Example — If $y = \tan^{-1}x$ then show that
$(1+x^2)y_{n+2} + 2(n+1)x\,y_{n+1} + n(n+1)y_n = 0$.

Solution:— Given that,

$$y = \tan^{-1}x$$

$$\Rightarrow y_1 = \frac{1}{1+x^2}.$$

$$\Rightarrow (1+x^2)y_1 = 1$$

Again differentiating,

$$\frac{d}{dx}\left[(1+x^2)\,y_1\right] = \frac{d}{dx}[1]$$

$$\Rightarrow (1+x^2)y_2 + y_1\frac{d}{dx}(1+x^2) = 0.$$

$$\Rightarrow (1+x^2)y_2 + 2xy_1 = 0. \quad \cdots (1)$$

Applying Leibnitz theorem in equation (1),

$$\frac{d^n}{dx^n}\left[(1+x^2)y_2\right] + \frac{d^n}{dx^n}\left[2xy_1\right] = 0.$$

$$\Rightarrow \frac{d^n}{dx^n}(y_2)\cdot(1+x^2) + {}^nC_1\frac{d^{n-1}}{dx^{n-1}}(y_2)\frac{d}{dx}(1+x^2) + {}^nC_2\frac{d^{n-2}}{dx^{n-2}}(y_2)\frac{d^2}{dx^2}(1+x^2)$$

$$+\frac{d^n}{dx^n}(y_1)\cdot 2x + {}^nC_1\frac{d^{n-1}}{dx^{n-1}}(y_1)\frac{d}{dx}(2x) = 0.$$

$$\Rightarrow (1+x^2)\, y_{n+2} + n(2x)\, y_{n+1} + \frac{n(n-1)}{2}\, y_n \cdot 2$$

$$+ y_{n+1} \cdot 2x + n\, y_n \cdot 2 = 0$$

$$\Rightarrow (1+x^2)\, y_{n+2} + 2x(n+1)\, y_{n+1} + (n^2-n+2n)\, y_n = 0$$

$$\Rightarrow (1+x^2)\, y_{n+2} + 2x(n+1)\, y_{n+1} + n(n+1)\, y_n = 0.$$

[Showed]

Parametric Equation :- Parametric equation is a type of equation that employs an independent variable called parameter and in which dependent variables are defined as continuous functions of the parameter

Parametric Differentiation :-

Example :- The equation of ellipse is specified by

$$x = a\cos t, \quad y = b\sin t .$$

The $\boxed{\text{slope}}$ of the ellipse is, (Gradient)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t .$$

Therefore, at $t = \frac{\pi}{6}$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = -\frac{b}{a}\cot\left(\frac{\pi}{6}\right)$$

$$= -\frac{\sqrt{3}b}{a}$$

CamScanner

* A tangent line to a parametric curve will be horizontal at those points where $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. i.e $\frac{dy}{dx} = 0$ at such points.

* A tangent line to a parametric curve will be vertical at those points where $\frac{dy}{dt} \neq 0$ and $\frac{dx}{dt} = 0$ i.e. there exists infinite slope.

* When $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ then we get indeterminate form of the slope. We call such a point a singular point.

Example: In a disastrous flight, an experimental paper airplane follows the trajectory of the particle

$$x = t - 3\sin t, \quad y = 4 - 3\cos t$$

but crashes into a wall at time $t = 10s$.

a) At what times was the plane flying horizontally?

b) At what times was the plane flying vertically?

Solution :-

a) The trajectory of the particle is given as

$$x = t - \sin 3t$$

$$y = 4 - 3\cos t$$

Here, $\frac{dy}{dt} = 3\sin t$

$$\frac{dx}{dt} = 1 - 3\cos 3t$$

Since the plane was flying horizontally

$$\frac{dy}{dt} = 3\sin t = 0$$

$$\Rightarrow \sin t = 0$$

$$\Rightarrow \sin t = \sin(0 + n\pi)$$

$$\Rightarrow t = n\pi$$

~~Here, $\frac{dx}{dt} \neq 0$ at po~~

There are four solutions on the interval $0 \leq t \leq 10$.

Therefore, $t = 0, \pi, 2\pi, 3\pi$.

Here, $\frac{dx}{dt} \neq 0$ at points $t = 0, \pi, 2\pi, 3\pi$.

b) The airplane was flying vertically at those times when $\frac{dx}{dt}=0$ and $\frac{dy}{dt} \neq 0$. Setting

$$\frac{dx}{dt} = 0 \Rightarrow 1-3\cos t = 0$$

$$\Rightarrow \cos t = \frac{1}{3}.$$

This equation has three solutions in the time interval $0 \leq t \leq 10$.

$$t = \cos^{-1}\left(\frac{1}{3}\right),\ 2\pi - \cos^{-1}\left(\frac{1}{3}\right),\ 2\pi + \cos^{-1}\left(\frac{1}{3}\right)$$

Here $\frac{dy}{dt} = 3\sin t$ is not zero at these points,

Ans.

Local Linear Approximation :-

A function that is differentiable at x_0 is sometimes said to be locally linear at x_0.

The line that best approximates the graph of f in the vicinity of $P(x_0, f(x_0))$ is the tangent line to the graph of f at x_0, given by the equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

$$\Rightarrow f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

This is called the local linear approximation of f at x_0.

Problem: Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 1$.

Solution: Given that,

$$f(x) = \sqrt{x}.$$

$$f(x_0) = \sqrt{1} = 1$$

Again $$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(x_0) = \frac{1}{2}.$$

Thus, the local linear approximation at $x_0 = 1$ is

$$f(x) \simeq f(x_0) + f'(x_0)(x - x_0)$$

$$\Rightarrow \sqrt{x} \simeq 1 + \frac{1}{2}(x-1)$$

Ans.

Problem: (H.W) Find the local linear approximation of $f(x) = \sin x$ at $x_0 = 0$.

Example: Find the gradient at the point (1, 2) on the curve whose equation is given by

$$x^3 - 5xy^2 + y^3 + 1 = 0$$

Solution: Given that

$$x^3 - 5xy^2 + y^3 + 1 = 0$$

$$\Rightarrow \frac{d}{dx}(x^3 - 5xy^2 + y^3 + 1) = 0.$$

$$\Rightarrow 3x^2 - 5y^2 - 10xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow (-10xy + 3y^2)\frac{dy}{dx} = -3x^2 + 5y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{5y^2 - 3x^2}{3y^2 - 10xy}.$$

At point (1, 2)

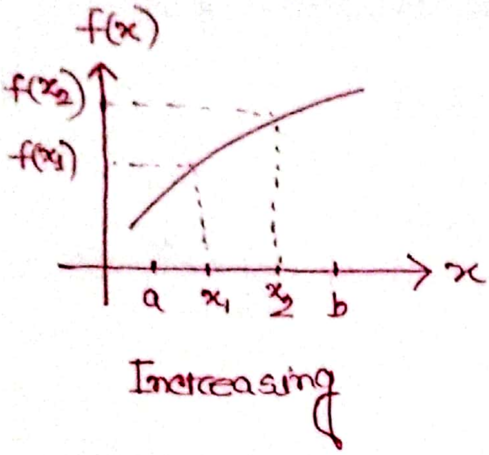
$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{5(2)^2 - 3(1)^2}{3(2)^2 - 10.1.2} = \frac{20-3}{12-20}$$

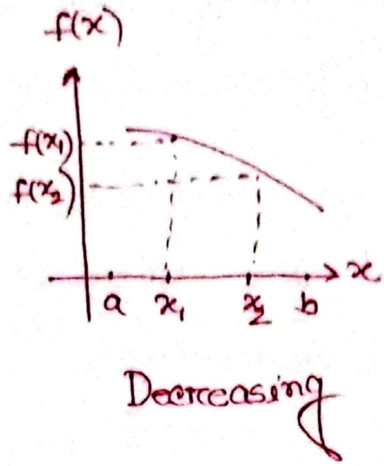
$$= \frac{-17}{8}$$

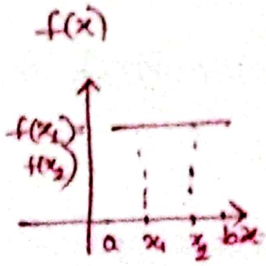
Ans.

Increasing and Decreasing Functions :-

Let f be defined on an interval and let x_1 and x_2 denote points in that interval.







a) The function $f(x)$ is increasing if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

b) The function $f(x)$ is decreasing if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

c) The function $f(x)$ is constant if $f(x_1) = f(x_2)$ whenever $x_1 < x_2$.

Theorem :- Let f be a function that is continuous on a closed interval on the open interval (a,b)

a) If $f'(x) > 0$ for every value of x in (a,b), then f is increasing on $[a,b]$

b) If $f'(x)<0$ for every value of x in (a,b), then f is decreasing on $[a,b]$.

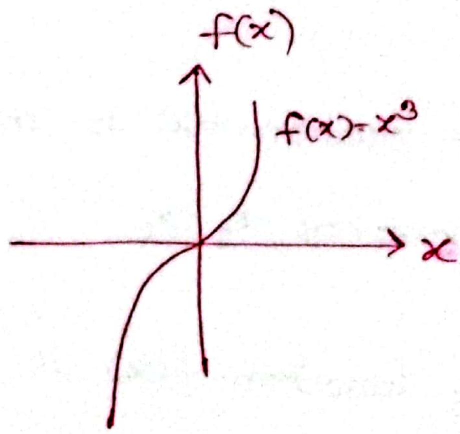
c) If $f'(x)=0$ for every value of x in (a,b), then f is constant on $[a,b]$.

Example: Find the intervals on which $y=x^3$ is increasing or decreasing.

Solution: Given that,

$f(x) = x^3$

$f'(x) - 3x^2$



Thus. $f'(x) > 0$ if $x<0$

$f(x) > 0$ if $x>0$

Since f is continuous everywhere f is increasing on $(-\infty, 0]$ and f is increasing on $[0, +\infty)$. Therefore f is increasing on $(-\infty, +\infty)$.

Ans.