Quiz 1 (Section : 17)
MAT110 : Differential Calculus and Coordinate Geometry
BRAC University

Date: 13/06/2023 Time: 40 minutes Total Mark: 30

Name:

ID:

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1. Find [4+5+3]

   (a) $\lim_{x\to 0} \dfrac{\sin(\pi \cos^2 x)}{x^2}$ (b) $\lim_{x\to 1}(1-x^2)^{\frac{1}{\ln(1-x)}}$ (c) $\lim_{x\to\infty} \dfrac{3^x+3^{-x}}{3^x-3^{-x}}$

2. Find the derivative of $f(x)g(x)$ by the idea of limit definition. [5]

3. A function $h(x)$ is defined as: [6]

$$h(x) = \begin{cases} \sqrt{|x|}, & \text{if } x \geq 0 \\ -\sqrt{|x|}, & \text{if } x < 0 \end{cases}$$

   Find the continuity and differentiability of the function at $x = 0$.

4. Find the derivative of the following functions :

   (a) $f(t) = e^t \ln t$ (b) $S(y) = \dfrac{y7^y}{y^2+1}$ [3+4]

Please start writing from here

1.

a) $\lim_{x \to 0} \dfrac{\sin(\pi \cos^2 x)}{x^2}$ $\quad \dfrac{0}{0}$-form

$= \lim_{x \to 0} \dfrac{\cos(\pi \cos^2 x) \cdot \pi(-2\cos x \sin x)}{2x}$

$= \lim_{x \to 0} \dfrac{-2\pi \cos x \sin x \cos(\pi \cos^2 x)}{2x}$

$= \lim_{x \to 0} \dfrac{-\pi \sin x \cos x \cos(\pi \cos^2 x)}{x}$

$= \lim_{x \to 0} \dfrac{-\pi \left(\sin x \cos x \left(-\sin(\pi \cos^2 x)\right) \cdot 2\pi \cos x(-\sin x) + \cos(\pi \cos^2 x)\left(-\sin^2 x + \cos^2 x\right) \right)}{x}$

$= \lim_{x \to 0} \cancel{\pi \left(\sin(\pi \cos^2 x) \sin x \cos x \right.}$

$= \dfrac{-\pi\left(+\cos(\pi \cdot 1)\left(-\sin^2 0 + \cos^2 0\right)\right)}{1}$

$= \dfrac{\pi}{1}$

$= \pi$

b) $\lim_{x \to 1} (1-x^2)^{\frac{1}{\ln(1-x)}}$

Let, $y = (1-x^2)^{\frac{1}{\ln(1-x)}}$

$\Rightarrow \ln y = \ln\left\{(1-x^2)^{1/\ln(1-x)}\right\}$

$\Rightarrow \ln y = \frac{1}{\ln(1-x)} \ln(1-x^2)$

$\Rightarrow \lim_{x \to 1} \ln y = \lim_{x \to 1} \frac{\ln(1-x^2)}{\ln(1-x)}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$\Rightarrow \lim_{x \to 1} \ln y = \lim_{x \to 1} \frac{\frac{1}{1-x^2}(-2x)}{\frac{1}{1-x}(-1)}$

$\Rightarrow \lim_{x \to 1} \ln y = \lim_{x \to 1} \left(\frac{-2x}{1-x^2}\right)\left(-(1-x)\right)$

$\Rightarrow \lim_{x \to 1} \ln y = \lim_{x \to 1} \frac{2x}{1+x}$

$\Rightarrow \lim_{x \to 1} \ln y = \frac{2}{2}$

$\Rightarrow \lim_{x \to 1} \ln y = 1.$

$\Rightarrow \lim_{x \to 1} e^{\ln y} = e^1 \Rightarrow \lim_{x \to 1} y e^{\ln} = e \Rightarrow \lim_{x \to 1} y = e$ Ans.

) $\lim_{x \to \infty} \dfrac{3^x + 3^{-x}}{3^x - 3^{-x}}$

$= \lim_{x \to \infty} \dfrac{3^x(1+3^{-2x})}{3^x(1-3^{-2x})}$

$= \lim_{x \to \infty} \dfrac{1+3^{-2x}}{1-3^{-2x}}$

$= \dfrac{1}{1} = 1.$

2. By limit definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

$$\frac{d}{dx}\left[f(x)\, g(x)\right] = \lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x)}{h}.$$

$$= \lim_{h \to 0} \frac{f(x+h)\, g(x+h) + f(x)\, g(x+h) - f(x)\, g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{g(x+h)\left(f(x+h) - f(x)\right) + f(x)\left(g(x+h) - g(x)\right)}{h}$$

$$= \lim_{h \to 0} \frac{f(x)\left(g(x+h) - g(x)\right) + g(x+h)\left(f(x+h) - f(x)\right)}{h}.$$

$$= f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + g(x+h) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$= f(x)g'(x) + g(x)f'(x)$

$= f(x)g'(x) + f'(x)g(x)$.

Ans.

3. Given that

$$f(x) = \begin{cases} \sqrt{|x|}, & x \geq 0. \\ -\sqrt{|x|}, & x < 0. \end{cases}$$

~~$f(0) = \sqrt{|x|}$~~

$f(0) = \sqrt{|0|} = 0$.

As x approaches from left

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -\sqrt{|x|} = -\sqrt{|0|} = 0$$

~~lim~~ As x approaches from right

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{|x|} = \sqrt{|0|} = 0.$$

Then, $\lim_{x \to 0} f(x)$ exists and $\lim_{x \to 0} f(x) = 0$.

Again, $f(0) = 0 = \lim_{x \to 0} f(x)$.

So the function is continuous at $x = 0$.

Now

$$Lf'(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^-} \frac{-\sqrt{|0+h|} - \sqrt{|0|}}{h}$$

$$= \lim_{h \to 0^-} \frac{-\sqrt{|h|} - 0}{h}$$

$$= \lim_{h \to 0^-} \frac{-\sqrt{h}}{h} = -\lim_{h \to 0^-} \frac{1}{\sqrt{h}} = -\infty.$$

$$Rf'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{\sqrt{|0+h|} - \sqrt{|0|}}{h}$$

$$= \lim_{h \to 0^+} \frac{\sqrt{|h|} - 0}{h}$$

$$= \lim_{h \to 0^+} \frac{\sqrt{h}}{h}$$

$$= \lim_{h \to 0^+} \frac{1}{\sqrt{h}}$$

$$= +\infty.$$

Therefore, the function is not differentiable at $x=0$.

4. Given that

a) $$f(t) = e^t \ln t.$$

$$\Rightarrow \frac{df(t)}{dt} = e^t \frac{d}{dt}(\ln t) + \ln t \frac{d}{dt}(e^t)$$

$$= \frac{e^t}{t} + e^t \ln t.$$

b) $$S(y) = \frac{y \cdot 7^y}{y^2+1}.$$

$$= \frac{(y^2+1)\frac{d}{dy}(y\,7^y) - (y \cdot 7^y)\frac{d}{dy}(y^2+1)}{(y^2+1)^2}.$$

$$= \frac{(y^2+1)(7^y + y\,7^y \ln y) - 2y^2 \cdot 7^y}{(y^2+1)^2}.$$

$$= \frac{(7^y + 7^y y \ln y)(y^2+1) - 2y^2 7^y}{(y^2+1)^2}.$$

Ans.