Lecture - 02

Limits of Rational Functions :-

Our technique for determining the end behavior of a rational function is to divide each term in the numerator and denominator by the highest power of x that occurs in the denominator.

Example :- Find $\lim_{x \to +\infty} \frac{3x+5}{6x-8}$.

Solution :- $\lim_{x \to +\infty} \frac{3x+5}{6x-8}$.

We divide each term in the numerator and denominator by the highest power of x that occurs in the denominator namely $x^1 = x$. We obtain

$$\lim_{x \to +\infty} \frac{3x+5}{6x-8} = \lim_{x \to +\infty} \frac{3+\frac{5}{x}}{6-\frac{8}{x}}$$

$$= \frac{\lim_{x \to +\infty} \left(3+\frac{5}{x}\right)}{\lim_{x \to +\infty} \left(6-\frac{8}{x}\right)}$$

$$= \frac{\lim_{x \to +\infty} 3 + \lim_{x \to +\infty} \frac{5}{x}}{\lim_{x \to +\infty} 6 - \lim_{x \to +\infty} \frac{8}{x}}$$

$$= \frac{3 + 5 \lim_{x \to +\infty} \frac{1}{x}}{6 - 8 \lim_{x \to +\infty} \frac{1}{x}} = \frac{3+0}{6-0} = \frac{1}{2}.$$

Ans.

~~Try yourself~~

Homework :- Find

a) $\lim_{x \to -\infty} \frac{4x^2 - x}{2x^3 - 5}$ b) $\lim_{x \to +\infty} \frac{5x^3 - 2x^2 + 1}{1 - 3x}$.

Ans. a) 0, b) $-\infty$.

Example :- Find. $\lim_{x \to +\infty} \frac{\sqrt{x^2+2}}{3x-6}$.

Solution :- Let us find out $\lim_{x \to +\infty} \frac{\sqrt{x^2+2}}{3x-6}$.

It would be helpful to manipulate the function so that the powers of x are transformed to powers of $\frac{1}{x}$. This can be achieved by dividing the numerator and denominator by $|x|$ and using the fact that $\sqrt{x^2} = |x|$.

$$\lim_{x\to+\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x\to+\infty} \frac{\frac{\sqrt{x^2+2}}{|x|}}{\frac{3x-6}{|x|}}.$$

$$= \lim_{x\to+\infty} \frac{\frac{\sqrt{x^2+2}}{\sqrt{x^2}}}{\frac{3x-6}{x}}$$

$$= \lim_{x\to+\infty} \frac{\sqrt{1+\frac{2}{x^2}}}{3-\frac{6}{x}}$$

$$= \frac{\sqrt{\lim_{x\to+\infty} 1 + \lim_{x\to+\infty} \frac{2}{x^2}}}{\lim_{x\to+\infty} 3 - \lim_{x\to+\infty} \frac{6}{x}}$$

$$= \frac{\sqrt{1+(2\cdot 0)}}{3-(6\cdot 0)}.$$

$$= \frac{1}{3}.$$

Ans.

Homework :- Find $\lim_{x\to-\infty} \frac{\sqrt{x^2+2}}{3x-6}$.

L' Hôpital's Rule :-

Some indeterminate forms are of following type :-

$$\frac{0}{0}, \frac{\infty}{\infty}, 0.\infty, 0^0, \infty^0, 1^\infty.$$

Applying L' Hôpital's Rule :-

1. Check that the limit of $\frac{f(x)}{g(x)}$ is an indeterminate ~~to~~ form of type $\frac{0}{0}$ / $\frac{\infty}{\infty}$.

2. Differentiate f and g separately.

3. Find the limit of $\frac{f'(x)}{g'(x)}$. If this limit is finite, $+\infty$ or $-\infty$, then it is equal to the limit of $\frac{f(x)}{g(x)}$.

Example :- Find $\lim_{x \to 0} \frac{\sin x}{x}$

Solution :- $\lim_{x \to 0} \frac{\sin x}{x}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \frac{\cos x}{1}$ [L' Hôpital's rule]

$= \frac{1}{1} = 1.$

Example :- Find $\lim_{x\to 0} \frac{\tan x}{x}$.

Solution :- $\lim_{x\to 0} \frac{\tan x}{x}$. $\left[\frac{0}{0}\text{-form}\right]$

$= \lim_{x\to 0} \frac{\sec^2 x}{1}$ [L' Hôpital's Rule].

$= \frac{1}{1}$

$= 1$.

Example :- Find $\lim_{x\to +\infty} \frac{x}{e^x}$.

Solution :- $\lim_{x\to +\infty} \frac{x}{e^x}$ $\left[\frac{\infty}{\infty}\text{ form}\right]$

$= \lim_{x\to +\infty} \frac{1}{e^x}$. [L' Hôpital's Rule]

$= 0$.

Example :- Find $\lim_{x\to +\infty} \frac{3x+5}{6x-8}$

Solution :- $\lim_{x\to +\infty} \frac{3x+5}{6x-8}$ $\left[\frac{\infty}{\infty}\text{ form}\right]$

$= \lim_{x\to +\infty} \frac{3}{6} = \frac{1}{2}$. [L' Hôpital's Rule]

Indeterminate Form of Type $0 \cdot \infty$

If we get the indeterminate form of type $0 \cdot \infty$ then we transform the type into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ of type.

Example: Find $\lim_{x \to \frac{\pi}{4}} (1 - \tan x) \sec 2x$.

Solution: $\lim_{x \to \frac{\pi}{4}} (1 - \tan x) \sec 2x \quad [0 \cdot \infty \text{ form}]$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(1 - \tan x)}{\frac{1}{\sec 2x}}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(1 - \tan x)}{\cos 2x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{-\sec^2 x}{-2\sin 2x} \quad [\text{L' Hôpital's rule}]$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\sec^2 x}{2\sin 2x}$$

$$= \frac{2}{2} = 1$$ Ans.

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Indeterminate forms of Type 0^0, ∞^0, 1^∞ :-

Indeterminate forms of these type can sometimes be evaluated by first introducing a dependent variable

$$y = [f(x)]^{g(x)}$$

$$\Rightarrow \ln y = g(x) \ln [f(x)]$$

Example :- Find $\lim\limits_{x \to 0} (1 + \sin x)^{1/x}$.

Solution :- Let $y = (1 + \sin x)^{1/x}$

$$\Rightarrow \ln y = \ln \left[(1 + \sin x)^{1/x} \right]$$

$$\Rightarrow \ln y = \frac{1}{x} \ln (1 + \sin x).$$

$$\Rightarrow \lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{1}{x} \ln (1 + \sin x)$$

$$\lim_{x \to 0} \frac{\ln(1 + \sin x)}{x} = \lim_{x \to 0} \frac{\frac{\cos x}{1 + \sin x}}{1}$$

$$= \frac{1}{1} = 1.$$

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Therefore, $\lim_{x\to 0} \ln y = 1$.

$$\Rightarrow \lim_{x\to 0} e^{\ln y} = e^{1}$$

$$\Rightarrow \lim_{x\to 0} y\, e^{\ln} = e$$

$$\Rightarrow \lim_{x\to 0} y = e \qquad [e^{\ln} = 1]$$

$$\Rightarrow \lim_{x\to 0} (1+\sin x)^{1/x} = e.$$

Ans.

Example :- Find $\lim_{x\to\infty} \left(1+\frac{2}{7x}\right)^{3x+5}$.

Continuity :-

A function f is said to be $\boxed{\text{continuous}}$ at $x=c$ if the following conditions are satisfied :-

1) $f(c)$ is defined

2) $\lim_{x\to c} f(x)$ exists

3) $\lim_{x\to c} f(x) = f(c)$

Example :- Determine whether the following func is continuous at $x=3$.

$$f(x)= \begin{cases} x+2, & -1\leqslant x<3 \\ 14-x^2, & 3\leqslant x<5. \end{cases}$$

Solution :- Given that,

$$f(x) = \begin{cases} x+2, & -1\leqslant x<3 \\ 14-x^2, & 3\leqslant x<5. \end{cases}$$

The value of the function at $x=3$ is,

$$f(3) = 14-3^2 = 14-9 = 5.$$

Now let's see if $\lim\limits_{x\to 3} f(x)$ exists.

As x approaches 3 from the left,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x+2 = 3+2=5.$$

As x approaches 3 from the right,

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 14-x^2 = 14-9 = 5.$$

The left hand limit and right hand limit exist and they are equal. Therefore $\lim_{x \to 3} f(x)$ exists and

$$\lim_{x \to 3} f(x) = 5.$$

Here we observe that

$$\lim_{x \to 3} f(x) = f(3) = 5.$$

Since the function $f(x)$ satisfies all the three conditions of continuity, therefore the function $f(x)$ is continuous at $x = 3$

Ans.

Homework :- A function $g(x)$ is defined as

$$g(x) = \begin{cases} 5 - x, & -1 \leq x \leq 2 \\ x^2 - 1, & 2 < x < 3 \end{cases}$$

Determine whether the function $g(x)$ is continuous at $x = 2$

Definition:- A function f is said to be continuous on a closed interval $[a,b]$ if the following conditions are satisfied :-

1) f is continuous on the open interval (a,b)

2) f is continuous from the right at a i.e.

$$\lim_{x \to a^+} f(x) = f(a)$$

3) f is continuous from the left at b i.e.

$$\lim_{x \to b^-} f(x) = f(b)$$

Problem:- A function is defined as $f(x) = \sqrt{x-2}$ Is the function continuous on the interval $[2, +\infty)$.

Solution:- Given that,

$$f(x) = \sqrt{x-2}$$

Let us consider the open interval $(2, +\infty)$. Let c be any point on $(2, +\infty)$.

The value of the function $f(x)$ at $x = c$.

$$f(c) = \sqrt{c-2}$$

Hence $\lim_{x\to c} f(x) = \lim_{x\to c} \sqrt{x-2} = \sqrt{c-2}$.

For any point $c \in (2,+\infty)$ the function is defined and is continuous. It is true for all points $c \in (2,+\infty)$.

Therefore the function $f(x)$ is continuous on the open interval $(2,+\infty)$.

As x approaches ~~from~~ 2 from the right.

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \sqrt{x-2} = 0.$$

At $x=2$, $f(2)=0$.

Therefore, $\lim_{x\to 2^+} f(x) = f(2)$. It means that the function $f(x)$ is continuous from the right at $x=2$.

As x approaches $+\infty$ from the left, there will be no point for which the function is not continuous.

~~A~~ The value of the function at the end point will approach $+\infty$.

~~So we can say~~

~~So we can say that~~ $\cancel{\lim_{x \to +\infty} f(x) = f}$

So we can say that the function $f(x)$ satisfies all the three conditions of continuity. Therefore the function $f(x)$ is continuous on the interval $[2, +\infty)$.

Homework:- What can you say about the continuity of the function $f(x) = \sqrt{9-x^2}$ on the closed interval $[-3, 3]$?