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1 LIMITS AND CONTINUITY

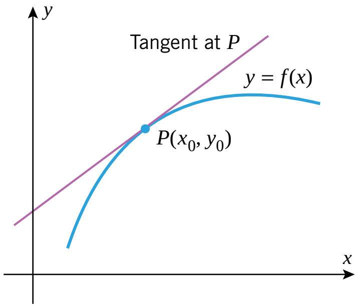
Air resistance prevents the velocity of a skydiver from increasing indefinitely. The velocity approaches a limit, called the "terminal velocity."

The development of calculus in the seventeenth century by Newton and Leibniz provided scientists with their first real understanding of what is meant by an "instantaneous rate of change" such as velocity and acceleration. Once the idea was understood conceptually, efficient computational methods followed, and science took a quantum leap forward. The fundamental building block on which rates of change rest is the concept of a "limit," an idea that is so important that all other calculus concepts are now based on it.

In this chapter we will develop the concept of a limit in stages, proceeding from an informal, intuitive notion to a precise mathematical definition. We will also develop theorems and procedures for calculating limits, and we will conclude the chapter by using the limits to study "continuous" curves.

1.1 LIMITS (AN INTUITIVE APPROACH)

The concept of a "limit" is the fundamental building block on which all calculus concepts are based. In this section we will study limits informally, with the goal of developing an intuitive feel for the basic ideas. In the next three sections we will focus on computational methods and precise definitions.



▲ **Figure 1.1.1**

Many of the ideas of calculus originated with the following two geometric problems:

THE TANGENT LINE PROBLEM Given a function f and a point $P(x_0, y_0)$ on its graph, find an equation of the line that is tangent to the graph at P (Figure 1.1.1).

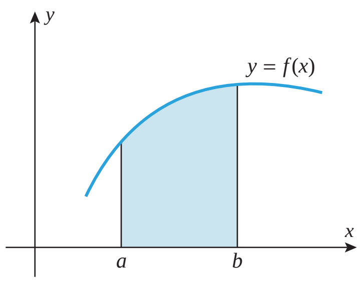
THE AREA PROBLEM Given a function f, find the area between the graph of f and an interval $[a, b]$ on the x-axis (Figure 1.1.2).

Traditionally, that portion of calculus arising from the tangent line problem is called ***differential calculus*** and that arising from the area problem is called ***integral calculus***. However, we will see later that the tangent line and area problems are so closely related that the distinction between differential and integral calculus is somewhat artificial.

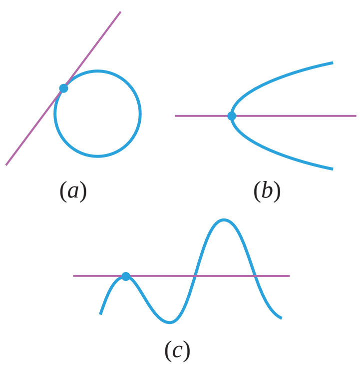
TANGENT LINES AND LIMITS

In plane geometry, a line is called *tangent* to a circle if it meets the circle at precisely one point (Figure 1.1.3*a*). Although this definition is adequate for circles, it is not appropriate for more general curves. For example, in Figure 1.1.3*b*, the line meets the curve exactly once but is obviously not what we would regard to be a tangent line; and in Figure 1.1.3*c*, the line appears to be tangent to the curve, yet it intersects the curve more than once.

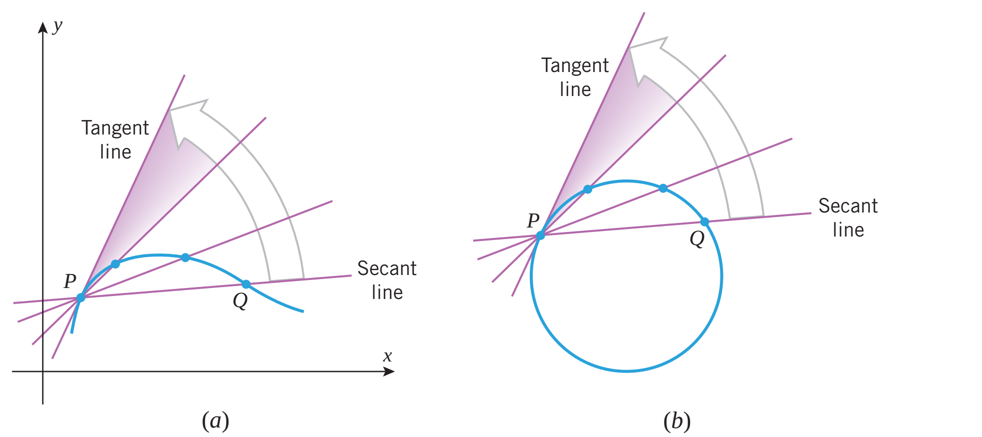
To obtain a definition of a tangent line that applies to curves other than circles, we must view tangent lines another way. For this purpose, suppose that we are interested in the tangent line at a point P on a curve in the xy-plane and that Q is any point that lies on the curve and is different from P. The line through P and Q is called a ***secant line*** for the curve at P. Intuition suggests that if we move the point Q along the curve toward P, then the secant line will rotate toward a *limiting position*. The line in this limiting position is what we will consider to be the ***tangent line*** at P (Figure 1.1.4*a*). As suggested by Figure 1.1.4*b*, this new concept of a tangent line coincides with the traditional concept when applied to circles.



▲ **Figure 1.1.2**



▲ **Figure 1.1.3**



▶ **Figure 1.1.4**

▶ Example 1 Find an equation for the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution. If we can find the slope $m_{\tan}$ of the tangent line at P, then we can use the point P and the point-slope formula for a line (Web Appendix G) to write the equation of the tangent line as

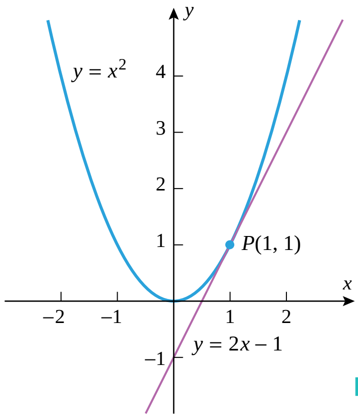
$$y - 1 = m_{\tan}(x - 1) \tag{1}$$

To find the slope $m_{\tan}$, consider the secant line through P and a point $Q(x, x^2)$ on the parabola that is distinct from P. The slope $m_{\sec}$ of this secant line is

$$m_{\sec} = \frac{x^2 - 1}{x - 1} \tag{2}$$

Why are we requiring that P and Q be distinct?

Figure 1.1.4*a* suggests that if we now let Q move along the parabola, getting closer and closer to P, then the limiting position of the secant line through P and Q will coincide with that of the tangent line at P. This in turn suggests that the value of $m_{\sec}$ will get closer and closer to the value of $m_{\tan}$ as P moves toward Q along the curve. However, to say that $Q(x, x^2)$ gets closer and closer to $P(1, 1)$ is algebraically equivalent to saying that x gets closer and closer to 1. Thus, the problem of finding $m_{\tan}$ reduces to finding the "limiting value" of $m_{\sec}$ in Formula (2) as x gets closer and closer to 1 (but with $x \neq 1$ to ensure that P and Q remain distinct).



▲ **Figure 1.1.5**

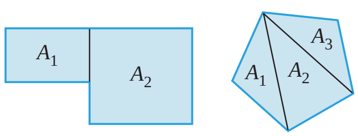
We can rewrite (2) as

$$m_{\text{sec}} = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{(x-1)} = x + 1$$

where the cancellation of the factor $(x - 1)$ is allowed because $x \neq 1$. It is now evident that m_{sec} gets closer and closer to 2 as x gets closer and closer to 1. Thus, $m_{\tan} = 2$ and (1) implies that the equation of the tangent line is

$$y - 1 = 2(x - 1) \quad \text{or equivalently} \quad y = 2x - 1$$

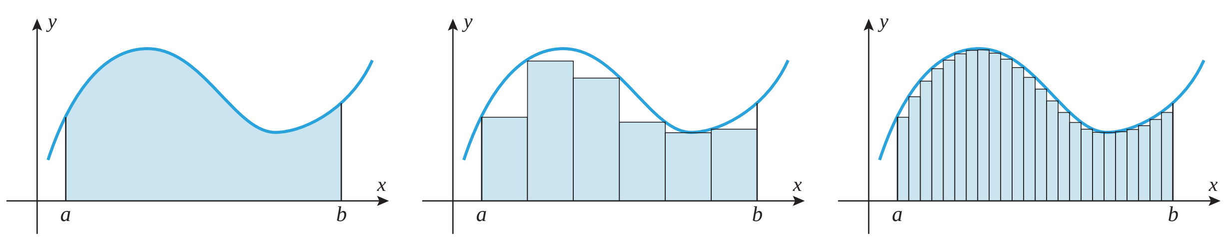
Figure 1.1.5 shows the graph of $y = x^2$ and this tangent line. ◀



▲ **Figure 1.1.6**

AREAS AND LIMITS

Just as the general notion of a tangent line leads to the concept of *limit*, so does the general notion of area. For plane regions with straight-line boundaries, areas can often be calculated by subdividing the region into rectangles or triangles and adding the areas of the constituent parts (Figure 1.1.6). However, for regions with curved boundaries, such as that in Figure 1.1.7*a*, a more general approach is needed. One such approach is to begin by approximating the area of the region by inscribing a number of rectangles of equal width under the curve and adding the areas of these rectangles (Figure 1.1.7*b*). Intuition suggests that if we repeat that approximation process using more and more rectangles, then the rectangles will tend to fill in the gaps under the curve, and the approximations will get closer and closer to the exact area under the curve (Figure 1.1.7*c*). This suggests that we can define the area under the curve to be the limiting value of these approximations. This idea will be considered in detail later, but the point to note here is that once again the concept of a limit comes into play.



(*a*) (*b*) (*c*)

▲ **Figure 1.1.7**

DECIMALS AND LIMITS

Limits also arise in the familiar context of decimals. For example, the decimal expansion of the fraction $\frac{1}{3}$ is

$$\frac{1}{3} = 0.33333\ldots \tag{3}$$

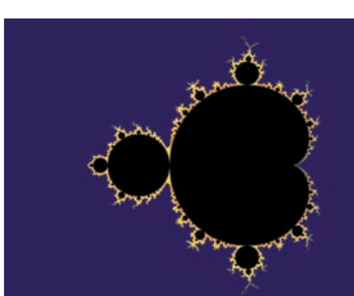
in which the dots indicate that the digit 3 repeats indefinitely. Although you may not have thought about decimals in this way, we can write (3) as

$$\frac{1}{3} = 0.33333\ldots = 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots \tag{4}$$

which is a sum with "infinitely many" terms. As we will discuss in more detail later, we interpret (4) to mean that the succession of finite sums

$$0.3, \quad 0.3 + 0.03, \quad 0.3 + 0.03 + 0.003, \quad 0.3 + 0.03 + 0.003 + 0.0003, \ldots$$

gets closer and closer to a limiting value of $\frac{1}{3}$ as more and more terms are included. Thus, limits even occur in the familiar context of decimal representations of real numbers.



*This figure shows a region called the **Mandelbrot Set**. It illustrates how complicated a region in the plane can be and why the notion of area requires careful definition.*

LIMITS

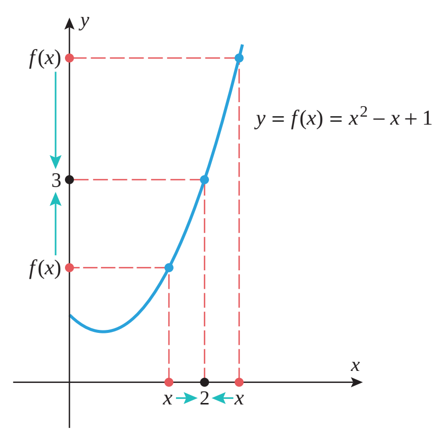
Now that we have seen how limits arise in various ways, let us focus on the limit concept itself.

The most basic use of limits is to describe how a function behaves as the independent variable approaches a given value. For example, let us examine the behavior of the function

$$f(x) = x^2 - x + 1$$

for x-values closer and closer to 2. It is evident from the graph and table in Figure 1.1.8 that the values of $f(x)$ get closer and closer to 3 as values of x are selected closer and closer to 2 on either the left or the right side of 2. We describe this by saying that the "limit of $x^2 - x + 1$ is 3 as x approaches 2 from either side," and we write

$$\lim_{x \to 2} (x^2 - x + 1) = 3 \tag{5}$$



x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001		3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

Left side → ← Right side

▲ **Figure 1.1.8**

This leads us to the following general idea.

1.1.1 LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L \tag{6}$$

which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as x approaches a." The expression in (6) can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a \tag{7}$$

Since x is required to be different from a in (6), the value of f at a, or even whether f is defined at a, has no bearing on the limit L. The limit describes the behavior of f *close to* a but not *at* a.

► Example 2 Use numerical evidence to make a conjecture about the value of

$$\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} \tag{8}$$

Solution. Although the function

$$f(x) = \frac{x-1}{\sqrt{x}-1} \tag{9}$$

is undefined at $x = 1$, this has no bearing on the limit. Table 1.1.1 shows sample x-values approaching 1 from the left side and from the right side. In both cases the corresponding values of $f(x)$, calculated to six decimal places, appear to get closer and closer to 2, and hence we conjecture that

$$\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} = 2$$

This is consistent with the graph of f shown in Figure 1.1.9. In the next section we will show how to obtain this result algebraically. ◄

TECHNOLOGY MASTERY

Use a graphing utility to generate the graph of the equation $y = f(x)$ for the function in (9). Find a window containing $x = 1$ in which all values of $f(x)$ are within 0.5 of $y = 2$ and one in which all values of $f(x)$ are within 0.1 of $y = 2$.

Table 1.1.1

x	0.99	0.999	0.9999	0.99999		1.00001	1.0001	1.001	1.01
$f(x)$	1.994987	1.999500	1.999950	1.999995		2.000005	2.000050	2.000500	2.004988

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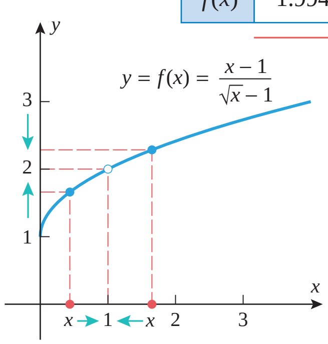


▲ **Figure 1.1.9**

► Example 3 Use numerical evidence to make a conjecture about the value of

$$\lim_{x \to 0} \frac{\sin x}{x} \tag{10}$$

Solution. With the help of a calculating utility set in radian mode, we obtain Table 1.1.2. The data in the table suggest that

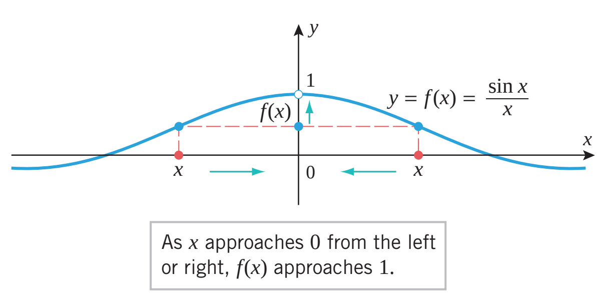
$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{11}$$

The result is consistent with the graph of $f(x) = (\sin x)/x$ shown in Figure 1.1.10. Later in this chapter we will give a geometric argument to prove that our conjecture is correct. ◄

Use numerical evidence to determine whether the limit in (11) changes if x is measured in degrees.

Table 1.1.2

x (RADIANS)	$y = \frac{\sin x}{x}$
±1.0	0.84147
±0.9	0.87036
±0.8	0.89670
±0.7	0.92031
±0.6	0.94107
±0.5	0.95885
±0.4	0.97355
±0.3	0.98507
±0.2	0.99335
±0.1	0.99833
±0.01	0.99998



► **Figure 1.1.10**

SAMPLING PITFALLS

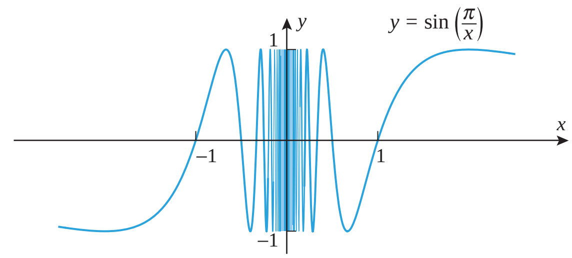
Numerical evidence can sometimes lead to incorrect conclusions about limits because of roundoff error or because the sample values chosen do not reveal the true limiting behavior. For example, one might *incorrectly* conclude from Table 1.1.3 that

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = 0$$

The fact that this is not correct is evidenced by the graph of f in Figure 1.1.11. The graph reveals that the values of f oscillate between -1 and 1 with increasing rapidity as $x \to 0$ and hence do not approach a limit. The data in the table deceived us because the x-values selected all happened to be x-intercepts for $f(x)$. This points out the need for having alternative methods for corroborating limits conjectured from numerical evidence.

Table 1.1.3

x	$\dfrac{\pi}{x}$	$f(x) = \sin\left(\dfrac{\pi}{x}\right)$
$x = \pm 1$	$\pm\pi$	$\sin(\pm\pi) = 0$
$x = \pm 0.1$	$\pm 10\pi$	$\sin(\pm 10\pi) = 0$
$x = \pm 0.01$	$\pm 100\pi$	$\sin(\pm 100\pi) = 0$
$x = \pm 0.001$	$\pm 1000\pi$	$\sin(\pm 1000\pi) = 0$
$x = \pm 0.0001$	$\pm 10{,}000\pi$	$\sin(\pm 10{,}000\pi) = 0$
$\vdots$	$\vdots$	$\vdots$

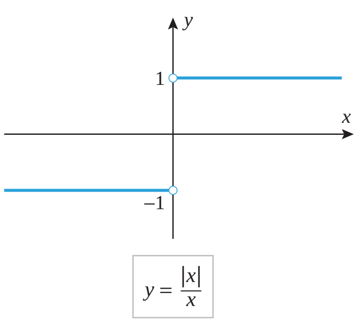


▲ **Figure 1.1.11**

ONE-SIDED LIMITS

The limit in (6) is called a ***two-sided limit*** because it requires the values of $f(x)$ to get closer and closer to L as values of x are taken from *either* side of $x = a$. However, some functions exhibit different behaviors on the two sides of an x-value a, in which case it is necessary to distinguish whether values of x near a are on the left side or on the right side of a for purposes of investigating limiting behavior. For example, consider the function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \tag{12}$$



▲ **Figure 1.1.12**

which is graphed in Figure 1.1.12. As x approaches 0 from the *right*, the values of $f(x)$ approach a limit of 1 [in fact, the values of $f(x)$ are exactly 1 for all such x], and similarly, as x approaches 0 from the *left*, the values of $f(x)$ approach a limit of -1. We denote these limits by writing

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1 \tag{13}$$

With this notation, the superscript "+" indicates a limit from the right and the superscript "−" indicates a limit from the left.

This leads to the general idea of a ***one-sided limit***.

As with two-sided limits, the one-sided limits in (14) and (15) can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a^+$$

and

$$f(x) \to L \quad \text{as} \quad x \to a^-$$

respectively.

1.1.2 ONE-SIDED LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\lim_{x \to a^+} f(x) = L \tag{14}$$

and if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x \to a^-} f(x) = L \tag{15}$$

Expression (14) is read "the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right." Similarly, expression (15) is read "the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left."

THE RELATIONSHIP BETWEEN ONE-SIDED LIMITS AND TWO-SIDED LIMITS

In general, there is no guarantee that a function f will have a two-sided limit at a given point a; that is, the values of $f(x)$ may not get closer and closer to any *single* real number L as $x \to a$. In this case we say that

$$\lim_{x\to a} f(x) \quad \textbf{\textit{does not exist}}$$

Similarly, the values of $f(x)$ may not get closer and closer to a single real number L as $x \to a^+$ or as $x \to a^-$. In these cases we say that

$$\lim_{x\to a^+} f(x) \quad \textbf{\textit{does not exist}}$$

or that

$$\lim_{x\to a^-} f(x) \quad \textbf{\textit{does not exist}}$$

In order for the two-sided limit of a function $f(x)$ to exist at a point a, the values of $f(x)$ must approach some real number L as x approaches a, and this number must be the same regardless of whether x approaches a from the left or the right. This suggests the following result, which we state without formal proof.

1.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is,

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

Example 4 Explain why

$$\lim_{x\to 0} \frac{|x|}{x}$$

does not exist.

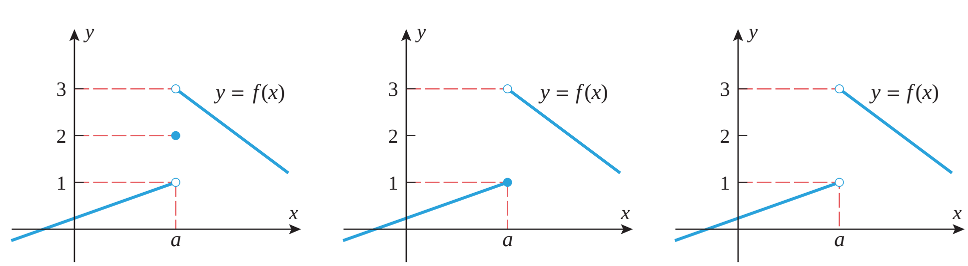
Solution. As x approaches 0, the values of $f(x) = |x|/x$ approach -1 from the left and approach 1 from the right [see (13)]. Thus, the one-sided limits at 0 are not the same. ◂

Example 5 For the functions in Figure 1.1.13, find the one-sided and two-sided limits at $x = a$ if they exist.

Solution. The functions in all three figures have the same one-sided limits as $x \to a$, since the functions are identical, except at $x = a$. These limits are

$$\lim_{x\to a^+} f(x) = 3 \quad \text{and} \quad \lim_{x\to a^-} f(x) = 1$$

In all three cases the two-sided limit does not exist as $x \to a$ because the one-sided limits are not equal. ◂



▸ **Figure 1.1.13**

Example 6 For the functions in Figure 1.1.14, find the one-sided and two-sided limits at $x = a$ if they exist.

Solution. As in the preceding example, the value of f at $x = a$ has no bearing on the limits as $x \to a$, so in all three cases we have

$$\lim_{x \to a^+} f(x) = 2 \quad \text{and} \quad \lim_{x \to a^-} f(x) = 2$$

Since the one-sided limits are equal, the two-sided limit exists and

$$\lim_{x \to a} f(x) = 2 \blacktriangleleft$$

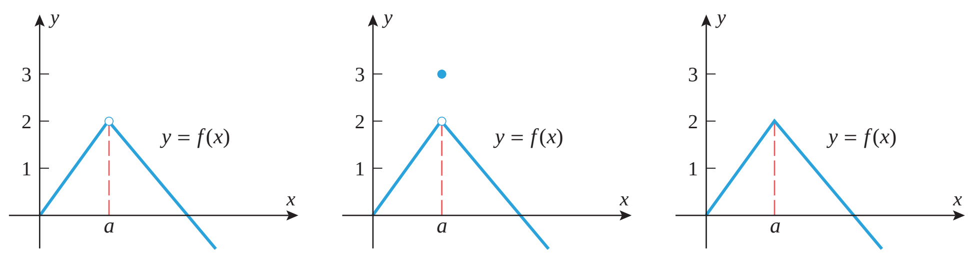


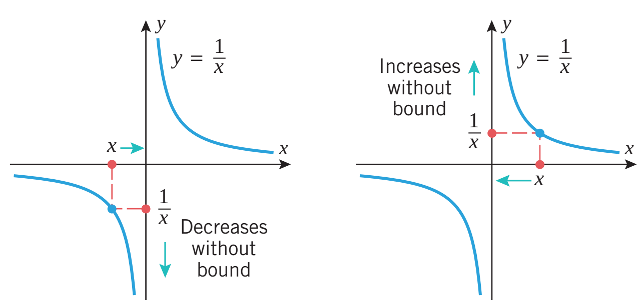
Figure 1.1.14

INFINITE LIMITS

Sometimes one-sided or two-sided limits fail to exist because the values of the function increase or decrease without bound. For example, consider the behavior of $f(x) = 1/x$ for values of x near 0. It is evident from the table and graph in Figure 1.1.15 that as x-values are taken closer and closer to 0 from the right, the values of $f(x) = 1/x$ are positive and increase without bound; and as x-values are taken closer and closer to 0 from the left, the values of $f(x) = 1/x$ are negative and decrease without bound. We describe these limiting behaviors by writing

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

The symbols $+\infty$ and $-\infty$ here are *not* real numbers; they simply describe particular ways in which the limits fail to exist. Do not make the mistake of manipulating these symbols using rules of algebra. For example, it is *incorrect* to write $(+\infty) - (+\infty) = 0$.



x	−1	−0.1	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01	0.1	1
$\frac{1}{x}$	−1	−10	−100	−1000	−10,000		10,000	1000	100	10	1

Left side → ← Right side

Figure 1.1.15

1.1.4 INFINITE LIMITS (AN INFORMAL VIEW) The expressions

$$\lim_{x \to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = +\infty$$

Similarly, the expressions

$$\lim_{x \to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = -\infty$$

▶ **Example 7** For the functions in Figure 1.1.16, describe the limits at $x = a$ in appropriate limit notation.

Solution (a). In Figure 1.1.16*a*, the function increases without bound as x approaches a from the right and decreases without bound as x approaches a from the left. Thus,

$$\lim_{x \to a^+} \frac{1}{x-a} = +\infty \quad \text{and} \quad \lim_{x \to a^-} \frac{1}{x-a} = -\infty$$

Solution (b). In Figure 1.1.16*b*, the function increases without bound as x approaches a from both the left and right. Thus,

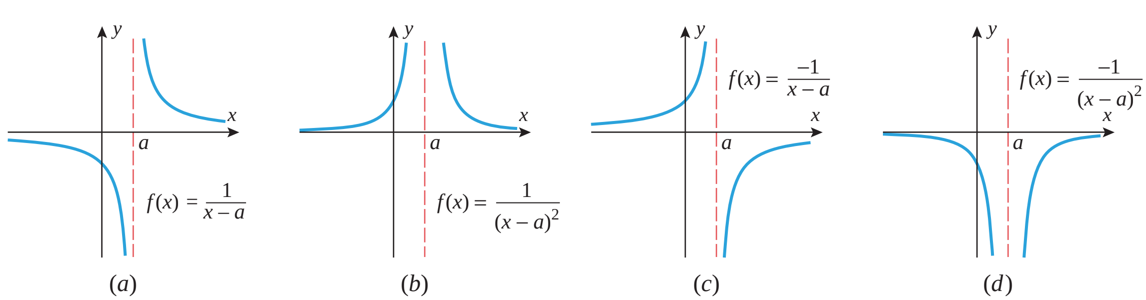
$$\lim_{x \to a} \frac{1}{(x-a)^2} = \lim_{x \to a^+} \frac{1}{(x-a)^2} = \lim_{x \to a^-} \frac{1}{(x-a)^2} = +\infty$$

Solution (c). In Figure 1.1.16*c*, the function decreases without bound as x approaches a from the right and increases without bound as x approaches a from the left. Thus,

$$\lim_{x \to a^+} \frac{-1}{x-a} = -\infty \quad \text{and} \quad \lim_{x \to a^-} \frac{-1}{x-a} = +\infty$$

Solution (d). In Figure 1.1.16*d*, the function decreases without bound as x approaches a from both the left and right. Thus,

$$\lim_{x \to a} \frac{-1}{(x-a)^2} = \lim_{x \to a^+} \frac{-1}{(x-a)^2} = \lim_{x \to a^-} \frac{-1}{(x-a)^2} = -\infty \quad ◀$$



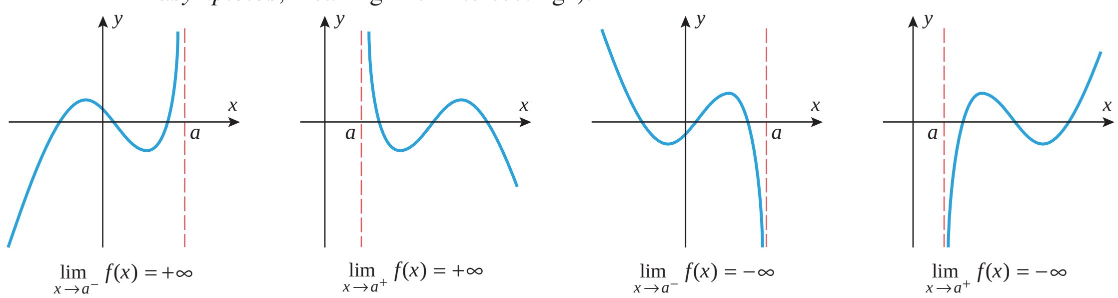
▲ **Figure 1.1.16**

VERTICAL ASYMPTOTES

Figure 1.1.17 illustrates geometrically what happens when any of the following situations occur:

$$\lim_{x \to a^-} f(x) = +\infty, \quad \lim_{x \to a^+} f(x) = +\infty, \quad \lim_{x \to a^-} f(x) = -\infty, \quad \lim_{x \to a^+} f(x) = -\infty$$

In each case the graph of $y = f(x)$ either rises or falls without bound, squeezing closer and closer to the vertical line $x = a$ as x approaches a from the side indicated in the limit. The line $x = a$ is called a ***vertical asymptote*** of the curve $y = f(x)$ (from the Greek word *asymptotos*, meaning "nonintersecting").

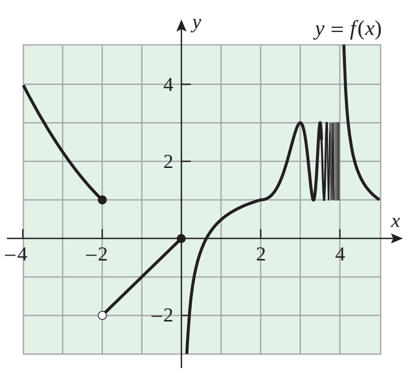


▲ **Figure 1.1.17**

In general, the graph of a single function can display a wide variety of limits.

▶ **Example 8** For the function f graphed in Figure 1.1.18, find

(a) $\lim_{x \to -2^-} f(x)$ (b) $\lim_{x \to -2^+} f(x)$ (c) $\lim_{x \to 0^-} f(x)$ (d) $\lim_{x \to 0^+} f(x)$
(e) $\lim_{x \to 4^-} f(x)$ (f) $\lim_{x \to 4^+} f(x)$ (g) the vertical asymptotes of the graph of f.



▲ **Figure 1.1.18**

Solution (a) and (b).

$$\lim_{x \to -2^-} f(x) = 1 = f(-2) \quad \text{and} \quad \lim_{x \to -2^+} f(x) = -2$$

Solution (c) and (d).

$$\lim_{x \to 0^-} f(x) = 0 = f(0) \quad \text{and} \quad \lim_{x \to 0+} f(x) = -\infty$$

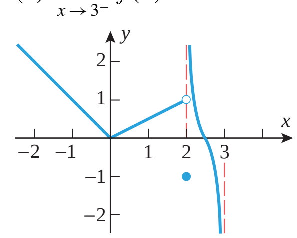
Solution (e) and (f).

$$\lim_{x \to 4^-} f(x) \text{ does not exist due to oscillation} \quad \text{and} \quad \lim_{x \to 4^+} f(x) = +\infty$$

Solution (g). The y-axis and the line $x = 4$ are vertical asymptotes for the graph of f. ◀

✔ QUICK CHECK EXERCISES 1.1 *(See page 79 for answers.)*

1. We write $\lim_{x \to a} f(x) = L$ provided the values of ________ can be made as close to ________ as desired, by taking values of ________ sufficiently close to ________ but not ________.
2. We write $\lim_{x \to a^-} f(x) = +\infty$ provided ________ increases without bound, as ________ approaches ________ from the left.
3. State what must be true about
$$\lim_{x \to a^-} f(x) \quad \text{and} \quad \lim_{x \to a^+} f(x)$$
in order for it to be the case that
$$\lim_{x \to a} f(x) = L$$
4. Use the accompanying graph of $y = f(x)$ $(-\infty < x < 3)$ to determine the limits.
(a) $\lim_{x \to 0} f(x) =$ ________
(b) $\lim_{x \to 2^-} f(x) =$ ________
(c) $\lim_{x \to 2^+} f(x) =$ ________
(d) $\lim_{x \to 3^-} f(x) =$ ________



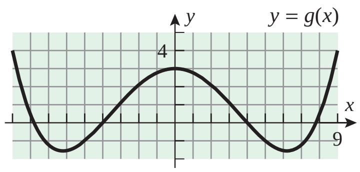
◀ **Figure Ex-4**

5. The slope of the secant line through $P(2, 4)$ and $Q(x, x^2)$ on the parabola $y = x^2$ is $m_{\text{sec}} = x + 2$. It follows that the slope of the tangent line to this parabola at the point P is ________.

EXERCISE SET 1.1 Graphing Utility C CAS

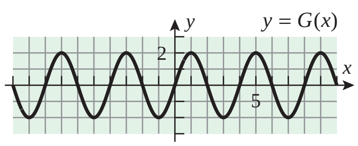
1–10 In these exercises, make reasonable assumptions about the graph of the indicated function outside of the region depicted. ■

1. For the function g graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} g(x)$ (b) $\lim_{x\to 0^+} g(x)$
(c) $\lim_{x\to 0} g(x)$ (d) $g(0)$.



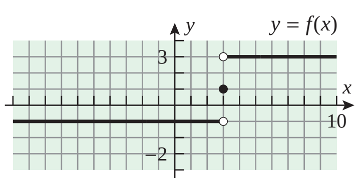
◀ **Figure Ex-1**

2. For the function G graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} G(x)$ (b) $\lim_{x\to 0^+} G(x)$
(c) $\lim_{x\to 0} G(x)$ (d) $G(0)$.



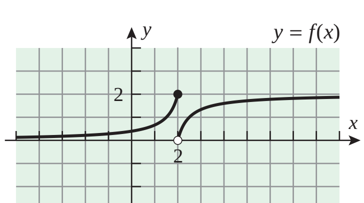
◀ **Figure Ex-2**

3. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$
(c) $\lim_{x\to 3} f(x)$ (d) $f(3)$.



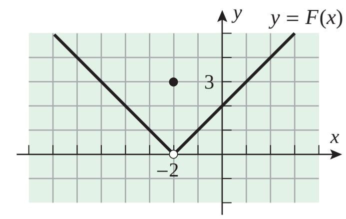
◀ **Figure Ex-3**

4. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$
(c) $\lim_{x\to 2} f(x)$ (d) $f(2)$.



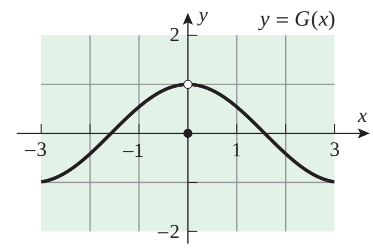
◀ **Figure Ex-4**

5. For the function F graphed in the accompanying figure, find
(a) $\lim_{x\to -2^-} F(x)$ (b) $\lim_{x\to -2^+} F(x)$
(c) $\lim_{x\to -2} F(x)$ (d) $F(-2)$.



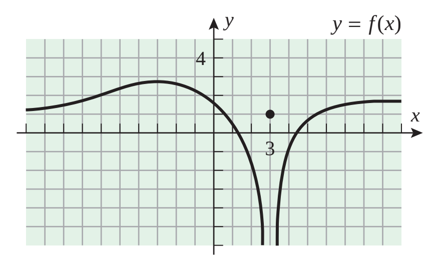
◀ **Figure Ex-5**

6. For the function G graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} G(x)$ (b) $\lim_{x\to 0^+} G(x)$
(c) $\lim_{x\to 0} G(x)$ (d) $G(0)$.



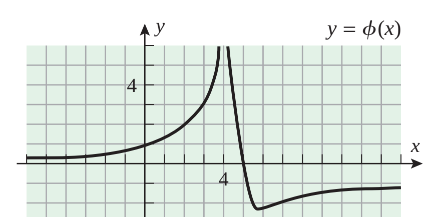
◀ **Figure Ex-6**

7. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$
(c) $\lim_{x\to 3} f(x)$ (d) $f(3)$.



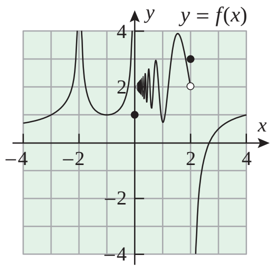
◀ **Figure Ex-7**

8. For the function ϕ graphed in the accompanying figure, find
(a) $\lim_{x\to 4^-} \phi(x)$ (b) $\lim_{x\to 4^+} \phi(x)$
(c) $\lim_{x\to 4} \phi(x)$ (d) $\phi(4)$.



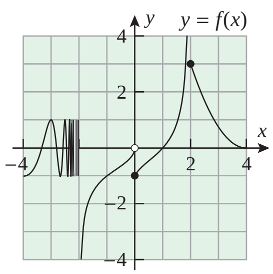
◀ **Figure Ex-8**

9. For the function f graphed in the accompanying figure on the next page, find
(a) $\lim_{x\to -2} f(x)$ (b) $\lim_{x\to 0^-} f(x)$
(c) $\lim_{x\to 0^+} f(x)$ (d) $\lim_{x\to 2^-} f(x)$
(e) $\lim_{x\to 2^+} f(x)$
(f) the vertical asymptotes of the graph of f.



◀ **Figure Ex-9**

10. For the function f graphed in the accompanying figure, find
(a) $\lim_{x \to -2^-} f(x)$ (b) $\lim_{x \to -2^+} f(x)$ (c) $\lim_{x \to 0^-} f(x)$
(d) $\lim_{x \to 0^+} f(x)$ (e) $\lim_{x \to 2^-} f(x)$ (f) $\lim_{x \to 2^+} f(x)$
(g) the vertical asymptotes of the graph of f.



◀ **Figure Ex-10**

11–12 (i) Complete the table and make a guess about the limit indicated. (ii) Confirm your conclusions about the limit by graphing a function over an appropriate interval. [*Note:* For the inverse trigonometric function, be sure to put your calculating and graphing utilities in radian mode.]

11. $f(x) = \dfrac{e^x - 1}{x}$; $\lim_{x \to 0} f(x)$

x	−0.01	−0.001	−0.0001	0.0001	0.001	0.01
$f(x)$						

▲ **Table Ex-11**

12. $f(x) = \dfrac{\sin^{-1} 2x}{x}$; $\lim_{x \to 0} f(x)$

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$						

▲ **Table Ex-12**

13–16 (i) Make a guess at the limit (if it exists) by evaluating the function at the specified x-values. (ii) Confirm your conclusions about the limit by graphing the function over an appropriate interval. (iii) If you have a CAS, then use it to find the limit. [*Note:* For the trigonometric functions, be sure to put your calculating and graphing utilities in radian mode.]

13. (a) $\lim_{x \to 1} \dfrac{x - 1}{x^3 - 1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 0, 0.5, 0.9, 0.99, 0.999$
(b) $\lim_{x \to 1^+} \dfrac{x + 1}{x^3 - 1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 1.0001$
(c) $\lim_{x \to 1^-} \dfrac{x + 1}{x^3 - 1}$; $x = 0, 0.5, 0.9, 0.99, 0.999, 0.9999$

14. (a) $\lim_{x \to 0} \dfrac{\sqrt{x+1} - 1}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$
(b) $\lim_{x \to 0^+} \dfrac{\sqrt{x+1} + 1}{x}$; $x = 0.25, 0.1, 0.001, 0.0001$
(c) $\lim_{x \to 0^-} \dfrac{\sqrt{x+1} + 1}{x}$; $x = -0.25, -0.1, -0.001, -0.0001$

15. (a) $\lim_{x \to 0} \dfrac{\sin 3x}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$
(b) $\lim_{x \to -1} \dfrac{\cos x}{x + 1}$; $x = 0, -0.5, -0.9, -0.99, -0.999, -1.5, -1.1, -1.01, -1.001$

16. (a) $\lim_{x \to -1} \dfrac{\tan(x + 1)}{x + 1}$; $x = 0, -0.5, -0.9, -0.99, -0.999, -1.5, -1.1, -1.01, -1.001$
(b) $\lim_{x \to 0} \dfrac{\sin(5x)}{\sin(2x)}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

17–20 True–False Determine whether the statement is true or false. Explain your answer.

17. If $f(a) = L$, then $\lim_{x \to a} f(x) = L$.

18. If $\lim_{x \to a} f(x)$ exists, then so do $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$.

19. If $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist, then so does $\lim_{x \to a} f(x)$.

20. If $\lim_{x \to a^+} f(x) = +\infty$, then $f(a)$ is undefined.

21–26 Sketch a possible graph for a function f with the specified properties. (Many different solutions are possible.)

21. (i) the domain of f is $[-1, 1]$
(ii) $f(-1) = f(0) = f(1) = 0$
(iii) $\lim_{x \to -1^+} f(x) = \lim_{x \to 0} f(x) = \lim_{x \to 1^-} f(x) = 1$

22. (i) the domain of f is $[-2, 1]$
(ii) $f(-2) = f(0) = f(1) = 0$
(iii) $\lim_{x \to -2^+} f(x) = 2$, $\lim_{x \to 0} f(x) = 0$, and $\lim_{x \to 1^-} f(x) = 1$

23. (i) the domain of f is $(-\infty, 0]$
(ii) $f(-2) = f(0) = 1$
(iii) $\lim_{x \to -2} f(x) = +\infty$

24. (i) the domain of f is $(0, +\infty)$
(ii) $f(1) = 0$
(iii) the y-axis is a vertical asymptote for the graph of f
(iv) $f(x) < 0$ if $0 < x < 1$

25. (i) $f(-3) = f(0) = f(2) = 0$

(ii) $\lim_{x \to -2^-} f(x) = +\infty$ and $\lim_{x \to -2^+} f(x) = -\infty$

(iii) $\lim_{x \to 1} f(x) = +\infty$

26. (i) $f(-1) = 0,\ f(0) = 1,\ f(1) = 0$

(ii) $\lim_{x \to -1^-} f(x) = 0$ and $\lim_{x \to -1^+} f(x) = +\infty$

(iii) $\lim_{x \to 1^-} f(x) = 1$ and $\lim_{x \to 1^+} f(x) = +\infty$

27–30 Modify the argument of Example 1 to find the equation of the tangent line to the specified graph at the point given. ■

27. the graph of $y = x^2$ at $(-1, 1)$

28. the graph of $y = x^2$ at $(0, 0)$

29. the graph of $y = x^4$ at $(1, 1)$

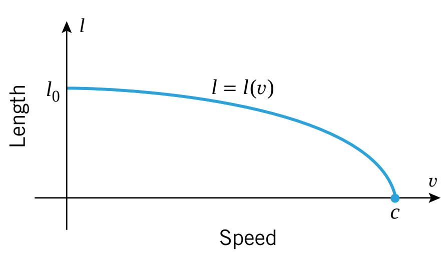
30. the graph of $y = x^4$ at $(-1, 1)$

FOCUS ON CONCEPTS

31. In the special theory of relativity the length l of a narrow rod moving longitudinally is a function $l = l(v)$ of the rod's speed v. The accompanying figure, in which c denotes the speed of light, displays some of the qualitative features of this function.

(a) What is the physical interpretation of l_0?

(b) What is $\lim_{v \to c^-} l(v)$? What is the physical significance of this limit?

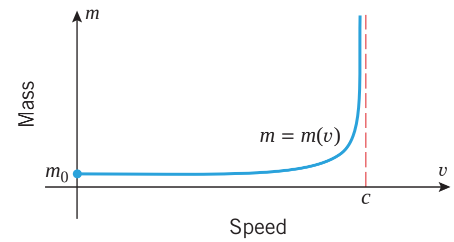


▲ **Figure Ex-31**

32. In the special theory of relativity the mass m of a moving object is a function $m = m(v)$ of the object's speed v. The accompanying figure, in which c denotes the speed of light, displays some of the qualitative features of this function.

(a) What is the physical interpretation of m_0?

(b) What is $\lim_{v \to c^-} m(v)$? What is the physical significance of this limit?



▲ **Figure Ex-32**

33. Let
$$f(x) = \left(1 + x^2\right)^{1.1/x^2}$$

(a) Graph f in the window
$$[-1, 1] \times [2.5, 3.5]$$
and use the calculator's trace feature to make a conjecture about the limit of $f(x)$ as $x \to 0$.

(b) Graph f in the window
$$[-0.001, 0.001] \times [2.5, 3.5]$$
and use the calculator's trace feature to make a conjecture about the limit of $f(x)$ as $x \to 0$.

(c) Graph f in the window
$$[-0.000001, 0.000001] \times [2.5, 3.5]$$
and use the calculator's trace feature to make a conjecture about the limit of $f(x)$ as $x \to 0$.

(d) Later we will be able to show that
$$\lim_{x \to 0} \left(1 + x^2\right)^{1.1/x^2} \approx 3.00416602$$
What flaw do your graphs reveal about using numerical evidence (as revealed by the graphs you obtained) to make conjectures about limits?

34. Writing Two students are discussing the limit of $\sqrt{x}$ as x approaches 0. One student maintains that the limit is 0, while the other claims that the limit does not exist. Write a short paragraph that discusses the pros and cons of each student's position.

35. Writing Given a function f and a real number a, explain informally why
$$\lim_{x \to 0} f(x + a) = \lim_{x \to a} f(x)$$
(Here "equality" means that either both limits exist and are equal or that both limits fail to exist.)

✔ QUICK CHECK ANSWERS 1.1

1. $f(x)$; L; x; a; a **2.** $f(x)$; x; a **3.** Both one-sided limits must exist and equal L. **4.** (a) 0 (b) 1 (c) $+\infty$ (d) $-\infty$ **5.** 4

1.2 COMPUTING LIMITS

In this section we will discuss techniques for computing limits of many functions. We base these results on the informal development of the limit concept discussed in the preceding section. A more formal derivation of these results is possible after Section 1.4.

SOME BASIC LIMITS

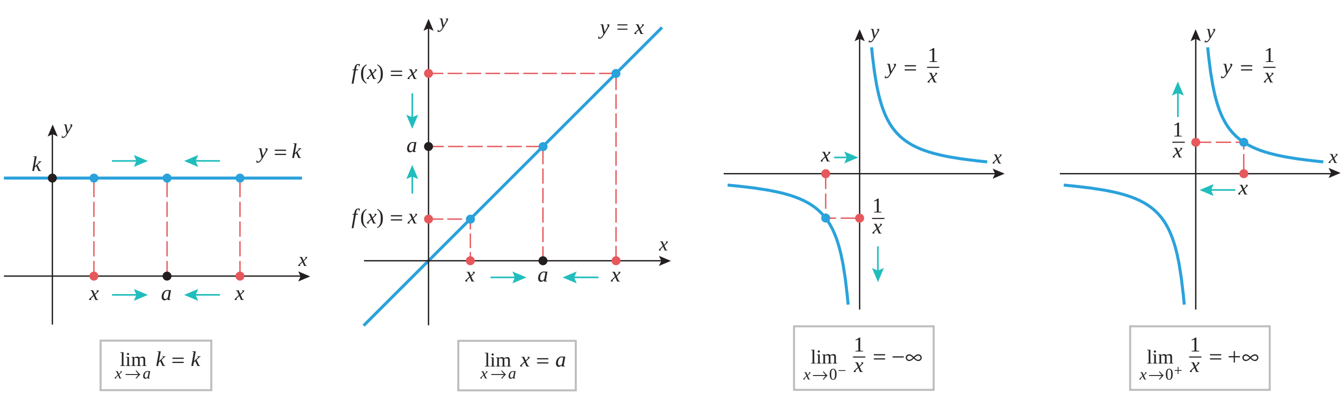
Our strategy for finding limits algebraically has two parts:

- First we will obtain the limits of some simple functions.
- Then we will develop a repertoire of theorems that will enable us to use the limits of those simple functions as building blocks for finding limits of more complicated functions.

We start with the following basic results, which are illustrated in Figure 1.2.1.

1.2.1 THEOREM *Let a and k be real numbers.*

(*a*) $\lim_{x \to a} k = k$ (*b*) $\lim_{x \to a} x = a$ (*c*) $\lim_{x \to 0^-} \dfrac{1}{x} = -\infty$ (*d*) $\lim_{x \to 0^+} \dfrac{1}{x} = +\infty$



▲ **Figure 1.2.1**

The following examples explain these results further.

▶ **Example 1** If $f(x) = k$ is a constant function, then the values of $f(x)$ remain fixed at k as x varies, which explains why $f(x) \to k$ as $x \to a$ for all values of a. For example,

$$\lim_{x \to -25} 3 = 3, \qquad \lim_{x \to 0} 3 = 3, \qquad \lim_{x \to \pi} 3 = 3 \;◀$$

Do not confuse the algebraic size of a number with its closeness to zero. For positive numbers, the smaller the number the closer it is to zero, but for negative numbers, the larger the number the closer it is to zero. For example, −2 is larger than −4, but it is closer to zero.

▶ **Example 2** If $f(x) = x$, then as $x \to a$ it must also be true that $f(x) \to a$. For example,

$$\lim_{x \to 0} x = 0, \qquad \lim_{x \to -2} x = -2, \qquad \lim_{x \to \pi} x = \pi \;◀$$

▶ **Example 3** You should know from your experience with fractions that for a fixed nonzero numerator, the closer the denominator is to zero, the larger the absolute value of the fraction. This fact and the data in Table 1.2.1 suggest why $1/x \to +\infty$ as $x \to 0^+$ and why $1/x \to -\infty$ as $x \to 0^-$. ◀

Table 1.2.1

	VALUES						CONCLUSION
x $1/x$	−1 −1	−0.1 −10	−0.01 −100	−0.001 −1000	−0.0001 −10,000	· · · · · ·	As $x \to 0^-$ the value of $1/x$ decreases without bound.
x $1/x$	1 1	0.1 10	0.01 100	0.001 1000	0.0001 10,000	· · · · · ·	As $x \to 0^+$ the value of $1/x$ increases without bound.

The following theorem, parts of which are proved in Appendix D, will be our basic tool for finding limits algebraically.

1.2.2 THEOREM *Let a be a real number, and suppose that*

$$\lim_{x \to a} f(x) = L_1 \quad \textit{and} \quad \lim_{x \to a} g(x) = L_2$$

That is, the limits exist and have values L_1 and L_2, respectively. Then:

(*a*) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2$

(*b*) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L_1 - L_2$

(*c*) $\lim_{x \to a} [f(x)g(x)] = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = L_1 L_2$

(*d*) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L_1}{L_2}$, *provided* $L_2 \neq 0$

(*e*) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L_1}$, *provided* $L_1 > 0$ *if n is even.*

Moreover, these statements are also true for the one-sided limits as $x \to a^-$ or as $x \to a^+$.

Theorem 1.2.2(e) remains valid for n even and $L_1 = 0$, provided $f(x)$ is nonnegative for x near a with $x \neq a$.

This theorem can be stated informally as follows:

(*a*) *The limit of a sum is the sum of the limits.*
(*b*) *The limit of a difference is the difference of the limits.*
(*c*) *The limit of a product is the product of the limits.*
(*d*) *The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero.*
(*e*) *The limit of an nth root is the nth root of the limit.*

For the special case of part (*c*) in which $f(x) = k$ is a constant function, we have

$$\lim_{x \to a} (kg(x)) = \lim_{x \to a} k \cdot \lim_{x \to a} g(x) = k \lim_{x \to a} g(x) \tag{1}$$

and similarly for one-sided limits. This result can be rephrased as follows:

A constant factor can be moved through a limit symbol.

Although parts (*a*) and (*c*) of Theorem 1.2.2 are stated for two functions, the results hold for any finite number of functions. Moreover, the various parts of the theorem can be used in combination to reformulate expressions involving limits.

▶ Example 4

$$\lim_{x\to a}[f(x)-g(x)+2h(x)]=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)+2\lim_{x\to a}h(x)$$

$$\lim_{x\to a}[f(x)g(x)h(x)]=\left(\lim_{x\to a}f(x)\right)\left(\lim_{x\to a}g(x)\right)\left(\lim_{x\to a}h(x)\right)$$

$$\lim_{x\to a}[f(x)]^3=\left(\lim_{x\to a}f(x)\right)^3$$

Take $g(x)=h(x)=f(x)$ in the last equation.

$$\lim_{x\to a}[f(x)]^n=\left(\lim_{x\to a}f(x)\right)^n$$

The extension of Theorem 1.2.2(*c*) in which there are n factors, each of which is $f(x)$

$$\lim_{x\to a}x^n=\left(\lim_{x\to a}x\right)^n=a^n$$

Apply the previous result with $f(x)=x$. ◀

LIMITS OF POLYNOMIALS AND RATIONAL FUNCTIONS AS $x\to a$

▶ Example 5 Find $\lim_{x\to 5}(x^2-4x+3)$.

Solution.

$$\lim_{x\to 5}(x^2-4x+3)=\lim_{x\to 5}x^2-\lim_{x\to 5}4x+\lim_{x\to 5}3$$

Theorem 1.2.2(*a*), (*b*)

$$=\lim_{x\to 5}x^2-4\lim_{x\to 5}x+\lim_{x\to 5}3$$

A constant can be moved through a limit symbol.

$$=5^2-4(5)+3$$

The last part of Example 4

$$=8$$ ◀

Observe that in Example 5 the limit of the polynomial $p(x)=x^2-4x+3$ as $x\to 5$ turned out to be the same as $p(5)$. This is not an accident. The next result shows that, in general, the limit of a polynomial $p(x)$ as $x\to a$ is the same as the value of the polynomial at a. Knowing this fact allows us to reduce the computation of limits of polynomials to simply evaluating the polynomial at the appropriate point.

1.2.3 THEOREM *For any polynomial*

$$p(x)=c_0+c_1x+\cdots+c_nx^n$$

and any real number a,

$$\lim_{x\to a}p(x)=c_0+c_1a+\cdots+c_na^n=p(a)$$

PROOF

$$\begin{aligned}
\lim_{x \to a} p(x) &= \lim_{x \to a} \left(c_0 + c_1 x + \cdots + c_n x^n\right) \\
&= \lim_{x \to a} c_0 + \lim_{x \to a} c_1 x + \cdots + \lim_{x \to a} c_n x^n \\
&= \lim_{x \to a} c_0 + c_1 \lim_{x \to a} x + \cdots + c_n \lim_{x \to a} x^n \\
&= c_0 + c_1 a + \cdots + c_n a^n = p(a) \quad \blacksquare
\end{aligned}$$

▶ Example 6 Find $\lim_{x \to 1} (x^7 - 2x^5 + 1)^{35}$.

Solution. The function involved is a polynomial (why?), so the limit can be obtained by evaluating this polynomial at $x = 1$. This yields

$$\lim_{x \to 1} (x^7 - 2x^5 + 1)^{35} = 0 \quad ◀$$

Recall that a rational function is a ratio of two polynomials. The following example illustrates how Theorems 1.2.2(d) and 1.2.3 can sometimes be used in combination to compute limits of rational functions.

▶ Example 7 Find $\lim_{x \to 2} \dfrac{5x^3 + 4}{x - 3}$.

Solution.

$$\begin{aligned}
\lim_{x \to 2} \frac{5x^3 + 4}{x - 3} &= \frac{\lim_{x \to 2} (5x^3 + 4)}{\lim_{x \to 2} (x - 3)} && \boxed{\text{Theorem 1.2.2}(d)} \\
&= \frac{5 \cdot 2^3 + 4}{2 - 3} = -44 && \boxed{\text{Theorem 1.2.3}} \quad ◀
\end{aligned}$$

The method used in the last example will not work for rational functions in which the limit of the denominator is zero because Theorem 1.2.2(d) is not applicable. There are two cases of this type to be considered—the case where the limit of the denominator is zero and the limit of the numerator is not, and the case where the limits of the numerator and denominator are both zero. If the limit of the denominator is zero but the limit of the numerator is not, then one can prove that the limit of the rational function does not exist and that one of the following situations occurs:

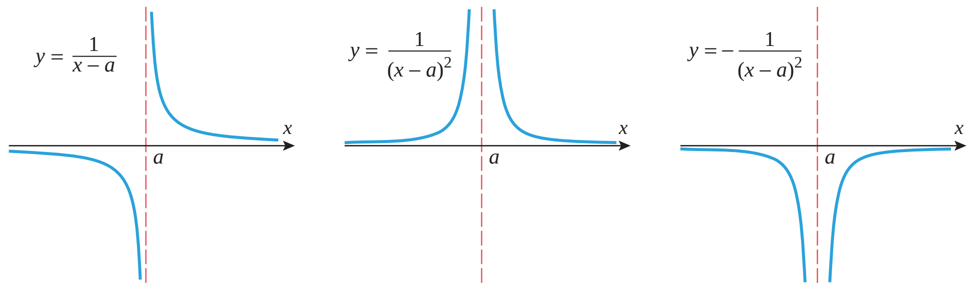
- The limit may be $-\infty$ from one side and $+\infty$ from the other.
- The limit may be $+\infty$.
- The limit may be $-\infty$.

Figure 1.2.2 illustrates these three possibilities graphically for rational functions of the form $1/(x - a)$, $1/(x - a)^2$, and $-1/(x - a)^2$.

▶ Example 8 Find

(a) $\lim_{x \to 4^+} \dfrac{2 - x}{(x - 4)(x + 2)}$ (b) $\lim_{x \to 4^-} \dfrac{2 - x}{(x - 4)(x + 2)}$ (c) $\lim_{x \to 4} \dfrac{2 - x}{(x - 4)(x + 2)}$

Solution. In all three parts the limit of the numerator is -2, and the limit of the denominator is 0, so the limit of the ratio does not exist. To be more specific than this, we need

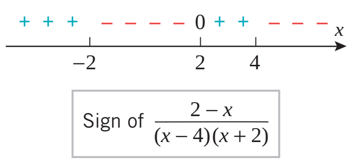


$$\lim_{x \to a^+} \frac{1}{x-a} = +\infty \qquad \lim_{x \to a^-} \frac{1}{x-a} = -\infty$$

$$\lim_{x \to a} \frac{1}{(x-a)^2} = +\infty$$

$$\lim_{x \to a} -\frac{1}{(x-a)^2} = -\infty$$

▲ **Figure 1.2.2**



▲ **Figure 1.2.3**

to analyze the sign of the ratio. The sign of the ratio, which is given in Figure 1.2.3, is determined by the signs of $2 - x$, $x - 4$, and $x + 2$. (The method of test points, discussed in Web Appendix E, provides a way of finding the sign of the ratio here.) It follows from this figure that as x approaches 4 from the right, the ratio is always negative; and as x approaches 4 from the left, the ratio is eventually positive. Thus,

$$\lim_{x \to 4^+} \frac{2-x}{(x-4)(x+2)} = -\infty \quad \text{and} \quad \lim_{x \to 4^-} \frac{2-x}{(x-4)(x+2)} = +\infty$$

Because the one-sided limits have opposite signs, all we can say about the two-sided limit is that it does not exist. ◀

In the case where $p(x)/q(x)$ is a rational function for which $p(a) = 0$ and $q(a) = 0$, the numerator and denominator must have one or more common factors of $x - a$. In this case the limit of $p(x)/q(x)$ as $x \to a$ can be found by canceling all common factors of $x - a$ and using one of the methods already considered to find the limit of the simplified function. Here is an example.

In Example 9(a), the simplified function $x - 3$ is defined at $x = 3$, but the original function is not. However, this has no effect on the limit as x *approaches* 3 since the two functions are identical if $x \neq 3$ (Exercise 50).

▶ Example 9 Find

(a) $\lim_{x \to 3} \frac{x^2 - 6x + 9}{x - 3}$ (b) $\lim_{x \to -4} \frac{2x + 8}{x^2 + x - 12}$ (c) $\lim_{x \to 5} \frac{x^2 - 3x - 10}{x^2 - 10x + 25}$

Solution (a). The numerator and the denominator both have a zero at $x = 3$, so there is a common factor of $x - 3$. Then

$$\lim_{x \to 3} \frac{x^2 - 6x + 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)^2}{x - 3} = \lim_{x \to 3} (x - 3) = 0$$

Solution (b). The numerator and the denominator both have a zero at $x = -4$, so there is a common factor of $x - (-4) = x + 4$. Then

$$\lim_{x \to -4} \frac{2x + 8}{x^2 + x - 12} = \lim_{x \to -4} \frac{2(x+4)}{(x+4)(x-3)} = \lim_{x \to -4} \frac{2}{x - 3} = -\frac{2}{7}$$

Solution (c). The numerator and the denominator both have a zero at $x = 5$, so there is a common factor of $x - 5$. Then

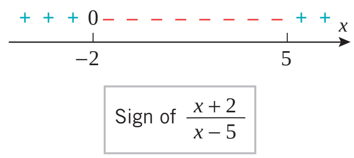
$$\lim_{x \to 5} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x \to 5} \frac{(x-5)(x+2)}{(x-5)(x-5)} = \lim_{x \to 5} \frac{x + 2}{x - 5}$$

However,

$$\lim_{x \to 5} (x + 2) = 7 \neq 0 \quad \text{and} \quad \lim_{x \to 5} (x - 5) = 0$$

so

$$\lim_{x \to 5} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x \to 5} \frac{x + 2}{x - 5}$$



▲ **Figure 1.2.4**

does not exist. More precisely, the sign analysis in Figure 1.2.4 implies that

$$\lim_{x \to 5^+} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x \to 5^+} \frac{x + 2}{x - 5} = +\infty$$

and

$$\lim_{x \to 5^-} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x \to 5^-} \frac{x + 2}{x - 5} = -\infty \blacktriangleleft$$

Discuss the logical errors in the following statement: An indeterminate form of type $0/0$ must have a limit of zero because zero divided by anything is zero.

A quotient $f(x)/g(x)$ in which the numerator and denominator both have a limit of zero as $x \to a$ is called an ***indeterminate form of type* $\mathbf{0/0}$**. The problem with such limits is that it is difficult to tell by inspection whether the limit exists, and, if so, its value. Informally stated, this is because there are two conflicting influences at work. The value of $f(x)/g(x)$ would tend to zero as $f(x)$ approached zero if $g(x)$ were to remain at some fixed nonzero value, whereas the value of this ratio would tend to increase or decrease without bound as $g(x)$ approached zero if $f(x)$ were to remain at some fixed nonzero value. But with both $f(x)$ and $g(x)$ approaching zero, the behavior of the ratio depends on precisely how these conflicting tendencies offset one another for the particular f and g.

Sometimes, limits of indeterminate forms of type $0/0$ can be found by algebraic simplification, as in the last example, but frequently this will not work and other methods must be used. We will study such methods in later sections.

The following theorem summarizes our observations about limits of rational functions.

1.2.4 THEOREM *Let*

$$f(x) = \frac{p(x)}{q(x)}$$

be a rational function, and let a be any real number.

(*a*) *If* $q(a) \neq 0$, *then* $\lim_{x \to a} f(x) = f(a)$.

(*b*) *If* $q(a) = 0$ *but* $p(a) \neq 0$, *then* $\lim_{x \to a} f(x)$ *does not exist.*

LIMITS INVOLVING RADICALS

▶ **Example 10** Find $\lim_{x \to 1} \dfrac{x - 1}{\sqrt{x} - 1}$.

Solution. In Example 2 of Section 1.1 we used numerical evidence to conjecture that this limit is 2. Here we will confirm this algebraically. Since this limit is an indeterminate form of type $0/0$, we will need to devise some strategy for making the limit (if it exists) evident. One such strategy is to rationalize the denominator of the function. This yields

$$\frac{x - 1}{\sqrt{x} - 1} = \frac{(x - 1)(\sqrt{x} + 1)}{(\sqrt{x} - 1)(\sqrt{x} + 1)} = \frac{(x - 1)(\sqrt{x} + 1)}{x - 1} = \sqrt{x} + 1 \quad (x \neq 1)$$

Confirm the limit in Example 10 by factoring the numerator.

Therefore,

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}=\lim_{x\to 1}(\sqrt{x}+1)=2 \blacktriangleleft$$

LIMITS OF PIECEWISE-DEFINED FUNCTIONS

For functions that are defined piecewise, a two-sided limit at a point where the formula changes is best obtained by first finding the one-sided limits at that point.

Example 11 Let

$$f(x)=\begin{cases}1/(x+2), & x<-2\\ x^2-5, & -2<x\le 3\\ \sqrt{x+13}, & x>3\end{cases}$$

Find

$$\text{(a)}\ \lim_{x\to -2} f(x) \qquad \text{(b)}\ \lim_{x\to 0} f(x) \qquad \text{(c)}\ \lim_{x\to 3} f(x)$$

Solution (a). We will determine the stated two-sided limit by first considering the corresponding one-sided limits. For each one-sided limit, we must use that part of the formula that is applicable on the interval over which x varies. For example, as x approaches -2 from the left, the applicable part of the formula is

$$f(x)=\frac{1}{x+2}$$

and as x approaches -2 from the right, the applicable part of the formula near -2 is

$$f(x)=x^2-5$$

Thus,

$$\lim_{x\to -2^-} f(x)=\lim_{x\to -2^-}\frac{1}{x+2}=-\infty$$

$$\lim_{x\to -2^+} f(x)=\lim_{x\to -2^+}(x^2-5)=(-2)^2-5=-1$$

from which it follows that $\lim_{x\to -2} f(x)$ does not exist.

Solution (b). The applicable part of the formula is $f(x)=x^2-5$ on both sides of 0, so there is no need to consider one-sided limits here. We see directly that

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}(x^2-5)=0^2-5=-5$$

Solution (c). Using the applicable parts of the formula for $f(x)$, we obtain

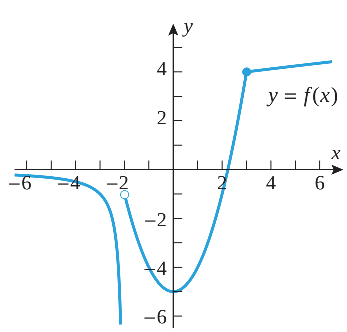
$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(x^2-5)=3^2-5=4$$

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}\sqrt{x+13}=\sqrt{\lim_{x\to 3^+}(x+13)}=\sqrt{3+13}=4$$

Since the one-sided limits are equal, we have

$$\lim_{x\to 3} f(x)=4$$

We note that the limit calculations in parts (a), (b), and (c) are consistent with the graph of f shown in Figure 1.2.5. ◀



▲ **Figure 1.2.5**

✔ QUICK CHECK EXERCISES 1.2 (See page 88 for answers.)

1. In each part, find the limit by inspection.
(a) $\lim_{x \to 8} 7 =$ ______
(b) $\lim_{y \to 3^+} 12y =$ ______
(c) $\lim_{x \to 0^-} \dfrac{x}{|x|} =$ ______
(d) $\lim_{w \to 5} \dfrac{w}{|w|} =$ ______
(e) $\lim_{z \to 1^-} \dfrac{1}{1-z} =$ ______

2. Given that $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = 2$, find the limits.
(a) $\lim_{x \to a} [3f(x) + 2g(x)] =$ ______
(b) $\lim_{x \to a} \dfrac{2f(x)+1}{1 - f(x)g(x)} =$ ______
(c) $\lim_{x \to a} \dfrac{\sqrt{f(x)+3}}{g(x)} =$ ______

3. Find the limits.
(a) $\lim_{x \to -1} (x^3 + x^2 + x)^{101} =$ ______
(b) $\lim_{x \to 2^-} \dfrac{(x-1)(x-2)}{x+1} =$ ______
(c) $\lim_{x \to -1^+} \dfrac{(x-1)(x-2)}{x+1} =$ ______
(d) $\lim_{x \to 4} \dfrac{x^2-16}{x-4} =$ ______

4. Let

$$f(x) = \begin{cases} x+1, & x \le 1 \\ x-1, & x > 1 \end{cases}$$

Find the limits that exist.
(a) $\lim_{x \to 1^-} f(x) =$ ______
(b) $\lim_{x \to 1^+} f(x) =$ ______
(c) $\lim_{x \to 1} f(x) =$ ______

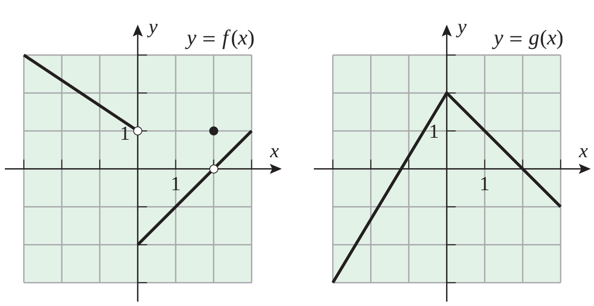
EXERCISE SET 1.2

1. Given that

$$\lim_{x \to a} f(x) = 2, \quad \lim_{x \to a} g(x) = -4, \quad \lim_{x \to a} h(x) = 0$$

find the limits.
(a) $\lim_{x \to a} [f(x) + 2g(x)]$
(b) $\lim_{x \to a} [h(x) - 3g(x) + 1]$
(c) $\lim_{x \to a} [f(x)g(x)]$
(d) $\lim_{x \to a} [g(x)]^2$
(e) $\lim_{x \to a} \sqrt[3]{6 + f(x)}$
(f) $\lim_{x \to a} \dfrac{2}{g(x)}$

2. Use the graphs of f and g in the accompanying figure to find the limits that exist. If the limit does not exist, explain why.
(a) $\lim_{x \to 2} [f(x) + g(x)]$
(b) $\lim_{x \to 0} [f(x) + g(x)]$
(c) $\lim_{x \to 0^+} [f(x) + g(x)]$
(d) $\lim_{x \to 0^-} [f(x) + g(x)]$
(e) $\lim_{x \to 2} \dfrac{f(x)}{1 + g(x)}$
(f) $\lim_{x \to 2} \dfrac{1 + g(x)}{f(x)}$
(g) $\lim_{x \to 0^+} \sqrt{f(x)}$
(h) $\lim_{x \to 0^-} \sqrt{f(x)}$



▲ **Figure Ex-2**

3–30 Find the limits. ■

3. $\lim_{x \to 2} x(x-1)(x+1)$

4. $\lim_{x \to 3} x^3 - 3x^2 + 9x$

5. $\lim_{x \to 3} \dfrac{x^2 - 2x}{x+1}$

6. $\lim_{x \to 0} \dfrac{6x - 9}{x^3 - 12x + 3}$

7. $\lim_{x \to 1^+} \dfrac{x^4 - 1}{x - 1}$

8. $\lim_{t \to -2} \dfrac{t^3 + 8}{t + 2}$

9. $\lim_{x \to -1} \dfrac{x^2 + 6x + 5}{x^2 - 3x - 4}$

10. $\lim_{x \to 2} \dfrac{x^2 - 4x + 4}{x^2 + x - 6}$

11. $\lim_{x \to -1} \dfrac{2x^2 + x - 1}{x + 1}$

12. $\lim_{x \to 1} \dfrac{3x^2 - x - 2}{2x^2 + x - 3}$

13. $\lim_{t \to 2} \dfrac{t^3 + 3t^2 - 12t + 4}{t^3 - 4t}$

14. $\lim_{t \to 1} \dfrac{t^3 + t^2 - 5t + 3}{t^3 - 3t + 2}$

15. $\lim_{x \to 3^+} \dfrac{x}{x - 3}$

16. $\lim_{x \to 3^-} \dfrac{x}{x - 3}$

17. $\lim_{x \to 3} \dfrac{x}{x - 3}$

18. $\lim_{x \to 2^+} \dfrac{x}{x^2 - 4}$

19. $\lim_{x \to 2^-} \dfrac{x}{x^2 - 4}$

20. $\lim_{x \to 2} \dfrac{x}{x^2 - 4}$

21. $\lim_{y \to 6^+} \dfrac{y + 6}{y^2 - 36}$

22. $\lim_{y \to 6^-} \dfrac{y + 6}{y^2 - 36}$

23. $\lim_{y \to 6} \dfrac{y + 6}{y^2 - 36}$

24. $\lim_{x \to 4^+} \dfrac{3 - x}{x^2 - 2x - 8}$

25. $\lim_{x \to 4^-} \dfrac{3 - x}{x^2 - 2x - 8}$

26. $\lim_{x \to 4} \dfrac{3 - x}{x^2 - 2x - 8}$

27. $\lim_{x \to 2^+} \dfrac{1}{|2 - x|}$

28. $\lim_{x \to 3^-} \dfrac{1}{|x - 3|}$

29. $\lim_{x \to 9} \dfrac{x - 9}{\sqrt{x} - 3}$

30. $\lim_{y \to 4} \dfrac{4 - y}{2 - \sqrt{y}}$

31. Let

$$f(x) = \begin{cases} x - 1, & x \le 3 \\ 3x - 7, & x > 3 \end{cases}$$

(cont.)

Find
(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$ (c) $\lim_{x\to 3} f(x)$.

32. Let

$$g(t) = \begin{cases} t-2, & t < 0 \\ t^2, & 0 \le t \le 2 \\ 2t, & t > 2 \end{cases}$$

Find
(a) $\lim_{t\to 0} g(t)$ (b) $\lim_{t\to 1} g(t)$ (c) $\lim_{t\to 2} g(t)$.

33–36 True–False Determine whether the statement is true or false. Explain your answer.

33. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then so does $\lim_{x\to a}[f(x)+g(x)]$.

34. If $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a}[f(x)/g(x)]$ does not exist.

35. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and are equal, then $\lim_{x\to a}[f(x)/g(x)] = 1$.

36. If $f(x)$ is a rational function and $x = a$ is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

37–38 First rationalize the numerator and then find the limit.

37. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$

38. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x}$

39. Let

$$f(x) = \frac{x^3-1}{x-1}$$

(a) Find $\lim_{x\to 1} f(x)$.
(b) Sketch the graph of $y = f(x)$.

40. Let

$$f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & x \ne -3 \\ k, & x = -3 \end{cases}$$

(a) Find k so that $f(-3) = \lim_{x\to -3} f(x)$.
(b) With k assigned the value $\lim_{x\to -3} f(x)$, show that $f(x)$ can be expressed as a polynomial.

FOCUS ON CONCEPTS

41. (a) Explain why the following calculation is incorrect.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right) = \lim_{x\to 0^+}\frac{1}{x} - \lim_{x\to 0^+}\frac{1}{x^2}$$
$$= +\infty - (+\infty) = 0$$

(b) Show that $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right) = -\infty$.

42. (a) Explain why the following argument is incorrect.

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{x^2+2x}\right) = \lim_{x\to 0}\frac{1}{x}\left(1-\frac{2}{x+2}\right)$$
$$= \infty \cdot 0 = 0$$

(b) Show that $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{x^2+2x}\right) = \frac{1}{2}$.

43. Find all values of a such that

$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{a}{x^2-1}\right)$$

exists and is finite.

44. (a) Explain informally why

$$\lim_{x\to 0^-}\left(\frac{1}{x}+\frac{1}{x^2}\right) = +\infty$$

(b) Verify the limit in part (a) algebraically.

45. Let $p(x)$ and $q(x)$ be polynomials, with $q(x_0) = 0$. Discuss the behavior of the graph of $y = p(x)/q(x)$ in the vicinity of $x = x_0$. Give examples to support your conclusions.

46. Suppose that f and g are two functions such that $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a}[f(x)+g(x)]$ does not exist. Use Theorem 1.2.2. to prove that $\lim_{x\to a} g(x)$ does not exist.

47. Suppose that f and g are two functions such that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a}[f(x)+g(x)]$ exist. Use Theorem 1.2.2 to prove that $\lim_{x\to a} g(x)$ exists.

48. Suppose that f and g are two functions such that

$$\lim_{x\to a} g(x) = 0 \quad \text{and} \quad \lim_{x\to a}\frac{f(x)}{g(x)}$$

exists. Use Theorem 1.2.2 to prove that $\lim_{x\to a} f(x) = 0$.

49. Writing According to Newton's Law of Universal Gravitation, the gravitational force of attraction between two masses is inversely proportional to the square of the distance between them. What results of this section are useful in describing the gravitational force of attraction between the masses as they get closer and closer together?

50. Writing Suppose that f and g are two functions that are equal except at a finite number of points and that a denotes a real number. Explain informally why both

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist and are equal, or why both limits fail to exist. Write a short paragraph that explains the relationship of this result to the use of "algebraic simplification" in the evaluation of a limit.

✔ QUICK CHECK ANSWERS 1.2

1. (a) 7 (b) 36 (c) -1 (d) 1 (e) $+\infty$ **2.** (a) 7 (b) -3 (c) 1 **3.** (a) -1 (b) 0 (c) $+\infty$ (d) 8
4. (a) 2 (b) 0 (c) does not exist

1.3 LIMITS AT INFINITY; END BEHAVIOR OF A FUNCTION

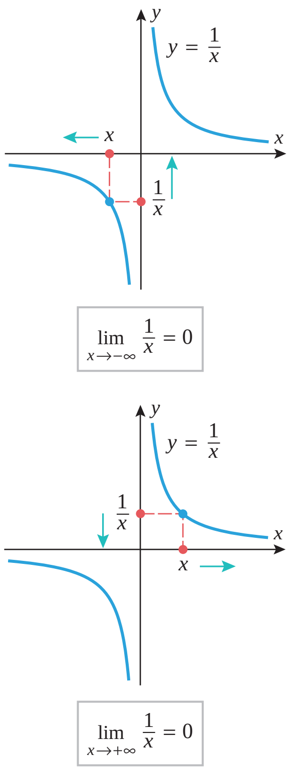
Up to now we have been concerned with limits that describe the behavior of a function $f(x)$ as x approaches some real number a. In this section we will be concerned with the behavior of $f(x)$ as x increases or decreases without bound.

LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

If the values of a variable x increase without bound, then we write $x \to +\infty$, and if the values of x decrease without bound, then we write $x \to -\infty$. The behavior of a function $f(x)$ as x increases without bound or decreases without bound is sometimes called the ***end behavior*** of the function. For example,

$$\lim_{x\to-\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to+\infty} \frac{1}{x} = 0 \tag{1–2}$$

are illustrated numerically in Table 1.3.1 and geometrically in Figure 1.3.1.



▲ **Figure 1.3.1**

Table 1.3.1

	VALUES						CONCLUSION
x	-1	-10	-100	-1000	$-10{,}000$	$\cdots$	As $x \to -\infty$ the value of $1/x$
$1/x$	-1	-0.1	-0.01	-0.001	-0.0001	$\cdots$	increases toward zero.
x	1	10	100	1000	$10{,}000$	$\cdots$	As $x \to +\infty$ the value of $1/x$
$1/x$	1	0.1	0.01	0.001	0.0001	$\cdots$	decreases toward zero.

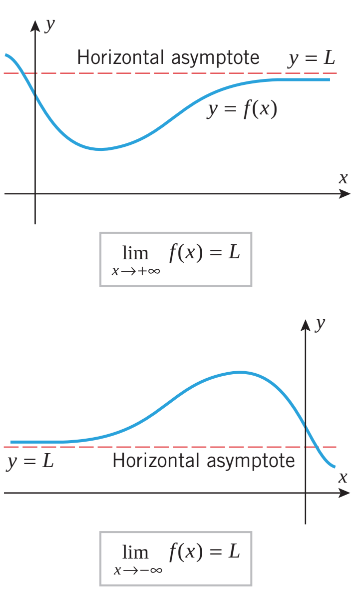
In general, we will use the following notation.

1.3.1 LIMITS AT INFINITY (AN INFORMAL VIEW) If the values of $f(x)$ eventually get as close as we like to a number L as x increases without bound, then we write

$$\lim_{x\to+\infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to +\infty \tag{3}$$

Similarly, if the values of $f(x)$ eventually get as close as we like to a number L as x decreases without bound, then we write

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to -\infty \tag{4}$$



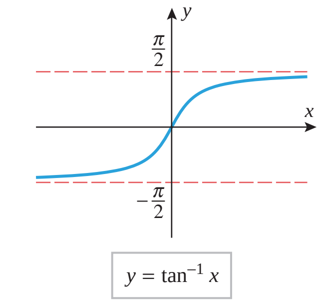
▲ **Figure 1.3.2**

Figure 1.3.2 illustrates the end behavior of a function f when

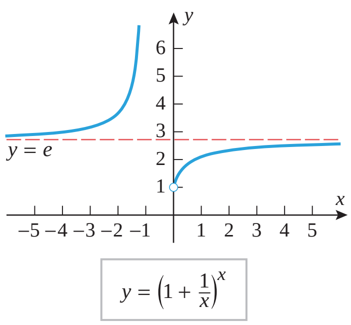
$$\lim_{x\to+\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

In the first case the graph of f eventually comes as close as we like to the line $y = L$ as x increases without bound, and in the second case it eventually comes as close as we like to the line $y = L$ as x decreases without bound. If either limit holds, we call the line $y = L$ a ***horizontal asymptote*** for the graph of f.

▶ **Example 1** It follows from (1) and (2) that $y = 0$ is a horizontal asymptote for the graph of $f(x) = 1/x$ in both the positive and negative directions. This is consistent with the graph of $y = 1/x$ shown in Figure 1.3.1. ◀



▲ **Figure 1.3.3**



▲ **Figure 1.3.4**

▶ **Example 2** Figure 1.3.3 is the graph of $f(x) = \tan^{-1} x$. As suggested by this graph,

$$\lim_{x\to+\infty} \tan^{-1} x = \frac{\pi}{2} \quad \text{and} \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2} \tag{5–6}$$

so the line $y = \pi/2$ is a horizontal asymptote for f in the positive direction and the line $y = -\pi/2$ is a horizontal asymptote in the negative direction. ◀

▶ **Example 3** Figure 1.3.4 is the graph of $f(x) = (1 + 1/x)^x$. As suggested by this graph,

$$\lim_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x = e \quad \text{and} \quad \lim_{x\to-\infty} \left(1 + \frac{1}{x}\right)^x = e \tag{7–8}$$

so the line $y = e$ is a horizontal asymptote for f in both the positive and negative directions. ◀

LIMIT LAWS FOR LIMITS AT INFINITY

It can be shown that the limit laws in Theorem 1.2.2 carry over without change to limits at $+\infty$ and $-\infty$. Moreover, it follows by the same argument used in Section 1.2 that if n is a positive integer, then

$$\lim_{x\to+\infty} (f(x))^n = \left(\lim_{x\to+\infty} f(x)\right)^n \qquad \lim_{x\to-\infty} (f(x))^n = \left(\lim_{x\to-\infty} f(x)\right)^n \tag{9–10}$$

provided the indicated limit of $f(x)$ exists. It also follows that constants can be moved through the limit symbols for limits at infinity:

$$\lim_{x\to+\infty} kf(x) = k \lim_{x\to+\infty} f(x) \qquad \lim_{x\to-\infty} kf(x) = k \lim_{x\to-\infty} f(x) \tag{11–12}$$

provided the indicated limit of $f(x)$ exists.

Finally, if $f(x) = k$ is a constant function, then the values of f do not change as $x \to +\infty$ or as $x \to -\infty$, so

$$\lim_{x\to+\infty} k = k \qquad \lim_{x\to-\infty} k = k \tag{13–14}$$

▶ **Example 4**

(a) It follows from (1), (2), (9), and (10) that if n is a positive integer, then

$$\lim_{x\to+\infty} \frac{1}{x^n} = \left(\lim_{x\to+\infty} \frac{1}{x}\right)^n = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = \left(\lim_{x\to-\infty} \frac{1}{x}\right)^n = 0$$

(b) It follows from (7) and the extension of Theorem 1.2.2(e) to the case $x \to +\infty$ that

$$\lim_{x\to+\infty} \left(1 + \frac{1}{2x}\right)^x = \lim_{x\to+\infty} \left[\left(1 + \frac{1}{2x}\right)^{2x}\right]^{1/2}$$

$$= \left[\lim_{x\to+\infty} \left(1 + \frac{1}{2x}\right)^{2x}\right]^{1/2} = e^{1/2} = \sqrt{e} \text{ ◀}$$

INFINITE LIMITS AT INFINITY

Limits at infinity, like limits at a real number a, can fail to exist for various reasons. One such possibility is that the values of $f(x)$ increase or decrease without bound as $x \to +\infty$ or as $x \to -\infty$. We will use the following notation to describe this situation.

1.3.2 INFINITE LIMITS AT INFINITY (AN INFORMAL VIEW) If the values of $f(x)$ increase without bound as $x \to +\infty$ or as $x \to -\infty$, then we write

$$\lim_{x \to +\infty} f(x) = +\infty \quad \text{or} \quad \lim_{x \to -\infty} f(x) = +\infty$$

as appropriate; and if the values of $f(x)$ decrease without bound as $x \to +\infty$ or as $x \to -\infty$, then we write

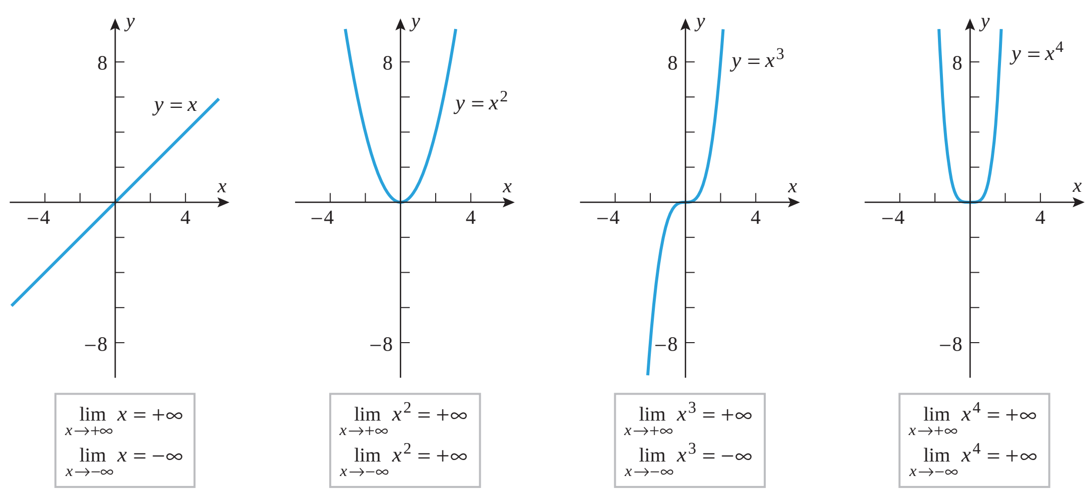
$$\lim_{x \to +\infty} f(x) = -\infty \quad \text{or} \quad \lim_{x \to -\infty} f(x) = -\infty$$

as appropriate.

LIMITS OF x^n AS $x \to \pm\infty$

Figure 1.3.5 illustrates the end behavior of the polynomials x^n for $n = 1, 2, 3$, and 4. These are special cases of the following general results:

$$\lim_{x \to +\infty} x^n = +\infty, \quad n = 1, 2, 3, \ldots \qquad \lim_{x \to -\infty} x^n = \begin{cases} -\infty, & n = 1, 3, 5, \ldots \\ +\infty, & n = 2, 4, 6, \ldots \end{cases} \tag{15–16}$$



▲ **Figure 1.3.5**

Multiplying x^n by a positive real number does not affect limits (15) and (16), but multiplying by a negative real number reverses the sign.

▶ **Example 5**

$$\lim_{x \to +\infty} 2x^5 = +\infty, \qquad \lim_{x \to -\infty} 2x^5 = -\infty$$

$$\lim_{x \to +\infty} -7x^6 = -\infty, \qquad \lim_{x \to -\infty} -7x^6 = -\infty \blacktriangleleft$$

LIMITS OF POLYNOMIALS AS $x \to \pm\infty$

There is a useful principle about polynomials which, expressed informally, states:

The end behavior of a polynomial matches the end behavior of its highest degree term.

More precisely, if $c_n \neq 0$, then

$$\lim_{x\to-\infty}(c_0 + c_1x + \cdots + c_nx^n) = \lim_{x\to-\infty} c_nx^n \tag{17}$$

$$\lim_{x\to+\infty}(c_0 + c_1x + \cdots + c_nx^n) = \lim_{x\to+\infty} c_nx^n \tag{18}$$

We can motivate these results by factoring out the highest power of x from the polynomial and examining the limit of the factored expression. Thus,

$$c_0 + c_1x + \cdots + c_nx^n = x^n\left(\frac{c_0}{x^n} + \frac{c_1}{x^{n-1}} + \cdots + c_n\right)$$

As $x\to-\infty$ or $x\to+\infty$, it follows from Example 4(a) that all of the terms with positive powers of x in the denominator approach 0, so (17) and (18) are certainly plausible.

▶ Example 6

$$\lim_{x\to-\infty}(7x^5 - 4x^3 + 2x - 9) = \lim_{x\to-\infty} 7x^5 = -\infty$$

$$\lim_{x\to-\infty}(-4x^8 + 17x^3 - 5x + 1) = \lim_{x\to-\infty} -4x^8 = -\infty \blacktriangleleft$$

LIMITS OF RATIONAL FUNCTIONS AS $x\to\pm\infty$

One technique for determining the end behavior of a rational function is to divide each term in the numerator and denominator by the highest power of x that occurs in the denominator, after which the limiting behavior can be determined using results we have already established. Here are some examples.

▶ Example 7 Find $\displaystyle\lim_{x\to+\infty}\frac{3x+5}{6x-8}$.

Solution. Divide each term in the numerator and denominator by the highest power of x that occurs in the denominator, namely, $x^1 = x$. We obtain

$$\lim_{x\to+\infty}\frac{3x+5}{6x-8} = \lim_{x\to+\infty}\frac{3+\frac{5}{x}}{6-\frac{8}{x}} \quad \boxed{\text{Divide each term by } x.}$$

$$= \frac{\displaystyle\lim_{x\to+\infty}\left(3+\frac{5}{x}\right)}{\displaystyle\lim_{x\to+\infty}\left(6-\frac{8}{x}\right)} \quad \boxed{\text{Limit of a quotient is the quotient of the limits.}}$$

$$= \frac{\displaystyle\lim_{x\to+\infty}3+\lim_{x\to+\infty}\frac{5}{x}}{\displaystyle\lim_{x\to+\infty}6-\lim_{x\to+\infty}\frac{8}{x}} \quad \boxed{\text{Limit of a sum is the sum of the limits.}}$$

$$= \frac{3+5\displaystyle\lim_{x\to+\infty}\frac{1}{x}}{6-8\displaystyle\lim_{x\to+\infty}\frac{1}{x}} = \frac{3+0}{6-0} = \frac{1}{2} \quad \boxed{\text{A constant can be moved through a limit symbol; Formulas (2) and (13).}} \blacktriangleleft$$

Example 8 Find

$$\text{(a)} \lim_{x\to-\infty} \frac{4x^2-x}{2x^3-5} \qquad \text{(b)} \lim_{x\to+\infty} \frac{5x^3-2x^2+1}{1-3x}$$

Solution (a). Divide each term in the numerator and denominator by the highest power of x that occurs in the denominator, namely, x^3. We obtain

$$\lim_{x\to-\infty} \frac{4x^2-x}{2x^3-5} = \lim_{x\to-\infty} \frac{\dfrac{4}{x}-\dfrac{1}{x^2}}{2-\dfrac{5}{x^3}}$$

Divide each term by x^3.

$$= \frac{\displaystyle\lim_{x\to-\infty}\left(\frac{4}{x}-\frac{1}{x^2}\right)}{\displaystyle\lim_{x\to-\infty}\left(2-\frac{5}{x^3}\right)}$$

Limit of a quotient is the quotient of the limits.

$$= \frac{\displaystyle\lim_{x\to-\infty}\frac{4}{x}-\lim_{x\to-\infty}\frac{1}{x^2}}{\displaystyle\lim_{x\to-\infty}2-\lim_{x\to-\infty}\frac{5}{x^3}}$$

Limit of a difference is the difference of the limits.

$$= \frac{\displaystyle 4\lim_{x\to-\infty}\frac{1}{x}-\lim_{x\to-\infty}\frac{1}{x^2}}{\displaystyle 2-5\lim_{x\to-\infty}\frac{1}{x^3}} = \frac{0-0}{2-0} = 0$$

A constant can be moved through a limit symbol; Formula (14) and Example 4.

Solution (b). Divide each term in the numerator and denominator by the highest power of x that occurs in the denominator, namely, $x^1 = x$. We obtain

$$\lim_{x\to+\infty} \frac{5x^3-2x^2+1}{1-3x} = \lim_{x\to+\infty} \frac{5x^2-2x+\dfrac{1}{x}}{\dfrac{1}{x}-3} \tag{19}$$

In this case we cannot argue that the limit of the quotient is the quotient of the limits because the limit of the numerator does not exist. However, we have

$$\lim_{x\to+\infty} 5x^2-2x = +\infty, \qquad \lim_{x\to+\infty}\frac{1}{x} = 0, \qquad \lim_{x\to+\infty}\left(\frac{1}{x}-3\right) = -3$$

Thus, the numerator on the right side of (19) approaches $+\infty$ and the denominator has a finite *negative* limit. We conclude from this that the quotient approaches $-\infty$; that is,

$$\lim_{x\to+\infty} \frac{5x^3-2x^2+1}{1-3x} = \lim_{x\to+\infty} \frac{5x^2-2x+\dfrac{1}{x}}{\dfrac{1}{x}-3} = -\infty$$

A QUICK METHOD FOR FINDING LIMITS OF RATIONAL FUNCTIONS AS $x\to+\infty$ OR $x\to-\infty$

Since the end behavior of a polynomial matches the end behavior of its highest degree term, one can reasonably conclude:

The end behavior of a rational function matches the end behavior of the quotient of the highest degree term in the numerator divided by the highest degree term in the denominator.

▶ Example 9 Use the preceding observation to compute the limits in Examples 7 and 8.

Solution.

$$\lim_{x \to +\infty} \frac{3x+5}{6x-8} = \lim_{x \to +\infty} \frac{3x}{6x} = \lim_{x \to +\infty} \frac{1}{2} = \frac{1}{2}$$

$$\lim_{x \to -\infty} \frac{4x^2-x}{2x^3-5} = \lim_{x \to -\infty} \frac{4x^2}{2x^3} = \lim_{x \to -\infty} \frac{2}{x} = 0$$

$$\lim_{x \to +\infty} \frac{5x^3-2x^2+1}{1-3x} = \lim_{x \to +\infty} \frac{5x^3}{(-3x)} = \lim_{x \to +\infty} \left(-\frac{5}{3}x^2\right) = -\infty \blacktriangleleft$$

LIMITS INVOLVING RADICALS

▶ Example 10 Find

$$\text{(a)} \lim_{x \to +\infty} \frac{\sqrt{x^2+2}}{3x-6} \qquad \text{(b)} \lim_{x \to -\infty} \frac{\sqrt{x^2+2}}{3x-6}$$

In both parts it would be helpful to manipulate the function so that the powers of x are transformed to powers of $1/x$. This can be achieved in both cases by dividing the numerator and denominator by $|x|$ and using the fact that $\sqrt{x^2} = |x|$.

Solution (a). As $x \to +\infty$, the values of x under consideration are positive, so we can replace $|x|$ by x where helpful. We obtain

$$\lim_{x \to +\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x \to +\infty} \frac{\dfrac{\sqrt{x^2+2}}{|x|}}{\dfrac{3x-6}{|x|}} = \lim_{x \to +\infty} \frac{\dfrac{\sqrt{x^2+2}}{\sqrt{x^2}}}{\dfrac{3x-6}{x}}$$

$$= \lim_{x \to +\infty} \frac{\sqrt{1+\dfrac{2}{x^2}}}{3-\dfrac{6}{x}} = \frac{\lim\limits_{x \to +\infty} \sqrt{1+\dfrac{2}{x^2}}}{\lim\limits_{x \to +\infty} \left(3-\dfrac{6}{x}\right)}$$

$$= \frac{\sqrt{\lim\limits_{x \to +\infty} \left(1+\dfrac{2}{x^2}\right)}}{\lim\limits_{x \to +\infty} \left(3-\dfrac{6}{x}\right)} = \frac{\sqrt{\left(\lim\limits_{x \to +\infty} 1\right) + \left(2 \lim\limits_{x \to +\infty} \dfrac{1}{x^2}\right)}}{\left(\lim\limits_{x \to +\infty} 3\right) - \left(6 \lim\limits_{x \to +\infty} \dfrac{1}{x}\right)}$$

$$= \frac{\sqrt{1+(2 \cdot 0)}}{3-(6 \cdot 0)} = \frac{1}{3}$$

TECHNOLOGY MASTERY

It follows from Example 10 that the function

$$f(x) = \frac{\sqrt{x^2+2}}{3x-6}$$

has an asymptote of $y = \frac{1}{3}$ in the positive direction and an asymptote of $y = -\frac{1}{3}$ in the negative direction. Confirm this using a graphing utility.

Solution (b). As $x \to -\infty$, the values of x under consideration are negative, so we can replace $|x|$ by $-x$ where helpful. We obtain

$$\lim_{x \to -\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x \to -\infty} \frac{\dfrac{\sqrt{x^2+2}}{|x|}}{\dfrac{3x-6}{|x|}} = \lim_{x \to -\infty} \frac{\dfrac{\sqrt{x^2+2}}{\sqrt{x^2}}}{\dfrac{3x-6}{(-x)}}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{1+\dfrac{2}{x^2}}}{-3+\dfrac{6}{x}} = -\frac{1}{3} \blacktriangleleft$$

Example 11 Find

$$\text{(a)} \lim_{x\to+\infty}(\sqrt{x^6+5}-x^3) \qquad \text{(b)} \lim_{x\to+\infty}(\sqrt{x^6+5x^3}-x^3)$$

Solution. Graphs of the functions $f(x)=\sqrt{x^6+5}-x^3$, and $g(x)=\sqrt{x^6+5x^3}-x^3$ for $x\geq 0$, are shown in Figure 1.3.6. From the graphs we might conjecture that the requested limits are 0 and $\frac{5}{2}$, respectively. To confirm this, we treat each function as a fraction with a denominator of 1 and rationalize the numerator.

$$\begin{aligned}
\lim_{x\to+\infty}(\sqrt{x^6+5}-x^3) &= \lim_{x\to+\infty}(\sqrt{x^6+5}-x^3)\left(\frac{\sqrt{x^6+5}+x^3}{\sqrt{x^6+5}+x^3}\right)\\
&= \lim_{x\to+\infty}\frac{(x^6+5)-x^6}{\sqrt{x^6+5}+x^3} = \lim_{x\to+\infty}\frac{5}{\sqrt{x^6+5}+x^3}\\
&= \lim_{x\to+\infty}\frac{\dfrac{5}{x^3}}{\sqrt{1+\dfrac{5}{x^6}}+1} \qquad \boxed{\sqrt{x^6}=x^3 \text{ for } x>0}\\
&= \frac{0}{\sqrt{1+0}+1} = 0
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to+\infty}(\sqrt{x^6+5x^3}-x^3) &= \lim_{x\to+\infty}(\sqrt{x^6+5x^3}-x^3)\left(\frac{\sqrt{x^6+5x^3}+x^3}{\sqrt{x^6+5x^3}+x^3}\right)\\
&= \lim_{x\to+\infty}\frac{(x^6+5x^3)-x^6}{\sqrt{x^6+5x^3}+x^3} = \lim_{x\to+\infty}\frac{5x^3}{\sqrt{x^6+5x^3}+x^3}\\
&= \lim_{x\to+\infty}\frac{5}{\sqrt{1+\dfrac{5}{x^3}}+1} \qquad \boxed{\sqrt{x^6}=x^3 \text{ for } x>0}\\
&= \frac{5}{\sqrt{1+0}+1} = \frac{5}{2} \blacktriangleleft
\end{aligned}$$

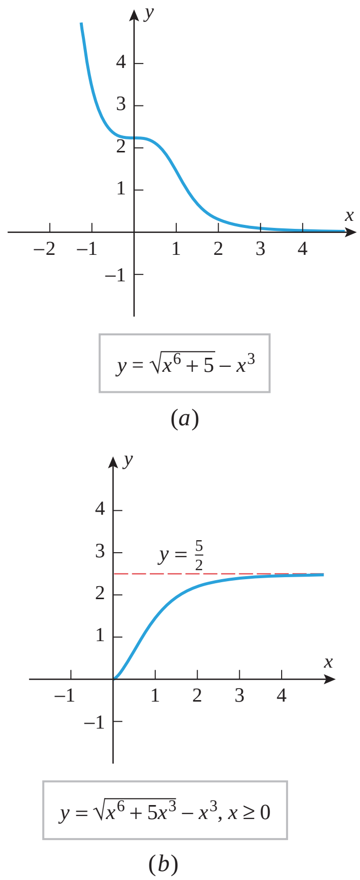


Figure 1.3.6

We noted in Section 1.1 that the standard rules of algebra do not apply to the symbols $+\infty$ and $-\infty$. Part (b) of Example 11 illustrates this. The terms $\sqrt{x^6+5x^3}$ and x^3 both approach $+\infty$ as $x\to+\infty$, but their difference does not approach 0.

END BEHAVIOR OF TRIGONOMETRIC, EXPONENTIAL, AND LOGARITHMIC FUNCTIONS

Consider the function $f(x)=\sin x$ that is graphed in Figure 1.3.7. For this function the limits as $x\to+\infty$ and as $x\to-\infty$ fail to exist not because $f(x)$ increases or decreases without bound, but rather because the values vary between -1 and 1 without approaching some specific real number. In general, the trigonometric functions fail to have limits as $x\to+\infty$ and as $x\to-\infty$ because of periodicity. There is no specific notation to denote this kind of behavior.

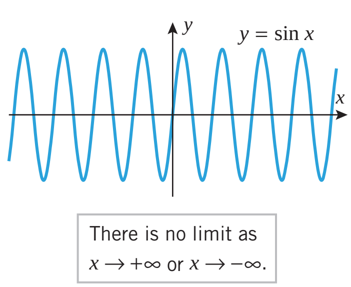


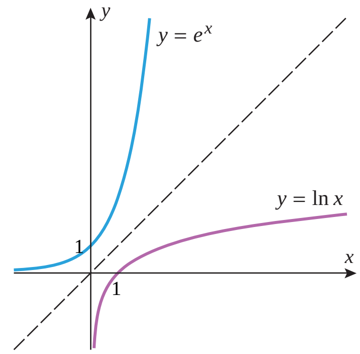
Figure 1.3.7

In Section 0.5 we showed that the functions e^x and $\ln x$ both increase without bound as $x\to+\infty$ (Figures 0.5.8 and 0.5.9). Thus, in limit notation we have

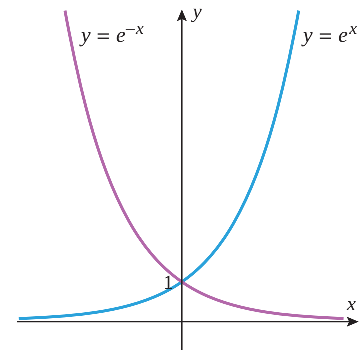
$$\lim_{x\to+\infty}\ln x=+\infty \qquad \lim_{x\to+\infty}e^x=+\infty \tag{20–21}$$

For reference, we also list the following limits, which are consistent with the graphs in Figure 1.3.8:

$$\lim_{x\to-\infty}e^x=0 \qquad \lim_{x\to 0^+}\ln x=-\infty \tag{22–23}$$



▲ **Figure 1.3.8**



▲ **Figure 1.3.9**

Finally, the following limits can be deduced by noting that the graph of $y = e^{-x}$ is the reflection about the y-axis of the graph of $y = e^x$ (Figure 1.3.9).

$$\lim_{x\to+\infty} e^{-x} = 0 \qquad \lim_{x\to-\infty} e^{-x} = +\infty \tag{24–25}$$

✔ QUICK CHECK EXERCISES 1.3 *(See page 100 for answers.)*

1. Find the limits.

(a) $\lim_{x\to-\infty}(3-x) =$ ______

(b) $\lim_{x\to+\infty}\left(5-\dfrac{1}{x}\right) =$ ______

(c) $\lim_{x\to+\infty}\ln\left(\dfrac{1}{x}\right) =$ ______

(d) $\lim_{x\to+\infty}\dfrac{1}{e^x} =$ ______

2. Find the limits that exist.

(a) $\lim_{x\to-\infty}\dfrac{2x^2+x}{4x^2-3} =$ ______

(b) $\lim_{x\to+\infty}\dfrac{1}{2+\sin x} =$ ______

(c) $\lim_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^x =$ ______

3. Given that

$$\lim_{x\to+\infty} f(x) = 2 \quad \text{and} \quad \lim_{x\to+\infty} g(x) = -3$$

find the limits that exist.

(a) $\lim_{x\to+\infty}[3f(x) - g(x)] =$ ______

(b) $\lim_{x\to+\infty}\dfrac{f(x)}{g(x)} =$ ______

(c) $\lim_{x\to+\infty}\dfrac{2f(x)+3g(x)}{3f(x)+2g(x)} =$ ______

(d) $\lim_{x\to+\infty}\sqrt{10-f(x)g(x)} =$ ______

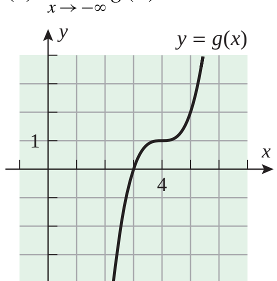
4. Consider the graphs of $1/x$, $\sin x$, $\ln x$, e^x, and e^{-x}. Which of these graphs has a horizontal asymptote?

EXERCISE SET 1.3 Graphing Utility

1–4 In these exercises, make reasonable assumptions about the end behavior of the indicated function. ■

1. For the function g graphed in the accompanying figure, find

(a) $\lim_{x\to-\infty} g(x)$ (b) $\lim_{x\to+\infty} g(x)$.

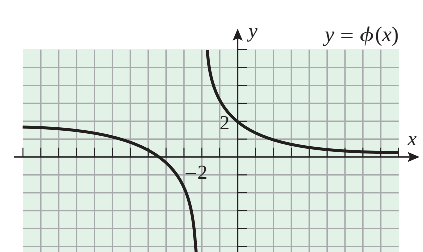


◀ **Figure Ex-1**

2. For the function ϕ graphed in the accompanying figure, find

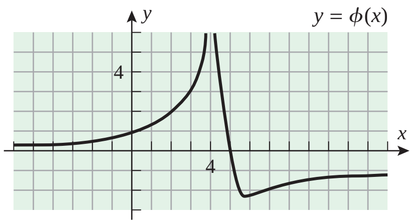
(a) $\lim_{x\to-\infty} \phi(x)$

(b) $\lim_{x\to+\infty} \phi(x)$.



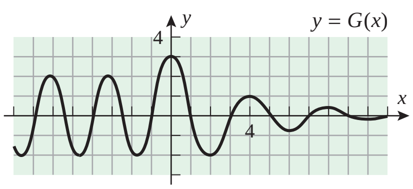
◀ **Figure Ex-2**

3. For the function ϕ graphed in the accompanying figure, find
(a) $\lim_{x\to-\infty} \phi(x)$ (b) $\lim_{x\to+\infty} \phi(x)$.



◀ **Figure Ex-3**

4. For the function G graphed in the accompanying figure, find
(a) $\lim_{x\to-\infty} G(x)$ (b) $\lim_{x\to+\infty} G(x)$.



◀ **Figure Ex-4**

5. Given that
$$\lim_{x\to+\infty} f(x) = 3, \quad \lim_{x\to+\infty} g(x) = -5, \quad \lim_{x\to+\infty} h(x) = 0$$
find the limits that exist. If the limit does not exist, explain why.
(a) $\lim_{x\to+\infty} [f(x) + 3g(x)]$
(b) $\lim_{x\to+\infty} [h(x) - 4g(x) + 1]$
(c) $\lim_{x\to+\infty} [f(x)g(x)]$ (d) $\lim_{x\to+\infty} [g(x)]^2$
(e) $\lim_{x\to+\infty} \sqrt[3]{5 + f(x)}$ (f) $\lim_{x\to+\infty} \dfrac{3}{g(x)}$
(g) $\lim_{x\to+\infty} \dfrac{3h(x)+4}{x^2}$ (h) $\lim_{x\to+\infty} \dfrac{6f(x)}{5f(x)+3g(x)}$

6. Given that
$$\lim_{x\to-\infty} f(x) = 7 \quad \text{and} \quad \lim_{x\to-\infty} g(x) = -6$$
find the limits that exist. If the limit does not exist, explain why.
(a) $\lim_{x\to-\infty} [2f(x) - g(x)]$ (b) $\lim_{x\to-\infty} [6f(x) + 7g(x)]$
(c) $\lim_{x\to-\infty} [x^2 + g(x)]$ (d) $\lim_{x\to-\infty} [x^2 g(x)]$
(e) $\lim_{x\to-\infty} \sqrt[3]{f(x)g(x)}$ (f) $\lim_{x\to-\infty} \dfrac{g(x)}{f(x)}$
(g) $\lim_{x\to-\infty} \left[f(x) + \dfrac{g(x)}{x}\right]$ (h) $\lim_{x\to-\infty} \dfrac{xf(x)}{(2x+3)g(x)}$

7. (a) Complete the table and make a guess about the limit indicated.
$$f(x) = \tan^{-1}\left(\frac{1}{x}\right) \qquad \lim_{x\to0^+} f(x)$$

x	0.1	0.01	0.001	0.0001	0.00001	0.000001
$f(x)$						

(b) Use Figure 1.3.3 to find the exact value of the limit in part (a).

8. Complete the table and make a guess about the limit indicated.
$$f(x) = x^{1/x} \qquad \lim_{x\to+\infty} f(x)$$

x	10	100	1000	10,000	100,000	1,000,000
$f(x)$						

9–40 Find the limits. ■

9. $\lim_{x\to+\infty} (1 + 2x - 3x^5)$

10. $\lim_{x\to+\infty} (2x^3 - 100x + 5)$

11. $\lim_{x\to+\infty} \sqrt{x}$

12. $\lim_{x\to-\infty} \sqrt{5-x}$

13. $\lim_{x\to+\infty} \dfrac{3x+1}{2x-5}$

14. $\lim_{x\to+\infty} \dfrac{5x^2-4x}{2x^2+3}$

15. $\lim_{y\to-\infty} \dfrac{3}{y+4}$

16. $\lim_{x\to+\infty} \dfrac{1}{x-12}$

17. $\lim_{x\to-\infty} \dfrac{x-2}{x^2+2x+1}$

18. $\lim_{x\to+\infty} \dfrac{5x^2+7}{3x^2-x}$

19. $\lim_{x\to+\infty} \dfrac{7-6x^5}{x+3}$

20. $\lim_{t\to-\infty} \dfrac{5-2t^3}{t^2+1}$

21. $\lim_{t\to+\infty} \dfrac{6-t^3}{7t^3+3}$

22. $\lim_{x\to-\infty} \dfrac{x+4x^3}{1-x^2+7x^3}$

23. $\lim_{x\to+\infty} \sqrt[3]{\dfrac{2+3x-5x^2}{1+8x^2}}$

24. $\lim_{s\to+\infty} \sqrt[3]{\dfrac{3s^7-4s^5}{2s^7+1}}$

25. $\lim_{x\to-\infty} \dfrac{\sqrt{5x^2-2}}{x+3}$

26. $\lim_{x\to+\infty} \dfrac{\sqrt{5x^2-2}}{x+3}$

27. $\lim_{y\to-\infty} \dfrac{2-y}{\sqrt{7+6y^2}}$

28. $\lim_{y\to+\infty} \dfrac{2-y}{\sqrt{7+6y^2}}$

29. $\lim_{x\to-\infty} \dfrac{\sqrt{3x^4+x}}{x^2-8}$

30. $\lim_{x\to+\infty} \dfrac{\sqrt{3x^4+x}}{x^2-8}$

31. $\lim_{x\to+\infty} (\sqrt{x^2+3} - x)$

32. $\lim_{x\to+\infty} (\sqrt{x^2-3x} - x)$

33. $\lim_{x\to-\infty} \dfrac{1-e^x}{1+e^x}$

34. $\lim_{x\to+\infty} \dfrac{1-e^x}{1+e^x}$

35. $\lim_{x\to+\infty} \dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

36. $\lim_{x\to-\infty} \dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

37. $\lim_{x\to+\infty} \ln\left(\dfrac{2}{x^2}\right)$

38. $\lim_{x\to0^+} \ln\left(\dfrac{2}{x^2}\right)$

39. $\lim_{x\to+\infty} \dfrac{(x+1)^x}{x^x}$

40. $\lim_{x\to+\infty} \left(1+\dfrac{1}{x}\right)^{-x}$

41–44 True–False Determine whether the statement is true or false. Explain your answer. ■

41. We have $\lim_{x\to+\infty} \left(1+\dfrac{1}{x}\right)^{2x} = (1+0)^{+\infty} = 1^{+\infty} = 1$.

42. If $y = L$ is a horizontal asymptote for the curve $y = f(x)$, then

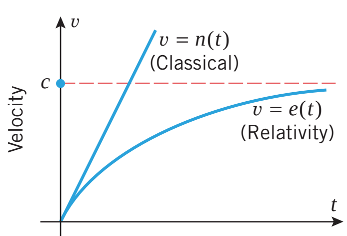
$$\lim_{x \to -\infty} f(x) = L \quad \text{and} \quad \lim_{x \to +\infty} f(x) = L$$

43. If $y = L$ is a horizontal asymptote for the curve $y = f(x)$, then it is possible for the graph of f to intersect the line $y = L$ infinitely many times.

44. If a rational function $p(x)/q(x)$ has a horizontal asymptote, then the degree of $p(x)$ must equal the degree of $q(x)$.

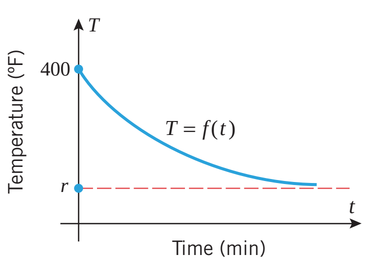
FOCUS ON CONCEPTS

45. Assume that a particle is accelerated by a constant force. The two curves $v = n(t)$ and $v = e(t)$ in the accompanying figure provide velocity versus time curves for the particle as predicted by classical physics and by the special theory of relativity, respectively. The parameter c represents the speed of light. Using the language of limits, describe the differences in the long-term predictions of the two theories.



◀ **Figure Ex-45**

46. Let $T = f(t)$ denote the temperature of a baked potato t minutes after it has been removed from a hot oven. The accompanying figure shows the temperature versus time curve for the potato, where r is the temperature of the room.
(a) What is the physical significance of $\lim_{t \to 0^+} f(t)$?
(b) What is the physical significance of $\lim_{t \to +\infty} f(t)$?



◀ **Figure Ex-46**

47. Let

$$f(x) = \begin{cases} 2x^2 + 5, & x < 0 \\ \dfrac{3 - 5x^3}{1 + 4x + x^3}, & x \geq 0 \end{cases}$$

Find
(a) $\lim_{x \to -\infty} f(x)$ (b) $\lim_{x \to +\infty} f(x)$.

48. Let

$$g(t) = \begin{cases} \dfrac{2 + 3t}{5t^2 + 6}, & t < 1{,}000{,}000 \\ \dfrac{\sqrt{36t^2 - 100}}{5 - t}, & t > 1{,}000{,}000 \end{cases}$$

Find
(a) $\lim_{t \to -\infty} g(t)$ (b) $\lim_{t \to +\infty} g(t)$.

49. Discuss the limits of $p(x) = (1 - x)^n$ as $x \to +\infty$ and $x \to -\infty$ for positive integer values of n.

50. In each part, find examples of polynomials $p(x)$ and $q(x)$ that satisfy the stated condition and such that $p(x) \to +\infty$ and $q(x) \to +\infty$ as $x \to +\infty$.
(a) $\displaystyle\lim_{x \to +\infty} \frac{p(x)}{q(x)} = 1$ (b) $\displaystyle\lim_{x \to +\infty} \frac{p(x)}{q(x)} = 0$
(c) $\displaystyle\lim_{x \to +\infty} \frac{p(x)}{q(x)} = +\infty$ (d) $\displaystyle\lim_{x \to +\infty} [p(x) - q(x)] = 3$

51. (a) Do any of the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ have horizontal asymptotes?
(b) Do any of the trigonometric functions have vertical asymptotes? Where?

52. Find

$$\lim_{x \to +\infty} \frac{c_0 + c_1 x + \cdots + c_n x^n}{d_0 + d_1 x + \cdots + d_m x^m}$$

where $c_n \neq 0$ and $d_m \neq 0$. [*Hint:* Your answer will depend on whether $m < n$, $m = n$, or $m > n$.]

FOCUS ON CONCEPTS

53–54 These exercises develop some versions of the ***substitution principle***, a useful tool for the evaluation of limits.

53. (a) Explain why we can evaluate $\lim_{x \to +\infty} e^{x^2}$ by making the substitution $t = x^2$ and writing

$$\lim_{x \to +\infty} e^{x^2} = \lim_{t \to +\infty} e^t = +\infty$$

(b) Suppose $g(x) \to +\infty$ as $x \to +\infty$. Given any function $f(x)$, explain why we can evaluate $\lim_{x \to +\infty} f[g(x)]$ by substituting $t = g(x)$ and writing

$$\lim_{x \to +\infty} f[g(x)] = \lim_{t \to +\infty} f(t)$$

(Here, "equality" is interpreted to mean that either both limits exist and are equal or that both limits fail to exist.)
(c) Why does the result in part (b) remain valid if $\lim_{x \to +\infty}$ is replaced everywhere by one of $\lim_{x \to -\infty}$, $\lim_{x \to c}$, $\lim_{x \to c^-}$, or $\lim_{x \to c^+}$?

54. (a) Explain why we can evaluate $\lim_{x \to +\infty} e^{-x^2}$ by making the substitution $t = -x^2$ and writing

$$\lim_{x \to +\infty} e^{-x^2} = \lim_{t \to -\infty} e^t = 0$$

(cont.)

(b) Suppose $g(x) \to -\infty$ as $x \to +\infty$. Given any function $f(x)$, explain why we can evaluate $\lim_{x\to+\infty} f[g(x)]$ by substituting $t = g(x)$ and writing

$$\lim_{x\to+\infty} f[g(x)] = \lim_{t\to-\infty} f(t)$$

(Here, "equality" is interpreted to mean that either both limits exist and are equal or that both limits fail to exist.)

(c) Why does the result in part (b) remain valid if $\lim_{x\to+\infty}$ is replaced everywhere by one of $\lim_{x\to-\infty}$, $\lim_{x\to c}$, $\lim_{x\to c^-}$, or $\lim_{x\to c^+}$?

55–62 Evaluate the limit using an appropriate substitution.

55. $\lim_{x\to 0^+} e^{1/x}$

56. $\lim_{x\to 0^-} e^{1/x}$

57. $\lim_{x\to 0^+} e^{\csc x}$

58. $\lim_{x\to 0^-} e^{\csc x}$

59. $\lim_{x\to+\infty} \dfrac{\ln 2x}{\ln 3x}$ [*Hint:* $t = \ln x$]

60. $\lim_{x\to+\infty} [\ln(x^2 - 1) - \ln(x + 1)]$ [*Hint:* $t = x - 1$]

61. $\lim_{x\to+\infty} \left(1 - \dfrac{1}{x}\right)^{-x}$ [*Hint:* $t = -x$]

62. $\lim_{x\to+\infty} \left(1 + \dfrac{2}{x}\right)^{x}$ [*Hint:* $t = x/2$]

63. Let $f(x) = b^x$, where $0 < b$. Use the substitution principle to verify the asymptotic behavior of f that is illustrated in Figure 0.5.1. [*Hint:* $f(x) = b^x = (e^{\ln b})^x = e^{(\ln b)x}$]

64. Prove that $\lim_{x\to 0}(1 + x)^{1/x} = e$ by completing parts (a) and (b).

(a) Use Equation (7) and the substitution $t = 1/x$ to prove that $\lim_{x\to 0^+}(1 + x)^{1/x} = e$.

(b) Use Equation (8) and the substitution $t = 1/x$ to prove that $\lim_{x\to 0^-}(1 + x)^{1/x} = e$.

65. Suppose that the speed v (in ft/s) of a skydiver t seconds after leaping from a plane is given by the equation $v = 190(1 - e^{-0.168t})$.

(a) Graph v versus t.

(b) By evaluating an appropriate limit, show that the graph of v versus t has a horizontal asymptote $v = c$ for an appropriate constant c.

(c) What is the physical significance of the constant c in part (b)?

66. The population p of the United States (in millions) in year t can be modeled by the function

$$p(t) = \frac{525}{1 + 1.1e^{-0.02225(t-1990)}}$$

(a) Based on this model, what was the U.S. population in 1990?

(b) Plot p versus t for the 200-year period from 1950 to 2150.

(c) By evaluating an appropriate limit, show that the graph of p versus t has a horizontal asymptote $p = c$ for an appropriate constant c.

(d) What is the significance of the constant c in part (c) for the population predicted by this model?

67. (a) Compute the (approximate) values of the terms in the sequence

$$1.01^{101},\ 1.001^{1001},\ 1.0001^{10001},\ 1.00001^{100001},\ 1.000001^{1000001},\ 1.0000001^{10000001}\ \ldots$$

What number do these terms appear to be approaching?

(b) Use Equation (7) to verify your answer in part (a).

(c) Let $1 \le a \le 9$ denote a positive integer. What number is approached more and more closely by the terms in the following sequence?

$$1.01^{a0a},\ 1.001^{a00a},\ 1.0001^{a000a},\ 1.00001^{a0000a},\ 1.000001^{a00000a},\ 1.0000001^{a000000a}\ \ldots$$

(The powers are positive integers that begin and end with the digit a and have 0's in the remaining positions).

68. Let $f(x) = \left(1 + \dfrac{1}{x}\right)^x$.

(a) Prove the identity

$$f(-x) = \frac{x}{x-1} \cdot f(x-1)$$

(b) Use Equation (7) and the identity from part (a) to prove Equation (8).

69–73 The notion of an asymptote can be extended to include curves as well as lines. Specifically, we say that curves $y = f(x)$ and $y = g(x)$ are ***asymptotic as*** $x \to +\infty$ provided

$$\lim_{x\to+\infty} [f(x) - g(x)] = 0$$

and are ***asymptotic as*** $x \to -\infty$ provided

$$\lim_{x\to-\infty} [f(x) - g(x)] = 0$$

In these exercises, determine a simpler function $g(x)$ such that $y = f(x)$ is asymptotic to $y = g(x)$ as $x \to +\infty$ or $x \to -\infty$. Use a graphing utility to generate the graphs of $y = f(x)$ and $y = g(x)$ and identify all vertical asymptotes.

69. $f(x) = \dfrac{x^2 - 2}{x - 2}$ [*Hint:* Divide $x - 2$ into $x^2 - 2$.]

70. $f(x) = \dfrac{x^3 - x + 3}{x}$

71. $f(x) = \dfrac{-x^3 + 3x^2 + x - 1}{x - 3}$

72. $f(x) = \dfrac{x^5 - x^3 + 3}{x^2 - 1}$

73. $f(x) = \sin x + \dfrac{1}{x - 1}$

74. **Writing** In some models for learning a skill (e.g., juggling), it is assumed that the skill level for an individual increases with practice but cannot become arbitrarily high. How do concepts of this section apply to such a model?

75. Writing In some population models it is assumed that a given ecological system possesses a ***carrying capacity*** L. Populations greater than the carrying capacity tend to decline toward L, while populations less than the carrying capacity tend to increase toward L. Explain why these assumptions are reasonable, and discuss how the concepts of this section apply to such a model.

✔ QUICK CHECK ANSWERS 1.3

1. (a) $+\infty$ (b) 5 (c) $-\infty$ (d) 0 **2.** (a) $\frac{1}{2}$ (b) does not exist (c) e **3.** (a) 9 (b) $-\frac{2}{3}$ (c) does not exist (d) 4
4. $1/x$, e^x, and e^{-x} each has a horizontal asymptote.

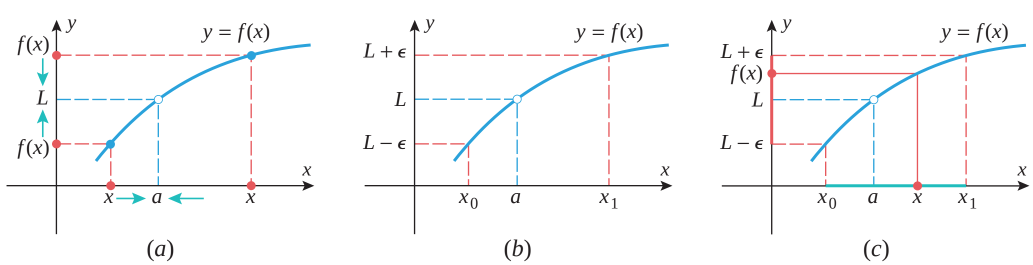
1.4 LIMITS (DISCUSSED MORE RIGOROUSLY)

In the previous sections of this chapter we focused on the discovery of values of limits, either by sampling selected x-values or by applying limit theorems that were stated without proof. Our main goal in this section is to define the notion of a limit precisely, thereby making it possible to establish limits with certainty and to prove theorems about them. This will also provide us with a deeper understanding of some of the more subtle properties of functions.

MOTIVATION FOR THE DEFINITION OF A TWO-SIDED LIMIT

The statement $\lim_{x\to a} f(x) = L$ can be interpreted informally to mean that we can make the value of $f(x)$ as close as we like to the real number L by making the value of x sufficiently close to a. It is our goal to make the informal phrases "as close as we like to L" and "sufficiently close to a" mathematically precise.

To do this, consider the function f graphed in Figure 1.4.1a for which $f(x)\to L$ as $x\to a$. For visual simplicity we have drawn the graph of f to be increasing on an open interval containing a, and we have intentionally placed a hole in the graph at $x = a$ to emphasize that f need not be defined at $x = a$ to have a limit there.



▲ **Figure 1.4.1**

Next, let us choose any positive number ϵ and ask how close x must be to a in order for the values of $f(x)$ to be within ϵ units of L. We can answer this geometrically by drawing horizontal lines from the points $L + \epsilon$ and $L - \epsilon$ on the y-axis until they meet the curve $y = f(x)$, and then drawing vertical lines from those points on the curve to the x-axis (Figure 1.4.1b). As indicated in the figure, let x_0 and x_1 be the points where those vertical lines intersect the x-axis.

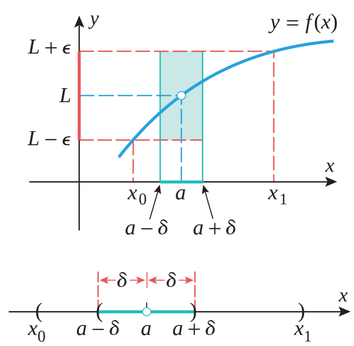
Now imagine that x gets closer and closer to a (from either side). Eventually, x will lie inside the interval (x_0, x_1), which is marked in green in Figure 1.4.1c; and when this happens, the value of $f(x)$ will fall between $L - \epsilon$ and $L + \epsilon$, marked in red in the figure. Thus, we conclude:

If $f(x) \to L$ as $x \to a$, then for any positive number ϵ, we can find an open interval (x_0, x_1) on the x-axis that contains a and has the property that for each x in that interval (except possibly for $x = a$), the value of $f(x)$ is between $L - \epsilon$ and $L + \epsilon$.

What is important about this result is that it holds no matter how small we make ϵ. However, making ϵ smaller and smaller forces $f(x)$ *closer and closer* to L—which is precisely the concept we were trying to capture mathematically.

Observe that in Figure 1.4.1 the interval (x_0, x_1) extends farther on the right side of a than on the left side. However, for many purposes it is preferable to have an interval that extends the same distance on both sides of a. For this purpose, let us choose any positive number δ that is smaller than both $x_1 - a$ and $a - x_0$, and consider the interval

$$(a - \delta, a + \delta)$$



▲ **Figure 1.4.2**

This interval extends the same distance δ on both sides of a and lies inside of the interval (x_0, x_1) (Figure 1.4.2). Moreover, the condition

$$L - \epsilon < f(x) < L + \epsilon \tag{1}$$

holds for every x in this interval (except possibly $x = a$), since this condition holds on the larger interval (x_0, x_1).

Since (1) can be expressed as

$$|f(x) - L| < \epsilon$$

and the condition that x lies in the interval $(a - \delta, a + \delta)$, but $x \neq a$, can be expressed as

$$0 < |x - a| < \delta$$

we are led to the following precise definition of a two-sided limit.

The definitions of one-sided limits require minor adjustments to Definition 1.4.1. For example, for a limit from the right we need only assume that $f(x)$ is defined on an interval (a, b) extending to the right of a and that the ϵ condition is met for x in an interval $a < x < a + \delta$ extending to the right of a. A similar adjustment must be made for a limit from the left. (See Exercise 27.)

1.4.1 LIMIT DEFINITION Let $f(x)$ be defined for all x in some open interval containing the number a, with the possible exception that $f(x)$ need not be defined at a. We will write

$$\lim_{x \to a} f(x) = L$$

if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta$$

This definition, which is attributed to the German mathematician Karl Weierstrass and is commonly called the "epsilon-delta" definition of a two-sided limit, makes the transition from an informal concept of a limit to a precise definition. Specifically, the informal phrase "as close as we like to L" is given quantitative meaning by our ability to choose the positive number ϵ arbitrarily, and the phrase "sufficiently close to a" is quantified by the positive number δ.

In the preceding sections we illustrated various numerical and graphical methods for *guessing* at limits. Now that we have a precise definition to work with, we can actually confirm the validity of those guesses with mathematical proof. Here is a typical example of such a proof.

▶ Example 1 Use Definition 1.4.1 to prove that $\lim_{x\to 2}(3x-5)=1$.

Solution. We must show that given any positive number ϵ, we can find a positive number δ such that

$$|\underbrace{(3x-5)}_{f(x)} - \underbrace{1}_{L}| < \epsilon \quad \text{if} \quad 0 < |x - \underbrace{2}_{a}| < \delta \tag{2}$$

There are two things to do. First, we must *discover* a value of δ for which this statement holds, and then we must *prove* that the statement holds for that δ. For the discovery part we begin by simplifying (2) and writing it as

$$|3x-6| < \epsilon \quad \text{if} \quad 0 < |x-2| < \delta$$

Next we will rewrite this statement in a form that will facilitate the discovery of an appropriate δ:

$$\begin{aligned} 3|x-2| &< \epsilon \quad \text{if} \quad 0 < |x-2| < \delta \\ |x-2| &< \epsilon/3 \quad \text{if} \quad 0 < |x-2| < \delta \end{aligned} \tag{3}$$

It should be self-evident that this last statement holds if $\delta = \epsilon/3$, which completes the discovery portion of our work. Now we need to prove that (2) holds for this choice of δ. However, statement (2) is equivalent to (3), and (3) holds with $\delta = \epsilon/3$, so (2) also holds with $\delta = \epsilon/3$. This proves that $\lim_{x\to 2}(3x-5)=1$. ◀

This example illustrates the general form of a limit proof: We *assume* that we are given a positive number ϵ, and we try to *prove* that we can find a positive number δ such that

$$|f(x) - L| < \epsilon \quad \text{if} \quad 0 < |x-a| < \delta \tag{4}$$

This is done by first discovering δ, and then proving that the discovered δ works. Since the argument has to be general enough to work for all positive values of ϵ, the quantity δ has to be expressed as a function of ϵ. In Example 1 we found the function $\delta = \epsilon/3$ by some simple algebra; however, most limit proofs require a little more algebraic and logical ingenuity. Thus, if you find our ensuing discussion of "ϵ-δ" proofs challenging, do not become discouraged; the concepts and techniques are intrinsically difficult. In fact, a precise understanding of limits evaded the finest mathematical minds for more than 150 years after the basic concepts of calculus were discovered.

Karl Weierstrass (1815–1897) Weierstrass, the son of a customs officer, was born in Ostenfelde, Germany. As a youth Weierstrass showed outstanding skills in languages and mathematics. However, at the urging of his dominant father, Weierstrass entered the law and commerce program at the University of Bonn. To the chagrin of his family, the rugged and congenial young man concentrated instead on fencing and beer drinking. Four years later he returned home without a degree. In 1839 Weierstrass entered the Academy of Münster to study for a career in secondary education, and he met and studied under an excellent mathematician named Christof Gudermann. Gudermann's ideas greatly influenced the work of Weierstrass. After receiving his teaching certificate, Weierstrass spent the next 15 years in secondary education teaching German, geography, and mathematics. In addition, he taught handwriting to small children. During this period much of Weierstrass's mathematical work was ignored because he was a secondary schoolteacher and not a college professor. Then, in 1854, he published a paper of major importance that created a sensation in the mathematics world and catapulted him to international fame overnight. He was immediately given an honorary Doctorate at the University of Königsberg and began a new career in college teaching at the University of Berlin in 1856. In 1859 the strain of his mathematical research caused a temporary nervous breakdown and led to spells of dizziness that plagued him for the rest of his life. Weierstrass was a brilliant teacher and his classes overflowed with multitudes of auditors. In spite of his fame, he never lost his early beer-drinking congeniality and was always in the company of students, both ordinary and brilliant. Weierstrass was acknowledged as the leading mathematical analyst in the world. He and his students opened the door to the modern school of mathematical analysis.

[Image: http://commons.wikimedia.org/wiki/File:Karl_Weierstrass.jpg]

Example 2 Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Solution. Note that the domain of $\sqrt{x}$ is $0 \le x$, so it is valid to discuss the limit as $x \to 0^+$. We must show that given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\sqrt{x} - 0| < \epsilon \quad \text{if} \quad 0 < x - 0 < \delta$$

or more simply,

$$\sqrt{x} < \epsilon \quad \text{if} \quad 0 < x < \delta \tag{5}$$

In Example 2 the limit from the left and the two-sided limit do not exist at $x = 0$ because $\sqrt{x}$ is defined only for nonnegative values of x.

But, by squaring both sides of the inequality $\sqrt{x} < \epsilon$, we can rewrite (5) as

$$x < \epsilon^2 \quad \text{if} \quad 0 < x < \delta \tag{6}$$

It should be self-evident that (6) is true if $\delta = \epsilon^2$; and since (6) is a reformulation of (5), we have shown that (5) holds with $\delta = \epsilon^2$. This proves that $\lim_{x \to 0^+} \sqrt{x} = 0$. ◀

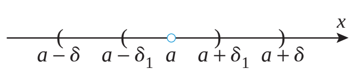
THE VALUE OF δ IS NOT UNIQUE

In preparation for our next example, we note that the value of δ in Definition 1.4.1 is not unique; once we have found a value of δ that fulfills the requirements of the definition, then any *smaller* positive number δ_1 will also fulfill those requirements. That is, if it is true that

$$|f(x) - L| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta$$

then it will also be true that

$$|f(x) - L| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta_1$$



▲ **Figure 1.4.3**

This is because $\{x : 0 < |x - a| < \delta_1\}$ is a subset of $\{x : 0 < |x - a| < \delta\}$ (Figure 1.4.3), and hence if $|f(x) - L| < \epsilon$ is satisfied for all x in the larger set, then it will automatically be satisfied for all x in the subset. Thus, in Example 1, where we used $\delta = \epsilon/3$, we could have used any smaller value of δ such as $\delta = \epsilon/4$, $\delta = \epsilon/5$, or $\delta = \epsilon/6$.

Example 3 Prove that $\lim_{x \to 3} x^2 = 9$.

Solution. We must show that given any positive number ϵ, we can find a positive number δ such that

$$|x^2 - 9| < \epsilon \quad \text{if} \quad 0 < |x - 3| < \delta \tag{7}$$

Because $|x - 3|$ occurs on the right side of this "if statement," it will be helpful to factor the left side to introduce a factor of $|x - 3|$. This yields the following alternative form of (7):

$$|x + 3||x - 3| < \epsilon \quad \text{if} \quad 0 < |x - 3| < \delta \tag{8}$$

If you are wondering how we knew to make the restriction $\delta \le 1$, as opposed to $\delta \le 5$ or $\delta \le \frac{1}{2}$, for example, the answer is that 1 is merely a convenient choice—any restriction of the form $\delta \le c$ would work equally well.

We wish to bound the factor $|x + 3|$. If we knew, for example, that $\delta \le 1$, then we would have $-1 < x - 3 < 1$, so $5 < x + 3 < 7$, and consequently $|x + 3| < 7$. Thus, if $\delta \le 1$ and $0 < |x - 3| < \delta$, then

$$|x + 3||x - 3| < 7\delta$$

It follows that (8) will be satisfied for any positive δ such that $\delta \le 1$ and $7\delta < \epsilon$. We can achieve this by taking δ to be the minimum of the numbers 1 and $\epsilon/7$, which is sometimes written as $\delta = \min(1, \epsilon/7)$. This proves that $\lim_{x \to 3} x^2 = 9$. ◀

LIMITS AS $x \to \pm\infty$

In Section 1.3 we discussed the limits

$$\lim_{x \to +\infty} f(x) = L \quad \text{and} \quad \lim_{x \to -\infty} f(x) = L$$

from an intuitive point of view. The first limit can be interpreted to mean that we can make the value of $f(x)$ as close as we like to L by taking x sufficiently large, and the second can be interpreted to mean that we can make the value of $f(x)$ as close as we like to L by taking x sufficiently far to the left of 0. These ideas are captured in the following definitions and are illustrated in Figure 1.4.4.

1.4.2 DEFINITION Let $f(x)$ be defined for all x in some infinite open interval extending in the positive x-direction. We will write

$$\lim_{x\to+\infty} f(x) = L$$

if given any number $\epsilon > 0$, there corresponds a positive number N such that

$$|f(x) - L| < \epsilon \quad \text{if} \quad x > N$$

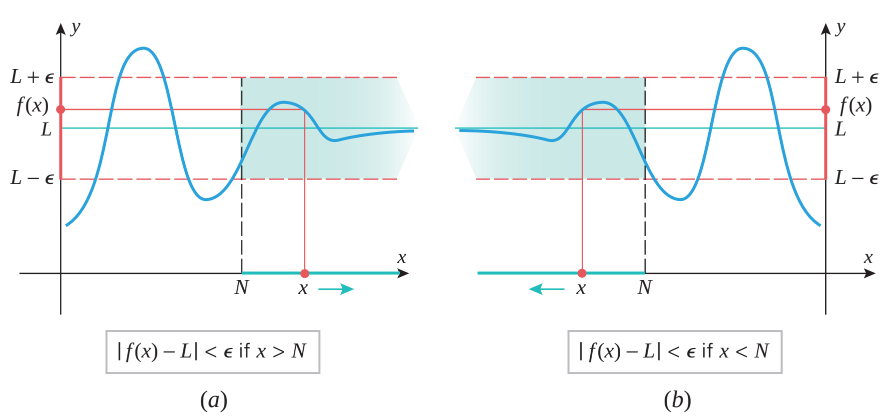
1.4.3 DEFINITION Let $f(x)$ be defined for all x in some infinite open interval extending in the negative x-direction. We will write

$$\lim_{x\to-\infty} f(x) = L$$

if given any number $\epsilon > 0$, there corresponds a negative number N such that

$$|f(x) - L| < \epsilon \quad \text{if} \quad x < N$$

To see how these definitions relate to our informal concepts of these limits, suppose that $f(x)\to L$ as $x\to+\infty$, and for a given ϵ let N be the positive number described in Definition 1.4.2. If x is allowed to increase indefinitely, then eventually x will lie in the interval $(N, +\infty)$, which is marked in green in Figure 1.4.4*a*; when this happens, the value of $f(x)$ will fall between $L-\epsilon$ and $L+\epsilon$, marked in red in the figure. Since this is true for all positive values of ϵ (no matter how small), we can force the values of $f(x)$ as close as we like to L by making N sufficiently large. This agrees with our informal concept of this limit. Similarly, Figure 1.4.4*b* illustrates Definition 1.4.3.



▲ **Figure 1.4.4**

▶ **Example 4** Prove that $\displaystyle\lim_{x\to+\infty} \frac{1}{x} = 0.$

Solution. Applying Definition 1.4.2 with $f(x) = 1/x$ and $L = 0$, we must show that given $\epsilon > 0$, we can find a number $N > 0$ such that

$$\left|\frac{1}{x} - 0\right| < \epsilon \quad \text{if} \quad x > N \tag{9}$$

Because $x \to +\infty$ we can assume that $x > 0$. Thus, we can eliminate the absolute values in this statement and rewrite it as

$$\frac{1}{x} < \epsilon \quad \text{if} \quad x > N$$

or, on taking reciprocals,

$$x > \frac{1}{\epsilon} \quad \text{if} \quad x > N \tag{10}$$

It is self-evident that $N = 1/\epsilon$ satisfies this requirement, and since (10) and (9) are equivalent for $x > 0$, the proof is complete. ◀

INFINITE LIMITS

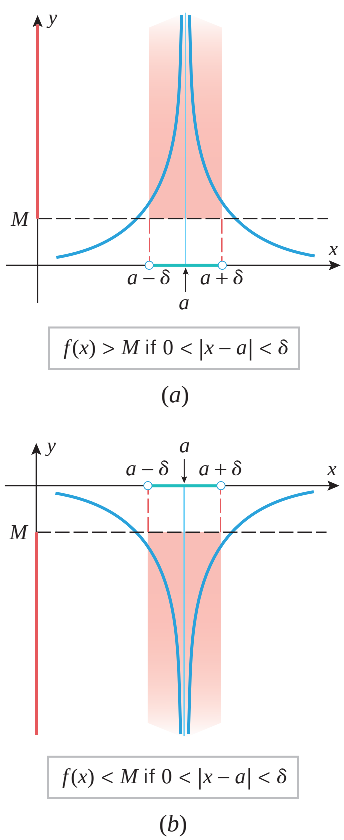
In Section 1.1 we discussed limits of the following type from an intuitive viewpoint:

$$\lim_{x \to a} f(x) = +\infty, \qquad \lim_{x \to a} f(x) = -\infty \tag{11}$$

$$\lim_{x \to a^+} f(x) = +\infty, \qquad \lim_{x \to a^+} f(x) = -\infty \tag{12}$$

$$\lim_{x \to a^-} f(x) = +\infty, \qquad \lim_{x \to a^-} f(x) = -\infty \tag{13}$$

Recall that each of these expressions describes a particular way in which the limit fails to exist. The $+\infty$ indicates that the limit fails to exist because $f(x)$ increases without bound, and the $-\infty$ indicates that the limit fails to exist because $f(x)$ decreases without bound. These ideas are captured more precisely in the following definitions and are illustrated in Figure 1.4.5.



▲ **Figure 1.4.5**

> **1.4.4 DEFINITION** Let $f(x)$ be defined for all x in some open interval containing a, except that $f(x)$ need not be defined at a. We will write
>
> $$\lim_{x \to a} f(x) = +\infty$$
>
> if given any positive number M, we can find a number $\delta > 0$ such that $f(x)$ satisfies
>
> $$f(x) > M \quad \text{if} \quad 0 < |x - a| < \delta$$

> **1.4.5 DEFINITION** Let $f(x)$ be defined for all x in some open interval containing a, except that $f(x)$ need not be defined at a. We will write
>
> $$\lim_{x \to a} f(x) = -\infty$$
>
> if given any negative number M, we can find a number $\delta > 0$ such that $f(x)$ satisfies
>
> $$f(x) < M \quad \text{if} \quad 0 < |x - a| < \delta$$

How would you define these limits?

$$\lim_{x \to a^+} f(x) = +\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$
$$\lim_{x \to a^-} f(x) = +\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$
$$\lim_{x \to +\infty} f(x) = +\infty \qquad \lim_{x \to +\infty} f(x) = -\infty$$
$$\lim_{x \to -\infty} f(x) = +\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

To see how these definitions relate to our informal concepts of these limits, suppose that $f(x) \to +\infty$ as $x \to a$, and for a given M let δ be the corresponding positive number described in Definition 1.4.4. Next, imagine that x gets closer and closer to a (from either side). Eventually, x will lie in the interval $(a - \delta, a + \delta)$, which is marked in green in Figure 1.4.5a; when this happens the value of $f(x)$ will be greater than M, marked in red in the figure. Since this is true for any positive value of M (no matter how large), we can force the values of $f(x)$ to be as large as we like by making x sufficiently close to a. This agrees with our informal concept of this limit. Similarly, Figure 1.4.5*b* illustrates Definition 1.4.5.

Example 5 Prove that $\lim_{x \to 0} \frac{1}{x^2} = +\infty$.

Solution. Applying Definition 1.4.4 with $f(x) = 1/x^2$ and $a = 0$, we must show that given a number $M > 0$, we can find a number $\delta > 0$ such that

$$\frac{1}{x^2} > M \quad \text{if} \quad 0 < |x - 0| < \delta \tag{14}$$

or, on taking reciprocals and simplifying,

$$x^2 < \frac{1}{M} \quad \text{if} \quad 0 < |x| < \delta \tag{15}$$

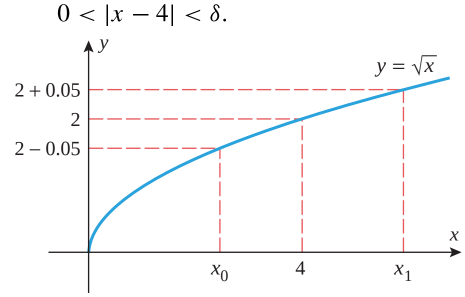
But $x^2 < 1/M$ if $|x| < 1/\sqrt{M}$, so that $\delta = 1/\sqrt{M}$ satisfies (15). Since (14) is equivalent to (15), the proof is complete. ◀

✔ QUICK CHECK EXERCISES 1.4 (See page 109 for answers.)

1. The definition of a two-sided limit states: $\lim_{x \to a} f(x) = L$ if given any number ________ there is a number ________ such that $|f(x) - L| < \epsilon$ if ________.
2. Suppose that $f(x)$ is a function such that for any given $\epsilon > 0$, the condition $0 < |x - 1| < \epsilon/2$ guarantees that $|f(x) - 5| < \epsilon$. What limit results from this property?
3. Suppose that ϵ is any positive number. Find the largest value of δ such that $|5x - 10| < \epsilon$ if $0 < |x - 2| < \delta$.
4. The definition of limit at $+\infty$ states: $\lim_{x \to +\infty} f(x) = L$ if given any number ________ there is a positive number ________ such that $|f(x) - L| < \epsilon$ if ________.
5. Find the smallest positive number N such that for each $x > N$, the value of $f(x) = 1/\sqrt{x}$ is within 0.01 of 0.

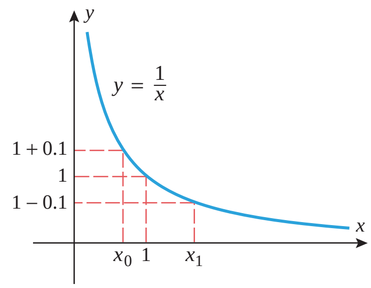
EXERCISE SET 1.4 Graphing Utility

1. (a) Find the largest open interval, centered at the origin on the x-axis, such that for each x in the interval the value of the function $f(x) = x + 2$ is within 0.1 unit of the number $f(0) = 2$.
   (b) Find the largest open interval, centered at $x = 3$, such that for each x in the interval the value of the function $f(x) = 4x - 5$ is within 0.01 unit of the number $f(3) = 7$.
   (c) Find the largest open interval, centered at $x = 4$, such that for each x in the interval the value of the function $f(x) = x^2$ is within 0.001 unit of the number $f(4) = 16$.
2. In each part, find the largest open interval, centered at $x = 0$, such that for each x in the interval the value of $f(x) = 2x + 3$ is within ϵ units of the number $f(0) = 3$.
   (a) $\epsilon = 0.1$ (b) $\epsilon = 0.01$
   (c) $\epsilon = 0.0012$
3. (a) Find the values of x_0 and x_1 in the accompanying figure.
   (b) Find a positive number δ such that $|\sqrt{x} - 2| < 0.05$ if $0 < |x - 4| < \delta$.



▲ **Figure Ex-3**

4. (a) Find the values of x_0 and x_1 in the accompanying figure on the next page.
   (b) Find a positive number δ such that $|(1/x) - 1| < 0.1$ if $0 < |x - 1| < \delta$.



Not drawn to scale ◀ **Figure Ex-4**

5. Generate the graph of $f(x) = x^3 - 4x + 5$ with a graphing utility, and use the graph to find a number δ such that $|f(x) - 2| < 0.05$ if $0 < |x - 1| < \delta$. [*Hint:* Show that the inequality $|f(x) - 2| < 0.05$ can be rewritten as $1.95 < x^3 - 4x + 5 < 2.05$, and estimate the values of x for which $x^3 - 4x + 5 = 1.95$ and $x^3 - 4x + 5 = 2.05$.]

6. Use the method of Exercise 5 to find a number δ such that $|\sqrt{5x+1} - 4| < 0.5$ if $0 < |x - 3| < \delta$.

7. Let $f(x) = x + \sqrt{x}$ with $L = \lim_{x\to 1} f(x)$ and let $\epsilon = 0.2$. Use a graphing utility and its trace feature to find a positive number δ such that $|f(x) - L| < \epsilon$ if $0 < |x - 1| < \delta$.

8. Let $f(x) = (\sin 2x)/x$ and use a graphing utility to conjecture the value of $L = \lim_{x\to 0} f(x)$. Then let $\epsilon = 0.1$ and use the graphing utility and its trace feature to find a positive number δ such that $|f(x) - L| < \epsilon$ if $0 < |x| < \delta$.

9–16 A positive number ϵ and the limit L of a function f at a are given. Find a number δ such that $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$. ■

9. $\lim\limits_{x\to 4} 2x = 8;\ \epsilon = 0.1$

10. $\lim\limits_{x\to 3} (5x - 2) = 13;\ \epsilon = 0.01$

11. $\lim\limits_{x\to 3} \dfrac{x^2 - 9}{x - 3} = 6;\ \epsilon = 0.05$

12. $\lim\limits_{x\to -1/2} \dfrac{4x^2 - 1}{2x + 1} = -2;\ \epsilon = 0.05$

13. $\lim\limits_{x\to 2} x^3 = 8;\ \epsilon = 0.001$

14. $\lim\limits_{x\to 4} \sqrt{x} = 2;\ \epsilon = 0.001$

15. $\lim\limits_{x\to 5} \dfrac{1}{x} = \dfrac{1}{5};\ \epsilon = 0.05$

16. $\lim\limits_{x\to 0} |x| = 0;\ \epsilon = 0.05$

17–26 Use Definition 1.4.1 to prove that the limit is correct. ■

17. $\lim\limits_{x\to 2} 3 = 3$

18. $\lim\limits_{x\to 4} (x + 2) = 6$

19. $\lim\limits_{x\to 5} 3x = 15$

20. $\lim\limits_{x\to -1} (7x + 5) = -2$

21. $\lim\limits_{x\to 0} \dfrac{2x^2 + x}{x} = 1$

22. $\lim\limits_{x\to -3} \dfrac{x^2 - 9}{x + 3} = -6$

23. $\lim\limits_{x\to 1} f(x) = 3$, where $f(x) = \begin{cases} x + 2, & x \neq 1 \\ 10, & x = 1 \end{cases}$

24. $\lim\limits_{x\to 2} f(x) = 5$, where $f(x) = \begin{cases} 9 - 2x, & x \neq 2 \\ 49, & x = 2 \end{cases}$

25. $\lim\limits_{x\to 0} |x| = 0$

26. $\lim\limits_{x\to 2} f(x) = 5$, where $f(x) = \begin{cases} 9 - 2x, & x < 2 \\ 3x - 1, & x > 2 \end{cases}$

FOCUS ON CONCEPTS

27. Give rigorous definitions of $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

28. Consider the statement that $\lim_{x\to a} |f(x) - L| = 0$.
(a) Using Definition 1.4.1, write down precisely what this limit statement means.
(b) Explain why your answer to part (a) shows that

$$\lim_{x\to a} |f(x) - L| = 0 \quad \text{if and only if} \quad \lim_{x\to a} f(x) = L$$

29. (a) Show that

$$|(3x^2 + 2x - 20) - 300| = |3x + 32| \cdot |x - 10|$$

(b) Find an upper bound for $|3x + 32|$ if x satisfies $|x - 10| < 1$.
(c) Fill in the blanks to complete a proof that

$$\lim_{x\to 10} [3x^2 + 2x - 20] = 300$$

Suppose that $\epsilon > 0$. Set $\delta = \min(1, \underline{\hspace{2cm}})$ and assume that $0 < |x - 10| < \delta$. Then

$$\begin{aligned} \left|(3x^2 + 2x - 20) - 300\right| &= |3x + 32| \cdot |x - 10| \\ &< \underline{\hspace{1.5cm}} \cdot |x - 10| \\ &< \underline{\hspace{1.5cm}} \cdot \underline{\hspace{1.5cm}} \\ &= \epsilon \end{aligned}$$

30. (a) Show that

$$\left|\frac{28}{3x+1} - 4\right| = \left|\frac{12}{3x+1}\right| \cdot |x - 2|$$

(b) Is $\left|12/(3x+1)\right|$ bounded if $|x - 2| < 4$? If not, explain; if so, give a bound.
(c) Is $\left|12/(3x+1)\right|$ bounded if $|x - 2| < 1$? If not, explain; if so, give a bound.
(d) Fill in the blanks to complete a proof that

$$\lim_{x\to 2} \left[\frac{28}{3x+1}\right] = 4$$

Suppose that $\epsilon > 0$. Set $\delta = \min(1, \underline{\hspace{2cm}})$ and assume that $0 < |x - 2| < \delta$. Then

$$\begin{aligned} \left|\frac{28}{3x+1} - 4\right| &= \left|\frac{12}{3x+1}\right| \cdot |x - 2| \\ &< \underline{\hspace{1.5cm}} \cdot |x - 2| \\ &< \underline{\hspace{1.5cm}} \cdot \underline{\hspace{1.5cm}} \\ &= \epsilon \end{aligned}$$

31–36 Use Definition 1.4.1 to prove that the stated limit is correct. In each case, to show that $\lim_{x\to a} f(x) = L$, factor $|f(x) - L|$ in the form

$$|f(x) - L| = |\text{“something”}| \cdot |x - a|$$

and then bound the size of |“something”| by putting restrictions on the size of δ. ■

31. $\lim_{x\to 1} 2x^2 = 2$ [*Hint:* Assume $\delta \le 1$.]

32. $\lim_{x\to 3} (x^2 + x) = 12$ [*Hint:* Assume $\delta \le 1$.]

33. $\lim_{x\to -2} \dfrac{1}{x+1} = -1$

34. $\lim_{x\to 1/2} \dfrac{2x+3}{x} = 8$

35. $\lim_{x\to 4} \sqrt{x} = 2$

36. $\lim_{x\to 2} x^3 = 8$

37. Let

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$$

Use Definition 1.4.1 to prove that $\lim_{x\to 0} f(x) = 0$.

38. Let

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

Use Definition 1.4.1 to prove that $\lim_{x\to 0} f(x)$ does not exist. [*Hint:* Assume $\lim_{x\to 0} f(x) = L$ and apply Definition 1.4.1 with $\epsilon = \frac{1}{2}$ to conclude that $|1 - L| < \frac{1}{2}$ and $|L| = |0 - L| < \frac{1}{2}$. Then show $1 \le |1 - L| + |L|$ and derive a contradiction.]

39. (a) Find the smallest positive number N such that for each x in the interval $(N, +\infty)$, the value of the function $f(x) = 1/x^2$ is within 0.1 unit of $L = 0$.
(b) Find the smallest positive number N such that for each x in the interval $(N, +\infty)$, the value of $f(x) = x/(x+1)$ is within 0.01 unit of $L = 1$.
(c) Find the largest negative number N such that for each x in the interval $(-\infty, N)$, the value of the function $f(x) = 1/x^3$ is within 0.001 unit of $L = 0$.
(d) Find the largest negative number N such that for each x in the interval $(-\infty, N)$, the value of the function $f(x) = x/(x+1)$ is within 0.01 unit of $L = 1$.

40. In each part, find the smallest positive value of N such that for each x in the interval $(N, +\infty)$, the function $f(x) = 1/x^3$ is within ϵ units of the number $L = 0$.
(a) $\epsilon = 0.1$ (b) $\epsilon = 0.01$ (c) $\epsilon = 0.001$

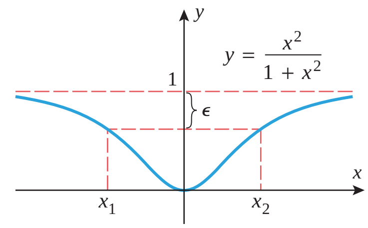
41. (a) Find the values of x_1 and x_2 in the accompanying figure.
(b) Find a positive number N such that

$$\left| \frac{x^2}{1+x^2} - 1 \right| < \epsilon$$

for $x > N$.
(c) Find a negative number N such that

$$\left| \frac{x^2}{1+x^2} - 1 \right| < \epsilon$$

for $x < N$.



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◀ **Figure Ex-41**

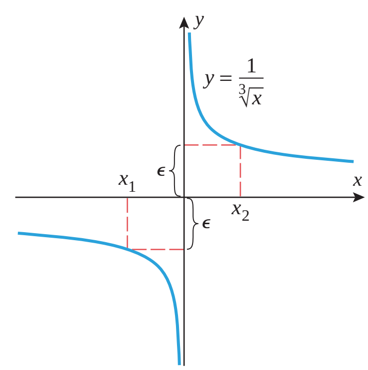
42. (a) Find the values of x_1 and x_2 in the accompanying figure.
(b) Find a positive number N such that

$$\left| \frac{1}{\sqrt[3]{x}} - 0 \right| = \left| \frac{1}{\sqrt[3]{x}} \right| < \epsilon$$

for $x > N$.
(c) Find a negative number N such that

$$\left| \frac{1}{\sqrt[3]{x}} - 0 \right| = \left| \frac{1}{\sqrt[3]{x}} \right| < \epsilon$$

for $x < N$.



◀ **Figure Ex-42**

43–46 A positive number ϵ and the limit L of a function f at $+\infty$ are given. Find a positive number N such that $|f(x) - L| < \epsilon$ if $x > N$. ■

43. $\lim_{x\to +\infty} \dfrac{1}{x^2} = 0;\ \epsilon = 0.01$

44. $\lim_{x\to +\infty} \dfrac{1}{x+2} = 0;\ \epsilon = 0.005$

45. $\lim_{x\to +\infty} \dfrac{x}{x+1} = 1;\ \epsilon = 0.001$

46. $\lim_{x\to +\infty} \dfrac{4x-1}{2x+5} = 2;\ \epsilon = 0.1$

47–50 A positive number ϵ and the limit L of a function f at $-\infty$ are given. Find a negative number N such that $|f(x) - L| < \epsilon$ if $x < N$. ■

47. $\lim_{x\to -\infty} \dfrac{1}{x+2} = 0;\ \epsilon = 0.005$

48. $\lim_{x\to -\infty} \dfrac{1}{x^2} = 0;\ \epsilon = 0.01$

49. $\lim_{x\to -\infty} \dfrac{4x-1}{2x+5} = 2;\ \epsilon = 0.1$

50. $\lim_{x\to-\infty} \dfrac{x}{x+1} = 1;\ \epsilon = 0.001$

51–56 Use Definition 1.4.2 or 1.4.3 to prove that the stated limit is correct. ■

51. $\lim_{x\to+\infty} \dfrac{1}{x^2} = 0$

52. $\lim_{x\to+\infty} \dfrac{1}{x+2} = 0$

53. $\lim_{x\to-\infty} \dfrac{4x-1}{2x+5} = 2$

54. $\lim_{x\to-\infty} \dfrac{x}{x+1} = 1$

55. $\lim_{x\to+\infty} \dfrac{2\sqrt{x}}{\sqrt{x}-1} = 2$

56. $\lim_{x\to-\infty} 2^x = 0$

57. (a) Find the largest open interval, centered at the origin on the x-axis, such that for each x in the interval, other than the center, the values of $f(x) = 1/x^2$ are greater than 100.

(b) Find the largest open interval, centered at $x = 1$, such that for each x in the interval, other than the center, the values of the function $f(x) = 1/|x-1|$ are greater than 1000.

(c) Find the largest open interval, centered at $x = 3$, such that for each x in the interval, other than the center, the values of the function $f(x) = -1/(x-3)^2$ are less than -1000.

(d) Find the largest open interval, centered at the origin on the x-axis, such that for each x in the interval, other than the center, the values of $f(x) = -1/x^4$ are less than $-10{,}000$.

58. In each part, find the largest open interval centered at $x = 1$, such that for each x in the interval, other than the center, the value of $f(x) = 1/(x-1)^2$ is greater than M.

(a) $M = 10$ (b) $M = 1000$ (c) $M = 100{,}000$

59–64 Use Definition 1.4.4 or 1.4.5 to prove that the stated limit is correct. ■

59. $\lim_{x\to 3} \dfrac{1}{(x-3)^2} = +\infty$

60. $\lim_{x\to 3} \dfrac{-1}{(x-3)^2} = -\infty$

61. $\lim_{x\to 0} \dfrac{1}{|x|} = +\infty$

62. $\lim_{x\to 1} \dfrac{1}{|x-1|} = +\infty$

63. $\lim_{x\to 0} \left(-\dfrac{1}{x^4}\right) = -\infty$

64. $\lim_{x\to 0} \dfrac{1}{x^4} = +\infty$

65–70 Use the definitions in Exercise 27 to prove that the stated one-sided limit is correct. ■

65. $\lim_{x\to 2^+} (x+1) = 3$

66. $\lim_{x\to 1^-} (3x+2) = 5$

67. $\lim_{x\to 4^+} \sqrt{x-4} = 0$

68. $\lim_{x\to 0^-} \sqrt{-x} = 0$

69. $\lim_{x\to 2^+} f(x) = 2$, where $f(x) = \begin{cases} x, & x > 2 \\ 3x, & x \le 2 \end{cases}$

70. $\lim_{x\to 2^-} f(x) = 6$, where $f(x) = \begin{cases} x, & x > 2 \\ 3x, & x \le 2 \end{cases}$

71–74 Write out the definition for the corresponding limit in the marginal note on page 105, and use your definition to prove that the stated limit is correct. ■

71. (a) $\lim_{x\to 1^+} \dfrac{1}{1-x} = -\infty$ (b) $\lim_{x\to 1^-} \dfrac{1}{1-x} = +\infty$

72. (a) $\lim_{x\to 0^+} \dfrac{1}{x} = +\infty$ (b) $\lim_{x\to 0^-} \dfrac{1}{x} = -\infty$

73. (a) $\lim_{x\to+\infty} (x+1) = +\infty$ (b) $\lim_{x\to-\infty} (x+1) = -\infty$

74. (a) $\lim_{x\to+\infty} (x^2-3) = +\infty$ (b) $\lim_{x\to-\infty} (x^3+5) = -\infty$

75. According to Ohm's law, when a voltage of V volts is applied across a resistor with a resistance of R ohms, a current of $I = V/R$ amperes flows through the resistor.

(a) How much current flows if a voltage of 3.0 volts is applied across a resistance of 7.5 ohms?

(b) If the resistance varies by ± 0.1 ohm, and the voltage remains constant at 3.0 volts, what is the resulting range of values for the current?

(c) If temperature variations cause the resistance to vary by $\pm\delta$ from its value of 7.5 ohms, and the voltage remains constant at 3.0 volts, what is the resulting range of values for the current?

(d) If the current is not allowed to vary by more than $\epsilon = \pm 0.001$ ampere at a voltage of 3.0 volts, what variation of $\pm\delta$ from the value of 7.5 ohms is allowable?

(e) Certain alloys become ***superconductors*** as their temperature approaches absolute zero ($-273°$C), meaning that their resistance approaches zero. If the voltage remains constant, what happens to the current in a superconductor as $R \to 0^+$?

76. **Writing** Compare informal Definition 1.1.1 with Definition 1.4.1.

(a) What portions of Definition 1.4.1 correspond to the expression "values of $f(x)$ can be made as close as we like to L" in Definition 1.1.1? Explain.

(b) What portions of Definition 1.4.1 correspond to the expression "taking values of x sufficiently close to a (but not equal to a)" in Definition 1.1.1? Explain.

77. **Writing** Compare informal Definition 1.3.1 with Definition 1.4.2.

(a) What portions of Definition 1.4.2 correspond to the expression "values of $f(x)$ eventually get as close as we like to a number L" in Definition 1.3.1? Explain.

(b) What portions of Definition 1.4.2 correspond to the expression "as x increases without bound" in Definition 1.3.1? Explain.

✔ QUICK CHECK ANSWERS 1.4

1. $\epsilon > 0$; $\delta > 0$; $0 < |x-a| < \delta$ **2.** $\lim_{x\to 1} f(x) = 5$ **3.** $\delta = \epsilon/5$ **4.** $\epsilon > 0$; N; $x > N$ **5.** $N = 10{,}000$

1.5 CONTINUITY

A thrown baseball cannot vanish at some point and reappear someplace else to continue its motion. Thus, we perceive the path of the ball as an unbroken curve. In this section, we translate "unbroken curve" into a precise mathematical formulation called continuity, and develop some fundamental properties of continuous curves.

Joseph Helfenberger/iStockphoto

A baseball moves along a "continuous" trajectory after leaving the pitcher's hand.

DEFINITION OF CONTINUITY

Intuitively, the graph of a function can be described as a "continuous curve" if it has no breaks or holes. To make this idea more precise we need to understand what properties of a function can cause breaks or holes. Referring to Figure 1.5.1, we see that the graph of a function has a break or hole if any of the following conditions occur:

- The function f is undefined at c (Figure 1.5.1a).
- The limit of $f(x)$ does not exist as x approaches c (Figures 1.5.1b, 1.5.1c).
- The value of the function and the value of the limit at c are different (Figure 1.5.1d).

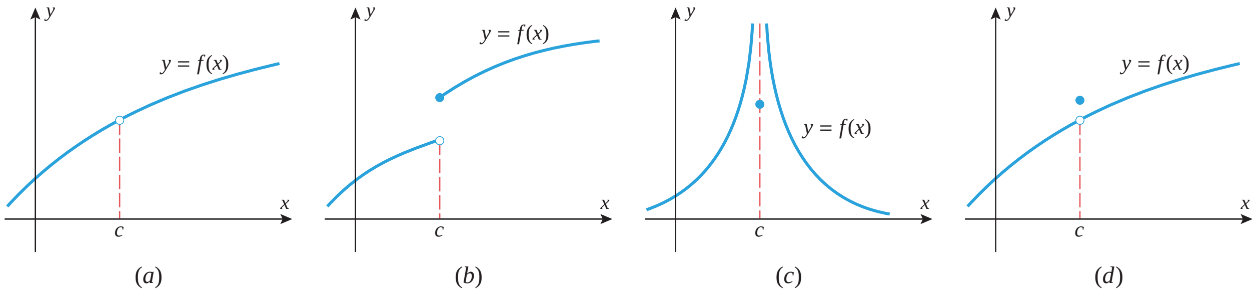


Figure 1.5.1

This suggests the following definition.

The third condition in Definition 1.5.1 actually implies the first two, since it is tacitly understood in the statement

$$\lim_{x\to c} f(x) = f(c)$$

that the limit exists and the function is defined at c. Thus, when we want to establish continuity at c our usual procedure will be to verify the third condition only.

1.5.1 DEFINITION A function f is said to be ***continuous at*** $\boldsymbol{x = c}$ provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$.

If one or more of the conditions of this definition fails to hold, then we will say that f has a ***discontinuity at*** $\boldsymbol{x = c}$. Each function drawn in Figure 1.5.1 illustrates a discontinuity at $x = c$. In Figure 1.5.1a, the function is not defined at c, violating the first condition of Definition 1.5.1. In Figure 1.5.1b, the one-sided limits of $f(x)$ as x approaches c both exist but are not equal. Thus, $\lim_{x\to c} f(x)$ does not exist, and this violates the second condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1b has a ***jump discontinuity*** at c. In Figure 1.5.1c, the one-sided limits of $f(x)$ as x approaches c are infinite. Thus, $\lim_{x\to c} f(x)$ does not exist, and this violates the second condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1c has an ***infinite discontinuity*** at c. In Figure 1.5.1d, the function is defined at c and $\lim_{x\to c} f(x)$ exists, but these two values are not equal, violating the third condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1d has a ***removable discontinuity*** at c. Exercises 33 and 34 help to explain why discontinuities of this type are given this name.

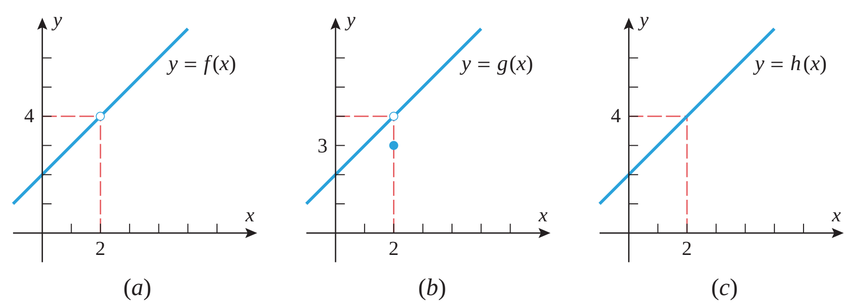
Example 1 Determine whether the following functions are continuous at $x = 2$.

$$f(x) = \frac{x^2 - 4}{x - 2}, \qquad g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 3, & x = 2, \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

Solution. In each case we must determine whether the limit of the function as $x \to 2$ is the same as the value of the function at $x = 2$. In all three cases the functions are identical, except at $x = 2$, and hence all three have the same limit at $x = 2$, namely,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

The function f is undefined at $x = 2$, and hence is not continuous at $x = 2$ (Figure 1.5.2a). The function g is defined at $x = 2$, but its value there is $g(2) = 3$, which is not the same as the limit as x approaches 2; hence, g is also not continuous at $x = 2$ (Figure 1.5.2b). The value of the function h at $x = 2$ is $h(2) = 4$, which is the same as the limit as x approaches 2; hence, h is continuous at $x = 2$ (Figure 1.5.2c). (Note that the function h could have been written more simply as $h(x) = x + 2$, but we wrote it in piecewise form to emphasize its relationship to f and g.) ◀



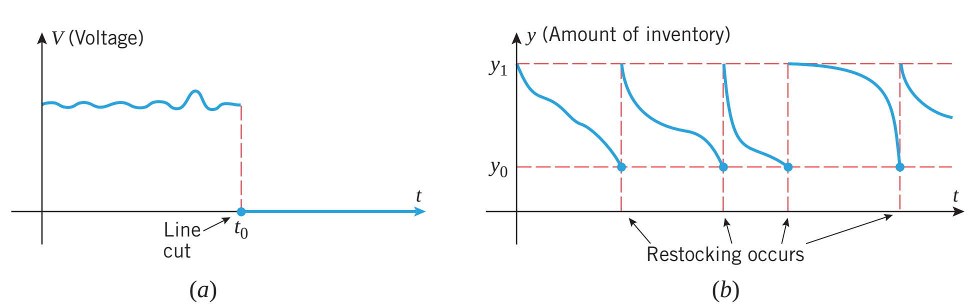
▲ **Figure 1.5.2**

Chris Hondros/Getty Images

A poor connection in a transmission cable can cause a discontinuity in the electrical signal it carries.

CONTINUITY IN APPLICATIONS

In applications, discontinuities often signal the occurrence of important physical events. For example, Figure 1.5.3a is a graph of voltage versus time for an underground cable that is accidentally cut by a work crew at time $t = t_0$ (the voltage drops to zero when the line is cut). Figure 1.5.3b shows the graph of inventory versus time for a company that restocks its warehouse to y_1 units when the inventory falls to y_0 units. The discontinuities occur at those times when restocking occurs.

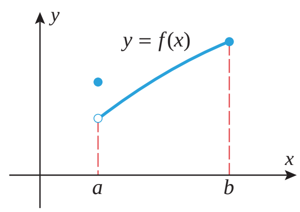


▲ **Figure 1.5.3**

CONTINUITY ON AN INTERVAL

If a function f is continuous at each number in an open interval (a, b), then we say that f is ***continuous on* (*a, b*)**. This definition applies to infinite open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the case where f is continuous on $(-\infty, +\infty)$, we will say that f is ***continuous everywhere***.

Because Definition 1.5.1 involves a two-sided limit, that definition does not generally apply at the endpoints of a closed interval $[a, b]$ or at the endpoint of an interval of the form $[a, b)$, $(a, b]$, $(-\infty, b]$, or $[a, +\infty)$. To remedy this problem, we will agree that a function is continuous at an endpoint of an interval if its value at the endpoint is equal to the appropriate one-sided limit at that endpoint. For example, the function graphed in Figure 1.5.4 is continuous at the right endpoint of the interval $[a, b]$ because



▲ **Figure 1.5.4**

$$\lim_{x \to b^-} f(x) = f(b)$$

but it is not continuous at the left endpoint because

$$\lim_{x \to a^+} f(x) \neq f(a)$$

In general, we will say a function f is ***continuous from the left*** at c if

$$\lim_{x \to c^-} f(x) = f(c)$$

and is ***continuous from the right*** at c if

$$\lim_{x \to c^+} f(x) = f(c)$$

Using this terminology we define continuity on a closed interval as follows.

Modify Definition 1.5.2 appropriately so that it applies to intervals of the form $[a, +\infty)$, $(-\infty, b]$, $(a, b]$, and $[a, b)$.

1.5.2 DEFINITION A function f is said to be ***continuous on a closed interval* [*a, b*]** if the following conditions are satisfied:

1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

▶ **Example 2** What can you say about the continuity of the function $f(x) = \sqrt{9 - x^2}$?

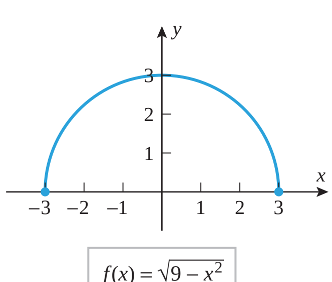
Solution. Because the natural domain of this function is the closed interval $[-3, 3]$, we will need to investigate the continuity of f on the open interval $(-3, 3)$ and at the two endpoints. If c is any point in the interval $(-3, 3)$, then it follows from Theorem 1.2.2(*e*) that

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{\lim_{x \to c} (9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

which proves f is continuous at each point in the interval $(-3, 3)$. The function f is also continuous at the endpoints since

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x \to 3^-} (9 - x^2)} = 0 = f(3)$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x \to -3^+} (9 - x^2)} = 0 = f(-3)$$



▲ **Figure 1.5.5**

Thus, f is continuous on the closed interval $[-3, 3]$ (Figure 1.5.5). ◀

SOME PROPERTIES OF CONTINUOUS FUNCTIONS

The following theorem, which is a consequence of Theorem 1.2.2, will enable us to reach conclusions about the continuity of functions that are obtained by adding, subtracting, multiplying, and dividing continuous functions.

1.5.3 THEOREM *If the functions f and g are continuous at c, then*

(*a*) *$f + g$ is continuous at c.*

(*b*) *$f - g$ is continuous at c.*

(*c*) *fg is continuous at c.*

(*d*) *f/g is continuous at c if $g(c) \neq 0$ and has a discontinuity at c if $g(c) = 0$.*

We will prove part (*d*). The remaining proofs are similar and will be left to the exercises.

PROOF First, consider the case where $g(c) = 0$. In this case $f(c)/g(c)$ is undefined, so the function f/g has a discontinuity at c.

Next, consider the case where $g(c) \neq 0$. To prove that f/g is continuous at c, we must show that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)} \tag{1}$$

Since f and g are continuous at c,

$$\lim_{x \to c} f(x) = f(c) \quad \text{and} \quad \lim_{x \to c} g(x) = g(c)$$

Thus, by Theorem 1.2.2(*d*)

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{f(c)}{g(c)}$$

which proves (1). ■

CONTINUITY OF POLYNOMIALS AND RATIONAL FUNCTIONS

The general procedure for showing that a function is continuous everywhere is to show that it is continuous at an *arbitrary* point. For example, we know from Theorem 1.2.3 that if $p(x)$ is a polynomial and a is *any* real number, then

$$\lim_{x \to a} p(x) = p(a)$$

This shows that polynomials are continuous everywhere. Moreover, since rational functions are ratios of polynomials, it follows from part (*d*) of Theorem 1.5.3 that rational functions are continuous at points other than the zeros of the denominator, and at these zeros they have discontinuities. Thus, we have the following result.

1.5.4 THEOREM

(*a*) *A polynomial is continuous everywhere.*

(*b*) *A rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero.*

TECHNOLOGY MASTERY

If you use a graphing utility to generate the graph of the equation in Example 3, there is a good chance you will see the discontinuity at $x = 2$ but not at $x = 3$. Try it, and explain what you think is happening.

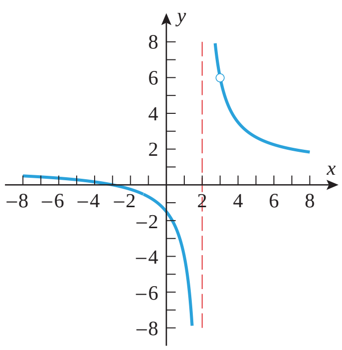
Example 3 For what values of x is there a discontinuity in the graph of

$$y = \frac{x^2 - 9}{x^2 - 5x + 6}?$$

Solution. The function being graphed is a rational function, and hence is continuous at every number where the denominator is nonzero. Solving the equation

$$x^2 - 5x + 6 = 0$$

yields discontinuities at $x = 2$ and at $x = 3$ (Figure 1.5.6). ◀



$y = \dfrac{x^2 - 9}{x^2 - 5x + 6}$

▲ **Figure 1.5.6**

Example 4 Show that $|x|$ is continuous everywhere (Figure 0.1.9).

Solution. We can write $|x|$ as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

so $|x|$ is the same as the polynomial x on the interval $(0, +\infty)$ and is the same as the polynomial $-x$ on the interval $(-\infty, 0)$. But polynomials are continuous everywhere, so $x = 0$ is the only possible discontinuity for $|x|$. Since $|0| = 0$, to prove the continuity at $x = 0$ we must show that

$$\lim_{x \to 0} |x| = 0 \tag{2}$$

Because the piecewise formula for $|x|$ changes at 0, it will be helpful to consider the one-sided limits at 0 rather than the two-sided limit. We obtain

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 \quad \text{and} \quad \lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Thus, (2) holds and $|x|$ is continuous at $x = 0$. ◀

CONTINUITY OF COMPOSITIONS

The following theorem, whose proof is given in Appendix J, will be useful for calculating limits of compositions of functions.

In words, Theorem 1.5.5 states that a limit symbol can be moved through a function sign provided the limit of the expression inside the function sign exists and the function is continuous at this limit.

1.5.5 THEOREM *If $\lim_{x \to c} g(x) = L$ and if the function f is continuous at L, then $\lim_{x \to c} f(g(x)) = f(L)$. That is,*

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$$

This equality remains valid if $\lim_{x \to c}$ is replaced everywhere by one of $\lim_{x \to c^+}$, $\lim_{x \to c^-}$, $\lim_{x \to +\infty}$, or $\lim_{x \to -\infty}$.

In the special case of this theorem where $f(x) = |x|$, the fact that $|x|$ is continuous everywhere allows us to write

$$\lim_{x \to c} |g(x)| = \left|\lim_{x \to c} g(x)\right| \tag{3}$$

provided $\lim_{x \to c} g(x)$ exists. Thus, for example,

$$\lim_{x \to 3} |5 - x^2| = \left|\lim_{x \to 3} (5 - x^2)\right| = |-4| = 4$$

The following theorem is concerned with the continuity of compositions of functions; the first part deals with continuity at a specific number and the second with continuity everywhere.

1.5.6 THEOREM

(*a*) *If the function g is continuous at c, and the function f is continuous at g(c), then the composition f ∘ g is continuous at c.*

(*b*) *If the function g is continuous everywhere and the function f is continuous everywhere, then the composition f ∘ g is continuous everywhere.*

PROOF We will prove part (*a*) only; the proof of part (*b*) can be obtained by applying part (*a*) at an arbitrary number c. To prove that $f \circ g$ is continuous at c, we must show that the value of $f \circ g$ and the value of its limit are the same at $x = c$. But this is so, since we can write

$$\lim_{x\to c}(f\circ g)(x) = \underbrace{\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)}_{\text{Theorem 1.5.5}} = \underbrace{f(g(c))}_{g \text{ is continuous at } c.} = (f\circ g)(c) \quad \blacksquare$$

Can the absolute value of a function that is not continuous everywhere be continuous everywhere? Justify your answer.

We know from Example 4 that the function $|x|$ is continuous everywhere. Thus, if $g(x)$ is continuous at c, then by part (*a*) of Theorem 1.5.6, the function $|g(x)|$ must also be continuous at c; and, more generally, if $g(x)$ is continuous everywhere, then so is $|g(x)|$. Stated informally:

The absolute value of a continuous function is continuous.

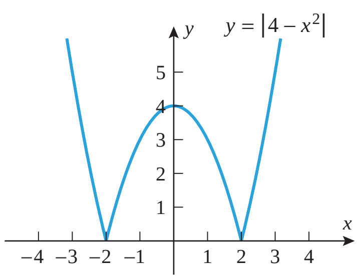


Figure 1.5.7

For example, the polynomial $g(x) = 4 - x^2$ is continuous everywhere, so we can conclude that the function $|4 - x^2|$ is also continuous everywhere (Figure 1.5.7).

THE INTERMEDIATE-VALUE THEOREM

Figure 1.5.8 shows the graph of a function that is continuous on the closed interval $[a, b]$. The figure suggests that if we draw any horizontal line $y = k$, where k is between $f(a)$ and $f(b)$, then that line will cross the curve $y = f(x)$ at least once over the interval $[a, b]$. Stated in numerical terms, if f is continuous on $[a, b]$, then the function f must take on every value k between $f(a)$ and $f(b)$ at least once as x varies from a to b. For example, the polynomial $p(x) = x^5 - x + 3$ has a value of 3 at $x = 1$ and a value of 33 at $x = 2$. Thus, it follows from the continuity of p that the equation $x^5 - x + 3 = k$ has at least one solution in the interval $[1, 2]$ for every value of k between 3 and 33. This idea is stated more precisely in the following theorem.

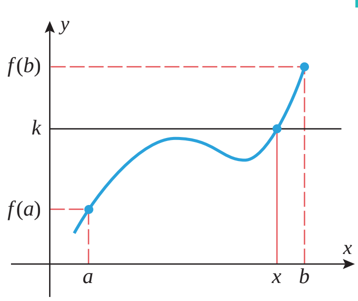


Figure 1.5.8

1.5.7 THEOREM (*Intermediate-Value Theorem*) *If f is continuous on a closed interval [a, b] and k is any number between f(a) and f(b), inclusive, then there is at least one number x in the interval [a, b] such that f(x) = k.*

Although this theorem is intuitively obvious, its proof depends on a mathematically precise development of the real number system, which is beyond the scope of this text.

APPROXIMATING ROOTS USING THE INTERMEDIATE-VALUE THEOREM

A variety of problems can be reduced to solving an equation $f(x) = 0$ for its roots. Sometimes it is possible to solve for the roots exactly using algebra, but often this is not possible and one must settle for decimal approximations of the roots. One procedure for approximating roots is based on the following consequence of the Intermediate-Value Theorem.

1.5.8 THEOREM *If f is continuous on $[a, b]$, and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).*

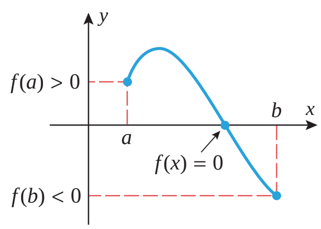


Figure 1.5.9

This result, which is illustrated in Figure 1.5.9, can be proved as follows.

PROOF Since $f(a)$ and $f(b)$ have opposite signs, 0 is between $f(a)$ and $f(b)$. Thus, by the Intermediate-Value Theorem there is at least one number x in the interval $[a, b]$ such that $f(x) = 0$. However, $f(a)$ and $f(b)$ are nonzero, so x must lie in the interval (a, b), which completes the proof. ■

Before we illustrate how this theorem can be used to approximate roots, it will be helpful to discuss some standard terminology for describing errors in approximations. If x is an approximation to a quantity x_0, then we call

$$\epsilon = |x - x_0|$$

the ***absolute error*** or (less precisely) the ***error*** in the approximation. The terminology in Table 1.5.1 is used to describe the size of such errors.

Table 1.5.1

ERROR	DESCRIPTION
$\lvert x - x_0\rvert \le 0.1$	x approximates x_0 with an error of at most 0.1.
$\lvert x - x_0\rvert \le 0.01$	x approximates x_0 with an error of at most 0.01.
$\lvert x - x_0\rvert \le 0.001$	x approximates x_0 with an error of at most 0.001.
$\lvert x - x_0\rvert \le 0.0001$	x approximates x_0 with an error of at most 0.0001.
$\lvert x - x_0\rvert \le 0.5$	x approximates x_0 to the nearest integer.
$\lvert x - x_0\rvert \le 0.05$	x approximates x_0 to 1 decimal place (i.e., to the nearest tenth).
$\lvert x - x_0\rvert \le 0.005$	x approximates x_0 to 2 decimal places (i.e., to the nearest hundredth).
$\lvert x - x_0\rvert \le 0.0005$	x approximates x_0 to 3 decimal places (i.e., to the nearest thousandth).

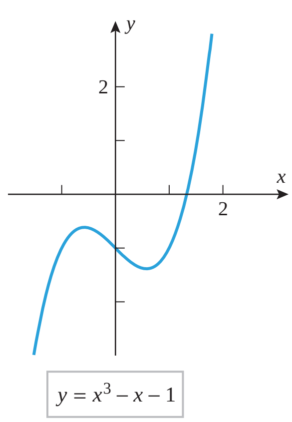


Figure 1.5.10

▶ **Example 5** The equation

$$x^3 - x - 1 = 0$$

cannot be solved algebraically very easily because the left side has no simple factors. However, if we graph $p(x) = x^3 - x - 1$ with a graphing utility (Figure 1.5.10), then we are led to conjecture that there is one real root and that this root lies inside the interval $[1, 2]$. The existence of a root in this interval is also confirmed by Theorem 1.5.8, since $p(1) = -1$ and $p(2) = 5$ have opposite signs. Approximate this root to two decimal-place accuracy.

Solution. Our objective is to approximate the unknown root x_0 with an error of at most 0.005. It follows that if we can find an interval of length 0.01 that contains the root, then the midpoint of that interval will approximate the root with an error of at most $\frac{1}{2}(0.01) = 0.005$, which will achieve the desired accuracy.

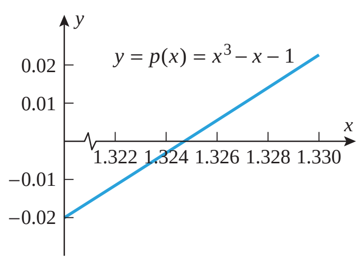
We know that the root x_0 lies in the interval $[1, 2]$. However, this interval has length 1, which is too large. We can pinpoint the location of the root more precisely by dividing the interval $[1, 2]$ into 10 equal parts and evaluating p at the points of subdivision using a calculating utility (Table 1.5.2). In this table $p(1.3)$ and $p(1.4)$ have opposite signs, so we know that the root lies in the interval $[1.3, 1.4]$. This interval has length 0.1, which is still too large, so we repeat the process by dividing the interval $[1.3, 1.4]$ into 10 parts and evaluating p at the points of subdivision; this yields Table 1.5.3, which tells us that the root is inside the interval $[1.32, 1.33]$ (Figure 1.5.11). Since this interval has length 0.01, its midpoint 1.325 will approximate the root with an error of at most 0.005. Thus, $x_0 \approx 1.325$ to two decimal-place accuracy. ◂

Table 1.5.2

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$p(x)$	−1	−0.77	−0.47	−0.10	0.34	0.88	1.50	2.21	3.03	3.96	5

Table 1.5.3

x	1.3	1.31	1.32	1.33	1.34	1.35	1.36	1.37	1.38	1.39	1.4
$p(x)$	−0.103	−0.062	−0.020	0.023	0.066	0.110	0.155	0.201	0.248	0.296	0.344



▲ **Figure 1.5.11**

REMARK

To say that x approximates x_0 to n decimal places does *not* mean that the first n decimal places of x and x_0 will be the same when the numbers are rounded to n decimal places. For example, $x = 1.084$ approximates $x_0 = 1.087$ to two decimal places because $|x - x_0| = 0.003$ (< 0.005). However, if we round these values to two decimal places, then we obtain $x \approx 1.08$ and $x_0 \approx 1.09$. Thus, if you approximate a number to n decimal places, then you should display that approximation to at least $n + 1$ decimal places to preserve the accuracy.

TECHNOLOGY MASTERY

Use a graphing or calculating utility to show that the root x_0 in Example 5 can be approximated as $x_0 \approx 1.3245$ to three decimal-place accuracy.

✔ QUICK CHECK EXERCISES 1.5 *(See page 120 for answers.)*

1. What three conditions are satisfied if f is continuous at $x = c$?

2. Suppose that f and g are continuous functions such that $f(2) = 1$ and $\lim_{x\to 2}[f(x) + 4g(x)] = 13$. Find
(a) $g(2)$
(b) $\lim_{x\to 2} g(x)$.

3. Suppose that f and g are continuous functions such that $\lim_{x\to 3} g(x) = 5$ and $f(3) = -2$. Find $\lim_{x\to 3}[f(x)/g(x)]$.

4. For what values of x, if any, is the function

$$f(x) = \frac{x^2 - 16}{x^2 - 5x + 4}$$

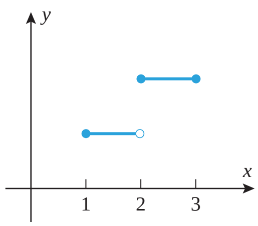
discontinuous?

5. Suppose that a function f is continuous everywhere and that $f(-2) = 3$, $f(-1) = -1$, $f(0) = -4$, $f(1) = 1$, and $f(2) = 5$. Does the Intermediate-Value Theorem guarantee that f has a root on the following intervals?
(a) $[-2, -1]$ (b) $[-1, 0]$ (c) $[-1, 1]$ (d) $[0, 2]$

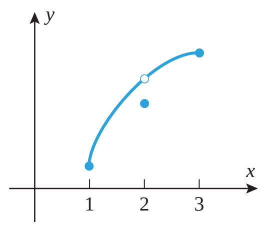
EXERCISE SET 1.5 Graphing Utility

1–4 Let f be the function whose graph is shown. On which of the following intervals, if any, is f continuous?
(a) $[1, 3]$ (b) $(1, 3)$ (c) $[1, 2]$
(d) $(1, 2)$ (e) $[2, 3]$ (f) $(2, 3)$
For each interval on which f is not continuous, indicate which conditions for the continuity of f do not hold. ■

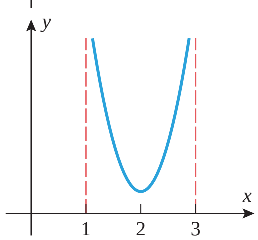
1.



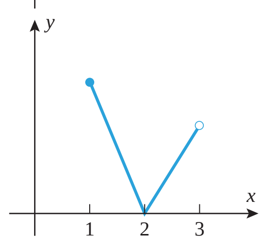
2.



3.



4.



5. Consider the functions

$$f(x) = \begin{cases} 1, & x \neq 4 \\ -1, & x = 4 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 4x - 10, & x \neq 4 \\ -6, & x = 4 \end{cases}$$

In each part, is the given function continuous at $x = 4$?
(a) $f(x)$ (b) $g(x)$ (c) $-g(x)$ (d) $|f(x)|$
(e) $f(x)g(x)$ (f) $g(f(x))$ (g) $g(x) - 6f(x)$

6. Consider the functions

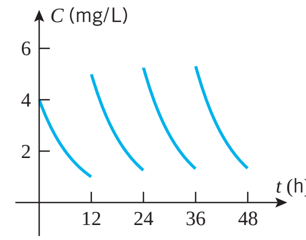
$$f(x) = \begin{cases} 1, & 0 \le x \\ 0, & x < 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0, & 0 \le x \\ 1, & x < 0 \end{cases}$$

In each part, is the given function continuous at $x = 0$?
(a) $f(x)$ (b) $g(x)$ (c) $f(-x)$ (d) $|g(x)|$
(e) $f(x)g(x)$ (f) $g(f(x))$ (g) $f(x) + g(x)$

FOCUS ON CONCEPTS

7. In each part sketch the graph of a function f that satisfies the stated conditions.
(a) f is continuous everywhere except at $x = 3$, at which point it is continuous from the right.
(b) f has a two-sided limit at $x = 3$, but it is not continuous at $x = 3$.
(c) f is not continuous at $x = 3$, but if its value at $x = 3$ is changed from $f(3) = 1$ to $f(3) = 0$, it becomes continuous at $x = 3$.
(d) f is continuous on the interval $[0, 3)$ and is defined on the closed interval $[0, 3]$; but f is not continuous on the interval $[0, 3]$.

8. The accompanying figure models the concentration C of medication in the bloodstream of a patient over a 48-hour period of time. Discuss the significance of the discontinuities in the graph.



◀ **Figure Ex-8**

9. A student parking lot at a university charges \$2.00 for the first half hour (or any part) and \$1.00 for each subsequent half hour (or any part) up to a daily maximum of \$10.00.
(a) Sketch a graph of cost as a function of the time parked.
(b) Discuss the significance of the discontinuities in the graph to a student who parks there.

10. In each part determine whether the function is continuous or not, and explain your reasoning.
(a) The Earth's population as a function of time.
(b) Your exact height as a function of time.
(c) The cost of a taxi ride in your city as a function of the distance traveled.
(d) The volume of a melting ice cube as a function of time.

11–22 Find values of x, if any, at which f is not continuous. ■

11. $f(x) = 5x^4 - 3x + 7$

12. $f(x) = \sqrt[3]{x - 8}$

13. $f(x) = \dfrac{x + 2}{x^2 + 4}$

14. $f(x) = \dfrac{x + 2}{x^2 - 4}$

15. $f(x) = \dfrac{x}{2x^2 + x}$

16. $f(x) = \dfrac{2x + 1}{4x^2 + 4x + 5}$

17. $f(x) = \dfrac{3}{x} + \dfrac{x - 1}{x^2 - 1}$

18. $f(x) = \dfrac{5}{x} + \dfrac{2x}{x + 4}$

19. $f(x) = \dfrac{x^2 + 6x + 9}{|x| + 3}$

20. $f(x) = \left| 4 - \dfrac{8}{x^4 + x} \right|$

21. $f(x) = \begin{cases} 2x + 3, & x \le 4 \\ 7 + \dfrac{16}{x}, & x > 4 \end{cases}$

22. $f(x) = \begin{cases} \dfrac{3}{x - 1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$

23–28 True–False Determine whether the statement is true or false. Explain your answer. ■

23. If $f(x)$ is continuous at $x = c$, then so is $|f(x)|$.

24. If $|f(x)|$ is continuous at $x = c$, then so is $f(x)$.

25. If f and g are discontinuous at $x = c$, then so is $f + g$.

26. If f and g are discontinuous at $x = c$, then so is fg.

27. If $\sqrt{f(x)}$ is continuous at $x = c$, then so is $f(x)$.

28. If $f(x)$ is continuous at $x = c$, then so is $\sqrt{f(x)}$.

29–30 Find a value of the constant k, if possible, that will make the function continuous everywhere. ■

29. (a) $f(x) = \begin{cases} 7x - 2, & x \le 1 \\ kx^2, & x > 1 \end{cases}$

(b) $f(x) = \begin{cases} kx^2, & x \le 2 \\ 2x + k, & x > 2 \end{cases}$

30. (a) $f(x) = \begin{cases} 9 - x^2, & x \ge -3 \\ k/x^2, & x < -3 \end{cases}$

(b) $f(x) = \begin{cases} 9 - x^2, & x \ge 0 \\ k/x^2, & x < 0 \end{cases}$

31. Find values of the constants k and m, if possible, that will make the function f continuous everywhere.
$$f(x) = \begin{cases} x^2 + 5, & x > 2 \\ m(x+1) + k, & -1 < x \le 2 \\ 2x^3 + x + 7, & x \le -1 \end{cases}$$

32. On which of the following intervals is
$$f(x) = \frac{1}{\sqrt{x-2}}$$
continuous?
(a) $[2, +\infty)$ (b) $(-\infty, +\infty)$ (c) $(2, +\infty)$ (d) $[1, 2)$

33–36 A function f is said to have a ***removable discontinuity*** at $x = c$ if $\lim_{x \to c} f(x)$ exists but f is not continuous at $x = c$, either because f is not defined at c or because the definition for $f(c)$ differs from the value of the limit. This terminology will be needed in these exercises. ■

33. (a) Sketch the graph of a function with a removable discontinuity at $x = c$ for which $f(c)$ is undefined.
(b) Sketch the graph of a function with a removable discontinuity at $x = c$ for which $f(c)$ is defined.

34. (a) The terminology *removable discontinuity* is appropriate because a removable discontinuity of a function f at $x = c$ can be "removed" by redefining the value of f appropriately at $x = c$. What value for $f(c)$ removes the discontinuity?
(b) Show that the following functions have removable discontinuities at $x = 1$, and sketch their graphs.
$$f(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad g(x) = \begin{cases} 1, & x > 1 \\ 0, & x = 1 \\ 1, & x < 1 \end{cases}$$
(c) What values should be assigned to $f(1)$ and $g(1)$ to remove the discontinuities?

35–36 Find the values of x (if any) at which f is not continuous, and determine whether each such value is a removable discontinuity. ■

35. (a) $f(x) = \dfrac{|x|}{x}$ (b) $f(x) = \dfrac{x^2 + 3x}{x + 3}$

(c) $f(x) = \dfrac{x - 2}{|x| - 2}$

36. (a) $f(x) = \dfrac{x^2 - 4}{x^3 - 8}$ (b) $f(x) = \begin{cases} 2x - 3, & x \le 2 \\ x^2, & x > 2 \end{cases}$

(c) $f(x) = \begin{cases} 3x^2 + 5, & x \ne 1 \\ 6, & x = 1 \end{cases}$

37. (a) Use a graphing utility to generate the graph of the function $f(x) = (x + 3)/(2x^2 + 5x - 3)$, and then use the graph to make a conjecture about the number and locations of all discontinuities.
(b) Check your conjecture by factoring the denominator.

38. (a) Use a graphing utility to generate the graph of the function $f(x) = x/(x^3 - x + 2)$, and then use the graph to make a conjecture about the number and locations of all discontinuities.
(b) Use the Intermediate-Value Theorem to approximate the locations of all discontinuities to two decimal places.

39. Prove that $f(x) = x^{3/5}$ is continuous everywhere, carefully justifying each step.

40. Prove that $f(x) = 1/\sqrt{x^4 + 7x^2 + 1}$ is continuous everywhere, carefully justifying each step.

41. Prove:
(a) part (*a*) of Theorem 1.5.3
(b) part (*b*) of Theorem 1.5.3
(c) part (*c*) of Theorem 1.5.3.

42. Prove part (*b*) of Theorem 1.5.4.

43. (a) Use Theorem 1.5.5 to prove that if f is continuous at $x = c$, then $\lim_{h \to 0} f(c + h) = f(c)$.
(b) Prove that if $\lim_{h \to 0} f(c + h) = f(c)$, then f is continuous at $x = c$. [*Hint:* What does this limit tell you about the continuity of $g(h) = f(c + h)$?]
(c) Conclude from parts (a) and (b) that f is continuous at $x = c$ if and only if $\lim_{h \to 0} f(c + h) = f(c)$.

44. Prove: If f and g are continuous on $[a, b]$, and $f(a) > g(a)$, $f(b) < g(b)$, then there is at least one solution of the equation $f(x) = g(x)$ in (a, b). [*Hint:* Consider $f(x) - g(x)$.]

FOCUS ON CONCEPTS

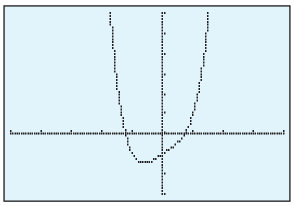
45. Give an example of a function f that is defined on a closed interval, and whose values at the endpoints have opposite signs, but for which the equation $f(x) = 0$ has no solution in the interval.

46. Let f be the function whose graph is shown in Exercise 2. For each interval, determine (i) whether the hypothesis of the Intermediate-Value Theorem is satisfied, and (ii) whether the conclusion of the Intermediate-Value Theorem is satisfied.
(a) $[1, 2]$ (b) $[2, 3]$ (c) $[1, 3]$

47. Show that the equation $x^3 + x^2 - 2x = 1$ has at least one solution in the interval $[-1, 1]$.

48. Prove: If $p(x)$ is a polynomial of odd degree, then the equation $p(x) = 0$ has at least one real solution.

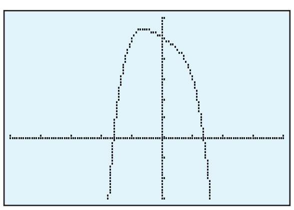
49. The accompanying figure shows the graph of the equation $y = x^4 + x - 1$. Use the method of Example 5 to approximate the x-intercepts with an error of at most 0.05.



$[-5, 4] \times [-3, 6]$
$x\text{Scl} = 1, y\text{Scl} = 1$

◀ **Figure Ex-49**

50. The accompanying figure shows the graph of the equation $y = 5 - x - x^4$. Use the method of Example 5 to approximate the roots of the equation $5 - x - x^4 = 0$ to two decimal-place accuracy.



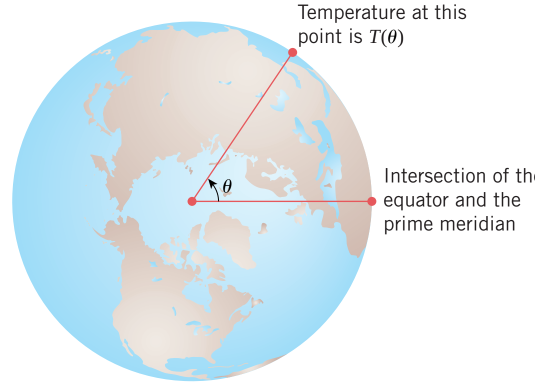
$[-5, 4] \times [-3, 6]$
$x\text{Scl} = 1, y\text{Scl} = 1$

◀ **Figure Ex-50**

51. Use the fact that $\sqrt{5}$ is a solution of $x^2 - 5 = 0$ to approximate $\sqrt{5}$ with an error of at most 0.005.

52. A sprinter, who is timed with a stopwatch, runs a hundred yard dash in 10 s. The stopwatch is reset to 0, and the sprinter is timed jogging back to the starting block. Show that there is at least one point on the track at which the reading on the stopwatch during the sprint is the same as the reading during the return jog. [*Hint:* Use the result in Exercise 44.]

53. Prove that there exist points on opposite sides of the equator that are at the same temperature. [*Hint:* Consider the accompanying figure, which shows a view of the equator from a point above the North Pole. Assume that the temperature $T(\theta)$ is a continuous function of the angle θ, and consider the function $f(\theta) = T(\theta + \pi) - T(\theta)$.]



▲ **Figure Ex-53**

54. Let R denote an elliptical region in the xy-plane, and define $f(z)$ to be the area within R that is on, or to the left of, the vertical line $x = z$. Prove that f is a continuous function of z. [*Hint:* Assume the ellipse is between the horizontal lines $y = a$ and $y = b$, $a < b$. Argue that $|f(z_1) - f(z_2)| \le (b - a) \cdot |z_1 - z_2|$.]

55. Let R denote an elliptical region in the plane. For any line L, prove there is a line perpendicular to L that divides R in half by area. [*Hint:* Introduce coordinates so that L is the x-axis. Use the result in Exercise 54 and the Intermediate-Value Theorem.]

56. Suppose that f is continuous on the interval $[0, 1]$ and that $0 \le f(x) \le 1$ for all x in this interval.
(a) Sketch the graph of $y = x$ together with a possible graph for f over the interval $[0, 1]$.
(b) Use the Intermediate-Value Theorem to help prove that there is at least one number c in the interval $[0, 1]$ such that $f(c) = c$.

57. Writing It is often assumed that changing physical quantities such as the height of a falling object or the weight of a melting snowball, are continuous functions of time. Use specific examples to discuss the merits of this assumption.

58. Writing The Intermediate-Value Theorem (Theorem 1.5.7) is an example of what is known as an "existence theorem." In your own words, describe how to recognize an existence theorem, and discuss some of the ways in which an existence theorem can be useful.

✔ QUICK CHECK ANSWERS 1.5

1. $f(c)$ is defined; $\lim_{x\to c} f(x)$ exists; $\lim_{x\to c} f(x) = f(c)$ **2.** (a) 3 (b) 3 **3.** $-2/5$ **4.** $x = 1, 4$
5. (a) yes (b) no (c) yes (d) yes

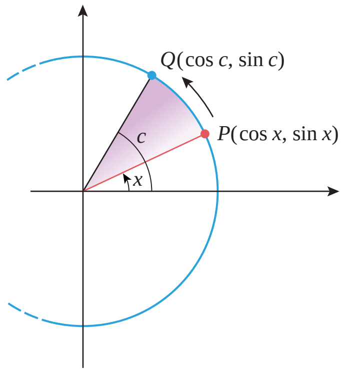
1.6 CONTINUITY OF TRIGONOMETRIC, EXPONENTIAL, AND INVERSE FUNCTIONS

In this section we will discuss the continuity properties of trigonometric functions, exponential functions, and inverses of various continuous functions. We will also discuss some important limits involving such functions.

CONTINUITY OF TRIGONOMETRIC FUNCTIONS

Recall from trigonometry that the graphs of $\sin x$ and $\cos x$ are drawn as continuous curves. We will not formally prove that these functions are continuous, but we can motivate this fact by letting c be a fixed angle in radian measure and x a variable angle in radian measure. If, as illustrated in Figure 1.6.1, the angle x approaches the angle c, then the point $P(\cos x, \sin x)$ moves along the unit circle toward $Q(\cos c, \sin c)$, and the coordinates of P approach the corresponding coordinates of Q. This implies that

$$\lim_{x\to c} \sin x = \sin c \quad \text{and} \quad \lim_{x\to c} \cos x = \cos c \tag{1}$$



As x approaches c the point P approaches the point Q.

Figure 1.6.1

Thus, $\sin x$ and $\cos x$ are continuous at the arbitrary point c; that is, these functions are continuous everywhere.

The formulas in (1) can be used to find limits of the remaining trigonometric functions by expressing them in terms of $\sin x$ and $\cos x$; for example, if $\cos c \neq 0$, then

$$\lim_{x\to c} \tan x = \lim_{x\to c} \frac{\sin x}{\cos x} = \frac{\sin c}{\cos c} = \tan c$$

Thus, we are led to the following theorem.

Theorem 1.6.1 implies that the six basic trigonometric functions are continuous on their domains. In particular, $\sin x$ and $\cos x$ are continuous everywhere.

1.6.1 THEOREM *If c is any number in the natural domain of the stated trigonometric function, then*

$$\lim_{x\to c} \sin x = \sin c \qquad \lim_{x\to c} \cos x = \cos c \qquad \lim_{x\to c} \tan x = \tan c$$

$$\lim_{x\to c} \csc x = \csc c \qquad \lim_{x\to c} \sec x = \sec c \qquad \lim_{x\to c} \cot x = \cot c$$

Example 1 Find the limit

$$\lim_{x\to 1} \cos\left(\frac{x^2-1}{x-1}\right)$$

Solution. Since the cosine function is continuous everywhere, it follows from Theorem 1.5.5 that

$$\lim_{x\to 1} \cos(g(x)) = \cos\left(\lim_{x\to 1} g(x)\right)$$

provided $\lim_{x\to 1} g(x)$ exists. Thus,

$$\lim_{x\to 1} \cos\left(\frac{x^2-1}{x-1}\right) = \lim_{x\to 1} \cos(x+1) = \cos\left(\lim_{x\to 1}(x+1)\right) = \cos 2 \blacktriangleleft$$

CONTINUITY OF INVERSE FUNCTIONS

Since the graphs of a one-to-one function f and its inverse f^{-1} are reflections of one another about the line $y = x$, it is clear geometrically that if the graph of f has no breaks or holes in it, then neither does the graph of f^{-1}. This, and the fact that the range of f is the domain of f^{-1}, suggests the following result, which we state without formal proof.

To paraphrase Theorem 1.6.2, *the inverse of a continuous function is continuous.*

1.6.2 THEOREM *If f is a one-to-one function that is continuous at each point of its domain, then f^{-1} is continuous at each point of its domain; that is, f^{-1} is continuous at each point in the range of f.*

▶ Example 2 Use Theorem 1.6.2 to prove that $\sin^{-1} x$ is continuous on the interval $[-1, 1]$.

Solution. Recall that $\sin^{-1} x$ is the inverse of the restricted sine function whose domain is the interval $[-\pi/2, \pi/2]$ and whose range is the interval $[-1, 1]$ (Definition 0.4.6 and Figure 0.4.13). Since $\sin x$ is continuous on the interval $[-\pi/2, \pi/2]$, Theorem 1.6.2 implies $\sin^{-1} x$ is continuous on the interval $[-1, 1]$. ◀

Arguments similar to the solution of Example 2 show that each of the inverse trigonometric functions defined in Section 0.4 is continuous at each point of its domain.

When we introduced the exponential function $f(x) = b^x$ in Section 0.5, we assumed that its graph is a curve without breaks, gaps, or holes; that is, we assumed that the graph of $y = b^x$ is a continuous curve. This assumption and Theorem 1.6.2 imply the following theorem, which we state without formal proof.

1.6.3 THEOREM *Let $b > 0$, $b \neq 1$.*

(a) The function b^x is continuous on $(-\infty, +\infty)$.

(b) The function $\log_b x$ is continuous on $(0, +\infty)$.

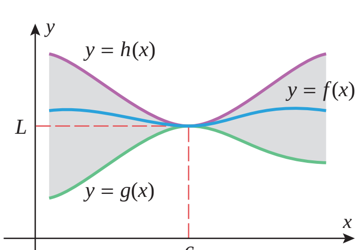
▶ Example 3 Where is the function $f(x) = \dfrac{\tan^{-1} x + \ln x}{x^2 - 4}$ continuous?

Solution. The fraction will be continuous at all points where the numerator and denominator are both continuous and the denominator is nonzero. Since $\tan^{-1} x$ is continuous everywhere and $\ln x$ is continuous if $x > 0$, the numerator is continuous if $x > 0$. The denominator, being a polynomial, is continuous everywhere, so the fraction will be continuous at all points where $x > 0$ and the denominator is nonzero. Thus, f is continuous on the intervals $(0, 2)$ and $(2, +\infty)$. ◀

OBTAINING LIMITS BY SQUEEZING

In Section 1.1 we used numerical evidence to conjecture that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{2}$$



▲ **Figure 1.6.2**

However, this limit is not easy to establish with certainty. The limit is an indeterminate form of type $0/0$, and there is no simple algebraic manipulation that one can perform to obtain the limit. Later in the text we will develop general methods for finding limits of indeterminate forms, but in this particular case we can use a technique called *squeezing*.

The method of squeezing is used to prove that $f(x) \to L$ as $x \to c$ by "trapping" or "squeezing" f between two functions, g and h, whose limits as $x \to c$ are known with *certainty* to be L. As illustrated in Figure 1.6.2, this forces f to have a limit of L as well. This is the idea behind the following theorem, which we state without proof.

1.6.4 THEOREM (*The Squeezing Theorem*) *Let f, g, and h be functions satisfying*

$$g(x) \le f(x) \le h(x)$$

for all x in some open interval containing the number c, with the possible exception that the inequalities need not hold at c. If g and h have the same limit as x approaches c, say

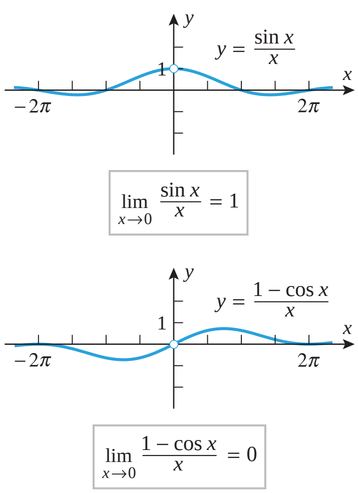
$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

then f also has this limit as x approaches c, that is,

$$\lim_{x\to c} f(x) = L$$

The Squeezing Theorem also holds for one-sided limits and limits at $+\infty$ and $-\infty$. How do you think the hypotheses would change in those cases?

To illustrate how the Squeezing Theorem works, we will prove the following results, which are illustrated in Figure 1.6.3.



▲ **Figure 1.6.3**

1.6.5 THEOREM

$$(a)\quad \lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad (b)\quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

PROOF (*a*) In this proof we will interpret x as an angle in radian measure, and we will assume to start that $0 < x < \pi/2$. As illustrated in Figure 1.6.4, the area of a sector with central angle x and radius 1 lies between the areas of two triangles, one with area $\frac{1}{2}\tan x$ and the other with area $\frac{1}{2}\sin x$. Since the sector has area $\frac{1}{2}x$ (see marginal note), it follows that

$$\frac{1}{2}\tan x \ge \frac{1}{2}x \ge \frac{1}{2}\sin x$$

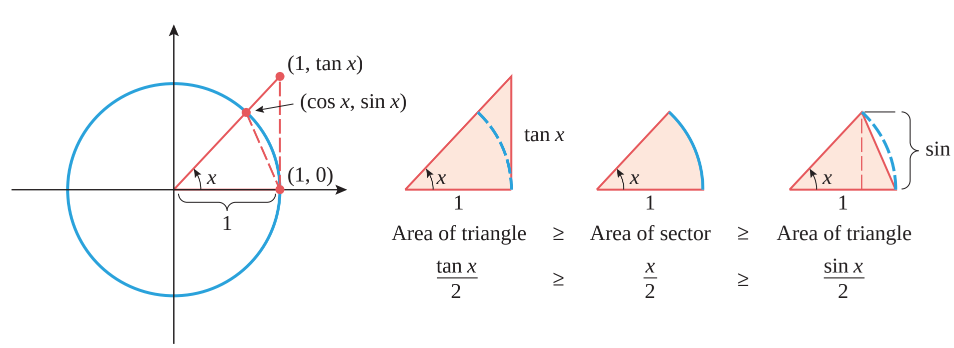
Multiplying through by $2/(\sin x)$ and using the fact that $\sin x > 0$ for $0 < x < \pi/2$, we obtain

$$\frac{1}{\cos x} \ge \frac{x}{\sin x} \ge 1$$

Next, taking reciprocals reverses the inequalities, so we obtain

$$\cos x \le \frac{\sin x}{x} \le 1 \tag{3}$$

which squeezes the function $(\sin x)/x$ between the functions $\cos x$ and 1. Although we derived these inequalities by assuming that $0 < x < \pi/2$, they also hold for $-\pi/2 < x < 0$ [since replacing x by $-x$ and using the identities $\sin(-x) = -\sin x$, and $\cos(-x) = \cos x$



▶ **Figure 1.6.4**

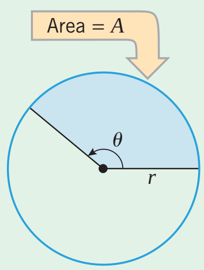
Recall that the area A of a sector of radius r and central angle θ is

$$A = \frac{1}{2}r^2\theta$$

This can be derived from the relationship

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

which states that the area of the sector is to the area of the circle as the central angle of the sector is to the central angle of the circle.



leaves (3) unchanged]. Finally, since

$$\lim_{x\to 0} \cos x = 1 \quad \text{and} \quad \lim_{x\to 0} 1 = 1$$

the Squeezing Theorem implies that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

PROOF (*b*) For this proof we will use the limit in part (*a*), the continuity of the sine function, and the trigonometric identity $\sin^2 x = 1 - \cos^2 x$. We obtain

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0}\left[\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\right] = \lim_{x\to 0}\frac{\sin^2 x}{(1+\cos x)x}$$
$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right) = (1)\left(\frac{0}{1+1}\right) = 0 \quad \blacksquare$$

▶ Example 4 Find

$$\text{(a)} \lim_{x\to 0}\frac{\tan x}{x} \qquad \text{(b)} \lim_{\theta\to 0}\frac{\sin 2\theta}{\theta} \qquad \text{(c)} \lim_{x\to 0}\frac{\sin 3x}{\sin 5x}$$

Solution (a).

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right) = \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{1}{\cos x}\right) = (1)(1) = 1$$

Solution (b). The trick is to multiply and divide by 2, which will make the denominator the same as the argument of the sine function [just as in Theorem 1.6.5(*a*)]:

$$\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta} = \lim_{\theta\to 0} 2\cdot\frac{\sin 2\theta}{2\theta} = 2\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta}$$

Now make the substitution $x = 2\theta$, and use the fact that $x \to 0$ as $\theta \to 0$. This yields

$$\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta} = 2\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta} = 2\lim_{x\to 0}\frac{\sin x}{x} = 2(1) = 2$$

Solution (c).

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 5x} = \lim_{x\to 0}\frac{\dfrac{\sin 3x}{x}}{\dfrac{\sin 5x}{x}} = \lim_{x\to 0}\frac{3\cdot\dfrac{\sin 3x}{3x}}{5\cdot\dfrac{\sin 5x}{5x}} = \frac{3\cdot 1}{5\cdot 1} = \frac{3}{5} \quad ◀$$

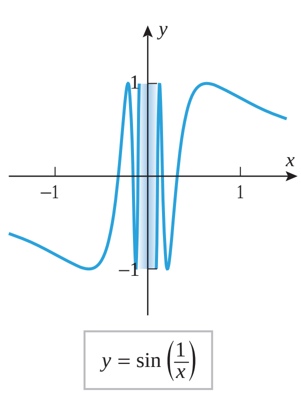
TECHNOLOGY MASTERY

Use a graphing utility to confirm the limits in Example 4, and if you have a CAS, use it to obtain the limits.

▶ Example 5 Discuss the limits

$$\text{(a)} \lim_{x\to 0}\sin\left(\frac{1}{x}\right) \qquad \text{(b)} \lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$$

Solution (a). Let us view $1/x$ as an angle in radian measure. As $x \to 0^+$, the angle $1/x$ approaches $+\infty$, so the values of $\sin(1/x)$ keep oscillating between -1 and 1 without approaching a limit. Similarly, as $x \to 0^-$, the angle $1/x$ approaches $-\infty$, so again the values of $\sin(1/x)$ keep oscillating between -1 and 1 without approaching a limit. These conclusions are consistent with the graph shown in Figure 1.6.5. Note that the oscillations become more and more rapid as $x \to 0$ because $1/x$ increases (or decreases) more and more rapidly as x approaches 0.



▲ **Figure 1.6.5**

Confirm (4) by considering the cases $x > 0$ and $x < 0$ separately.

Solution (b). Since

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

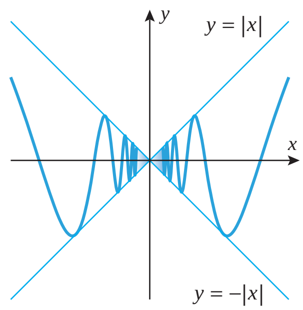
it follows that if $x \neq 0$, then

$$-|x| \le x \sin\left(\frac{1}{x}\right) \le |x| \tag{4}$$

Since $|x| \to 0$ as $x \to 0$, the inequalities in (4) and the Squeezing Theorem imply that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$$

This is consistent with the graph shown in Figure 1.6.6. ◀



$y = x\sin\left(\frac{1}{x}\right)$

▲ **Figure 1.6.6**

REMARK It follows from part (b) of this example that the function

$$f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at $x = 0$, since the value of the function and the value of the limit are the same at 0. This shows that the behavior of a function can be very complex in the vicinity of $x = c$, even though the function is continuous at c.

✔ QUICK CHECK EXERCISES 1.6 *(See page 128 for answers.)*

1. In each part, is the given function continuous on the interval $[0, \pi/2)$?
(a) $\sin x$ (b) $\cos x$ (c) $\tan x$ (d) $\csc x$

2. Evaluate
(a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$
(b) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x}$.

3. Suppose a function f has the property that for all real numbers x
$$3 - |x| \le f(x) \le 3 + |x|$$
From this we can conclude that $f(x) \to$ ______ as $x \to$ ______.

4. In each part, give the largest interval on which the function is continuous.
(a) e^x (b) $\ln x$ (c) $\sin^{-1} x$ (d) $\tan^{-1} x$

EXERCISE SET 1.6 Graphing Utility

1–8 Find the discontinuities, if any. ■

1. $f(x) = \sin(x^2 - 2)$

2. $f(x) = \cos\left(\dfrac{x}{x-\pi}\right)$

3. $f(x) = |\cot x|$

4. $f(x) = \sec x$

5. $f(x) = \csc x$

6. $f(x) = \dfrac{1}{1+\sin^2 x}$

7. $f(x) = \dfrac{1}{1 - 2\sin x}$

8. $f(x) = \sqrt{2 + \tan^2 x}$

9–14 Determine where f is continuous. ■

9. $f(x) = \sin^{-1} 2x$

10. $f(x) = \cos^{-1}(\ln x)$

11. $f(x) = \dfrac{\ln(\tan^{-1} x)}{x^2 - 9}$

12. $f(x) = \exp\left(\dfrac{\sin x}{x}\right)$

13. $f(x) = \dfrac{\sin^{-1}(1/x)}{x}$

14. $f(x) = \ln|x| - 2\ln(x+3)$

15–16 In each part, use Theorem 1.5.6(*b*) to show that the function is continuous everywhere.

15. (a) $\sin(x^3 + 7x + 1)$ (b) $|\sin x|$ (c) $\cos^3(x + 1)$

16. (a) $|3 + \sin 2x|$ (b) $\sin(\sin x)$
(c) $\cos^5 x - 2\cos^3 x + 1$

17–40 Find the limits.

17. $\lim_{x\to+\infty} \cos\left(\frac{1}{x}\right)$

18. $\lim_{x\to+\infty} \sin\left(\frac{\pi x}{2-3x}\right)$

19. $\lim_{x\to+\infty} \sin^{-1}\left(\frac{x}{1-2x}\right)$

20. $\lim_{x\to+\infty} \ln\left(\frac{x+1}{x}\right)$

21. $\lim_{x\to 0} e^{\sin x}$

22. $\lim_{x\to+\infty} \cos(2\tan^{-1} x)$

23. $\lim_{\theta\to 0} \frac{\sin 3\theta}{\theta}$

24. $\lim_{h\to 0} \frac{\sin h}{2h}$

25. $\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta^2}$

26. $\lim_{\theta\to 0} \frac{\sin^2\theta}{\theta}$

27. $\lim_{x\to 0} \frac{\tan 7x}{\sin 3x}$

28. $\lim_{x\to 0} \frac{\sin 6x}{\sin 8x}$

29. $\lim_{x\to 0^+} \frac{\sin x}{5\sqrt{x}}$

30. $\lim_{x\to 0} \frac{\sin^2 x}{3x^2}$

31. $\lim_{x\to 0} \frac{\sin x^2}{x}$

32. $\lim_{h\to 0} \frac{\sin h}{1-\cos h}$

33. $\lim_{t\to 0} \frac{t^2}{1-\cos^2 t}$

34. $\lim_{x\to 0} \frac{x}{\cos\left(\frac{1}{2}\pi - x\right)}$

35. $\lim_{\theta\to 0} \frac{\theta^2}{1-\cos\theta}$

36. $\lim_{h\to 0} \frac{1-\cos 3h}{\cos^2 5h - 1}$

37. $\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right)$

38. $\lim_{x\to 0} \frac{x^2 - 3\sin x}{x}$

39. $\lim_{x\to 0} \frac{2-\cos 3x - \cos 4x}{x}$

40. $\lim_{x\to 0} \frac{\tan 3x^2 + \sin^2 5x}{x^2}$

41–42 (a) Complete the table and make a guess about the limit indicated. (b) Find the exact value of the limit.

41. $f(x) = \frac{\sin(x-5)}{x^2-25}$; $\lim_{x\to 5} f(x)$

x	4	4.5	4.9	5.1	5.5	6
$f(x)$						

Table Ex-41

42. $f(x) = \frac{\sin(x^2+3x+2)}{x+2}$; $\lim_{x\to -2} f(x)$

x	−2.1	−2.01	−2.001	−1.999	−1.99	−1.9
$f(x)$						

Table Ex-42

43–46 True–False Determine whether the statement is true or false. Explain your answer.

43. Suppose that for all real numbers x, a function f satisfies
$$|f(x) + 5| \le |x + 1|$$
Then $\lim_{x\to -1} f(x) = -5$.

44. For $0 < x < \pi/2$, the graph of $y = \sin x$ lies below the graph of $y = x$ and above the graph of $y = x\cos x$.

45. If an invertible function f is continuous everywhere, then its inverse f^{-1} is also continuous everywhere.

46. Suppose that M is a positive number and that for all real numbers x, a function f satisfies
$$-M \le f(x) \le M$$
Then
$$\lim_{x\to 0} xf(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} \frac{f(x)}{x} = 0$$

FOCUS ON CONCEPTS

47. In an attempt to verify that $\lim_{x\to 0}(\sin x)/x = 1$, a student constructs the accompanying table.
(a) What mistake did the student make?
(b) What is the exact value of the limit illustrated by this table?

x	−0.01	−0.001	0.001	0.01
$\sin x/x$	0.017453	0.017453	0.017453	0.017453

Table Ex-47

48. In the circle in the accompanying figure, a central angle of measure θ radians subtends a chord of length $c(\theta)$ and a circular arc of length $s(\theta)$. Based on your intuition, what would you conjecture is the value of $\lim_{\theta\to 0^+} c(\theta)/s(\theta)$? Verify your conjecture by computing the limit.

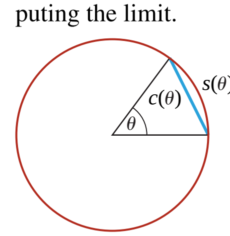


Figure Ex-48

49. Find a nonzero value for the constant k that makes
$$f(x) = \begin{cases} \dfrac{\tan kx}{x}, & x < 0 \\ 3x + 2k^2, & x \ge 0 \end{cases}$$
continuous at $x = 0$.

50. Is
$$f(x) = \begin{cases} \dfrac{\sin x}{|x|}, & x \ne 0 \\ 1, & x = 0 \end{cases}$$
continuous at $x = 0$? Explain.

51. In parts (a)–(c), find the limit by making the indicated substitution.
(a) $\lim_{x\to+\infty} x \sin \frac{1}{x}$; $t = \frac{1}{x}$
(b) $\lim_{x\to-\infty} x\left(1 - \cos \frac{1}{x}\right)$; $t = \frac{1}{x}$
(c) $\lim_{x\to\pi} \frac{\pi - x}{\sin x}$; $t = \pi - x$

52. Find $\lim_{x\to 2} \frac{\cos(\pi/x)}{x - 2}$. $\left[Hint\text{: Let } t = \frac{\pi}{2} - \frac{\pi}{x}.\right]$

53. Find $\lim_{x\to 1} \frac{\sin(\pi x)}{x - 1}$.

54. Find $\lim_{x\to\pi/4} \frac{\tan x - 1}{x - \pi/4}$.

55. Find $\lim_{x\to\pi/4} \frac{\cos x - \sin x}{x - \pi/4}$.

56. Suppose that f is an invertible function, $f(0) = 0$, f is continuous at 0, and $\lim_{x\to 0}(f(x)/x)$ exists. Given that $L = \lim_{x\to 0}(f(x)/x)$, show

$$\lim_{x\to 0} \frac{x}{f^{-1}(x)} = L$$

[*Hint:* Apply Theorem 1.5.5 to the composition $h \circ g$, where

$$h(x) = \begin{cases} f(x)/x, & x \neq 0 \\ L, & x = 0 \end{cases}$$

and $g(x) = f^{-1}(x)$.]

57–60 Apply the result of Exercise 56, if needed, to find the limits. ■

57. $\lim_{x\to 0} \frac{x}{\sin^{-1} x}$

58. $\lim_{x\to 0} \frac{\tan^{-1} x}{x}$

59. $\lim_{x\to 0} \frac{\sin^{-1} 5x}{x}$

60. $\lim_{x\to 1} \frac{\sin^{-1}(x - 1)}{x^2 - 1}$

FOCUS ON CONCEPTS

61. Use the Squeezing Theorem to show that

$$\lim_{x\to 0} x \cos \frac{50\pi}{x} = 0$$

and illustrate the principle involved by using a graphing utility to graph the equations $y = |x|$, $y = -|x|$, and $y = x\cos(50\pi/x)$ on the same screen in the window $[-1, 1] \times [-1, 1]$.

62. Use the Squeezing Theorem to show that

$$\lim_{x\to 0} x^2 \sin\left(\frac{50\pi}{\sqrt[3]{x}}\right) = 0$$

and illustrate the principle involved by using a graphing utility to graph the equations $y = x^2$, $y = -x^2$, and $y = x^2 \sin(50\pi/\sqrt[3]{x})$ on the same screen in the window $[-0.5, 0.5] \times [-0.25, 0.25]$.

63. In Example 5 we used the Squeezing Theorem to prove that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$$

Why couldn't we have obtained the same result by writing

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = \lim_{x\to 0} x \cdot \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$
$$= 0 \cdot \lim_{x\to 0} \sin\left(\frac{1}{x}\right) = 0?$$

64. Sketch the graphs of the curves $y = 1 - x^2$, $y = \cos x$, and $y = f(x)$, where f is a function that satisfies the inequalities

$$1 - x^2 \leq f(x) \leq \cos x$$

for all x in the interval $(-\pi/2, \pi/2)$. What can you say about the limit of $f(x)$ as $x \to 0$? Explain.

65. Sketch the graphs of the curves $y = 1/x$, $y = -1/x$, and $y = f(x)$, where f is a function that satisfies the inequalities

$$-\frac{1}{x} \leq f(x) \leq \frac{1}{x}$$

for all x in the interval $[1, +\infty)$. What can you say about the limit of $f(x)$ as $x \to +\infty$? Explain your reasoning.

66. Draw pictures analogous to Figure 1.6.2 that illustrate the Squeezing Theorem for limits of the forms $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

67. (a) Use the Intermediate-Value Theorem to show that the equation $x = \cos x$ has at least one solution in the interval $[0, \pi/2]$.
(b) Show graphically that there is exactly one solution in the interval.
(c) Approximate the solution to three decimal places.

68. (a) Use the Intermediate-Value Theorem to show that the equation $x + \sin x = 1$ has at least one solution in the interval $[0, \pi/6]$.
(b) Show graphically that there is exactly one solution in the interval.
(c) Approximate the solution to three decimal places.

69. In the study of falling objects near the surface of the Earth, the ***acceleration g due to gravity*** is commonly taken to be a constant 9.8 m/s^2. However, the elliptical shape of the Earth and other factors cause variations in this value that depend on latitude. The following formula, known as the World Geodetic System 1984 (WGS 84) Ellipsoidal Gravity Formula, is used to predict the value of g at a latitude of ϕ degrees (either north or south of the equator):

$$g = 9.7803253359 \frac{1 + 0.0019318526461 \sin^2 \phi}{\sqrt{1 - 0.0066943799901 \sin^2 \phi}} \text{ m/s}^2$$

(cont.)

(a) Use a graphing utility to graph the curve $y = g(\phi)$ for $0° \le \phi \le 90°$. What do the values of g at $\phi = 0°$ and at $\phi = 90°$ tell you about the WGS 84 ellipsoid model for the Earth?
(b) Show that $g = 9.8\ \text{m/s}^2$ somewhere between latitudes of 38° and 39°.

70. Writing In your own words, explain the *practical value* of the Squeezing Theorem.

71. Writing A careful examination of the proof of Theorem 1.6.5 raises the issue of whether the proof might actually be a circular argument! Read the article "A Circular Argument" by Fred Richman in the March 1993 issue of *The College Mathematics Journal*, and write a short report on the author's principal points.

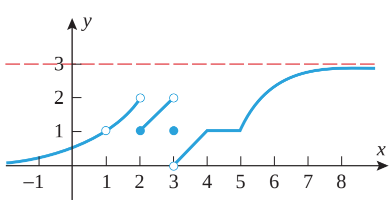
✔ QUICK CHECK ANSWERS 1.6

1. (a) yes (b) yes (c) yes (d) no **2.** (a) 1 (b) 0 **3.** 3; 0 **4.** (a) $(-\infty, +\infty)$ (b) $(0, +\infty)$ (c) $[-1, 1]$ (d) $(-\infty, +\infty)$

CHAPTER 1 REVIEW EXERCISES

Graphing Utility C CAS

1. For the function f graphed in the accompanying figure, find the limit if it exists.
(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 2} f(x)$ (c) $\lim_{x\to 3} f(x)$
(d) $\lim_{x\to 4} f(x)$ (e) $\lim_{x\to +\infty} f(x)$ (f) $\lim_{x\to -\infty} f(x)$
(g) $\lim_{x\to 3^+} f(x)$ (h) $\lim_{x\to 3^-} f(x)$ (i) $\lim_{x\to 0} f(x)$



◀ **Figure Ex-1**

2. In each part, complete the table and make a conjecture about the value of the limit indicated. Confirm your conjecture by finding the limit analytically.

(a) $f(x) = \dfrac{x-2}{x^2-4}$; $\lim_{x\to 2^+} f(x)$

x	2.00001	2.0001	2.001	2.01	2.1	2.5
$f(x)$						

(b) $f(x) = \dfrac{\tan 4x}{x}$; $\lim_{x\to 0} f(x)$

x	−0.01	−0.001	−0.0001	0.0001	0.001	0.01
$f(x)$						

3. (a) Approximate the value for the limit
$$\lim_{x\to 0} \frac{3^x - 2^x}{x}$$
to three decimal places by constructing an appropriate table of values.
(b) Confirm your approximation using graphical evidence.

C **4.** Approximate
$$\lim_{x\to 3} \frac{2^x - 8}{x - 3}$$
both by looking at a graph and by calculating values for some appropriate choices of x. Compare your answer with the value produced by a CAS.

5–10 Find the limits. ■

5. $\displaystyle\lim_{x\to -1} \frac{x^3 - x^2}{x - 1}$ **6.** $\displaystyle\lim_{x\to 1} \frac{x^3 - x^2}{x - 1}$

7. $\displaystyle\lim_{x\to -3} \frac{3x + 9}{x^2 + 4x + 3}$ **8.** $\displaystyle\lim_{x\to 2^-} \frac{x + 2}{x - 2}$

9. $\displaystyle\lim_{x\to +\infty} \frac{(2x-1)^5}{(3x^2+2x-7)(x^3-9x)}$

10. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x^2+4} - 2}{x^2}$

11. In each part, find the horizontal asymptotes, if any.
(a) $y = \dfrac{2x - 7}{x^2 - 4x}$ (b) $y = \dfrac{x^3 - x^2 + 10}{3x^2 - 4x}$
(c) $y = \dfrac{2x^2 - 6}{x^2 + 5x}$

12. In each part, find $\lim_{x\to a} f(x)$, if it exists, where a is replaced by $0, 5^+, -5^-, -5, 5, -\infty$, and $+\infty$.
(a) $f(x) = \sqrt{5 - x}$
(b) $f(x) = \begin{cases} (x-5)/|x-5|, & x \ne 5 \\ 0, & x = 5 \end{cases}$

13–20 Find the limits. ■

13. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{\tan 3x}$ **14.** $\displaystyle\lim_{x\to 0} \frac{x \sin x}{1 - \cos x}$

15. $\displaystyle\lim_{x\to 0} \frac{3x - \sin(kx)}{x}$, $k \ne 0$

16. $\displaystyle\lim_{\theta\to 0} \tan\left(\frac{1 - \cos\theta}{\theta}\right)$

17. $\displaystyle\lim_{t\to \pi/2^+} e^{\tan t}$ **18.** $\displaystyle\lim_{\theta\to 0^+} \ln(\sin 2\theta) - \ln(\tan\theta)$

19. $\lim_{x \to +\infty} \left(1 + \frac{3}{x}\right)^{-x}$ **20.** $\lim_{x \to +\infty} \left(1 + \frac{a}{x}\right)^{bx}, \quad a, b > 0$

21. If \$1000 is invested in an account that pays 7% interest compounded n times each year, then in 10 years there will be $1000(1 + 0.07/n)^{10n}$ dollars in the account. How much money will be in the account in 10 years if the interest is compounded quarterly ($n = 4$)? Monthly ($n = 12$)? Daily ($n = 365$)? Determine the amount of money that will be in the account in 10 years if the interest is compounded *continuously*, that is, as $n \to +\infty$.

22. (a) Write a paragraph or two that describes how the limit of a function can fail to exist at $x = a$, and accompany your description with some specific examples.
(b) Write a paragraph or two that describes how the limit of a function can fail to exist as $x \to +\infty$ or $x \to -\infty$, and accompany your description with some specific examples.
(c) Write a paragraph or two that describes how a function can fail to be continuous at $x = a$, and accompany your description with some specific examples.

23. (a) Find a formula for a rational function that has a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 2$.
(b) Check your work by using a graphing utility to graph the function.

24. Paraphrase the ϵ-δ definition for $\lim_{x \to a} f(x) = L$ in terms of a graphing utility viewing window centered at the point (a, L).

25. Suppose that $f(x)$ is a function and that for any given $\epsilon > 0$, the condition $0 < |x - 2| < \frac{3}{4}\epsilon$ guarantees that $|f(x) - 5| < \epsilon$.
(a) What limit is described by this statement?
(b) Find a value of δ such that $0 < |x - 2| < \delta$ guarantees that $|8f(x) - 40| < 0.048$.

26. The limit
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$
ensures that there is a number δ such that
$$\left|\frac{\sin x}{x} - 1\right| < 0.001$$
if $0 < |x| < \delta$. Estimate the largest such δ.

27. In each part, a positive number ϵ and the limit L of a function f at a are given. Find a number δ such that $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.
(a) $\lim_{x \to 2} (4x - 7) = 1;\ \epsilon = 0.01$
(b) $\lim_{x \to 3/2} \frac{4x^2 - 9}{2x - 3} = 6;\ \epsilon = 0.05$
(c) $\lim_{x \to 4} x^2 = 16;\ \epsilon = 0.001$

28. Use Definition 1.4.1 to prove the stated limits are correct.
(a) $\lim_{x \to 2} (4x - 7) = 1$ (b) $\lim_{x \to 3/2} \frac{4x^2 - 9}{2x - 3} = 6$

29. Suppose that f is continuous at x_0 and that $f(x_0) > 0$. Give either an ϵ-δ proof or a convincing verbal argument to show that there must be an open interval containing x_0 on which $f(x) > 0$.

30. (a) Let
$$f(x) = \frac{\sin x - \sin 1}{x - 1}$$
Approximate $\lim_{x \to 1} f(x)$ by graphing f and calculating values for some appropriate choices of x.
(b) Use the identity
$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha - \beta}{2}\cos\frac{\alpha + \beta}{2}$$
to find the exact value of $\lim_{x \to 1} f(x)$.

31. Find values of x, if any, at which the given function is not continuous.
(a) $f(x) = \frac{x}{x^2 - 1}$ (b) $f(x) = |x^3 - 2x^2|$
(c) $f(x) = \frac{x + 3}{|x^2 + 3x|}$

32. Determine where f is continuous.
(a) $f(x) = \frac{x}{|x| - 3}$ (b) $f(x) = \cos^{-1}\left(\frac{1}{x}\right)$
(c) $f(x) = e^{\ln x}$

33. Suppose that
$$f(x) = \begin{cases} -x^4 + 3, & x \le 2 \\ x^2 + 9, & x > 2 \end{cases}$$
Is f continuous everywhere? Justify your conclusion.

34. One dictionary describes a continuous function as "one whose value at each point is closely approached by its values at neighboring points."
(a) How would you explain the meaning of the terms "neighboring points" and "closely approached" to a nonmathematician?
(b) Write a paragraph that explains why the dictionary definition is consistent with Definition 1.5.1.

35. Show that the conclusion of the Intermediate-Value Theorem may be false if f is not continuous on the interval $[a, b]$.

36. Suppose that f is continuous on the interval $[0, 1]$, that $f(0) = 2$, and that f has no zeros in the interval. Prove that $f(x) > 0$ for all x in $[0, 1]$.

37. Show that the equation $x^4 + 5x^3 + 5x - 1 = 0$ has at least two real solutions in the interval $[-6, 2]$.

CHAPTER 1 MAKING CONNECTIONS

In Section 1.1 we developed the notion of a tangent line to a graph at a given point by considering it as a limiting position of secant lines through that point (Figure 1.1.4a). In these exercises we will develop an analogous idea in which secant lines are replaced by "secant circles" and the tangent line is replaced by a "tangent circle" (called the ***osculating circle***). We begin with the graph of $y = x^2$.

1. Recall that there is a unique circle through any three noncollinear points in the plane. For any positive real number x, consider the unique "secant circle" that passes through the fixed point $O(0, 0)$ and the variable points $Q(-x, x^2)$ and $P(x, x^2)$ (see the accompanying figure). Use plane geometry to explain why the center of this circle is the intersection of the y-axis and the perpendicular bisector of segment OP.

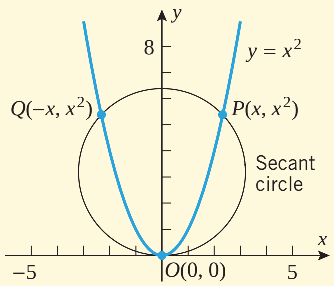


Figure Ex-1

2. (a) Let $(0, C(x))$ denote the center of the circle in Exercise 1 and show that

$$C(x) = \tfrac{1}{2}x^2 + \tfrac{1}{2}$$

(b) Show that as $x \to 0^+$, the secant circles approach a limiting position given by the circle that passes through the origin and is centered at $\left(0, \frac{1}{2}\right)$. As shown in the accompanying figure, this circle is the osculating circle to the graph of $y = x^2$ at the origin.

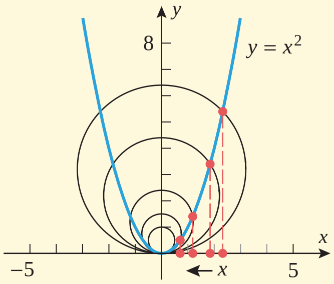


Figure Ex-2

3. Show that if we replace the curve $y = x^2$ by the curve $y = f(x)$, where f is an even function, then the formula for $C(x)$ becomes

$$C(x) = \frac{1}{2}\left[f(0) + f(x) + \frac{x^2}{f(x) - f(0)}\right]$$

[Here we assume that $f(x) \neq f(0)$ for positive values of x close to 0.] If $\lim_{x\to 0^+} C(x) = L \neq f(0)$, then we define the osculating circle to the curve $y = f(x)$ at $(0, f(0))$ to be the unique circle through $(0, f(0))$ with center $(0, L)$. If $C(x)$ does not have a finite limit different from $f(0)$ as $x \to 0^+$, then we say that the curve has no osculating circle at $(0, f(0))$.

4. In each part, determine the osculating circle to the curve $y = f(x)$ at $(0, f(0))$, if it exists.

(a) $f(x) = 4x^2$
(b) $f(x) = x^2 \cos x$
(c) $f(x) = |x|$
(d) $f(x) = x \sin x$
(e) $f(x) = \cos x$
(f) $f(x) = x^2 g(x)$, where $g(x)$ is an even continuous function with $g(0) \neq 0$
(g) $f(x) = x^4$