Example :- Find the intervals on which $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ is increasing or decreasing.

Solution : Given that,

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$$

$$f'(x) = 12x^3 + 12x^2 - 24x$$

$$= 12x(x^2 + x - 2) \qquad \Big| \quad x = 0, -2, 1$$

$$= 12x(x^2 + 2x - x - 2)$$

$$= 12x(x+2)(x-1)$$

The sign analysis of f' is given in the following table :-

Interval	$12x(x+2)(x-1)$	$f'(x)$	Conclusion
$x < -2$	$(-)(-)(-)$	$-$	Decreasing on $(-\infty, -2]$
$-2 < x < 0$	$(-)(+)(-)$	$+$	Increasing on $[-2, 0]$
$0 < x < 1$	$(+)(+)(-)$	$-$	Decreasing on $[0, 1]$
$x > 1$	$(+)(+)(+)$	$+$	Increasing on $[1, +\infty)$

Therefore, f is increasing on the intervals

$[-2, 0]$ and $[1, +\infty)$ and decreasing on the intervals $(-\infty, -2]$ and $[0, 1]$.

Ans.

Practice Problems :-

Exercise 3.5 → 5-10, 11-16.

Exercise 10.1 → 53, 54, 45-52.

Exercise 4.1 → 15-32 (a, b).

Exercise 4.2 → 7-14, 25-32, 33-50.

Concavity :- Although the sign of the derivative of f reveals where the graph of f is increasing or decreasing, it does not reveal the direction of curvature. On intervals where the graph of f has upward curvature we say that f is concave up and on intervals where the graph of f has downward curvature we say that f is concave down.

The following two points suggest two ways to characterize the concavity of a differentiable function on an open interval :-

1. f is concave up on an open interval if its tangent lines have increasing slopes on that interval and is concave down if they have decreasing slopes.

2. f is concave up on an open interval if its graph lies above its tangent lines on that interval and is concave down if it lies below its tangent lines.

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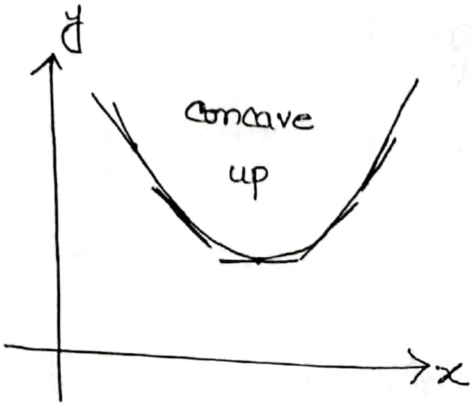
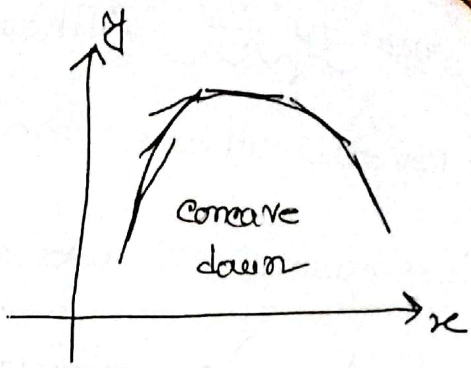


Increasing slopes



Decreasing slopes

Theorem :- Let f be twice differentiable on an open interval.

a) If $f''(x) > 0$ for every value of x in the open interval the f is concave up on that interval.

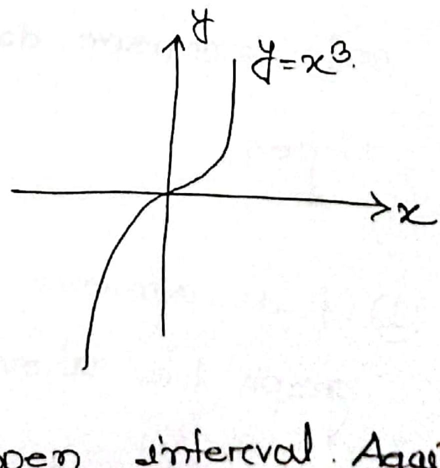
b) If $f''(x) < 0$ for every value of x in the open interval, then f is concave down on that interval.

Example :- Let,

$$f(x) = x^3.$$

$$f'(x) = 3x^2$$

$$f''(x) = 6x.$$



For $x < 0$, $f''(x) < 0$ on the open interval. Again for $x > 0$, $f''(x) > 0$ on the open interval.

Therefore, $f(x) = x^3$ is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, +\infty)$.

Inflection Point:- If f is continuous on an open interval containing a value of x_0 and if f changes the direction of its concavity at the point $(x_0, f(x_0))$, then we say that f has an inflection point at x_0 and we call the point $(x_0, f(x_0))$ on the graph of f an inflection point of f.

Example:- If the function $f(x)$ is given as $f(x) = x^3 - 3x^2 + 1$ then find the intervals on which

a) f is increasing

b) f is decreasing

c) f is concave up

d) f is concave down

e) Find all inflection points.

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Solution :— Given that,

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x-2).$$

$$f''(x) = 6x - 6 = 6(x-1)$$

The sign analysis of these derivatives is shown in the following tables :—

Interval	$(3x)(x-2)$	$f'(x)$	Conclusion
$x<0$	$(-)(-)$	$+$	f is increasing on $(-\infty, 0]$
$0<x<2$	$(+)(-)$	$-$	f is decreasing on $[0,2]$
$x>2$	$(+)(+)$	$+$	f is increasing on $[2,+\infty)$.

The inflection points are found by setting

$$f''(x) = 0.$$

$$\Rightarrow 6(x-1) = 0.$$

$$\Rightarrow x = 1$$

Interval	$6(x-1)$	$f''(x)$	Conclusion
$x<1$	$-$	$-$	concave down $(-\infty, 1)$
$x>1$	$+$	$+$	Concave up $(1, +\infty)$

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Example: $f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^4 + \frac{1}{6}x^3 + 3x^2 + 5x - 2$

$$f'(x) = \frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{6}x^2 + 6x + 5$$

$$f''(x) = x^3 - 4x^2 + x + 6.$$
$$= (x+1)(x-2)(x-3)$$

The inflection points are ~~fallo~~ found by setting

$$f''(x) = 0$$

$$\Rightarrow x^3 - 4x^2 + x + 6 = 0$$

$$\Rightarrow x^3 + x^2 - 5x^2 - 5x + 6x + 6 = 0$$

$$\Rightarrow x^2(x+1) - 5x(x+1) + 6(x+1) = 0$$

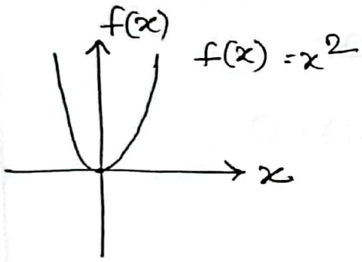
$$\Rightarrow (x+1)(x^2 - 5x + 6) = 0$$

$$\Rightarrow (x+1)(x-2)(x-3) = 0.$$

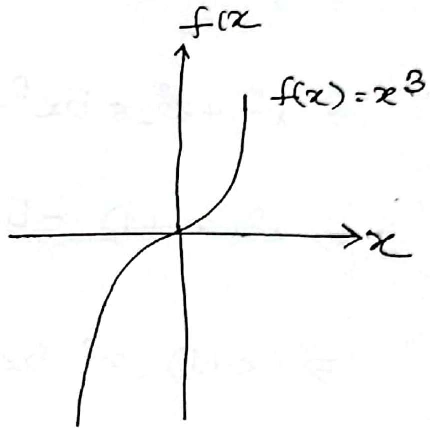
$$\Rightarrow x = -1, 2, 3.$$

Intervals	$(x+1)(x-2)(x-3)$	Sign of f'	Conclusion
$x < -1$	$(-)(-)(-)$	$-$	Concave down $(-\infty, -1)$
$-1 < x < 2$	$(+)(-)(-)$	$+$	Concave up $(-1, 2)$
$2 < x < 3$	$(+)(+)(-)$	$-$	Concave down $(2, 3)$
$x > 3$	$(+)(+)(+)$	$+$	Concave up $(3, +\infty)$

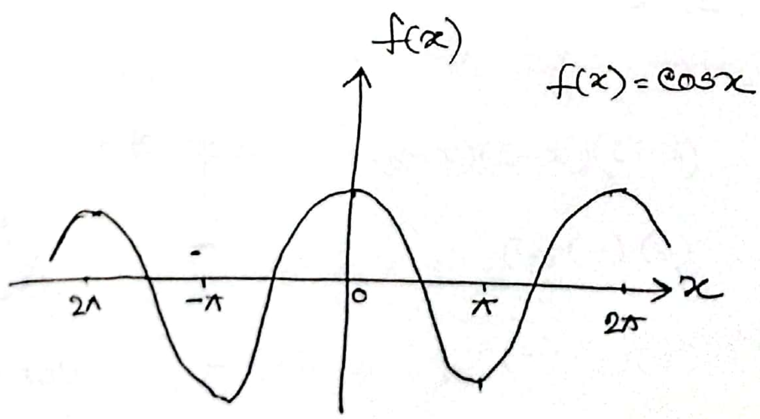
Relative Maxima and Minima :— A function f is said to have a relative maximum at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in the interval. Similarly, f is said to have a relative minimum at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is, $f(x_0) \leq f(x)$ for all x in the interval.



Relative minimum at $x=0$ but no relative maximum



No relative maximum or minimum



Relative maxima at all even multiples of π and relative minima at all odd multiples of π

Critical Point:- In general, we define a critical point for a function f to be a point in the domain of f at which either the graph of f has a horizontal tangent line or f is not differentiable. To distinguish between the two types of critical points we call x stationary point of f if $f'(x) = 0$.

Example:- Find all critical points of $f(x) = x^3 - 3x + 1$.

Solution:- The function f, being a polynomial, is differentiable everywhere, so its critical points are all stationary points. To find these points, we must solve the equation $f'(x) = 0$. Since

$$f'(x) = 3x^2 - 3 = 0$$

$$\Rightarrow 3(x^2 - 1) = 0$$

$$\Rightarrow x = -1, 1 .$$

Therefore, the critical points occur at $x = -1$, and $x = 1$

Example:- Find all critical points of $f(x) = 3x^{5/3} - 15x^{2/3}$.

Solution:- Given that,

$$f(x) = 3x^{5/3} - 15x^{2/3}$$

$$f'(x) = 5x^{2/3} - 10x^{-1/3}$$

$$= \frac{5(x-2)}{x^{1/3}}$$

We see from this that $f'(x) = 0$ if $x = 2$ and $f'(x)$ is undefined at $x = 0$. Thus $x = 0$ and $x = 2$ are critical points and $x = 2$ is a stationary point.

First Derivative Test:- Let us suppose that f is continuous at a critical point x_0.

a) If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, the f has a relative maximum at x_0.

b) If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending from x_0, then f has relative minimum at x_0

c) If $f'(x)$ has the same sign on an open interval extending left from x_0 as it does on an open interval extending right from x_0, then f does not have a relative maximum or minimum at x_0.

Example :- The function $f(x) = 3x^{5/3} - 15x^{2/3}$ has a relative maximum at $x=0$ and a relative minimum at $x=2$. Confirm this using first derivative test.

Solution :- Given that,

$$f(x) = 3x^{5/3} - 15x^{2/3}$$

$$f'(x) = \frac{5(x-2)}{x^{1/3}}$$

The sign of f' changes from + to − at $x=0$ so there is a relative maximum at that point. The sign changes from − to + at $x=2$, there is a relative minimum at that point.

Second Derivative Test :— let us suppose that f is twice differentiable at the point x_0.

a) If $f(x_0) = 0$ and $f''(x_0) > 0$, then f has a relative minimum at x_0.

b) If $f(x_0) = 0$ and $f''(x_0) < 0$, then f has a relative maximum at x_0.

c) If $f'(x_0) = 0$ and $f''(x_0) = 0$ then the test is inconclusive, i.e. f may have a relative maximum, a relative minimum, or neither at x_0.

Example :— Find the relative extrema of

$$f(x) = 3x^5 - 5x^3$$

Solution :— Given that

$$f(x) = \cancel{15}\ 3x^5 - 5x^3$$

$$f'(x) = 15x^4 - 15x^2$$

$$f''(x) = 60x^3 - 30x$$

$$= 30x(x^2 - 1)$$

Solving $f'(x) = 0$.

$\Rightarrow x = 0, -1, 1 \rightarrow$ Stationary Points

	$30x(2x^2-1)$	$f''(x)$	Second derivative Test
$x=-1$	-30	$-$	f has a relative maximum
$x=0$	0	0	Inconclusive
$x=1$	30	$+$	f has relative minimum

The test is inconclusive at $x=0$, so we will try the first derivative test at that point.

Interval	$15x^2(x+1)(x-1)$	$f'(x)$
$-1<x<0$	$(+)(+)(-)$	$-$
$0<x<1$	$(+)(+)(-)$	$-$

Since there is no sign change in f' at $x=0$, there is neither a relative maximum nor a relative minimum at that point.

General Guidelines for Sketching a Graph of a Function :-

1. Domain
2. Intercepts
3. Symmetry Test
4. Asymptotes
5. Intervals of increase and decrease.
6. Local maximum and minimum
7. Concavity and Points of inflection
8. Sketch the curve.