k}=dx_{i_1}\wedge dx_{i_2}\wedge dx_{i_3}$$

如果 $\alpha\in\wedge^k(R^n),\beta\in\wedge^l(R^n)$, $\alpha=\sum_I a_Idx^I,\beta=\sum_J b_Jdx^J$

$$\alpha\wedge\beta=\sum a_Ib_Jdx^I\wedge dx^J$$

$$\alpha\wedge\beta(v1,\ldots,v_{k+l})=$$
$$\frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}sgn(\sigma)\alpha(v_{\sigma(1)},\ldots,v_{\sigma(k)})\beta(v_{\sigma(k+1)},\ldots,v_{\sigma(k+l)})$$

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!}\mathcal{A}(\alpha \otimes \beta)$$

其中 $\otimes$ 是张量乘, $\mathcal{A}$ 是 anti-symmetrization 操作

内积, 对于 k-form α, 内积 $l_v\alpha$ 是 k-1 form.

$$l_v\alpha(v1,\ldots,v_{k-1}) = \alpha(v,v_1,\ldots,v_{k-1})$$

$$l_v(\alpha+\beta) = l_v\alpha + l_v\beta$$

$$l_{(v+w)}\alpha = l_v\alpha + l_w\alpha$$

$$l_v(\alpha\wedge\beta) = (l_v\alpha)\wedge\beta + (-1)^k\alpha\wedge(l_v\beta)$$

$$(l_ul_v + l_vl_u)\alpha = 0$$

外微分 exterior derivative, 是 differentiation for differential forms, 还有其他的定义方法, 如 Lie derivative of a form. 假设 α 是 n-form, d 是微分算子.

$$\alpha : T_pM \times T_pM \times \cdots \times T_pM \to R$$

$$d : \wedge^n(M) \to \wedge^{n+1}(M)$$

假设 f 是 zero-form, 则 exterior derivative of f 是

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

假设 α 是 one-form, 则 exterior derivative of α 是

$$d\alpha = \sum df_i \wedge dx_i$$

假设 $\omega = \sum f_{i_1\ldots i_n} dx_{i_1}\wedge\cdots\wedge dx_{i_n}$ 是 n-form, 则 exterior derivative of ω 是

$$d\omega = \sum df_{i_1\ldots i_n}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_n}$$

$$d(\sum \alpha_{i_1\ldots i_n} dx_{i_1}\wedge\cdots\wedge dx_{i_n}) = \sum\sum_{j=1}^{n} \frac{\partial \alpha_{i_1\ldots i_n}}{\partial x_{i_j}} dx_{i_j}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_n}$$

$$\begin{aligned} d\alpha(v,w) =& \langle d\langle\alpha,w\rangle, v\rangle - \langle d\langle\alpha,v\rangle,w\rangle \\ =& v[\alpha(w)] - w[\alpha(v)] \end{aligned}$$

$$\begin{aligned} dw(v_0,\ldots,v_k) &= \sum_i (-1)^i \langle d\langle w,(v0,\ldots,v_i,\ldots,v_k)\rangle, v_i\rangle \\ &= \sum_i (-1)^i v_i[w(v_0,\ldots,v_i,\ldots,v_k)] \end{aligned}$$

$$d\alpha(v,w) = v[\alpha(w)] - w[\alpha(v)] - \alpha([v,w])$$

$$\begin{aligned} d\alpha(v_0,\ldots,v_k) =& \sum_i (-1)^i v_i[\alpha(v_0,\ldots,v_i,\ldots,v_k)] \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([v_i,v_j],v_0,\ldots,v_i,\ldots,v_j,\ldots,v_k) \end{aligned}$$

假设 $f,g,h : R^3 \to R$

$$df\wedge dg\wedge dh = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{vmatrix} dx\wedge dy\wedge dz$$

push-forward of a vector 与 pull-back of a differential form. push forwards of vectors 允许将一个向量从一个流形" 移动" 到另外一个流形. 假设一个映射 $f : R^2 \to R^2$,

$$D_{(x_1,x_2)}f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}_{(x,y)}$$

$$D_{(x_1,x_2)}f : T_{(x_1,x_2)}R^2_{x_1x_2} \to T_{f(x_1,x_2)}R^2_{f_1f_2}$$

$$f : R^2 \to R^2$$

$$p \mapsto f(p)$$

$$T_pf : T_pR^2_{x_1x_2} \to T_{f(p)}R^2_{f_1f_2}$$

$$v_p \mapsto T_pfv_p$$

$$T_pf = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}|_p & \frac{\partial f_1}{\partial x_2}|_p & \cdots & \frac{\partial f_1}{\partial x_n}|_p \\ \frac{\partial f_2}{\partial x_1}|_p & \frac{\partial f_2}{\partial x_2}|_p & \cdots & \frac{\partial f_2}{\partial x_n}|_p \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}|_p & \frac{\partial f_n}{\partial x_2}|_p & \cdots & \frac{\partial f_n}{\partial x_n}|_p \end{pmatrix}$$

$$f : M \to N$$

$$T_pf : T_pM \to T_{f(p)}N$$

differential form 的 pull-back, 假设 w 是 k-form

$$(T^*fw)(v_1,v_2,\ldots,v_k) = w(Tfv_1,Tfv_2,\ldots,Tfv_k)$$

$$f : M \to N$$

$$T_pf : T_pM \to T_{f(p)}N$$

$$T_p^*f : T_{f(p)}^*N \to T_p^*M$$

对于映射 $\phi : R^n_{(x1,\ldots,x_n)} \to R^n_{(\phi_1,\ldots,\phi_n)}$

$$\begin{aligned} &T^*\phi(d\phi_1\wedge\cdots\wedge d\phi_n) \\ &= \begin{vmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_n} \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \frac{\partial\phi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \cdots \\ \frac{\partial\phi_n}{\partial x_1} & \frac{\partial\phi_n}{\partial x_2} & \cdots & \frac{\partial\phi_n}{\partial x_n} \end{vmatrix} dx_1\wedge\cdots\wedge dx_n \end{aligned}$$

假设 R^3 上向量场 $F = Pe_1 + Qe_2 + Re_3$, 其中 $P,Q,R : R^3 \to R$, 定义算子 $\nabla = \frac{\partial}{\partial x}e_1 + \frac{\partial}{\partial y}e_2 + \frac{\partial}{\partial z}e_3$

$$div\ F = \nabla\cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\int_{\partial V} F \cdot dS = \int_V div\ F dV$$

$$curl\ F = \nabla \times F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})e_1 + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})e_2 + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})e_3$$

$$\int_{\partial S} F \cdot ds = \int_S curl\ F dS$$

$$grad\ f \cdot u = u[f] = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$$

$$\flat : T_pM \to T_p^*M$$
$$v^i \frac{\partial}{\partial x^i} \mapsto v_i dx^i$$

$$\sharp : T_p^*M \to T_pM$$
$$\alpha_i dx^i \mapsto \alpha^i \frac{\partial}{\partial x^i}$$

$$* : \wedge^0(R^3) \to \wedge^3(R^3)$$
$$* : \wedge^1(R^3) \to \wedge^2(R^3)$$
$$* : \wedge^2(R^3) \to \wedge^1(R^3)$$
$$* : \wedge^3(R^3) \to \wedge^0(R^3)$$

$$\begin{aligned} &* 1 = dx^1 \wedge dx^2 \wedge dx^3 = dx \wedge dy \wedge dz \\ &* dx = dy \wedge dz, *dy = dz \wedge dx, *dz = dx \wedge dy \\ &* dy \wedge dz = dx, *dz \wedge dx = dy, *dx \wedge dy = dz, \\ &* dx \wedge dy \wedge dz = 1 \end{aligned}$$

$$\begin{aligned} (* \circ \flat)F &= *(F^\flat) \\ &= *((P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z})^\flat) \\ &= *(Pdx + Qdy + Rdz) \\ &= Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \end{aligned}$$

$$\begin{tikzcd} C(\mathbb{R}^3) \arrow[r, "\text{grad}"] \arrow[d, "\text{id}"] & T\mathbb{R}^3 \arrow[r, "\text{curl}"] \arrow[d, "\flat"] & T\mathbb{R}^3 \arrow[r, "\text{div}"] \arrow[d, "*\circ\flat"] & C(\mathbb{R}^3) \arrow[d, "*"] \\ \bigwedge^0(\mathbb{R}^3) \arrow[r, "d"] & \bigwedge^1(\mathbb{R}^3) \arrow[r, "d"] & \bigwedge^2(\mathbb{R}^3) \arrow[r, "d"] & \bigwedge^3(\mathbb{R}^3) \end{tikzcd}$$

$$grad\ f = (df)^\sharp$$

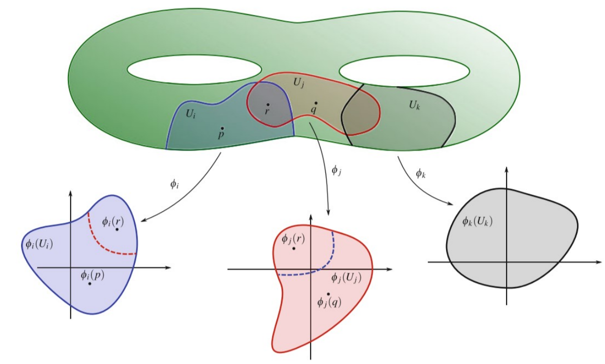
$$curl\ F = [*(df\flat)]^\sharp$$

$$div\ F = *d(*(F^\flat))$$

Fund. Thm. Line Integrals	Stokes' Theorem	Divergence Theorem
$f(c(b)) - f(c(a)) = \int_C \nabla f \cdot d\mathbf{s}$	$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$	$\int_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F}\, dV$
$\Downarrow$	$\Downarrow$	$\Downarrow$
$\int_{\partial C} \alpha = \int_C d\alpha$	$\int_{\partial S} \alpha = \int_S d\alpha$	$\int_{\partial V} \alpha = \int_V d\alpha.$

定义

$$U_i : \text{coordinate neighborhood}$$
$$\phi_i : U_i \to R^n : \text{coordinate map}$$
$$(U_i, \phi_i) : \text{coordinate patch/chart}$$
$$\{(U_i, \phi_i)\} : \text{coordinate system/atlas}$$
$$\phi_j \circ \phi_i^{-1} : R^n \to R^n : \text{transition function}$$



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3 数字几何处理