Lecture 03 MAT-110

Differentiability :-

A function $f(x)$ is said to be differentiable at x_0 if

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

exists.

The function f' is defined by the form

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is called the derivative of f with respect to x. The domain f' consists of all x in the domain of f for which the limit exists.

Example :- Use the limit definition to find the derivative of $f(x) = \sin x$ at $x = \frac{\pi}{2}$.

Solution :- The derivative of the function using limit definition can be written as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \cdot \lim_{h \to 0} \left(\frac{\sin h}{h} \right)$$

$$= \sin x \times 0 + \cos x \times 1.$$

$$= \cos x.$$

At $x = \frac{\pi}{2}$ $\quad f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0.$

Ans.

Homework :- Find the derivative of $f(x) = \sqrt{x}$ at $x = 9$.

* A function is not differentiable at :

1) Corner points → At corner points the slope of the secant lines have different limits from the left and from the right and hence the two sided limit that defines the derivative does not exist.

2) Vertical tangency → At a point of vertical tangency the slope of the secant line approach $+\infty$ or $-\infty$ from the left and from the right.

Problem :- Prove that $f(x) = |x|$ is not differentiable at $x=0$.

Solution :- The derivative of $f(x)$ is defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

$$\Rightarrow f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}.$$

$$= \lim_{h\to 0} \frac{|h| - 0}{h}.$$

$$= \lim_{h \to 0} \frac{|h|}{h}.$$

Let's calculate the left hand limit and right hand limit.

~~$Lf'(0) = li$~~

The function $|h|$ is defined by

$$|h| = \begin{cases} h, & h \geq 0. \\ -h, & h < 0. \end{cases}$$

Then,

$$Lf'(0) = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h}.$$

$$= \lim_{h \to 0^-} -1 = -1.$$

$$Rf'(0) = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h}$$

$$= \lim_{h \to 0^+} 1 = 1.$$

Since the left hand limit and right hand limit of $f'(0)$ are not equal, then the limit

$\lim_{h\to 0} \frac{|h|}{h}$ does not exist.

Therefore, $f(x) = |x|$ is not differentiable at $x=0$.

[Proved]

Problem :- Show that

$$f(x) = \begin{cases} x^2+2, & x \leq 1 \\ x+2, & x > 1 \end{cases}$$

is continuous but not differentiable at $x = 1$.

Solution :- Given that

$$f(x) = \begin{cases} x^2+2, & x \leq 1 \\ x+2, & x > 1 \end{cases}$$

At $x=1$, the value of the function is,

$$f(1) = 1^2 + 2 = 3.$$

Now let's see if ~~the~~ $\lim_{x\to 1} f(x)$ exists.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2+2 = 1+2 = 3.$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x+2 = 1+2 = 3.$$

Therefore $\lim_{x \to 1} f(x)$ exists and $\lim_{x \to 1} f(x) = 3$.

We observe that the limiting value of the function ~~at x~~ as $x \to 1$ and the functional value at $x=1$ ~~are eq~~ are equal.

$$\lim_{x \to 1} f(x) = 3 = f(1).$$

Since the function $f(x)$ satisfies all the three conditions, the function $f(x)$ is continuous at $x=1$.

To show the differentiability of the function $f(x)$ at $x=1$, we set

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}.$$

$$\cancel{= \lim_{h \to 0}}$$

Since the behavior of the function changes at $x=1$, we have to calculate the left hand limit and right hand limit at $x=1$.

$$Lf'(1) = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^-} \frac{(1+h)^2+2-(1^2+2)}{h}.$$

$$= \lim_{h\to 0^-} \frac{2h+h^2}{h} = \lim_{h\to 0^-} 2+h$$

$$= 2.$$

$$Rf'(1) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{1+h+2-3}{h}$$

$$= \lim_{h\to 0^+} \frac{h}{h} = 1.$$

Since $Lf'(1) \neq Rf'(1)$. so the function is not differentiable.

[Showed]

Techniques of Differentiation :-

We should learn ~~some~~ basic differentiation of some known functions from the reference book. You can find them on internet also.

Theorem :- If a function f is differentiable at x_0, then f is continuous at x_0. If f is continuous at x_0, then it ~~is not~~ can not be said that, ~~a f~~ the function is differentiable at x_0.

Rules of Differentiation :-

Sum Rule :-

$$\frac{d}{dx}\left(Af(x) + Bg(x)\right) = A\frac{df}{dx} + B\frac{dg}{dx}$$

Product Rule :-

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}$$

Quotient Rule :-

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{[g(x)]^2} \quad [g(x) \neq 0]$$

Example :- Evaluate, $\frac{dy}{dx}$ of $y = (4x^2-1)(7x^3+x)$ by product rule of differentiation.

Solution :- Given that,

$$y = \underbrace{(4x^2-1)}_{u}\underbrace{(7x^3+x)}_{v}$$

$$\therefore \frac{dy}{dx} = (4x^2-1)\frac{d}{dx}(7x^3+x) + (7x^3+x)\frac{d}{dx}(4x^2-1)$$

$$= (4x^2-1)(21x^2+1) + (7x^3+x).8x$$

$$= 84x^4 - 21x^2 + 4x^2 - 1 + 56x^4 + 8x^2$$

$$= 140x^4 - 9x^2 - 1.$$

Ans.

Example :- Evaluate $\frac{dy}{dx}$ of $y = \frac{x^3+2x^2-1}{x+5}$ by quotient rule of differentiation.

Solution :- Given that,

$$y = \frac{x^3+2x^2-1}{x+5}$$

$$\therefore \frac{dy}{dx} = \frac{(x+5)\frac{d}{dx}(x^3+2x^2-1) - (x^3+2x^2-1)\frac{d}{dx}(x+5)}{(x+5)^2}$$

$$= \frac{(x+5)(3x^2+4x) - (x^3+2x^2-1)}{(x+5)^2}.$$

$$= \frac{3x^3+15x^2+4x^2+20x-x^3-2x^2+1}{(x+5)^2}$$

$$= \frac{2x^3+17x^2+20x+1}{(x+5)^2}.$$

Practice Sheet :-

Chapter 1.2 → 31, 37, 38, 39, 40

Chapter 1.5 → 11 - 22, 29, 30

Chapter 2.2 → 9, 14, 46, 47-50,

Chapter 2.3 → 1-8, 24,

Practice Sheet

Exercise 2.4 – (5-20), 21, 24

Exercise 2.6 – Example → 5, 6,
Ex – 3, 4, 7-40, 43-54.

Exercise 2.3 → 29, 30, 31, 32, 41-48

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