Lecture - 1

Limit: If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a, (but not equal to a) then we write

$$\lim_{x \to a} f(x) = L.$$

Example: $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}.$

x	$f(x)$	x	$f(x)$
2.99	5.99	3.01	6.01
2.9999	5.9999	3.0001	6.0001
2.99999	5.99999	3.00001	6.00001 .

Here we observe that the function $f(x)$ is undefined at $x = 3$. But when we move closer to 3, the value of the function moves toward 6. That means the limiting value of the function is 6 when the value of x moves toward 3.

Problem 1: $\lim_{x \to 5} (x^2 - 4x + 3)$

$$= \lim_{x \to 5} x^2 - \lim_{x \to 5} 4x + \lim_{x \to 5} 3.$$

$$= 5^2 - 4 \times 5 + 3 = 8.$$

Therefore, the value of the function approaches ~~8~~ to 8 when x approaches to 5.

Problem 2:- $\lim_{x \to 2} \frac{5x^3+4}{x-3}$

$$= \lim_{x \to 2} \frac{5x^3+4}{x-3}$$

$$= \frac{5 \cdot 2^3+4}{2-3} = \frac{44}{-1} = -44.$$

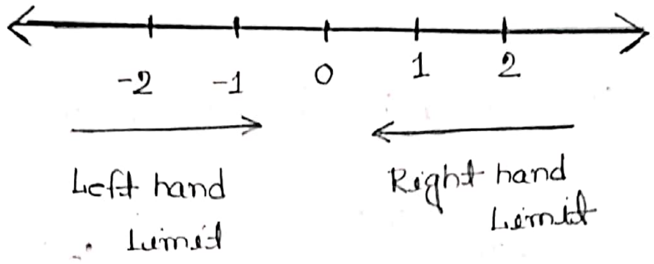
Left Hand Limits:- If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (less than a), then we write

$$\lim_{x \to a^-} f(x) = L.$$

Right Hand Limits:- If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (greater than a) then we write

$$\lim_{x \to a^+} f(x) = L.$$

Real Line



* These left hand and right hand limits are often called one sided limits.

Relationship Between One-sided and Two-sided Limits :-

The two sided limits ~~exists~~ of a function $f(x)$ exists at 'a' if and only if both of the one-sided limits exist at 'a' and have the same value i.e.

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

Example :- Explain why $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist.

Solution :- Let us assume that

$$f(x) = \frac{|x|}{x}$$

Let us find out the left hand limits and right hand limit at $x = 0$.

To find out the left hand limit we set,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x}.$$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}.$$

$$= \lim_{x \to 0^-} \frac{-x}{x}.$$

$$= \lim_{x \to 0^-} (-1)$$

$$= -1.$$

and right hand limit we set.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x}.$$

$$= \lim_{x \to 0^+} \frac{x}{x}.$$

$$= \lim_{x \to 0^+} 1.$$

$$= 1$$

Since left hand limit and right han limit are not equal, therefore $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Ans.

Example :- A function $f(x)$ is defined as

$$f(x) = \begin{cases} x-1, & x \leq 3 \\ 3x-7, & x > 3. \end{cases}$$

Find a) $\lim\limits_{x \to 3} f(x)$.

Solution :- Given that,

$$f(x) = \begin{cases} x-1, & x \leq 3 \\ 3x-7, & x > 3. \end{cases}$$

As x approaches 3 from the right,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 3x-7 = 3.3-7 = 2.$$

As x approaches 3 from the left,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x-1 = 3-1 = 2$$

Here we observe that the left hand limit and right hand limit exist and they are equal.

Therefore, $\lim_{x \to 3} f(x) = 2$.

Ans.

Example: A function $g(t)$ is defined as

$$g(t) = \begin{cases} t-2, & t<0 \\ t^2 & 0 \leq t \leq 2 \\ 2t & t>2 \end{cases}$$

Find a) $\lim_{t \to 0} g(t)$ b) $\lim_{t \to 1} g(t)$ c) $\lim_{t \to 2} g(t)$.

Solution: Given that

$$g(t) = \begin{cases} t-2, & t<0 \\ t^2, & 0 \leq t \leq 2 \\ 2t, & t>2. \end{cases}$$

a) $\lim_{t \to 0} g(t)$.

As t approaches 0 from the left.

$$\lim_{t \to 0^-} g(t) = \lim_{t \to 0^-} t-2 = -2.$$

As x approaches from the right

$$\lim_{t \to 0^+} g(t) = \lim_{t \to 0^+} t^2 = 0.$$

The left hand limit and right hand limit exist but they are not equal. Therefore, $\lim_{t \to 0} g(t)$ ~~exists.~~ does not exist.

b) $\lim_{t \to 1} g(t)$.

When t approaches 1 from ~~bot~~ the left, the corresponding function is $g(t) = t^2$. Again when t approaches 1 from the right, the corresponding function is the same as before. So the left hand limit and right hand limit exist and they are equal i.e.

$$\lim_{t \to 1} g(t) = \lim_{t \to 1} t^2 = 1.$$

c) $\lim_{t \to 2} g(t)$

As t approaches 2 from the left,

$$\lim_{t \to 2^-} g(t) = \lim_{t \to 2^-} t^2 = 2^2 = 4.$$

As t approaches 2 from the right

$$\lim_{t \to 2^+} g(t) = \lim_{t \to 2^+} 2t = 4.$$

Since the left hand limit and right hand limit exist and they are equal therefore

$$\lim_{t \to 2} g(t) = 4$$

Ans.

Homework :- A function $f(x)$ is defined as

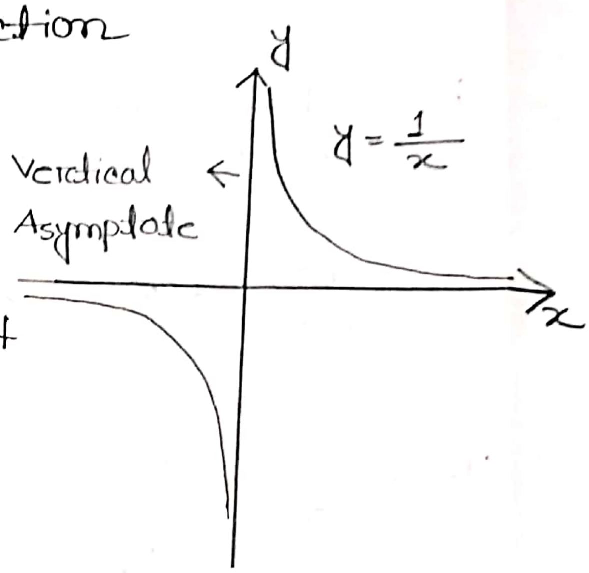
$$f(x) = \begin{cases} \dfrac{1}{x+2}, & x < -2 \\ x^2 - 5, & -2 < x \leq 3 \\ \sqrt{x+13}, & x > 3 \end{cases}$$

Find a) $\lim_{x \to -2} f(x)$ b) $\lim_{x \to 0} f(x)$ c) $\lim_{x \to 3} f(x)$.

Horizontal and Vertical Asymptotes :-

Let us consider the function

$$y = f(x) = \frac{1}{x}.$$



Now we calculate the limit of the function f(x) and $x = 0$.

Then $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{1}{x} = +\infty$.

In each case, the function f(x) either rises or falls without bound getting closer and closer to the vertical line $x=0$. as x approaches 0 from the sides indicated in the limit. The line $\boxed{x=0}$ is called the $\boxed{\text{vertical asymptote}}$ of the curve $y=f(x)$.

Now let us calculate the limit of the function $f(x)$ when x approaches $+\infty$ and $-\infty$.

Then $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \frac{1}{x} = 0$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x} = 0.$$

Here $\lim\limits_{x \to +\infty} f(x) = 0$ and $\lim\limits_{x \to -\infty} f(x) = 0$. Therefore $\boxed{y=0}$ is the $\boxed{\text{horizontal asymptote}}$ of the curve $y = f(x)$.

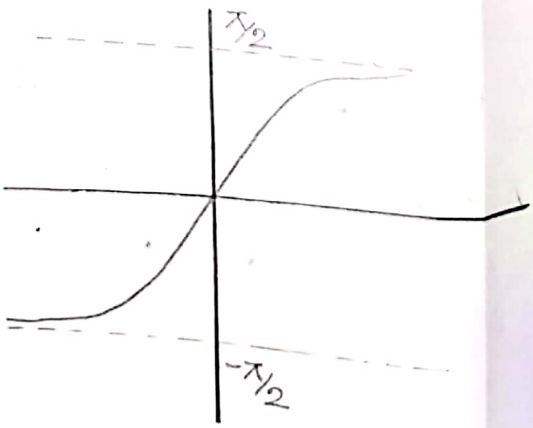
Example :- Find the horizontal asymptote of $f(x) = \tan^{-1}x$.

Solution :- When x approaches $+\infty$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \tan^{-1}x = \frac{\pi}{2}.$$

When x approaches $-\infty$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2}.$$



Therefore $y = \frac{\pi}{2}$ is a horizontal asymptote for f in the positive direction and the line $y = -\frac{\pi}{2}$ is a horizontal asymptote in the negative direction.